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COMP90038
Algorithms and Complexity

Lecture 10: Decrease-and-Conquer-by-a-Factor
(with thanks to Harald Søndergaard)

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Decrease-and-Conquer

- **Last lecture:** to solve a problem of size n, try to express the solution in terms of a solution to the same problem of size n–1.

- A simple example was sorting: To sort an array of length n, just:

   1. sort the first n – 1 items, then
   2. locate the cell A[j] that should hold the last item, right-shift all elements to its right, then place the last element in A[j].

- This led to an $O(n^2)$ algorithm called **insertion sort**. We can implement the idea either with recursion or iteration (we chose iteration).

Decrease-and-Conquer by-a-Factor

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- We now look at better utilization of the approach, often leading to methods with logarithmic time behaviour or better!

- **Decrease-by-a-constant-factor** is exemplified by binary search.

- **Decrease-by-a-variable-factor** is exemplified by interpolation search.

- Let us look at these and other instances.

Binary Search

- This is a well-known approach for searching for an element *k* in a sorted array.

- Start by comparing against the array's middle element A[*m*]. If A[*m*] = *k* we are done.

- If A[m] > k, search the sub-array up to A[m – 1] recursively.

- If A[m] < k, search the sub-array from A[m + 1] recursively.

$$\underbrace{A[0]\ \cdots\ A[m-1]}_{\text{search here if } A[m] > k} \quad \overset{\overset{=k?}{\downarrow}}{\color{blue}{A[m]}} \quad \underbrace{A[m+1]\ \cdots\ A[n-1]}_{\text{search here if } A[m] < k}$$

Binary Search

- We have already seen a recursive formulation in Lecture 4. Here is an iterative one.

function $\textsc{BinSearch}(A[\cdot], n, k)$
    $lo \leftarrow 0$
    $hi \leftarrow n - 1$
    while $lo \leq hi$ **do**
        $m \leftarrow \lfloor (lo + hi)/2 \rfloor$
        if $A[m] = k$ **then**
            return m
        if $A[m] > k$ **then**
            $hi \leftarrow m - 1$
        else
            $lo \leftarrow m + 1$
    return -1

Example:
Binarv Search in Sorted Array

```
function BinSearch(A[·], n, k)
    lo ← 0
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            hi ← m − 1
        else
            lo ← m + 1
    return −1
```

A:

A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

Example:
Binarv Search in Sorted Array

function $\text{BinSearch}(A[\cdot], n, k)$
- $lo \leftarrow 0$
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    - $lo \leftarrow m + 1$
- **return** -1

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BinSearch(A,7,41)

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Binarv Search in Sorted Array

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k: 41

A:

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    return -1

k: 41

lo: 0

hi: 6

A:

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0	1	2	3	4	5	6

BinSearch(A,7,41)

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k: 41

lo: 0

hi: 6

m: 3

A:

0	1	2	3	4	5	6
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BinSearch(A,7,41)

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Binarv Search in Sorted Array

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Example: Binarv Search in Sorted Array

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k: 41

lo: 4

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k: 41

lo: 4

hi: 6

m: 5

A:

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Example:
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```

k: 41

lo: 4

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4	9	13	22	41	83	96
0	1	2	3	4	5	6

BinSearch(A,7,41)

Example:
Binarv Search in Sorted Array

$$
\begin{array}{l}
\textbf{function } \textsc{BinSearch}(A[\cdot], n, k) \\
\quad lo \leftarrow 0 \\
\quad hi \leftarrow n - 1 \\
\quad \textbf{while } lo \leq hi \textbf{ do} \\
\quad\quad m \leftarrow \lfloor (lo + hi)/2 \rfloor \\
\quad\quad \textbf{if } A[m] = k \textbf{ then} \\
\quad\quad\quad \textbf{return } m \\
\quad\quad \textbf{if } A[m] > k \textbf{ then} \\
\quad\quad\quad hi \leftarrow m - 1 \\
\quad\quad \textbf{else} \\
\quad\quad\quad lo \leftarrow m + 1 \\
\quad \textbf{return } -1
\end{array}
$$

k: 41

lo: 4

hi: 4

m: 5

A:

0	1	2	3	4	5	6
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BinSearch(A,7,41)

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BinSearch(A,7,41)

Complexity of Binary Search

- Worst-case input to binary sarch:
  - When *k* is not in the array
- In that case, its complexity is given by the following recursive equation:

$$C(n) = \begin{cases} 1 & \text{if } n = 1 \\ C(\lfloor n/2 \rfloor) + 1 & \text{if } n > 1 \end{cases}$$

- A closed form is: $C(n) = \lfloor \log_2 n \rfloor + 1$
- In the worst case, searching for k in an array of size 1,000,000 requires 20 comparisons.
- The average-case time complexity is also Θ(log n)

Russian Peasant Multiplication

- A way of doing multiplication.

- For even *n*:

$$n \cdot m = \frac{n}{2} \cdot 2m$$

- For odd *n:*

$$n \cdot m = \frac{n-1}{2} \cdot 2m + m$$

- Thus, ~halve *n* repeatedly, until n = 1. Add up all odd values of *m*

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n

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n	m

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n	m
81	

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n	m
81	92

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n	m	
81	92	92

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n	m	
81	92	92
40		

Russian Peasant Multiplication

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n	m	
81	92	92
40	184	

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n	m	
81	92	92
40	184	

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n	m	
81	92	92
40	184	
20		

Russian Peasant Multiplication

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- For even *n*:

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- Thus, ~halve *n* repeatedly, until n = 1. Add up all odd values of *m*

n	m	
81	92	92
40	184	
20	368	

Russian Peasant Multiplication

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- A way of doing multiplication.

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- Thus, ~halve n repeatedly, until n = 1. Add up all odd values of m

n	m	
81	92	92
40	184	
20	368	

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n	m	
81	92	92
40	184	
20	368	
10		

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n	m	
81	92	92
40	184	
20	368	
10	736	

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n	m	
81	92	92
40	184	
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n	m	
81	92	92
40	184	
20	368	
10	736	
5		

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n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	

Russian Peasant Multiplication

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n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472

Russian Peasant Multiplication

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n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2		

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n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2	2944	

Russian Peasant Multiplication

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n	m	
81	92	92
40	184	
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2	2944	

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n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2	2944	
1		

Russian Peasant Multiplication

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- Thus, ~halve *n* repeatedly, until n = 1. Add up all odd values of *m*

n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2	2944	
1	5888	

Russian Peasant Multiplication

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n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2	2944	
1	5888	5888

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n	m	
81	92	92
40	184	
20	368	
10	736	
5	1472	1472
2	2944	
1	5888	5888

= 7452

Finding the Median

- Given an array, an important problem is how to find the **median**, that is, an array value which is no larger than half the elements and no smaller than half.

A:	9	23	3	41	22	8	46
	0	1	2	3	4	5	6

- More generally, we would like to solve the problem of finding the kth smallest element. (e.g. when k=3)

A:	9	23	3	41	22	8	46
	0	1	2	3	4	5	6

- If the array is sorted, the solution is straight-forward, so one approach is to start by sorting (as we'll soon see, this can be done in time O (n log n)).

A:	3	8	9	22	23	41	46
	0	1	2	3	4	5	6

- However, sorting the array seems like overkill.

A Detour via Partitioning

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- Partitioning an array around some pivot element p means reorganizing the array so that all elements to the left of p are no greater than p, while those to the right are no smaller.

