(12) (a) $f(k) = -\frac{6}{3^{k-1}}, k > 0$; (b) $f(k) = 6\left[\frac{1}{2^{k-1}} - \frac{1}{3^{k-1}}\right], k \geq 1$

$= -12 \cdot 2^{-k}, k \leq 0$

(13) $-8 \cdot 2^k - k$

$k < 0 \quad k > 0$

(14) $f(k) = 1 + k \cdot 2^{k+1}, k \geq 0$

(15) $f(k) = 3^{k+2} - 2^{k+3} - k \cdot 2^{k+1}, k \geq 0$

(16) Hint : $\frac{F(z)}{z} = \frac{z}{z-1} - \frac{1}{z-(1/2)}$

(i) $2 - \left(\frac{1}{2}\right)^k, k \geq 0$; (ii) $-2 + \frac{1}{2^k}, k < 0$; (iii) -2 when $k < 0$ and $-\frac{1}{2^k}$ when $k \geq 0$.]

EXERCISE - VII

Theory

1. Define Z-transform.
2. State convolution theorem for Z-transform. (M.U. 2008)

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CHAPTER 4

Probability Distributions

1. Introduction

In this chapter we shall study two standard probability distributions. Study of such theoretical distributions is the foundation for the development of further topics. The first distribution is discrete while the second is continuous.

2. Poisson Distribution

Poisson distribution was discovered by the French Mathematician Poisson in 1837. Poisson distribution is **the limiting case of the binomial distribution** under the following conditions :

(i) n, the number of trials is infinitely large *i.e.* $n \to \infty$.

(ii) p, the probability of success in each trial is constant and infinitely small *i.e.* $p \to 0$.

(iii) np, the average success is finite say m, *i.e.* $np = m$.

Siméon Denis Poisson (1781 - 1840)

A French mathematician geometer and physicist was born in Pithviers, Loiret in France. He was a favourite student of Lagrange and was treated like a son by Laplace. Lagrange and Laplace were his doctoral advisors. In 1806 he became full professor at École Polytechnique, in Paris succeeding Fourier. His notable students were Dirichlet and Liouville. As a teacher of mathematics Poisson is said to have been extremely successful. As a scientific worker, his productivity has rarely, if ever, been equalled. Inspite of his many official duties, he published more than three hundred works, several of them extensive treatises and many of them memoires dealing with the most abtruse branches of pure mathematics, applied mathematics, mathematical physics and rational mechanics. His memoires on the theory of electricity and magnetism created a new branch of mathematical physics. He made important advances in planetary theory. Poisson is known for Poisson distribution, Poisson process, Poisson equation, Poisson Kernel, Poisson regression, Poisson summation formula, Poisson ratio, Euler-Poisson-Darboux equation, Conway-Maxwell-Poisson distribution.

(a) To Derive Poisson Distribution

Consider $p(x) = {}^nC_x p^x q^{n-x}$

$$= {}^nC_x \left(\frac{p}{q}\right)^x q^n = {}^nC_x \left(\frac{p}{1-p}\right)^x (1-p)^n$$

Putting $p = \frac{m}{n}$,

$$p(x) = \frac{n(n-1)(n-2)\ldots\ldots(n-x+1)}{x!} \frac{(m/n)^x}{[1-(m/n)]^x}\left[1-\frac{m}{n}\right]^n$$

$$p(x) = \frac{\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right]\ldots\ldots\left[1-\frac{x-1}{n}\right]m^x\left[1-\frac{m}{n}\right]^n}{x!\,\left[1-\frac{m}{n}\right]^x}$$

Since $\lim_{n\to\infty}\left[1-\frac{m}{n}\right]^n = e^{-m}$ and $\lim_{n\to\infty}\left[1-\frac{m}{n}\right]^x = 1$

Taking the limits of both sides as $n \to \infty$

$$p(x) = \frac{m^x \cdot e^{-m}}{x!}$$

Thus, the limit of the Binomial random variable is the Poisson random variable.

(b) Definition

A random variable X is said to follow **Poisson distribution** if the probability of x is given by

$$\boxed{P(X = x) = \frac{e^{-m}m^x}{x!}, \quad x = 0, 1, 2, \ldots\ldots}$$

and m (> 0) is called the parameter of the distribution.

Remarks ....

1. The sum of the probabilities is 1.

$$\sum_{x=0} P(X = x) = \sum_{x=0}\frac{e^{-m}m^x}{x!} = e^{-m}\sum\frac{m^x}{x!}$$

$$= e^{-m}\left[1 + m + \frac{m^2}{2!} + \ldots\ldots\right] = e^{-m} \cdot e^m = 1.$$

2. Poisson distribution occurs where the probability of occurrence p is very small and the number of trials n is very large and where the probability of occurrence only can be known e.g. the number of accidents, the number of deaths by a disease, the number of printing mistakes on a page etc. In these cases we can only observe the number of successes but the number of failures cannot be observed. We can observe how many accidents occur; we cannot observe how many times accidents do not occur.

(c) When do we get Poisson distribution ?

As seen before we get a Poisson distribution if the following conditions are satisfied.

1. The number of trials n is infinitely large i.e. $n \to \infty$.
2. A trial results in only two ways-success or failure.
3. If p and q are probabilities of success and failure, then $p + q = 1$.

4. These probabilities are mutually exclusive, exhaustive but not necessarily equally likely. (i.e., p is not necessarily equal to 1 / 2.)
5. The probability p of success is very small i.e. $p \to 0$.
6. $n \to \infty$ and $p \to 0$ such that $np = m$, (> 0) a constant.

(d) Uses

Poisson distribution is used in problems involving :

(i) the number of deaths due to a disease such as heart attack, cancer etc.

(ii) the number of accidents during a week or a month etc.

(iii) the number of phone-calls received at a particular telephone exchange during a period of time.

(iv) the number of cars passing a particular point on a road during a period of time.

(v) the number of printing mistakes on a page of a book etc.

Note ....

Since the Poisson distribution is the limiting case of Binomial distribution, we can calculate binomial probabilities approximately by using Possion distribution whenever n is large and p is small. (See Ex. 21, page 4-13 and Ex. 22, page 4-13)

(e) Moments of the Poisson's Distribution (M.U. 2010)

We obtain below the first two moments about the origin.

$$\mu_1' = E(x) = \sum p_i x_i$$

$$= \sum_{x=0}\frac{e^{-m}m^x}{x!}x = \sum_{x=1}\frac{e^{-m}m^x}{(x-1)!} = m\,e^{-m}\sum\frac{m^{x-1}}{(x-1)!}$$

$$= m\,e^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \ldots\ldots\right]$$

$$= m\,e^{-m} \cdot e^m = m$$

Hence, $\boxed{\text{mean} = m}$

$$\mu_2' = E(x^2) = \sum p_i x_i^2 = \sum_{x=0} e^{-m}\cdot\frac{m^x}{x!}\cdot x^2$$

We can write $x^2 = x + x(x-1)$

$$\mu_2' = \sum_{x=0} e^{-m}\cdot\frac{m^x}{x!}[x + x(x-1)]$$

$$= \sum_{x=0} e^{-m}\cdot\frac{m^x \cdot x}{x!} + \sum_{x=0} e^{-m}\cdot\frac{m^x}{x!}x\cdot(x-1)$$

$$= m\,e^{-m}\sum_{x=1}\frac{m^{x-1}}{(x-1)!} + m^2 e^{-m}\sum_{x=2}\frac{m^{x-2}}{(x-2)!}$$

$$= m\,e^{-m}\cdot e^m + m^2 e^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \ldots\ldots\right]$$

$$= m\,e^{-m}\cdot e^m + m^2 e^{-m}\cdot e^m = m + m^2$$

$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = m + m^2 - m^2 = m$

$\therefore \quad \boxed{\textbf{Variance} = m}$

Thus, the mean and variance of the Poisson's distribution are both equal to m.

(f) Moment Generating Function (M.U. 2014)

The m.g.f. about the origin is,

$$M_0(t) = E(e^{tx}) = \Sigma p(x) e^{tx} = \Sigma \frac{e^{-m} \cdot m^x}{x!} \cdot e^{tx}$$

$$= e^{-m} \Sigma \frac{(m e^t)^x}{x!} = e^{-m} \cdot e^{me^t}$$

$$\boxed{M_0(t) = e^{m(e^t - 1)}} \quad \text{.................... (A)}$$

(Note that $\Sigma \frac{k^x}{x!} = 1 + k + \frac{k^2}{2!} + \frac{k^3}{3!} + ... = e^k$.

We shall often meet expressions of this type in the discussion of Poisson's distribution.)

Now $\frac{d}{dt}\{m_0(t)\} = e^{m(e^t-1)} \cdot me^t$

$\therefore \quad \left[\frac{d}{dt} M_0(t)\right]_{t=0} = m \quad \therefore \ \mu_1' = m$

$$\frac{d^2}{dt^2}\{M_0(t)\} = m\left[e^t \cdot e^{m(e^t-1)} \cdot me^t + e^{m(e^t-1)} \cdot e^t\right]$$

$\therefore \quad \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = m[m+1] = m^2 + m \quad \therefore \ \mu_2' = m^2 + m$

$\because \quad \mu_2 = \mu_2' - \mu_1'^2 = m^2 + m - m^2 \quad \therefore \ \mu_2 = m$

We may denote the logarithm of the m.g.f. (A) by $L(t)$. Then

$$L(t) = \log M_0(t) = \log[e^{m(e^t-1)}] \quad \therefore \ L(t) = m(e^t - 1)$$

Differentiating both sides w.r.t. t

$L'(t) = me^t \quad \therefore \ L'(0) = m$

$L''(t) = me^t \quad \therefore \ L''(0) = m$

Hence, mean $\mu = L'(0) = m$ and variance $\mu_2 = L''(0) = m$.

Example : If the moment generating function about the origin of a discrete random variable is X is $e^{4(e^t-1)}$, find $P(X = \mu + \sigma)$ where μ and σ are mean and standard deviation of X. **(M.U. 2007, 09)**

Sol. : We know that m.g.f. of a Poisson distribution with mean m is

$$M_0(t) = e^{m(e^t-1)}$$

Comparing this with the given m.g.f. we see that X is a Poisson variate with mean $m = 4$ and variance $\sigma^2 = m = 4$.

Hence, the Poisson distribution is

$$P(X = x) = \frac{e^{-4}(4)^x}{x!}$$

But $m + \sigma = 4 + 2 = 6$.

$$\therefore \ P(X = 6) = \frac{e^{-4} \cdot 4^6}{6!} = 0.1.$$

(g) Additive property of Independent Poisson distributions

If two independent variates have Poisson distribution with means m_1 and m_2 then their sum also is a Poisson distribution with mean $m_1 + m_2$.

Proof : Let $M_1(t)$ and $M_2(t)$ be the m.g.f. s of the two Poisson variates X_1 and X_2 and let $M(t)$ be the m.g.f. of their sum.

Now $\quad M_1(t) = E(e^{tX_1})$ and $M_2(t) = E(e^{tX_2})$

$\quad M_1(t) = e^{m_1(e^t-1)}$ and $M_2(t) = e^{m_2(e^t-1)}$

Since, X_1 and X_2 are independent, m.g.f. of $X_1 + X_2$ is

$$M(t) = E[e^{t(X_1+X_2)}] = E(e^{tX_1}) \cdot E(e^{tX_2})$$

$$= e^{m_1(e^t-1)} \cdot e^{m_2(e^t-1)} = e^{(m_1+m_2)(e^t-1)}$$

But this is the m.g.f. of the Poisson distribution with mean $m_1 + m_2$. Hence, the result.

Notes ....

1. Although the sum of two Poisson variates is a Poisson variate, the difference between two Poisson variates is not a Poisson variate. If X, Y are two Poisson variates then.

$$M_{(X-Y)}(t) = M_{X+(-Y)}(t) = M_X(t) \cdot M_{(-Y)}(t)$$

$$= M_X(t) \cdot M_Y(-t) \qquad [\because \ M_{cx}(t) = M_x(ct)]$$

$$= e^{m_1(e^t-1)} \cdot e^{m_2(e^{-t}-1)} = e^{m_1(e^t-1)+m_2(e^{-t}-1)}$$

But this cannot be put in the form $e^{m(e^t-1)}$ and hence $(X - Y)$ is not a Poisson variate.

2. If $X_1, X_2, ..., X_n$ are n independent Poisson variates with parameters $m_1, m_2, ... m_n$ then $Y = X_1 + X_2 + ... + X_n$ is also a Poisson variate with parameter $m_1 + m_2 + ... + m_n$.

3. If X_1 and X_2 are independent Poisson variates with parameter m_1, m_2 then $Y = a_1X_1 + a_2X_2$ is **not** a Poisson variate.

(You can very easily prove these two results.)

(h) Recurrence Relation For Probabilities

We have for Poisson distribution

$$p(x) = \frac{e^{-m} \cdot m^x}{x!} \qquad \therefore \ p(x+1) = \frac{e^{-m} \cdot m^{x+1}}{(x+1)!}$$

$$\therefore \ \frac{p(x+1)}{p(x)} = \frac{m^{x+1}}{(x+1)!} \cdot \frac{x!}{m^x} = \frac{m}{x+1} \qquad \boxed{\therefore \ p(x+1) = \frac{m}{x+1} \cdot p(x)}$$

If we know $p(0) = e^{-m}$, we can find the probabilities of $x = 1, 2, 3$ .....

Thus, $p(1) = m \cdot p(0)$, $p(2) = \frac{m}{2} p(1)$, $p(3) = \frac{m}{3} p(2)$ and so on.

Since, expected frequency of x *i.e.* $f(x)$ is $Np(x)$, we have from the above relation.

$$Np(x+1) = \frac{m}{x+1} \cdot Np(x) \qquad \boxed{\therefore\ f(x+1) = \frac{m}{x+1} f(x)}$$

This relation can be used to find expected frequencies. This is called **fitting a Poisson distribution**.

Example 1 : Find out the fallacy if any in the following statement "The mean of a Poisson distribution is 2 and the variance is 3".

Sol. : In a Poisson distribution the mean and variance are same. Hence, the above statement is false.

Example 2 : If the mean of the Poisson distribution is 4, find $P(m - 2\sigma < X < m + 2\sigma)$. **(M.U. 2005)**

Sol. : For Poisson distribution mean = variance = m.

Hence, $m = 4$ and $\sigma = 2$

$$\therefore\ P(m - 2\sigma < X < m + 2\sigma) = P(0 < X < 8) = P[X = 1, 2, 3, ..., 7]$$

But $P(X) = e^{-m} \frac{m^x}{x!} = e^{-4} \frac{4^x}{x!}$

$$\therefore\ \text{Required probability} = e^{-4}\left[\frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} + \frac{4^5}{5!} + \frac{4^6}{6!} + \frac{4^7}{7!}\right] = 0.93$$

Example 3 : If the variance of a Poisson distribution is 2, find the probabilities of $r = 1, 2, 3, 4$ from the recurrence relation of Poisson distribution. **(M.U. 2002)**

Sol. : We have $P(x) = e^{-m} \frac{m^x}{x!}$

Since variance = m = 2 by data.

$P(x) = e^{-2} \frac{2^x}{x!}$ when $x = 0$, $P(0) = e^{-2}$

Now, the recurrence relation is $P(x+1) = \frac{m}{x+1} P(x)$

Putting $x = 0$, $P(1) = \frac{2}{1} P(0) = 2e^{-2}$

Putting $x = 1$, $P(2) = \frac{2}{2} P(1) = \frac{2}{2} \cdot 2e^{-2} = e^{-2}$

Putting $x = 2$, $P(3) = \frac{2}{3} P(3) = \frac{2}{3} P(2) = \frac{2}{3} e^{-2}$

Putting $x = 3$, $P(4) = \frac{2}{4} P(3) = \frac{1}{3} e^{-2}$

Example 4 : Find out the fallacy if any in the following statement.

"If X is a Poisson variate such that $P(X = 2) = 9P(X = 4) + 90P(X = 6)$ then mean of $X = 1$." **(M.U. 1997)**

Sol. : Let m be the mean of X. $\quad \therefore\ P(X = x) = e^{-m} \frac{m^x}{x!}$

By data $e^{-m} \cdot \frac{m^2}{2!} = 9 \cdot e^{-m} \cdot \frac{m^4}{4!} + 90 \cdot e^{-m} \cdot \frac{m^6}{6!}$

$\therefore\ \frac{1}{2} = \frac{3m^2}{8} + \frac{m^4}{8} \qquad \therefore\ m^4 + 3m^2 - 4 = 0$

$\therefore\ (m^2 + 4)(m^2 - 1) = 0 \qquad \therefore\ m^2 = -4$ or $m^2 = 1$

$\therefore$ The mean is 1 since $m > 0$.

