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# When the greedy algorithm fails

Jørgen Bang-Jensen[a], Gregory Gutin[b], Anders Yeo[b]

[a] *Department of Mathematics and Computer Science, University of Southern Denmark, Odense, Denmark*
[b] *Department of Computer Science, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK*

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**Abstract**

We provide a characterization of the cases when the greedy algorithm may produce the unique worst possible solution for the problem of finding a minimum weight base in an independence system when the weights are taken from a finite range. We apply this theorem to TSP and the minimum bisection problem. The practical message of this paper is that the greedy algorithm should be used with great care, since for many optimization problems its usage seems impractical even for generating a starting solution (that will be improved by a local search or another heuristic).


*Keywords:* Greedy algorithm; Traveling salesman problem; Combinatorial optimization

## 1. Introduction

The greedy algorithm is one of the simplest algorithms in combinatorial optimization. The greedy paradigm is often used in combinatorial optimization theory and practice. In our view, this phenomenon can be explained by the fact that it is widely assumed that while the greedy algorithm rarely outputs optimal solutions, it often provides some kind of ‘approximation’, i.e., it provides solutions that are significantly better than the worst ones. This assumption seem to be justified by numerous results on ‘good’ behavior of the greedy algorithm, see, e.g., [1] for results on Euclidean TSP, max SAT, etc.

However, several experimental and theoretical results question this assumption. For example, the experimental results for the asymmetric TSP presented in [7] led its authors to the conclusion that the greedy algorithm ‘might be said to self-destruct’ and that it should not be used even as ‘a general-purpose starting tour generator’. The theorem in [6] on the greedy algorithm for the asymmetric TSP confirms the above conclusion: for every $n \geqslant 2$ there exist instances of the asymmetric TSP with $n$ vertices for which the greedy algorithm produces the unique worst tour. We show in Theorem 4.3 that this result can be strengthened, i.e., there are TSP instances that have an exponentially large number of optimal tours, which are $f(n)$ times shorter than the unique worst tour, where $f(n)$ is any function in $n$, and yet the greedy algorithms produces the unique worst tour. It is worth noting that there are many heuristics for the asymmetric TSP that always produce a tour, which is better than at least an $\Omega(1/n)$ part of all tours, see, e.g., [5,9–11].

The authors of [4] generalized the above-mentioned theorem from [6] to a wide class of uniform independence families (these families are defined in the next section). As a consequence of the main theorem in [4], it is shown in [4] that even for the polynomially solvable assignment problem the greedy algorithm may produce the unique worst possible solution. The authors of [4] posed the problem of obtaining results, which show that the greedy algorithm fails on other combinatorial optimization problems.

*E-mail addresses:* jbj@imada.sdu.dk (J. Bang-Jensen), gutin@cs.rhul.ac.uk (G. Gutin), ander@cs.rhul.ac.uk (A. Yeo).

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The main theorem of [4] is applicable only to combinatorial optimization problems with unrestricted weights. At the same time, in some combinatorial optimization problems, the weights are restricted. For example, in TSP(1,$B$) [2,8,12] only weights $\{1, 2, \ldots, B\}$ are available. TSP(1,2) has applications in the frequency assignment problems in mobile and radio networks, see, e.g., [3]. The obvious restriction in length of the memory units of computers indicates that we may always assume that the maximum weight in an optimization problem is restricted.

The purpose of this paper is to extend the main theorem of [4] to the case of restricted weights. Unlike the main theorem in [4] whose inequality conditions are sufficient but not necessary, our main theorem *completely characterizes* all independence families and finite range weight functions for which the greedy algorithm may find the unique worst possible solution.

We also provide some applications of this theorem to particular combinatorial optimization problems including TSP with restricted weights. The first two theorems in Section 4 strengthen the greedy algorithm theorem in [6] by showing the following results: For every $n \geqslant 3$ there exists an instance of the symmetric TSP (the asymmetric TSP) with weights restricted to the set $\{1, 2, \ldots, n-1\}$ ($\{1, 2, \ldots, \lceil \frac{n+1}{2} \rceil\}$) for which the greedy algorithm may find the unique worst possible tour. The same result, but with only weights $\{1, 2\}$ available, is proved for the minimum bisection problem, see Proposition 5.1.

