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COMP90038
Algorithms and Complexity

Lecture 4: Analysis of Algorithms
(with thanks to Harald Søndergaard)

Toby Murray

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Last Time: Time Complexity

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- Measure **input size** by natural number *n*
- Measure **execution time** as number of **basic operations** performed
- **Time complexity** $t(n)$ for an algorithm: number of **basic operations** as a function of *n*
- How to **compare** different $t(n)$?
  - Asymptotic growth rate
  - $O(\mathrm{g}(\mathrm{n}))$, $\Omega(g(n))$, $\Theta(g(n))$

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- Hov
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Problem	Size Measure	Basic Operation
Search in a list of n items	n	Key comparison
Multiply two matrices of floats	Matrix size (rows x columns)	Float multiplication
Compute a^n	log n	Float multiplication
Graph problem	Number of nodes and edges	Visiting a node

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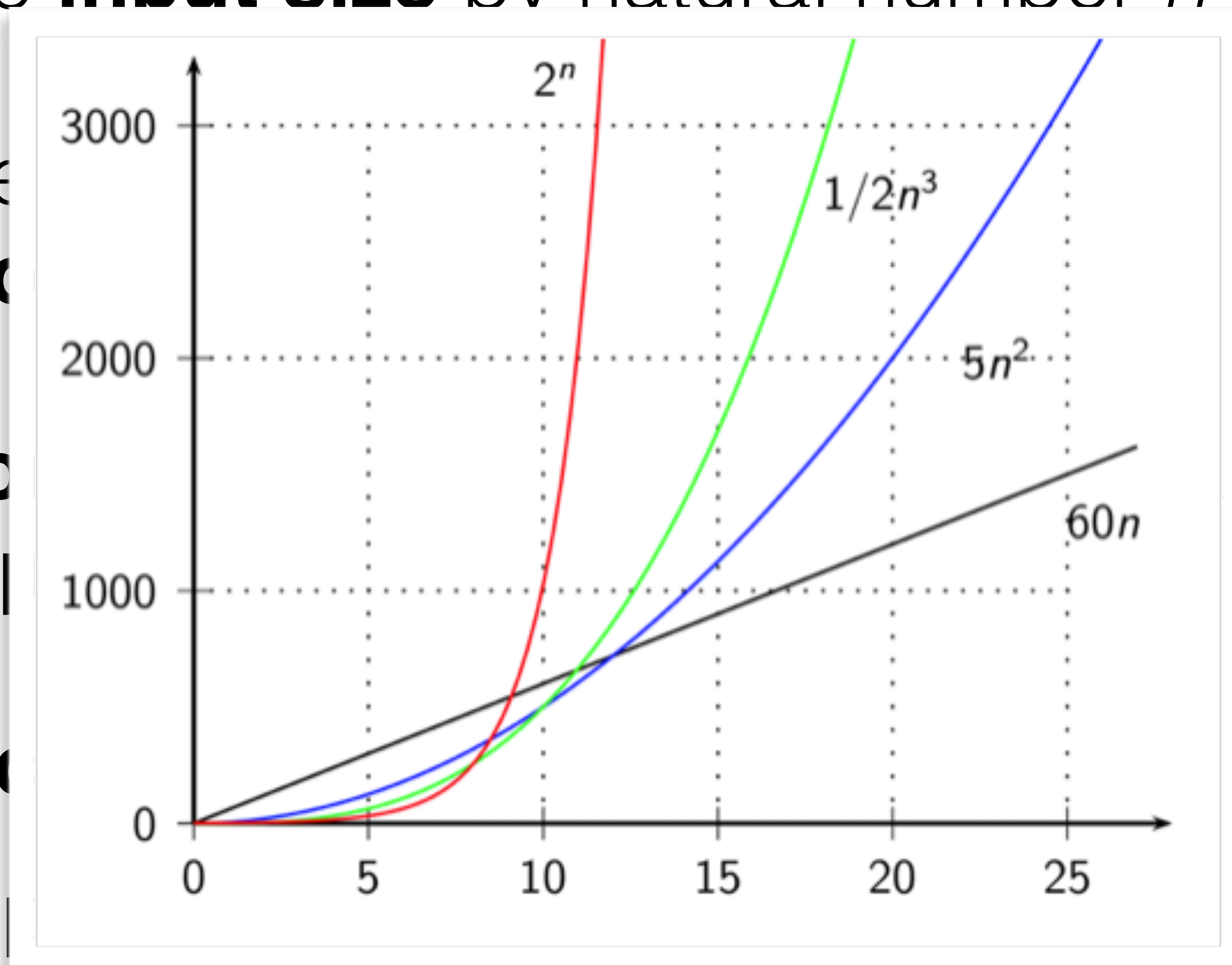

