COMP90038
Algorithms and Complexity

Lecture 19: Warshall and Floyd
(with thanks to Harald Søndergaard & Michael Kirley)

Casey Myers
Casey.Myers@unimelb.edu.au
David Caro Building (Physics) 274

Review from Lecture 18: Dynamic Programming

- Dynamic programming is an algorithm design technique that is sometimes applicable when we want to solve a recurrence relation and the recursion involves overlapping instances.
- In Lecture 16 we achieved a spectacular performance improvement in the calculation of Fibonacci numbers by switching from a naïve top-down algorithm to one that solved, and tabulated, smaller sub-problems.
- The bottom-up approach used the tabulated results, rather than solving overlapping sub-problems repeatedly.
- That was a particularly simple example of dynamic programming.

Review from Lecture 18: Dynamic Programming

- Since all values S(1) to S(n) need to be found anyway, we may as well proceed from the bottom up, storing intermediate results in an array S as we go.

- Given an array C that holds the coin values, the recurrence relation tells us what to do:

function $\textsc{CoinRow}(C[1..n])$
    $S[0] \leftarrow 0$
    $S[1] \leftarrow C[1]$
    for $i \leftarrow 2$ to n **do**
        $S[i] \leftarrow max\{S[i-1], S[i-2] + C[i]\}$
    return $S[n]$

Review from Lecture 18: Dynamic Programming

- We can say the same thing formally, as a recurrence relation:

$$S(i) = max\{S(i-1), S(i-2) + v_i\}$$

- This holds for $i > 1$.

- We need two base cases: $S(0) = 0$ and $S(1) = v_1$.

Review from Lecture 18: Dynamic Programming

- In Lecture 5 we looked at the knapsack problem.
- Given n items with
  - weights: $w_1, w_2, \cdots, w_n$
  - values: $v_1, v_2, \cdots, v_n$
  - knapsack of weight capacity W.
- Find the most valuable selection of items that will fit in the knapsack.
- We assume that all entities involved are positive integers.

Review from Lecture 18: Dynamic Programming

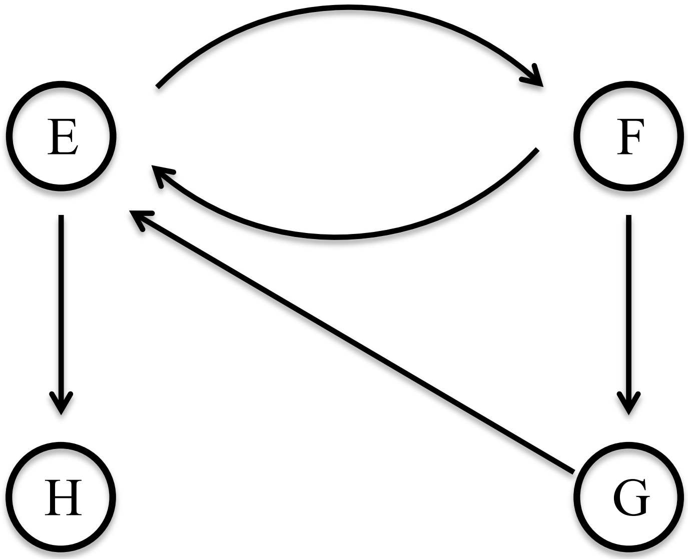
- First fill the leftmost column and top row, then proceed row by row:

for $i \leftarrow 0$ to n **do**
  $K[i, 0] \leftarrow 0$
for $j \leftarrow 1$ to W **do**
  $K[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
  for $j \leftarrow 1$ to W **do**
    if $j < w_i$ **then**
      $K[i, j] \leftarrow K[i-1, j]$
    else
      $K[i, j] \leftarrow max(K[i-1, j], K[i-1, j-w_i] + v_i)$
return $K[n, W]$

- The algorithm has time (and space) complexity $\Theta(nW)$.

Directed Graphs

- A directed graph:

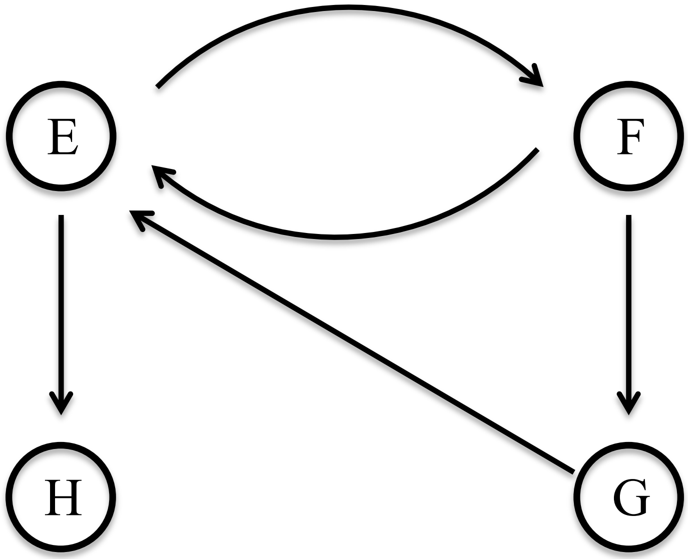


- And the adjacency matrix for this graph:

$$\begin{array}{cc} & \begin{array}{cccc} E & F & G & H \end{array} \\ \begin{array}{c} E \\ F \\ G \\ H \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Directed Graphs

- A directed graph:



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$$\begin{array}{c} \\ E \\ F \\ G \\ H \end{array} \begin{array}{c} \begin{array}{cccc} E & F & G & H \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Dynamic Programming and Graphs

- In the last lecture we looked at dynamic programming.
- We now apply dynamic programming to two graph problems: For dynamic programming to be useful, the optimality principle must hold:
  - Computing the transitive closure of a directed graph; and
  - Finding shortest distances in weighted directed graphs.

Warshall's Algorithm

- Warshall's algorithm computes the transitive closure of a binary relation (or a directed graph), presented as a matrix.

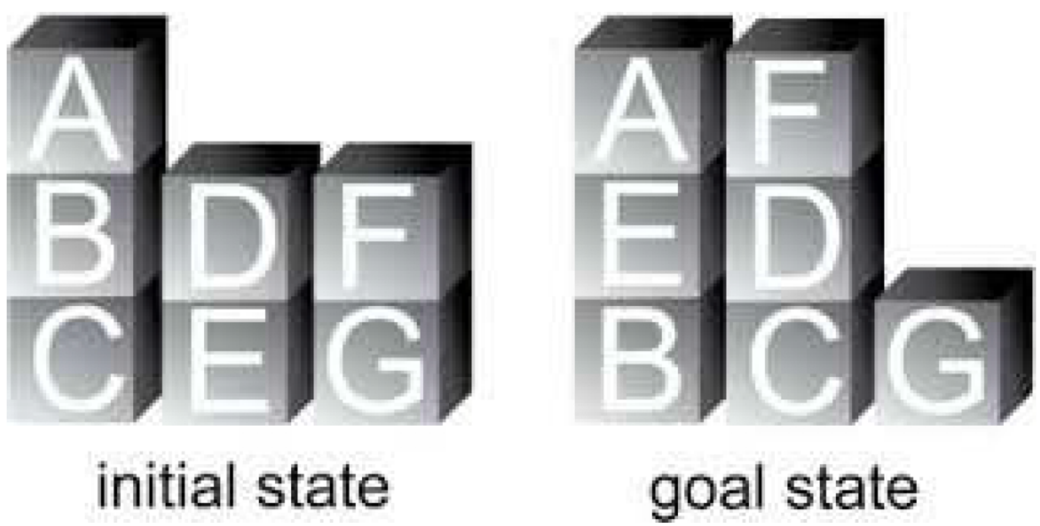
- An edge (a, z) is in the transitive closure of graph G iff there is a path in G from a to z.

- Warshall's algorithm was not originally thought of as an instance of dynamic programming but it fits the bill.

Transitive Closure over a State Space

- Transitive closure is important in all sorts of applications where we want to see of some “goal start” is reachable from some “initial state”.

- For example is there a sequence of steps that will allow the containers of a ship to be reorganised in a certain way?



Reasoning about Transitive Closure

- Assume the nodes of graph G are numbered from 1 to n.
- Ask the question: Is there a path from node i to node j using only nodes that are no larger than some k as "stepping stones"?

Reasoning about Transitive Closure

- Assume the nodes of graph G are numbered from 1 to n.
- Ask the question: Is there a path from node i to node j using only nodes that are no larger than some k as "stepping stones"?
- Such a path either uses node k as a stepping stone, or it doesn't.
- So an answer is: There is such a path if and only if we can

  step from i to j using only nodes $\leq k-1$, or

  step from i to k using only nodes $\leq k-1$, and then
  step from k to j using only nodes $\leq k-1$.

Warshall's Algorithm

- If G's adjacency matrix is A then we can express the recurrence relation as

$$R_{ij}^{0} = A[i,j]$$

$$R_{ij}^{k} = R_{ij}^{k-1} \text{ or } \left(R_{ik}^{k-1} \text{ and } R_{kj}^{k-1}\right)$$

Warshall's Algorithm

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Use the existing path created in the previous step

Warshall's Algorithm

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$$R_{ij}^{0} = A[i,j]$$

$$R_{ij}^{k} = R_{ij}^{k-1} \text{ or } \left(R_{ik}^{k-1} \text{ and } R_{kj}^{k-1}\right)$$

Or create a new path using *k* as an intermediate step

Warshall's Algorithm

- If G's adjacency matrix is A then we can express the recurrence relation as

$$R_{ij}^{0} = A[i,j]$$
$$R_{ij}^{k} = R_{ij}^{k-1} \text{ or } \left(R_{ik}^{k-1} \text{ and } R_{kj}^{k-1}\right)$$

- This gives us a dynamic programming algorithm:

function $\textsc{Warshall}(A[1..n, 1..n])$
  $R^0 \leftarrow A$
  for $k \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to n **do**
      for $j \leftarrow 1$ to n **do**
        $R^k[i,j] \leftarrow R^{k-1}[i,j]$ **or** $(R^{k-1}[i,k]$ **and** $R^{k-1}[k,j])$
  return R^n

Warshall's Algorithm

- If we allow input A to be used for the output, we can simplify things.

