P-01 → LIMITS & CONTINUITY

(1) $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$

$\lim_{x \to a} \frac{(a+2x-3x)}{(3a+x-4x)} \times \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})}$

$\lim_{x \to a} \frac{(a-x)}{(a-x)} \times \frac{(\sqrt{3a+x} + 2\sqrt{x})}{(\sqrt{a+2x} + \sqrt{3x})}$

$\frac{1}{3} \frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$

$\frac{1}{3} \times \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$

$\frac{1}{3} \times \frac{4\sqrt{a}}{2\sqrt{3}\sqrt{a}}$

$= \frac{2}{3\sqrt{3}}.$

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(2) $\lim\limits_{y \to 0} \left[\dfrac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

$\lim\limits_{y \to 0} \left[\dfrac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \dfrac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right]$

$\lim\limits_{y \to 0} \dfrac{a+y-a}{y\sqrt{a+y}\left(\sqrt{a+y}+\sqrt{a}\right)}$

$\lim\limits_{y \to 0} \dfrac{y}{y\sqrt{a+y}\left(\sqrt{a+y}+\sqrt{a}\right)}$

$\lim\limits_{y \to 0} \dfrac{1}{\sqrt{a+0}\left(\sqrt{a+0}+\sqrt{a}\right)}$

$\dfrac{1}{\sqrt{a}\left(\sqrt{a}+\sqrt{a}\right)}$

$\dfrac{1}{\sqrt{2}\left(2\sqrt{a}\right)} = \dfrac{1}{2a}$.

(3) $\lim_{x \to \pi/6} \dfrac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

By substituting $x - \frac{\pi}{6} = h$

$x = h + \pi/6$

where $h \to 0$

$$\lim_{h \to 0} \frac{\cos(h + \pi/6) - \sqrt{3}\sin(h + \pi/6)}{\pi - 6(\cancel{h} + \pi/6)}$$

{Using $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$.}

$$\lim_{h \to 0} \frac{(\cos h \cdot \cos \pi/6 - \sin h \cdot \sin \pi/6 - \sqrt{3}\sin h \cdot \cos \pi/6 + \cos h \cdot \sin \pi/6)}{\pi - 6\left(\frac{6h + \pi}{6}\right)}$$

$\cos \pi/6 = \cos 30^\circ = \sqrt{3}/2$

$\sin \pi/6 = \sin 30^\circ = 1/\sqrt{2}$

$$\lim_{h \to 0} \frac{\cos h \cdot \sqrt{3}/2 - \sin h \frac{1}{\sqrt{2}} - \sqrt{3}\left(\sin h \cdot \sqrt{3}/2 + \cos h \cdot \frac{1}{2}\right)}{\pi - 6h + \pi}$$

$$\lim_{h \to 0} \frac{\left(\cos \sqrt{3}/2\, h - \sin h/2 - \sin 3h/2 - \cos \sqrt{3}/2\, h\right)}{-6h}$$

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$$\lim_{h\to 0} \frac{-\sin\frac{4h}{2}}{-6h}$$

$$\lim_{h\to 0} \frac{\sin 4h}{3+2h}$$

$$\frac{1}{3}\lim_{h\to 0} \frac{\sin h}{h} = \frac{1}{3}\times 1 = 1/3.$$

(4) $$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$$

By rationalizing numerator & denominator both.

$$\lim_{x\to\infty}\left[\frac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\times\frac{\sqrt{x^2+5}+\sqrt{x^2-3}}{\sqrt{x^2+5}+\sqrt{x^2-3}}\times\frac{\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+3}+\sqrt{x^2+1}}\right]$$

$$\lim_{x\to\infty}\left[\frac{(x^2+5-x^2+3)}{(x^2+3-x^2-1)}\times\frac{(\sqrt{x^2+3}+\sqrt{x^2+1}}{\sqrt{x^2+5}+\sqrt{x^2-3}}\right]$$

$$\lim_{x\to\infty}\frac{8}{2}\left[\frac{(\sqrt{x^2+3}+\sqrt{x^2+1})}{(\sqrt{x^2+5}+\sqrt{x^2-3})}\right]$$

$$4\lim_{x\to\infty}\frac{\sqrt{x^2(1+3/x^2)}+\sqrt{x^2(1+1/x^2)}}{\sqrt{x^2(1+5/x^2)}+\sqrt{x^2(1-3/x^2)}}$$

After applying limit,
we get;

$\Longrightarrow 4.$

5) If $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 < x \leq \pi/2$

$= \dfrac{\cos x}{x - 2\pi}$, for $\pi/2 < x < \pi$ } at $x = \pi/2$

$f(\pi/2) = \dfrac{\sin 2(\pi/2)}{\sqrt{1-\cos 2(\pi/2)}}$ $\therefore f(\pi/2) = 0.$

f at $x = \pi/2$ define.

$\lim\limits_{x \to \pi/2} f(x) = \lim\limits_{x \to \pi/2^{\pm}} \dfrac{\cos x}{\pi - 2x}$

By substituting method,

$x - \pi/2 = h$

$x = h + \pi/2$
where $h \to 0$.

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$$\lim_{h\to 0} \frac{\cos\left(h+\pi/2\right)}{\pi - 2\left(h+\pi/2\right)}$$

$$\lim_{h\to 0} \frac{\cos(h+\pi/2)}{\pi - 2\left(\frac{2h+\pi}{2}\right)}$$

$$\lim_{h\to 0} \frac{\cos(h+\pi/2)}{-2h}$$ using $\cos(A+B) = \cos A\cos B - \sin A \sin B$.

$$\lim_{h\to 0} \frac{\cos h\cdot\cos\pi/2 - \sin h\cdot\sin\pi/2}{-2h}$$

$$\lim_{h\to 0} \frac{\cos h\cdot 0 - \sin h}{-2h}$$

$$\lim_{h\to 0} \frac{\cancel{-}\sin h}{\cancel{-}2h}$$

$$\frac{1}{2}\lim_{h\to 0}\frac{\sin h}{h} = \frac{1}{2}$$ //.

$$\lim_{x\to\pi/2^-} f(x) = \lim_{x\to\pi/2^-} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$$

Using $\sin 2x = 2\sin x\cdot\cos x$.

$$\lim_{x \to \pi/2^-} \frac{2 \cdot \sin x \cdot \cos x}{\sqrt{2 \sin^2 x}}$$

$$\lim_{x \to \pi/2^-} \frac{2 \cos x}{\sqrt{2}}$$

$$\frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cdot \cos x .$$

$\therefore$ LHL $\neq$ RHL .

$\therefore f$ is not continuous at $x = \pi/2$.

(5)

(ii) $f(x) = \dfrac{x^2 - 9}{x - 3}$; $0 < x < 3$

$= x + 3$; $3 \le x \le 6$

$= \dfrac{x^2 - 9}{x + 3}$; $1 \le x < 9$

at $x = 3$ & $x = 6$.

at $x = 3$

$$f(3) = \frac{x^2 - 4}{x - 3} = 0$$

f. ~~at~~ $x = 3$ defined.

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(2) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x + 3$

$f(3) = x + 3 = 3 + 3 = 6$

f is defined at $x = 3$.

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+3) = 6$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - 9}{x - 3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}} = x+$

LHL = RHL

$\therefore$ f is continuous at $x = 3$.

for $x = 6$.

$f(6) = \frac{x^2 - 9}{x + 3} = \frac{36 - 9}{6 + 3} = \frac{27}{9} = 3$

$\lim_{x \to 6^+} \frac{x^2 - 9}{x + 3}$

$\lim_{x \to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$

$\lim_{x \to 6^+} (x - 3) = 6 - 3 = 3$

$\lim_{x \to 6^-} (x + 3) = 3 + 6 = 9$

$\therefore$ LHL $\neq$ RHL

f^n is not continuous.

Q6. $f(x) = \frac{1-\cos 4x}{x^2}$, $x<0$
$= k$, $x=0$ } at $x=0$

→ f is continuous at $x=0$.

$\lim_{x\to 0} f(x) = f(0)$

$\lim_{x\to 0} \frac{1-\cos 4x}{x^2} = k$

$\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = k$

$2\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = k$

$2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2 = k$

$2(2)^2 = k$

$\therefore k = 8$

(ii) $f(x) = (\sec^2 x)^{\cot^2 x}$

~~Using~~ $\tan^2 x = 1+\sec^2 x$

$\therefore \sec^2 x = 1+\tan^2 x$

& $\cot^2 x = \frac{1}{\tan^2 x}$

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$$\lim_{x \to 0} (\sec^2 x)^{\cot^2 x}$$

$$\lim_{x \to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$$

wkt,

$$\lim_{x \to 0} (1+px)^{1/px} = e$$

$$= e$$

$$K = e$$

(iii) $f(x) = \frac{\sqrt{3} - \tan x}{\pi - 3x}$, $x \neq \pi/3$; $= K$, $x = \pi/3$ } at $x = \pi/3$

$x - \pi/3 - h$

$x = h + \pi/3$

when $h \to 0$

$$f(\pi/3 + h) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

$$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

Using $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A . \tan B}$.

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$$\lim_{h \to 0} \frac{\dfrac{\sqrt{3} - \tan \pi/3 + \tan h}{1 - \tan \pi/3 \cdot \tan h}}{\pi - \cancel{\pi} \cdot - 3h}$$

$$\lim_{h \to 0} \frac{\dfrac{\sqrt{3}(1 - \tan \pi/3 \cdot \tan h) - (\tan \pi/3 + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

Using $\tan \pi/3 = \tan 60° = \sqrt{3}$.

$$\lim_{h \to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3} \times \sqrt{3} \cdot \tan h) - (\sqrt{3} + \tan h)}{1 - \tan \pi/3 \cdot \tan h}}{-3h}$$

$$\lim_{h \to 0} \frac{\dfrac{(\sqrt{3} - 3\tan h - \sqrt{3} - \tan h)}{1 - \sqrt{3} \tan h}}{-3h}.$$

$$\frac{4}{3} \lim_{h \to 0} \frac{\tan h}{h} \cdot \lim_{h \to 0} \frac{1}{(1 - \sqrt{3} \tan h)}$$

$$\Rightarrow \frac{4}{3}\left(\frac{1}{1}\right) = \frac{4}{3}.$$

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(10)

(2) $f(x) = \dfrac{1-\cos 3x}{x \cdot \tan x}$, $x \neq 0$
$= 9$, $x = 0$ } at $x = 0$

$f(x) = \dfrac{1-\cos 3x}{x\tan x}$

$\lim_{x \to 0} \dfrac{2\sin^2 3/2 x}{x \cdot \tan x}$

$\lim_{x \to 0} \dfrac{\dfrac{2\sin^2 3x/2}{x^2} \cdot x^2}{\dfrac{x \cdot \tan x}{x^2} \cdot x^2}$

$= 2 \lim_{x \to 0} \dfrac{(3/2)^2}{1} = 2 \times 9/4 = 9/2$

$\lim_{x \to 0} f(x) = 9/2$, $9 = f(0)$

$\therefore$ f is not continuous at $x = 0$

Redefine function

$f(x) = \begin{cases} \dfrac{1-\cos 3x}{x\tan x} & , x \neq 0 \\ = 9/2 & , x = 0 \end{cases}$

Now, $\lim_{x \to 0} f(x) = f(0)$

f has removable discontinuity at $x = 0$.