$\therefore$ The statement is correct.

Example 5 : A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1·5. Calculate the proportion of days on which (i) neither car is used, (ii) some demand is refused. **(M.U. 1996, 98)**

Sol. : We have $P(x) = e^{-m} \frac{m^x}{x!} = \frac{e^{-1.5} \cdot (1.5)^x}{x!}$, $x = 0, 1, 2, ...$

(i) Probability that there is no demand is

$$P(X = 0) = e^{-1.5} \frac{(1.5)^0}{0!} = 0.2231$$

(ii) Probability that some demand is refused means there was demand for more than two cars.

$$\therefore\ P(X > 2) = P(X = 3) + P(X = 4) + ...$$
$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[e^{-1.5} \frac{(1.5)^0}{0!} + e^{-1.5} \frac{(1.5)^1}{1!} + e^{-1.5} \frac{(1.5)^2}{2!}\right]$$
$$= 1 - [0.2231 + 0.3347 + 0.2510] = 0.1912.$$

$\therefore$ Proportion of days on which (i) neither car is used is 0·2231.

(ii) some demand is refused is 0·1912.

Example 6 : If a random variable X follows Poisson distribution such that $P(X = 1) = 2P(X = 2)$, find the mean and the variance of the distribution. Also find $P(X = 3)$. **(M.U. 2002, 05, 16)**

Sol. : Let the parameter of the Poisson distribution be m.

$$\therefore\ P(X = x) = \frac{e^{-m} \times m^x}{x!}$$

We are given that $P(X = 1) = 2P(X = 2)$

$\therefore\ \frac{e^{-m} \times m^1}{1!} = \frac{2e^{-m} \times m^2}{2!} \qquad \therefore\ m = 1$

$\therefore$ The mean and the variance = 1.

Now, $P(X = 3) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-1} \cdot 1^3}{3!} = 0.0613.$

Example 7 : *A hospital switch board receives an average of 4 emergency calls in a 10 minutes interval. What is the probability that (i) there are atleast 2 emergency calls, (ii) there are exactly 3 emergency call in an interval of 10 minutes ?*

Sol. : We have $P(x) = \dfrac{e^{-m} \cdot m^x}{x!}$. Here, $m = 4$.

(i) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{e^{-4} \cdot 4^0}{0!} + \frac{e^{-4} \cdot 4^1}{1!} + \frac{e^{-4} \cdot 4^2}{2!}$$

$$= e^{-4}(1 + 4 + 8) = 0{\cdot}238$$

(ii) $P(X = 3) = \dfrac{e^{-m} \cdot m^x}{x!} = \dfrac{e^{-4} \cdot 4^3}{3!} = 0{\cdot}195$

Example 8 : A variable X follows a Poisson distribution with variance 3. Calculate (i) $P(X = 2)$, (ii) $P(X \ge 4)$. **(M.U. 1996)**

Sol. : We have $P(X = x) = \dfrac{e^{-m} \times m^x}{x!}$, $x = 0, 1, 2, \ldots$

$\therefore \ P(X = 2) = \dfrac{e^{-3} \times 3^2}{2!} = 0{\cdot}224$

$\therefore \ P(X \ge 4) = 1 - [P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)]$

$= 1 - 0{\cdot}647 = 0{\cdot}353.$

Example 9 : If X and Y are independent Poisson variates with mean m_1 and m_2, find the probability that $X + Y = k$.

Sol. : Since X, Y are independent Poisson variates with parameters m_1 and m_2, $Z = X + Y$ is also a Poisson variate with parameter $m_1 + m_2$.

$\therefore \ P(Z = k) = \dfrac{e^{-(m_1+m_2)}(m_1 + m_2)^k}{k!}$, $k = 0, 1, 2$

Example 10 : If X, Y are independent Poisson variates with mean 2 and 3, find the variance of $3X - 2Y$.

Sol. : For Poisson variate mean and variance are equal. Hence, Var. $X = 2$ and Var. $Y = 3$.

Since, X, Y are independent

Var. $(3X - 2Y) = 9$ Var. $(X) + 4$ Var. (Y)

$= 9(2) + 4(3) = 30.$

Example 11 : If X, Y are independent Poisson variates such that $P(X = 1) = P(X = 2)$ and $P(Y = 2) = P(Y = 3)$, find the variance of $2X - 3Y$.

Sol. : Let the parameter of X and Y be m_1 and m_2

$\therefore \ P(X = 1) = P(X = 2)$ gives $\dfrac{e^{-m_1} m_1}{1} = \dfrac{e^{-m_1} m_1^2}{2!}$

$\therefore \ 2e^{-m_1} m_1 - e^{-m_1} m_1^2 = 0 \quad \therefore \ e^{-m_1} m_1 (2 - m_1) = 0 \quad \therefore \ m_1 = 2$

$P(Y = 2) = P(Y = 3)$ gives $\dfrac{e^{-m_2} m_2^2}{2!} = \dfrac{e^{-m_2} m_2^3}{3!}$

$\therefore \ 3e^{-m_2} m_2^2 - e^{-m_2} m_2^3 = 0 \quad \therefore \ e^{-m_2} m_2^2 (3 - m_2) = 0 \quad \therefore \ m_2 = 3$

$\therefore$ Var. $(X) = m_1 = 2$; Var. $(Y) = m_2 = 3$

Since, X and Y are Independent

$V(2X - 3Y) = 4V(X) + 9V(Y)$

$= 4(2) + 9(3) = 35.$

Example 12 : If X_1, X_2, X_3 are three independent Poisson variates with parameters $m_1 = 1$, $m_2 = 2$, $m_3 = 3$ respectively, find (i) $P[(X_1 + X_2 + X_3) \ge 3]$ and (ii) $P[X_1 = 1 / (X_1 + X_2 + X_3) = 3]$. **(M.U. 2005)**

Sol. : By additive property of Poisson distribution $Z = X_1 + X_2 + X_3$ is also a Poisson distribution with parameter $m = m_1 + m_2 + m_3 = 6$.

$\therefore \ P(Z \ge 3) = 1 - P(Z \le 2)$

$$= 1 - \sum_{Z=0}^{2} \frac{e^{-6} 6^z}{z!} = 1 - \left(e^{-6} + 6e^{-6} + \frac{6^2 e^{-6}}{z!}\right)$$

$= 1 - 25e^{-6} = 1 - 25(0{\cdot}002478) = 0{\cdot}938$

By definition of conditional probability.

$$P[X_1 = 1 / (X_1 + X_2 + X_3) = 3] = \frac{P(X_1 = 1 \text{ and } X_2 + X_3 = 2)}{P[(X_1 + X_2 + X_3) = 3]}$$

Now, X_1 is a Poisson variate with parameter $m_1 = 1$, $X_2 + X_3$ is a Poisson variate with parameter $m_2 + m_3 = 2 + 3 = 5$, $X_1 + X_2 + X_3$ is a Poisson variate with parameter $m_1 + m_2 + m_3 = 1 + 2 + 3 = 6$.

$$\therefore \ P[X_1 = 1 / (X_1 + X_2 + X_3) = 3] = \frac{\left(e^{-1} \cdot \frac{1}{1}\right)\left(e^{-5} \cdot \frac{5^2}{2!}\right)}{e^{-6} \cdot \frac{6^3}{3!}} = \frac{25}{72}$$

Example 13 : An insurance company found that only 0·01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that no more than two of its clients are involved in such accident next year ? **(M.U. 2002)**

Sol. : We have $p = \dfrac{0{\cdot}01}{100} = 0{\cdot}0001$, $n = 1000$.

$\therefore \ m = np = 1000 \times 0{\cdot}0001 = 0{\cdot}1$

$\therefore \ P(X = x) = e^{-0.1} \dfrac{(0{\cdot}1)^x}{x!}$

$\therefore \ P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$$\therefore \ P(X \le 2) = e^{-0.1}\left[\frac{(0{\cdot}1)^0}{0!} + \frac{(0{\cdot}1)^1}{1!} + \frac{(0{\cdot}1)^2}{2!}\right] = 0{\cdot}9998$$

Example 14 : Find the probability that atmost 4 defective bulbs will be found in a box of 200 bulbs if it is known that 2 percent of the bulbs are defective. (Given $e^{-4} = 0{\cdot}0183$).

Sol. : Since probability of a defective bulb is small we can use Poisson distribution.

We have, $\therefore\ m = np = 200 \times 0{\cdot}02 = 4$

$$\therefore\ P(X = x) = \frac{e^{-4} \times 4^x}{x!}$$

$$\therefore\ P(X \le 4) = p(0) + p(1) + p(2) + p(3) + p(4)$$

$$= \frac{e^{-4} \times 4^0}{0!} + \frac{e^{-4} \times 4^1}{1!} + \frac{e^{-4} \times 4^2}{2!} + \frac{e^{-4} \times 4^3}{3!} + \frac{e^{-4} \times 4^4}{4!}$$

$$= e^{-4}\left[1 + \frac{4}{1!} + \frac{16}{2!} + \frac{64}{3!} + \frac{256}{4!}\right]$$

$$= e^{-4} \times \frac{103}{3} = 0{\cdot}0183 \times \frac{103}{3} = 0{\cdot}6283.$$

Example 15 : Using Poisson distribution find the approximate value of

$^{300}C_2\ (0.02)^2\ (0{\cdot}98)^{298} + {}^{300}C_3\ (0{\cdot}02)^3\ (0{\cdot}98)^{297}$ **(M.U. 2004)**

Sol. : Clearly the above probabilities are the probabilities of Binomial distribution. Comparing them with

$$P(X = x) = {}^nC_x p^x q^{n-x}$$

We see that $n = 300$, $p = 0{\cdot}02$, $q = 0{\cdot}98$, $x = 2$ and 3.

Now, Binomial distribution is related to Poisson distribution where $m = np$.

Hence, $m = 300 \times 0{\cdot}02 = 0{\cdot}6$.

$\therefore$ Corresponding Poisson distribution is given by

$$P(X = x) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-6} 6^x}{x!}$$

$$\therefore\ P(X = 2) + P(X = 3) = \frac{e^{-6} \cdot 6^2}{2} + \frac{e^{-6} \cdot 6^3}{3}$$

$$= 0{\cdot}04462 + 0{\cdot}08923 = 0{\cdot}1338$$

Example 16 : Find the probability that at most 4 defective bulbs will be found in a box of 200 bulbs if it is known that 2 percent of the bulbs are defective. (Given $e^{-4} = 0{\cdot}0183$). **(M.U. 1997)**

Sol. : Since probability of a defective bulbs is small we can use Poisson distribution.

We have, $\therefore\ m = np = 200 \times 0{\cdot}02 = 4$

$$\therefore\ P(X = x) = \frac{e^{-4} \times 4^x}{x!}$$

$$\therefore\ P(X \le 4) = p(0) + p(1) + p(2) + p(3) + p(4)$$

$$= \frac{e^{-4} \times 4^0}{0!} + \frac{e^{-4} \times 4^1}{1!} + \frac{e^{-4} \times 4^2}{2!} + \frac{e^{-4} \times 4^3}{3!} + \frac{e^{-4} \times 4^4}{4!}$$

$$\therefore\ P(X \le 4) = e^{-4}\left[1 + \frac{4}{1!} + \frac{16}{2!} + \frac{64}{3!} + \frac{256}{4!}\right]$$

$$= e^{-4} \times \frac{103}{3} = 0{\cdot}0183 \times \frac{103}{3} = 0{\cdot}6283.$$

Example 17 : The number of accidents in a year attributed to taxi drivers in a city follows Poisson distribution with mean 3. Out of 1,000 taxi drivers, find approximately the number of drivers

with (i) no accident in a year, (ii) more than 3 accidents in a year. (Given : $e^{-1} = 0{\cdot}3679$, $e^{-2} = 0{\cdot}1353$, $e^{-3} = 0{\cdot}0498$) **(M.U. 1998, 2018)**

Sol. : For a Poisson variate

$$P(X = x) = \frac{e^{-m} \times m^x}{x!},\quad x = 0, 1, 2, \ldots..$$

We are given $m = 3$

$$\therefore\ P(X = x) = \frac{e^{-3} \times (3)^x}{x!},\quad x = 0, 1, 2, \ldots.$$

$$\therefore\ P(X = 0) = \frac{e^{-3} \times (3)^0}{0!} = e^{-3} = 0{\cdot}0498$$

$$P(X = 1) = \frac{e^{-3} \times (3)^1}{1!} = 0{\cdot}0498 \times 3 = 0{\cdot}1494$$

$$P(X = 2) = \frac{e^{-3} \times (3)^2}{2!} = 0{\cdot}0498 \times \frac{9}{2} = 0{\cdot}2241$$

$\therefore$ Expected number of drivers with no accidents

$= N \times p(0) = 1{,}000 \times 0{\cdot}0498 = 49{\cdot}8 = 50$ nearly.

$\therefore\ p$ (0, 1, 2, 3 accidents) $= p(0) + p(1) + p(2) + p(3)$

$= 0{\cdot}0498 + 0{\cdot}1494 + 0{\cdot}2241 = 0{\cdot}4233$

$\therefore\ p$ (more than 3 accidents) $= 1 - 0{\cdot}4233 = 0{\cdot}5767$.

Expected number of drivers with more than 3 accidents

$= Np = 1{,}000 \times 0{\cdot}5767$

$= 576{\cdot}7 = 577$ nearly.

Example 18 : In a certain factory turning out blades, there is a small chance 1 / 500 for any blade to be defective. The blades are supplied in packets of 10. Use the Poisson distribution to calculate the approximate number of packets containing no defective, one defective, two defective blades in a consignment of 10,000 packets. (Given $e^{-0{\cdot}02} = 0{\cdot}9802$)

Sol. : We have, $n = 10$, $p = \frac{1}{500}$. $\therefore\ m = np = 10 \times \frac{1}{500} = 0{\cdot}02$

$$\therefore\ P(X = x) = \frac{e^{-0{\cdot}02} \times (0{\cdot}02)^x}{x!}$$

$$\therefore\ P(X = 0) = \frac{e^{-0{\cdot}02} \times (0{\cdot}02)^0}{0!} = e^{-0{\cdot}02} = 0{\cdot}9802$$

$$P(X = 1) = \frac{e^{-0{\cdot}02} \times (0{\cdot}02)^1}{1!} = e^{-0{\cdot}02} \times 0{\cdot}02 = 0{\cdot}0196$$

$$P(X = 2) = \frac{e^{-0{\cdot}02} \times (0 \cdot 02)^2}{2!} = e^{-0{\cdot}02} \times 0{\cdot}0002 = 0{\cdot}0002$$

$\therefore$ Expected freq. of no defective $= 10000 \times 0{\cdot}9802 = 9802$

Expected freq. of one defective $= 10000 \times 0{\cdot}0196 = 196$

Expected freq. of two defective $= 10000 \times 0{\cdot}0002 = 2.$

Example 19 : Fit a Poisson distribution to the following data

No. of deaths :	0,	1,	2,	3,	4.
Frequencies :	123,	59,	14,	3,	1.

(M.U. 2000, 01, 04)

Sol. : Fitting Poisson distribution means finding expected frequencies of $X = 0, 1, 2, 3, 4$.

Now, mean $= \frac{\Sigma f_i x_i}{\Sigma f_i} = m$

$\therefore$ Mean $= 123(0) + 59(1) + 14(2) + 3(3) + 1(4)$

$= \frac{100}{200} = 0.5$

$\therefore$ Poisson distribution of X is

$$P(X = x) = \frac{e^{-m} \times m^x}{x!} = \frac{e^{-0.5} \times (0.5)^x}{x!} \quad \text{.......... (1)}$$

Expected frequency $= N \times p(x)$

$$= 200 \times \frac{e^{-0.5} \times (0.5)^x}{x!}$$

Putting $x = 0, 1, 2, 3, 4$ we get the expected frequencies as 121, 61, 15, 2, 1.

Or putting $x = 0$ in (1).

$$P(X = 0) = e^{-0.5} \frac{(0.5)^0}{0!} = 0.6065$$

$\therefore$ Expected frequency $f(0) = Np = 200 \times 0.6065 = 121$.