- Namely, if $R^{k-1}[i,k]$ (that is, $A[i,k]$) is 0, then the assignment is doing nothing. And if it is 1, and if $A[k,j]$ is also 1, then $A[i,j]$ gets set to 1. So:

```
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            if A[i, k] then
                if A[k, j] then
                    A[i, j] ← 1
```

- But now we notice that $A[i,k]$ does not depend on j, so testing it can be moved outside the innermost loop.

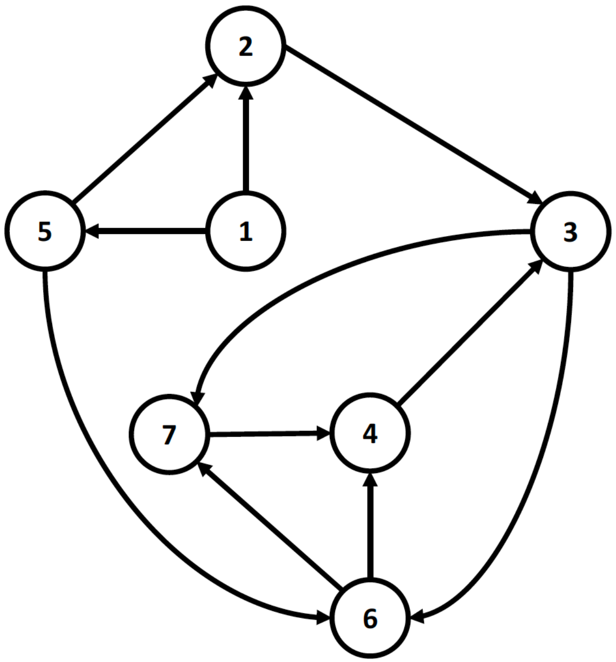
Warshall's Algorithm

- This leads to a better version of Warshall's algorithm:

for $k \leftarrow 1$ to n **do**
  for $i \leftarrow 1$ to n **do**
    if $A[i,k]$ **then**
      for $j \leftarrow 1$ to n **do**
        if $A[k,j]$ **then**
          $A[i,j] \leftarrow 1$

- If each row in the matrix is represented as a bit-string, the innermost for loop (and *j*) can be gotten rid of–instead of iterating, just apply the "bitwise or" of row k to row i.

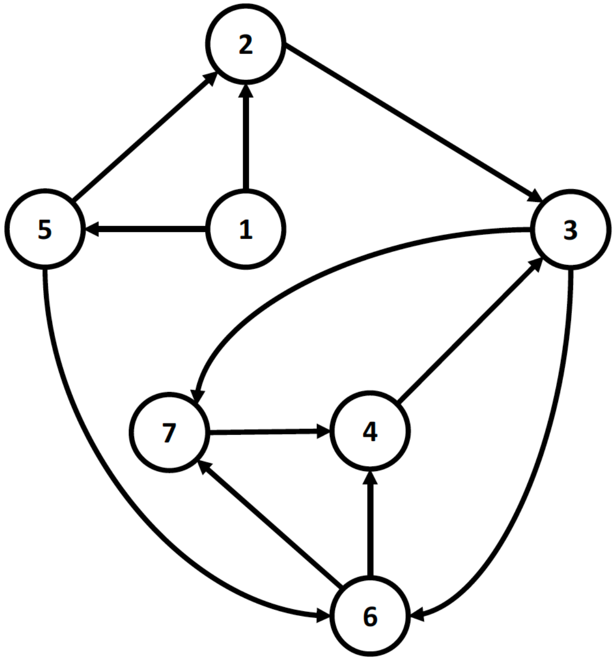
Example of Running Warshall's Algorithm



k = 1: nothing, not change

$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

Example of Running Warshall's Algorithm

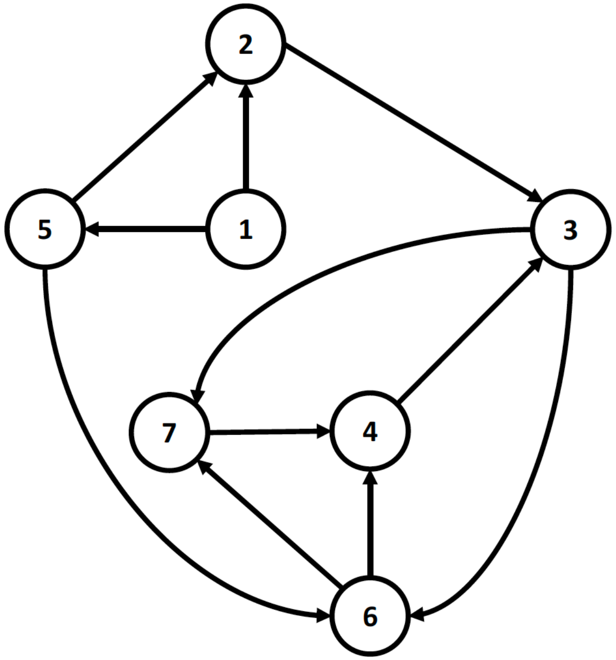


$$\begin{array}{c c} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

k = 2: i values A[i, k]: 1, 5
j values A[k, j]: 3

A[1,3] = 1
A[5,3] = 1

Example of Running Warshall's Algorithm

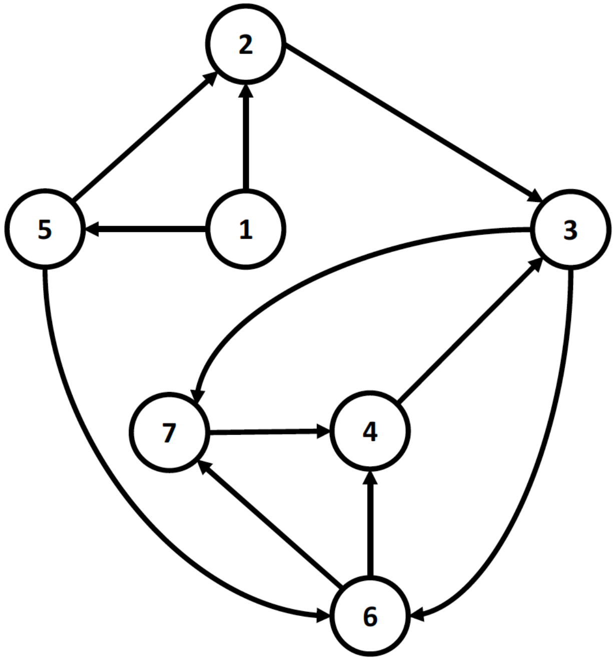


$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix}
0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\end{array}
$$

k = 3: i values $\mathrm{A}[i, k]$: 1, 2, 4, 5
j values $\mathrm{A}[k, j]$: 6,7

$\mathrm{A}[1,6] = 1, \mathrm{A}[1,7] = 1$
$\mathrm{A}[2,6] = 1, \mathrm{A}[2,7] = 1$
$\mathrm{A}[4,6] = 1, \mathrm{A}[4,7] = 1$
$\mathrm{A}[5,6] = 1, \mathrm{A}[5,7] = 1$

Example of Running Warshall's Algorithm

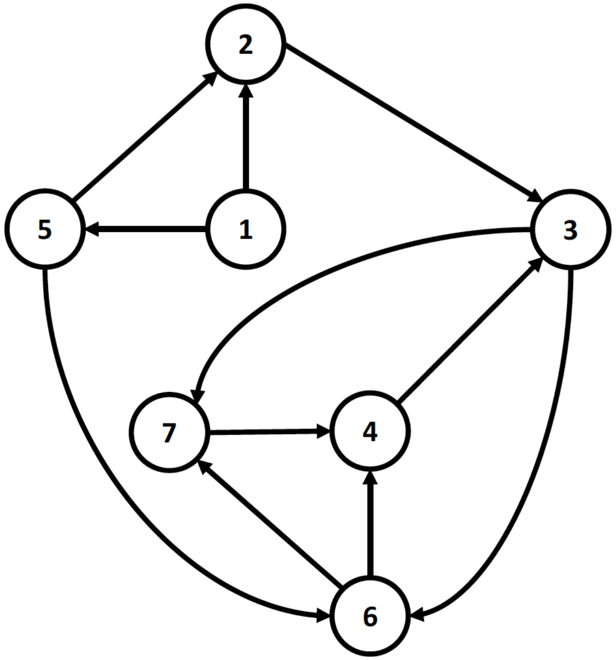


$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

k = 4: i values A[i, k]: 6, 7
j values A[k, j]: 3,6,7

$A[6,3] = 1, A[6,6] = 1, A[6,7] = 1$
$A[7,3] = 1, A[7,6] = 1, A[7,7] = 1$

Example of Running Warshall's Algorithm

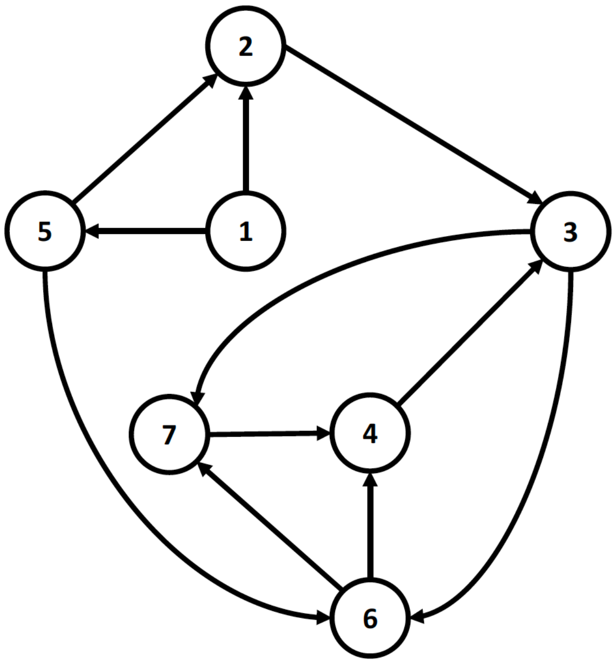


$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & {\color{red}4} & 5 & 6 & 7 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
2 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
4 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
5 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
6 & 0 & 0 & {\color{red}1} & 1 & 0 & {\color{red}1} & 1 \\
7 & 0 & 0 & {\color{red}1} & 1 & 0 & {\color{red}1} & {\color{red}1}
\end{array}
$$