A:

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0	1	2	3	4	5	6

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Partitioning around the pivot 9

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Partitioning around the pivot 9

A:

3	8	9	23	22	41	46
0	1	2	3	4	5	6

Lomuto Partitioning

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```
function LomutoPartition(A[·], lo, hi)
    p ← A[lo]
    s ← lo
    for i ← lo + 1 to hi do
        if A[i] < p then
            s ← s + 1
            swap(A[s], A[i])
    swap(A[lo], A[s])
    return s
```

lo						hi
9	23	3	41	22	8	46
0	1	2	3	4	5	6

Lomuto Partitioning

function LomutoPartition($A[\cdot], lo, hi$)
    $p \leftarrow A[lo]$
    $s \leftarrow lo$
    for $i \leftarrow lo + 1$ **to** hi **do**
        if $A[i] < p$ **then**
            $s \leftarrow s + 1$
            $swap(A[s], A[i])$
    $swap(A[lo], A[s])$
    return s

lo						hi
9	23	3	41	22	8	46
0	1	2	3	4	5	6
s						

Lomuto Partitioning

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lo						hi
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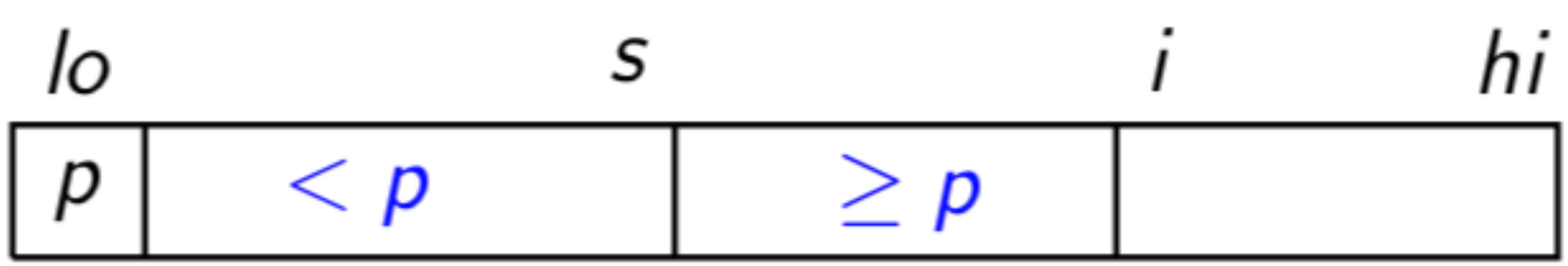

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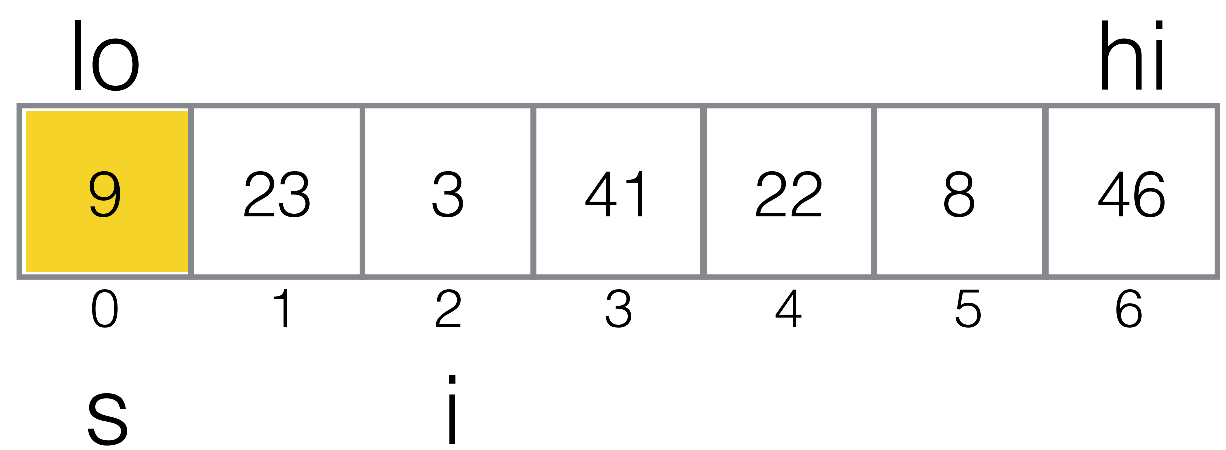

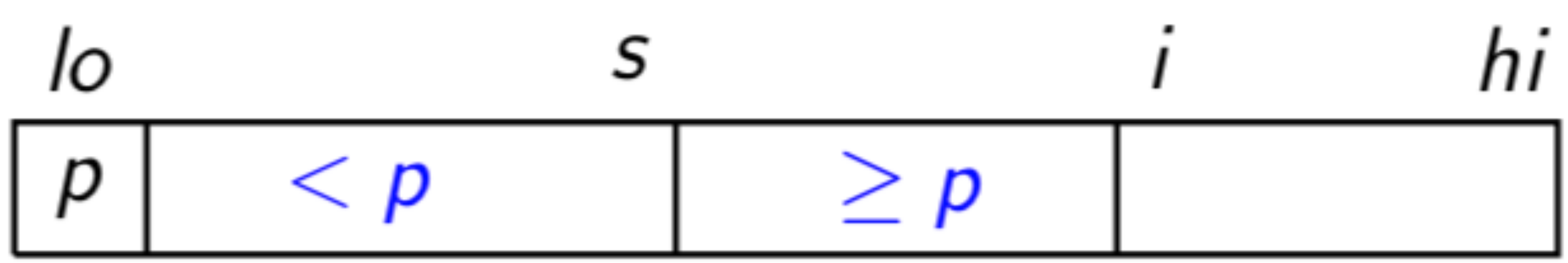

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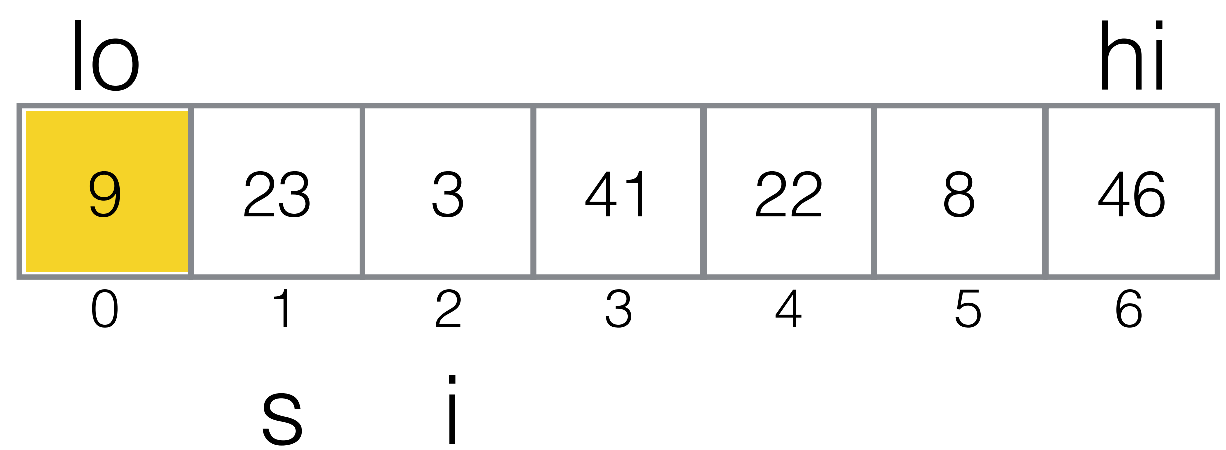

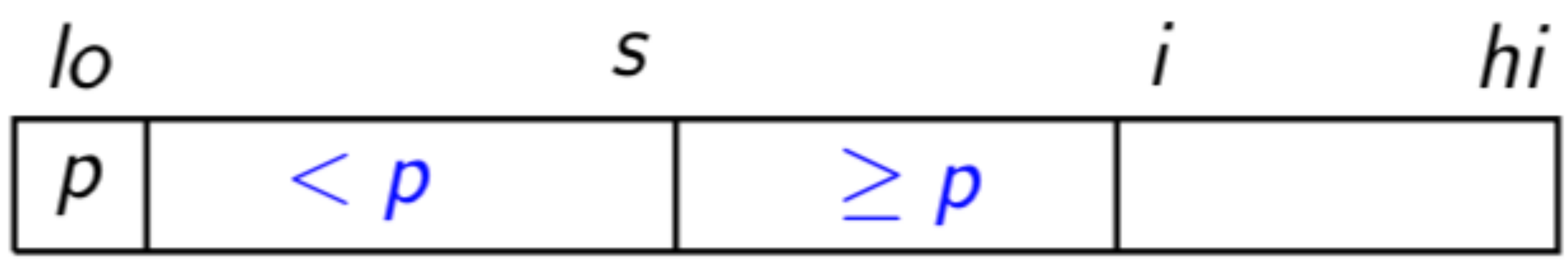

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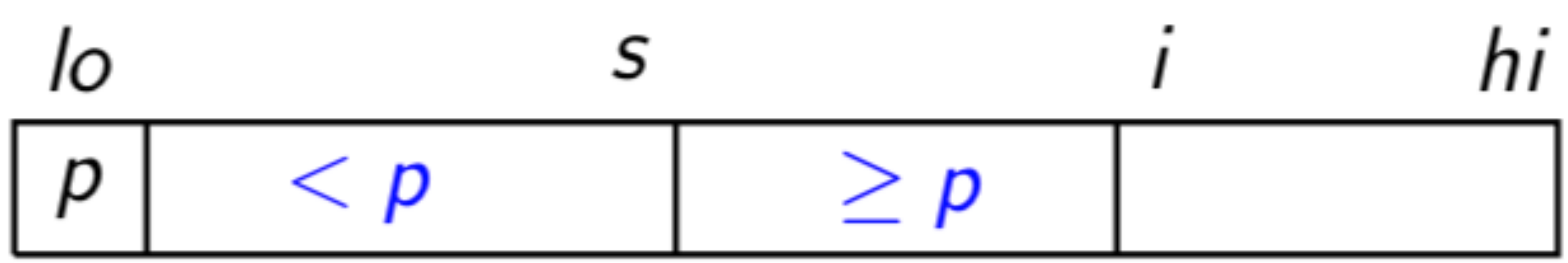

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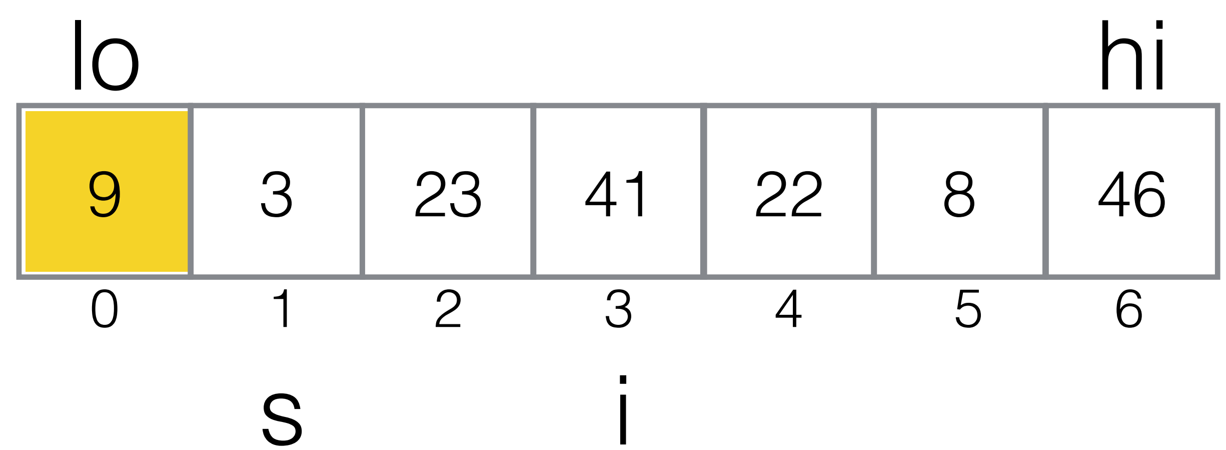

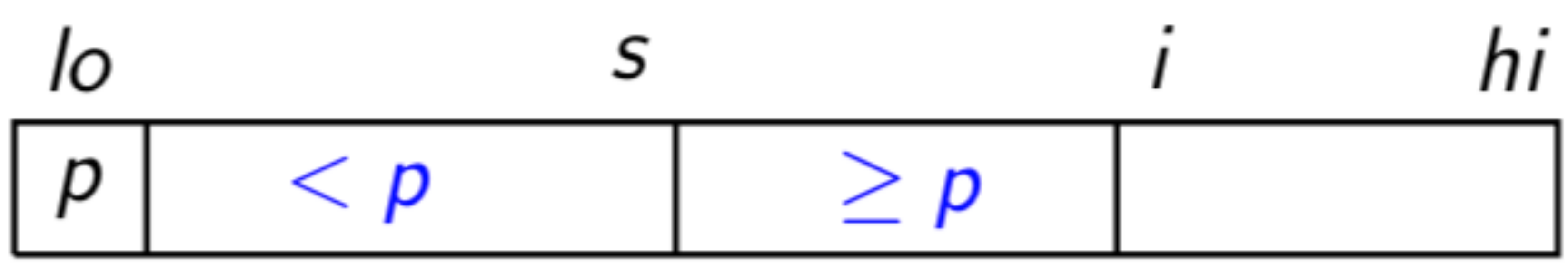

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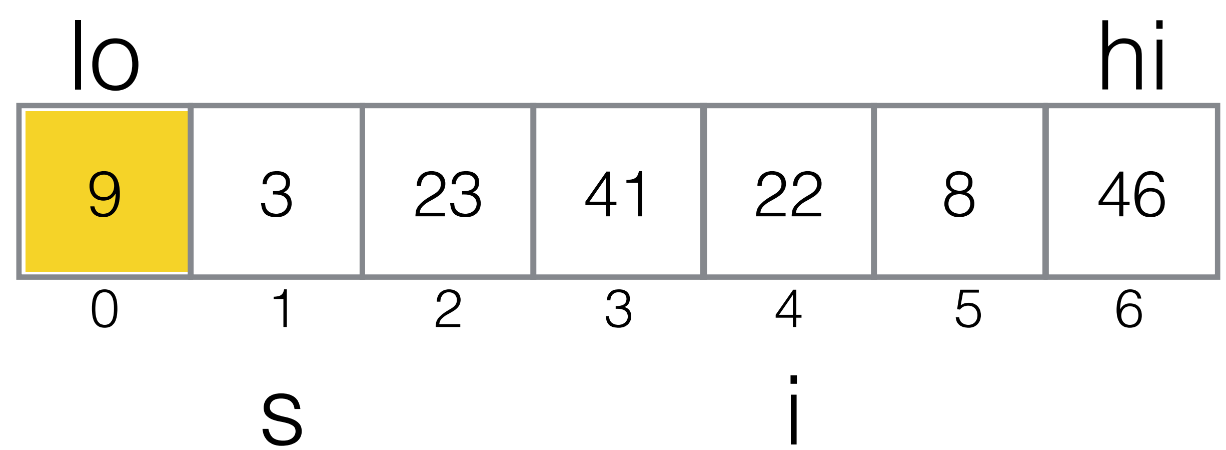

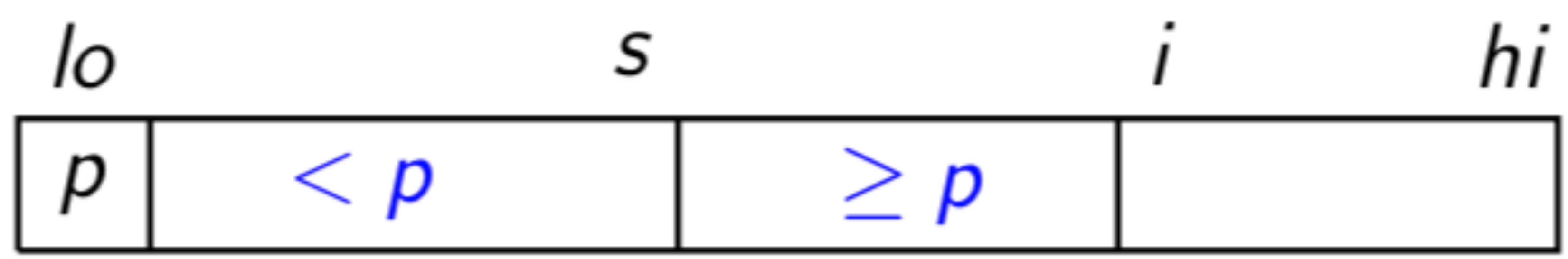

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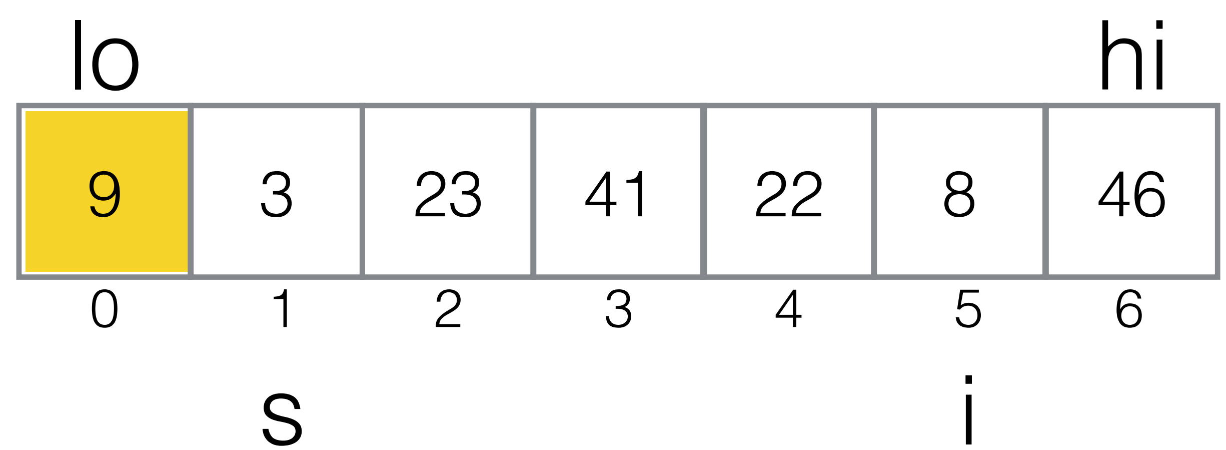

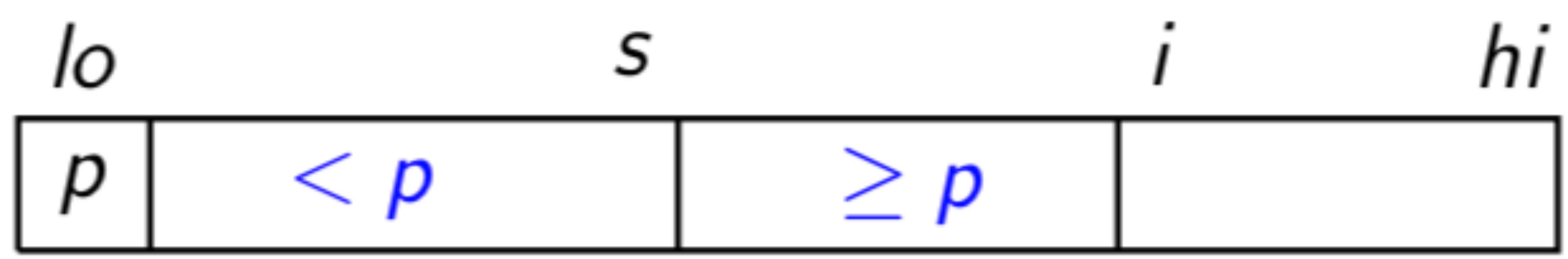