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$$f(n) < g(n) \text{ iff } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

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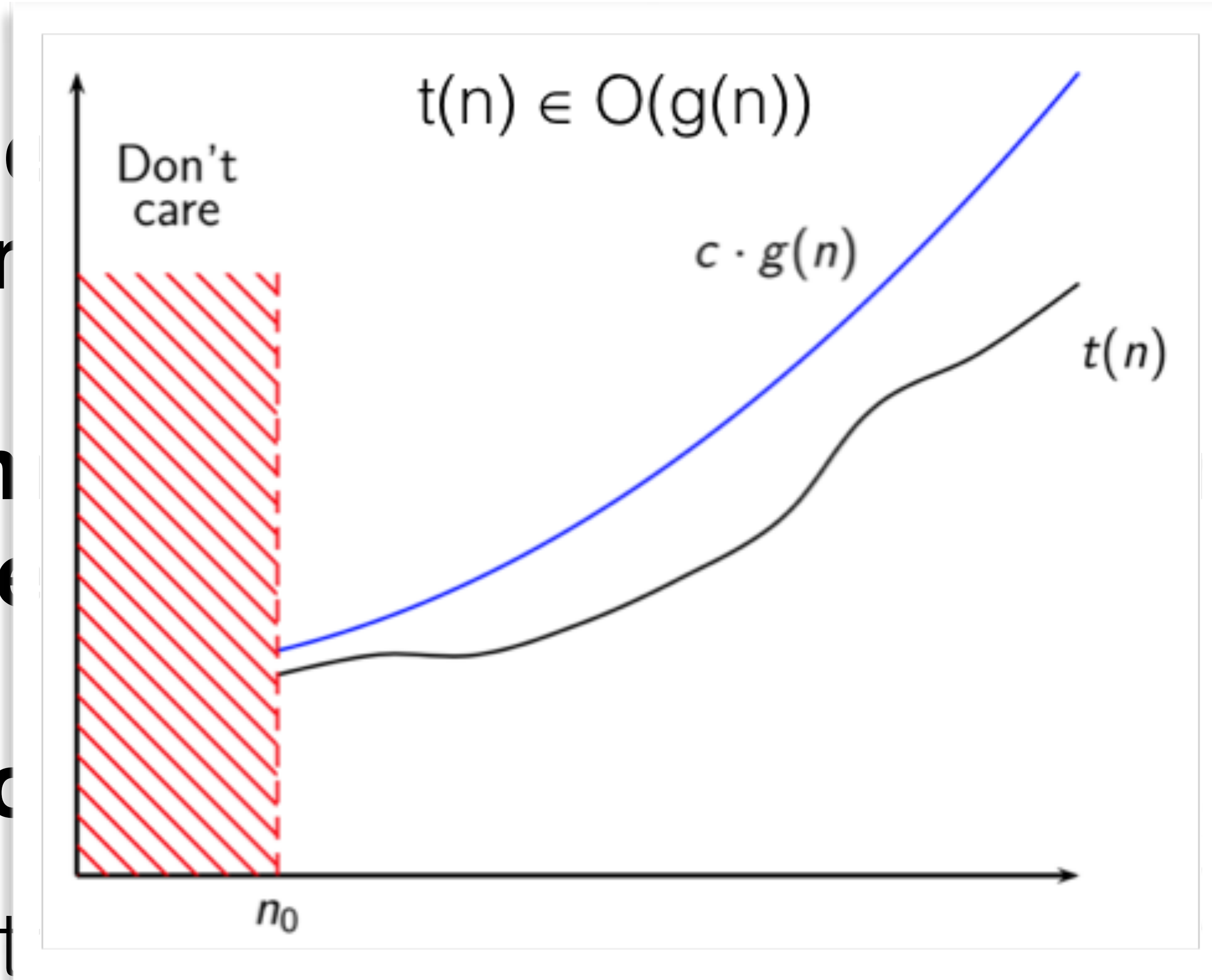