(ii) $f(x) = (e^{3x} - 1) \sin x^{\circ} / x^2$; $x \neq 0$
$= \pi/6$; $x = 0$ } at $x = 0$.

$$\lim_{x \to 0} \frac{(e^{x} - 1) \sin (\pi x / 180)}{x^2}$$

$$\lim_{x \to 0} \frac{3 \cdot e^{3x} - 1}{3x} \lim_{x \to 0} \sin\left(\frac{\pi x}{180}\right) / x$$

$$3 \cdot \lim_{x \to 0} \left(\frac{e^{3x} - 1}{3x}\right) \lim_{x \to 0} \sin \cancel{x} (\cancel{\pi} / 180) / x$$

$$3 \cdot \log (\pi / 180) = \pi / 60 = f(6).$$

f is continuous at $x = 0$.

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(8) $f(x) = \frac{e^{x^2} - \cos x}{x^2}$, $x = 0$

is continuous at $x = 0$.

$\therefore$ Given, f is cts. at $x = 0$.

$\lim_{x \to 0} f(x) = f(0)$

$= \lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2} = f(0)$

$\lim_{x \to 0} \frac{e^{x^2} - 1}{x} + \lim_{x \to 0} \frac{1 - \cos x}{x^2}$

$\log e + \lim_{x \to 0} \left(\frac{2 \sin^2 x/2}{x}\right)^2$

Multiply with 2 on numerator & denominator.

$\Rightarrow 1 + 2 \times \frac{1}{4} = 3/2 = f(0)$

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(9) $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x}$; $x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$$\lim_{x \to \pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$$

$$\lim_{x \to \pi/2} \left(\frac{2 - 1 + \sin x}{\cos^2 x \left(\sqrt{2} + \sqrt{1+\sin x}\right)} \right)$$

$$\lim_{x \to \pi/2} \frac{1 + \sin x}{1 - \sin^2 x \left(\sqrt{2} + \sqrt{1+\sin x}\right)}$$

$$\lim_{x \to \pi/2} \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)\left(\sqrt{2} + \sqrt{1+\sin x}\right)}$$

$$\lim_{x \to \pi/2} \frac{1}{(1 - \sin x)\left(\sqrt{2} + \sqrt{1+\sin x}\right)}$$

$$= \frac{1}{2(\sqrt{2} + \sqrt{2})}$$

$$= \frac{1}{2(2\sqrt{2})}$$

$$= \frac{1}{4\sqrt{2}}.$$

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$\therefore f(\pi/2) = \dfrac{1}{4\sqrt{2}}.$

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PRACTICAL - 02

Topic : Derivatives.

1\. Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable :

i) $\cot x$.

$f(x) = \cot(x)$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\tan x - 1/\tan a}{x - a}$$

$$= \lim_{x \to a} \frac{\tan a - \tan x}{(x - a)\tan x \cdot \tan a}$$

Put $x - a = h$

$x = h + a$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(a+h-a)\ \tan(a+h) \cdot \tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times (\tan(a+h) \cdot \tan a)}$$

FORMULA : $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$

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$$\tan A - \tan B = \tan(A-B)\,(1+\tan A \cdot \tan B)$$

$$= \lim_{h\to 0} \frac{\tan(a-a-h) - (1+\tan a \tan(a+h))}{h \times \tan(a+h)\cdot \tan a}$$

$$= \lim_{h\to 0} \frac{-\tan h}{h} \times \frac{1+\tan a \tan(a+h)}{\tan(a+h)\cdot \tan a}$$

$$= -1 \times \frac{1+\tan^2 a}{\tan^2 a}$$

$$= \frac{-\sec^2 a}{\tan^2 a}$$

$$= \frac{-1}{\cancel{\cos^2 a}} \times \frac{\cancel{\cos^2 a}}{\sin^2 a}$$

$$= -\frac{1}{\sin^2 a} = -\operatorname{cosec}^2 a$$

$\therefore Df(a) = -\operatorname{cosec}^2 a$

$\therefore f$ is differentiable $\forall a \in \mathbb{R}$

(ii) $\operatorname{cosec} x$.

$f(x) = \operatorname{cosec} x$

$$Df(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x\to a} \frac{1/\sin x - 1/\sin a}{x-a}$$

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$= \lim_{x \to a} \dfrac{\sin a - \sin x}{(x-a)\sin a \sin x}$

Put ~~sin~~ $x - a = h$

$\therefore x = a + h$

as $x \to a$, $h \to 0$.

$Df(h) = \lim_{h \to 0} \dfrac{\sin a - \sin(a+h)}{(a+h-a)\sin a \cdot \sin(a+h)}$

$= \lim_{h \to 0} \dfrac{2\cos\left(\frac{a+a+h}{2}\right) \cdot \sin\left(\frac{a-a-h}{2}\right)}{h \times \sin a \cdot \sin(a+h)}$

$\left\{ \text{FROM FORMULA: } \sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{C-D}{2}\right) \right\}$

$= \lim_{h \to 0} \dfrac{-\sin h/2}{h/2} \times 1/2 \times \dfrac{2\cos\left(\frac{2a+h}{2}\right)}{\sin a \cdot \sin(a+h)}$

$= -1/2 \times \dfrac{2\cos\left(\frac{2a+a}{2}\right)}{\sin(a+a)}$

$= -\dfrac{\cos a}{\sin^2 a} = -\cot a \cdot \cancel{c}\operatorname{cosec} a$ //

(iii) $\sec x$

$f(x) = \sec x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x - a}$$

$$= \lim_{x \to a} \frac{1/\cos x - 1/\cos a}{x - a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(x-a)(\cos a \cdot \cos x)}$$

Put $x = a + h$

$\therefore x - a = h$

as $x \to a$, $h \to 0$

$$Df(h) = \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{h \times \cos a \cdot \cos(a+h)}$$

FORMULA:

$$-2\sin\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{a+a+h}{2}\right) \cdot \sin\left(\frac{a-a+h}{2}\right)}{h \times \cos a \cdot \cos(a+h)}$$

$$= \lim_{h \to 0} \frac{-2\sin\left(\frac{2a+h}{2}\right) \cdot \sin(-h/2)}{\cos a \cdot \cos(a+h) \times -h/2} \times -1/2$$

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$= \frac{1}{2} \times 2 \frac{\sin a}{\cos a \times \cos a}$

$= \tan a \cdot \sec a.$

Q2. If $f(x) = 4x+1$, $x \leq 2$
$= x^2+5$, $x > 0$ at $x=2$, then find f^n is differentiable or not.

Sol^n:

LHD:

$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$

$= \lim_{x \to 2^-} \frac{4x+1-(4(2)+1)}{x-2}$

$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$

$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$

$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$

$= \cancel{\lim_{x \to 2^-}} \frac{4(x-2)}{(x-2)} = 4$

$Df(2^-) = 4$

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RHD:

$$Df(2^+) = \lim_{x\to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x\to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 2+2 = 4$$

$$Df(2^+) = 4$$

$\therefore$ LHD = RHD

$\therefore$ f^n is differentiable at $x=2$.

Q3. If $f(x) = 4x+7$, $x<3$
$\qquad = x^2+3x+1$, $x \geq 3$ at $x=3$, then find f is differentiable or not.

Soln:

RHD:

$$Df(3^+) = \lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+3x+1-(3^2+3\times 3+1)}{x-3}$$

$$= \lim_{x\to 3^+} \frac{x^2+6x-3x-18}{x-3}$$

$= \cancel{\lim}$

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$$= \lim_{x \to 3^+} \frac{x(x+6) - 3(x+6)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= 3 + 6 = 9.$$

LHD:

$$Df(3^-) = \lim_{x \to 3^-} \frac{f(x) - f(3)}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x+7-19}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= 4.$$

LHD $\neq$ RHD

$\therefore$ F^n is not differentiable at $x = 3$.

Q4. If $f(x) = 8x - 5$, $x \leq 2$

$= 3x^2 - 4x + 7$, $x > 2$ at $x = 2$. Then find f is differentiable ~~at~~ or not.

Sol^n:

RHD:

$$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2}$$

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RHD

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - 11}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x - 2}$$

$$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= \lim_{x \to 2^+} (3x+2)$$

$$= 3(2)+2 = 8.$$

LHD:

$$Df(2-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - 11}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x - 2}$$

$$= \lim_{x \to 2^-} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 8.$$

$\therefore$ LHD = RHD.

$\therefore$ F^n is differentiable at $x = 2$.

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PRACTICAL-03

Topic: Application of Derivatives.

Q1. Find the intervals in which function is increasing or decreasing :

(i) $f(x) = x^3 - 5x - 11$.
(ii) $f(x) = x^2 - 4x$.
(iii) $f(x) = 2x^3 - x^2 - 20x + 4$.
(iv) $f(x) = x^3 - 27x + 5$.
(v) $f(x) = 69 - 24x - 9x^2 + 2x^3$.

Q2. Find the intervals in which function is concave upwards or concave downwards:

(i) $y = 3x^2 - 2x^3$.
(ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$.
(iii) $y = x^3 - 27x + 5$.
(iv) $y = 69 - 24x - 9x^2 + 2x^3$
(v) $y = 2x^3 + x^2 - 20x + 4$.

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Sol^n:

Q1.