k = 5: i values A$[i, k]$: 1
j values A$[k, j]$: 2,3,6,7

$A[1,2] = 1, A[1,3] = 1,$
$A[1,6] = 1, A[1,7] = 1$

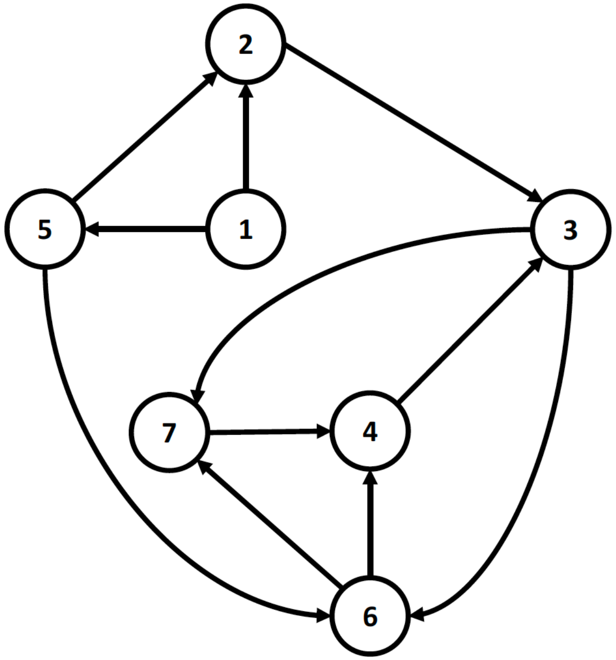
Example of Running Warshall's Algorithm



k = 6: i values A[i, k]: 1,2,3,4,5,6,7
j values A[k, j]: 3,4,6,7

$$\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}
\end{array}$$

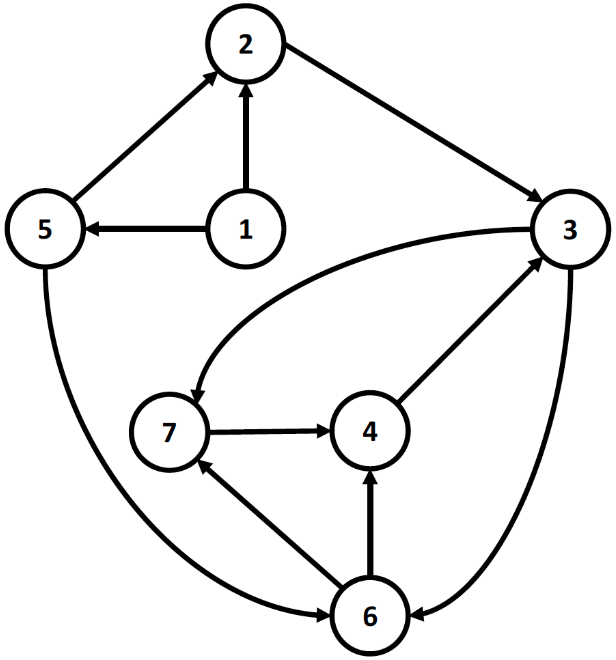
Example of Running Warshall's Algorithm



k = 7: i values A[i, k]: 1,2,3,4,5,6,7
j values A[k, j]: 3,4,6,7

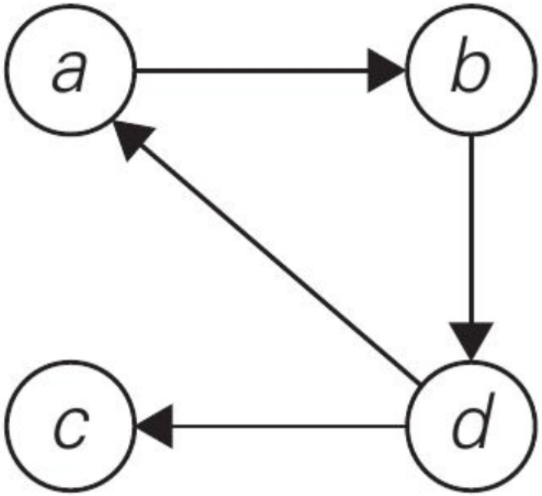
$$\begin{array}{c|ccccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
3 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
4 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
5 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
6 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
7 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{array}$$

Example of Running Warshall's Algorithm



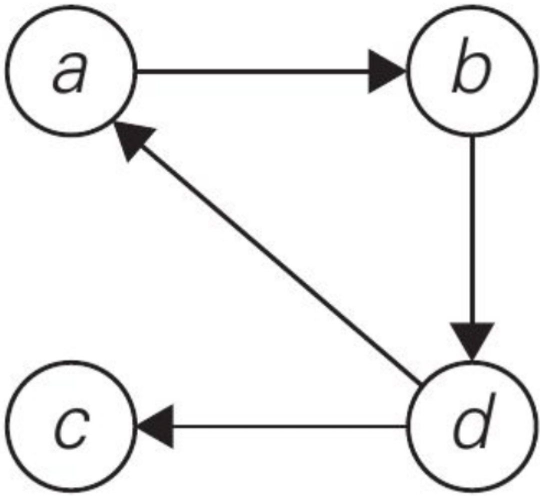
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \color{red}{7} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example of Running Warshall's Algorithm



$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example of Running Warshall's Algorithm: Extended



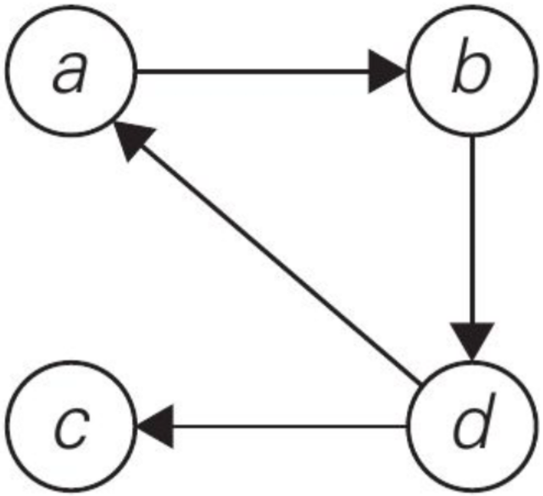
k = 1: i values $A[i,k]$: 4
j values $A[k,j]$: 2

$A[4,2] = 1$

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Example of Running Warshall's Algorithm: Extended



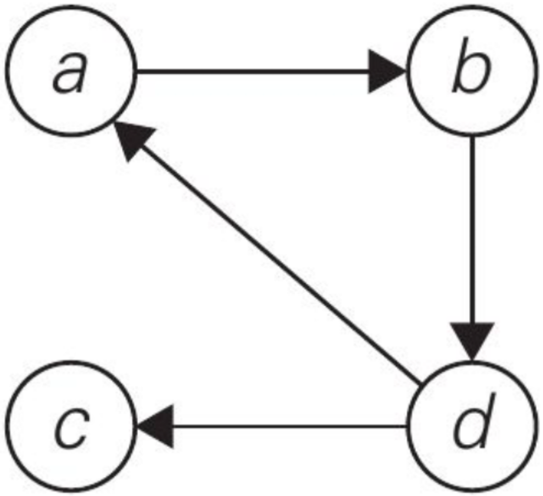
k = 2: i values $\mathrm{A}[i, k]$: 1, 4
j values $\mathrm{A}[k, j]$: 4

$\mathrm{A}[1,4] = 1, \mathrm{A}[4,4] = 1$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 0 \end{array}$$

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 1 \\ b & 0 & 0 & 0 & 1 \\ c & 0 & 0 & 0 & 0 \\ d & 1 & 1 & 1 & 1 \end{array}$$

Example of Running Warshall's Algorithm: Extended



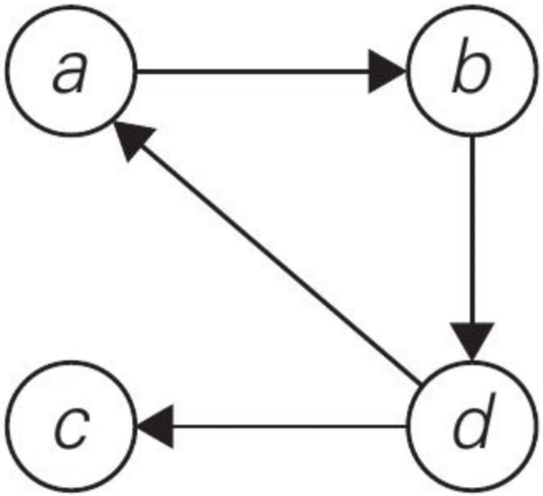
k = 3: i values $\mathrm{A}[i, k]$: 4

j values $\mathrm{A}[k, j]$: $-$

$$\begin{array}{c c} & \begin{array}{cccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \end{array} \\ \begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\begin{array}{c c} & \begin{array}{cccc} \mathrm{a} & \mathrm{b} & \mathrm{c} & \mathrm{d} \end{array} \\ \begin{array}{c} \mathrm{a} \\ \mathrm{b} \\ \mathrm{c} \\ \mathrm{d} \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Example of Running Warshall's Algorithm: Extended

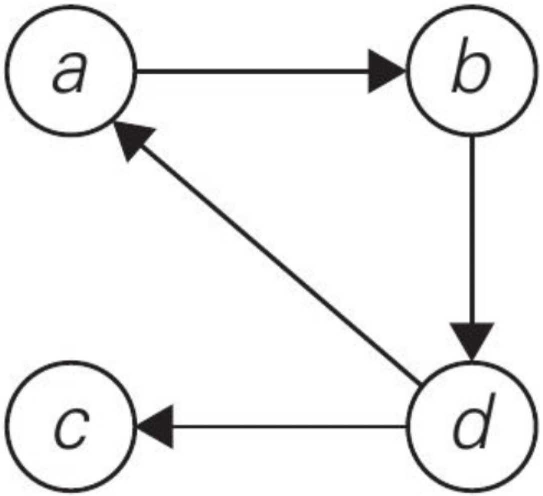


k = 4: i values A$[i, k]$: 1,2, 4
j values A$[k, j]$: 1,2,3,4

$$A[1,1] = 1, A[1,2] = 1, A[1,3] = 1, A[1,4] = 1$$
$$A[2,1] = 1, A[2,2] = 1, A[2,3] = 1, A[2,4] = 1$$
$$A[4,1] = 1, A[4,2] = 1, A[4,3] = 1, A[4,4] = 1$$

$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Example of Running Warshall's Algorithm



$$
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}
\end{array}
\Rightarrow
\begin{array}{c c}
 & \begin{array}{cccc} a & b & c & d \end{array} \\
\begin{array}{c} a \\ b \\ c \\ d \end{array} &
\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}
\end{array}
$$

Correction from lecture slide

Analysis of Warshall's Algorithm

- The analysis is straightforward.