$$\ldots) \text{ iff } \lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

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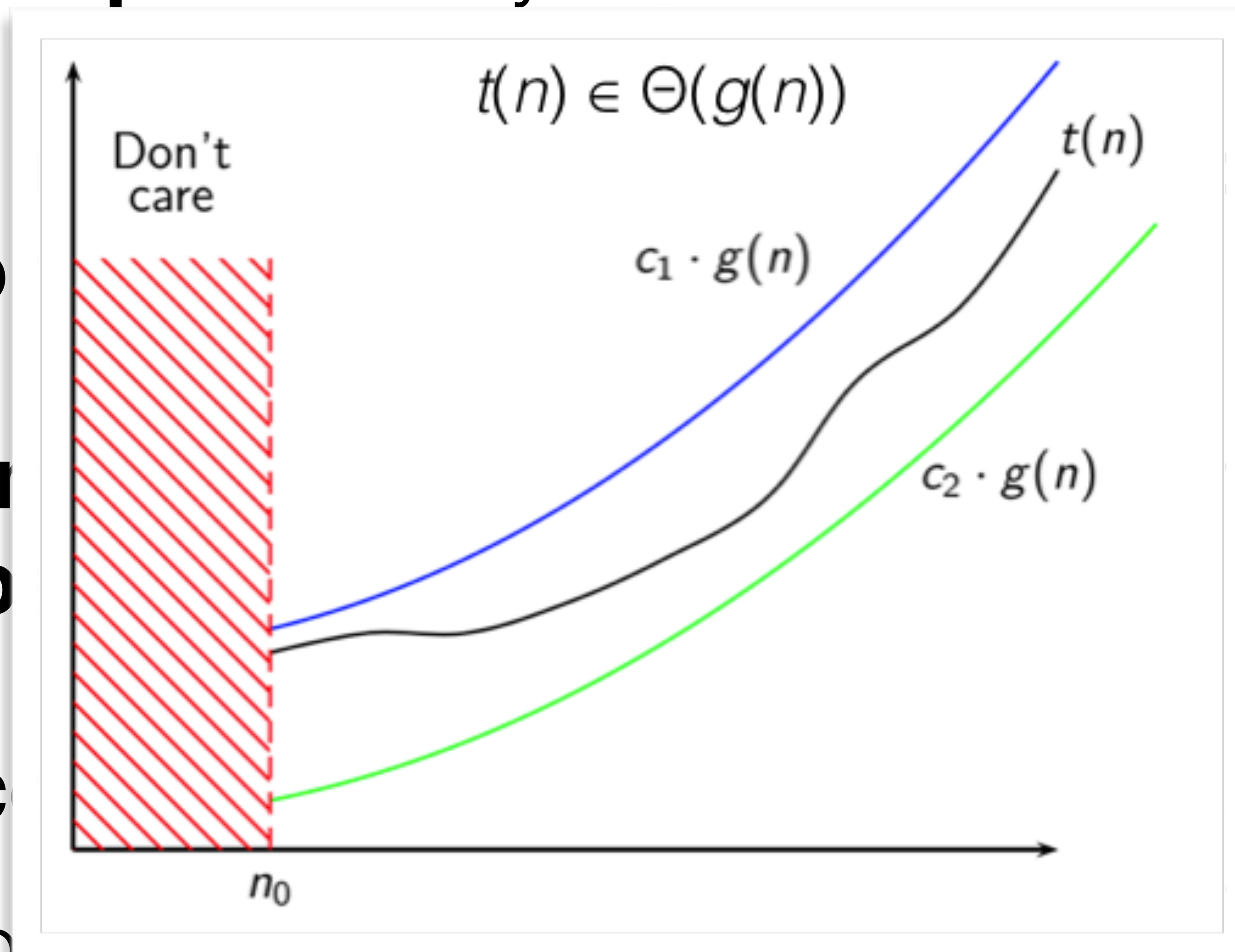

...) iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

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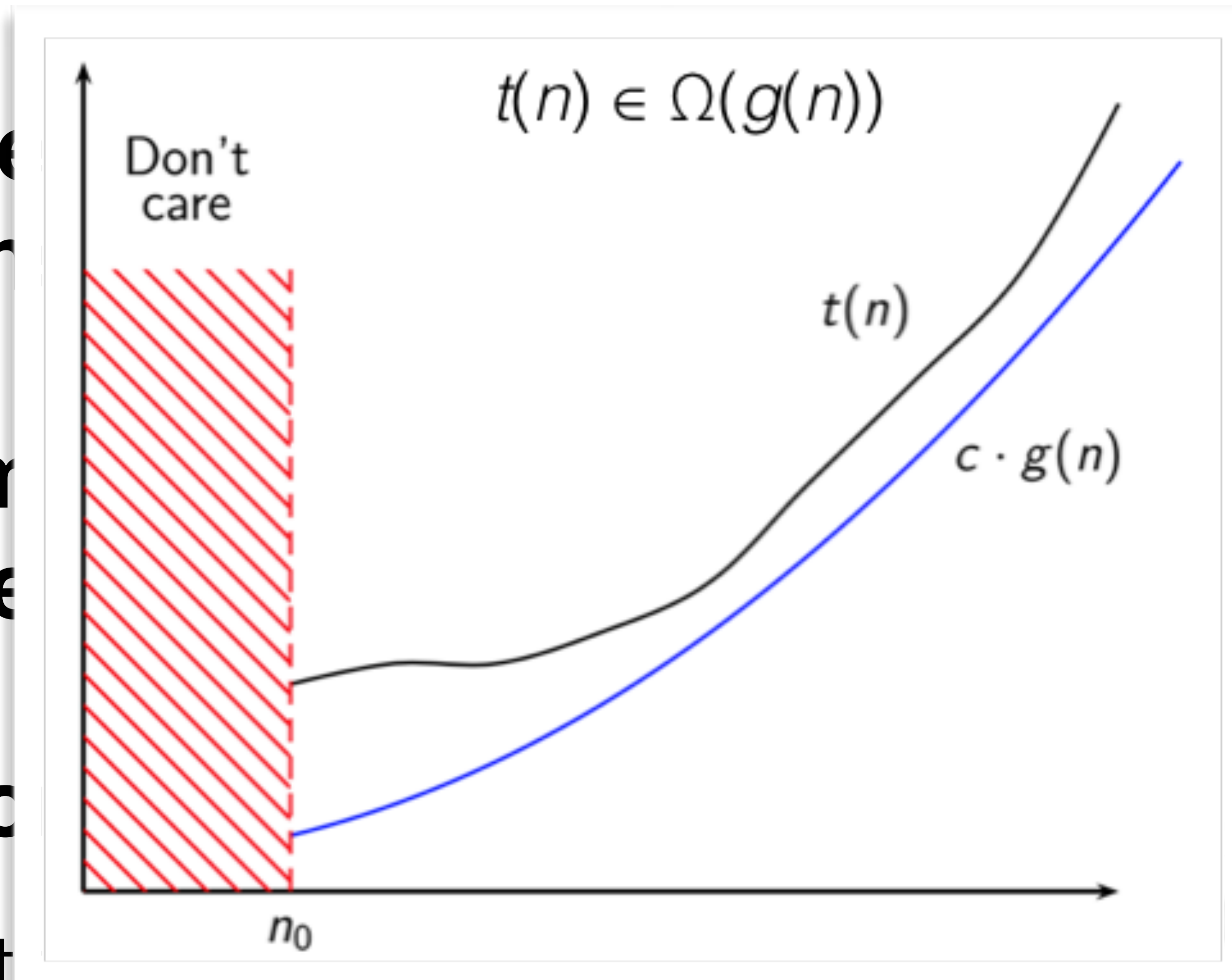

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$$f(n) < g(n) \text{ iff } \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Establishing Growth Rate

- In the last lecture we proved $t(n) \in O(g(n))$ for some cases of t and g, using the definition of O directly:

  $n > n_0 \Rightarrow t(n) < c \cdot g(n)$ for some c and n_0.

- A more common approach uses

- $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f \text{ grows asymptotically slower than } g \\ c & f \text{ and } g \text{ have same order of growth} \\ \infty & f \text{ grows asymptotically faster than } g \end{cases}$$

- Use this to show that $1000n \in O(n^2)$

$1000n \in O(n^2)$

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$$\lim_{n \to \infty} \frac{1000n}{n^2}$$

$1000n \in O(n^2)$

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$1000n \in O(n^2)$

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$$\lim_{n \to \infty} \frac{1000\cancel{n}}{\cancel{n^2}}$$

$$= \lim_{n \to \infty} \frac{1000}{n}$$

$$= 0$$

$1000n \in O(n^2)$

$$\lim_{n \to \infty} \frac{1000\cancel{n}}{n^{\cancel{2}}}$$

$$= \lim_{n \to \infty} \frac{1000}{n}$$

$$= 0$$

So $1000n$ grows asymptotically slower than n^2

$1000n \in O(n^2)$

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So $1000n$ grows asymptotically slower than n^2

Thus $1000n \in O(n^2)$

O, Ω, Θ

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What this tells us about how $f(n)$ relates to $O(g(n))$, $\Omega(g(n))$ and $\Theta(g(n))$

- $$\lim_{n \to \infty} \frac{f(n)}{g(n)} = \begin{cases} 0 & f \text{ grows asymptotically slower than } g \\ c & f \text{ and } g \text{ have same order of growth} \\ \infty & f \text{ grows asymptotically faster than } g \end{cases}$$

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L'Hôpital's Rule

$$\lim_{h \to \infty} \frac{t(n)}{g(n)} = \lim_{h \to \infty} \frac{t'(n)}{g'(n)}$$

where t' and g' are the derivatives of t and g

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Example:
Finding Largest Element in an Array