The main *practical message* of this paper is that the greedy algorithm should be used with great care, since for many optimization problems its usage seems impractical even for generating a starting solution (that will be improved by a local search or another heuristic). Whenever possible, more robust alternatives to simple greedy approaches should be considered.

## 2. Terminology and notation

An *independence system* is a pair consisting of a finite set $E$ and a family $\mathscr{F}$ of subsets (called *independent sets*) of $E$ such that (I1) and (I2) are satisfied.

(I1) the empty set is in $\mathscr{F}$;
(I2) If $X \in \mathscr{F}$ and $Y$ is a subset of $X$, then $Y \in \mathscr{F}$.

All maximal sets of $\mathscr{F}$ are called *bases*. An independence system is *uniform* if all its bases are of the same cardinality.

Many combinatorial optimization problems can be formulated as follows. We are given an independence system $(E, \mathscr{F})$, a set $W \subseteq \mathbb{Z}_+$ and a weight function $w$ that assigns a weight $w(e) \in W$ to every element of $E$ ($\mathbb{Z}_+$ is the set of non-negative integers). The weight $w(S)$ of $S \in \mathscr{F}$ is defined as the sum of the weights of the elements of $S$. It is required to find a base $B \in \mathscr{F}$ of minimum weight. We will consider only such problems and call them the $(E, \mathscr{F}, W)$-*optimization problems*.

If $S \in \mathscr{F}$, then let $I(S) = \{x : S \cup \{x\} \in \mathscr{F}\} - S$. This means that $I(S)$ consists of those elements from $E - S$, which can be added to $S$, in order to have an independent set of size $|S| + 1$. Note that by (I2) $I(S) \neq \emptyset$ for every independent set $S$ which is not a base.

*The greedy algorithm* tries to construct a minimum weight base as follows: it starts from an empty set $X$, and at every step it takes the current set $X$ and adds to it a minimum weight element $e \in I(X)$, the algorithm stops when a base is built.

We assume that the greedy algorithm may choose any element among equally weighted elements in $I(X)$. Thus, when we say that the greedy algorithm *may construct* a base $B$, we mean that $B$ is built provided the appropriate choices between elements of the same weight are made.

An *ordered partitioning* of an ordered set $Z = \{z_1, z_2, \ldots, z_k\}$ is a collection of subsets $A_1, A_2, \ldots, A_q$ of $Z$ satisfying that if $z_r \in A_i$ and $z_s \in A_j$ where $1 \leqslant i < j \leqslant q$ then $r < s$. Some of the sets $A_i$ may be empty and $\cup_{i=1}^{q} A_i = Z$.

The complete undirected (directed) graph on $n$ vertices will be denoted by $K_n$ ($\overleftrightarrow{K}_n$).

## 3. Characterization

In the following theorem, we characterize all independence systems $(E, \mathscr{F})$ for which there is a finite range assignment of weights to the elements of $E$ such that the greedy algorithm solving the $(E, \mathscr{F}, \{1, 2, \ldots, r\})$-optimization problem may construct the unique worst possible solution.

**Theorem 3.1.** *Let $(E, \mathscr{F})$ be independence system and let $r \geqslant 2$ be a natural number. There exists a weight assignment $w: E \to \{1, 2, \ldots, r\}$ such that the greedy algorithm may produce the unique worst possible base if and only if $\mathscr{F}$ contains some base $B$ with the property that for some ordering $x_1, \ldots, x_k$ of the elements of $B$ and some ordered partitioning $A_1, A_2, \ldots, A_r$ of $x_1, \ldots, x_k$ the following holds for every base $B' \neq B$ of $\mathscr{F}$:*

$$\sum_{j=0}^{r-1} |I(A_{0,j}) \cap B'| < \sum_{j=1}^{r} j \times |A_j|, \tag{1}$$

*where $A_{0,j} = A_0 \cup \cdots \cup A_j$ and $A_0 = \emptyset$.*

**Proof.** We may assume that $\cup_{F \in \mathscr{F}} F = E$ as the weight of elements not contained in any base is immaterial.