- Warshall's algorithm, as it is usually presented, is $\Theta(n^3)$, and there is no difference between the best, average, and worst cases.

- The algorithm has an incredibly tight inner loop, making it ideal for dense graphs.

- However, it is not the best transitive-closure algorithm to use for sparse graphs. For sparse graphs, you may be better off just doing DFS from each node v in turn, keeping track of which nodes are reached from v.

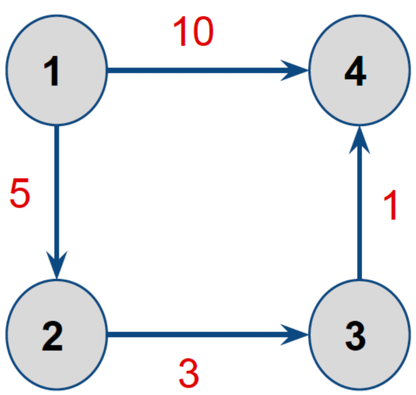
Floyd's Algorithm: All-Pairs Shortest-Paths

- Floyd's algorithm solves the all-pairs shortest-path problem for weighted graphs with positive weights.
- It works for directed as well as undirected graphs.
- (It also works, in some circumstances, when there are non-positive weights in the graph, but not always.)
- We assume we are given a weight matrix W that holds all the edges' weights (for technical reasons, if there is no edge from node i to node j, we let $W[i,j] = \infty$).
- We will construct the distance matrix D, step by step.

Floyd's Algorithm

- We can use the same problem decomposition as we used to derive Warshall's algorithm. Again assume nodes are numbered 1 to n.

- This time ask the question: What is the shortest path from node i to node j using only nodes $\leq k$ as "stepping stones"?

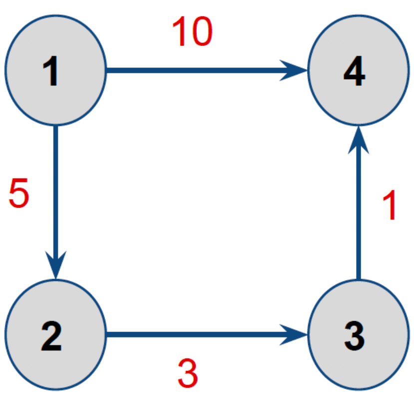


$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 5 & \infty & 10 \\ \infty & 0 & 3 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

Floyd's Algorithm

- We can use the same problem decomposition as we used to derive Warshall's algorithm. Again assume nodes are numbered 1 to n.

- This time ask the question: What is the shortest path from node i to node j using only nodes $\leq k$ as "stepping stones"?



$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 5 & \infty & 9 \\ \infty & 0 & 3 & \infty \\ \infty & \infty & 0 & 1 \\ \infty & \infty & \infty & 0 \end{bmatrix} \end{array}$$

Floyd's Algorithm

- We can use the same problem decomposition as we used to derive Warshall's algorithm. Again assume nodes are numbered 1 to n.
- This time ask the question: What is the shortest path from node i to node j using only nodes $\leq k$ as "stepping stones"?
- We either use node k as a stepping stone, or we avoid it. So again, we can

  step from i to j using only nodes $\leq k-1$, or

  step from i to k using only nodes $\leq k-1$, and then
  step from k to j using only nodes $\leq k-1$.

Floyd's Algorithm

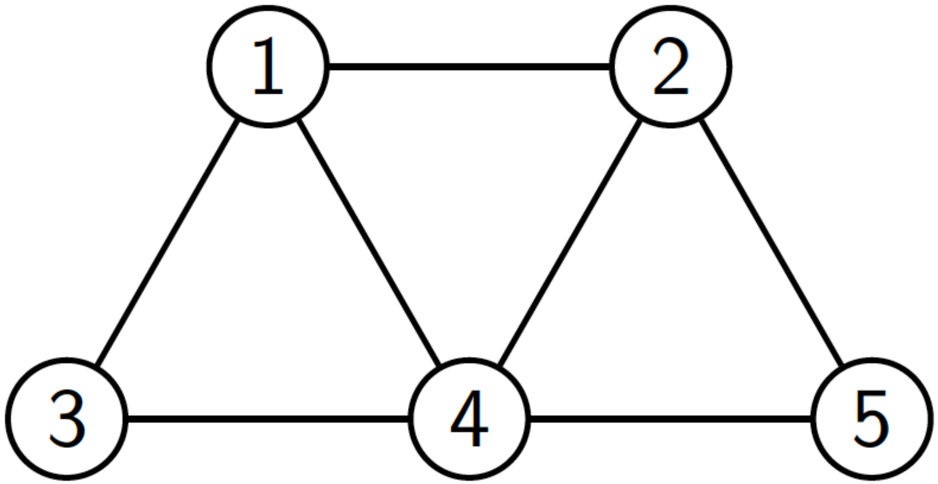
- If G's weight matrix is W then we can express the recurrence relation for minimal distances as follows:

$$D_{ij}^{0} = W[i,j]$$
$$D_{ij}^{k} = \min\left(D_{ij}^{k-1},\ \ D_{ik}^{k-1} + \ D_{kj}^{k-1}\right)$$

- And then the algorithm follows easily:

function $\text{F}\scriptstyle{\text{LOYD}}$$(W[1..n, 1..n])$
  $D \leftarrow W$
  for $k \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to n **do**
      for $j \leftarrow 1$ to n **do**
        $D[i,j] \leftarrow min(D[i,j], D[i,k] + D[k,j])$
  return D

Example of Running Floyd's Algorithm



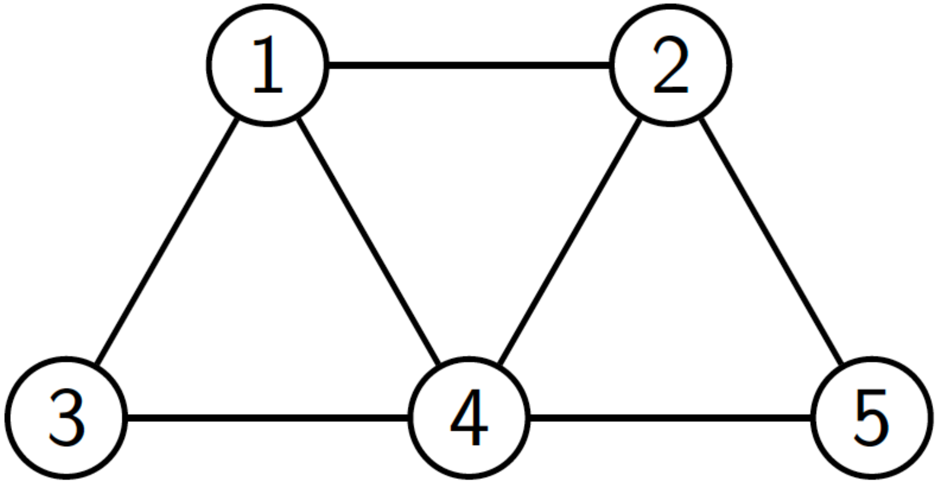
- The initial distance matrix (for the unweighted graph above).

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
0 & 1 & 1 & 1 & \infty \\
1 & 0 & \infty & 1 & 1 \\
1 & \infty & 0 & 1 & \infty \\
1 & 1 & 1 & 0 & 1 \\
\infty & 1 & \infty & 1 & 0
\end{bmatrix}
\end{array}
$$

k = 1: i values $\mathrm{D}[i,k]$: 2,3,4
j values $\mathrm{D}[k,j]$: 2,3,4

$\mathrm{D}[2,3] = 1 + 1 = 2$
$\mathrm{D}[3,2] = 1 + 1 = 2$

Example of Running Floyd's Algorithm

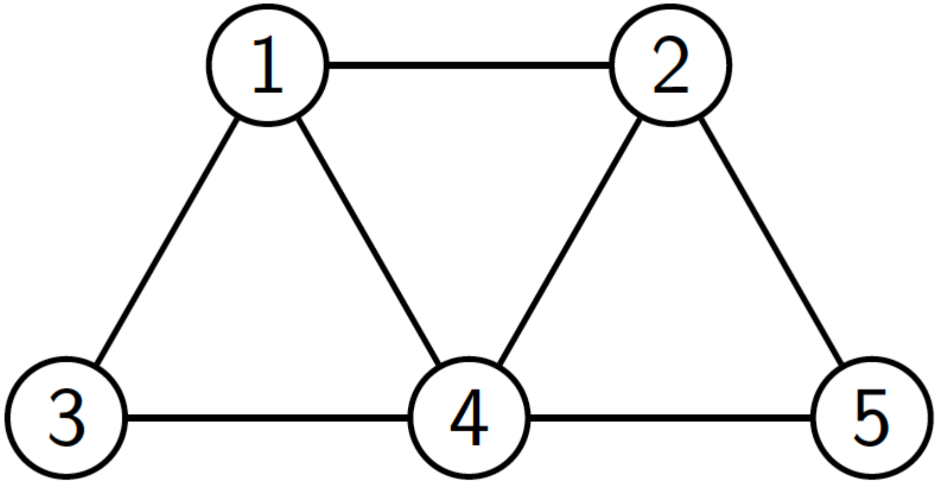


- Distance matrix after first round ($k = 1$).