But $f(x+1) = \frac{m}{x+1} \cdot f(x) = \frac{0.5}{x+1} \cdot f(x)$

Putting $x = 0$, $f(1) = \frac{0.5}{1} \cdot 121 = 61$. Putting $x = 1$, $f(2) = \frac{0.5}{2} \cdot 60 = 15$.

Putting $x = 2$, $f(3) = \frac{0.5}{3} \cdot 15 = 3$. Putting $x = 3$, $f(4) = \frac{0.5}{4} \cdot 3 = 1$.

Example 20 : Letters were received in an office on each of 100 days. Fit a Poisson distribution and find the expected frequencies for $x = 0$ and 1. (Given : $e^{-4} = 0.0183$).

Number of letters :	0,	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Frequency :	1,	4,	15,	22,	21,	20,	8,	6,	2,	0,	1.

Sol. : $\Sigma f_i = 1 + 4 + 15 + ... + 1 = 100$

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i} = \frac{1 \times 0 + 4 \times 1 + 15 \times 2 + ... + 1 \times 10}{100} = \frac{400}{100} = 4$$

$$\therefore P(X = x) = \frac{e^{-m} \cdot m^x}{x!} = \frac{e^{-4} 4^x}{x!} \quad \therefore p(0) = e^{-4} = 0.0183$$

Now, expected frequency $f(0) = Np = 100 \times 0.0183 = 1.83 = 2$

But $f(x+1) = \frac{m}{x+1} \cdot f(x) \quad \therefore f(1) = \frac{4}{1}(1.83) = 7.32 = 7$

Example 21 : A transmission channel has a per-digit error probability $p = 0.01$. Calculate the probability of more than 1 error in 10 received digits using (i) Binomial Distribution, (ii) Poisson Distribution.

Also find the m.g.f. in each case. **(M.U. 2004)**

Sol. : (i) Binomial Distribution : We have $p = 0.01$, $q = 1 - p = 0.99$, $n = 10$.

$$P(X = x) = {}^nC_x p^x q^{n-x} = 10^{10}C_x (0.01)^x (0.99)^{10-x}$$

$$\therefore P(X > 1) = 1 - P(X \le 1) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - {}^{10}C_0 (0.01)^0 (0.99)^{10} - {}^{10}C_1 (0.01)^1 (0.99)^9$$

$$= 1 - 0.9044 - 0.09135 = 0.00425$$

(ii) **Poisson Distribution :** We have $m = np = 10(0.01) = 0.1$.

$$\therefore P(X = x) = e^{-m} \cdot \frac{m^x}{x!} = e^{-0.1} \frac{(0.1)^x}{x!}$$

$$\therefore P(X > 1) = 1 - P(X \le 1) = 1 - P(X = 0) - P(X = 1)$$

$$= 1 - e^{-0.1} \frac{(0.1)^0}{0!} - e^{-0.1} \frac{(0.1)^1}{1!}$$

$$= 1 - 0.9048 - 0.0905 = 0.0047$$

(iii) M.G.F. of Binomial Distribution is given by

$$M_0(t) = (q + pe^t)$$

Here, $q = 0.9$, $p = 0.01$, $n = 10$

$$\therefore M_0(t) = (0.99 + 0.01 \cdot e^t)^{10}$$

(iv) M.G.F. of Poisson Distribution is given by

$$M_0(t) = e^{m(e^t - 1)}$$ [See (A), page 4-4]

Here, $m = 0.1$. $\therefore M_0(t) = e^{0.1(e^t - 1)}$

Example 22 : It is known that the probability of an item produced by a certain machine will be defective is 0.05. If the produced items are sent to the market in packets of 20, find the number of packets containing (i) at least, (ii) exactly and (iii) at most 2 defective items in a consignment of 1000 packets using (a) Binomial distribution, (b) Poisson approximation to the Binomial distribution. **(M.U. 2004, 05, 06)**

Sol. : We have P (defective) $= 0.05$, P (non-defective) $= 0.95$, $n = 20$ and $N = 1000$.

(I) ***By Binomial Distribution***

$$P(X = x) = {}^{20}C_x (0.05)^x (0.95)^{20-x}$$

$$P(X = 0) = {}^{20}C_0 (0.05)^0 (0.95)^{20} = 0.36$$

$$P(X = 1) = {}^{20}C_1 (0.05)^1 (0.95)^{19} = 0.38$$

$$P(X = 2) = {}^{20}C_2 (0.05)^2 (0.95)^{18} = 0.19$$

No. of packets containing at least 2 defective

$$= N[1 - P(X = 0) - P(X = 1)]$$

$$= 1000[1 - 0.36 - 0.38] = 260$$

No. of packets containing exactly 2 defective

$$= N\,P(X=2) = 1000 \times 0{\cdot}19 = 190$$

No. of packets containing at most 2 defective

$$= N\,[\,P(X=0) + P(X=1) + P(X=2)]$$

$$= 1000\,[\,0{\cdot}36 + 0{\cdot}38 + 0{\cdot}19\,] = 930$$

(ii) *By Poisson Distribution*

Since, $m = np = 20 \times 0{\cdot}05 = 1$

$$P(X=x) = e^{-1}\frac{(1)^x}{x!}, \quad x = 0, 1, 2, \ldots..$$

$$\therefore \quad P(x=0) = e^{-1}\frac{(1)^0}{0!} = 0{\cdot}37\,;$$

$$P(x=1) = e^{-1}\frac{(1)^1}{1!} = 0{\cdot}37\,;$$

$$P(x=2) = e^{-1}\frac{(1)^2}{2!} = 0{\cdot}1839$$

$\therefore$ No. of packets containing at least 2 defective

$$= N\,[1 - P(X=0) - P(X=1)]$$

$$= 1000\,[\,1 - 0{\cdot}37 - 0{\cdot}37\,] = 260$$

No. of packets containing exactly 2 defective

$$= N \cdot P(X=2) = 100 \times 0.1839 = 180$$

No. of packets containing atmost 2 defective

$$= N\,[\,P(X=0) + P(X=1) + P(X=2)]$$

$$= 1000\,[\,0{\cdot}37 + 0{\cdot}37 + 0{\cdot}1839\,] = 924.$$

EXERCISE - I

(A) 1. "Can we have a Poisson distribution with mean 4 and variance 5 ? Give reasoning for your answer. [**Ans. :** No]

2. Find the mean and variance of the following distribution

$$P(X=x) = \frac{e^{-3} \times (3)^x}{x!}, \quad x = 0, 1, 2, \ldots.$$ [**Ans. :** Mean = variance = 3]

3. The mean and the variance of a probability distribution is 2. Write down the distribution.

(M.U. 2002, 05) [**Ans. :** $P(X=x) = \dfrac{e^{-2} \times (2)^x}{x!}, \quad x = 0, 1, 2, \ldots.$]

4. In a Poisson distribution $P(x=3)$ is $2/3$ of $P(X=4)$. Find the mean and the standard deviation. **(M.U. 2007)** [**Ans. :** $m = 6$, $\sqrt{6}$]

5. In a Poisson distribution the probability $p(x)$ for $x = 0$ is 20 percent. Find the mean of the distribution. [**Ans. :** $m = 2{\cdot}9957$]

6. If X is a Poisson variate and $P(X=0) = 6P(X=3)$, find $P(X=2)$. **(M.U. 1998)**

[**Ans. :** $m = 1\,;\ e^{-1}/2$]

7. If a random variable X follows Poisson distribution such that $P(X=2) = 9\,P(X=4) + 90\,P(X=6)$, find the mean and the variance of X.

[**Ans. :** mean = variance = $m = 1$]

8. If X is a Poisson variate such that $P(X=1) = P(X=2)$, find $E(X^2)$. **(M.U. 2004)**

[**Ans. :** 6]

9. The probability that a Poisson variable X takes a positive value is $1 - e^{-1{\cdot}5}$. Find the variance and the probability that X lies between $-1{\cdot}5$ and $1{\cdot}5$.

(**Hint :** $P(X>0) = 1 - P(X=0) = 1 - e^{-1{\cdot}5}$

$\therefore\ 1 - e^{-m} = 1 - e^{-1{\cdot}5} \quad \therefore\ m = 1{\cdot}5$

$P(-1{\cdot}5 < X < 1{\cdot}5) = P(X=0) + P(X=1) = 2{\cdot}5\,(e^{-1{\cdot}5})$)

10. If X is a Poisson variate with mean 4 and Y is a Poisson variate with mean 5, what is the mean of the variate $X + Y$? [**Ans. :** 9]

11. If X and Y are Poisson variates with mean 2 and 4 respectively, find $P[(X+Y) \geq 4]$.

[**Ans. :** 0·45]

12. If X and Y are Poisson variates with parameters 3 and 4, find the variance of $4X - 3Y$.

[**Ans. :** 84]

13. If X and Y are independent Poisson variates such that $P(X=1) = P(X=2)$ and $P(Y=2) = P(Y=3)$, find the variance of $3X - 4Y$.

[**Ans. :** 66]

14. If the mean of the Poisson distribution is 2. Find the probabilities of $x = 1, 2, 3, 4$, from the recurrence relation of probability. **(M.U. 2004)**

[**Ans. :** $m = 2$. $p(x+1) = \dfrac{m}{x+1}\,p(x)$

$P(0) = e^{-2},\ p(1) = 2e^{-2},\ p(2) = e^{-2},\ p(3) = \dfrac{2}{3}e^{-2},\ p(4) = \dfrac{1}{3}e^{-2}.$]

15. If the variance of the Poisson distribution is 1·2. Find the probabilities for $r = 1, 2, 3, 4$ from the recurrence relation. **(M.U. 2004)** [**Ans. :** $m = 1{\cdot}2$, $P(0) = e^{-1{\cdot}2}$ etc.]

16. In sampling a large number of parts manufactured by a machine, the mean number of defectives in a sample of 20 is 2. Out of 100 such samples, how many would you expect to contain 3 defectives (i) using the Binomial distribution, (ii) Poisson distribution. **(M.U. 2004, 15)**

[**Ans. :** **(i)** $p = 0{\cdot}1,\ q = 0{\cdot}9,\ p(x=3) = {}^{20}C_3\,(0{\cdot}1)^3\,(0{\cdot}9)^{17} = 0.1901$. No. $= Np = 190$.

(ii) $m = np = 20 \times \dfrac{1}{10} = 2,\ P(X=3) = e^{-2}\cdot\dfrac{2^3}{3!} = 0{\cdot}18$. No. $= Np = 180$]

17. If X_1, X_2, X_3 are three independent Poisson variates with parameters 1, 2, 3 respectively, find $P[(X_1 + X_2 + X_3) \leq 3]$ and $P[X_1 = 1 / (X_1 + X_2 + X_3) = 3]$ [**Ans. :** **(i)** 0·15, **(ii)** 25/72]

18. Using Poisson distribution, find the approximate value of

${}^{300}C_2\,(0{\cdot}03)^2\,(0{\cdot}97)^{298} + {}^{300}C_3\,(0{\cdot}03)^3\,(0{\cdot}97)^{297}$. [**Ans. :** 0·1338]

19. If 2 percent bulbs are known to be defective bulbs, find the probability that in a lot of 300 bulbs there will be 2 or 3 defective bulbs, using (i) Binomial distribution, (ii) Poisson distribution.

[**Ans. :** **(i)** 0·1319, **(ii)** 0·1338]

8. In a certain factory producing certain articles the probability that an article is defective is 1 / 400. The articles are supplied in packets of 10. Find approximately the number of packets in a consignment of 20,000 packets containing (i) no defective, (ii) one defective and (ii) two defective blades using (a) Binomial Distribution, (b) Poisson approximation to Binomial distribution.

[**Ans. :** (a) (I) 19506, (II) 489, (III) 6; (b) (I) 19506, (II) 488, (III) 6]

(D) 1. Fit a Poisson distribution to the following data

X :	0,	1,	2,	3,	4.	Total
f :	192,	100,	24,	3,	1.	320

[**Ans. :** $m = 0{\cdot}5$, Frequencies : 194, 97, 24, 4, 1]

2. Fit a Poisson distribution to the following data.

X :	0,	1,	2,	3,	4.	Total
f :	122,	60,	15,	2,	1.	200

(M.U. 2001)

[**Ans. :** $m = 0{\cdot}595$, Frequencies : 121, 61, 15, 3, 0]

3. The following mistakes per page were observed in a book.

No. of mistakes :	0,	1,	2,	3,	4.	Total
No. of pages :	211,	90,	19,	5,	0.	325

Fit a Poisson distribution. **(M.U. 1997, 2001)**

[**Ans. :** $m = 0{\cdot}44$, Frequencies : 209, 92, 20, 3, 1]

4. Fit a Poisson distribution to the following data.

No. of defects per piece :	0,	1,	2,	3,	4.	Total
No. of pieces :	43,	40,	25,	10,	2.	120

(M.U. 1997)

[**Ans. :**

No. of defects per piece :	0,	1,	2,	3,	4.	Total
No. of pieces :	42,	44,	24,	8,	2.	120]

5. Fit a Poisson distribution to the following data

X :	0,	1,	2,	3,	4,	5,	6,	7.
f :	314,	335,	204,	86,	29,	9,	3,	0.

(M.U. 2004)

[**Ans. :**

X :	0,	1,	2,	3,	4,	5,	6,	7
f :	295,	354,	212,	85,	26,	6,	1,	1]

6. Fit a Poisson distribution to the following data.

X :	0,	1,	2,	3,	4,	5,	6,	7,	8
f :	56,	156,	132,	92,	37,	22,	4,	0,	1

(M.U. 2004, 07, 09, 15)

[**Ans. :**

X :	0,	1,	2,	3,	4,	5,	6,	7,	8
f :	70,	137,	135,	89,	44,	17,	6,	2,	0]

7. Fir a Poisson distribution to the following data.

X :	0,	1,	2,	3,	4,	5	Total
f :	142,	156,	69,	27,	5,	1	400

(M.U. 2002, 03, 04, 11)

[**Ans. :**

X :	0,	1,	2,	3,	4,	5	Total
f :	147,	147,	74,	24,	6,	2	400]

3. Normal Distribution

Normal distribution is one of the most important and commonly used continuous distribution. It was first developed by DeMoivre but it is credited to Gauss who referred to it first in 1809. A large number of continuous variates follow this distribution, hence, the name 'normal'.

A renowned mathematician Poincare' has remarked that "there must be something mysterious about the normal distribution because mathematicians think it is a law of nature whereas physicists are convinced that it is a mathematical theorem."

1. Definition : A continuous random variable X is said to follow **normal distribution** with parameter m (called mean) and σ^2 (called variance), if its probability density function is given by

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}} \qquad \begin{array}{l} -\infty < x < \infty \\ -\infty < m < \infty,\ \sigma^2 > 0 \end{array}$$

Remarks ....

(I) A continuous random variate X following normal distribution with mean m and standard deviation σ is referred to as $X \sim N(m, \sigma)$.

(II) If X is a normal variate with parameter m, σ, then $\boxed{Z = \frac{X-m}{\sigma}}$ is also a normal variate with mean = 0 and standard deviation = 1.

It is called **Standard Normal Variate.**

2. Importance of Normal Distribution

(I) The variables such as height, weight, intelligence etc. follow normal distribution.

(II) Many other distributions occurring in practice such as Binomial, Poisson etc. can be approximated by normal distribution.

(III) Many of the distributions of sample statistic *e.g.* Sample Mean, Sample Variance tend to normal distribution for large samples.

(Iv) Normal distribution has wide applications in Statistical Quality Control.

(v) Errors in measurements of physical quantities follow normal distribution.

(vi) It is also useful in psychological and educational research.

Abraham De Moivre (1667 - 1754)

A French mathematician who made important contributions to statistics, theory of probability and trigonometry. The concept of statistically independent events was first developed by De Moivre. Through the use of complex number he transformed trigonometry from a branch of geometry to a branch of analysis. His treatise on probability has influenced the development of probability theory.