Suppose $B$ is a base and that $B = \{x_1, \ldots, x_k\}$ is an ordering that satisfies (1) with respect to the ordered partitioning $A_1, \ldots, A_r$ of $B$. Let $w : E \to \{1, 2, \ldots, r\}$ be the weight function that assigns weight $s$ to $x$ precisely when $x \in I(A_{0,s-1}) - I(A_{0,s})$. By this assignment every element of $A_i$ is assigned weight $i$ and hence the weight of $B$ is given by

$$w(B) = \sum_{j=1}^{r} j \times |A_j|. \tag{2}$$

Now let $B'$ be any base distinct from $B$. By (I2) every element in $I(A_{0,j}) \cap B'$ is also in $I(A_{0,i}) \cap B'$ for all $0 \leqslant i < j$. Thus it follows from the definition of $w$ that $z \in B'$ has weight $j + 1$ precisely if it belongs to each of the sets $I(A_{0,i})$, for $i = 0, 1, 2, \ldots, j$ but $z \notin I(A_{0,j+1})$. Thus, we can write $w(B')$ as follows:

$$w(B') = \sum_{j=0}^{r-1} |I(A_{0,j}) \cap B'|. \tag{3}$$

It follows from (1) that $B$ is the worst possible base; hence it remains to show that the greedy algorithm may produce $B$. This is clearly the case if $x_{j+1}$ has minimum weight in $I(\{x_1, x_2, \ldots, x_j\})$, for all $j = 0, 1, 2, \ldots, k - 1$. Thus, assume that this is not the case for some $j$, and let $z \in I(\{x_1, x_2, \ldots, x_j\})$ be some element with $s = w(z) < w(x_{j+1})$. By the definition of the weight function we see that $z \notin I(A_{0,s})$. Therefore, the fact that $A_1, \ldots, A_r$ is an ordered partitioning of $x_1, \ldots, x_k$ and $z \in I(\{x_1, x_2, \ldots, x_j\})$ implies that $\{x_1, x_2, \ldots, x_j\} \subset A_{0,s}$. This in turn implies that $\{x_1, x_2, \ldots, x_j, x_{j+1}\} \subseteq A_{0,s}$, and, thus, $w(x_{j+1}) \leqslant s = w(z)$, a contradiction. Therefore, the greedy algorithm may produce $B$.

To prove the other direction assume that $w : E \to \{1, 2, \ldots, r\}$ is a weight function with respect to which the greedy algorithm may produce a base $B$ such that $w(B) > w(B')$ for every base $B' \neq B$. Let $A_i = \{x \in B : w(x) = i\}$. Note that $A_i$ may be empty for some $i$. Clearly $w(B)$ is then given by (2). Let $B'$ be any base different from $B$. For each $j \in \{0, 1, 2, \ldots, r - 1\}$ we have $w(z) \geqslant j + 1$ for every $z \in I(A_1 \cup \cdots \cup A_j) \cap B'$ since the greedy algorithm extends $A_1 \cup \cdots \cup A_j$ by elements from the first non-empty $A_\ell$, $\ell \geqslant j + 1$, all of which have weight $\ell \geqslant j + 1$. This implies that we have

$$\begin{aligned} w(B) &> w(B') \\ &\geqslant \sum_{j=0}^{r-1} |I(A_{0,j}) \cap B'|, \end{aligned}$$

implying that (1) holds. □

The following theorem can be deduced from Theorem 3.1. However, it has a shorter proof, which is presented.

**Theorem 3.2.** *Let $(E, \mathscr{F})$ be a uniform independence system. For every choice of distinct natural numbers $a, b$ there exists a weight function $w : E \to \{a, b\}$ such that the greedy algorithm may produce the unique worst base if and only if $\mathscr{F}$ contains a base $B = \{x_1, x_2, \ldots, x_k\}$ such that for some $1 \leqslant i < k$ the following holds*:

(a) *If $B'$ is a base such that $\{x_1, \ldots, x_i\} \subseteq B'$ then $B' = B$.*
(b) *If $B'$ is a base such that $\{x_{i+1}, \ldots, x_k\} \subseteq B'$ then $B' = B$.*

**Proof.** Suppose $B = \{x_1, x_2, \ldots, x_k\}$ is a base satisfying (a) and (b), and assume that $a < b$. Let all elements in $\{x_{i+1}, x_{i+2}, \ldots, x_k\}$ have weight $b$, and all other elements have weight $a$. Clearly $B$ is the unique worst solution, as by (b) no other base contains all the elements of weight $b$. Moreover, the greedy algorithm may produce $B$, since if it starts by picking $\{x_1, x_2, \ldots, x_i\}$ then it has to produce $B$, by (a).

