COMP90038
Algorithms and Complexity

Lecture 6: Recursion
(with thanks to Harald Søndergaard)

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Recursion

- We've already seen some examples
- A very natural approach when the data structure is recursive (e.g. lists, trees)
- But also examples of naturally recursive array processing algorithms
- Next week we'll express **depth first graph traversal** recursively (the natural way); later we'll meet other examples of recursion too

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\text{FAC}(n)$
  if $n = 0$ **then**
    return 1
  return $\text{FAC}(n-1) * n$

function $\text{FAC}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return *result*

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
  if $n = 0$ **then**
    return 1
  return $\textsc{Fac}(n-1) * n$

$F(5)$

function $\textsc{Fac}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad \textbf{if } n = 0 \textbf{ then} \\
\quad\quad \textbf{return } 1 \\
\quad \textbf{return } \textsc{Fac}(n-1) * n
\end{array}
$$

$$F(5) = F(4) \cdot 5$$

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad result \leftarrow 1 \\
\quad \textbf{while } n > 0 \textbf{ do} \\
\quad\quad result \leftarrow result * n \\
\quad\quad n \leftarrow n - 1 \\
\quad \textbf{return } result
\end{array}
$$

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
  if $n = 0$ **then**
    return 1
  return $\textsc{Fac}(n-1) * n$

$$F(5) = F(4) \cdot 5$$
$$= (F(3) \cdot 4) \cdot 5$$

function $\textsc{Fac}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad \textbf{if } n = 0 \textbf{ then} \\
\quad\quad \textbf{return } 1 \\
\quad \textbf{return } \textsc{Fac}(n-1) * n
\end{array}
$$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad \mathit{result} \leftarrow 1 \\
\quad \textbf{while } n > 0 \textbf{ do} \\
\quad\quad \mathit{result} \leftarrow \mathit{result} * n \\
\quad\quad n \leftarrow n - 1 \\
\quad \textbf{return } \mathit{result}
\end{array}
$$

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
  if $n = 0$ **then**
    return 1
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\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

function $\textsc{Fac}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

```
function FAC(n)
    if n = 0 then
        return 1
    return FAC(n − 1) ∗ n
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

```
function FAC(n)
    result ← 1
    while n > 0 do
        result ← result ∗ n
        n ← n − 1
    return result
```

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad \textbf{if } n = 0 \textbf{ then} \\
\quad\quad \textbf{return } 1 \\
\quad \textbf{return } \textsc{Fac}(n-1) * n
\end{array}
$$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
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&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad \mathit{result} \leftarrow 1 \\
\quad \textbf{while } n > 0 \textbf{ do} \\
\quad\quad \mathit{result} \leftarrow \mathit{result} * n \\
\quad\quad n \leftarrow n - 1 \\
\quad \textbf{return } \mathit{result}
\end{array}
$$

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

$$
\begin{array}{l}
\textbf{function } \text{FAC}(n) \\
\quad \textbf{if } n = 0 \textbf{ then} \\
\quad\quad \textbf{return } 1 \\
\quad \textbf{return } \text{FAC}(n-1) * n
\end{array}
$$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

$$
\begin{array}{l}
\textbf{function } \text{FAC}(n) \\
\quad \mathit{result} \leftarrow 1 \\
\quad \textbf{while } n > 0 \textbf{ do} \\
\quad\quad \mathit{result} \leftarrow \mathit{result} * n \\
\quad\quad n \leftarrow n - 1 \\
\quad \textbf{return } \mathit{result}
\end{array}
$$

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
    if $n = 0$ **then**
        return 1
    return $\textsc{Fac}(n-1) * n$

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\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
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&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

function $\textsc{Fac}(n)$
    $result \leftarrow 1$
    while $n > 0$ **do**
        $result \leftarrow result * n$
        $n \leftarrow n - 1$
    return $result$

n: 5

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
  if $n = 0$ **then**
    return 1
  return $\textsc{Fac}(n-1) * n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

function $\textsc{Fac}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

n: 5

result: 1

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

```
function Fac(n)
    if n = 0 then
        return 1
    return Fac(n − 1) ∗ n
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

```
function Fac(n)
    result ← 1
    while n > 0 do
        result ← result ∗ n
        n ← n − 1
    return result
```

n: 5

result: 5

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

```
function Fac(n)
    if n = 0 then
        return 1
    return Fac(n − 1) ∗ n
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
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&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

```
function Fac(n)
    result ← 1
    while n > 0 do
        result ← result ∗ n
        n ← n − 1
    return result
```

n: 4

result: 5

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad \textbf{if } n = 0 \textbf{ then} \\
\quad\quad \textbf{return } 1 \\
\quad \textbf{return } \textsc{Fac}(n-1) * n
\end{array}
$$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

$$
\begin{array}{l}
\textbf{function } \textsc{Fac}(n) \\
\quad \mathit{result} \leftarrow 1 \\
\quad \textbf{while } n > 0 \textbf{ do} \\
\quad\quad \mathit{result} \leftarrow \mathit{result} * n \\
\quad\quad n \leftarrow n - 1 \\
\quad \textbf{return } \mathit{result}
\end{array}
$$

n: 4

result: 20

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
    if $n = 0$ **then**
        return 1
    return $\textsc{Fac}(n-1) * n$

$$
\begin{aligned}
F(5) &= F(4)\cdot 5\\
&= (F(3)\cdot 4)\cdot 5\\
&= ((F(2)\cdot 3)\cdot 4)\cdot 5\\
&= (((F(1)\cdot 2)\cdot 3)\cdot 4)\cdot 5\\
&= ((((F(0)\cdot 1)\cdot 2)\cdot 3)\cdot 4)\cdot 5\\
&= ((((1\cdot 1)\cdot 2)\cdot 3)\cdot 4)\cdot 5\\
&= 120
\end{aligned}
$$

function $\textsc{Fac}(n)$
    $result \leftarrow 1$
    while $n > 0$ **do**
        $result \leftarrow result * n$
        $n \leftarrow n - 1$
    return $result$

n: 3

result: 20

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

```
function FAC(n)
    if n = 0 then
        return 1
    return FAC(n − 1) ∗ n
```