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function $\textsc{MaxElement}(A[\cdot], n)$
  $max \leftarrow A[0]$
  for $i \leftarrow 1$ to $n-1$ **do**
    if $A[i] > max$ **then**
      $max \leftarrow A[i]$
  return max

(where n is length of the array)

A:

23	12	42	6	69	18	3
0	1	2	3	4	5	6

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max: 42

A[i]

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A[i]

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A[i]

max: 69

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23	12	42	6	69	18	3
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A[i]

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Example: Finding Largest Element in an Array

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Basic operation: comparison "A[i] > max"

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Count the number of basic operations executed for an array of size n:

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Size of input, *n*: length of the array

Basic operation: comparison "A[i] > max"

Count the number of basic operations executed for an array of size *n:*

$$C(n) = \sum_{i=1}^{n-1} 1$$

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Size of input, *n*: length of the array

Basic operation: comparison "A[i] > max"

Count the number of basic operations executed for an array of size *n:*

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1)$$

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Size of input, n: length of the array

Basic operation: c

$$\sum_{i=l}^{u} 1 = \underbrace{1+1+\cdots+1}_{u-l+1 \text{ times}} = u - l + 1$$

Count the number of basic

for an array of size n:

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1)$$

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Basic operation: c

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Count the number of basic

for an array of size *n:*

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Count the number of basic
for an array of size n:

$$C(n) = \sum_{i=1}^{n-1} 1 = ((n-1) - 1 + 1) = n - 1 \in \Theta(n)$$

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
 for $i \leftarrow 0$ to $n - 2$ **do**
  $min \leftarrow i$
  for $j \leftarrow i + 1$ to $n - 1$ **do**
   if $A[j] < A[min]$ **then**
    $min \leftarrow j$
  swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

↑
A[i]

Example: Selection Sort

function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

min: 0

↑
A[i]

Example: Selection Sort

function SelSort($A[\cdot]$, n)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 0

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 1

Example: Selection Sort

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function $\text{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 1

Example: Selection Sort

function SELSORT($A[\cdot], n$)
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 1

Example: Selection Sort

function SELSORT($A[\cdot], n$)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] (at index 0) A[j] (at index 3)

min: 3

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i]   A[j]

min: 3

Example: Selection Sort

function SELSORT($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6
↑ A[i]					↑ A[j]	

min: 3

Example: Selection Sort

function $\text{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

23	12	42	6	69	18	3
0	1	2	3	4	5	6
A[i]						A[j]

min: 3

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

23	12	42	6	69	18	3
0	1	2	3	4	5	6

A[i] A[j]

min: 6

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

↑
A[i]

min: 6

Example: Selection Sort

function SELSORT($A[\cdot]$, n)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i]

min: 6

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

↑
A[i]

Example: Selection Sort

function SELSORT($A[\cdot], n$)
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i]

min: 1

Example: Selection Sort

function SelSort($A[\cdot], n$)
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i] A[j]

min: 1

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i] A[j]

min: 1

Example: Selection Sort

function SELSORT($A[\cdot]$, n)
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i] A[j]

min: 3

Example: Selection Sort

function SELSORT($A[\cdot], n$)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6

min: 3

A[i] A[j]

Example: Selection Sort

function SelSort($A[\cdot], n$)
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	12	42	6	69	18	23
0	1	2	3	4	5	6
	A[i]				A[j]	

min: 3

Example: Selection Sort

```
function SELSORT(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

3	12	42	6	69	18	23
0	1	2	3	4	5	6

A[i]   A[j]

min: 3

Example: Selection Sort

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function SELSORT($A[\cdot], n$)
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

3	6	42	12	69	18	23
0	1	2	3	4	5	6

↑
A[i]

min: 3

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ **do**
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ **do**
            if $A[j] < A[min]$ **then**
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

3	6	42	12	69	18	23
0	1	2	3	4	5	6

A[i]

min: 3

Example: Selection Sort

```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

Example: Selection Sort

$$
\begin{array}{l}
\textbf{function } \textsc{SelSort}(A[\cdot], n) \\
\quad \textbf{for } i \leftarrow 0 \text{ to } n-2 \textbf{ do} \\
\quad\quad min \leftarrow i \\
\quad\quad \textbf{for } j \leftarrow i+1 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad \textbf{if } A[j] < A[min] \textbf{ then} \\
\quad\quad\quad\quad min \leftarrow j \\
\quad\quad \text{swap } A[i] \text{ and } A[min]
\end{array}
$$

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

C(n) =

Example: Selection Sort

function SelSort($A[\cdot], n$)
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1$$