We now just need to show that if the greedy algorithm does pick the unique worst solution, say $B = \{x_1, x_2, \ldots, x_k\}$ for some weight function $w$: $E \to \{a, b\}$, then (a) and (b) hold. We may assume that $\{x_1, x_2, \ldots, x_i\}$ are all the elements in $B$ of weight $a$. All other elements in $B$ have weight $b$. If (a) does not hold then there is some base $B' \neq B$, such that $\{x_1, x_2, \ldots, x_i\} \subseteq B'$. If there is another element in $B'$ of weight $a$, then the greedy would not have produced $B$, and if there is not, then $w(B') = w(B)$, a contradiction. Hence (a) holds. If (b) does not hold, then clearly there is another base of weight greater than or equal to $B$, a contradiction. Thus (b) holds, and the theorem is proved. □

## 4. Applications to TSP

In this section, we apply the general results and approaches from the previous section to the symmetric and asymmetric traveling salesman problems (STSP and ATSP). Let $E$ be the set of edges (arcs) in $K_n$ ($\overleftrightarrow{K}_n$) and let $\mathscr{H}$ be the collection of sets of edges (arcs) such that every such set is a subset of edges (arcs) of a Hamilton undirected (directed) cycle in $K_n$ ($\overleftrightarrow{K}_n$). The usually formulated STSP and ATSP can be considered as the $(E, \mathscr{H}, \mathbb{Z}_+)$-optimization problems. As we discussed in Section 1, also $(E, \mathscr{H}, W)$-optimization problems, which are restricted versions of STSP and ATSP, are of interest, where $W \subset \mathbb{Z}_+$.

For all these problems the bases are Hamilton cycles (called *tours* in the TSP literature) in the corresponding graph. Thus, we will use the terms 'base' and 'tour' interchangeably in the rest of this section.

**Theorem 4.1.** *Consider restricted versions of STSP as $(E, \mathscr{H}, W)$-optimization problems.*

(a) *If $n \geqslant 4$ and $|W| \leqslant \lfloor \frac{n-1}{2} \rfloor$, then the greedy algorithm never produces the unique worst possible base.*
(b) *If $n \geqslant 3$, $r \geqslant n-1$ and $W = \{1, 2, \ldots, r\}$, then there exists a weight function $w: E \to \{1, 2, \ldots, r\}$ such that the greedy algorithm may produce the unique worst possible base.*

**Proof.** To prove (a) suppose that $B = \{x_1, x_2, \ldots, x_n\}$ is a base produced by the greedy algorithm and that its elements were chosen by the algorithm in the order $x_1, x_2, \ldots, x_n$. Also assume that $B$ is the unique worst base.

Since there are at most $\lfloor \frac{n-1}{2} \rfloor$ different weights, there must exist vertex disjoint edges $x_i, x_j$ of the same weight $k$. Let $x_i$ have endvertices $u$ and $v$ and $x_j$ have endvertices $x$ and $y$. Without loss of generality, we may assume that $B' = B \cup \{uy, vx\} - \{uv, xy\}$ is a base. We must have $w(uy), w(xv) \geqslant k$ since otherwise the greedy algorithm would have chosen one of these edges instead of $x_i$ or $x_j$ in the step just before the first of $x_i, x_j$ was chosen. Thus, $w(B') \geqslant w(B)$, a contradiction.

To prove (b) let $B$ be an arbitrary base and fix an ordering $\{x_1, x_2, \ldots, x_n\}$ of $B$ such that $v_0, x_1, v_1, x_2, v_2, \ldots, v_{n-1}x_n, v_0$ is a tour, where $x_i = v_{i-1}v_i$. Let $A_1, A_2, \ldots, A_{n-1}$, be the ordered partitioning of $\{x_1, x_2, \ldots, x_n\}$ such that $A_1 = \{x_1, x_2\}$ and $A_j = \{x_{j+1}\}$ for $j = 2, 3, \ldots, n-1$.