```
function FAC(n)
    result ← 1
    while n > 0 do
        result ← result ∗ n
        n ← n − 1
    return result
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

n: 3

result: 60

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

```
function Fac(n)
    if n = 0 then
        return 1
    return Fac(n − 1) ∗ n
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

```
function Fac(n)
    result ← 1
    while n > 0 do
        result ← result ∗ n
        n ← n − 1
    return result
```

n: 2

result: 60

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

```
function FAC(n)
    if n = 0 then
        return 1
    return FAC(n − 1) ∗ n
```

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

```
function FAC(n)
    result ← 1
    while n > 0 do
        result ← result ∗ n
        n ← n − 1
    return result
```

n: 2

result: 120

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\text{FAC}(n)$
  if $n = 0$ **then**
    return 1
  return $\text{FAC}(n-1) * n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

function $\text{FAC}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

n: 1

result: 120

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
  if $n = 0$ **then**
    return 1
  return $\textsc{Fac}(n-1) * n$

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\begin{aligned}
F(5) &= F(4) \cdot 5 \\
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&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 120
\end{aligned}
$$

function $\textsc{Fac}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

n: 0

result: 120

Example: Factorial

- n!: we can use recursion (left) or iteration (right)

function $\textsc{Fac}(n)$
  if $n = 0$ **then**
    return 1
  return $\textsc{Fac}(n-1) * n$

$$
\begin{aligned}
F(5) &= F(4)\cdot 5 \\
&= (F(3)\cdot 4)\cdot 5 \\
&= ((F(2)\cdot 3)\cdot 4)\cdot 5 \\
&= (((F(1)\cdot 2)\cdot 3)\cdot 4)\cdot 5 \\
&= ((((F(0)\cdot 1)\cdot 2)\cdot 3)\cdot 4)\cdot 5 \\
&= ((((1\cdot 1)\cdot 2)\cdot 3)\cdot 4)\cdot 5 \\
&= 120
\end{aligned}
$$

function $\textsc{Fac}(n)$
  $result \leftarrow 1$
  while $n > 0$ **do**
    $result \leftarrow result * n$
    $n \leftarrow n - 1$
  return $result$

n: 0

result: 120

Iterative version normally preferred since it is constant space

Example: Fibonacci Numbers

- To generate the *n*th number of sequence: 1 1 2 3 5 8 13 21 34 55 …