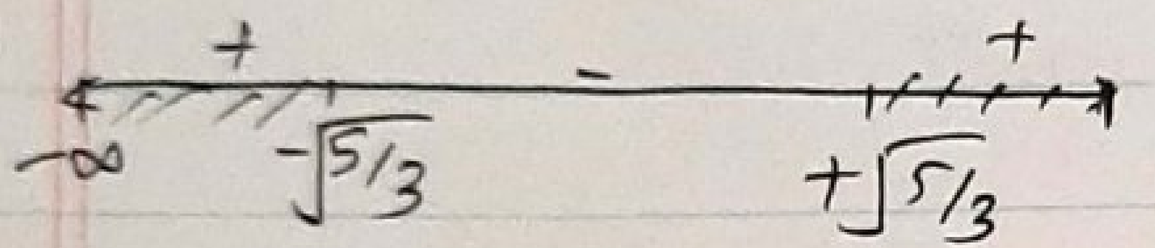
(i) To find increasing,

$f'(x)>0$

$f'(x) = 3x^2 - 5 > 0$

$= 3x^2 > 5$

$\therefore x = \pm\sqrt{5/3}$



$x \in (-\infty, -\sqrt{5/3}) \cup (+\sqrt{5/3}, +\infty)$

To find decreasing,

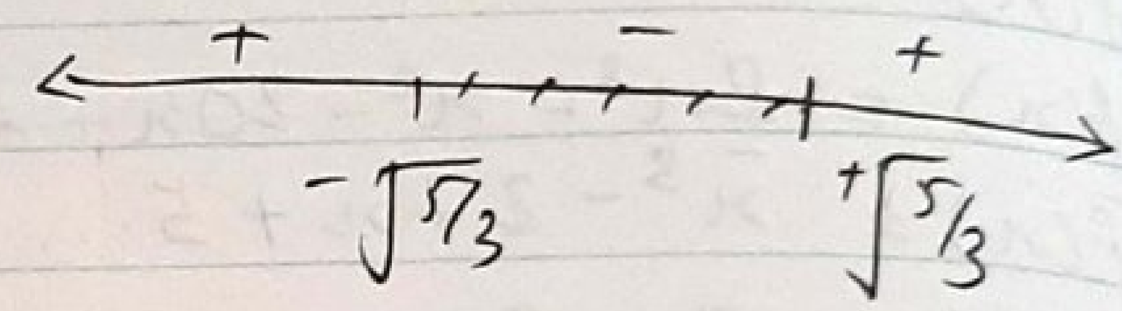
$f'(x) < 0$

$f'(x) < 0$

$= 3x^2 - 5 < 0$

$= 3x^2 < 5$

$x = \pm\sqrt{5/3}$



$x \in (-\sqrt{5/3}, +\sqrt{5/3})$.

(ii) $f(x) = x^2 - 4x$

To find increasing,

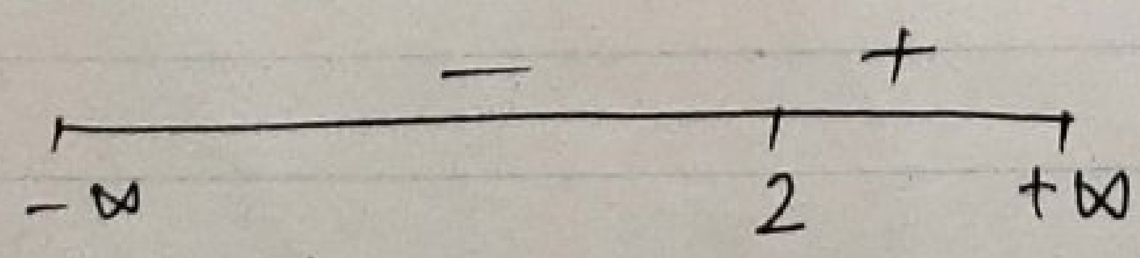
$f'(x)>0$

$f'(x) = 2x - 4 > 0$

$= 2(x-2) > 0$

$= x - 2 > 0$

$x = 2$



$x \in (2, \infty)$

To find decreasing,

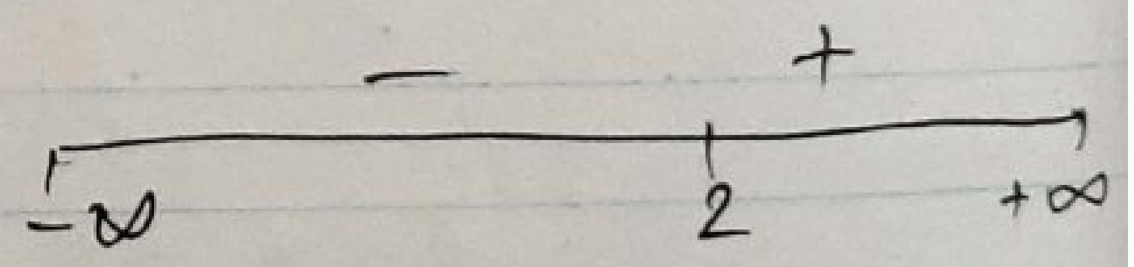
$f'(x) < 0$

$f'(x) < 0$

$= 2x - 4 < 0$

$= x - 2 < 0$

$\therefore x = 2$



$x \in (-2, -\infty)$

(iii) $f(x) = 2x^3 - x^2 - 20x + 4$

$f'(x) > 0$
for increasing.

$f'(x) > 0$
$f'(x) = 6x^2 - 2x - 20 > 0$
$= 6x^2 - 12x + 10x - 20 > 0$
$= 6x(x-2) + 10(x-2) > 0$
$= 6x + 10 > 0, (x-2) > 0$
$\therefore x = -5/3, \ x = 2$

+   −   +

$-5/3$ 2

$x \in (-5/3, +\infty) \cup (2, +\infty)$

$f'(x) < 0$
for decreasing.

$f'(x) < 0$
$f'(x) = 6x^2 - 2x - 20 < 0$
$= 6x(x-2) + 10(x-2) < 0$
$= 6x + 10 < 0, x - 2 < 0$
$\therefore x = -5/3, x = 2$

+   −   +

$-5/3$ 2

$x \in (-5/3, 2)$

(iv) $f(x) = x^3 - 27x + 5$

$f'(x) > 0$
for increasing.

$f'(x) = 3x^2 - 27 > 0$
$= x^2 - 9 > 0$
$= (x-3)(x+3) > 0$
$= x = \pm 3$

+   −   +

-3 $+3$

$x \in (-3, -\infty) \cup (+3, +\infty)$

$f'(x) < 0$
for decreasing.

$f'(x) = 3x^2 - 27 < 0$
$= x^2 - 9 < 0$
$= (x-3)(x+3) < 0$
$= x = \pm 3$

+   −   +

-3 $+3$

$x \in (-3, +3)$

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(v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

$f'(x) > 0$

for increasing

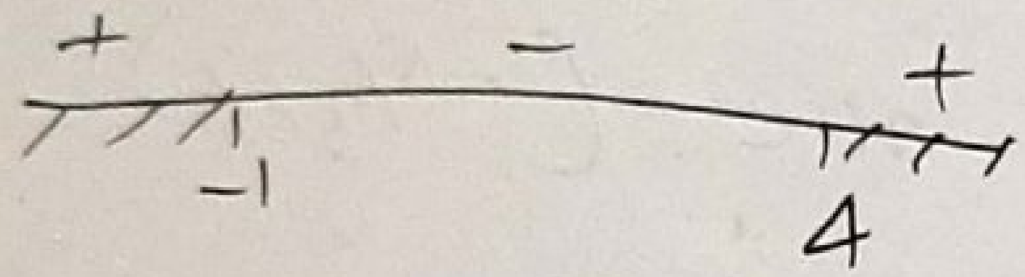
$f'(x) = 6x^2 - 18x - 24 > 0$

$= x^2 - 3x - 4 > 0$

$= x^2 - 4x + x - 4 > 0$

$= x(x-4) + 1(x-4) > 0$

$\therefore (x+1) > 0, \; x - 4 > 0$



$x \in (-1, -\infty) \cup (4, +\infty)$

$f'(x) < 0$

for decreasing

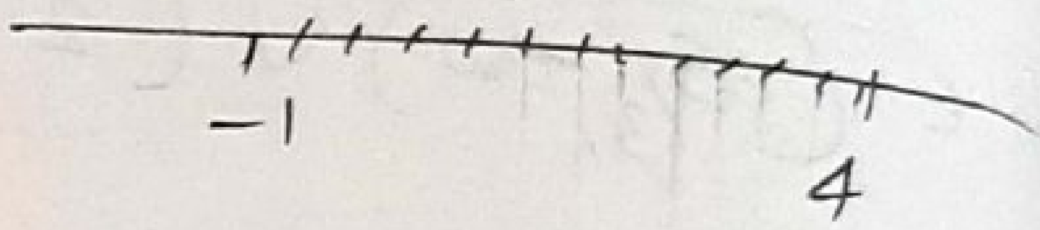
$f'(x) = 6x^2 - 18x - 24 < 0$

$= x^2 - 3x - 4 < 0$

$= x^2 - 4x + x - 4 < 0$

$= x(x-4) + 1(x-4) < 0$

$x + 1 < 0, \; x - 4 < 0$



$x \in (-1, 4)$

Q2.

(i) $y = 3x^2 - 2x^3$

$\boxed{\text{Case I}}$ f^n is concave upwards iff $f''(y) > 0$,

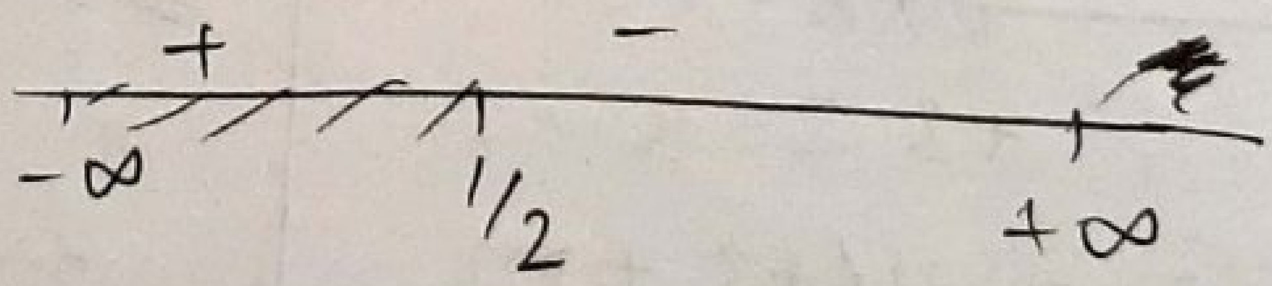
$f'(y) = 6x - 6x^2$

$f''(y) = 6 - 12x$

$f''(y) > 0$

$6 - 12x > 0$

$x = 1/2$



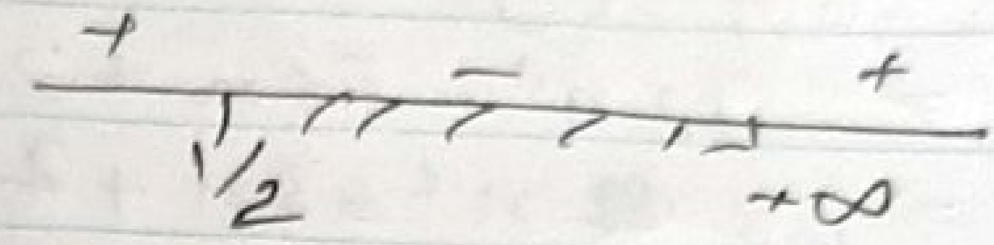
$\therefore x \in (-\infty, 1/2)$

$\boxed{\text{Case II}}$: f^n is concave downwards iff $f''(y) < 0$.

$f''(y) < 0$

$6\cancel{x} - 12x < 0$

$\cancel{x} = 1/2$



$\therefore x \in (1/2, +\infty)$

(ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$\boxed{\text{Case I}}$: f^n is concave upwards iff $f''(y) > 0$.