Example: Selection Sort

function SelSort($A[\cdot], n$)
  for $i \leftarrow 0$ to $n - 2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i + 1$ to $n - 1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n - 1 - i)$$

Example: Selection Sort

function $\textsc{SelSort}(A[\cdot], n)$
    for $i \leftarrow 0$ to $n - 2$ do
        $min \leftarrow i$
        for $j \leftarrow i + 1$ to $n - 1$ do
            if $A[j] < A[min]$ then
                $min \leftarrow j$
        swap $A[i]$ and $A[min]$

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n - 1 - i) = \sum_{i=0}^{n-2} (n - 1) - \sum_{i=0}^{n-2} i$$

Example: Selection Sort

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function $\textsc{SelSort}(A[\cdot], n)$
  for $i \leftarrow 0$ to $n-2$ **do**
    $min \leftarrow i$
    for $j \leftarrow i+1$ to $n-1$ **do**
      if $A[j] < A[min]$ **then**
        $min \leftarrow j$
    swap $A[i]$ and $A[min]$

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n-1-i) = \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2}$$

Example: Selection Sort

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```
function SelSort(A[·], n)
    for i ← 0 to n − 2 do
        min ← i
        for j ← i + 1 to n − 1 do
            if A[j] < A[min] then
                min ← j
        swap A[i] and A[min]
```

Input size *n*: length of the array

Basic operation: comparison A[j] < A[min]

$$C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} (n-1-i) = \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i$$

$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{n(n-1)}{2} \in \Theta(n^2)$$

Example: Matrix Multiplication

function MatrixMult($A[\cdot,\cdot]$, $B[\cdot,\cdot]$, n) ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

i: 0

j: 0

$$\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix} \qquad \begin{bmatrix} 0 & \\ & \end{bmatrix}$$

A B C

Example: Matrix Multiplication

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function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 0 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

THE UNIVERSITY OF MELBOURNE

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 0

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 40 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$) $\quad \triangleright$ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

i: 0

j: 0

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 40 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices

  for $i \leftarrow 0$ to $n-1$ **do**

    for $j \leftarrow 0$ to $n-1$ **do**

      $C[i,j] \leftarrow 0.0$

      for $k \leftarrow 0$ to $n-1$ **do**

        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$

  return C

i: 0

j: 0

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\quad \triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \quad \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \quad \underset{C}{\begin{bmatrix} 103 & \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

$$\underbrace{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}}_{B} \qquad \underbrace{\begin{bmatrix} 103 & 0 \\ & \end{bmatrix}}_{C}$$

Example: Matrix Multiplication

function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$) $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

k: 0

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 0 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

$$
\begin{array}{l}
\textbf{function } \textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n) \qquad \triangleright \text{ For } n \times n \text{ matrices} \\
\quad \textbf{for } i \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad \textbf{for } j \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad C[i,j] \leftarrow 0.0 \\
\quad\quad\quad \textbf{for } k \leftarrow 0 \text{ to } n-1 \textbf{ do} \\
\quad\quad\quad\quad C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j] \\
\quad \textbf{return } C
\end{array}
$$

i: 0

j: 1

k: 0

$$
\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}}
\underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}}
\qquad
\underset{C}{\begin{bmatrix} 103 & 10 \\ & \end{bmatrix}}
$$

Example: Matrix Multiplication

function $\text{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \quad \underset{C}{\begin{bmatrix} 103 & 10 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

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function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

i: 0

j: 1

k: 1

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix}} \underset{B}{\begin{bmatrix} 8 & 2 \\ 9 & 6 \end{bmatrix}} \qquad \underset{C}{\begin{bmatrix} 103 & 52 \\ & \end{bmatrix}}$$

Example: Matrix Multiplication

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function $\text{MATRIXMULT}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
      for $k \leftarrow 0$ to $n-1$ **do**
        $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
  return C

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

Basic operation: multiplication A[i,k] * B[k,j]

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\quad\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
    for $j \leftarrow 0$ to $n-1$ **do**