```
function FIB(n)
    if n = 0 then
        return 1
    if n = 1 then
        return 1
    return FIB(n − 1) + FIB(n − 2)
```

Follows the mathematical definition of Fibonacci numbers very closely.

Easy to understand

But performs lots of redundant computation

Basic operation: addition

Complexity is **exponential** in *n*

Fibonacci Again

- Of course we only need to remember the latest two items. Recursive version: left; iterative version: right

function $\text{FIB}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\text{FIB}(n - 1, a + b, a)$

function $\text{FIB}(n)$
  $a \leftarrow 1$
  $b \leftarrow 0$
  while $n > 0$ **do**
    $t \leftarrow a$
    $a \leftarrow a + b$
    $b \leftarrow t$
    $n \leftarrow n - 1$
  return a

Time complexity of both solutions is linear in n

(There is a cleverer, still recursive, way which is $O(\log n)$.)

Fibonacci Again

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- Of course we only need to remember the latest two items. Recursive version: left; iterative version: right

function $\text{FIB}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\text{FIB}(n - 1, a + b, a)$

Initial call: Fib(n, 1, 0)

Time complexity of both solutions is linear in n

function $\text{FIB}(n)$
  $a \leftarrow 1$
  $b \leftarrow 0$
  while $n > 0$ **do**
    $t \leftarrow a$
    $a \leftarrow a + b$
    $b \leftarrow t$
    $n \leftarrow n - 1$
  return a

(There is a cleverer, still recursive, way which is $O(\log n)$.)

Tracing Recursive Fibonacci

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function $\textsc{Fib}(n, a, b)$
    if $n = 0$ **then**
        return a
    return $\textsc{Fib}(n-1, a+b, a)$

Fibonacci
Sequence:
1 1 2 3 5 8 ...

Tracing Recursive Fibonacci

function $\text{FIB}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\text{FIB}(n - 1, a + b, a)$

Initial call: Fib(*5*, 1, 0)