$f'(y) = 4x^3 - 18x^2 + 24x + 5$

$f''(y) = 12x^2 - 36x + 24$

$\therefore f''(y) > 0$

$\therefore 12x^2 - 36x + 24 > 0$

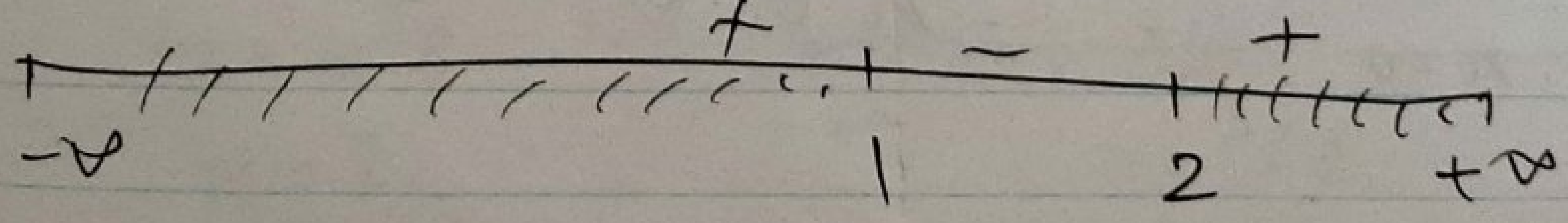
$\therefore x^2 - 3x + 2 > 0$

$\therefore x^2 - 2x - x + 2 > 0$

$\therefore x(x-2) - 1(x-2) > 0$

$\underbrace{(x-1)}_{>0}(x-2) > 0$

$x = 1, x = 2$



$\therefore x \in (1, -\infty) \cup (2, +\infty)$.

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Case II − $f''(y) < 0$, then it is concave downwards

$\therefore f''(y) < 0$

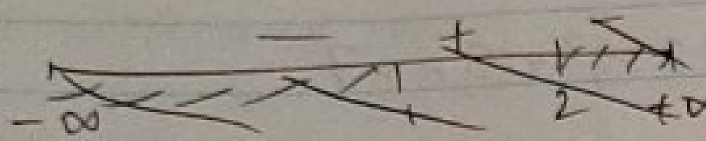
$\Rightarrow 12x^2 - 36x + 24 < 0$

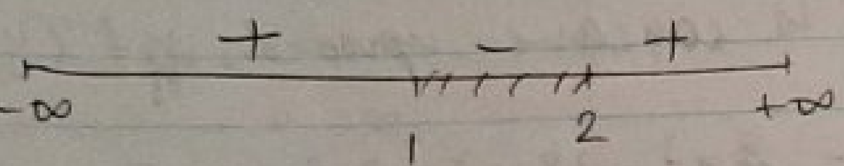
$\Rightarrow x^2 - 3x + 2 < 0$

$\Rightarrow x^2 - 2x - x + 2 < 0$

$\Rightarrow x(x-2) - 1(x-2) < 0$

$\therefore x = 1, \ x = 2$





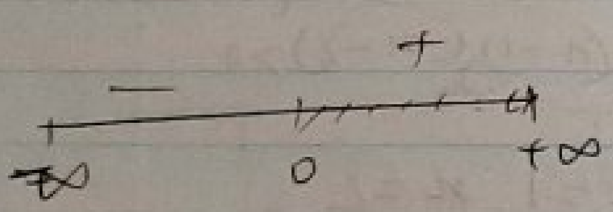
$\therefore x \in (1, 2)$

(iii) $y = x^3 - 27x + 5$

$f''(y) > 0$ for concavity upwards.

$f'(y) = 3x^2 - 27$

$f''(y) = 6x$



$f''(y) > 0$

$6x > 0$

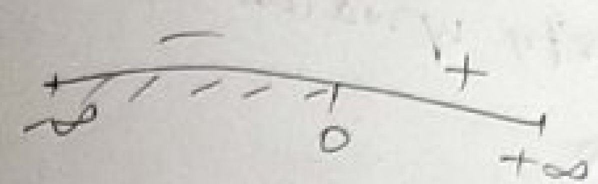
$\therefore x = 0$

$\therefore x \in (0, +\infty)$

$f''(y) < 0$ for concavity downwards.

$f''(y) = 6x < 0$

$\therefore x = 0$



$\therefore x \in (0, -\infty)$

(iv) $y = 69 - 24x - 9x^2 + 2x^3$

$f''(y) > 0$ for concavity upwards.

$f'(y) = -24 - 18x + 6x^2$

$f''(y) = 12x - 18$

$f''(y) > 0$

$12x - 18 > 0$

$12x > 18$

$x = 18/12 = 3/2$

$\therefore x \in (3/2, +\infty)$

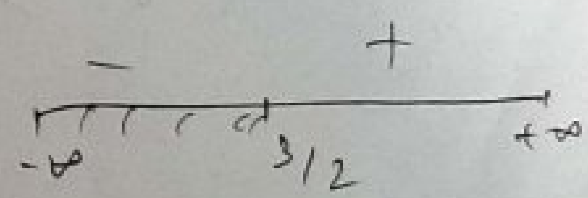
$f''(y) < 0$ for concavity downwards.

$f''(y) < 0$

$12x - 18 < 0$

$12x < 18$

$x = 3/2$



$\therefore x \in (3/2, -\infty)$

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(v) $y = 2x^3 + x^2 - 20x + 4$

$f''(y) > 0$ for concavity upwards.

$f'(y) = 6x^2 + 2x - 20$

$f''(y) = 12x + 2$

$\therefore f''(y) > 0$

$\therefore 12x + 2 > 0$

$12x = -2$

$x = -2/12 = -1/6$

$\therefore x \in (-1/6, +\infty)$

$f''(y) < 0$ for concavity downwards.

$\therefore f''(y) < 0$

$\therefore 12x + 2 < 0$

$\therefore 12x = -2$

$\therefore x = -1/6$

$\therefore x \in (-1/6, -\infty)$.

18/12/19

Topic: APPLICATION OF DERIVATIVES & NEWTON'S METHOD.

Q1. Find the maximum and minimum values of the following functions.

(i) $f(x) = x^2 + \frac{16}{x^2}$.

(ii) $f(x) = 3 - 5x^3 + 3x^5$.

(iii) $f(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Q2. Find the root of the following equations by Newton's Method. (Take Iteration only) correct upto 4 decimal.

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (Take $x_0 = 0$)

(ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$.

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

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Solutions:

Q1

(i) $f(x) = x^2 + \frac{16}{x^2}$

Let's take $f'(x)=0$ to find critical points.

$f'(x)=0$

$2x + \left(-\frac{32}{x^3}\right) = 0$

$2x - \frac{32}{x^3} = 0$

$x = \frac{16}{x^3}$

$x^4 = 16$

$x^2 = 4$

$\therefore \boxed{x = \pm 2}$

$f''(x) > 0$ for minima/minimum value.

$f''(2) > 0$

$2 + \frac{64}{x^4} > 0$

$2 \times \frac{64}{(2)^4} > 0$

$2 \times \frac{64}{64} > 0$

$\therefore x > 0$

$\therefore f''(+2) > 0$ has minima at $x = 2$.

$\therefore f(2) = 4 + \frac{16}{4}$

$= 4+4$

$\therefore \boxed{f(2) = 8}$

$f''(-2) < 0$

$f''(-2) = (-2)^2 \mp \frac{32}{(-2)^2}$

$= 4 \mp \frac{32}{16}$

$\therefore \boxed{f''(-2) = -4}$ is the maxima/maximum value for $x=-2$.

(ii) $f(x) = 3 - 5x^3 + 3x^5$

Let $f'(x) = 0$ to find critical points.

$f'(x) = 0$

$-15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2(x^2-1) = 0$

$x^2 = 0$, $x = \pm 1$

$f''(0) > 0$

$-30x + 60x^3 > 0$

$-30x(1-2x^2) > 0$

$x < 0$, $1-2x^2 > 0$

$2x^2 > 1$

$x^2 > 1/2$

$\therefore x = \pm 1/\sqrt{2}$

$f''(x) > 0$

$\Rightarrow -30x + 60x^3 > 0$

$= -30(1) + 60(1)^3 > 0$

$= -30 + 60 > 0$

$f''(1) = 30 > 0$

$\therefore f$ is minimum at $f''(1)$ with value

$\therefore \boxed{f''(1) = 30}$

$f''(x) < 0$

$\Rightarrow -30x + 60x^3 < 0$

$= -30(-1) + 60(-1)^3 < 0$

$= 30 - 60 < 0$

$f''(-1) = -30 < 0$

$\therefore f$ is maximum at $f''(-1)$ with value

$\boxed{f''(-1) = -30}$.

(iii) $f(x) = x^3 - 3x^2 + 1$ in $[-1/2, 4]$.

Let $f(x) = 0$ ~~to~~ find critical point

$f'(x) = 0$

[illegible]

[illegible]

$\therefore 3x^2 - 6x = 0$

$\therefore 3x(x-2) = 0$

$3x = 0$, $\boxed{x = 0}$ | $x - 2 = 0$, $\boxed{x = +2}$

$f''(x) > 0$ for minima –

$\Rightarrow 6x - 6 > 0$

$\Rightarrow 6(x-1) > 0$

$f''(2) \Rightarrow (x-1)6 > 0$

$= 6(2-1) > 0$

$= 6 > 0$

$f''(2) = 6$ is minimum at $f''(2)$.

$f''(x) < 0$ for maxima –

$\Rightarrow 6x - 6 < 0$

$\Rightarrow 6(x-1) < 0$

$f''(0) \Rightarrow 6(0-1) < 0$

$6(-1) < 0$

$-6 < 0$

$f''(2) = -6$ is maximum at $f''(2)$.

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ at $[-2, 3]$

→ Let $f'(x) = 0$ to find critical pts.

$f'(x) = 0$

$6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - 2x + x - 2 = 0$

$x(x-2) + 1(x-2) = 0$

$x + 1 = 0$, $\boxed{x = -1}$ | $x - 2 = 0$, $\boxed{x = 2}$

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$f''(x) < 0$ for maxima —

$f''(x) = 12x - 6 < 0$

$= 6(2x - 1) < 0$

$f''(-1) = 6(-2-1) < 0$

$= 6(-3) < 0$

$f''(-1) = -18 < 0$

$f''(-1) = -18$ is maximum value.

$f''(x) > 0$ for minima —

$f''(x) = 12x - 6 > 0$

$= 6(2x-1) > 0$

$f''(2) = 6(4-1) > 0$

$= 6 \times 3 > 0$

$f''(2) = 18 > 0$

$f''(2) = 18$ is minimum value.

Q2.