(a) Mean and Variance of the Normal Distribution

(I) By definition mean is given by

$$\text{Mean} = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_{-\infty}^{\infty} [(x-m)+m] f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-m) f(x)\,dx + m\int_{-\infty}^{\infty} f(x)\,dx$$

But the first integral is zero, because the first moment about the mean is zero and the second integral is unity.

$$\therefore \quad \boxed{\text{Mean} = m}$$

(II) Now, $\text{Var}(X) = E(X-m)^2$

$$= \int_{-\infty}^{\infty} (x-m)^2 f(x)\,dx$$

$$= \int_{-\infty}^{\infty} (x-m)^2 \cdot \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

Now, put $\frac{x-m}{\sigma} = t \quad \therefore\ dx = \sigma\,dt$

$$\therefore \quad \text{Var}(X) = \int_{-\infty}^{\infty} \sigma^2 \cdot t^2 \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}t^2} dt = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t\, d\left(-e^{-\frac{1}{2}t^2}\right)$$

Now, integrating by parts

$$\text{Var}(X) = \frac{\sigma^2}{\sqrt{2\pi}}\left[\left[t\left(-e^{-\frac{1}{2}t^2}\right)\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2} dt\right]$$

[By Gamma Functions, $\int_{-\infty}^{\infty} e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$. Refer to Applied Mathematics - II]

$$= \frac{\sigma^2}{\sqrt{2\pi}}\left[0 + \sqrt{2\pi}\right] = \sigma^2$$

$$\therefore \quad \boxed{\text{Var.}(X) = \sigma^2}$$

Note ....

A normal variate with mean m and standard deviation σ is shortly denoted as $N(m, \sigma)$.

Remark ....

Here we simply note that mean, median and mode of the normal distribution are equal to m

(M.U. 2009)

$$\therefore \quad \boxed{\text{Mean} = \text{Median} = \text{Mode} = m}$$

(b) Moment Generating Function of Normal Distribution

By definition,

$$M_0(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

Putting $\frac{x-m}{\sigma} = z, \quad \frac{dx}{\sigma} = dz$

$$M_0(t) = \int_{-\infty}^{\infty} e^{t(m+\sigma z)} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z^2} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{mt} \cdot e^{-\frac{1}{2}(z^2 - 2t\sigma z)} dz$$

Now $z^2 - 2t\sigma z = (z - t\sigma)^2 - t^2\sigma^2$

$$\therefore \quad M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{mt} \cdot e^{-\frac{1}{2}(z-t\sigma)^2} \cdot e^{\frac{1}{2}t^2\sigma^2} dz$$

Putting $u = z - t\sigma$

$$M_0(t) = \frac{1}{\sqrt{2\pi}} \cdot e^{mt + \frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} du$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{mt + \frac{t^2\sigma^2}{2}} \cdot 2\int_{0}^{\infty} e^{-\frac{1}{2}u^2} du$$

$$= e^{mt + \frac{t^2\sigma^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} \qquad \left[\because \int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]$$

$$\therefore \quad \boxed{M_0(t) = e^{mt + \frac{t^2\sigma^2}{2}}}$$

Now, $M_0(t) = e^{mt} \cdot e^{t^2\sigma^2/2} = \left[1 + mt + \frac{m^2t^2}{2!} + \ldots\right]\left[1 + \frac{t^2\sigma^2}{2!} + \frac{t^4\sigma^4}{4!} + \ldots\right]$

$\therefore \quad \mu_1' =$ Coefficient of $t = m$,

$\mu_2' =$ Coefficient of $\frac{t^2}{2!} = m^2 + \sigma^2$

$\therefore \quad$ Mean $= \mu_1' = m$;

Variance $= \mu_2' - \mu_1'^2 = m^2 + \sigma^2 - m^2 = \sigma^2$.

Corollary : Since mean of the standard normal variate is zero and the standard deviation is unity, putting $m = 0$ and $\sigma = 1$ in the above m.g.f., we get the moment generating function of the standard normal variate as

$$\boxed{M_0(t) = e^{t^2/2}}$$

Moments of Normal Distribution : The m.g.f. about mean is given by

$$M_{\bar{x}}(t) = E[e^{t(x-m)}] = E(e^{tx} \cdot e^{-mt})$$
$$= e^{-mt}E(e^{tx}) = e^{-mt}M_0(t)$$
$$= e^{-mt} \cdot e^{mt+(t^2\sigma^2/2)} = e^{t^2\sigma^2/2}$$
$$= 1 + (t^2\sigma^2/2) + \frac{(t^2\sigma^2/2)^2}{2!} + \frac{(t^2\sigma^2/2)^3}{3!} + \ldots\ldots$$

Now, μ_r = coefficient of $\frac{t^r}{r!}$.

Since, no odd power of t appears in the above expansion, **moments of odd powers about mean of a normal distribution are zero.**

$$\therefore \quad \boxed{\mu_{2n+1} = 0, \quad n = 0, 1, 2, \ldots\ldots} \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

Moments of even power

$$\mu_{2n} = \text{coefficient of } \frac{t^{2n}}{(2n)!} = \frac{(\sigma^2/2)^n}{n!} \cdot (2n)!$$
$$= \frac{\sigma^{2n}}{2^n n!}[2n(2n-1)(2n-2)\ldots\ldots 4 \cdot 3 \cdot 2 \cdot 1]$$
$$= \frac{\sigma^{2n}}{2^n \cdot n!}[2 \cdot n(2n-1) \cdot 2(n-1)(2n-3)\ldots\ldots 4 \cdot 3 \cdot 2 \cdot 1]$$
$$= \frac{\sigma^{2n}}{2^n \cdot n!} 2^n n!(2n-1)(2n-3)\ldots\ldots 5 \cdot 3 \cdot 1$$

$$\boxed{\mu_{2n} = 1 \cdot 3 \cdot 5 \ldots\ldots (2n-1) \cdot \sigma^{2n}} \qquad \ldots\ldots\ldots (2)$$

(c) Moment Generating Function of Standard Normal Variate (M.U. 2006)

We shall here find the moment generating function of Standard Normal Variate and from the m.g.f., we shall obtain the mean and variance of S.N.V. $Z = \frac{X-m}{\sigma}$.

Let $f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$ $\quad -\infty < x < \infty$, $-\infty < m < \infty$, $\sigma^2 > 0$

Now putting, $\frac{x-m}{\sigma} = z$, $dx = \sigma\, dz$, we get $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$.

$$\therefore \quad M_0(t) = E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz} f(z)\,dz = \int_{-\infty}^{\infty} e^{tz} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2tz)} dz.$$

Now, $z^2 - 2tz = (z-t)^2 - t^2$

$$\therefore \quad M_0(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-t)^2} \cdot e^{t^2/2} dz$$

Now, put $z - t = u$, $dz = du$

$$\therefore \quad M_0(t) = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2} \cdot du = \frac{e^{t^2/2}}{\sqrt{2\pi}} \int_0^{\infty} 2 \cdot e^{-u^2/2} \cdot du$$
$$= \frac{e^{t^2/2}}{\sqrt{2\pi}} \cdot 2 \cdot \sqrt{\frac{\pi}{2}} = e^{t^2/2} = 1 + \frac{t^2}{2} + \frac{(t^2/2)^2}{2!} + \ldots\ldots$$

$\therefore$ Mean = coefficient of $t = 0$. Variance = coefficient of $\frac{t^2}{2!} = 1$.

$\therefore$ Mean and standard deviation of Standard Normal Variate are zero and one respectively.

Alter : We shall obtain mean and variance of Standard Normal Variate from the definitions.

(i) Mean $\bar{Z} = E(Z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z \cdot e^{-z^2/2} dz$

But the function on the r.h.s. is an odd function, hence, the definite integral is zero.

$\therefore$ Mean = 0.

(ii) $$\mu_2' = E(z^2) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} z^2 \cdot e^{-z^2/2} dz$$

Since the function on the r.h.s. is an even function,

$$\mu_2' = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^2 \cdot e^{-z^2/2} dz$$

Now, put $\frac{z^2}{2} = t$ $\quad \therefore z = \sqrt{2t}$ $\quad \therefore dz = \frac{\sqrt{2}}{2\sqrt{t}} dt$

$$\therefore \quad \mu_2' = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} 2t \cdot e^{-t} \frac{\sqrt{2}}{2} \cdot \frac{dt}{\sqrt{t}} = \frac{2}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{1/2} dt = \frac{2}{\sqrt{\pi}} \left\lceil \frac{3}{2} \right.$$
$$= \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \left\lceil \frac{1}{2} \right. = \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \cdot \sqrt{\pi} = 1$$

$\therefore$ Variance $= \mu_2' - \mu_1'^2 = 1 - 0 = 1$.

(d) Linear Combination (Additive Property)

Theorem : Let X_i, $i = 1, 2, 3, \ldots$ n be n independent normal variates with mean m_i and variance σ_i^2. Let their linear combination be denoted by Y i.e.

$$Y = a_1X_1 + a_2X_2 + \ldots\ldots + a_nX_n$$

Then Y is also a normal variate with mean m and variance σ^2 where

$$m = a_1m_1 + a_2m_2 + \ldots\ldots + a_n m_n \quad \text{and} \quad \sigma^2 = a_1\sigma_1^2 + a_2\sigma_2^2 + \ldots\ldots + a_n \sigma_n^2.$$

Proof : We know that m.g.f. of a normal variate with mean m and variance σ^2 is given by

$$M_0(t) = e^{mt + t^2\sigma^2/2}$$

Further $M_Y(t) = M_{a_1X_1+a_2X_2+\ldots+a_nX_n}$

$$= M_{a_1X_1}(t) \cdot M_{a_2X_2}(t) \ldots\ldots M_{a_nX_n}(t)$$

(since $X_1, X_2, \ldots\ldots X_n$ are independent variates.)

$$= M_{X_1}(a_1t) \cdot M_{X_2}(a_2t) \ldots\ldots M_{X_n}(a_nt) \quad [\because M_{cX}(t) = M_X(ct)]$$

$$= e^{a_1m_1t + a_1^2\sigma_1^2t^2/2} \cdot e^{a_2m_2t + a_2^2\sigma_2^2t^2/2} \cdot \ldots\ldots$$

$$= e^{(a_1m_1 + a_2m_2 + \ldots)t + (a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots)t^2/2}$$

But from its form this is m.g.f. of a normal variate whose mean is $(a_1m_1 + a_2m_2 + \ldots\ldots)$ and whose variance is $(a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \ldots\ldots)$

Remarks ....

1. This property is known as **additive property** of normal distribution.

2. If $a_3 = a_4 = \ldots\ldots = a_n = 0$ then $Y = a_1X_1 + a_2X_2$ is a normal variate with mean $a_1m_1 + a_2m_2$ and variance $= a_1^2\sigma_1^2 + a_2^2\sigma_2^2$.

3. If $a_1 = a_2 = 1$, $a_3 = a_4 = \ldots\ldots = a_n = 0$, then $Y = X_1 + X_2$ is a normal variate with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$.

4. If $a_1 = 1$ $a_2 = -1$, $a_3 = a_4 = \ldots\ldots = a_n = 0$ then $Y = X_1 - X_2$ is also a normal variate with mean $m_1 - m_2$ and variance $\sigma_1^2 + \sigma_2^2$.

5. Comparing Normal Distribution with Poisson Distribution we find that **sum** of two Normal or Poisson Variates is a Normal or Poisson variate. But although difference of two normal variates is a normal variate, the difference of two Poisson variates is **not** a Poisson variate.

6. If X_i ; $i = 1, 2, 3, \ldots\ldots n$ are n independent, identical normal varieties all with the same mean m and same standard deviation σ and if we put $a_1 = a_2 = a_3 = \ldots\ldots = a_n = 1/n$ then $Y = X_1 + X_2 + \ldots\ldots + X_n$ is a normal variate with mean

$$m = \frac{m + m + \ldots\ldots + m}{n} = m$$

and $$\sigma^2 = \frac{1}{n^2}\sigma^2 + \frac{1}{n^2}\sigma^2 + \ldots\ldots + \frac{1}{n^2}\sigma^2 = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

In other words, **Y is a normal variate with mean m and standard deviation** $\sigma/\sqrt{n}$.

(e) Area Property

" If X is a normal variate with mean m and variance σ^2 and Z is standard normal variate (with mean zero and variance one) then the area under the normal curve of X between X = m and X = x_1 is equal to the area under the S.N. Curve of Z between Z = 0 to Z = z_1 (say, corresponding to x_1)."

Proof : Consider a normal variate X with mean m and variance σ^2.

Then $$P(m \le X \le x_1) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_m^{x_1} e^{-\frac{1}{2}\left(\frac{X-m}{\sigma}\right)^2} dx$$

Now, put $\dfrac{X-m}{\sigma} = Z$

When $X = m$, $Z = 0$; when $X = x_1$, $Z = \dfrac{x_1 - m}{\sigma} = z_1$ say

$$\therefore P(m \le X \le x_1) = P(0 \le Z \le z_1) = \frac{1}{\sqrt{2\pi}}\int_0^{z_1} e^{-\frac{1}{2}z^2} dz$$

Thus, the area under the normal curve from m to x_1 is equal to the area under the standard normal curve from 0 to z_1.

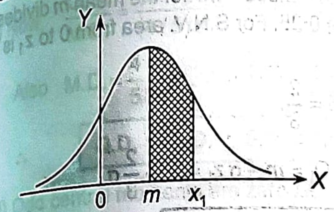

Fig. 4.1

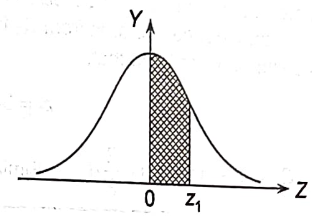

Fig. 4.2

The integral $\dfrac{1}{\sqrt{2\pi}}\int_0^{z_1} e^{-\frac{1}{2}z^2} dz$ is denoted by $\int_0^{z_1}\phi(z)\,dz$ and is known as **normal probability integral.** The areas under standard normal curve from $z = 0$ to various values of z_1 have been calculated and are given in the table at the end of the book.

Reverse Problem : In some problems we know that X is a normal variate with mean m and standard deviation σ. We are required to find the value of $X = x_1$ corresponding to a given probability. Suppose, we want to find of $X = x_1$ such that $P(X > x_1) = \alpha$. In this case also we consider

S.N.V. $Z = \dfrac{X - m}{\sigma}$

We now consult the area table and find the value of $Z = z_1$ for which area to the right is α *i.e.* area between $Z = 0$ to $Z = z_1$ is $(0{\cdot}5 - \alpha)$. From this value of z_1 we get X using

$$z_1 = \frac{x_1 - m}{\sigma}$$

i.e. $x_1 = m + z_1\sigma$.

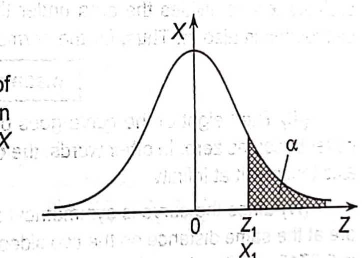

Fig. 4.3

Remarks ....

1. Since standard normal curve is symmetrical about the y-axis it is enough to find the areas to the right. The areas to the left of y-axis at equal distances will be equal.

2. The total area under the curve is unity. Hence, because of symmetry the area under S.N.V. to the right of the y-axis is 0·5.

3. To find the probability that X will lie between x_1 and x_2 $(x_1 < x_2)$, we find the corresponding values of S.N.V. Z (from $Z = \dfrac{X - m}{\sigma}$) say z_1 and z_2 and find the area from z_1 to z_2 and S.N. Curve. The required probability is this area.

$$P(x_1 \le X \le x_2) = P(z_1 \le Z \le z_2)$$
$$= \text{area between } Z = z_1 \text{ and } Z = z_2 \text{ under the S.N. curve.}$$

(f) The Quartile Deviation of a Normal Distribution is (2 / 3) S.D.

Let X be a normal variate with mean m and variance σ^2. Consider the standard normal variate

$$Z = \frac{X - m}{\sigma}$$

When $X = m$, $Z = 0$ and when $X = Q_3$ the third quartile let $Z = \dfrac{Q_3 - m}{\sigma} = z_1$.

We know that for Normal variate mean = median = mode = m. *i.e.* the mean m divides the area into two equal parts. Hence, the area from Q_1 to m is 0·25. For S.N.V. area from 0 to z_1 is 0·25. For area 0·25 we find from the table that,

$$z_1 = 0{\cdot}6745 = \frac{2}{3}.$$

$$\therefore\ Q_3 = m + \sigma z_1 = m + \frac{2}{3}\sigma. \quad \text{Similarly, } Q_1 = m - \sigma z_1 = m - \frac{2}{3}\sigma.$$

$$\therefore \quad \boxed{\text{Quartile Deviation } = \frac{Q_3 - Q_1}{2} = \frac{2}{3}\sigma.}$$

(g) Properties of the Normal Distribution

We summaries below important properties of normal distribution which we have proved above. They are given in terms of the properties of the normal curve.