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 & 1 & \infty \\ 2 & 1 & 0 & 2 & 1 & 1 \\ 3 & 1 & 2 & 0 & 1 & \infty \\ 4 & 1 & 1 & 1 & 0 & 1 \\ 5 & \infty & 1 & \infty & 1 & 0 \end{array}$$

k = 2: i values D[i, k]: 1,3,4,5
j values D[k, j]: 1,3,4,5
$D[1,5] = 1 + 1 = 2$
$D[3,5] = 2 + 1 = 3$
$D[5,1] = 1 + 1 = 2$

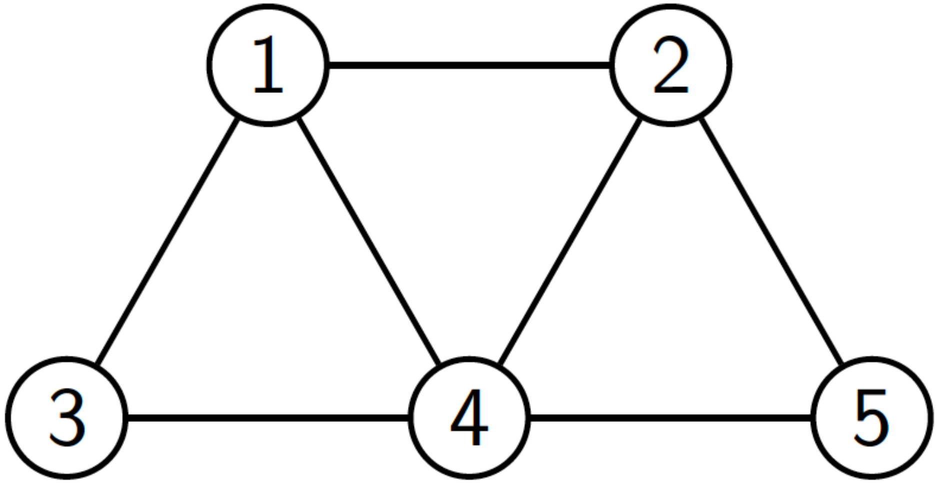
Example of Running Floyd's Algorithm



- Distance matrix after second round ($k = 2$).
- In this example, no change happens in the following round ($k = 3$).

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & \color{red}{2} \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & \color{red}{3} \\ 1 & 1 & 1 & 0 & 1 \\ \color{red}{2} & 1 & \color{red}{3} & 1 & 0 \end{bmatrix}
\end{array}
$$

Example of Running Floyd's Algorithm

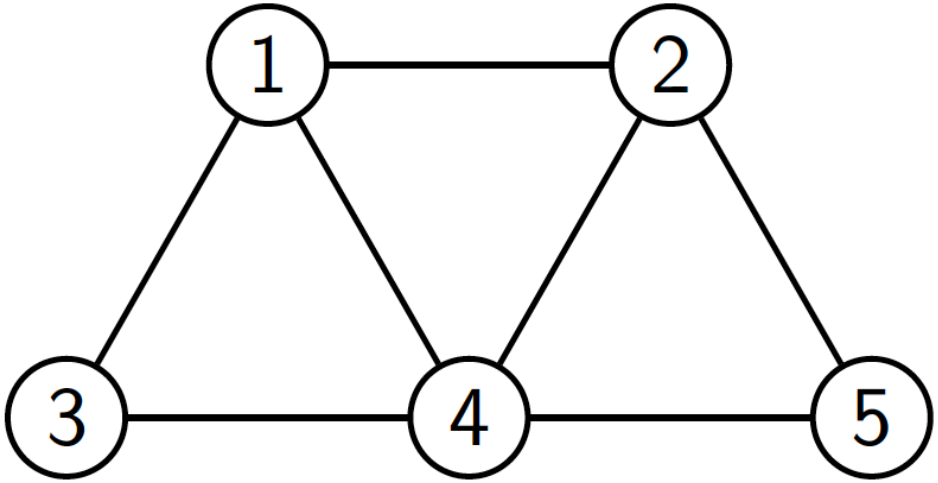


- Distance matrix after second round ($k = 2$).
- In this example, no change happens in the following round ($k = 3$).

k $= 4$: i values D$[i, k]$: 1,2,3,5
j values D$[k, j]$: 1,2,3,5
D$[5,3] = 1 + 1 = 2$

$$
\begin{array}{cc}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
0 & 1 & 1 & 1 & 2 \\
1 & 0 & 2 & 1 & 1 \\
1 & 2 & 0 & 1 & 3 \\
1 & 1 & 1 & 0 & 1 \\
2 & 1 & 3 & 1 & 0
\end{bmatrix}
\end{array}
$$

Example of Running Floyd's Algorithm

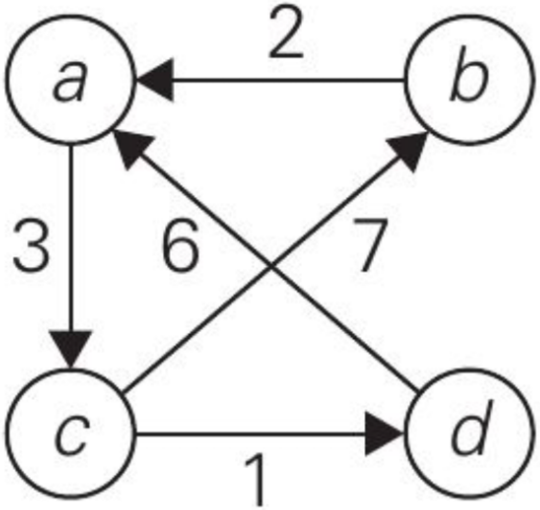


- Distance matrix after fourth round ($k = 4$).

$$\begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix} \end{array}$$

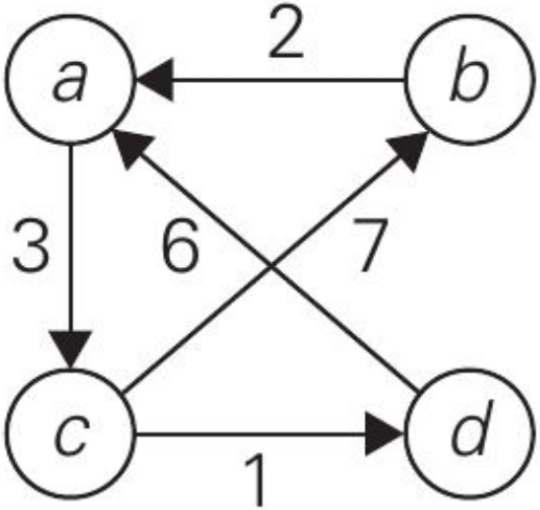
- In this example, no further change happens for $k = 5$, so this is the final result.

Example of Running Floyd's Algorithm



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \end{array}$$

Example of Running Floyd's Algorithm: Extended



k = 1: i values D[i, k]: 2,4

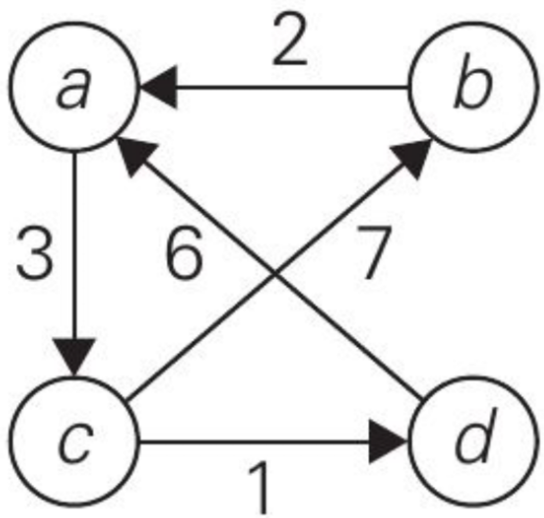
j values D[k, j]: 3

D[2,3] = 2 + 3 = 5

D[4,3] = 3 + 6 = 9

$$\begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{array} \begin{array}{c} \begin{array}{cccc} \text{a} & \text{b} & \text{c} & \text{d} \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{array} \begin{array}{c} \begin{array}{cccc} \text{a} & \text{b} & \text{c} & \text{d} \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \end{array}$$

Example of Running Floyd's Algorithm: Extended

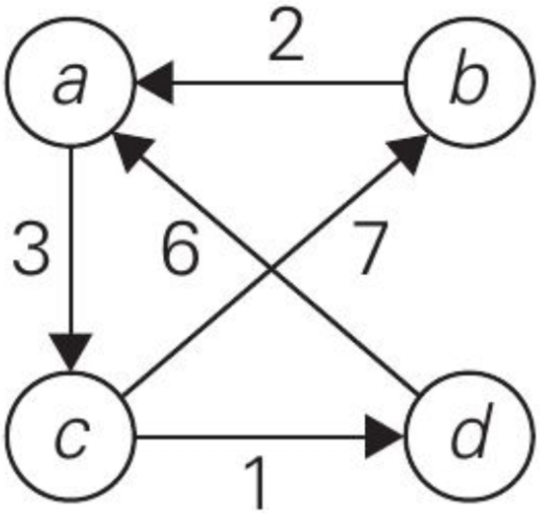


k = 2: i values D[i, k]: 3
j values D[k, j]: 1,3
D[3,1] = 7 + 2 = 9

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ 9 & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \end{array}$$

Example of Running Floyd's Algorithm: Extended



k = 3: i values D[i, k]: 1,2,4

j values D[k, j]: 1,2,4

D[1,2] = 3 + 7 = 10, D[1,4] = 3 + 1 = 4

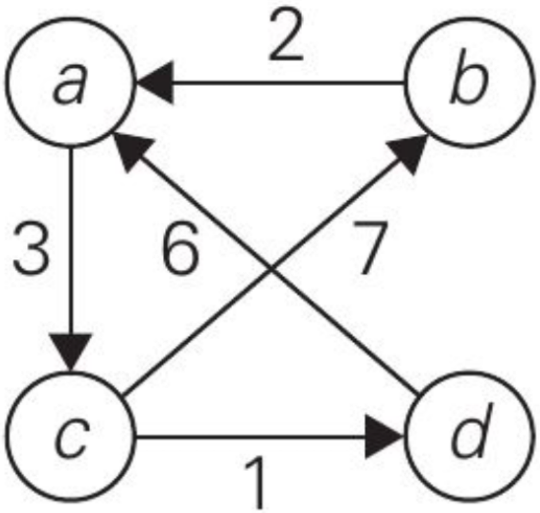
D[2,1] = 5 + 9 = 14, D[2,4] = 5 + 1 = 6

D[4,1] = 9 + 9 = 18, D[4,2] = 9 + 7 = 16

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ 9 & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c|c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 9 & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \end{array}$$