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

→ $x_0 = 0$

$f'(x) = 3x^2 - 6x - 55$

$f'(x_0) = -55$

$f(x_0) = 9.5$

$x_1 = x_0 - \frac{f'(x_0)}{f(x_0)}$

$= 0 - \frac{9.5}{-55}$

$= +\frac{9.5}{55}$

$= 0.1727$

$f(x_1) = 0.0051 + 0.0894 - 9.4985 + 9.5000$

$= 0.0960$

$f'(x_1) = 0.0894 - 55 + 1.0362$

$= -53.8744$

$x_2 = x_1 - \frac{f'(x_1)}{f(x_1)} = 0.1727 - \frac{0.0960}{-53.8744}$

$= 0.1727 + 0.0017$

$= 0.1744$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{-5.5}{9.5} = \underline{5.7894 = x_1}$

$f(x_1) = 194.0442 - 100.5514 - 318.4170 + 9.5$

$= -215.4242$

$f'(x_1) = 100.5514 + 11.7894 - 55$

$= 57.3408$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5.7894 - \frac{-215.4242}{57.3408}$

$\underline{x_2 = 9.5463}$

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$x_2 = 9.5463$

$$f(x_2) = 869.9719 - 273.3955 - 525.0465 + 9.5$$
$$= 81.0299$$

$$f'(x_2) = 273.3955 + 57.2778 + 9.5$$
$$= 340.1733$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 9.5463 - \frac{81.0299}{340.1733}$$

$$x_3 = 9.3080$$

$$f(x_3) = 806.4345 - 259.9165 - 511.9400 + 9.5$$
$$= -544.911$$

$$f'(x_3) = 259.9165 - 55.8480 - 55$$
$$= 260.7645$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 9.3080 - \frac{-544.911}{260.7645}$$

$$x_4 = 11.3976$$

(ii) $f(x) = x^3 - 4x - 9$ in $[2,3]$

$$f(2) = 8 - 8 - 9$$
$$= -9$$
$$f(3) = 27 - 12 - 9$$
$$= 15 - 9$$
$$= 6$$

$x_0 = 3$ → initial approximation.

By Newton's method –

$f(x_0) = f(3) = 6$ $\quad f'(x) = 3x^2 - 4$

$f'(x_0) = 102$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{6}{102} = 3 - 0.0588 = \boxed{2.9411 = x_1}$$

$$f(x_1) = 25.4407 - 11.7644 - 9$$
$$= 4.6763$$

$$f'(x_2) = 25.9502 - 4$$
$$= 21.9502$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_2)} = 2.9411 - \frac{4.6763}{21.9502}$$

$$\boxed{x_2 = 2.7280}$$

$f(x_2) = 20.3017 - 9 - 10.9120$
$= 0.3897$
$f'(x_2) = 22.3259 - 4$
$= 18.3259$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7280 - \frac{0.3897}{18.3259}$

$\boxed{x_3 = 2.7067}$

$f(x_3) = 19.8298 - 9 - 10.8268 = 0.0130$
$f'(x_3) = 21.9786 - 4 = 17.9786$

$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.7067 - \frac{0.0130}{17.9786}$

$\boxed{x_4 = 2.7060}$

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1,2]$
$f(1) = 1 - 1.8 - 10 + 17$
$= 17 - 11 - 1.8$
$= 4.2$
$f(2) = 8 - 7.2 + 20 + 17$
$= 37.8$
$x_0 = 1 \longrightarrow$ Initial approximation.

By Newton's Method.

$f(x_0) = 4.2$ $\qquad f'(x) = 3x^2 - 3.6x - 10$
$f'(x_0) = -10.6$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \frac{4.2}{-10.6}$

$\boxed{x_1 = 1.3962}$

$f(x_1) = 2.7217 - 3.5088 - 13.9620 + 17$
$= 2.2509$
$f'(x_1) = 5.8481 - 10 - 5.0263$
$= -9.1782$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.3962 - \frac{2.2509}{-9.1782}$

$\therefore \boxed{x_2 = 1.6414}$

$f(x_2) = 4.422 - 4.8495 - 16.4140 + 17$
$= 0.1585$
$f'(x_2) = 8.0825 - 5.9090 - 10$
$= -7.8265$

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$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.6414 - \frac{0.1585}{-7.8265}$$

$$\boxed{x_3 = 1.6616}$$

$$f(x_3) = 4.5875 - 4.9696 + 17 - 16.6160$$
$$= 0.0019$$

$$f'(x_3) = 8.2827 - 5.9817 - 10$$
$$= -7.6990$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$
$$= 1.6616 - \frac{0.0019}{-7.6990}$$

$$\therefore \boxed{x_4 = 1.6618}$$

Topic: Integration

Q. Solve the following:

(i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$ (ii) $\int (4e^{3x}+1)\,dx$ (iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

(iv) $\int \sqrt{x}\,(x^2-1)\,dx$ (v) $\int t^7 \sin(2t^4)\,dt$

(vi) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$ (vii) $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

(viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx$ (ix) $\int e^{\cos^2 x} \sin 2x\,dx$

(x) $\int \left(\frac{x^2-2x}{x^3-3x^2+1}\right)dx$

Solutions:

Q1.

(i) $\int \frac{dx}{\sqrt{x^2+2x-3}} = \int \frac{1}{\sqrt{x^2-2x-1-4}}\,dx$

$= \int \frac{1}{\sqrt{(x+1)^2-(2)^2}} = I = \ln\left|(x+1)\sqrt{x^2+2x-3}\right| + C$

(2) $I = \int (4e^{3x} + 1)\, dx$

$= 4\int e^{3x} dx + \int 1\, dx$

$= \frac{4}{3} e^{3x} + x + c$ //.

(3) $I = \int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

$= 2\int x^2 - 3\int \sin x + 5\int \sqrt{x}$

$= \frac{2x^3}{3} + 3\cos x + \frac{10x^{3/2}}{3} + c$ //.

(4) $I = \int \frac{x^3 + 3x + 4}{\sqrt{x}}\, dx$

$\Rightarrow$ Put $\sqrt{x} \to t$

$\frac{1}{2\sqrt{x}} = \frac{dt}{dx}$

$\frac{dx}{\sqrt{x}} = 2dt$.

$I = \int \frac{(\sqrt{x})^2 + 3(\sqrt{x})^2 + 4}{\sqrt{x}}\, dx$

$= 2\int t^6 + 3t^2 + 4\, dt$

$= 2\int (t^6 + 3t^2 + 4)\, dt$

$I = 2\left(\frac{t^7}{7} + \frac{3t^3}{3} + 4t\right) + c$ //.

(5) $I = \int t^7 \sin(2t^7)\, dt$

$I = \int t^4 \sin(2t^4) \times t^3 \cdot dt$

Put $t^4 = x$

$\therefore 4t^3 = \frac{dx}{dt}$

$\therefore t^3 dt = 1/4\, dx$

$I = 1/4 \int x \cdot \sin(2x)\, dx$.

$I = 1/4 \left(x\sin 2x\, dx - \int \left(\frac{dx}{dt}\right) \int \sin 2x\, dx\right)$

$= 1/4 \left[\frac{-x\cos 2x}{2} + 1/2 \int \cos 2x\, dx\right]$

$= 1/16 \sin 2x - \frac{x\cos 2x}{8} + c$

$\therefore I = 1/16 \sin(2t^4) - \frac{t^4 \cos(2t^4)}{8} + c$ //.

(6) $I = \int \sqrt{x}\,(x^2 - 1)\, dx$

$I = \int x^2 \cdot \sqrt{x}\, dx - \int \sqrt{x}\, dx$

$= \int x^{5/2} dx - \int x^{1/2} dx$

$\therefore I = \frac{2}{7} x^{7/2} - \frac{2}{3} x^{3/2} + c$ //

(7) $\int \frac{1}{x^3} \cdot \sin\left(1/x^2\right) dx$

Put $1/x^2 = t$

$-2/x^3 = \frac{dt}{dx}$

$I = \frac{\cos t}{2} + c$

$I = \frac{\cos(1/x^2)}{2} + c$ //.

(8) $I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}} \cdot dx$

Put $\sin x = t$

$\therefore \cos x \cdot dx = dt$.

$I = \int \frac{1}{t^{2/3}} dt$

$I = \int t^{-2/3} \cdot dt$

$I = \frac{t^{-2/3+1}}{-2/3+1} + c$

$I = 3t^{1/3} + c$

$\therefore I = 3\sqrt[3]{\sin x} + c$ //.

(9) $I = \int e^{\cos^2 x} * \sin 2x \ . dx$

Put $\cos^2 x = t$

$\therefore -2\cos x \cdot \sin x = \frac{dt}{dx}$

$\therefore \sin 2x \cdot dx = -dt$

$I = -\int e^t \cdot dt$

$I = -e^t + c$

$\therefore I = -e^{\cos^2 x} + c$ //.

(10) $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1}\right) dx$

$x^3 - 3x^2 + 1 = t$

$\therefore 3x^2 - 6x = \frac{dt}{dx}$

$\therefore (x^2 - 2x)dx = \frac{dt}{3}$

$\therefore I = 1/3 \int \left(\frac{1}{t}\right) dt$

$= 1/3 \log |t| + c$

$\therefore I = 1/3 \log |x^3 - 3x^2 + 1| + c$ //.

PRACTICAL - 06

Topic: APPLICATION OF INTEGRATION & NUMERICAL INTEGRATION.

Q1. Find length of following curve -

(1) $x = t\sin t,\ y = 1\cdot\cos t,\ t \in [0, 2\pi]$

(2) $y = \sqrt{4-x^2},\ x \in [-2, 2]$

(3) $y = x^{3/2}$ in $[0, 4]$

(4) $x = 3\sin t,\ y = 3\cos t,\ t \in [0, 2\pi]$

(5) $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y \in [1, 2]$

Q2. Using Simpson's Rule, solve following -

(1) $\int_0^2 e^{x^2}\,dx$ with $n = 4$.

(2) $\int_0^4 x^2\,dx$ with $n = 4$.