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0	1	2	3	4	5	6
		s			i	

lo	s	i	hi
p	$< p$	$\geq p$	

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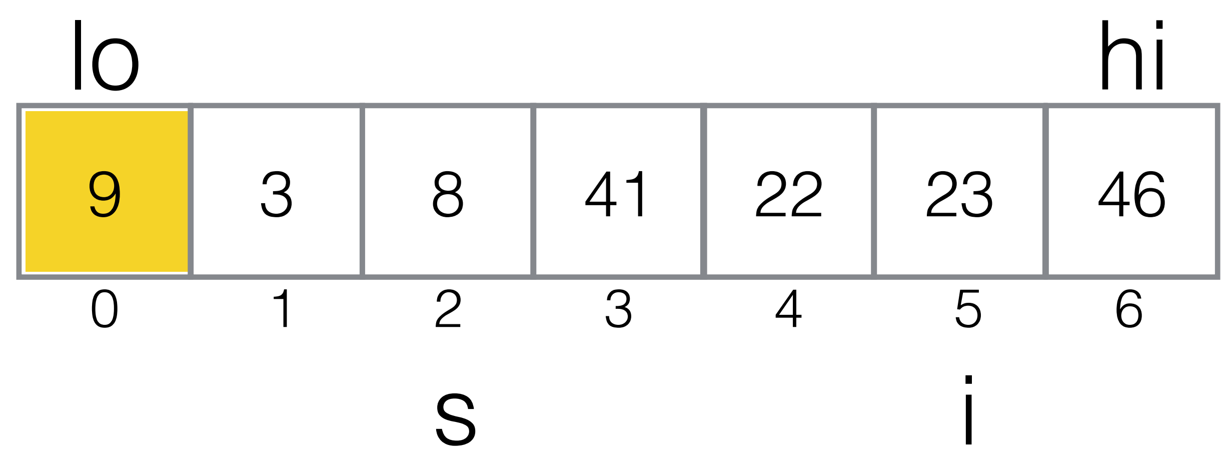

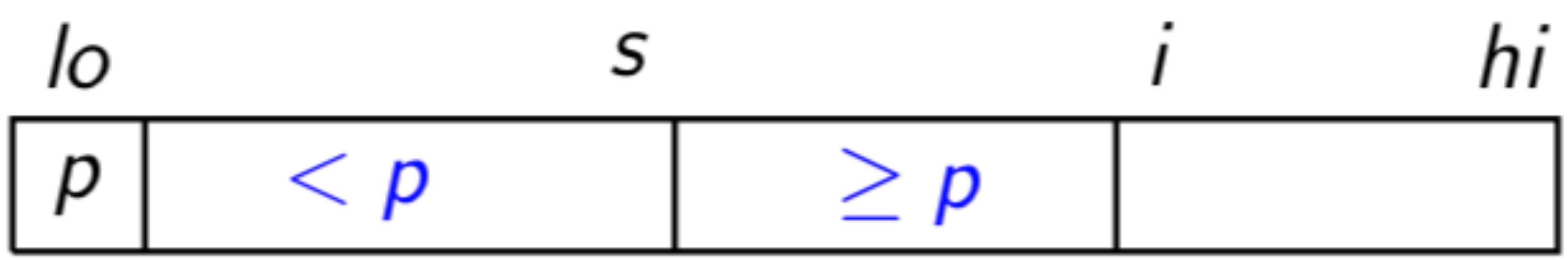

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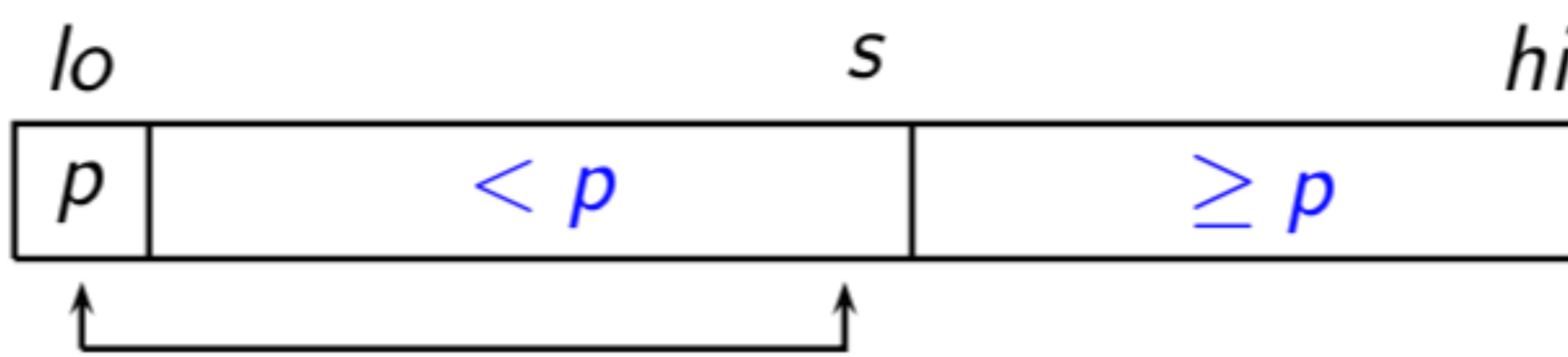

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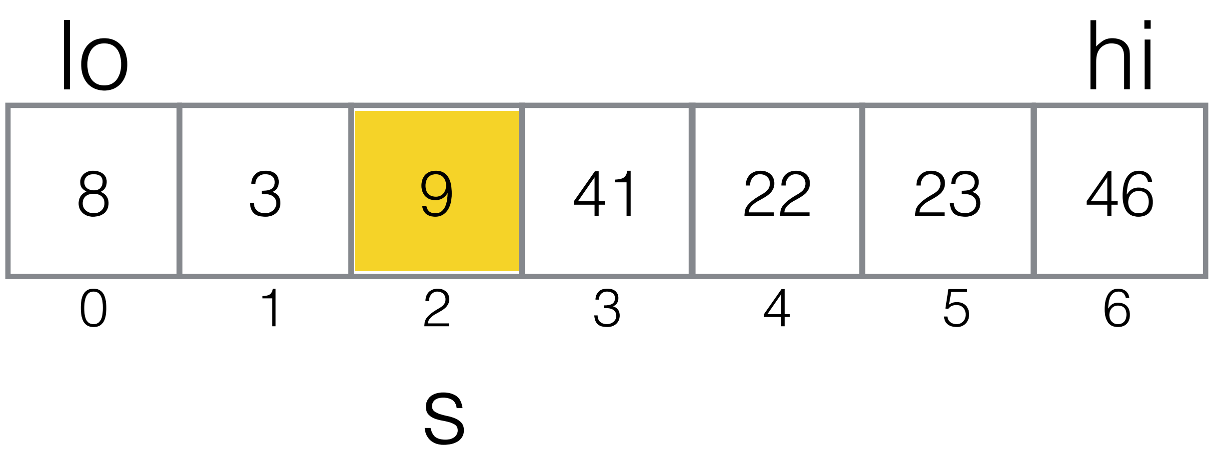

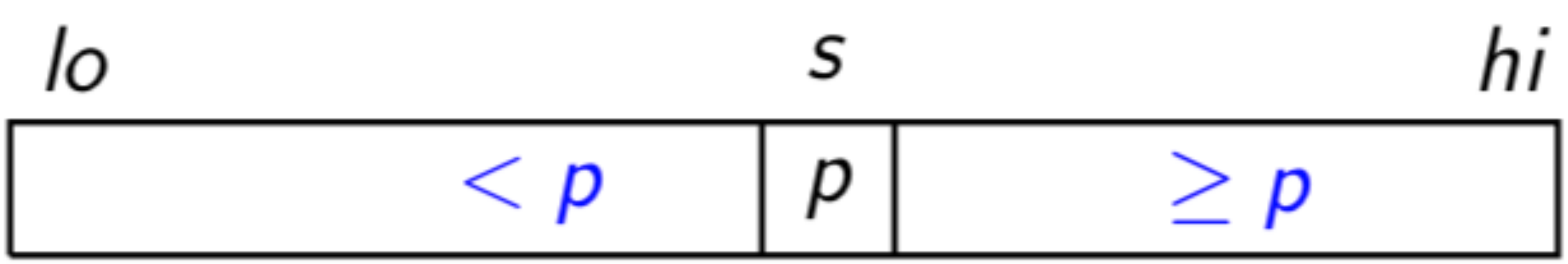

Finding the *k*th-smallest Element

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function QuickSelect($A[\cdot]$, lo, hi, k)
    $s \leftarrow$ LomutoPartition(A, lo, hi)
    if $s - lo = k - 1$ **then**
        return $A[s]$
    else
        if $s - lo > k - 1$ **then**
            QuickSelect(A, lo, $s - 1$, k)
        else
            QuickSelect(A, $s + 1$, hi, $(k - 1) - (s - lo)$)

lo						hi
8	3	9	41	22	23	46
0	1	2	3	4	5	6
		s				

Example: Find the Sixth Smallest Element