Fibonacci Sequence:
1 1 2 3 5 8 ...

Tracing Recursive Fibonacci

function $\text{Fib}(n, a, b)$
    if $n = 0$ **then**
        return a
    return $\text{Fib}(n - 1, a + b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)

Fibonacci Sequence:
1 1 2 3 5 8 ...

Tracing Recursive Fibonacci

function $\text{FIB}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\text{FIB}(n-1, a+b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)
= Fib(4,1,1)

Fibonacci
Sequence:
1 1 2 3 5 8 ...

Tracing Recursive Fibonacci

function $\textsc{Fib}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\textsc{Fib}(n-1, a+b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)
= Fib(4,1,1)
= Fib(3,2,1)

Fibonacci Sequence:
1 1 2 3 5 8 ...

Tracing Recursive Fibonacci

function $\text{FIB}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\text{FIB}(n-1, a+b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)
= Fib(4,1,1)
= Fib(3,2,1)
= Fib(2,3,2)

Fibonacci
Sequence:
1 1 2 3 5 8 …

Tracing Recursive Fibonacci

function $\text{FIB}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\text{FIB}(n-1, a+b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)
= Fib(4,1,1)
= Fib(3,2,1)
= Fib(2,3,2)
= Fib(1,5,3)

Fibonacci Sequence:
1 1 2 3 5 8 …

Tracing Recursive Fibonacci

function $\textsc{Fib}(n, a, b)$
    if $n = 0$ **then**
        return a
    return $\textsc{Fib}(n - 1, a + b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)
= Fib(4,1,1)
= Fib(3,2,1)
= Fib(2,3,2)
= Fib(1,5,3)
= Fib(0,8,5)

Fibonacci Sequence:
1 1 2 3 5 8 ...

Tracing Recursive Fibonacci

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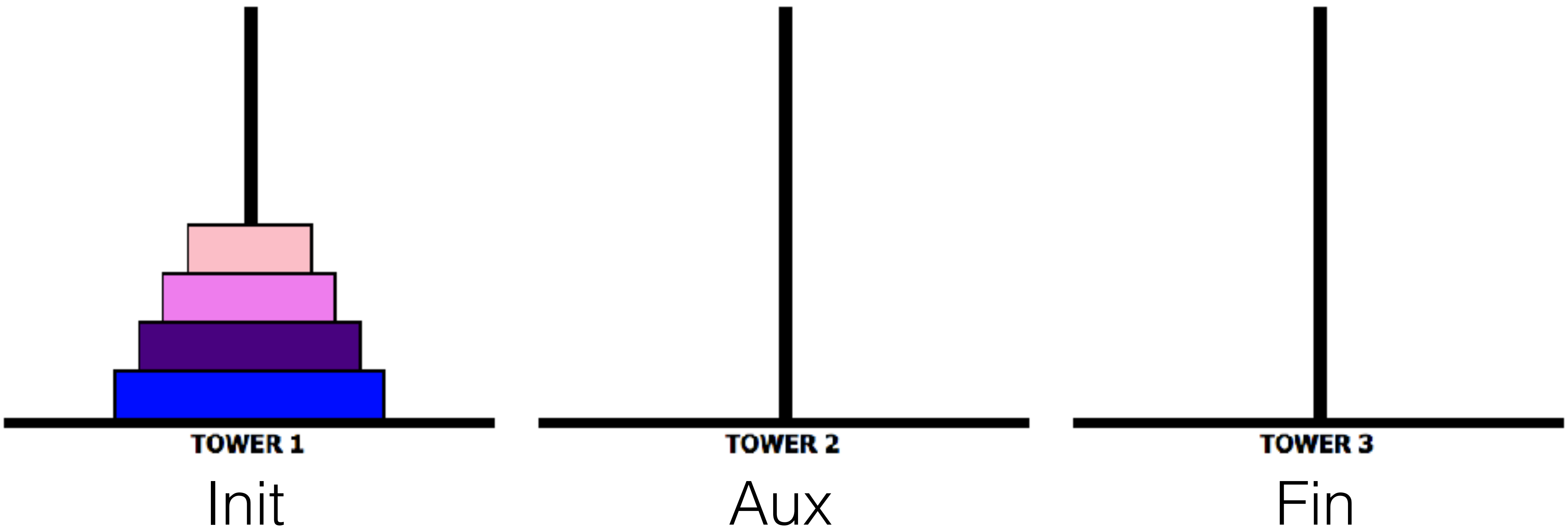
function $\mathrm{FIB}(n, a, b)$
  if $n = 0$ **then**
    return a
  return $\mathrm{FIB}(n-1, a+b, a)$

Initial call: Fib(*5*, 1, 0)

Fib(5,1,0)
= Fib(4,1,1)
= Fib(3,2,1)
= Fib(2,3,2)