We will show that (1) of Theorem 3.1 holds w.r.t. the given ordering of $B$ and $A_1, A_2, \ldots, A_{n-1}$. By the choice of the ordering of $B$ (corresponding to the tour $v_0, x_1, v_1, x_2, v_2, \ldots, v_n x_n, v_0$) it follows that:

$$|I(A_{0,0}) \cap B| = n \quad \text{and} \quad |I(A_{0,j}) \cap B| = n - j - 1 \quad \text{for } j = 1, 2, \ldots, n-2. \tag{4}$$

This implies that

$$\sum_{j=0}^{n-2} |I(A_{0,j}) \cap B| = \frac{n(n-1)}{2} + 1. \tag{5}$$

Let $B'$ be a base different from $B$. We claim that

$$|I(A_{0,j}) \cap B'| \leqslant |I(A_{0,j}) \cap B| \quad \text{for } j = 0, 1, \ldots, n-2. \tag{6}$$

This clearly holds for $j = 0$. To see that it holds for $j = 1, 2, \ldots, n-2$, it suffices to observe that no edge incident to the vertices $v_1, \ldots, v_j$ belongs to $I(A_{0,j}) \cap B'$. Hence at least $j+1$ edges of $B'$ do not belong to $I(A_{0,j})$ and (6) follows from (4).

We claim that we will have strict inequality at least once in (6). Assume that this is not true.

Observe that unless the vertices $v_1, v_2, \ldots, v_j$ induce a connected component in the tour $B'$ (that is, when we delete these vertices we get a path) we will have $|I(A_{0,j}) \cap B'| < n - j - 1$. Furthermore none of the edges $v_0 v_{j+1}$, $j = 1, \ldots, n-2$ can belong to $B'$. This is because the edge $v_0 v_{j+1}$ cannot belong to $I(A_{0,j})$, implying again that if it was in $B'$ we would have $|I(A_{0,j}) \cap B'| < n - j - 1$. But then $B'$ must contain the edges $v_0 v_1$ and $v_0 v_n$ and using that the set $v_1, v_2, \ldots, v_j$ induces a connected component in $B'$ for $j = 1, 2, \ldots, n-2$ we conclude that $B' = B$, a contradiction. Thus we have shown that we have strict inequality in (6) and now it follows from Theorem 3.1 that we can assign weights from $\{1, 2, \ldots, n-1\}$ to the edges of $K_n$ so that the greedy algorithm may find the unique worst tour. □

**Remarks.** 1. Notice that while $W$ in part (a) of Theorem 4.1 is an arbitrary set of cardinality at most $\lfloor (n-1)/2 \rfloor$, $W$ in part (b) is the set with elements $1, 2, \ldots, r$. We have to restrict the elements in $W$ in part (b) because we use Theorem 3.1.

2. It follows from the way we proved (a) that no greedy tour containing two vertex disjoint edges of the same cost can be the unique worst possible. Hence if $B$ is a 'greedy' base, which is also the unique worst possible, then there are at most two edges of cost $k$ for any $k$ in the range of $w$ and furthermore such edges must be consecutive on the tour $B$.

3. The proof of (b) does not work if we replace $n-1$ by $n/2$. This is because in this case we cannot guarantee that a base $B'$ which has equality in (6) must use the edges $v_0 v_1$ and $v_{n-1} v_0$. Consider, for example, the case when $n = 6$ and

$B = v_0, x_1, v_1, x_2, v_2, \ldots, v_5, x_6, v_0$. Then the suggested assignment used in the proof of Theorem 3.1 would give $w(x_1) = w(x_2) = 1$, $w(x_3) = w(x_4) = 2$ and $w(x_5) = w(x_6) = 3$ and $B$ would have weight 12. On the other hand it is easy to check that the tour which visits the vertices in the order $v_0, v_3, v_1, v_2, v_4, v_5, v_0$ also has weight 12, implying that $B$ is not the unique worst base.

For ATSP we can in fact determine the exact borderline for the complete failure of the greedy algorithm.