function $\textsc{QuickSelect}(A[\cdot], lo, hi, k)$
    $s \leftarrow \textsc{LomutoPartition}(A, lo, hi)$
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k: 6

lo						hi
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0	1	2	3	4	5	6

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lo						hi
8	3	9	41	22	23	46
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		s				

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k: 3

			lo			hi
8	3	9	41	22	23	46
0	1	2	3	4	5	6

Example: Find the Sixth Smallest Element

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    $p \leftarrow A[lo]$
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			lo			hi
8	3	9	41	22	23	46
0	1	2	3	4	5	6
			s	i		

Example: Find the Sixth Smallest Element

function LOMUTOPARTITION($A[\cdot], lo, hi$)

$\quad p \leftarrow A[lo]$

$\quad s \leftarrow lo$

$\quad$ **for** $i \leftarrow lo + 1$ **to** hi **do**

$\quad\quad$ **if** $A[i] < p$ **then**

$\quad\quad\quad s \leftarrow s + 1$

$\quad\quad\quad swap(A[s], A[i])$

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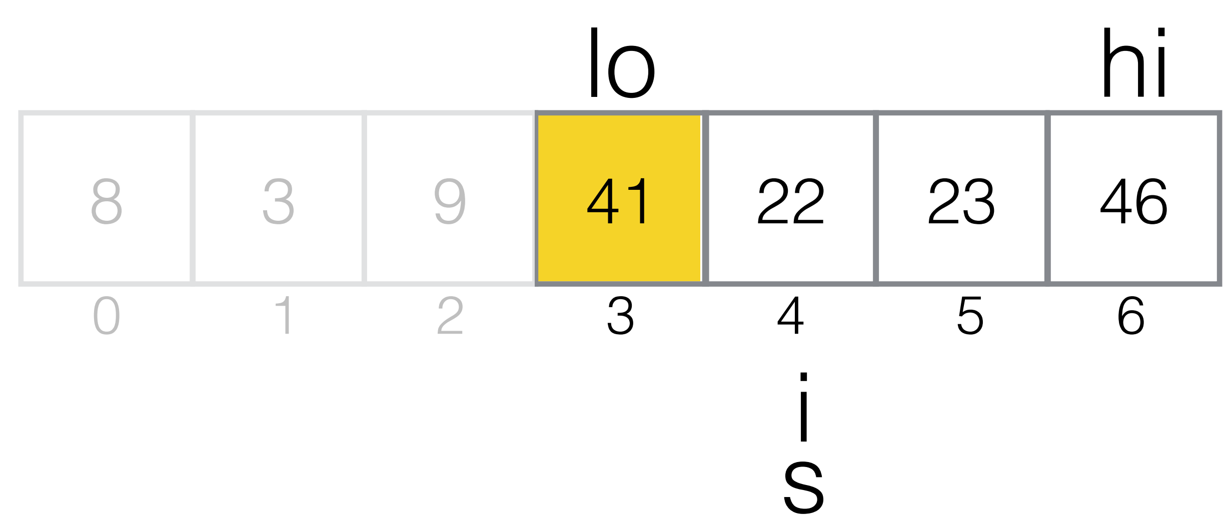
$\quad$ **return** s



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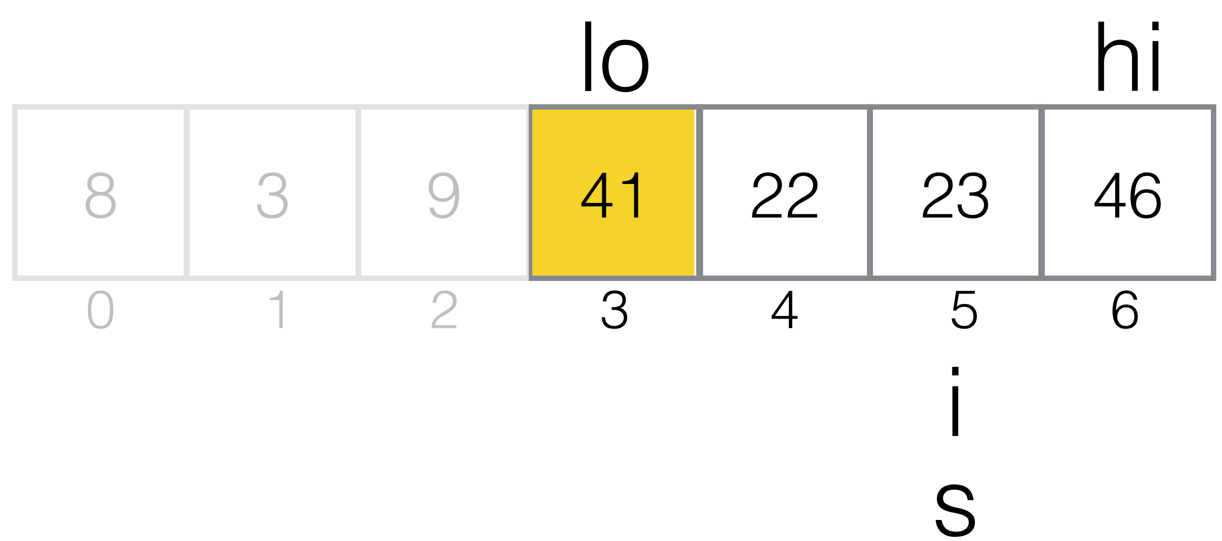

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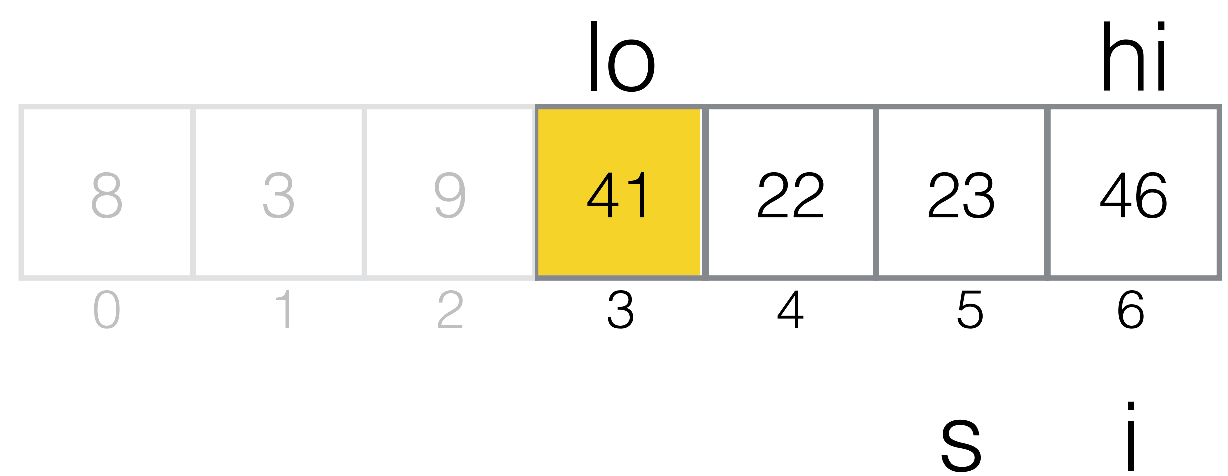

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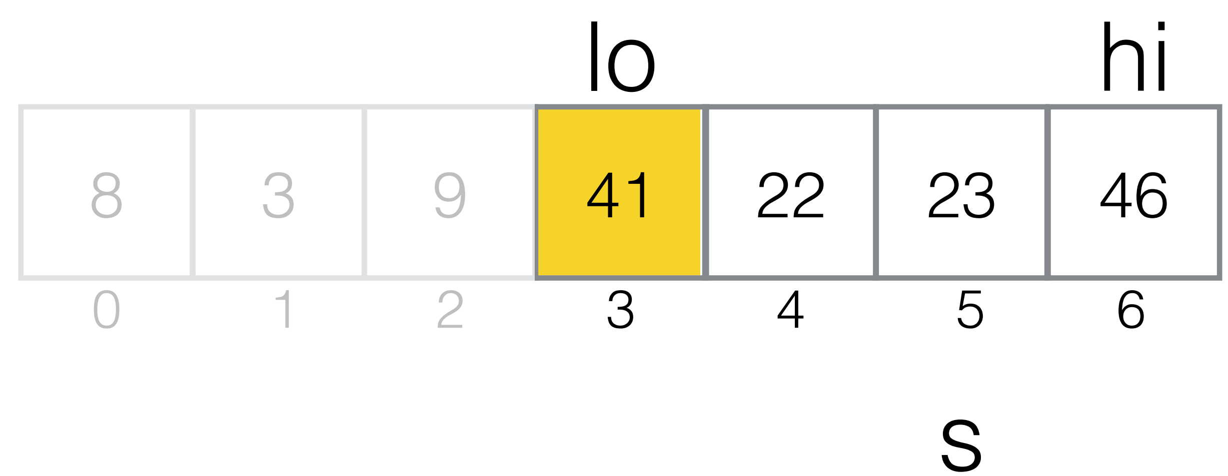

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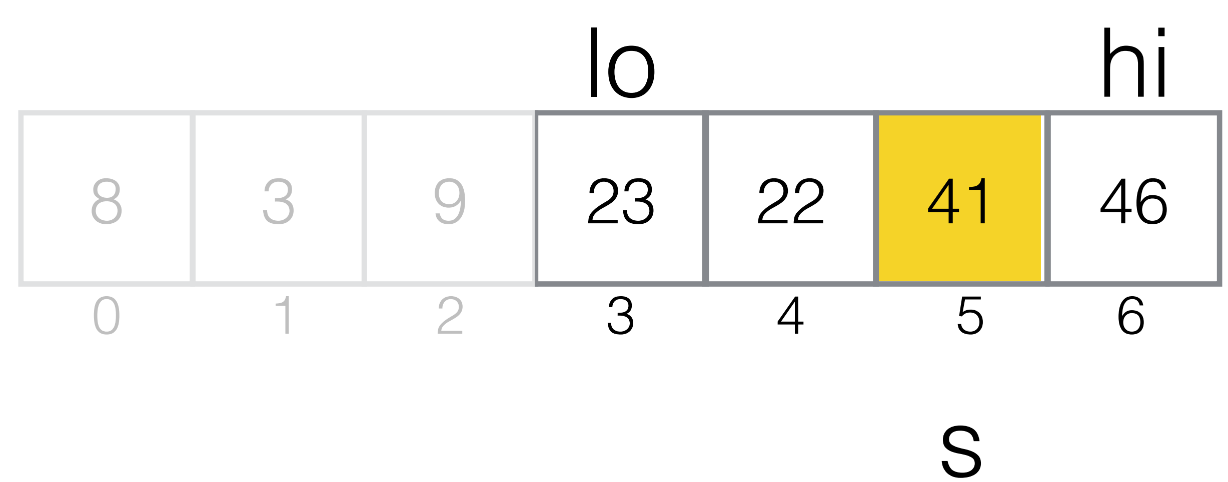

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      QuickSelect(A, $s + 1$, hi, $(k - 1) - (s - lo)$)

k: 3

			lo			hi
8	3	9	23	22	41	46
0	1	2	3	4	5	6
					s	

Example: Find the Sixth Smallest Element

function $\text{QUICKSELECT}(A[\cdot], lo, hi, k)$
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k: 3

			lo			hi
8	3	9	23	22	41	46
0	1	2	3	4	5	6
					s	

returns 41!

QuickSelect Complexity