      $C[i,j] \leftarrow 0.0$
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  return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) =$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \qquad\qquad 1$$

Example: Matrix Multiplication

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function MatrixMult($A[\cdot,\cdot], B[\cdot,\cdot], n$) ▷ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
        for $j \leftarrow 0$ to $n-1$ **do**
            $C[i,j] \leftarrow 0.0$
            for $k \leftarrow 0$ to $n-1$ **do**
                $C[i,j] \leftarrow C[i,j] + A[i,k] \cdot B[k,j]$
    return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

function $\text{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\quad\triangleright$ For $n \times n$ matrices
  for $i \leftarrow 0$ to $n-1$ **do**
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  return C

Basic operation: multiplication A[i,k] * B[k,j]

$$M(n) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

function $\textsc{MatrixMult}(A[\cdot,\cdot], B[\cdot,\cdot], n)$ $\triangleright$ For $n \times n$ matrices
    for $i \leftarrow 0$ to $n-1$ **do**
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Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1)$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n$$

Example: Matrix Multiplication

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

Example: Matrix Multiplication

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$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

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$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right)$$

$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} c a_i = c \sum_{i=l}^{u} a_i$$

Example: Matrix Multiplication

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} ca_i = c \sum_{i=l}^{u} a_i$$

$$= n^3$$

Example: Matrix Multiplication

$$\sum_{i=l}^{u} 1 = \underbrace{1 + 1 + \cdots + 1}_{u-l+1 \text{ times}} = u - l + 1$$

$$M(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} 1$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} ((n-1) - 0 + 1) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} (n \cdot 1)$$

$$= \sum_{i=0}^{n-1} \left(n \cdot \sum_{j=0}^{n-1} 1 \right) = \sum_{i=0}^{n-1} n^2$$

$$\sum_{i=l}^{u} ca_i = c \sum_{i=l}^{u} a_i$$

$$= n^3$$

$$\in \Theta(n^3)$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

$F(5)$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$F(5) = F(4) \cdot 5$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$\begin{aligned} F(5) &= F(4) \cdot 5 \\ &= (F(3) \cdot 4) \cdot 5 \end{aligned}$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

$$
\begin{aligned}
F(5) &= F(4) \cdot 5 \\
&= (F(3) \cdot 4) \cdot 5 \\
&= ((F(2) \cdot 3) \cdot 4) \cdot 5 \\
&= (((F(1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((F(0) \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= ((((1 \cdot 1) \cdot 2) \cdot 3) \cdot 4) \cdot 5 \\
&= 5!
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) =$$

Analysing Recursive Algorithms

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function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$
$$M(n) =$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$M(0) = \quad 0$$
$$M(n) = \qquad\qquad 1$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
    else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$
\begin{aligned}
M(0) &= \quad 0 \\
M(n) &= \qquad\qquad\qquad + 1
\end{aligned}
$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$\begin{aligned} M(0) &= 0 \\ M(n) &= M(n-1) + 1 \end{aligned}$$

Analysing Recursive Algorithms

function $\mathrm{F}(n)$
  if $n = 0$ **then return** 1
  else return $\mathrm{F}(n-1) \cdot n$

Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$\begin{aligned} M(0) = &\quad 0 \\ M(n) = &\quad M(n-1) + 1 \end{aligned}$$

Need to express M(n) in **closed form** (i.e. non-recursively)

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function $\mathrm{F}(n)$
    if $n = 0$ **then return** 1
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Basic operation: multiplication

We express the cost **recursively** (as a **recurrence relation**)

$$\begin{aligned} M(0) &= \quad 0 \\ M(n) &= \quad M(n-1) + 1 \end{aligned}$$

Need to express M(n) in **closed form** (i.e. non-recursively)

Try: **“telescoping”** aka **“backward substitution”**

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

Telescoping (aka Backward Substitution)

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$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ? $M(n-1) = M((n-1)-1) + 1$

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$M(n) =$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$M(n) = (M(n-2) + 1) + 1$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1)+1 \\ &= M(n-2)+1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$M(n) = (M(n-2) + 1) + 1$$
$$= M(n-2) + 2$$
$$= (M(n-3) + 1) + 2$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \end{aligned}$$

Telescoping
(aka Backward Substitution)

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$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \\ &= (M(n-3) + 1) + 2 \\ &= M(n-3) + 3 \\ &\cdots \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\dots \\ &= M(n-n)+n \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\ldots \\ &= M(n-n)+n \\ &= M(0)+n \end{aligned}$$