**Theorem 4.2.** *Consider restricted versions of ATSP as $(E, \mathscr{H}, W)$-optimization problems. Let $n \geqslant 3$.*

(a) *If $|W| \leqslant \lfloor \frac{n-1}{2} \rfloor$, then the greedy algorithm never produces the unique worst possible base.*
(b) *For every $r \geqslant \lceil \frac{n+1}{2} \rceil$ there exists a weight function $w: E(\overleftrightarrow{K}_n) \to \{1, 2, \ldots, r\}$ such that the greedy algorithm may produce the unique worst possible base.*

**Proof.** Since the proof is similar to that of Theorem 4.1 we will only give a few hints. To prove (a) observe that since there are at most $\lfloor \frac{n-1}{2} \rfloor$ different weights, there will be three arcs with the same weight in any base $B$. By deleting three such arcs and adding three different ones not in $B$ we obtain a new base $B'$ and as above we can argue that $w(B') \geqslant w(B)$.

To prove (b) we consider a tour $v_0, x_1, v_1, x_2, v_2, \ldots, v_n, x_n, v_0$, fix the base $B = \{x_1, x_2, \ldots, x_n\}$ and take the ordered partition to be $A_1, \ldots, A_{\lfloor \frac{n+1}{2} \rfloor}$ where $A_i = \{x_{2i-1}, x_{2i}\}$, $i = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and $A_{\lfloor \frac{n+1}{2} \rfloor} = \{x_{n-1}, x_n\}$ if $n$ is even and $A_{\lfloor \frac{n+1}{2} \rfloor} = \{x_n\}$ if $n$ is odd. Now arguing in a way similar to that in the proof of Theorem 4.1 we can show that equality holds in (6) if and only if $B' - A_{0,j}$ is a path with the same endvertices as $B - A_{0,j}$ for $j = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ (the key observation here is that if we have equality in (6), then the arc $v_0 v_1$ must belong to $B'$). Since $B' \neq B$ it follows that $B'$ cannot have equality in (6) for every $j = 0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and hence (1) holds and Theorem 3.1 implies that (b) holds. □

Some ideas used to prove Theorem 3.1 can be reutilized to show the following:

**Theorem 4.3.** *For each even $n \geqslant 4$ there exists an instance of STSP (ATSP) that has $\Omega(\frac{(n-1)!}{2^n n^{3/2}})$ optimal tours, each of which is $f(n)$ times shorter than the unique worst tour, where $f(n) \geqslant 1$ is an arbitrary function in $n$, and yet the greedy algorithms produces the unique worst tour.*

**Proof.** The proof is very similar for STSP and ATSP. Thus, we restrict ourselves to the STSP only, but we comment on the part, where there is some difference. Let $K_n$ be a complete graph on vertices $\{1, 2, \ldots, n\}$ and let edge $\{i, i+1\}$ be denoted by $e_i$ for $i = 1, 2, \ldots, n$, where $n + 1 = 1$.

Then $T = \{e_1, e_2, \ldots, e_n\}$ is a base. Let $T'$ be an arbitrary base distinct from $T$. It was proved in [4] that

$$\sum_{j=0}^{n-1} |I(e_1, e_2, \ldots, e_j) \cap T'| < n(n+1)/2. \tag{7}$$

Let $w'(e_i) = i(n+1)$ for each $e_i \in T$ and, for $e \notin T$, let $w'(e) = 1 + j(n+1)$ if $e \in I(e_1, e_2, \ldots, e_{j-1})$ but $e \notin I(e_1, e_2, \ldots, e_j)$.

Let $P(n)$ be the $w'$-weight of a $w'$-heaviest tour in $K_n$. Let $L = \{2, 3, \ldots, \frac{n}{2}+1\}$ and $R = \{\frac{n}{2}+2, \frac{n}{2}+3, \ldots, n\} \cup \{1\}$. We define the weights of $e \in E$ as follows: $w(e) = w'(e)$ unless both endvertices of $e$ are in $R$, in which case $w(e) = w'(e) + f(n)P(n)$.

Clearly, the greedy algorithm constructs $T$ and $w'(T) = n(n+1)^2/2$, $w(T) = w'(T) + (\frac{n}{2} - 1) f(n)P(n)$. Let $\mathscr{A}$ be the set of all tours alternating between $L$ and $R$ and containing edges $e' = \{\frac{n}{2}+1, \frac{n}{2}+2\}$ and $e'' = \{2, 1\}$.