(i) The normal curve is bell-shaped and symmetrical about the maximum ordinate at $x = m$, the *mean*. In other words, the curve is divided into two equal parts by this ordinate. The curve on one side of this ordinate is the mirror image of the curve on the other side.

(ii) The curve has maximum height at $x = m$. Hence the *mode* of the distribution is also m. The ordinate $x = m$ divides the area under the curve into two equal parts. Hence the *median* of the distribution is also m. Thus, for the normal distribution,

$$\boxed{\text{mean} = \text{median} = \text{mode} = m}$$

(iii) The height of the curve goes on decreasing on either side of the ordinate at $x = m$ but never becomes zero. In other words, the curve never intersects the x-axis at any finite point. The x-axis touches it at infinity.

(iv) Since the curve is symmetrical about mean, the first quartile Q_1 and the third quartile Q_3 are at the same distance on the two sides of the mean. The distance of any quartile from the mean is 0·6745 σ units. Hence,

$$Q_1 = m - 0{\cdot}6745\,\sigma \quad \text{or} \quad \boxed{Q_1 = m - \frac{2}{3}\sigma} \quad \text{and}$$

$$Q_3 = m + 0{\cdot}6745 \quad \text{or} \quad \boxed{Q_3 = m + \frac{2}{3}\sigma}$$

Hence, middle 50 % items lie between $m - \dfrac{2}{3}\sigma$ and $m + \dfrac{2}{3}\sigma$.

Further, the quartile deviation of a normal distribution is given by

$$\text{Q.D.} = \frac{Q_3 - Q_1}{2} = \frac{\left(m + \frac{2}{3}\sigma\right) - \left(m - \frac{2}{3}\sigma\right)}{2} = \frac{2}{3}\sigma.$$

$$\boxed{\therefore\ \text{Q.D.} = \frac{2}{3}\sigma}$$

(v) The mean of the distribution is m and the standard deviation is σ.

(vi) The mean deviation is given by $\boxed{\text{M.D.} = \dfrac{4}{5}\sigma}$

(vii) Now, $\dfrac{\text{Q.D.}}{\text{M.D.}} = \dfrac{(2/3)\sigma}{(4/5)\sigma} = \dfrac{2}{3} \times \dfrac{5}{4} = \dfrac{10}{12}$

Also $\text{M.D.} = \dfrac{4}{5}\sigma \quad \therefore \dfrac{\text{M.D.}}{\sigma} = \dfrac{4}{5} = \dfrac{12}{15}$

$$\therefore \quad \boxed{\text{Q.D. : M.D. : S.D.} = 10 : 12 : 15}$$

(viii) Odd central moments are zero *i.e.*

$\mu_{2r+1} = 0$ for (r = 0, 1, 2, ...)

and $\mu_{2r} = 1 \cdot 3 \cdot 5 \ldots\ldots (2r - 1)\sigma^{2r}$ (r = 0, 1, 2,.....)

i.e., $\mu_2 = \sigma^2,\ \mu_4 = 3\sigma^4,\ \mu_6 = 15\sigma^6$

(ix) The area under the normal curve is distributed as follows,

(a) The area between $x = m - \sigma$ and $x = m + \sigma$ is 68·27 %.

(b) The area between $x = m - 2\sigma$ and $x = m + 2\sigma$ is 95·45 %.

(c) The area between $x = m - 3\sigma$ and $x = m + 3\sigma$ is 99·73 %.

These areas under the normal curve and standard normal curve are shown below.

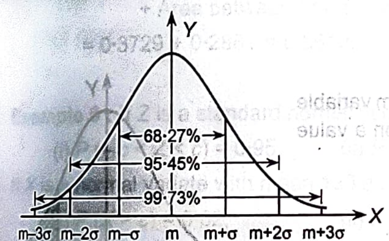

Fig. 4.4 (a)

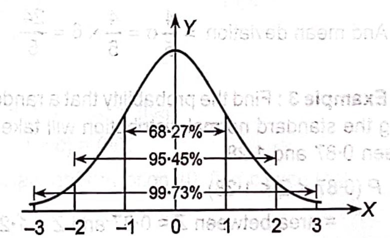

Fig. 4.4 (b)

(h) Normal Approximation To The Binomial Distribution

It can be proved, although we do not, that if X is a Binomial variate with parameter n and p (*i.e.* mean = np and S.D. = $\sqrt{npq}$ where $q = 1 - p$) then

$$Z = \frac{X - np}{\sqrt{npq}}$$

is a Standard Normal Variate if $n \to \infty$ (*i.e.* is large) and neither p nor q is small.

Remark ....

1. Normal distribution can be used in place of Binomial distribution when np and nq are both greater than 15.

2. Normal distribution can also be obtained from Poisson distribution when the parameter $m \to \infty$.

Type I

Example 1 : For a normal distribution the mean is 50 and the standard deviation is 15. Find (i) Q_1 and Q_3, (ii) mean deviation (Also the interquartile range).

Sol. : (i) For a normal distribution

$$Q_1 = m - \frac{2}{3}\sigma = 50 - \frac{2}{3}15 = 40$$

Again $$Q_3 = m + \frac{2}{3}\sigma = 50 + \frac{2}{3}15 = 60$$

(ii) The mean deviation of the normal distribution is,

$$\text{M.D.} = \frac{4}{5}\sigma = \frac{4}{5}\times 15 = 12$$

$\therefore$ Interquartile range $= Q_3 - Q_1 = 60 - 40 = 20.$

Example 2 : The first and the third quartiles of a normal distribution are 36 and 44. Find mean, standard deviation and the mean deviation.

Sol. : We have $Q_1 = m - \frac{2}{3}\sigma$ and $Q_3 = m + \frac{2}{3}\sigma$

$$\therefore\ 36 = m - \frac{2}{3}\sigma \quad \text{and} \quad 44 = m + \frac{2}{3}\sigma$$

Adding $80 = 2m \quad \therefore\ m = 40$

Then, $36 = 40 - \frac{2}{3}\sigma \quad \therefore\ \frac{2}{3}\sigma = 4 \quad \therefore\ \sigma = 6$

And mean deviation $= \frac{4}{5}\sigma = \frac{4}{5}\times 6 = \frac{24}{5}.$

Example 3 : Find the probability that a random variable having the standard normal distribution will take on a value between 0·87 and 1·28.

Sol. : $P(0{\cdot}87 < Z < 1{\cdot}28)$

= area between $Z = 0{\cdot}87$ and $Z = 1{\cdot}28$.

= (area from $Z = 0$ to $Z = 1{\cdot}28$)

− (area from $Z = 0$ to $Z = 0{\cdot}87$)

$= 0{\cdot}3997 - 0{\cdot}3078 = 0{\cdot}0919.$

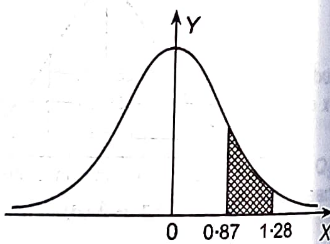

Fig. 4.5

Example 4 : Find the probability that a random variable having standard normal distribution will take a value between (i) 0·87 and 1·28, (ii) − 0·34 and 0·62.

Sol. : (I) As in the above example.

(II) Area from $Z = -0{\cdot}34$ to 0 is the same as $Z = 0$ to $Z = 0{\cdot}34$ and is 0·1331

Area from $Z = 0$ to $Z = 0{\cdot}62$ is 0·2324.

Required area is the sum of the two

$$\therefore\ P(-0{\cdot}34 < Z < 0{\cdot}62) = 0{\cdot}1331 + 0{\cdot}2324 = 0{\cdot}3655.$$

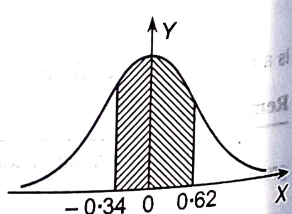

Fig. 4.6

Example 5 : For a normal variate with mean 2·5 and standard deviation 3·5, find the probability that (i) $2 \le X \le 4{\cdot}5$, (ii) $-1{\cdot}5 \le X \le 5{\cdot}5$.

Sol. : We have S.N.V. $Z = \frac{X - m}{\sigma} = \frac{X - 2{\cdot}5}{3{\cdot}5}$

(i) When $X = 2$, $Z = \frac{2 - 2{\cdot}5}{3{\cdot}5} = -0{\cdot}14$

When $X = 4{\cdot}5$, $X = \frac{4{\cdot}5 - 2{\cdot}5}{3{\cdot}5} = 0{\cdot}57$

$P(2 \le X \le 4{\cdot}5) = P(-0.14 \le Z \le 0{\cdot}57)$

= Area between ($Z = -0{\cdot}14$ and $Z = 0{\cdot}57$)

= Area between ($Z = 0$ and $Z = 0{\cdot}14$)

+ Area between ($Z = 0$ and $Z = 0{\cdot}57$)

$= 0{\cdot}0557 + 0{\cdot}2157 = 0{\cdot}2714.$

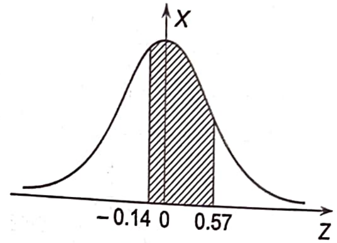

Fig. 4.7 (a)

(ii) When $X = -1{\cdot}5$, $Z = \frac{-1{\cdot}5 - 2{\cdot}5}{3{\cdot}5} = -1{\cdot}14$

When $X = 5{\cdot}3$, $Z = \frac{5{\cdot}3 - 2{\cdot}5}{3{\cdot}5} = 0{\cdot}8$

$P(-1{\cdot}5 \le X \le 5{\cdot}3) = P(-1{\cdot}14 \le Z \le 0{\cdot}8)$

= Area between ($Z = -1{\cdot}14$ and $Z = 0{\cdot}8$)

= Area between ($Z = 0$ and $Z = 1{\cdot}14$)

+ Area between ($Z = 0$ and $Z = 0{\cdot}8$)

$= 0{\cdot}3729 + 0{\cdot}2881 = 0.6610.$

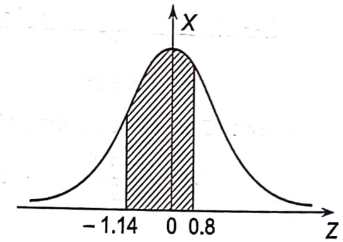

Fig. 4.7 (b)

Example 6 : If Z is a standard normal variate, find c such that

(i) $P(-c < Z < c) = 0{\cdot}95$, (ii) $P(|Z| > c) = 0{\cdot}01$.

If X is a normal variate with mean 120 and standard deviation 10, find c such that

(i) $P(X > c) = 0{\cdot}02$, (ii) $P(X < c) = 0{\cdot}05$.

Sol. : Consulting the table of S.N.V. we have to find the entry 0·475 and the corresponding value of Z. We find from the table that corresponding to the entry 0·4750, $Z = 1{\cdot}96$

Again consulting the table of S.N.V. we find the entry 0·495. Corresponding to the entry 0·495, $Z = 2{\cdot}58$. $\therefore\ c = 2{\cdot}58$.

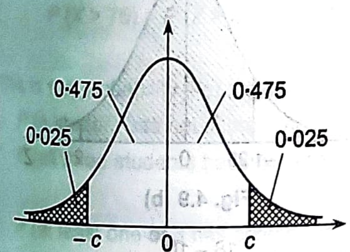

Fig. 4.8 (a)

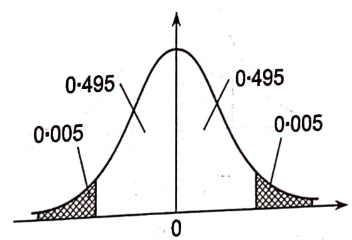

Fig. 4.8 (b)

If Z is a S.N.V. then $Z = \dfrac{X - m}{\sigma}$ $\quad \therefore \quad Z = \dfrac{X - 120}{10}$

$\therefore \quad P(X > c) = P(Z > c) = 0{\cdot}02.$

Corresponding to entry $0{\cdot}5 - 0{\cdot}02 = 0{\cdot}48$, $Z = 2{\cdot}05$.

$\therefore \quad 2{\cdot}05 = \dfrac{X - 120}{10} \qquad \therefore \quad X = 120 + 2{\cdot}05 \times 10 = 140{\cdot}5.$

Again $P(X < c) = P(Z < c) = 0{\cdot}05.$

Corresponding to entry $0{\cdot}5 - 0{\cdot}05 = 0{\cdot}45$, $Z = 1{\cdot}64$.

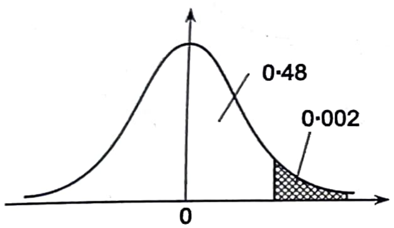

Fig. 4.8 (c)

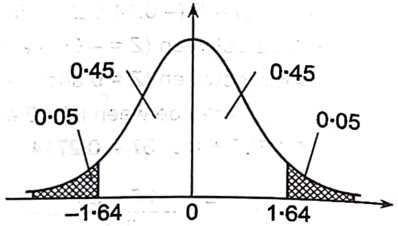

Fig. 4.8 (d)

$\therefore$ Since Z is less than c, c must be negative $\therefore \quad c = -1{\cdot}64.$

$\therefore \quad -1{\cdot}64 = \dfrac{X - 120}{10} \qquad \therefore \quad X = 120 - 10 \times 1{\cdot}64 = 103{\cdot}6$

Example 7 : If X is a normal variate with mean 10 and standard deviation 4, find

(i) $P(|X - 14| < 1)$, (ii) $P(5 \le X \le 18)$, (iii) $P(X \le 12)$. **(M.U. 2002, 09, 16)**

Sol. : We have $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 10}{4}$

(i) When $X = 14$, $Z = \dfrac{14 - 10}{4} = 1$

$\therefore \quad P(|X - 14| \le 1) = P(|Z| \le 1)$ = area between ($Z = -1$ and $Z = 1$)

= 2 (area between $Z = 0$ and $Z = 1$)

= 2 (0·3413) = 0·6826

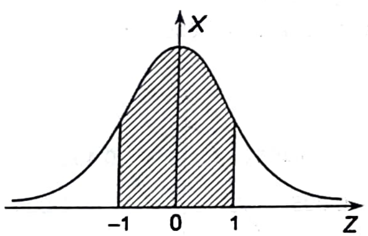

Fig. 4.9 (a)

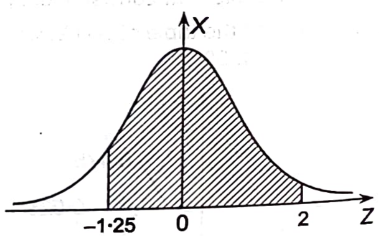

Fig. 4.9 (b)

(ii) When $X = 5$, $Z = \dfrac{5 - 10}{4} = -1{\cdot}25$. $\qquad$ When $X = 18$, $Z = \dfrac{18 - 10}{4} = 2$

$\therefore \quad P(5 \le X \le 18) = P(-1{\cdot}25 \le Z < 2)$

= area between $Z = -1{\cdot}25$ and $Z = 2$

= (area between $Z = 0$ and $Z = 1{\cdot}25$) + (are between $Z = 0$ and $Z = 2$)

= 0·3944 + 0·4772 = 0·8716

(iii) When $Z = 12$, $Z = \dfrac{12 - 10}{4} = 0{\cdot}5$

$\therefore \quad P(X \le 12) = P(Z \le 0{\cdot}5)$ = area upto $Z = 0 \le 0{\cdot}5$

= (area from $-\infty$ to $Z = 0$) + (area from $Z = 0$ to $Z = 0{\cdot}5$)

= 0·5 + 0·1915 = 0·6915.

Type II

Example 1 : In a factory turning out blades in mass production, it was found that in a packet of 100 blades on an average 16 blades are defective. Find the standard deviation of the defective blades. Can the distribution of defective blades be approximated to a normal distribution ? If so write its equation. **(M.U. 1999)**

Sol. : The distribution of defective blades is a Binomial distribution.