= Fib(1,5,3)
= Fib(0,8,5) = 8

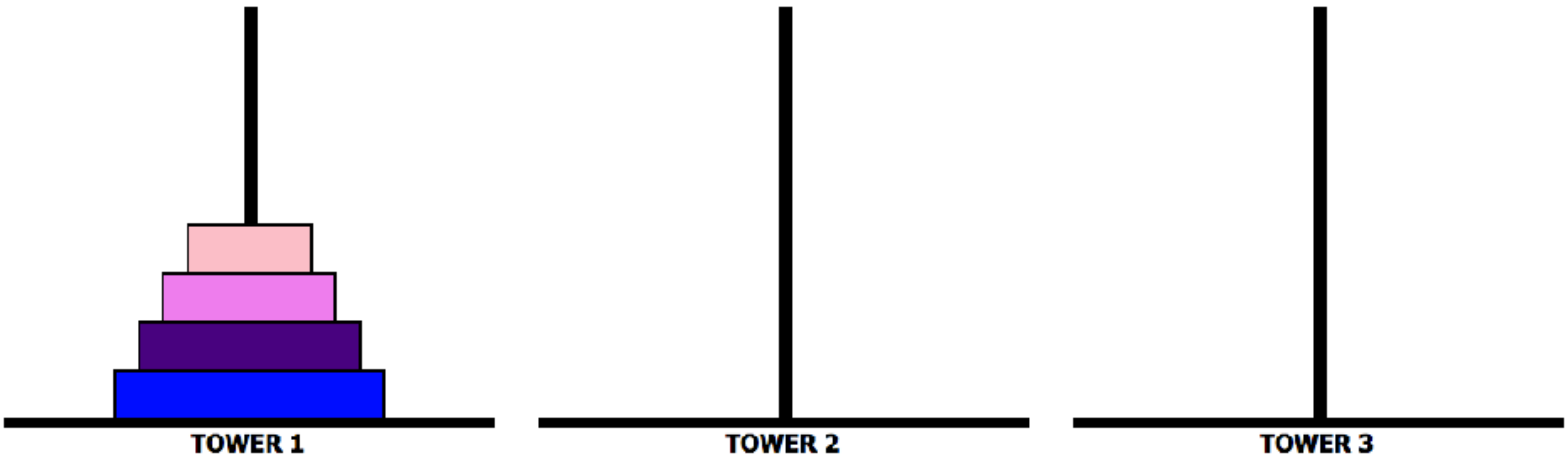
Fibonacci Sequence:
1 1 2 3 5 8 ...

Tower of Hanoi



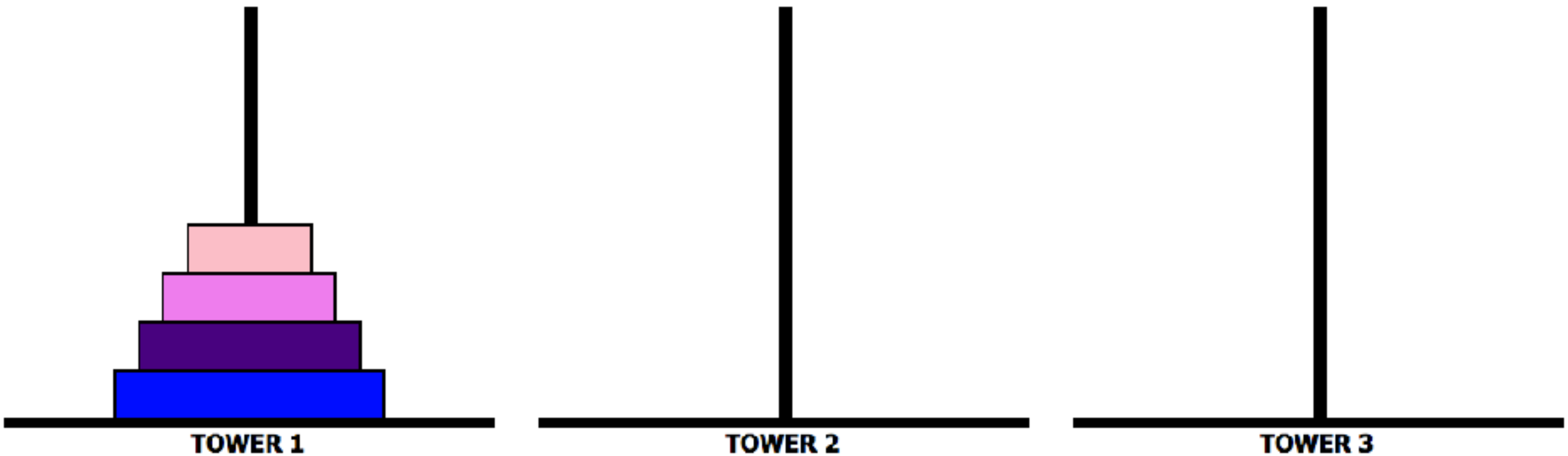
Move *n* disks from *Init* to *Fin*. A larger disk can never be placed on top of a smaller one.

Tower of Hanoi: Recursive Solution



Init Aux Fin

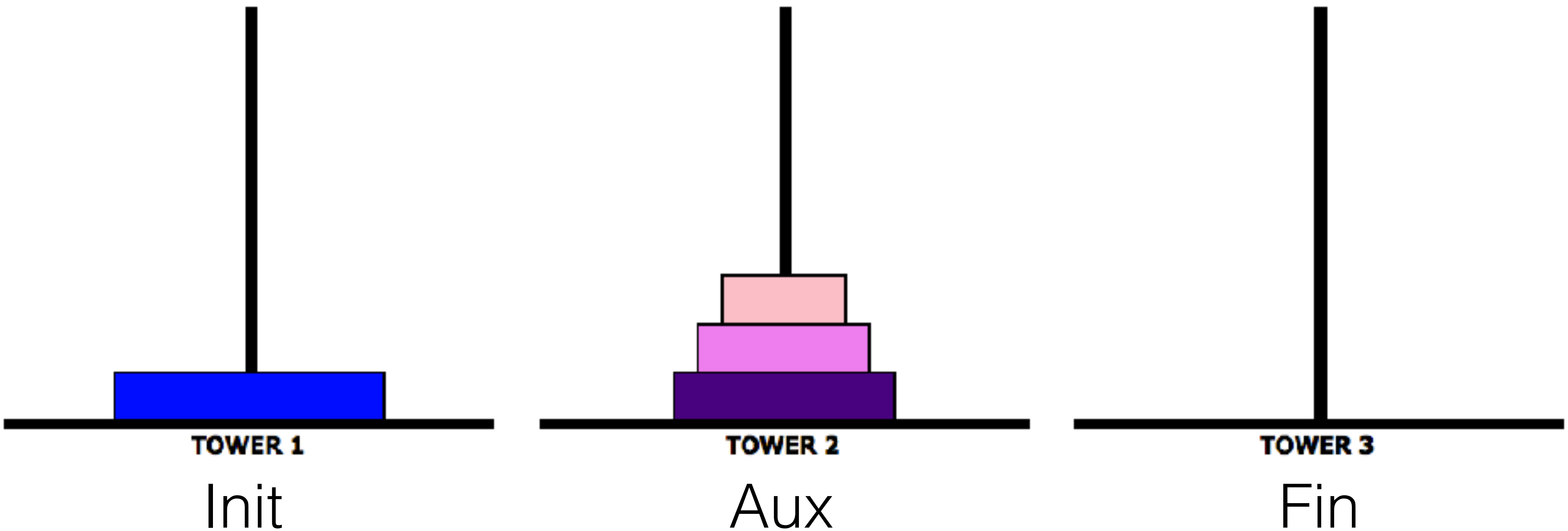
Tower of Hanoi: Recursive Solution



Init Aux Fin

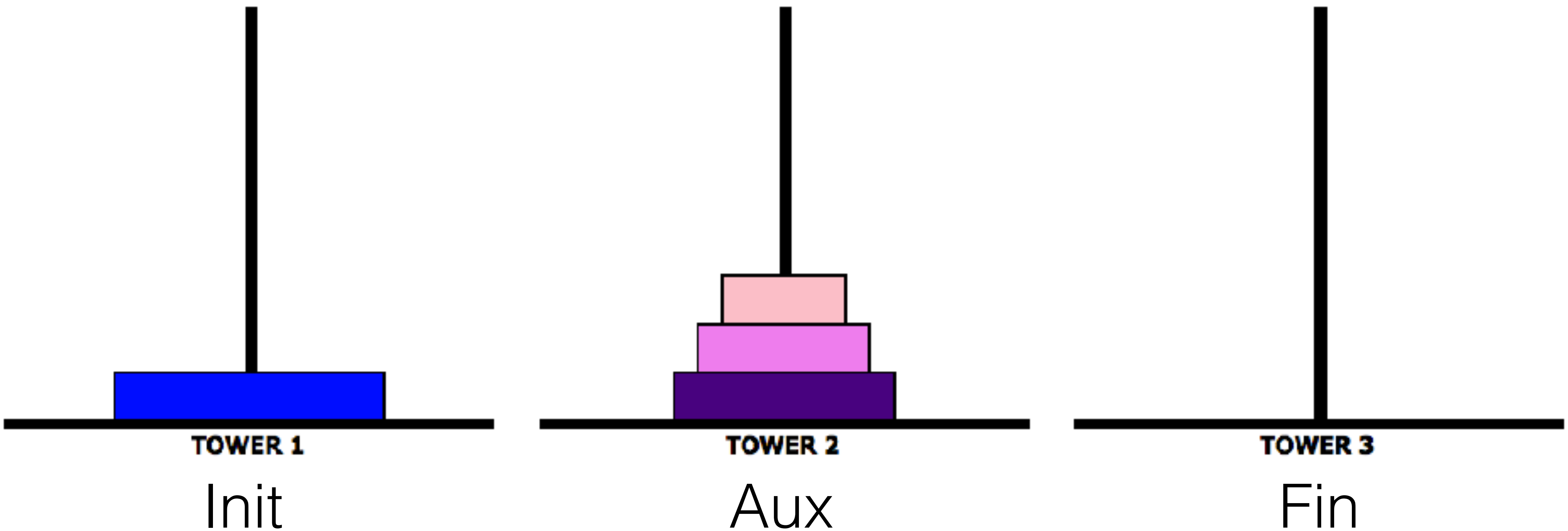
Move *n-1* disks from Init to Aux.

Tower of Hanoi: Recursive Solution



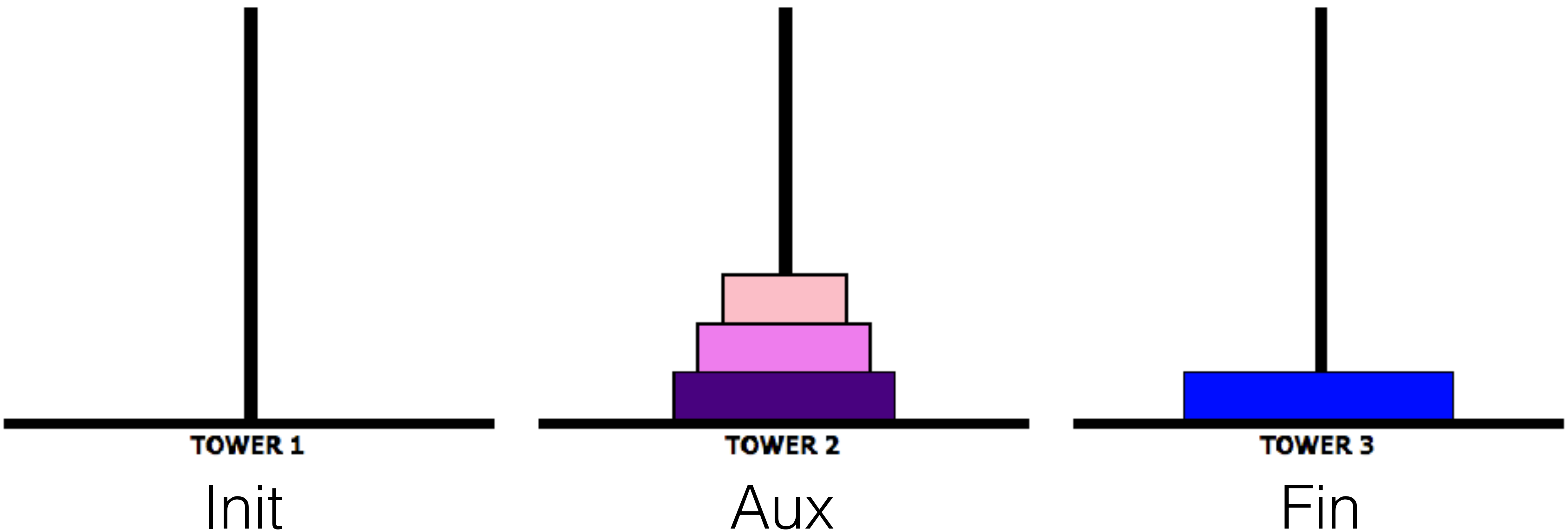
Move *n-1* disks from Init to Aux.

Tower of Hanoi: Recursive Solution



Move *n-1* disks from Init to Aux.
Then move the *n*th disk to Fin.

Tower of Hanoi: Recursive Solution



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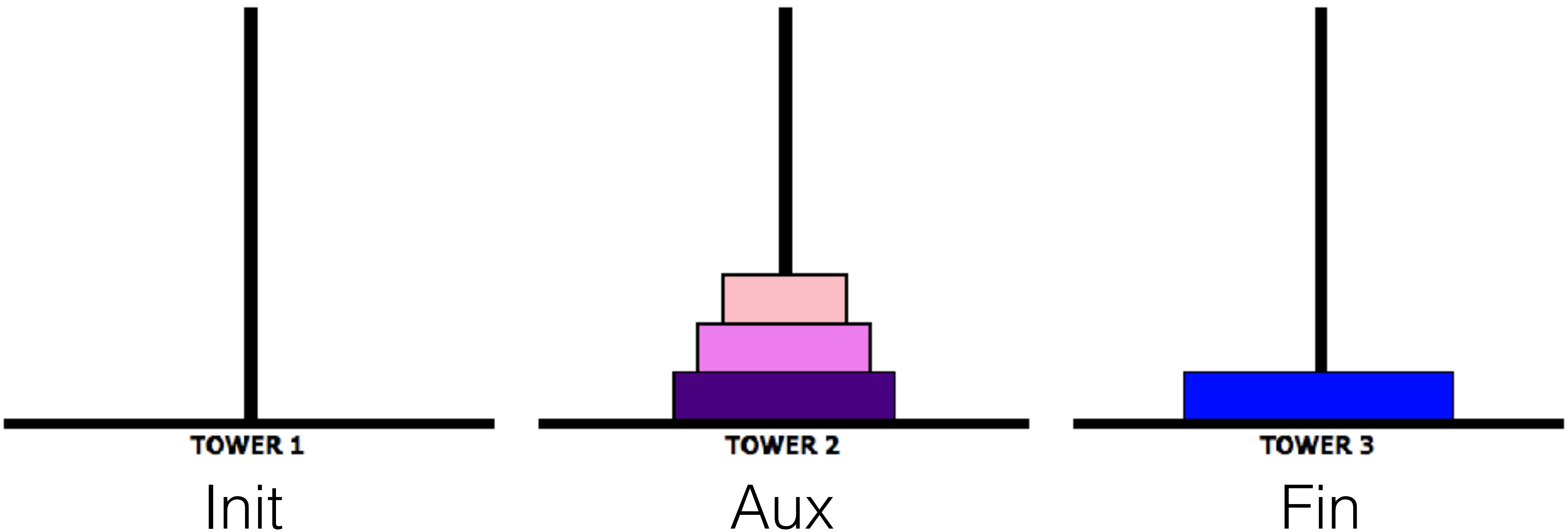
Tower of Hanoi: Recursive Solution



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Tower of Hanoi: Recursive Solution

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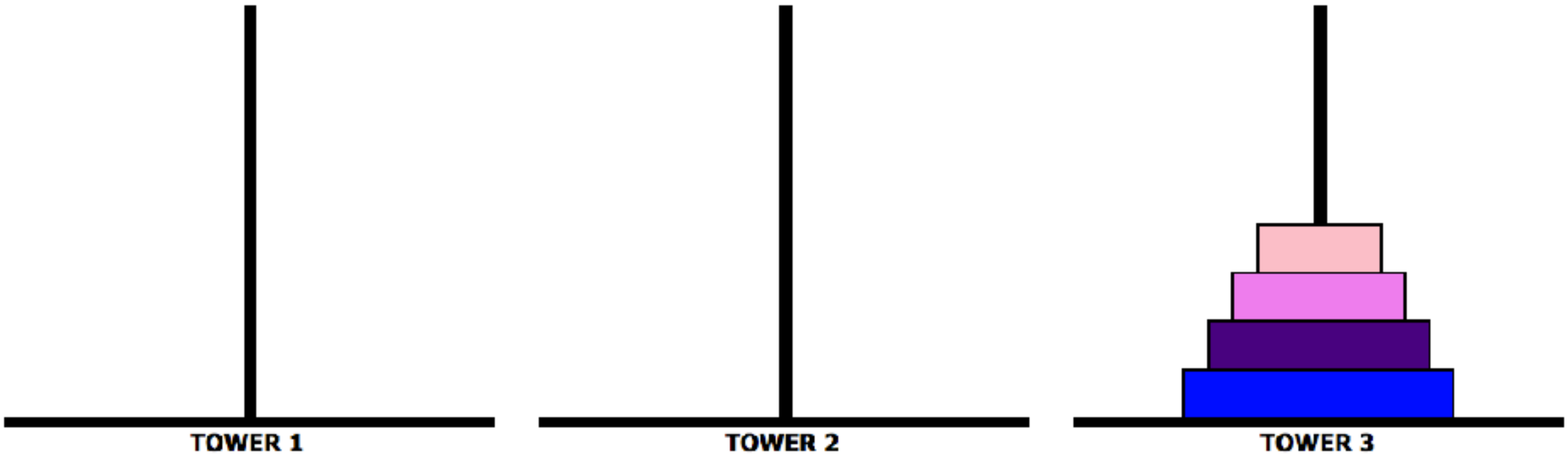

Init Aux Fin

Move *n-1* disks from Init to Aux.
Then move the *n*th disk to Fin.
Then move the *n-1* disks from Aux to Fin.

Tower Of Hanoi: Recursive Algorithm

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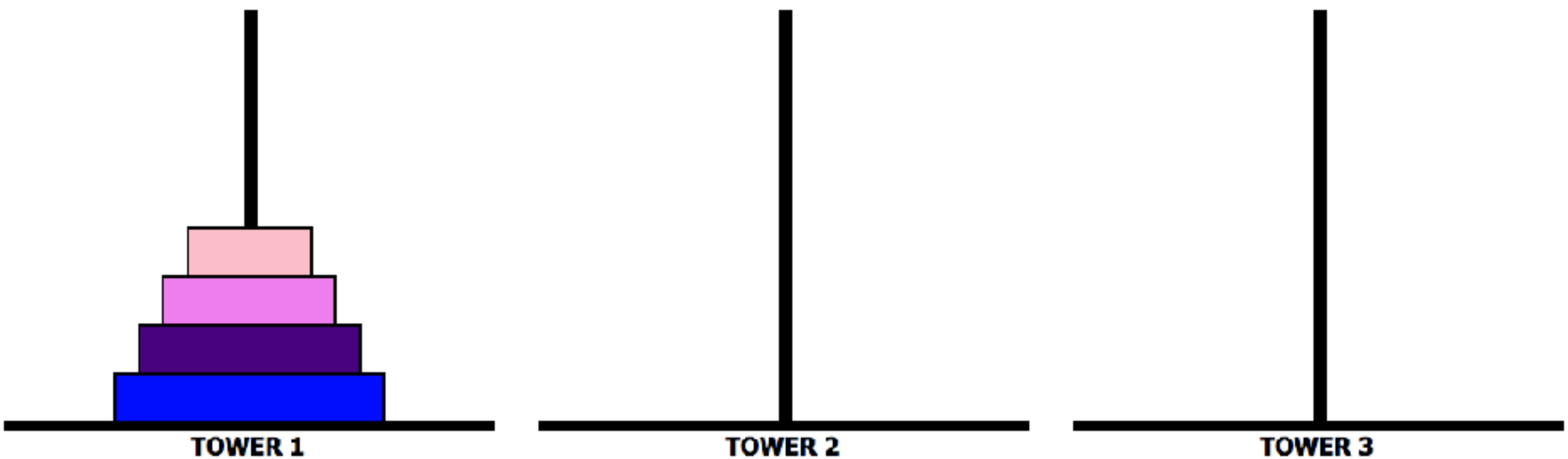
function $\text{HANOI}(n, init, aux, fin)$
  if $n > 0$ **then**
    $\text{HANOI}(n-1, init, fin, aux)$
    Move one disk from *init* to *fin*
    $\text{HANOI}(n-1, aux, init, fin)$



Init   Aux   Fin

Tower Of Hanoi: Recursive Algorithm

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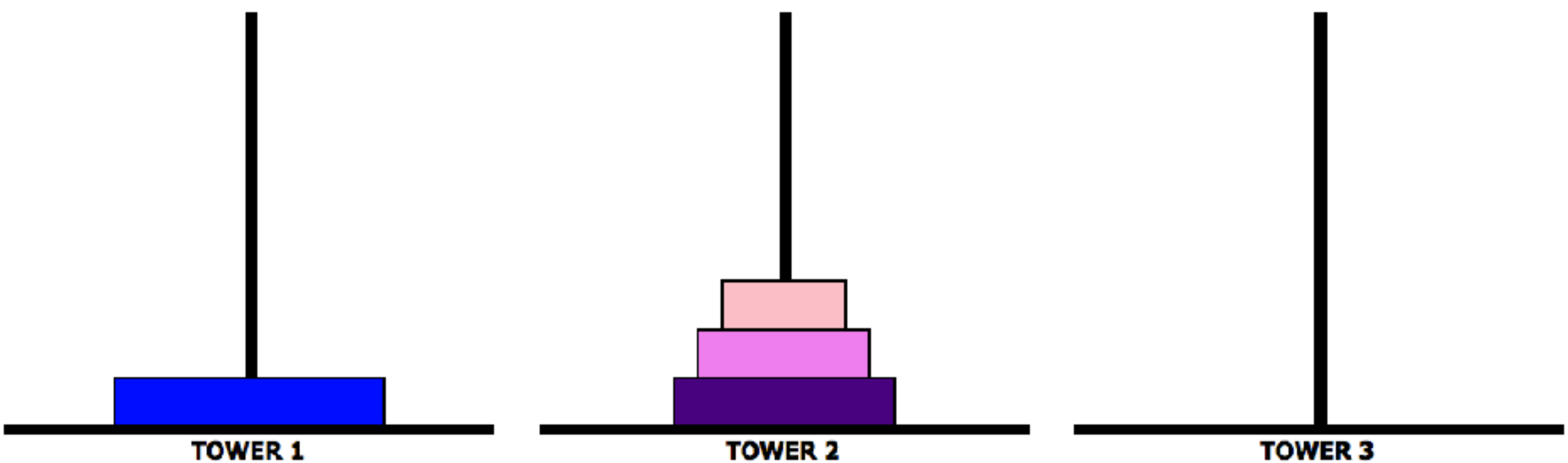

Init Aux Fin

Tower Of Hanoi: Recursive Algorithm

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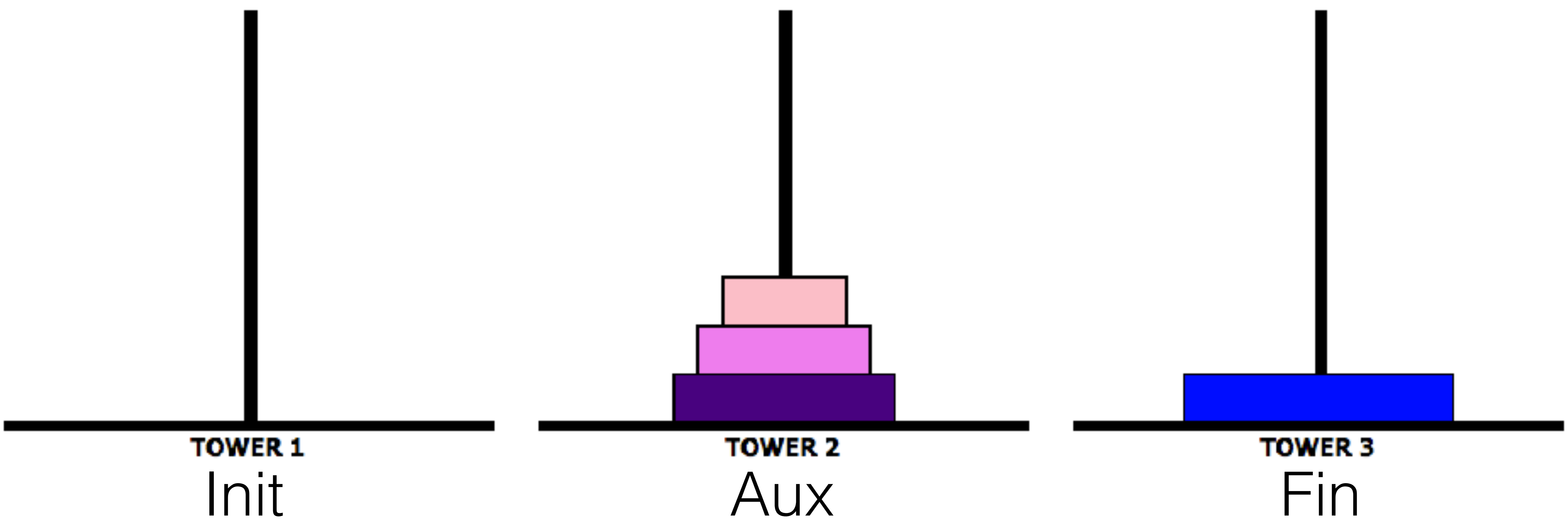
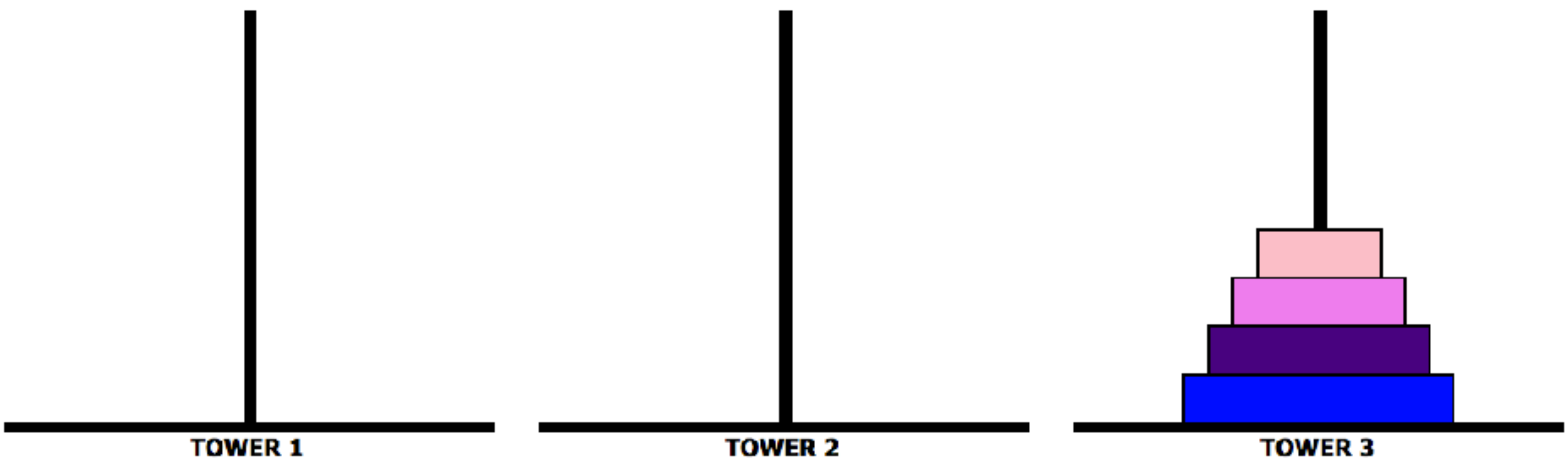


Tower Of Hanoi: Recursive Algorithm