Let $G$ be the induced subgraph of $K_n$ obtained from $K_n$ by deleting the vertices $\frac{n}{2}+1$ and $\frac{n}{2}+2$. Clearly, there are $[(\frac{n}{2}-2)!]^2$ tours alternating between $L$ and $R$ and containing the edge $e''$ in the graph $G$. To form a tour containing $e', e''$ in $K_n$ from a tour $C$ containing $e''$ in $G$, it suffices to insert the edge $e'$ into $C$ such that $e''$ remains in the tour. This can be done in $n-3$ ways. Hence, $|\mathscr{A}| = [(\frac{n}{2}-2)!]^2(n-3) = \Omega(\frac{(n-1)!}{2^n n^{3/2}})$. (Notice that for ATSP, the definition and cardinality of $\mathscr{A}$ are slightly different: $\mathscr{A}$ consists of tours alternating between $L$ and $R$ and containing arcs $(1, 2)$ and $(\frac{n}{2}+1, \frac{n}{2}+2)$, $|\mathscr{A}| = [(\frac{n}{2}-2)!]^2(\frac{n}{2}-1) = \Omega(\frac{(n-1)!}{2^n n^{3/2}})$.)

It is easy to verify that each cycle $H$ in $\mathscr{A}$ has the same weight and $w(T)/w(H) \geqslant f(n)$. It remains to prove that every $H \in \mathscr{A}$ is an optimal tour and $T$ is the unique worst tour.

Let $C$ be a tour alternating between $L$ and $R$. Observe that the sum of the weights of two edges of $C$ incident to a vertex $i \in L$ equals $2i(n+1)+2$ provided none of the two edges coincides with $e'$ or $e''$ (the only exception is when one of the edges is incident to vertex 1, in which case $n$ has to be subtracted; notice that there are exactly two edges of $C$ incident to vertex 1). Including $e'$ ($e''$) into $C$, we decrease the weight of $C$ by one. Thus, every tour $C$ alternating between $L$ and $R$ and not containing at least one of the edges $e', e''$ has weight larger than that of $H \in \mathscr{A}$. Every tour $C$ not alternating between $L$ and $R$ has an edge between vertices in $R$. Thus, $w(C) > w(H)$.

Let $C = \{e'_1, e'_2, \ldots, e'_n\}$ be a tour distinct from $T$. Assume that $w'(e'_i) \in \{a(n+1), a(n+1)+1\}$. Then clearly $e'_i \in I(e_1, e_2, \ldots, e_{a-1})$, but $e'_i \notin I(e_1, e_2, \ldots, e_a)$, so $e'_i$ lies in $I(e_1, e_2, \ldots, e_j) \cap C$, provided $j \leqslant a-1$. Thus, $e'_i$ is counted $a$ times in the sum in (7). Hence,

$$w'(C) = \sum_{i=1}^{n} w(e'_i) \leqslant n + (n+1) \sum_{j=0}^{n-1} |I(\{e_1, e_2, \ldots, e_j\}) \cap C|$$
$$\leqslant n + (n+1)(n(n+1)/2 - 1) = n - (n+1) + w'(T) < w'(T).$$

It remains to notice that $w(T) = w'(T) + (\frac{n}{2} - 1) f(n) P(n)$ and no tour contains more than $\frac{n}{2} - 1$ edges whose all endvertices are in $R$. □

## 5. Applications to other problems

Let $\mathscr{F}$ be the sets of those subsets $X$ of $E(K_{2n})$ which induce a bipartite graph. Then $(E(K_{2n}), \mathscr{F})$ is a uniform independence system and the bases of $(E(K_{2n}), \mathscr{F})$ correspond to copies of the complete balanced bipartite graph $K_{n,n}$ in $K_{2n}$. The $(E(K_{2n}), \mathscr{F}, \mathbb{Z}_+)$-optimization problem is called the minimum bisection problem [1].