We have $n = 100$, $p = \dfrac{16}{100} = 0{\cdot}16$

$\therefore \quad$ Mean $\bar{X} = np = 100 \times 0{\cdot}16 = 16$

$\therefore \quad q = 1 - p = 0{\cdot}84 \qquad \therefore \quad nq = 100 \times 0{\cdot}84 = 84.$

Since both np and nq are greater than 15 as stated in the remark 1 (page 4-27), the Binomial distribution can be approximated to Normal distribution.

Now, as seen above, $\bar{X} = 16$ and $\sigma = \sqrt{npq} = \sqrt{100 \times 0{\cdot}16 \times 0{\cdot}84} = 3{\cdot}67$.

$\therefore$ The equation of the normal distribution is

$$y = \frac{1}{\sqrt{2\pi}\cdot\sigma}\, e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\cdot(3{\cdot}67)}\cdot e^{-\frac{1}{2}\left(\frac{x-16}{3{\cdot}67}\right)^2}.$$

Example 2 : The marks obtained by students in a college are normally distributed with mean 65 and variance 25. If 3 students are selected at random from this college what is the probability that at least one of them would have scored more than 75 marks ? **(M.U. 2005)**

Sol. : We have S.N.V. $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 65}{5}$. $\qquad$ When $X = 75$, $Z = \dfrac{75 - 65}{5} = 2$

$\therefore \quad P(X > 75) = P(Z > 2) = 0{\cdot}5 -$ (area from $z = 0$ to $z = 2$)

$= 0{\cdot}5 - 0{\cdot}4772 = 0{\cdot}0228$

This is the probability that a student chosen at random has scored more than 75 marks.

$\therefore$ P (a student has not scored more than 75) = 1 – 0·0228 = 0·9772

P (all three students have not scored more than 75 marks) = 0·9772 × 0·9772 × 0·9772

= 0·93

$\therefore$ P (at least one of 3 has scored more than 75 marks) = 1 – 0·93 = 0·07.

Example 3 : For a normal variate X with mean 25 and standard deviation 10, find the area between (i) $X = 25, X = 35$, (ii) $X = 15, X = 35$ and also the area such that, (iii) $X \geq 15$, (iv) $X \geq 35$.

Sol. : S.N.V. $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 25}{10}$

(i) When $X = 25$, $Z = 0$, and when $X = 35$, $Z = 1$.

$\therefore$ Area (between $X = 25$ and $X = 35$) $=$ area (between $Z = 0$ and $Z = 1$)
$= 0{\cdot}3413$.

(ii) When $X = 15$, $Z = -1$ and when $X = 35$, $Z = 1$.

$\therefore$ Area between ($X = 15$ and $X = 35$) $=$ area between ($Z = -1$ and $Z = 1$)
$= 2$ (area between $Z = 0$ and $Z = 1$)
$= 2\,(0{\cdot}3413) = 0{\cdot}6826$.

(iii) When $X \geq 15$, $Z \geq -1$

$\therefore$ Area to the right of ($X = 15$)
$=$ area to the right of ($Z = -1$)
$=$ (area between $Z = -1$ to $Z = 0$) + (area to the right of $Z = 0$)
$= 0{\cdot}3413 + 0{\cdot}5 = 0{\cdot}8413$.

(iv) When $X \geq 35$, $z \geq 1$.

$\therefore$ Area to the right of ($X = 35$)
$=$ area to the right of ($Z = 1$)
$=$ (area to the right of $Z = 0$) $-$ (area between $Z = 0$ and $Z = 1$)
$= 0{\cdot}5 - 0{\cdot}3413 = 0{\cdot}1587$.

Example 4 : A normal population has mean 0.1 and standard deviation 2·1. Find the probability that the value of the mean of the sample of size 900 drawn from this population will be negative. **(M.U. 2004)**

Sol. : The mean $\overline{X}$ of the sample is a S.N.V. We have $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$ [Refer to § 4, page 5-3]

$\because\ \mu = 0.1,\ \sigma = 2{\cdot}1,\ n = 900 \quad \therefore\ Z < -\dfrac{0{\cdot}1}{2{\cdot}1/\sqrt{900}} = -1{\cdot}43$

$\therefore\ P(Z < -1{\cdot}43) = P(Z > 1{\cdot}43) = 0{\cdot}5 -$ (area from $Z = 0$ to $Z = 1{\cdot}43$)
$= 0{\cdot}5 - 0{\cdot}4236 = 0{\cdot}0764$.

Example 5 : A manufacturer knows from his experience that the resistance of resistors he produces is normal with $\mu = 100$ ohms and standard deviation $\sigma = 2$ ohms. What percentage of resistors will have resistance between 98 ohms and 102 ohms ? **(M.U. 1996, 10)**

Sol. : We have S.N.V. $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 100}{2}$

When $X = 98$, $Z = \dfrac{98 - 100}{2} = -1$. When $X = 102$, $Z = \dfrac{102 - 100}{2} = 1$.

$\therefore\ P(98 \leq X \leq 102) = P(-1 \leq Z \leq 1)$
$=$ Area between ($Z = -1$ and $Z = 1$)

$\therefore\ P(98 \leq X \leq 102)$ = Area from ($Z = -1$ to $Z = 0$) + Area from ($Z = 0$ to $Z = 1$)
$= 2$ Area from ($Z = 0$ to $Z = 1$)
$= 2 \times 0{\cdot}3413 = 0{\cdot}6826$.

$\therefore$ % of resistors having resistance between 98 and 102 = 68·26 %.

Type III

Example 1 : Monthly salary X in a big organisation is normally distributed with mean ₹ 3000 and standard deviation of ₹ 250. What should be the minimum salary of a worker in this organisation, so that the probability that he belongs to top 5% workers ? **(M.U. 2019)**

Sol. : We have $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 3000}{250}$.

We want to find Z_1 such that

$P(Z > z_1) = \dfrac{5}{100} = 0{\cdot}05$

Since $0{\cdot}5 - 0{\cdot}05 = 0{\cdot}45$, and corresponding to 0·45 the entry in the area table is 1·64. $\therefore\ z_1 = 1{\cdot}64$.

$\therefore\ 1{\cdot}64 = \dfrac{X - 3000}{250}$

$\therefore\ X = 3000 + 250 \times 1{\cdot}64 =$ ₹ 3410.

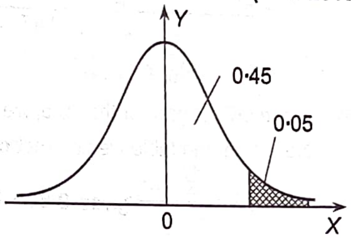

Fig. 4.10 (a)

Example 2 : The diameters of can tops produced by a machine are normally distributed with standard deviation of 0·05 cms. At what mean diameter the machine be set so that not more than 5% of the can tops produced by the machine have diameters exceeding 3 cms. ?

Sol. : Let X denote the diameter of the can tops. X is normally distributed with mean μ (unknown) and standard deviation $\sigma = 0{\cdot}05$. We are given that

$P(Z > z_1) = 0{\cdot}05$

Now, $0{\cdot}5 - 0{\cdot}05 = 0{\cdot}45$ and corresponding to 0·45 the entry in the area table is 1·64

$\therefore\ z_1 = 1{\cdot}64$

$Z = \dfrac{X - m}{\sigma}$ gives $1{\cdot}64 = \dfrac{3 - m}{0{\cdot}01}$

$\therefore\ m = 3 - 1{\cdot}64 \times 0{\cdot}01 = 2{\cdot}984$

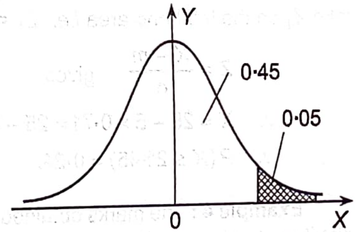

Fig. 4.10 (b)

Example 3 : If X is a normal variate with mean 25 and standard deviation 5, find the value (i) of $X = x_1$, such that $P(X \geq x_1) = 0{\cdot}32$, (ii) of $X = x_2$, such that $P(X \leq x_2) = 0{\cdot}73$, (iii) of $X = x_3$ such that $P(X \leq x_3) = 0{\cdot}24$.

Sol. : (i) Since 0·32 is less than 0·5. We have to find $Z = z_1$ = corresponding to area $= 0{\cdot}5 - 0{\cdot}32 = 0{\cdot}18$.

Now, from the table we find that corresponding to $Z = 0{\cdot}47$ the are under S.N.V. is 0·18.

$\therefore\ Z = \dfrac{X - m}{\sigma}$ gives $0{\cdot}47 = \dfrac{X - 25}{5}$

$\therefore \quad X = 25 + 5 \times 0{\cdot}47 = 25 + 2{\cdot}35 = 27{\cdot}35$

$\therefore \quad P(X \geq 27{\cdot}35) = 0{\cdot}32$ [See Fig. 4.11 (a)]

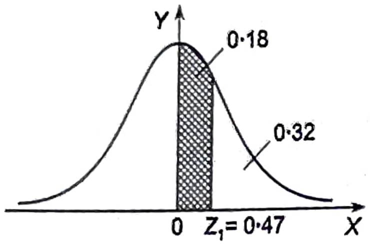

Fig. 4.11 (a)

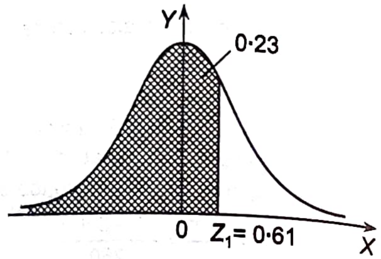

Fig. 4.11 (b)

(II) Since 0·73 is greater than 0·5, we have to find $Z = z_1$ corresponding to area $0{\cdot}73 - 0{\cdot}5 = 0{\cdot}23$. Now, from the table we find that corresponding to $Z = 0{\cdot}61$ the area under S.N.V. is 0·23.

$\therefore \quad Z = \dfrac{X - m}{\sigma}$ gives $0{\cdot}61 = \dfrac{X - 25}{5}$

$\therefore \quad X = 25 + 5 \times 0{\cdot}61 = 28{\cdot}05$

$\therefore \quad P(X \leq 28{\cdot}05) = 0{\cdot}73$ [See Fig. 4.11 (b)]

(III) Since 0·24 is less than 0·5, we have to find $Z = z_1$ corresponding to are $0{\cdot}5 - 0{\cdot}24 = 0{\cdot}26$.

Now, from the table, we find that corresponding to $Z = 0{\cdot}71$, the are under the S.N.V is 0·26.

Since, we want X less than the desired value, we must take Z_1 on the left hand area *i.e.* $Z_1 = -0{\cdot}71$.

$\therefore \quad Z = \dfrac{X - m}{\sigma}$ gives $-0{\cdot}71 = \dfrac{X - 25}{5}$

$\therefore \quad X = 25 - 5 \times 0{\cdot}71 = 25 - 3{\cdot}55 = 21{\cdot}45$

$\therefore \quad P(X \leq 21{\cdot}45) = 0{\cdot}24.$ [See Fig. 4.11 (c)]

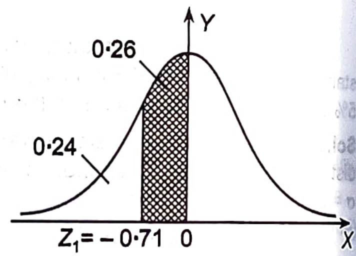

Fig. 4.11 (c)

Example 4 : The marks obtained by 1000 students in an examination are found to be normally distributed with mean 70 and standard deviation 5. Estimate the number of students whose marks will be (i) between 60 and 75, (ii) more than 75.

(M.U. 2003, 16)

Sol. : We have S.N.V. $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 70}{5}$

(i) When $X = 60$, $Z = \dfrac{60 - 70}{5} = -2$;

When $X = 75$, $Z = \dfrac{75 - 70}{5} = 1$.

$P(60 \leq X \leq 75) = P(-2 < Z < 1)$

$= \text{Area between } (Z = -2 \text{ and } Z = 1)$

$= \text{Area from } (Z = 0 \text{ to } Z = 2) + \text{area from } (Z = 0 \text{ to } Z = 1)$

$= 0{\cdot}4772 + 0{\cdot}3413 = 0{\cdot}8185$

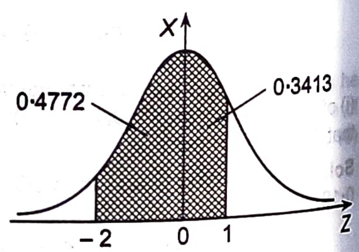

Fig. 4.12

$\therefore$ Number of students getting marks between 60 and 75

$= Np = 1000 \times 0{\cdot}8185 = 818$

(ii) $P(X \geq 75) = P(Z \geq 1)$

$= \text{Area to the right of } Z = 1$

$= 0{\cdot}5 - (\text{area between } Z = 0 \text{ and } Z = 1)$

$= 0{\cdot}5 - 0{\cdot}3413 = 0{\cdot}1587$

$\therefore$ Number of students getting more than 75 marks

$= Np = 1000 \times 0{\cdot}1587 = 159$

Example 5 : In an intelligence test administered to 1000 students, the average was 42 and standard deviation was 24. Find the number of students (i) exceeding the score 50 and (ii) between 30 and 54.

(M.U. 2016)

Sol. : We have S.N.V. $Z = \dfrac{X - m}{\sigma}$

By data, $m = 42$ and $\sigma = 24$.

$\therefore \quad Z = \dfrac{X - 42}{24}$

(i) When $X = 50$, $Z = \dfrac{50 - 42}{24} = \dfrac{1}{3} = 0{\cdot}33$

$P(Z \geq 50) = \text{area to the right of } 0{\cdot}33$

$= 0{\cdot}5 - (\text{area between } Z = 0 \text{ and } Z = 0{\cdot}33)$

$= 0{\cdot}5 - 0{\cdot}1293 = 0{\cdot}3707$

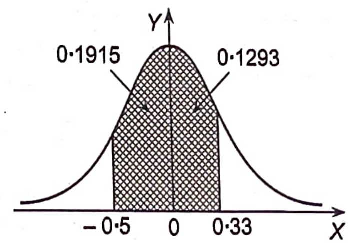

Fig. 4.13

(ii) When $X = 30$ and $X = 54$, we get

$Z = \dfrac{30 - 42}{24} = -0{\cdot}5$ and $Z = \dfrac{54 - 42}{24} = 0{\cdot}5$

$P(30 \leq Z \leq 54) = \text{area between } Z = -0{\cdot}5 \text{ to } Z = 0{\cdot}5$

$= 2\ (\text{area between } Z = 0 \text{ and } Z = 0{\cdot}5)$

$= 2\ (0{\cdot}1915) = 0{\cdot}3830$

Number of students getting more than 50 marks

$= Np = 1000 \times 0{\cdot}3707 = 371$

Number of students getting marks between 30 and 54

$= Np = 1000 \times 0{\cdot}383 = 383$

Type IV

Example 1 : If X_1 and X_2 are two independent random variates with means 30 and 25 and variances 16 and 12 and if $Y = 3X_1 - 2X_2$, find $P(60 \leq Y \leq 80)$. **(M.U. 2005)**

Sol. : Since X_1, X_2 are independent normal variates with means 30 and 25 and variances 16 and 12, $Y = 3X_1 - 2X_2$ is also a normal variate with mean

$$m = a_1 \overline{X}_1 + a_2 \overline{X}_2 = 3\,(30) + (-2)\,(25) = 90 - 50 = 40.$$

and variance $\sigma^2 = a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 = 9\,(16) + 4\,(12) = 192.$

S.N.V. $Z = \dfrac{Y - m}{\sigma} = \dfrac{Y - 40}{\sqrt{192}}$

When $Y = 60$, $Z = \dfrac{20}{\sqrt{192}} = 1{\cdot}44$

When $Y = 80$, $Z = \dfrac{40}{\sqrt{192}} = 2{\cdot}89$

$\therefore$ $P(60 \le Y \le 80) = P(1{\cdot}44 \le Z \le 2{\cdot}89)$

= area between $Z = 1{\cdot}44$ and $Z = 2{\cdot}89$

= (area from $Z = 0$ to $Z = 2{\cdot}89$) – (are from $Z = 0$ to $Z = 1{\cdot}44$)

$= 0{\cdot}4981 - 0.4251 = 0{\cdot}0730.$

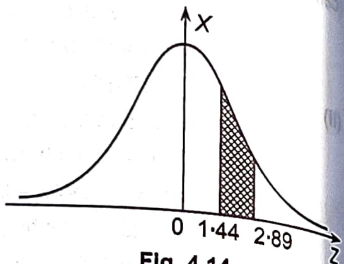

Fig. 4.14

Example 2 : Two independent random variates X and Y are distributed normally with mean and standard deviations (52, 3) and (50, 2) respectively. Find the probability that a randomly chosen pair of values of X and Y will differ by 1·7 or more.