Example of Running Floyd’s Algorithm: Extended



k $= 4$: i values $\mathrm{D}[i, k]$: 1,2,3

j values $\mathrm{D}[k, j]$: 1,2,3

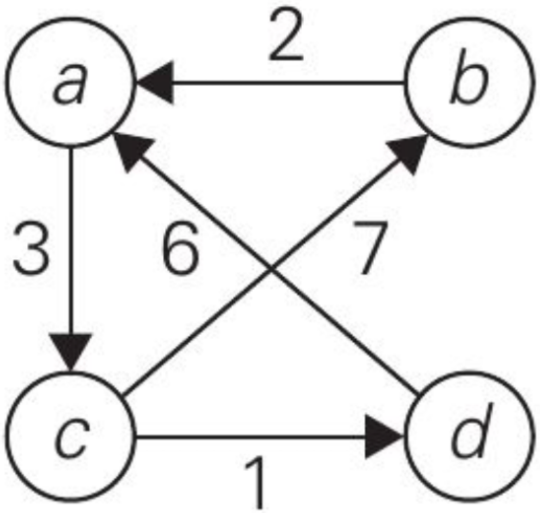
$\mathrm{D}[1,2] = 4 + 16 = 20, \mathrm{D}[1,3] = 4 + 9 = 15$

$\mathrm{D}[2,1] = 6 + 6 = 12, \mathrm{D}[2,3] = 6 + 9 = 15$

$\mathrm{D}[3,1] = 1 + 6 = 7, \mathrm{D}[3,2] = 1 + 16 = 17$

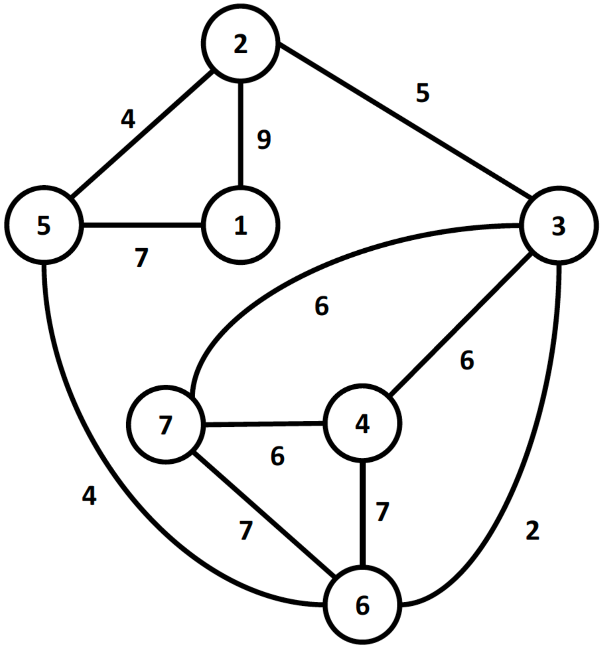
$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 9 & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 7 & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \end{array}$$

Example of Running Floyd's Algorithm



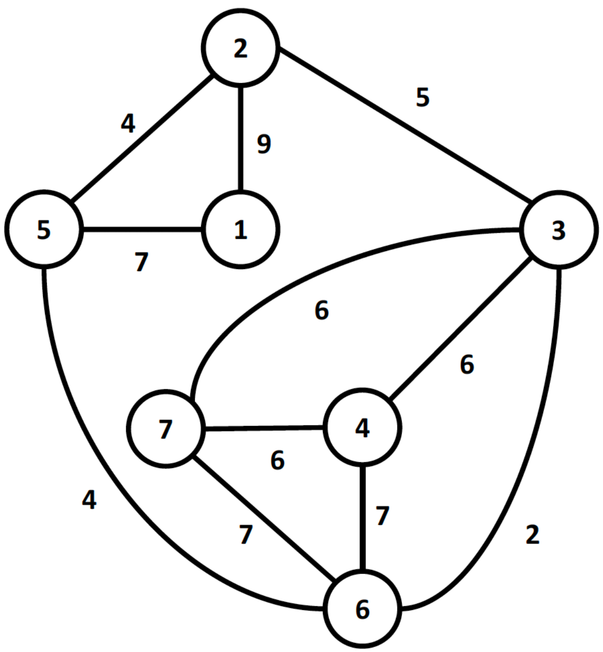
$$
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & \infty & 3 & \infty \\
b & 2 & 0 & \infty & \infty \\
c & \infty & 7 & 0 & 1 \\
d & 6 & \infty & \infty & 0
\end{array}
\quad \Rightarrow \quad
\begin{array}{c|cccc}
 & a & b & c & d \\
\hline
a & 0 & 10 & 3 & 4 \\
b & 2 & 0 & 5 & 6 \\
c & 7 & 7 & 0 & 1 \\
d & 6 & 16 & 9 & 0
\end{array}
$$

Example of Running Floyd's Algorithm



$$
\begin{array}{c c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix}
0 & 9 & \infty & \infty & 7 & \infty & \infty \\
9 & 0 & 5 & \infty & 4 & \infty & \infty \\
\infty & 5 & 0 & 6 & \infty & 2 & 6 \\
\infty & \infty & 6 & 0 & \infty & 7 & 6 \\
7 & 4 & \infty & \infty & 0 & 4 & \infty \\
\infty & \infty & 2 & 7 & 4 & 0 & 7 \\
\infty & \infty & 6 & 6 & \infty & 7 & 0
\end{bmatrix}
\end{array}
$$

Example of Running Floyd's Algorithm: Extended

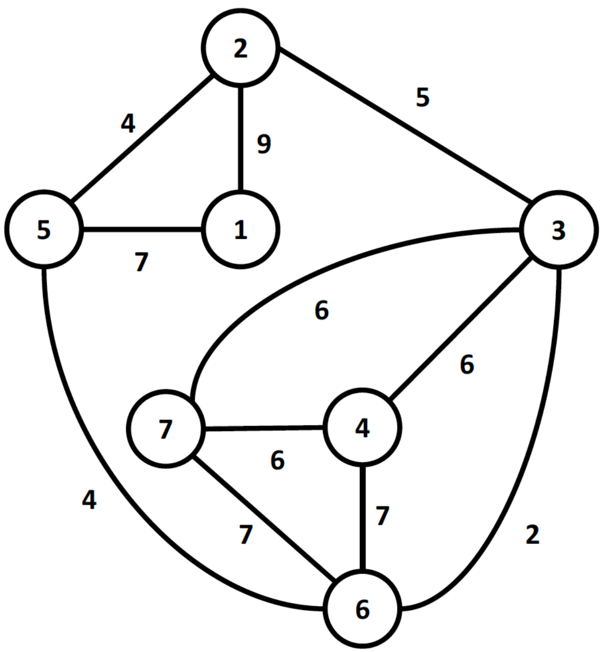


$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 0 & 9 & \infty & \infty & 7 & \infty & \infty \\ 9 & 0 & 5 & \infty & 4 & \infty & \infty \\ \infty & 5 & 0 & 6 & \infty & 2 & 6 \\ \infty & \infty & 6 & 0 & \infty & 7 & 6 \\ 7 & 4 & \infty & \infty & 0 & 4 & \infty \\ \infty & \infty & 2 & 7 & 4 & 0 & 7 \\ \infty & \infty & 6 & 6 & \infty & 7 & 0 \end{bmatrix} \end{array}$$

k = 1: i values D$[i,k]$: 2,5
j values D$[k,j]$: 2,5
D$[2,5] = 9 + 7 = 16$, D$[5,2] = 7 + 9 = 16$

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} & \begin{bmatrix} 0 & 9 & \infty & \infty & 7 & \infty & \infty \\ 9 & 0 & 5 & \infty & 4 & \infty & \infty \\ \infty & 5 & 0 & 6 & \infty & 2 & 6 \\ \infty & \infty & 6 & 0 & \infty & 7 & 6 \\ 7 & 4 & \infty & \infty & 0 & 4 & \infty \\ \infty & \infty & 2 & 7 & 4 & 0 & 7 \\ \infty & \infty & 6 & 6 & \infty & 7 & 0 \end{bmatrix} \end{array}$$

Example of Running Floyd's Algorithm: Extended



$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
0 & 9 & \infty & \infty & 7 & \infty & \infty \\
9 & 0 & 5 & \infty & 4 & \infty & \infty \\
\infty & 5 & 0 & 6 & \infty & 2 & 6 \\
\infty & \infty & 6 & 0 & \infty & 7 & 6 \\
7 & 4 & \infty & \infty & 0 & 4 & \infty \\
\infty & \infty & 2 & 7 & 4 & 0 & 7 \\
\infty & \infty & 6 & 6 & \infty & 7 & 0
\end{bmatrix}
\end{array}
$$

k $= 2$: i values $D[i,k]$: 1,3,5

j values $D[k,j]$: 1,3,5

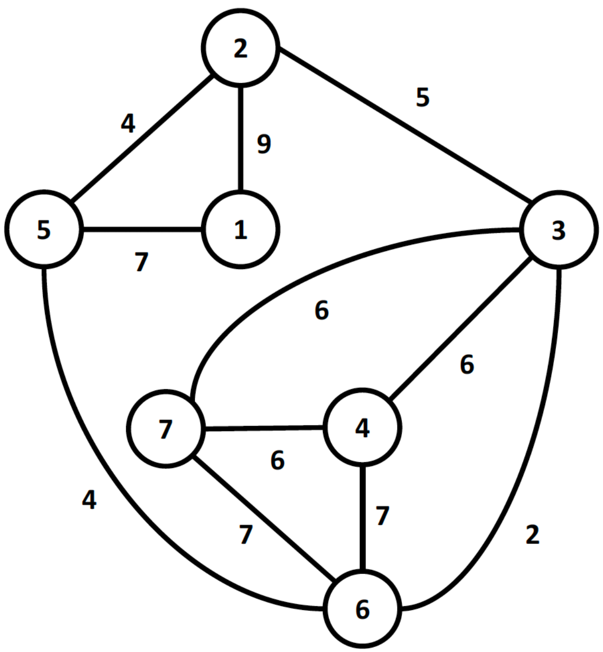
$D[1,3] = 9 + 5 = 14$, $D[1,5] = 9 + 4 = 13$