(3) $\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n = 6$.

solution

Q1. arc length $= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\,dx$

$= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt$

$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt$

$= \int_0^{2\pi} \sqrt{2}\,|\sin t/2|\,dt \quad \left(\because \sin^2 t/2 = \frac{1-\cos t}{2}\right)$

$= \int_0^{2\pi} 2\sin t/2\,dt \quad (\because \sin t/2 > 0,$ where $0 < t \le 2\pi)$

$= \left(-4\cos(t/2)\right)_0^{2\pi}$

$= (-4\cos\pi) - (-4\cos 0)$

$= 4 + 4 = 8$

(2) $y = \sqrt{4-x^2}$

$\therefore \frac{dy}{dx} = \frac{1}{\cancel{2}\sqrt{4-x^2}} \times (-\cancel{2}x)$

$L = \int_{-2}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$

$= \int_{-2}^{2} \sqrt{1 + \frac{x^2}{4-x^2}}\,dx$

$$= \int_{-2}^{2} \sqrt{\frac{4}{4-x^2}}\, dx$$

$$= 2\left[\sin^{-1}(x/2)\right]_{-2}^{2}$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 2\left[\pi/2 - (-\pi/2)\right]$$

$$\boxed{L = 2\pi \text{ units}}$$

(3) $y = x^{3/2}$, $x \in [0,4]$

$$\frac{dy}{dx} = \cancel{3}/2\ t^{3/2}$$

$$\therefore L = \int_0^4 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{1+\frac{9}{4}}\, dx$$

$$= 1/2 \int_0^4 \sqrt{4-9x}\, dx$$

$$= 1/2 \left[\frac{(4+9x)^{3/2}}{3/2}\cdot\frac{1}{9}\right]_0^4$$

$$= 1/27 \left[(4+9x)^{3/2}\right]_0^4$$

$$L = -1/27 \left[(4+0)^{3/2} - (4+36)^{3/2}\right] \text{ units}$$

(4) $\frac{dx}{dt} = 3\cos t$, $\frac{dy}{dt} = -3\sin t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= 3\int_0^{2\pi} \sqrt{1}\, dt$$

$$= 3 \left[t\right]_0^{2\pi}$$

$$= 3(2\pi - 0)$$

$$\boxed{L = 6\pi \text{ units}}$$

(5) $a = \frac{1}{6} y^3 + \frac{1}{2y}$

$\therefore \frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^4-1}{2y^2}$

$$L = \int_1^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2} \cdot dy.$$

$$= \int_1^2 \sqrt{1+\frac{(y^4-1)^2}{4y^2}} \cdot dy$$

$$= \int_1^2 \sqrt{\frac{(y^4-1)^2 + 4y}{4y}} \cdot dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}} \cdot dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2} \cdot dy$$

$$= \frac{1}{2} \int_1^2 y^2 dy + \frac{1}{2} \int_1^2 y^2 dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3} - \frac{y^{-2}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{17}{6}\right]$$

$$\therefore \boxed{L = \frac{17}{12} \text{ units}}$$

Q2.

(1) $\int_0^2 e^{x^2} dx$, $n = 4$

$h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

x	0	0.5	1	1.5	2
y	1	1.284	2.7183	9.4877	54.5982

$$\int_0^2 e^{x^2} dx = \frac{h}{3}\left[(Y_0+Y_2) + 4(Y_1+Y_3) + 2(Y_2)\right]$$

$$= \frac{0.5}{3}\left[(1+54.5982) + 4(1.284 + 9.4877) + 2(2.7183)\right]$$

$$= \frac{0.5}{3}\left[55.5982 + 43.0868 + 5.4366\right]$$

$$\therefore \boxed{\int_0^2 e^{x^2} dx = 17.3535}$$

(2) $\int_0^4 x^2\,dx,\ n=4$

$L = \frac{b-a}{n} = \frac{4-0}{4} = 1$

x	0	1	2	3	4
y	0	1	4	9	16

$\int_0^4 x^2\,dx = \frac{L}{3}\left[(Y_0+Y_4)+4(Y_1+Y_3)+2Y_2\right]$

$= 1/3\left[(0+16)+4(1+9)+2\times4\right]$

$= 1/3\left[16+40+8\right]$

$= 64/3$

$\therefore \boxed{\int_0^4 x^2\,dx = 21.333}$

(iii) $\int_0^{2\pi/3} \sqrt{\sin x}\,dx,\ n=6$

$L = \frac{2\pi/3-0}{6} = \frac{\pi}{18}$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$	$6\pi/18$
y	0	0.4167	0.584	0.707	0.801	0.8750	0.93

$\int_0^{\pi/3} \sqrt{\sin x}\,dx = \frac{L}{3}\left[(Y_0)+4(Y_1+Y_3+Y_5)+2(Y_2+Y_4)+Y_2\right]$

$= \frac{\pi}{8}\left(0+4(0.4167+0.707+0.875)+2(0.584+0.801)+0.584\right)$

$\therefore \boxed{\int_0^{\pi/3} \sqrt{\sin x}\,dx \approx 0.681}$

PRACTICAL-07

Topic: DIFFERENTIAL EQUATION:-

Q1 Solve the following differential equation:-

(1) $x \cdot \frac{dy}{dx} + y = e^x$

$x \cdot \frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{e^x}{x}$

$P(x) = 1/x, \; Q(x) = \frac{e^x}{x}$

$IF = e^{\int P dx} = e^{\int 1/x \, dx} = e^{\ln x}$

$IF = x$

$y(IF) = \int Q(x)(IF) dx + c$

$= \int \frac{e^x}{x} \cdot x \cdot dx + c$

$= \int e^x \cdot dx + c$

$\therefore \boxed{xy = e^x + c}$

(ii) $e^x \frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + \frac{2e^x y}{e^x} = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = 1/e^x$

$\frac{dy}{dx} + 2y = e^{-x}$

$P(x) = 2, \; Q(x) = e^{-x}$

$IF = e^{\int P dx} = e^{\int 2 dx} = e^{2x}$

$y(IF) = \int Q(x)(IF) dx + c$

$y \cdot e^{2x} = \int e^{-x} \cdot e^{2x} dx + c$

$= \int e^{-x+2x} \cdot dx + c$

$= \int e^x \cdot dx + c$

$\therefore \boxed{y \cdot e^{2x} = e^x + c}$

(iii) Sol^n:- $x \cdot \frac{dy}{dx} = \frac{\cos x}{x^2} - 2y$

$x \cdot \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$

$P(x) = 2/x$; $Q(x) = \frac{\cos x}{x^2}$

$IF = e^{\int P dx} = e^{\int 2/x \, dx}$

$= e^{2 \ln x}$

$IF = x^2$

$y(IF) = \int Q(x)(IF)\, dx + c$

$= \int \frac{\cos x}{x^2} \cdot x^2 \cdot dx + c$

$= \int \cos x + c$

$\therefore \boxed{x^2 y = \sin x + c}$

(iv) $x \cdot \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

Sol^n:- $\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$

$P(x) = 3/x$, $Q(x) = \sin x / x^3$

$IF = e^{\int P dx}$

$= e^{\int 3/x \, dx}$

$= e^{3 \ln x}$

$= e^{\ln x^3}$

$\therefore IF = x^3$

$y(IF) = \int Q(x)(IF)\, dx + c$

$= \int \frac{\sin x}{x^3} \cdot x^3 \cdot dx + c$

$= \int \sin x \cdot dx + c$

$\therefore \boxed{y x^3 = -\cos x + c}$

(5) $e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

$P(x) = 2,\ Q(x) = 2x/e^{2x} = 2xe^{-2x}$

$IF = e^{\int P(x)dx}$

$= e^{\int 2dx}$

$IF = e^{2x}$

$y(IF) = \int Q(x)(IF)dx + c$

$= \int 2x \cdot e^{-2x} \cdot e^{2x} dx + c$

$= \int 2x \cdot dx + c$

$\therefore \boxed{y \cdot e^{2x} = x^2 + c}$

(6) $\sec^2 x \cdot \tan y \cdot dx + \sec^2 y \cdot \tan x \cdot dy = 0$

$\sec^2 x \cdot \tan y \cdot dx = -\sec^2 y \cdot \tan x \cdot dy$

$\frac{\sec^2 x\, dx}{\tan x} = \frac{-\sec^2 y}{\tan y} \cdot dy$

$\therefore \log|\tan x| = -\log|\tan y| + c$

$\log|\tan x \cdot \tan y| = c$

$\therefore \boxed{\tan x \cdot \tan y = e^c}$

(7) $\frac{dy}{dx} = \sin^2(x-y+1)$

Put $x-y+1 = v$

Differentiate b.t.s.,

$x - y + 1 = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\therefore 1 - \frac{dv}{dx} = \frac{dy}{dx}$

$1 - \frac{dv}{dx} = \sin^2 v$

$\frac{dv}{dx} = 1 - \sin^2 v$

$\frac{dv}{\cos^2 v} = dx$

$\int \sec^2 v \cdot dv = \int dx$

$\tan x = x + c$

$\therefore \boxed{\tan(x+y-1) = x + c}$

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(8) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

Put $(2x+3y) = v$

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\therefore \frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\frac{(v-1)}{(v+2)}$

$\therefore \frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$\therefore dv = v-1+2$

$\therefore \frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$

$= \frac{3v+3}{v+2}$

$= 3\left(\frac{v+1}{v+2}\right)$

$= \int\left(\frac{v+2}{v+1}\right)dv = 3\int dx$

$\therefore \int \frac{v+2}{v+1}\,dv + \int\frac{1}{v+1} = 3x+c$

$v + \log|v+1| = 3x+c$

$\therefore 2x+3y+\log|2x+3y+1| = 3x+c$

$\therefore \boxed{3y = x - \log|2x+3y+1| + c.}$

08/07/2020

Topic : EULER'S METHOD.

* Using Euler's Method, find the following :

(1) $\frac{dy}{dx} = Y + e^{x} - 2$; $Y(0) = 2$, $h = 0.5$. find $Y(2)$.

(2) $\frac{dy}{dx} = 1 + y^2$, $Y(0) = 0$, $h = 0.2$. find $Y(1)$.

(3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $Y(0) = 1$, $h = 0.2$, find $Y(1)$.

(4) $\frac{dy}{dx} = 3x^2 + 1$, $Y(1) = 2$. find $Y(2)$.

For $h = 0.5$ & $h = 0.25$.

(5) $\frac{dy}{dx} = \sqrt{xy} + 2$, $Y(1) = 1$, find $Y(1.2)$ with $h = 0.2$.

Solutions :

(1) $\frac{dy}{dx} = Y + e^{x} - 2$, $Y(0) = 2$, $h = 0.5$,
$x_0 = 0$, $y_0 = 2$.

Sol^n:
$f(x) = y + e^{x} - 2$, $x_0 = 0$, $y_0 = 2$,
$h = 0.5$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$\therefore$ $y(2) = 98215$ is the value of $Y(2)$ according to Euler's method.

Formula: $Y_{n+1} = Y_n + h \cdot f(x_n, y_n)$.

(2) $\frac{dy}{dx} = 1 + y^2$

$Y(0) = 0$

$h = 0.2$

$x_0 = 0,\ y_0 = 0$

x_0	y_0	$f(x_0, y_0)$	$h \cdot f(x_0, y_0)$	Y_{n+1}
0	0	1	0.2	0.2
0.2	0.2	1.04	0.208	0.408
0.4	0.408	1.166	0.233	0.641
0.6	0.641	1.410	0.2821	0.9231
0.8	0.9231	1.852	0.3704	1.2935
1.0	$\boxed{1.2935}$			

$\therefore Y(1)$ according to Euler's method is 1.2935.