Telescoping
(aka Backward Substitution)

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$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\dots \\ &= M(n-n)+n \\ &= M(0)+n \\ &= n \end{aligned}$$

Telescoping
(aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1}$$

$$\boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\cdots \\ &= M(n-n)+n \\ &= M(0)+n \\ &= n \end{aligned}$$

Closed form:

Telescoping (aka Backward Substitution)

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$$M(n) = M(n-1) + 1$$

$$M(0) = 0$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2)+1)+1 \\ &= M(n-2)+2 \\ &= (M(n-3)+1)+2 \\ &= M(n-3)+3 \\ &\ldots \\ &= M(n-n)+n \\ &= M(0)+n \\ &= n \end{aligned}$$

Closed form:

$$M(n) = n$$

Telescoping (aka Backward Substitution)

$$\boxed{M(n) = M(n-1) + 1} \qquad \boxed{M(0) = 0}$$

What is M(n-1) ?

$$\begin{aligned} M(n-1) &= M((n-1)-1) + 1 \\ &= M(n-2) + 1 \end{aligned}$$

$$\begin{aligned} M(n) &= (M(n-2) + 1) + 1 \\ &= M(n-2) + 2 \\ &= (M(n-3) + 1) + 2 \\ &= M(n-3) + 3 \\ &\dots \\ &= M(n-n) + n \\ &= M(0) + n \\ &= n \end{aligned}$$

Closed form:

$$\boxed{M(n) = n}$$

Complexity:

Telescoping (aka Backward Substitution)

$$M(n) = M(n-1) + 1 \qquad M(0) = 0$$

What is M(n-1) ?

$$M(n-1) = M((n-1)-1) + 1$$
$$= M(n-2) + 1$$

$$\begin{aligned}
M(n) &= (M(n-2)+1)+1 \\
&= M(n-2)+2 \\
&= (M(n-3)+1)+2 \\
&= M(n-3)+3 \\
&\dots \\
&= M(n-n)+n \\
&= M(0)+n \\
&= n
\end{aligned}$$

Closed form:

$$M(n) = n$$

Complexity:

$$M(n) \in \Theta(n)$$

Example: Binary Search in Sorted Array

function $\textsc{BinSearch}(A[\cdot], lo, hi, key)$
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BinSearch(A,0,6,41)

BinSearch(A,4,6,41)

Example:
Binary Search in Sorted Array

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lo: 4

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Binary Search in Sorted Array

$$
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returns 4

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Basic operation: key comparison A[mid] = key

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$C(0) = 0$ $C(n) = \qquad + 1$

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$C(0) = 0$ $\qquad\qquad$ $C(n) = C(n/2) + 1$

Telescoping

A **smoothness rule** allows us to assume that n is a power of 2

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&= ((C(n/8) + 1) + 1) + 1 \\
&\dots \\
&= C(n/n) + \log_2 n
\end{aligned}$$

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&= 1 + \log_2 n
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$$C(n) \in \Theta(\log n)$$

Logarithmic Functions Have Same Rate of Growth

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In *O, Ω, Θ,* expressions we can just write “log” for any logarithmic function no matter what the base is

Asymptotically, all logarithmic behaviour is the same, since

$$\log_a x = (\log_a b)(\log_b x)$$

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So, e.g.: $\Theta(log_a n) = \Theta(log_b n)$

Also, since $\log n^c = c \cdot \log n$ $\quad \log n^c \in \Theta(\log n)$

Back to Euclid

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function $\textsc{Euclid}(m, n)$
    while $n \neq 0$ **do**
        $r \leftarrow m \text{ mod } n$
        $m \leftarrow n$
        $n \leftarrow r$
    return m

function $\textsc{Euclid}(m, n)$
    if $n = 0$ **then**
        return m
    return $\textsc{Euclid}(n, m \text{ mod } n)$

Back to Euclid

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Running time is **linear** in size (in bits) of input, i.e.

$$O(\log(m+n))$$

Back to Euclid

function $\text{EUCLID}(m, n)$
    while $n \neq 0$ **do**
        $r \leftarrow m \bmod n$
        $m \leftarrow n$
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    return m

function $\text{EUCLID}(m, n)$
    if $n = 0$ **then**
        return m
    return $\text{EUCLID}(n, m \bmod n)$

Running time is **linear** in size (in bits) of input, i.e.

$$O(\log(m + n))$$

since the value of *m* (and *n*) is at least halved in every two iterations.

Back to Euclid

```
function EUCLID(m, n)
    while n ≠ 0 do
        r ← m mod n
        m ← n
        n ← r
    return m
```

```
function EUCLID(m, n)
    if n = 0 then
        return m
    return EUCLID(n, m mod n)
```

Running time is **linear** in size (in bits) of input, i.e.

$$O(\log(m+n))$$

since the value of *m* (and *n*) is at least halved in every two iterations.

Why? After two iterations, m becomes m mod n; also

$$1 < n < m \implies m \bmod n < m/2$$

Summarising Reasoning with Big-Oh

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(for nested loops: count number of times outer loop is executed, multiply by cost of inner loop)

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See also Cormen's Appendix A or Levitin's Appendix A.

Levitin's Appendix B is a tutorial on recurrence relations.

The Road Ahead

- You'll get much more familiar with asymptotic analysis as we use it on algorithms we meet in this course.
- Next week we begin our study of algorithms by looking at **brute force** approaches