Sol. : If $U = X - Y$ then by the additive property of normal variates, U is a normal variate with mean $= 52 - 50 = 2$ and standard deviation $\sqrt{9+4} = \sqrt{13}$ *i.e.* $N(2, \sqrt{13})$. [See remark (4), page 4-24]

$\therefore$ $Z = \dfrac{U - m}{\sigma} = \dfrac{U - 2}{\sqrt{13}}$ is a S.N.V.

Now, P (X and Y will differ by 1.7 or more)

$= P(|X - Y| \ge 1{\cdot}7)$

$= P(|U| \ge 1{\cdot}7)$

$= 1 - P(|U| \le 1{\cdot}7)$

$= 1 - P(-1{\cdot}7 \le U \le 1{\cdot}7)$

Now, when $U = -1{\cdot}7$, $Z = \dfrac{-1.7 - 2}{\sqrt{13}} = -1.03$

and when $U = 1{\cdot}7$, $Z = \dfrac{1.7 - 2}{\sqrt{13}} = -0.08$

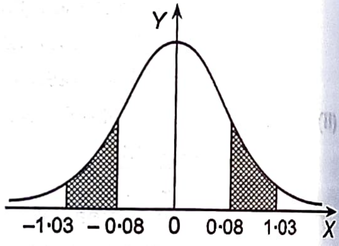

Fig. 4.15

$\therefore$ P (X and Y will differ by 1·7 or more)

$= 1 - P(-1{\cdot}03 \le Z \le -0{\cdot}08)$

= 1 – (area from $Z = 0{\cdot}08$ to $Z = 1{\cdot}03$)

= 1 – [(area from $Z = 0$ to $Z = 1{\cdot}03$) – (area from $Z = 0$ to $Z = 0{\cdot}08$)]

$= 1 - (0{\cdot}3485 - 0{\cdot}0319) = 1 - 0{\cdot}3766$

$= 0{\cdot}6234$

Example 3 : If X and Y are two independent random variates $N(3, 4)$ and $N(8, 5)$, find the probability that a point (X, Y) will lie between the lines $5X + 3Y = 8$ and $5X + 3Y = 15$.

Sol. : Since X is $N(3, 4)$ and Y is $N(8, 5)$ by additive property of normal distribution $U = 5X + 3Y$ follows a normal distribution with mean

$m = 5 \times 3 + 3 \times 8 = 39$ and $\sigma = \sqrt{25 \times 16 + 9 \times 25} = 25.$

P (the point (X, Y) lies between the lines $5X + 3Y = 8$ and $5X + 3Y = 15$) $= P(8 \le U \le 15)$)

Now, $Z = \dfrac{U - 39}{25}$ is a S.N.V.

When $U = 8$, $Z = \dfrac{8 - 39}{25} = -1{\cdot}24$

and when $U = 15$, $Z = \dfrac{15 - 39}{25} = -0{\cdot}96$

$\therefore$ P (the point lies between the two lines)

= area between $Z = -0{\cdot}96$ and $Z = -1{\cdot}24$

= area between $Z = 0{\cdot}96$ and $Z = 1{\cdot}24$

= (area from $z = 0$ to $z = 1{\cdot}.24$) – (area from $z = 0$ to $z = 0{\cdot}96$)

$= 0{\cdot}3925 - 0{\cdot}3315$

$= 0{\cdot}061$

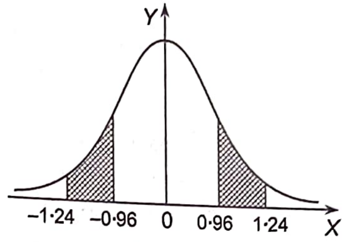

Fig. 4.16

Example 4 : If X and Y are two independent normal random variates such that their means are 8, 12 and standard deviations are 2 and $4\sqrt{3}$ respectively, find the value α such that $P[(2X - Y) \le 2\alpha] = P[(X + 2Y) \ge 3\alpha]$.

Sol. : By additive property of normal distribution $U = 2X - Y$ is a normal variate with

mean, $m = 2 \times 8 - 1 \times 12 = 4$

and standard deviation, $\sigma = \sqrt{2^2 \times 2^2 + 1^2 \times (4\sqrt{3})^2} = \sqrt{16 + 48} = 8$

$\therefore$ U is a S.N.V. with mean 4 and S.D. = 8.

$V = X + 2Y$ is a normal variate with mean $m = 1 \times 8 + 2 \times 12 = 32$ and standard deviation

$\sigma = \sqrt{1^2 \times 2^2 + 2^2 \times (4\sqrt{3})^2} = \sqrt{4 + 192} = 14$

$\therefore$ V is a S.N.V. with mean 32 and S.D. = 14.

Now, $P[(2X - Y) \le 2\alpha] = P[(X + 2Y) \ge 3\alpha]$

$\therefore$ $P(U \le 2\alpha) = P(V \ge 3\alpha)$

$\therefore$ $P\left(\dfrac{U - 4}{8} \le \dfrac{2\alpha - 4}{8}\right) = P\left(\dfrac{V - 32}{14} \ge \dfrac{3\alpha - 32}{14}\right)$

This means if Z is a S.N.V.

$P\left(Z \le \dfrac{2\alpha - 4}{8}\right) = P\left(Z \ge \dfrac{3\alpha - 32}{14}\right)$

By symmetry of normal distribution if $P(Z \ge z_1) = \alpha$ then $P(Z \le -z_1) = \alpha$.

$\therefore$ $P\left(Z \le \dfrac{2\alpha - 4}{8}\right) = P\left(Z \le -\left(\dfrac{3\alpha - 32}{14}\right)\right)$

$\therefore$ $\dfrac{2\alpha - 4}{8} = -\dfrac{(3\alpha - 32)}{14}$

$\therefore$ $\dfrac{\alpha - 2}{4} = -\dfrac{3\alpha - 32}{14}$

$\therefore$ $14a - 28 = -12a + 128$

$\therefore$ $26a = 156$

$\therefore$ $\alpha = 6.$

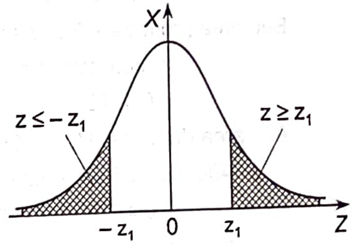

Fig. 4.17

Example 5 : In an examination marks obtained by students in Mathematics, Physics and Chemistry are normally distributed with means 51, 53 and 46 with standard deviation 15, 12, 16 respectively. Find the probability of securing total marks (i) 180 or above, (ii) 80 or below.

(M.U. 2005, 10)

Sol. : Let X_1, X_2, X_3 denote the marks obtained in the three subjects. Then X_1, X_2, X_3 are normal variates with mean 51, 53, 46 and variance $15^2, 12^2, 16^2$.

Assuming the variates to be independent, $Y = X_1 + X_2 + X_3$ is distributed normally with mean $m = 51 + 53 + 46 = 150$ and $\sigma^2 = 15^2 + 12^2 + 16^2 = 625 = 25^2$.

$\therefore$ S.N.V. $Z = \dfrac{Y - m}{\sigma} = \dfrac{Y - 150}{25}$

When $Y = 180$, $Z = \dfrac{180 - 150}{25} = \dfrac{30}{25} = 1{\cdot}2$.

$\therefore$ $P(Y \geq 150) = P(Z \geq 1{\cdot}2)$

= Area to the right of $Z = 1{\cdot}2$

= $0{\cdot}5$ – (area between $Z = 0$ and $Z = 1{\cdot}2$)

= $0{\cdot}5 - 0{\cdot}3849 = 0{\cdot}1151$.

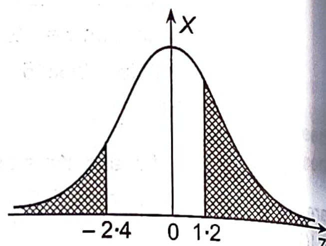

Fig. 4.18

When $Y = 80$, $Z = \dfrac{80 - 150}{25} = -\dfrac{70}{25} = -2{\cdot}4$

$\therefore$ $P(Y \leq 80) = P(Z \leq -2{\cdot}4)$

= Area to the left of $Z = -2{\cdot}4$

= $0{\cdot}5$ – area from $Z = 0$ to $Z = 2{\cdot}4$

= $0{\cdot}5 - 0{\cdot}4918 = 0{\cdot}0082$.

Type V

Example 1 : The incomes of a group of 10,000 persons were found to be normally distributed with mean ₹ 520 and standard deviation ₹ 60. Find (i) the number of persons having incomes between ₹ 400 and 550, (ii) the lowest income of the richest 500.

Sol. : S.N.V. $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 520}{60}$

(i) When $X = 400$, $Z = \dfrac{400 - 520}{60} = -2$.

When $X = 550$, $Z = \dfrac{550 - 520}{60} = 0{\cdot}5$

$\therefore$ $P(400 \leq X \leq 550)$ = Area (from $z = -2$ to $z = 0{\cdot}5$)

But, area (from $z = -2$ to $z = 0$)

= Area (from $z = 0$ to $z = 2$)

= $0{\cdot}4772$.

And area (from $z = 0$ to $z = 0{\cdot}5$) = $0{\cdot}1915$.

$\therefore$ $P(400 \leq x \leq 550)$ = Area (from $z = -2$ to $z = 0{\cdot}5$)

= $0{\cdot}4772 + 0{\cdot}1915 = 0{\cdot}6687$.

$\therefore$ The number of persons whose incomes are between ₹ 400 and ₹ 550 = $Np = 10{,}000 \times 0{\cdot}6687 = 6687$

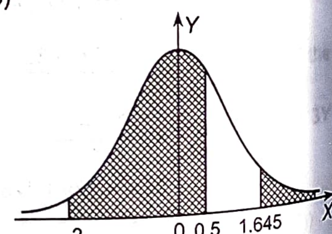

Fig. 4.19

(ii) If we have to consider the richest 500 persons then the probability that a person selected at random will be one of them

$= \dfrac{500}{10{,}000} = 0{\cdot}05$

This is a reverse problem. So far we have found the probability for a given value of Z. Here, we have to find the value of Z for a given probability. We have to find the value of Z to the right of which the area is 0·05. But area to the right of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) = $0{\cdot}5 - 0{\cdot}05 = 0{\cdot}45$

From the table we find that the area from $Z = 0$ to $Z = 1{\cdot}645$ is $0{\cdot}45$

$\therefore$ The required value of $Z = 1{\cdot}645$

But $Z = \dfrac{X - 520}{60}$ $\quad \therefore$ $1{\cdot}645 = \dfrac{X - 520}{60}$

$\therefore$ $X - 520 = 60 \times 1{\cdot}645$ $\quad \therefore$ $X = 520 + 98{\cdot}7 = ₹\ 618{\cdot}7$.

$\therefore$ The lowest income of the richest 500 is ₹ 618·7.

Example 2 : The income of a group of 10,000 persons was found to be normally distributed with mean of ₹ 750 and standard deviation of ₹ 50. What is the lowest income of richest 250 ?

Sol. : S.N.V. $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 750}{50}$.

If we have to consider the richest 250 persons then the probability that a person selected at random will be one of them is $= \dfrac{250}{10{,}000} = 0{\cdot}025$.

This again is a reverse problem. So far we have found the probability for a given value of Z. Here, we have to find the value of Z for a given probability. We have to find the value of Z to the right of which the area is 0·025. But the area to the right of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) = $0{\cdot}5 - 0{\cdot}025 = 0{\cdot}475$.

From the table we find that the area from $Z = 0$ to $Z = 1{\cdot}96$ is $0{\cdot}475$.

$\therefore$ The required value of $Z = 1{\cdot}96$

When $Z = 1{\cdot}96$, $1{\cdot}96 = \dfrac{X - 750}{50}$

$\therefore$ $X - 750 = 1{\cdot}96 \times 50$ $\quad \therefore$ $X = 848$

$\therefore$ The lowest income of richest 250 persons = ₹ 848.

Example 3 : In a competitive examination the top 15% of the students appeared will get grade A, while the bottom 20% will be declared fail. If the grades are normally distributed with mean % of marks 75 and S.D. 10, determine the lowest % of marks to receive grade A and the lowest % of marks that passes.

(M.U. 2014)

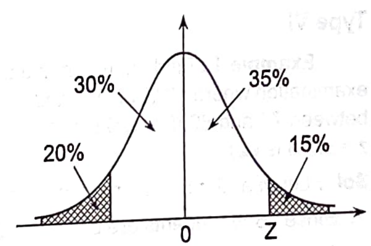

Fig. 4.20

Sol. : This is a reverse problem as above.

We have $Z = \dfrac{X - m}{\sigma} = \dfrac{X - 75}{10}$.

(i) Grade A is given for 15%. We have to find the value of Z to the right of which the area is 0·15. But the area to the right of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) = 0·5 − 0·15 = 0·35.

From the table, we find that the area between $Z = 0$ to $Z = 1{\cdot}04$ is 0·35.

$\therefore$ The required value of $Z = 1{\cdot}04$.

But $Z = \dfrac{X - 75}{10}$ $\quad \therefore \; 1{\cdot}04 = \dfrac{X - 75}{10}$ $\quad \therefore \; X = 75 + 10{\cdot}4 = 85{\cdot}4$

(ii) Lowest 20% students are declared fail. We have to find the value of Z to the left of which the area is 0·20. But the area to the left of $Z = 0$ is 0·5.

$\therefore$ Area from ($Z = 0$ to Z = this value) = 0·5 − 0·2 = 0·3.

From the table we find that the area between $Z = 0$ and $Z = 0{\cdot}84$ is 0·3.

But this ordinate is on the left and hence negative.

$\therefore$ The required value of $Z = -0{\cdot}84$.

But $Z = \dfrac{X - 75}{10}$ $\quad \therefore \; -0{\cdot}84 = \dfrac{X - 75}{10}$ $\quad \therefore \; X = 75 - 8{\cdot}4 = 66{\cdot}6$.

Example 4 : If the actual amount of coffee which a filling machine puts into 6 ounce jars is a random variable having normal distribution with standard deviation 0·05 ounce and if only 3% of the jars are to contain less than 6 ounce of coffee what must be the mean fill of these jars ?

(M.U. 2004, 07)

Sol. : Let $Z = \dfrac{X - m}{\sigma}$

We have $\sigma = 0{\cdot}05$, $X = 6$ $\quad \therefore \; Z = \dfrac{6 - m}{0{\cdot}05}$.

We want Z such that

$$P(Z \le 3) = P\left(\frac{6 - m}{0{\cdot}05}\right) = 0{\cdot}03$$

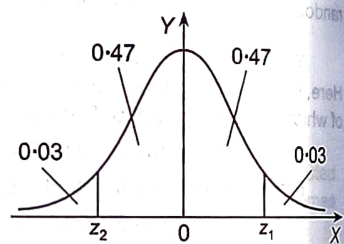

Fig. 4.21

From the table for area to be 0·47

$z_1 = 1{\cdot}808$ $\quad \therefore \; z_2 = -1{\cdot}808$

$\therefore \; \dfrac{6 - m}{0{\cdot}05} = -1{\cdot}808$

$\therefore \; m = 6 + 0{\cdot}05 \times 1{\cdot}808 = 6{\cdot}09$ ounce.

Type VI

Example 1 : Find the mean and the standard deviation of a normal distribution of marks in an examination where 58 % of the candidates obtained marks below 75, 4 % got above 80 and the rest between 75 and 80 (For a S.N.V. the area under the curve between $z = \pm 0{\cdot}2$ is 0·16 and between $z = \pm 1{\cdot}8$ is 92).

(M.U. 1999)

Sol. : Let m and σ be the mean and the standard deviation of the variate.

Since 58 % students are below 75, 58 − 50 = 8 % students are between 75 and m.

Since 4 % students are above 80, 50 − 4 = 46 % students are between m and 80.

We are given that area between $Z = \pm 0{\cdot}2$ is 0·16 and that between $Z = \pm 1{\cdot}8$ is 9·92.