(3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$

$Y(0) = 1$

$h = 0.2$

$x_0 = 0,\ y_0 = 1$

x_0	y_0	$f(x_0, y_0)$	$h \cdot f(x_0, y_0)$	Y_{n+1}
0	1	0	0	1
0.2	1	0.4472	0.0894	1.0894
0.4	1.0894	0.6059	0.1211	1.2105
0.6	1.2105	0.7040	0.1408	1.3513
0.8	1.3513	0.7694	0.1538	1.5051
1.0	$\boxed{1.5051}$			

$\therefore Y(1)$ according to Euler's method is 1.5051.

(4) $\frac{dy}{dx} = 3x^2 + 1$

$Y(1) = 2$

(i) $h = 0.25$

$x_0 = 1,\ y_0 = 2$.

x_0	y_0	$f(x_0, y_0)$	$h \cdot f(x_0, y_0)$	Y_{n+1}
1	2	4	1	3
1.25	3	5.6875	1.4218	4.4218
1.5	4.218	7.75	1.9375	6.1555
1.75	1.555	10.1875	2.5468	8.7023
2.0	$\boxed{8.7023}$			

$\therefore Y(2)$ according to Euler's method is 8.7023.

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(ii) $h = 0.5$
$x_0 = 1$, $y_0 = 2$.

x_0	y_0	$f(x_0, y_0)$	$h \cdot f(x_0, y_0)$	Y_{n+1}
1	2	4	2	4
1.5	4	7.75	3.875	7.875
2.0	$\boxed{7.875}$			

$\therefore Y(2)$ according to Euler's method is
7.875.

(5) $\frac{dy}{dx} = \sqrt{xy} + 2$;
$Y(1) = 1$
$h = 0.2$
$x_0 = 1$, $y_0 = 1$.

x_0	y_0	$f(x_0, y_0)$	$h \cdot f(x_0, y_0)$	Y_{n+1}
1	1	3.4142	0.6828	1.6828
1.2	$\boxed{1.6828}$			

$\therefore Y(1.2)$ according to Euler's method is
1.6828.

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PRACTICAL-09

TOPIC: LIMITS & PARTIAL ORDER DERIVATIVE.

Q1. Evaluate following limits :

(i) $\lim_{(x,y)\to(-4,-1)} \frac{x^3-3y+y^2-1}{xy-5}$ (ii) $\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+}$

(iii) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz}$.

Q2. Find f_x, f_y for each of the following f.

(i) $f(x,y) = xy\,e^{x^2+y^2}$ (ii) $f(x,y) = e^x \cos y$.

(iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

Q3. Using definition find values of f_x, f_y at $(0,0)$ for

$f(x,y) = \frac{2x}{1+y^2}$

Q4. Find all second order derivative partial of f.
Also, verify whether $f_{xy} = f_{yx}$.

(i) $f(x,y) = \frac{y^2 - xy}{x^2}$ ~~(ii) $f(x,y) = 1-x+y\sin x$ at $(\pi/2, 0)$~~

(ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$ (iii) $f(x,y) = \sin(xy) + e^{x+y}$

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Q5. Find the linearization of $f(x,y)$ at given point.

(i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$.

(ii) $f(x,y) = 1 - x + y\sin(x)$ at $(\pi/2, 0)$.

(iii) $f(x,y) = \log(x) + \log(y)$ at $(1,1)$.

SOLUTION:

Q1.

(i) $$\lim_{(x,y)\to(-4,-1)} \frac{x^3-3x+y^2-1}{xy+5}$$

$$\cancel{\lim_{(x,y)\to(-4,-1)} x^3-3x+y^2-1}$$

Apply limit;

$$= \frac{(-4)^3 - 3(-4) + (-1)^2 - 1}{(-4)(-1)+5} = \frac{-64+12+1-1}{4+5} = \frac{-52}{9}$$

(ii) $$\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$$

Apply limits;

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$$= \frac{(0+1)\left((2)^2+(0)^2-4(2)\right)}{2+3(0)} = \frac{1(4+0-8)}{2}$$

$$= \frac{4-8}{2} = -\frac{4}{2} = -2$$

(iii) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2y^2}$

Apply limit;

$$\Rightarrow \frac{(1)^2-(1)^2(1)^2}{(1)^3-(1)^2(1)(1)} = \frac{1-1}{1-1} = 0$$

$\therefore$ Limit does not exist.

Q2.

(i) $f(x,y) = xye^{x^2+y^2}$

$$f_x = y\,(e^{x+y^2}) + xy\,(e^{x^2+y^2}.2x)$$
$$= y \cdot e^{x^2+y^2} + 2x^2y \cdot e^{x^2+y^2}$$
$$f_y = x\,(1.e^{x^2+y^2}) + xy\,(e^{x^2+y^2}\cdot 2y)$$
$$= x \cdot e^{x^2+y^2} + 2xy^2 \cdot e^{x^2+y^2}$$
$$f_x = y \cdot e^{x^2+y^2} + 2x^2y \cdot e^{x^2+y^2}$$
$$f_y = x \cdot e^{x^2+y^2} + 2xy^2 \cdot e^{x^2+y^2}.$$

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(ii) $f(x,y) = e^x \cos y$

$f_y = \cos y \cdot e^x$

$f_y = e^x - \sin y$

$\therefore f_y = -\sin y \, e^x$

(iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = y^2 3x^2 - 3y^2x + 6 + 0$

$= 3x^2y^2 - 6xy$

$f_y = x^3 2y - 3x^2 + 3y^2$

$= 2x^3y - 3x^2 + 3y^2$

Q3.

(i) $f(x,y) = \dfrac{2x}{1+y^2}$

$f(x)[a,b] = \lim\limits_{h\to 0} \dfrac{f(a+h,b) - f(a,b)}{h}$

$f(y)[a,b] = \lim\limits_{h\to 0} \dfrac{f(a,b+h) - f(a,b)}{h}$

$\cancel{f(y)[a,b] = \lim\limits_{h\to 0} f(a,b+h)}$

According to given $(a,b) = (0,0)$

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$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h-0}{h} = 2$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$f_x = 2,\ f_y = 0$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{2h-0}{h} = 2$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$f_x = 2,\ f_y = 0.$

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Q4.

(i) $f(x,y) = \dfrac{y^2 - xy}{x^2}$

$f_{xx} = \dfrac{d^2 f}{dx^2}$ $\quad f_{yy} = \dfrac{d^2 f}{dy^2}$

Applying $\dfrac{u}{v}$ rule.

$$f_x = \frac{x^2(0-y) - (y^2 - xy)2x}{x^4}$$

$$= \frac{-x^2 y - 2xy^2 + 2x^2 y}{x^4}$$

$$\therefore f_x = \frac{x^2 y - 2xy^2}{x^4}$$

$$f_{xx} = \frac{x^4(2xy - 2y^2) - (x^2 y - 2xy^2)(4x^3)}{x^8}$$

$$= \frac{2x^5 y - 2x^4 y^2 - (4x^5 y - 8x^4 y^2)}{x^8}$$

$$= \frac{2x^5 y - 2x^4 y^2 - 4x^5 y + 8x^4 y^2}{x^8}$$

$$= \frac{+6x^4 y^2 - 2x^5 y}{x^8}$$

$$f_{xx} = \frac{6y^2 - 2xy}{x^4}$$

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$f_y = \frac{1}{x^2} [2y - x]$ $\quad \therefore f_y = \frac{2x}{x^2}$

$f_{yy} = \frac{1}{x^2} \cdot 2 = \frac{2}{x^2}$

$f_{xy} = \frac{2y-x}{x^2}$

$= \frac{x^2(-1) - (2y-x)(2x)}{x^4}$.

$= \frac{-x^2 - 4xy + 2x^2}{x^4}$

$= \frac{x^2 - 4xy}{x^4}$

$= \frac{x - 4y}{x^3}$

$\therefore f_{xy} = \frac{x-4y}{x^3}$

$\therefore f_{yx} = \frac{x^2 y - 2xy}{x^4}$

$= \frac{x^2 - 4xy}{x^4}$

$= \frac{x - 4y}{x^3}$

$\therefore f_{yx} = f_{xy}$

Hence, verified.

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Q5.

(i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$.

$$f_x = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x \qquad f_y = \frac{2y}{\sqrt{x^2+y^2}}$$

$$= \frac{x}{\sqrt{x^2+y^2}} \qquad \cancel{f_y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$f_x(1,1) = \frac{1}{\sqrt{2}} \qquad f_y(1,1) = \frac{1}{\sqrt{2}}$$

$$L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$= \sqrt{2} + \frac{x-1}{\sqrt{2}} + \frac{y-1}{\sqrt{2}}$$

$$= \sqrt{2} + \frac{x+y-2}{\sqrt{2}}$$

$$= \frac{x+y}{\sqrt{2}}.$$

(ii) $f(x,y) = 1 - x + y\sin x$ at $(\pi/2)$

$$f(\pi/2, 0) = 1 - \pi/2 + 0 + \sin \pi/2$$

$$f(\pi/2, 0) = \frac{2-\pi}{2}$$

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$f_x = -1 + y\cos x$ $\qquad f_y = \sin x$

$f_x(\pi/2, 0) = -1 + 0 \cdot \cos \pi/2$
$= -1$

$f_y(\pi/2, 0) = \sin \pi/2$
$= 1$

$L(x,y) = f(\pi/2, 0) + f_x(\pi/2, 0)(x - \pi/2) + f_y(\pi/2, 0)(y - 0)$

$= \frac{2-\pi}{2} + (-1(x - \pi/2) + 1(y))$

$= 1 - \pi/2 - x + \pi/2 + y$

$L(x,y) = 1 - x + y.$

$f(x,y) = \log x + \log y$

$f(1,1) = \log 1 + \log 1$
$= 0$

$f_x = 1/x \qquad f_y = 1/y$
$f_x(1,1) = 1 \qquad f_y(1,1) = 1$

$\therefore L(x,y) = f[1,1] + f[1,1](x-1) + f_y(1,1)(y-1)$
$= 0 + 1(x-1) + 1(y-1)$

$\therefore L(x,y) = x + y - 2.$

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PRACTICAL-10

Topic: Directional derivative, Gradient vector & maxima, minima, Tangent & normal vectors.

Q1. Find directional derivative of function at given pts. & in direction of given vector:

(i) $f(x,y) = x+2y-3$, $a=(1,-1)$, $u=3i-j$.

(ii) $f(x,y) = y^2-4x+1$, $a=(3,4)$, $u=i+5j$.

(iii) $f(x,y) = 2x+3y$, $a=(1,2)$, $u=3i+4j$.

Q2. Find gradient vector for the following function at given point:-

(i) $f(x,y) = x^y+y^x$, $a=(1,1)$.

(ii) $f(x,y) = (\tan^{-1}x)\cdot y^2$, $a=(1,-1)$.

(iii) $f(x,y,z) = xyz-e^{x+y+z}$, $a=(1,-1,0)$.

Q3. Find eqn of tangent & normal to each of curves at given points:

(1) $x^2\cos y + e^{xy} = 2$ at $(1,0)$.