$D[3,1] = 5 + 9 = 14$, $D[3,5] = 5 + 4 = 9$

$D[5,1] = 4 + 9 = 13$, $D[5,3] = 4 + 5 = 9$

$$
\begin{array}{c}
 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
0 & 9 & 14 & \infty & 7 & \infty & \infty \\
9 & 0 & 5 & \infty & 4 & \infty & \infty \\
14 & 5 & 0 & 6 & 9 & 2 & 6 \\
\infty & \infty & 6 & 0 & \infty & 7 & 6 \\
7 & 4 & 9 & \infty & 0 & 4 & \infty \\
\infty & \infty & 2 & 7 & 4 & 0 & 7 \\
\infty & \infty & 6 & 6 & \infty & 7 & 0
\end{bmatrix}
\end{array}
$$

Example of Running Floyd's Algorithm: Extended



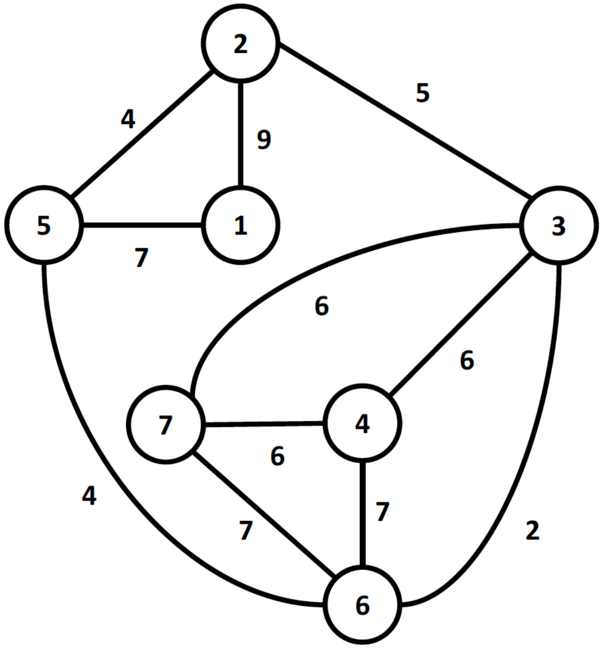
$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} &
\begin{bmatrix}
0 & 9 & 14 & \infty & 7 & \infty & \infty \\
9 & 0 & 5 & \infty & 4 & \infty & \infty \\
14 & 5 & 0 & 6 & 9 & 2 & 6 \\
\infty & \infty & 6 & 0 & \infty & 7 & 6 \\
7 & 4 & 9 & \infty & 0 & 4 & \infty \\
\infty & \infty & 2 & 7 & 4 & 0 & 7 \\
\infty & \infty & 6 & 6 & \infty & 7 & 0
\end{bmatrix}
\end{array}
$$

k = 3: i values D[i, k]: 1,2,4,5,6,7
j values D[k, j]: 1,2,4,5,6,7

D[1,2] = 19, D[1,4] = 20, D[1,5] = 23, D[1,6] = 16 , D[1,7] = 20
D[2,1] = 19, D[2,4] = 11, D[2,5] = 14, D[2,6] = 7 , D[2,7] = 11
D[4,1] = 20, D[4,2] = 11, D[4,5] = 15, D[4,6] = 8 , D[4,7] = 12
D[5,1] = 23, D[5,2] = 14, D[5,4] = 15, D[5,6] = 11 , D[5,7] = 15
D[6,1] = 16, D[6,2] = 7, D[6,4] = 8, D[6,5] = 11 , D[6,7] = 8
D[7,1] = 20, D[7,2] = 11, D[7,4] = 12, D[7,5] = 15 , D[7,6] = 8

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} &
\begin{bmatrix}
0 & 9 & 14 & 20 & 7 & 16 & 20 \\
9 & 0 & 5 & 11 & 4 & 7 & 11 \\
14 & 5 & 0 & 6 & 9 & 2 & 6 \\
20 & 11 & 6 & 0 & 15 & 7 & 6 \\
7 & 4 & 9 & 15 & 0 & 4 & 15 \\
16 & 7 & 2 & 7 & 4 & 0 & 7 \\
20 & 11 & 6 & 6 & 15 & 7 & 0
\end{bmatrix}
\end{array}
$$

Example of Running Floyd's Algorithm: Extended



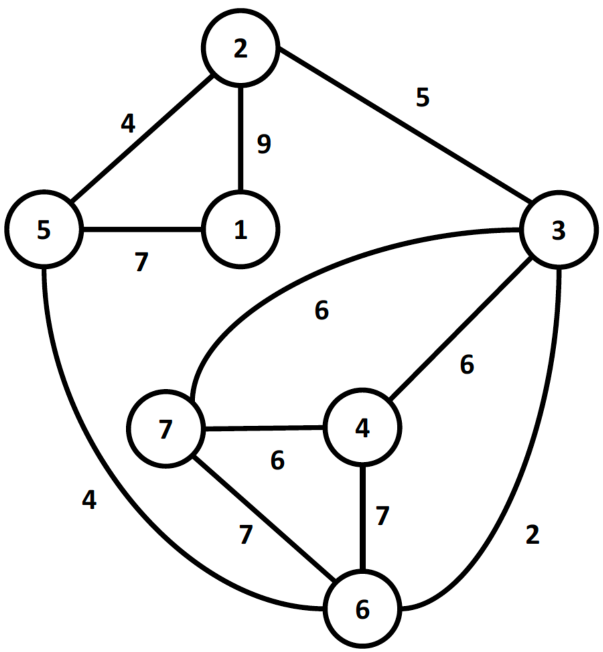
$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 0 & 9 & 14 & 20 & 7 & 16 & 20 \\ 2 & 9 & 0 & 5 & 11 & 4 & 7 & 11 \\ 3 & 14 & 5 & 0 & 6 & 9 & 2 & 6 \\ 4 & 20 & 11 & 6 & 0 & 15 & 7 & 6 \\ 5 & 7 & 4 & 9 & 15 & 0 & 4 & 15 \\ 6 & 16 & 7 & 2 & 7 & 4 & 0 & 7 \\ 7 & 20 & 11 & 6 & 6 & 15 & 7 & 0 \end{array}$$

k = 4: i values D[i, k]: 1,2,3,5,6,7
j values D[k, j]: 1,2,3,5,6,7

D[1,2] = 31, D[1,3] = 26, D[1,5] = 35, D[1,6] = 27 , D[1,7] = 26
D[2,1] = 31, D[2,3] = 17, D[2,5] = 26, D[2,6] = 18 , D[2,7] = 17
D[3,1] = 26, D[3,2] = 17, D[3,5] = 21, D[3,6] = 13 , D[3,7] = 12
D[5,1] = 35, D[5,2] = 26, D[5,3] = 21, D[5,6] = 22 , D[5,7] = 21
D[6,1] = 27, D[6,2] = 18, D[6,3] = 13, D[6,5] = 22 , D[6,7] = 13
D[7,1] = 26, D[7,2] = 17, D[7,3] = 12, D[7,5] = 21 , D[7,6] = 13

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 0 & 9 & 14 & 20 & 7 & 16 & 20 \\ 2 & 9 & 0 & 5 & 11 & 4 & 7 & 11 \\ 3 & 14 & 5 & 0 & 6 & 9 & 2 & 6 \\ 4 & 20 & 11 & 6 & 0 & 15 & 7 & 6 \\ 5 & 7 & 4 & 9 & 15 & 0 & 4 & 15 \\ 6 & 16 & 7 & 2 & 7 & 4 & 0 & 7 \\ 7 & 20 & 11 & 6 & 6 & 15 & 7 & 0 \end{array}$$

Example of Running Floyd's Algorithm: Extended



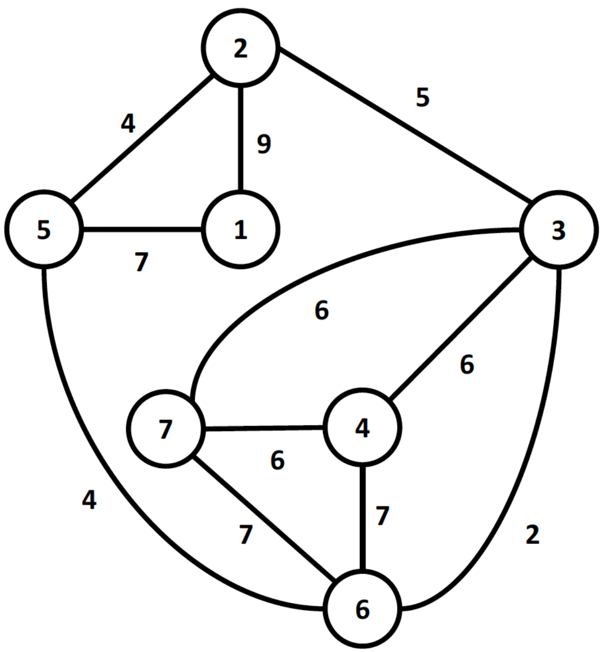
$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 9 & 14 & 20 & 7 & 16 & 20 \\
2 & 9 & 0 & 5 & 11 & 4 & 7 & 11 \\
3 & 14 & 5 & 0 & 6 & 9 & 2 & 6 \\
4 & 20 & 11 & 6 & 0 & 15 & 7 & 6 \\
5 & 7 & 4 & 9 & 15 & 0 & 4 & 15 \\
6 & 16 & 7 & 2 & 7 & 4 & 0 & 7 \\
7 & 20 & 11 & 6 & 6 & 15 & 7 & 0
\end{array}
$$

k = 5: i values D[i, k]: 1,2,3,4,6,7
j values D[k, j]: 1,2,3,4,6,7

D[1,2] = 11, D[1,3] = 16, D[1,4] = 22, D[1,6] = 11 , D[1,7] = 22
D[2,1] = 11, D[2,3] = 13, D[2,4] = 19, D[2,6] = 8 , D[2,7] = 19
D[3,1] = 16, D[3,2] = 13, D[3,4] = 24, D[3,6] = 13 , D[3,7] = 24
D[4,1] = 22, D[4,2] = 19, D[4,3] = 24, D[4,6] = 19 , D[4,7] = 30
D[6,1] = 11, D[6,2] = 8, D[6,3] = 13, D[6,4] = 19 , D[6,7] = 19
D[7,1] = 22, D[7,2] = 19, D[7,3] = 24, D[7,5] = 30 , D[7,6] = 19