- **Worst case** complexity for QuickSelect is quadratic,
- **Average-case** complexity is linear.

Interpolation Search

- If the elements of a sorted array are distributed reasonably evenly, we can do better than binary search!
- Think about how you search for an entry in the telephone directory: If you look for ‘Zobel’, you make a rough estimate of where to do the first probe—very close to the end of the directory.
- This is the idea in interpolation search.
- When searching for k in the array segment A[lo] to A[hi], take into account where k is, relative to A[lo] and A[hi].

Interpolation Search

A:

0	1	2	3	4	5	6
4	9	13	22	41	83	96

Suppose we are searching for k = 83

Interpolation Search

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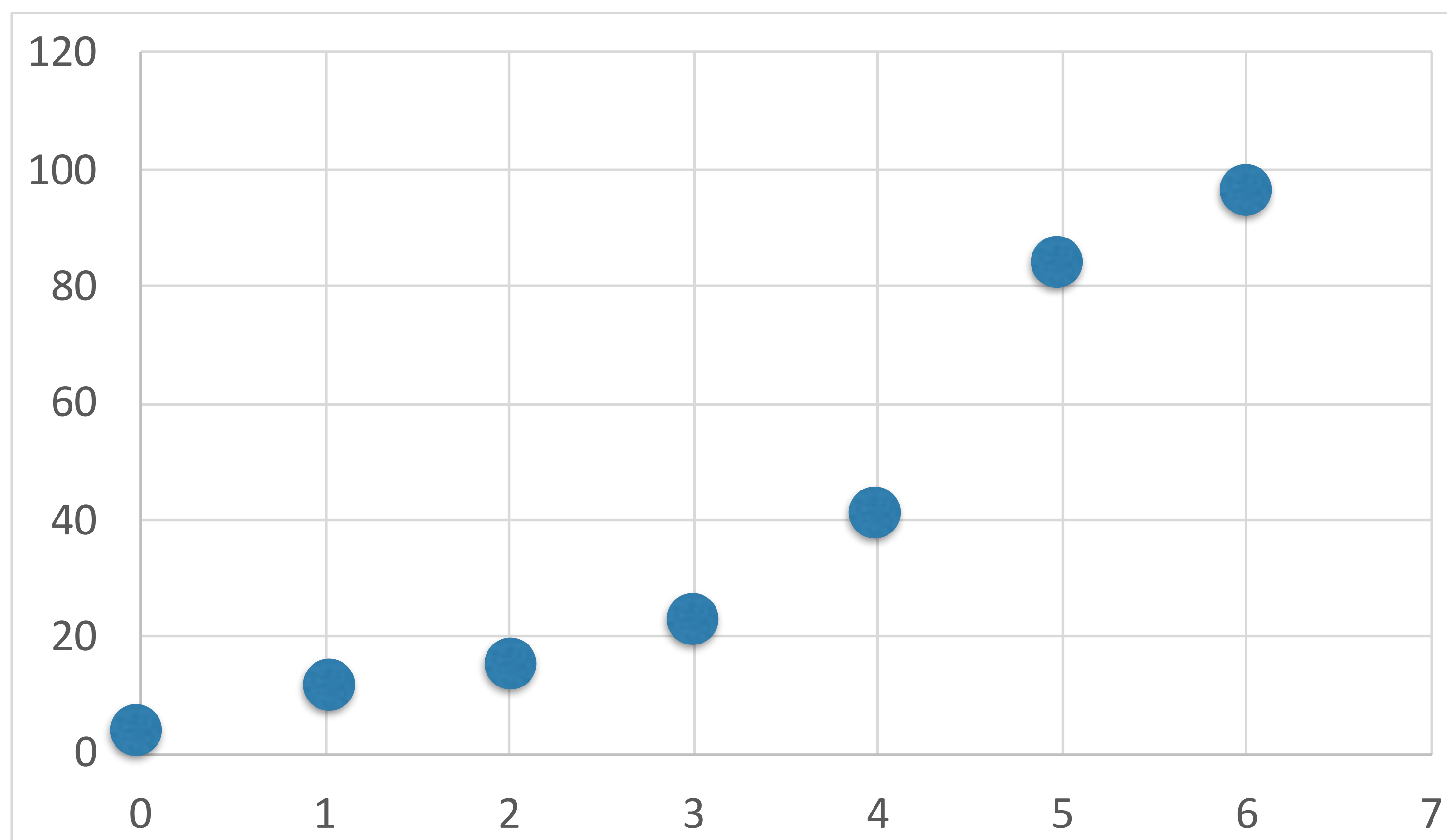
4	9	13	22	41	83	96
0	1	2	3	4	5	6

Suppose we are searching for k = 83



Interpolation Search

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A:	4	9	13	22	41	83	96