function $\text{HANOI}(n, \mathit{init}, \mathit{aux}, \mathit{fin})$
  if $n > 0$ **then**
    $\text{HANOI}(n-1, \mathit{init}, \mathit{fin}, \mathit{aux})$
    Move one disk from *init* to *fin*
    $\text{HANOI}(n-1, \mathit{aux}, \mathit{init}, \mathit{fin})$



Init Aux Fin

Tracing Tower of Hanoi Recursive Algorithm

http://vornlocher.de/tower.html

A Challenge: Coin Change Problem

- There are 6 different kinds of Australian coin
- In cents, their values are: 5, 10, 20, 50, 100, 200
- In how many different ways can I produce a handful of coins adding up to $4?
- This is not an easy problem!
- Key to solving it is to find a way to break it down into simpler sub-problems

Coin Change Problem: Decomposition

Coin Change Problem: Decomposition

Coin Change Problem: Decomposition

$4

Coin Change Problem: Decomposition

$4

made from

Coin Change Problem: Decomposition

$4

made from

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

Coin Change Problem: Decomposition

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$4

made from

Does the bag contain at least one $2 coin?

Yes

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

+

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

+

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

+ $2

Coin Change Problem: Decomposition

$4

made from

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Yes

+ $2

made from

Coin Change Problem: Decomposition

$4

made from

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Yes

+ $2

made from

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

No

+ $2

made from

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

+ $2

made from

No

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

No

+ $2

$4

made from

Coin Change Problem: Decomposition

$4

made from

Does the bag contain at least one $2 coin?