Hence, the area between $Z = 0$ and $Z = 0{\cdot}2$ is $\dfrac{0{\cdot}16}{2} = 0{\cdot}8$ and that between $Z = 0$ and $Z = +1{\cdot}8$ is $\dfrac{0{\cdot}92}{2} = 0{\cdot}46$.

In other words for area 0·08 (8 %), $Z = 0{\cdot}2$ and for area 0·46 (46 %), $Z = 1{\cdot}8$.

$\therefore \; \dfrac{75 - m}{\sigma} = 0{\cdot}2$ and $\dfrac{80 - m}{\sigma} = 1{\cdot}8$

$\therefore \; 75 - m = 0{\cdot}2\,\sigma$ and $80 - m = 1{\cdot}8\,\sigma$

Subtracting $-5 = -1{\cdot}6\,\sigma$ $\;\therefore\; \sigma = \dfrac{5}{1{\cdot}6} = 3{\cdot}125$

$\therefore \; m = 75 - 0{\cdot}2\,\sigma = 75 - 3{\cdot}125 \times 0{\cdot}2 = 74{\cdot}4$ mark.

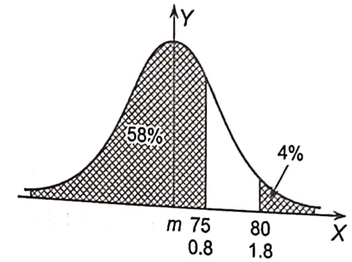

Fig. 4.22

Example 2 : Marks obtained by students in an examination follow normal distribution. If 30% of students got below 35 marks and 10% got above 60 marks, find the mean and standard deviation.

(M.U. 2016, 18)

Sol. : Let m and σ be the mean and the standard deviation of the distribution.

Since 30% students are below 35, 20% students are between 35 and m.

Since 10% students are above 60, 40% students are between m and 60.

From the table we find that,

0·2 area corresponds to $Z = 0{\cdot}525$

and 0·4 area corresponds to $Z = 1{\cdot}283$

But 0·2 area is to the left of m hence $Z = -0{\cdot}525$.

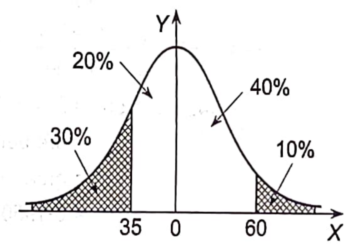

Fig. 4.23

$\therefore \; \dfrac{35 - m}{\sigma} = -0{\cdot}525$; $\quad \dfrac{60 - m}{\sigma} = 1{\cdot}283$

$\therefore \; 35 - m = -0{\cdot}525\,\sigma$ .... (1)

$\therefore \; 60 - m = 1{\cdot}283\,\sigma$

By subtraction, we get $\;25 = 1{\cdot}808\,\sigma$ $\;\therefore\; \sigma = \dfrac{25}{1{\cdot}808} = 13{\cdot}83$.

Putting this value of σ in (1), we get

$35 - m = -0{\cdot}525\,(13{\cdot}83)$

$\therefore \; m = 35 + 0{\cdot}525\,(13{\cdot}83) = 42{\cdot}26$

Hence, mean = 13·83 and standard deviation, $\sigma = 42{\cdot}26$.

Example 3 : In a distribution exactly normal 7 % of items are under 35 and 89 % are under 63. What are the mean and standard deviation ?

(M.U. 2004)

Sol. : Since 7 % items are below 35, 50 − 7 = 43 % items are between 35 and m, and since 89 % items are below 63, 89 − 50 = 39 % items are between m, and 63.

For area 0·43, $Z = 1{\cdot}48$.

Since $35 < m$, $Z = -1{\cdot}48$ and for area 0·39, $Z = 1{\cdot}23$.

$$\therefore \quad \frac{35-m}{\sigma} = -1.48 \quad \text{and} \quad \frac{63-m}{\sigma} = 1.23$$

$\therefore$ $35 - m = -1.48\,\sigma$

and $63 - m = 1.23\,\sigma$.

Subtracting $28 = 2.71\sigma$

$$\therefore \quad \sigma = \frac{28}{2.71} = 10.33$$

$$\therefore \quad = 35 + 1.48\sigma = 35 + 1.48 \times 10.33$$

$$= 35 + 15.3 = 50.3.$$

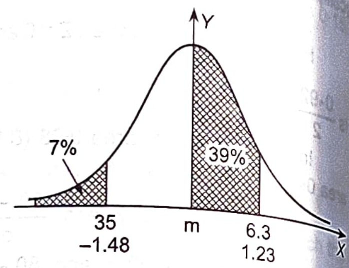

Fig. 4.24

Example 4 : In a distribution exactly normal 7% of items are under 35 and 89 % of the items are under 63. Find the probability that an item selected at random lies between 45 and 56.

(M.U. 2011, 15)

Sol. : As in the above example $m = 50.3$ and $\sigma = 10.33$.

$$\text{Now,} \quad Z = \frac{X-m}{\sigma} = \frac{X-50.3}{10.33}$$

$$\text{When } X = 45, \quad Z = \frac{45-50.3}{10.33} = -0.51$$

$$\text{When } X = 56, \quad Z = \frac{56-50.3}{10.33} = 0.55$$

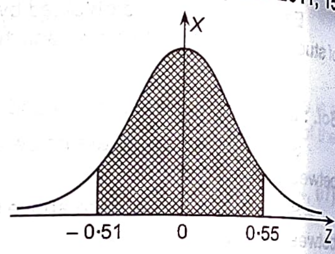

Fig. 4.25

$\therefore$ $P(45 \le X \le 56) = P(-0.51 \le Z \le 0.55)$

= are between ($Z = -0.51$ to $Z = 0.55$)

= area from 0 to 0.51 + area from 0 to 0.55

= 0.1950 + 0.2088 = 0.4038

Example 5 : A large number of automobile batteries have average life of 24 months. If 34 percent of them average between 22 and 26 months and 272 of them last longer than 29 months how many were in the group tested ? Assume the distribution to be normal. (For a normal curve 17 % and 36.4 % values lie between the mean and respective distance of 0.44 and 1.1 times standard deviation from the mean).

Sol. : Because of symmetry of the normal distribution and because $m = 24$, the area between 22 and 24 is equal to that between 24 and 26. This area is equal to 34/2 = 17% which corresponds to S.N.V. $Z = 0.44$ by data and $X = 26$.

$$\therefore \quad \frac{26-24}{\sigma} = 0.44 \qquad \therefore \quad \sigma = \frac{2}{0.44} = \frac{50}{11}$$

$$\text{Now when } X = 29, \quad Z = \frac{X-m}{\sigma} = \frac{29-24}{50/11} = \frac{11}{10} = 1.1.$$

By data area between $Z = 0$ and $Z = 1.1$ is 0.364.

$\therefore$ The area to the right of $Z = 1.1$ *i.e.* $X = 29$ is $= 0.5 - 0.364 = 0.136$ which is the probability that a battery will last longer than 29 months.

$$\text{But } p = \frac{f}{N} \text{ and } f = 272,\ p = 0.136. \qquad \therefore \quad N = \frac{f}{p} = \frac{272}{0.136} = 2000$$

Hence, 2000 batteries were tested.

Type VII

Example 1 : The probability that an electronic component will fail in less 1200 hours of continuous use is 0.25. Using normal approximation to Binomial distribution, find the probability that among 200 such components fewer than 45 will fail in less than 1200 hours of continuous use.

(M.U. 2005)

Sol. : While using continuous variate in place of discrete variate we must "spread" its values over a continuous scale. This we do by taking each integer k to represent the interval $k - (1/2)$ to $k + (1/2)$.

Now, we have $n = 200$, $p = 0.25$, $q = 0.75$.

$\therefore$ $m = np = 200 \times 0.25 = 50$

$$\sigma = \sqrt{npq} = \sqrt{200 \times 0.25 \times 0.75} = 6.12$$

$$Z = \frac{X-np}{\sqrt{npq}} = \frac{X-50}{6.12}$$

When $X = 44.5$ (*i.e.* $k - (1/2)$)

$$Z = \frac{44.5-50}{6.12} = -0.8986 = -0.9$$

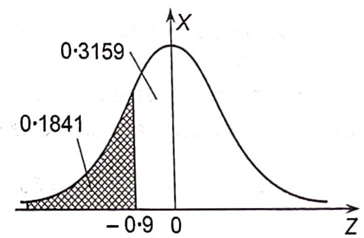

Fig. 4.26

For $Z = 0.9$, $p = 0.3159$.

$\therefore$ $P(X \le 44.5) = P(Z \le -0.9)$

$= 0.5 - 0.3159 = 0.1841.$

Example 2 : Determine in two different ways, the probability that by guess-work a student can correctly answer 25 to 30 questions in a multiple choice quiz consisting of 80 questions. Assume that in each question with four choices only one is correct and the student has no knowledge.

(M.U. 2004)

Sol. : (a) By using Normal approximation to Binomial Distribution

Mean $= m = np = 80 \times (1/4) = 20$

$$\text{S.D., } \sigma = \sqrt{npq} = \sqrt{80 \times \frac{1}{4} \times \frac{3}{4}} = 3.873$$

$$\therefore \quad Z = \frac{X-m}{\sigma} = \frac{X-20}{3.873}$$

Since from discrete (Binomial Distribution) we are approximating to continuous (Normal distribution) we extend the range 25 to 30 by half units on either side *i.e.* we take the range as 24.5 to 30.5.

$$\text{When } X = 24.5, \quad Z = \frac{24.5-20}{3.873} = 1.16. \qquad \text{When } X = 30.5, \quad Z = \frac{30.5-20}{3.873} = 2.71$$

$\therefore$ $P(24.5 \le X \le 30.5) = P(1.16 \le Z \le 2.71)$

= (area from $Z = 0$ to $Z = 2.71$) − (area from $Z = 0$ to $Z = 1.16$)

= 0.4966 − 0.6770

= 0.1196

(b) By using Binomial Distribution

$$\text{We have } p = \frac{1}{4},\ q = \frac{1}{4},\ n = 80$$

$\therefore\ P(X=x) = {}^nC_x\, p^x q^{n-x} = {}^{80}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{80-x}$

$\therefore$ The required probability $= \sum_{x=25}^{30} {}^{80}C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{80-x}$

$= 0{\cdot}0434 + 0{\cdot}0306 + 0{\cdot}0204 + 0{\cdot}0129 + 0{\cdot}0077 + 0{\cdot}0043$

$= 0{\cdot}1193$

(**Note** that the two values differ only by 0.0003.)

Example 3 : Using normal distribution, find the probability of getting 55 heads in the toss of 100 fair coins. (M.U. 2004)

(Compare the result with that obtained from Binomial distribution).

Sol. : Since the coins are fair, we have $p = 1/2$, $q = 1/2$. By data $n = 100$.

$\therefore\ m = np = 100 \times \frac{1}{2} = 50, \quad \sigma = \sqrt{npq} = \sqrt{100 \times \frac{1}{2} \times \frac{1}{2}} = 5$

Hence, we have S.N.V. $Z = \frac{X-m}{\sigma} = \frac{X-50}{5}$

When $X = 54{\cdot}5$, $\quad Z = \frac{54{\cdot}5 - 50}{5} = 0{\cdot}9$.

When $X = 55{\cdot}5$, $\quad Z = \frac{55{\cdot}5 - 50}{5} = 1{\cdot}1$

$\therefore\ P(0{\cdot}9 < Z < 1{\cdot}1)$ = area from $Z = 0{\cdot}9$ to $Z = 1{\cdot}1$

= (area from $Z = 0$ to $Z = 1{\cdot}1$) – (area from $Z = 0$ to $0{\cdot}9$)

$= 0{\cdot}3643 - 0{\cdot}3159 = 0{\cdot}0484$

Now, for the second part, we have

$$P(X=x) = {}^nC_x\, p^x q^{n-x} = {}^{100}C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{100-x}$$

$$\therefore\ P(X=55) = {}^{100}C_{55} \left(\frac{1}{2}\right)^{55} \left(\frac{1}{2}\right)^{45} = 0.04847$$

EXERCISE - II

Type I

1. Find k and the mean and standard deviation of the normal distribution given by

(i) $y = k\,e^{-\left(\frac{x^2}{18} - x + \frac{9}{2}\right)}$ [**Ans. :** $k = \frac{1}{3\sqrt{2\pi}}$, $m = 9$, $\sigma = 3$]

(ii) $y = k\,e^{-\left(\frac{x^2}{6} - x + \frac{3}{2}\right)}$ [**Ans. :** $k = \frac{1}{\sqrt{6 \cdot \pi}}$, $m = 3$, $\sigma = \sqrt{3}$]

2. Write down the equation of the curve of the normal distribution with mean 50 and standard deviation 9. What is the first quartile of the distribution? [**Ans. :** 44]

3. Write down the equation of the normal curve with mean 10 and variance 36. What is the quartile deviation of the distribution ? [**Ans. :** 4]

4. What is the Q.D. of a normal distribution with S.D. 9 ? [**Ans. :** 6]

Type II

1. If X is normally distributed with mean 10 and standard deviation 2, find $P(-3 \le X \le 12)$. [**Ans. :** 0·8413]

2. If X is normally distributed with mean 15 and standard deviation 6, find $P(-3 \le X \le 18)$ and $P(|X| \ge 16{\cdot}02)$. [**Ans. :** 0·6902; 0·5675]

3. If X is normally distributed with mean and standard deviation 4, find (i) $P(5 \le X \le 10)$, (ii) $P(X \ge 15)$, (iii) $P(10 \le X \le 15)$, (iv) $P(X \le 5)$. (M.U. 2003) [**Ans. :** (i) 0·3326, (ii) 0·003, (iii) 0·1557, (iv) 0·05987]

4. A normal distribution has mean 5 and standard deviation 3. What is the probability that the deviation from the mean of an item taken at random will be negative ? (M.U. 2004) [**Ans. :** 0·0575]

Type III

1. If Z is a S.N.V., find c such that (i) $P(-c < Z < c) = 0{\cdot}98$, (ii) $P(|Z| > c) = 0{\cdot}04$. [**Ans. :** (i) $c = 2{\cdot}33$, (ii) $c = 2{\cdot}05$]

2. If X is a normal variate with mean 30 and standard deviation 6, find the value of $X = x_1$ such that $P(X \ge x_1) = 0{\cdot}05$. [**Ans. :** $z_1 = 1.64$, $x_1 = 39{\cdot}84$]

3. If X is a normal variate with mean 25 and standard deviation 5, find the value of $X = x_1$ such that $P(X \le x_1) = 0{\cdot}01$. [**Ans. :** $z_1 = -2{\cdot}33$, $x_1 = 13{\cdot}35$]

Type IV

1. The first and third quartiles of a normal distribution are respectively 92 and 128. Find the mean and the standard deviation. [**Ans. :** 110, 27]

2. For a normal distribution the first quartile is 46 and the variance is 144. Find the (i) the mode, (ii) limits of central 50% items, (iii) mean deviation. [**Ans. :** (i) 54, (ii) 46; 62, (iii) 9·6]

3. The mean and the standard deviation of a normal distribution are 70 and 15. Find the quartile deviation and mean deviation. [**Ans. :** (i) 10, (ii) 12]

Type V

1. The weights of 4000 students are found to be normally distributed with mean 50 kgs. and standard deviation 5 kgs. Find the probability that a student selected at random will have weight (i) less than 45 kgs., (ii) between 45 and 60 kgs. [**Ans. :** (i) 0·1587, (ii) 0·8185]

2. The sizes of 10,000 items are normally distributed with mean 20 cms and standard deviation 4 cm. Find the probability that an item selected at random will have size between (i) 18 cms and 23 cms, (ii) above 26 cms. [**Ans. :** (i) 0·4649, (ii) 0·0668]

3. The daily sales of a firm are normally distributed with mean ₹ 8000 and variance of ₹ 10,000. (i) What is the probability that on a certain day the sales will be less than ₹ 8210 ? (ii) What is % of days on which the sales will be between ₹ 8100 and ₹ 8200 ? (M.U. 1999, 2001) [**Ans. :** (i) 0·5832, (ii) 14%]

Type VI

1. Mean and standard deviation of chest measurements of 1200 soldiers are 85 cms and 5 cms respectively. How many of them are expected to have their chest measurements exceeding 95 cms. assuming the measurements to follow the normal distribution ? (Area for S.N.V. z from z = 0 to z = 2 is 0·4772) [**Ans. :** 27]