	0	1	2	3	4	5	6

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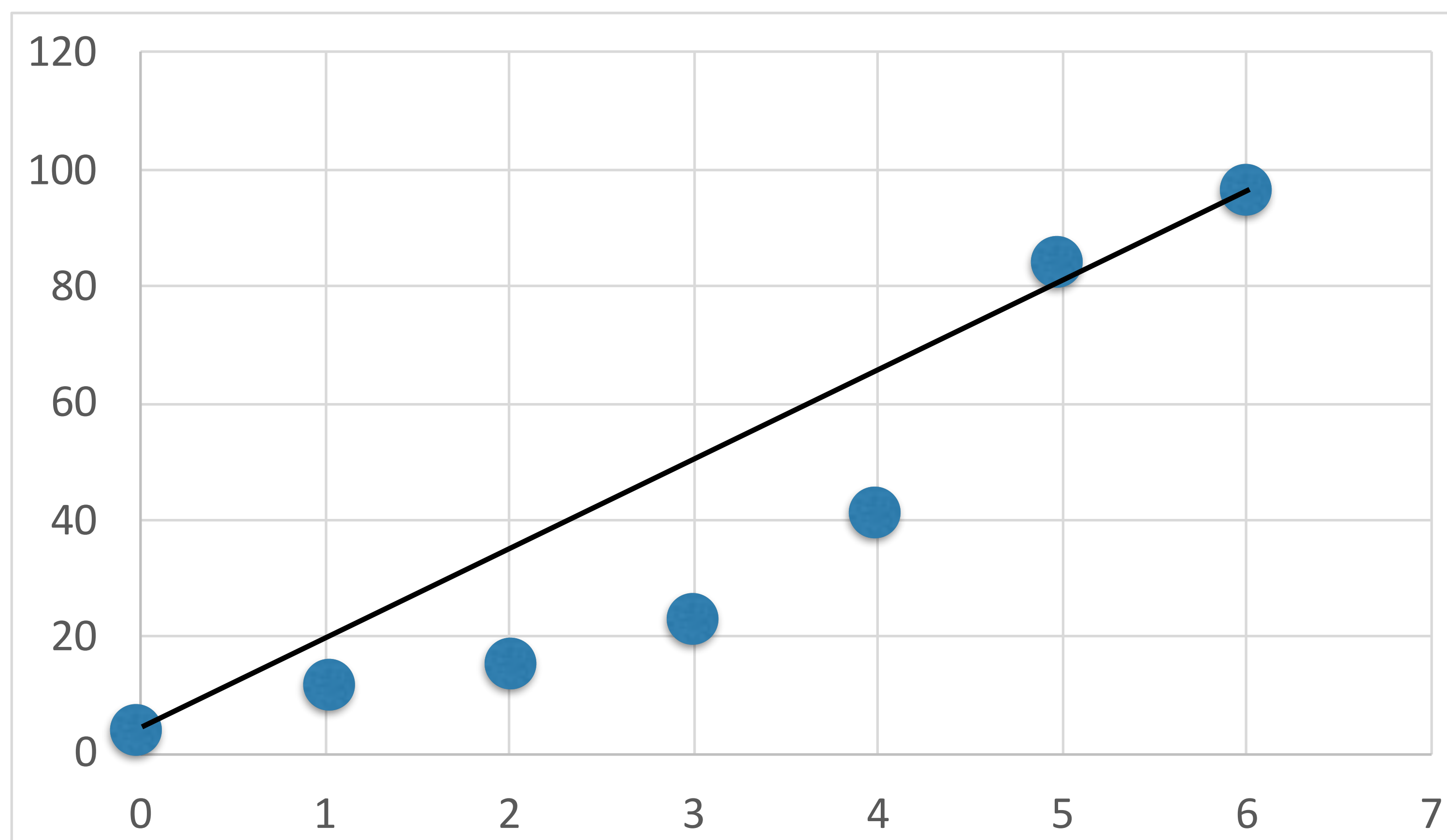


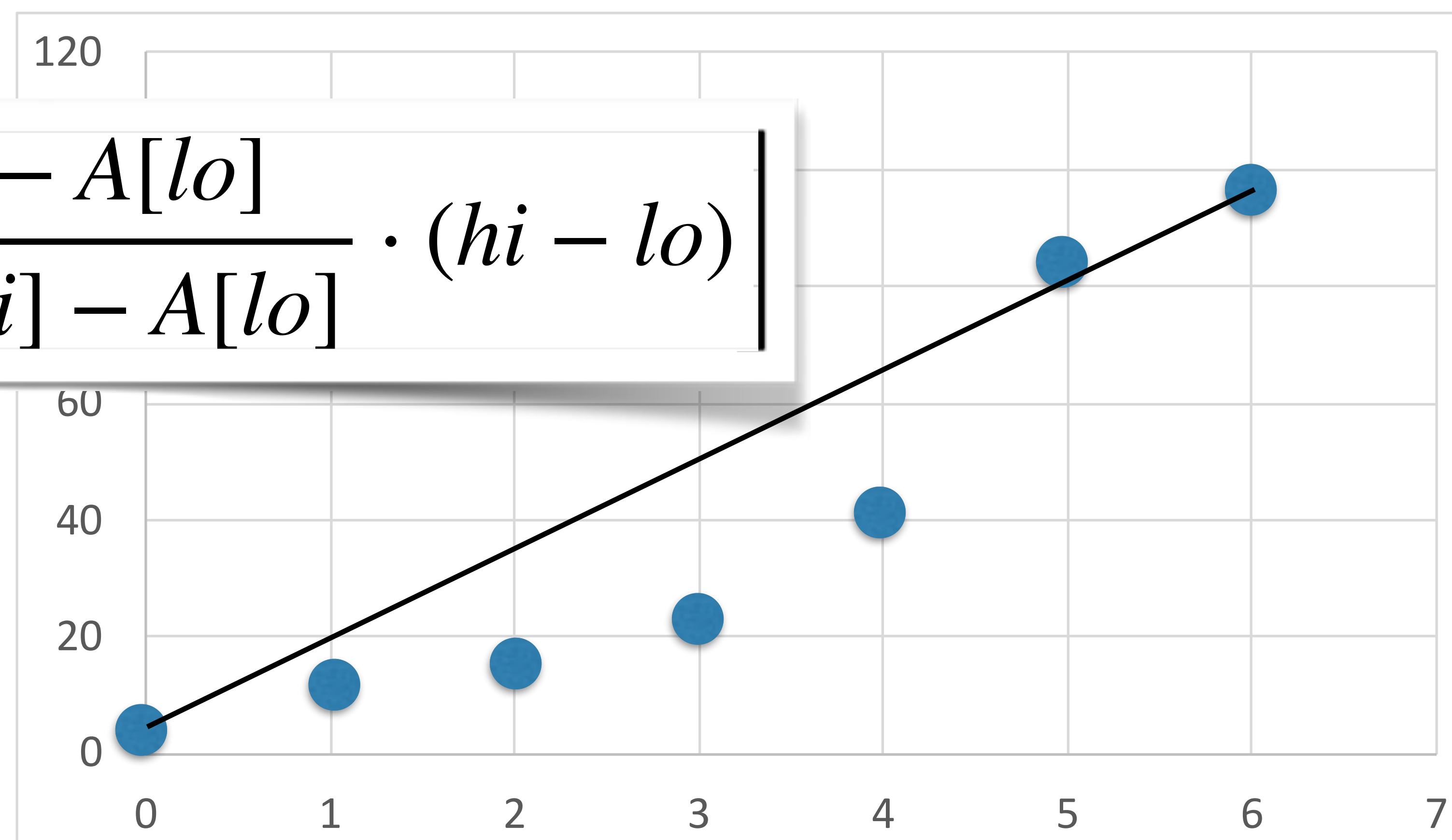
Interpolation Search

A:

4	9	13	22	41	83	96
0	1	2	3	4	5	6

Suppose we are searching for k = 83

$$mid = A[lo] + \left\lfloor \frac{k - A[lo]}{A[hi] - A[lo]} \cdot (hi - lo) \right\rfloor$$



Interpolation Search

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- Instead of computing the mid-point *m* as in binary search:

$$m \leftarrow \lfloor (lo + hi)/2 \rfloor$$

  we instead perform linear interpolation between the points (lo,A[lo]) and (hi,A[hi]). That is, we use:

$$m \leftarrow lo + \left\lfloor \frac{k - A[lo]}{A[hi] - A[lo]}(hi - lo) \right\rfloor$$

- Interpolation search has average complexity O(log log n)
- It is the right choice for large arrays when elements are **uniformly distributed**

Next Week

- Learn to divide and conquer!
- Read Levitin Chapter 5, but skip 5.4.