Yes

\+ $2

made from

No

$4

made from

Coin Change Problem: Decomposition

Coin Change Problem: Decomposition

+

Coin Change Problem: Decomposition

The number of ways of making $4 is therefore:

+

Coin Change Problem: Decomposition

The number of ways of making \$4 is therefore:

1 x the number of ways of making \$2 +

Coin Change Problem: Decomposition

The number of ways of making $4 is therefore:

1 x the number of ways of making $2 +

Coin Change Problem: Decomposition

The number of ways of making $4 is therefore:

1 x the number of ways of making $2

+

the number of ways of making $4 without using a $2 coin

Coin Change Problem: Partial Algorithm

function WAYS(amount, denominations)

// … base cases ….

d ← selectLargest(denominations)

return WAYS(amount – d, denominations) +

WAYS(amount, denominations \ {d})

For example:

Ways(400, {5,10,20,50,100,200}) =

Ways(200, {5,10,20,50,100,200}) +

Ways(400, {5,10,20,50,100})

Coin Change Problem: Base Cases

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or

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- Each time we recurse, we decrease either:
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  - amount base cases:

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or
  - demonisations (by removing an item from the set)
- Consider each of these separately.
  - amount base cases:
    - amount = 0:

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or
  - demonisations (by removing an item from the set)
- Consider each of these separately.
  - amount base cases:
    - amount = 0: there is **one** way (using no coins)

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or
  - demonisations (by removing an item from the set)
- Consider each of these separately.
  - amount base cases:
    - amount = 0: there is **one** way (using no coins)
    - amount < 0:

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or
  - demonisations (by removing an item from the set)
- Consider each of these separately.
  - amount base cases:
    - amount = 0: there is **one** way (using no coins)
    - amount < 0: there are **no ways** to make this

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or
  - demonisations (by removing an item from the set)
- Consider each of these separately.
  - amount base cases:
    - amount = 0: there is **one** way (using no coins)
    - amount < 0: there are **no ways** to make this
    - denominations = ∅ (and amount > 0):

Coin Change Problem: Base Cases

- Each time we recurse, we decrease either:
  - amount (by subtracting some quantity from it), or
  - demonisations (by removing an item from the set)

- Consider each of these separately.
  - amount base cases:
    - amount = 0: there is **one** way (using no coins)
    - amount < 0: there are **no ways** to make this
    - denominations = ∅ (and amount > 0): there are **no ways** to make this amount

Coin Change Problem: Full Recursive Algorithm

```
function WAYS(amount, denominations)
    if amount = 0 then
        return 1
    if amount < 0 then
        return 0
    if denominations = ∅ then
        return 0
    d ← selectLargest(denominations)
    return  WAYS(amount − d, denominations) +
            WAYS(amount, denominations \ {d})
```

Initial call: WAYS(amount, $\{5, 10, 20, 50, 100, 200\}$).

Recursive Solution and its Complexity

- Although our recursive algorithm is short and elegant, it is not the most efficient way of solving the problem.
- Its running time grows **exponentially** as you grow the input amount.
- More efficient solutions can be developed using **memoing** or **dynamic programming**—more about that later (around Week 10).

Next Time…

- Graphs, trees, graph traversal and allied algorithms.