(2) $x^2+y^2-2x+3y+2=0$ at $(2,-2)$.

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Q4. Find the equation of tangent & normal line to each of the following surfaces —

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(1) $x^2 - 2yz + 3y + xz = 7$ at $(2,1,0)$.

(2) $3xyz - x - y + z = -4$ at $(1,-1,2)$.

Q5. Find the local maxima & minima for the following functions —

(1) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$.

(2) $f(x,y) = 2x^4 + 3x^2y - y^2$.

(3) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$.

Solution :

Q1.

(i) $f(x,y) = x + 2y - 3$, $a = (1,-1)$, $u = 3i - j$.

$$|u| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$$

$\therefore$ unit vector $u = \dfrac{\vec{u}}{|u|} = \cancel{\dfrac{3\hat{i} + \hat{j}}{}}$

$$= \frac{3\hat{i}}{\sqrt{10}} - \frac{1\hat{j}}{\sqrt{10}} = \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$$

$\because a = (1,-1)$

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$f(a) = f(1,-1) = 1 + 2(-1) - 3$

$= 1 - 2 - 3$

$= -4$

$f(a, hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= f\left(1 + \frac{3}{\sqrt{10}}\right) + 2\left(-1, -\frac{h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} \cancel{+2} \cancel{(1} - \frac{2h}{\sqrt{10}} - 2 - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h \to 0} \frac{\cancel{-4} + \cancel{4} + \cancel{h}/\sqrt{10}}{\cancel{h}}$

$D_u f(a) = \frac{1}{\sqrt{10}}$

(ii) $f(x) = y^2 - 4x + 1$, $a = (3,4)$, $u = i + 5j$

$|\vec{u}| = \sqrt{1^2 + 5^2} = \sqrt{26}$

$\vec{u} = \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$f(a) = f(3,4) = (4)^2 - 4(3) + 1 = 5$

$f(a+hu) = f(3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right)$

$f(a+hu)$ for (x,y) $= \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$

$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$

$= \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - \frac{4h}{\sqrt{26}} + 5$

$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$

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$\therefore D_u f(a) = \frac{25h}{26} + \frac{36}{\sqrt{26}}$ //.

(iii) $2x + 3y$, $a = (1,2)$, $u = (3i + 4j)$

$|\vec{u}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\vec{u} = \left(\frac{3}{5}, \frac{4}{5}\right)$

$f(a) = f(1,2) = 2(1) + 3(2) = 8$

$f(a+hu) = f(1,2) + h(3/5, 4/5)$

$= f\left(1 + \frac{3h}{5}, 2 + \frac{4}{5}\right)$

$f(a+hu) = 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$

$= \frac{18h}{5} + 8$

$D_u f(a) = \lim_{h \to 0} \frac{18h/5 + 8 - \cancel{8}}{h}$

$= 18/5$ //.

Q2.

(i) $f(x,y) = x^y + y^x$, $a = (1,1)$

$f_x = y \cdot x^{y-1} + y^x \log y$

$f_y = x^y \log x + xy^{x-1}$

$\nabla f(x,y) = (f_x, f_y)$

$= (y x^{y-1} + y^2 \log y, x^y \log x + xy^{x-1})$

$f(1,1) = (\tan^{-1} x) \cdot y^2$; $a = (1,-1)$

$f_x = \frac{1}{1+x^2} \cdot y^2$

$f_y = 2y \tan^{-1} x$

$\nabla f(x,y) = (f_x, f_y)$

$= \left(\frac{y^2}{1+x^2}, 2y \tan^{-1} x\right)$

$f(1,-1) = \left(\frac{1}{2}, \tan^{-1}(1)(-2)\right)$

$= \left(1/2, -\pi/2\right)$

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(iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = f_x, f_y, f_z$

$= yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z}$

$f(1,-1,0) = ((-1)(0) - e^{1+(-1)+0}, (1)(0) - e^{1+(-1)+0}$

$(1)(-1) - e^{1+(-1)+0}$

$= 0 - e^0, 0 - e^0, -1 - e^0$

$= -1, -1, -2.$

Q3.

(i) $f_x = \cancel{\cos y}\, 2x + e^{xy} y$

$f_y = x^2(-\sin y) + e^{xy}.x$

$(x_0, y_0) = (1,0) = \therefore x_0 = 1, y_0 = 0$.

Eqn. of tangent:

$f_x(x - x_0) + f_y(y - y_0) = 0$

$f_x(x_0, y_0) = \cos 0 (2(1)) + e^0 . 0$

$= 1(2) + 0$

$= 2$

$f_y(x_0, y_0) = (1)^2(-\sin 0) + e^0 . 1$

$= 0 + 1.1$

$= 1$

$2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$\boxed{2x + y - 2 = 0}$

$\therefore 2x + y - 2 = 0$ is the required eqn. of tangent.

Eqn. of Normal $= ax + by + c = 0$

$= bx + ay + d = 0$

$(1) 1 + 2(y) + d = 0$

$= 1 + 2y + d = 0$

$= 1 + 2(0) + d = 0$

$d + 1 = 0$

$\therefore d = -1.$

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(ii) $f_x = 2x + 0 - 2 + 0 + 0$
$= 2x - 2$
$f_y = 0 + 2y - 0 + 3 + 0$
$= 2y + 3$

$(x_0, y_0) = (2, -2) \therefore x_0 = 2, y_0 = -2.$

$f_x(x_0 y_0) = 2(2) - 2 = 2$
$f_y(x_0, y_0) = 2(-2) + 3 = -1$

Eq^n of tangent –

$f_x(x - x_0) + f_y(y - y_0) = 0$
$2(x-2) + (-1(y+2)) = 0$
$2x - 2 - y - 2 = 0$
$\boxed{2x - y - 4 = 0}$

$\therefore$ It is the required eq^n of tangent.

Eq^n of normal:
$= ax + by + c = 0$
$bx + ay + d = 0$
$\Rightarrow -1(x) + 2(y) + d = 0$
$\Rightarrow -x + 2y + d = 0$ at $(2, -2)$
$\Rightarrow -2 + 4 + d = 0$
$\Rightarrow 6 + d = 0$
$\Rightarrow \boxed{d = 6}$

Q 4.

(i) $f_x = 2x - 0 + 0 + z$
$= 2x + z$
$f_y = 0 - 2z + 3 + 0$
$= 2z + 3$
$f_z = 0 - 2y + 0 + x$
$= -2y + x$
$(x_0, y_0, z_0) = (2, 1, 0)$ $\therefore x_0 = 2, y_0 = 1, z_0 = 0.$

Eq^n of tangent –
$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$
$= 4(x-2) + 3(y-1) + 0(z-0) = 0$
$= 4x - 8 + 3y - 3 = 0$
$= \boxed{4x + 3y - 11 = 0}$ is required eq^n of tangent

Eq^n. of normal at $(4, 3, -11)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$= \frac{x-2}{4} = \frac{y-1}{3} = \frac{z+11}{0}$$ //.

(ii) $3xyz - x - y + z = -4 \rightarrow$ at $(1,-1,2)$

$3xyz - x - y + z + 4 = 0 \nearrow$ at $(1,-1,2)$

$f_x = 3yz - 1 - 0 + 0 + 0$

$= 3yz - 1$

$f_y = 3xz - 0 - 1 + 0 + 0$

$= 3xz - 1$

$f_z = 3x\cancel{y} - 0 + 1 + 0 + 0$

$= 3xy + 1$

$(x_0, y_0, z_0) = (1,-1,2) \therefore x_0 = 1, y_0 = -1, z_0 = 2.$

$f_x(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$

$f_y(x_0, y_0, z_0) = 3(1)(2) - 1 = 5$

$f_z(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2.$

Eq^n of tangent —

$-7(x-1) + 5(y+1) - 2(z-2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$\therefore \boxed{-7x + 5y - 2z + 16 = 0}$ is the required eq^n of tangent.

Eq^n of normal at $(-7, 5, -2)$

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$= \frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2} \text{ //.}$$

Q5.

(i) $f_x = 6x - 3y + 6$

$f_y = 2y - 3x - 4$

$f_x = 0 \rightarrow$

$6x - 3y + 6 = 0$

$3(2x - y + 2) = 0$

$2x - y + 2 = 0$

$2x - y = -2$ — ①

$f_y = 0 \rightarrow$

$2y - 3x - 4 = 0$

$2y - 3x = 4$ — ②

X^{ly} eqn 1 with 2,

$4x - \cancel{2y} = -4$

$\cancel{2y} - 3x = 4$

$x = 0.$

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Substitute value of x in eqn. ①,

$2(0) - y = -2$

$-y = -2$

$\therefore y = 2$

$\therefore$ Critical pts. are $(0, 2)$.

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r > 0$,

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9$

$= 3 > 0$

$\therefore f$ has maximum at $(0, 2)$

$3x^2 + y^2 - 3xy + 6x - 4y$ at $(0, 2)$

$3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$

$0 + 4 + 0 - 0 - 8 = -4$

(ii) ~~f(x,y)~~

$f_x = 8x^3 + 6xy$

$f_y = 3x^2 - 2y$

$f_x = 0$

$8x^3 + 6xy = 0$

$2x(4x^2 + 3y) = 0$

$4x^2 + 3y = 0$ — ①

$f_y \rightarrow$

$3x^2 - 2y = 0$ — ②

$\times^{ly}$ eqn. ① with 3,

② with 4.

$12x^2 + 9y = 0$

$-12x^2 - 8y = 0$

$11y = 0$

$\therefore \boxed{y = 0}$

Substitute value of y in eqn ①

$4x^2 + 3(0) = 0$

$4x^2 = 0$

$x = 0$

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Critical pts. are (0,0)

$r = f_{xx} = 24x^2 + 6x$

$t = f_{yy} = 0 - 2 = -2$

$s = f_{xy} = 6x - 0 = 6(0) = 0$

r at (0,0),

$= 24(0) + 6(0) = 0$

$\therefore r = 0.$

$rt - s^2 = 0(-2) - (0)^2$

$= 0 - 0$

$= 0$

$r = 0, \quad rt - s^2 = 0$

{Nothing to say}.

(iii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$

$f_y = -2y + 8$

$f_x = 0 \quad \therefore 2x + 2 = 0$

$x = -2/3 \quad \therefore x = -1.$

$f_y = 0 \quad \therefore -2y + 8 = 0.$

$y = -8/-2$

$\therefore y = 4$

$\therefore$ Critical point is $(-1, 4)$.

$r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$

$rt - s^2 = 2(-2) - (0)^2$

$= -4 - 0$

$= -4 < 0.$

$f(x,y)$ at $(-1, 4)$

$(-1)^2 - (4)^2 + 2(-1) + 8(4) - 70$

$= 1 + 16 - 2 + 32 - 70$

$= 17 + 30 - 70$

$= 37 - 70 = 33$

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