$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
1 & 0 & 9 & 14 & 20 & 7 & 11 & 20 \\
2 & 9 & 0 & 5 & 11 & 4 & 7 & 11 \\
3 & 14 & 5 & 0 & 6 & 9 & 2 & 6 \\
4 & 20 & 11 & 6 & 0 & 15 & 7 & 6 \\
5 & 7 & 4 & 9 & 15 & 0 & 4 & 15 \\
6 & 11 & 7 & 2 & 7 & 4 & 0 & 7 \\
7 & 20 & 11 & 6 & 6 & 15 & 7 & 0
\end{array}
$$

Example of Running Floyd's Algorithm: Extended



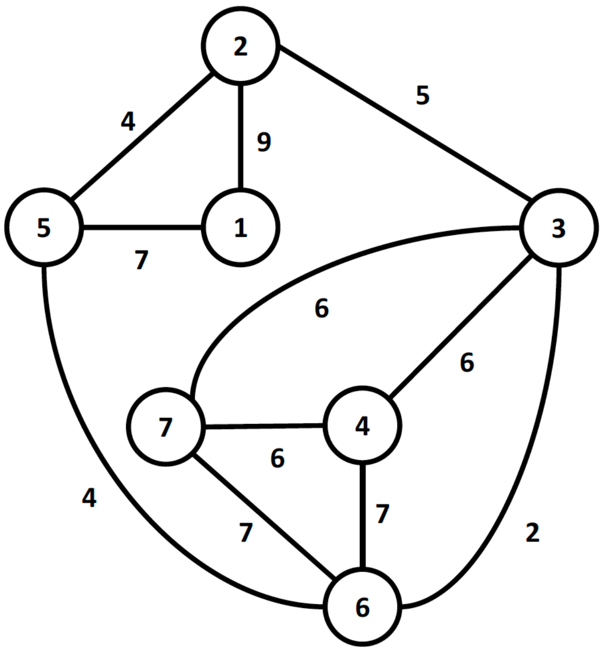
$$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 9 & 14 & 20 & 7 & 11 & 20 \\ 9 & 0 & 5 & 11 & 4 & 7 & 11 \\ 14 & 5 & 0 & 6 & 9 & 2 & 6 \\ 20 & 11 & 6 & 0 & 15 & 7 & 6 \\ 7 & 4 & 9 & 15 & 0 & 4 & 15 \\ 11 & 7 & 2 & 7 & 4 & 0 & 7 \\ 20 & 11 & 6 & 6 & 15 & 7 & 0 \end{bmatrix} \end{array}$$

k = 6: i values D[i, k]: 1,2,3,4,5,7
j values D[k, j]: 1,2,3,4,5,7

D[1,2] = 18, D[1,3] = 13, D[1,4] = 18, D[1,5] = 15 , D[1,7] = 18
D[2,1] = 18, D[2,3] = 9, D[2,4] = 14, D[2,5] = 11 , D[2,7] = 14
D[3,1] = 13, D[3,2] = 9, D[3,4] = 9, D[3,5] = 6 , D[3,7] = 9
D[4,1] = 18, D[4,2] = 14, D[4,3] = 9, D[4,5] = 11 , D[4,7] = 14
D[5,1] = 15, D[5,2] = 11, D[5,3] = 6, D[5,4] = 11 , D[5,7] = 11
D[7,1] = 18, D[7,2] = 14, D[7,3] = 9, D[7,4] = 14 , D[7,5] = 11

$$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} 0 & 9 & 13 & 18 & 7 & 11 & 18 \\ 9 & 0 & 5 & 11 & 4 & 7 & 11 \\ 13 & 5 & 0 & 6 & 6 & 2 & 6 \\ 18 & 11 & 6 & 0 & 11 & 7 & 6 \\ 7 & 4 & 6 & 11 & 0 & 4 & 11 \\ 11 & 7 & 2 & 7 & 4 & 0 & 7 \\ 18 & 11 & 6 & 6 & 11 & 7 & 0 \end{bmatrix} \end{array}$$

Example of Running Floyd's Algorithm: Extended



$$
\begin{array}{c|c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix}
0 & 9 & 13 & 18 & 7 & 11 & 18 \\
9 & 0 & 5 & 11 & 4 & 7 & 11 \\
13 & 5 & 0 & 6 & 6 & 2 & 6 \\
18 & 11 & 6 & 0 & 11 & 7 & 6 \\
7 & 4 & 6 & 11 & 0 & 4 & 11 \\
11 & 7 & 2 & 7 & 4 & 0 & 7 \\
18 & 11 & 6 & 6 & 11 & 7 & 0
\end{bmatrix}
\end{array}
$$

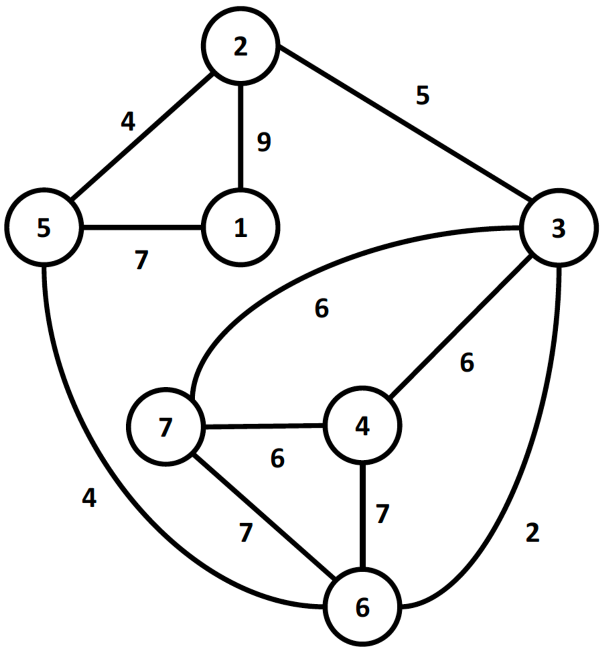
k = 7: i values D[i, k]: 1,2,3,4,5,6

j values D[k, j]: 1,2,3,4,5,6

D[1,2] = 29, D[1,3] = 24, D[1,4] = 24, D[1,5] = 29 , D[1,6] = 25
D[2,1] = 29, D[2,3] = 17, D[2,4] = 17, D[2,5] = 22 , D[2,6] = 18
D[3,1] = 24, D[3,2] = 17, D[3,4] = 12, D[3,5] = 17 , D[3,6] = 13
D[4,1] = 24, D[4,2] = 17, D[4,3] = 12, D[4,5] = 17 , D[4,6] = 13
D[5,1] = 29, D[5,2] = 22, D[5,3] = 17, D[5,4] = 17 , D[5,6] = 18
D[6,1] = 25, D[6,2] = 18, D[6,3] = 13, D[6,4] = 13 , D[6,5] = 18

$$
\begin{array}{c|c}
 & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} &
\begin{bmatrix}
0 & 9 & 13 & 18 & 7 & 11 & 18 \\
9 & 0 & 5 & 11 & 4 & 7 & 11 \\
13 & 5 & 0 & 6 & 6 & 2 & 6 \\
18 & 11 & 6 & 0 & 11 & 7 & 6 \\
7 & 4 & 6 & 11 & 0 & 4 & 11 \\
11 & 7 & 2 & 7 & 4 & 0 & 7 \\
18 & 11 & 6 & 6 & 11 & 7 & 0
\end{bmatrix}
\end{array}
$$

Example of Running Floyd's Algorithm

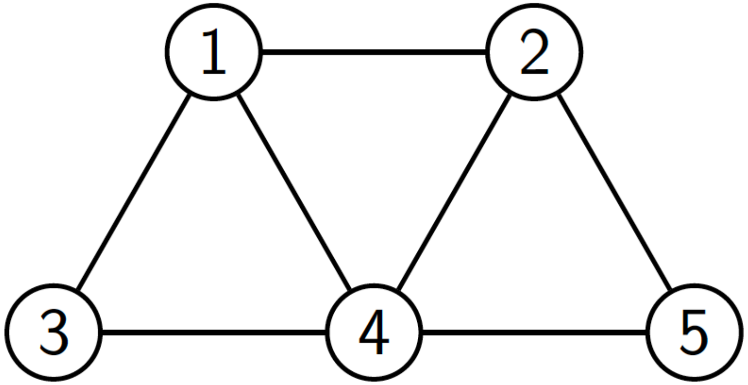


$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
0 & 9 & \infty & \infty & 7 & \infty & \infty \\
9 & 0 & 5 & \infty & 4 & \infty & \infty \\
\infty & 5 & 0 & 6 & \infty & 2 & 6 \\
\infty & \infty & 6 & 0 & \infty & 7 & 6 \\
7 & 4 & \infty & \infty & 0 & 4 & \infty \\
\infty & \infty & 2 & 7 & 4 & 0 & 7 \\
\infty & \infty & 6 & 6 & \infty & 7 & 0
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
0 & 9 & 13 & 18 & 7 & 11 & 18 \\
9 & 0 & 5 & 11 & 4 & 7 & 11 \\
13 & 5 & 0 & 6 & 6 & 2 & 6 \\
18 & 11 & 6 & 0 & 11 & 7 & 6 \\
7 & 4 & 6 & 11 & 0 & 4 & 11 \\
11 & 7 & 2 & 7 & 4 & 0 & 7 \\
18 & 11 & 6 & 6 & 11 & 7 & 0
\end{bmatrix}
\end{array}
$$

A Sub-Structure Property

- For a dynamic programming approach to be applicable, the problem must have a certain "sub-structure" property that allows for a compositional solution.
- Shortest-path problems have the property: if $x_1 - x_2 - \cdots - x_i - \cdots - x_n$ is a shortest path from x_1 to x_n then $x_1 - x_2 - \cdots - x_i$ is a shortest path from x_1 to x_i.
- Longest-path problems don't have that property. In our sample graph 1—3—4—2—5 is a longest path from 1 to 5, but 1—3—4—2 is not a longest path from 1 to 2 (since 1—3—4—5—2 is longer).



Coming Up Next

- Greedy techniques
  - Prim’s algorithm (Levitin Section 9.1)
  - Dijkstra’s algorithm (Levitin Section 9.3).