**Proposition 5.1.** *Let $n \geqslant 4$. The greedy algorithm for the $(E(K_{2n}), \mathscr{F}, W)$-optimization problem may produce the unique worst solution even if $|W| = 2$.*

**Proof.** Fix an arbitrary copy $B$ of $K_{n,n}$ in $K_{2n}$ and order the edges of $B$ as $B = \{e_1, e_2, \ldots, e_{n^2}\}$ so that the first $2n-1$ edges form a spanning tree $T$ in $K_{2n}$ and the last $2n-1$ edges form a spanning tree $T'$ in $K_{2n}$ (this is clearly possible when $n \geqslant 4$). Now consider any base (a copy of $K_{n,n}$) $B'$ which is different from $B$. Then both (a) and (b) of Theorem 3.2 must hold for $B'$ because as soon as a bipartite subgraph of $K_{2n}$ contains the edges of either $T$ or $T'$ the bipartition is fixed to be that of $B$. Thus it follows from Theorem 3.2 that there exists an assignment of weights, using only two weights such that the greedy algorithm will produce the unique worst solution. □

Let $A$ be the arc set of the complete digraph $\overleftrightarrow{K}_n$. Let $\mathscr{F}$ be the family of those subsets $X$ of $A$ for which the subdigraph $D[X]$ induced by the arcs in $X$ has maximum out-degree one and contains at least one vertex with out-degree zero. Then $(A, \mathscr{F})$ is an independence system and the bases of $(A, \mathscr{F})$ correspond to in-branchings of $\overleftrightarrow{K}_n$. It is not difficult to show that the greedy algorithm does not always find an optimal base, even if the arcs have only two different weights. On the other hand, we can prove that it never produces the unique worst solution either.

**Proposition 5.2.** *Let $n \geqslant 2$ and let $(A, \mathscr{F})$ be the independence system above. Then the greedy algorithm will never produce the unique worst possible solution for the $(A, \mathscr{F}, \{1, 2, \ldots, r\})$-optimization problem for any $r \geqslant 2$ and weight function $w$.*

**Proof.** Let $B$ be a base produced by the greedy algorithm. Observe that if $e$ is the last arc included in $B$ by the greedy algorithm, then $\{e, e'\} \subseteq I(B - e)$ for some $e' \neq e$. Then $w(e') \geqslant w(e)$ and $w(B \cup \{e'\} - \{e\}) \geqslant w(B)$ and $B$ is not the unique worst base. □

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## References

[1] G. Ausiello, P. Crescenzi, G. Gambosi, V. Kann, A. Marchetti-Spaccamela, M. Protasi, Complexity and Approximation, Springer, Berlin, 1999.
[2] L. Engebretsen, M. Karpinski, Approximation hardness of TSP with bounded metrics, Lecture Notes in Computer Science, Vol. 2076, Springer, Berlin, 2001, pp. 201–212.

[3] D. Fotakis, G. Pantziou, G. Pentaris, P. Spirakis, Frequency assignment in mobile and radio networks, Networks in distributed computing, DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Vol. 45, 1999, pp. 73–90.
[4] G. Gutin, A. Yeo, Anti-matroids, Oper. Res. Lett. 30 (2002) 97–99.
[5] G. Gutin, A. Yeo, Polynomial approximation algorithms for the TSP and the QAP with a factorial domination number, Discrete Appl. Math. 119 (2002) 107–116.
[6] G. Gutin, A. Yeo, A. Zverovich, Traveling salesman should not be greedy: domination analysis of greedy-type heuristics for the TSP, Discrete Appl. Math. 117 (2002) 81–86.
[7] D.S. Johnson, G. Gutin, L. McGeoch, A. Yeo, X. Zhang, A. Zverovitch, Experimental analysis of heuristics for ATSP, in: G. Gutin, A. Punnen (Eds.), The Traveling Salesman Problem and its Variations, Kluwer, Dordrecht, 2002.
[8] C.H. Papadimitriou, M. Yannakakis, The traveling salesman problem with distances one and two, Math. Oper. Res. 18 (1993) 1–11.
[9] A. Punnen, S. Kabadi, Domination analysis of some heuristics for the asymmetric traveling salesman problem, Discrete Appl. Math. 119 (2002).
[10] A. Punnen, F. Margot, S. Kabadi, TSP heuristics: domination analysis and complexity, Algorithmica 35 (2003) 111–127.
[11] V.I. Sarvanov, On the minimization of a linear from on a set of all $n$-elements cycles, Vestsi Akad. Navuk BSSR, Ser. Fiz.-Mat. Navuk 4 (1976) 17–21 (in Russian).
[12] S. Vishwanathan, An approximation algorithm for the asymmetric traveling salesman problem with distances one and two, Inform. Process. Lett. 44 (1992) 297–302.