Normal Modes of Mechanical Systems
APPM 3310 - Spring 2021

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Contents

1 Introduction

1.1 Abstract

In our write-up, we discuss the dynamics of 1-dimensional mass-spring chains. We use techniques from linear algebra to introduce the concept of *normal modes* and use this to gain insight into the behaviour of our systems. We discuss short and long mass-spring chains, and through this we introduce the concept of waves and talk about how they relate to musical instruments, and finally, we end with a discussion of periodic structures and how they lead to the formation of *bandgaps.*

1.2 Attribution

Sanjay worked on sections 1.4, 2.2, and 2.4, Matthew worked on section 2.1, Helen worked on section 2.3, and everyone worked on the remaining sections of the write-up.

1.3 Background

We will investigate the *normal modes* of mechanical systems. When studying the dynamics of a single-particle system, it is often straightforward to write down the equation of motion using Newton's second law. Then, this differential equation can be solved relatively easily, and, given the initial conditions, the behaviour of the particle can be determined at all future times. A simple example of this is a mass attached to a spring while oscillating in simple harmonic motion.

Multi-particle systems are far more complicated because the motion of a given particle is affected by what all the other particles are doing at any given instant. In other words, the system of differential equations arising from Newton's second law are often coupled, making them much harder to solve. Thankfully, in certain situations, such systems of differential equations can be solved efficiently using techniques from linear algebra. In such problems, the system of differential equations resulting from Newton's second law can be rewritten as an eigenvalue problem that can then be solved relatively easily. Further, this reformulation allows us to introduce the concept of normal modes. Normal mode solutions involve every particle of the system oscillating with the same frequency and phase. Once the normal mode solutions of a certain mechanical system are found, a linear combination can be used to express any possible motion of the system. In other words, the normal mode solutions can be stitched together to obtain the general solution.

In our write-up, we begin by studying the dynamics of a 1-dimensional mass-spring chain, and we find the general solution describing the motion of the chain in a rigorous manner using linear algebra techniques. In the process, we introduce the concept of *normal modes*. Then, we move to look at numerical examples of short mass-springs chains and try to understand their motion given different initial conditions. After this, we study long mass-chains and discuss how the normal modes of continuous systems arise. Next, we extend our discussion of continuous systems and discuss the normal modes that arise in musical instruments. This introduces the concept of waves. Finally, we discuss how periodic structures can lead to the formation of *bandgaps* and end with some interesting real-world applications.

1.4 Mathematical Formulation

We will begin our analysis by studying the dynamics of a linear, 1-dimensional, mass-spring chain comprised of N masses and N+1 connecting springs. Our ultimate goal is to re-formulate the differential equations that describe this physical system as an eigenvalue problem. This will not only allow us to find the general solution with ease, but it will also provide an opportunity to introduce the concept of a *normal mode*. This can then be used to gain valuable insight into the way coupled multi-particle systems behave. The overall setup of the system we will investigate is shown in figure (1). To start off, we will assume that the motion of each mass is parallel to the direction of the chain and therefore restricted to one dimension. Further, we will also assume that all the springs are linear and obey Hooke's law. To remind the reader, Hooke's law states that the restoring force provided by a spring is directly proportional to its extension from equilibrium. Finally, we will assume that the ends of the mass-spring chain are fixed in place. To keep things as general as possible, we will consider the case where the *ith* has mass $m_{(i)}$ and the spring connecting the *ith* mass and the *jth* mass has spring constant $k_{(i),(j)}$. It should be noted that $k_{(i),(i)} = 0$ (because a mass is not connected to itself via a spring), and $k_{(i),(j)} = 0$ when $j - i$ is not equal to +1 or -1 (because only neighbouring masses in the chain are connected to each other by a spring). Finally, we should also clarify that $k_{(i),(j)} = k_{(j),(i)}$ since they represent the same spring. When every spring is neither stretched nor compressed, and every mass is at rest, then the system is in *static equilibrium* and every mass is in its equilibrium position. We can denote the displacement of each mass from its equilibrium position as $x_{(i)}$ and take the convention that the positive direction points to the right. The details are summarized in figure (1).

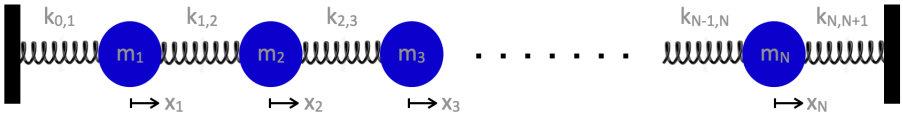


Figure 1: One Dimensional Mass-Spring Chain

We can now proceed to use Newton's second law to write down the differential equations governing the physical system at hand. Newton's second law states that the net force on a body is equal to the product of its mass and its acceleration (provided the mass of the body is constant and this is the case in our problem). This leads to the following equation:

$$F_{(i)} = m_{(i)}\ddot{x}_{(i)} \tag{1}$$

Since $F_{(i)}$ is the net force on $m_{(i)}$, we can apply Hooke's law to get the following relations:

$$F_{(i)} = (k_{(i),(i+1)} \cdot (x_{(i+1)} - x_{(i)})) - (k_{(i),(i-1)} \cdot (x_{(i)} - x_{(i-1)})) \tag{2}$$

$$m_{(i)}\ddot{x}_{(i)} = (k_{(i),(i+1)} \cdot (x_{(i+1)} - x_{(i)})) - (k_{(i),(i-1)} \cdot (x_{(i)} - x_{(i-1)})) \tag{3}$$

Now, equation (3) defines a system of N coupled linear homogeneous differential equations which we wish to solve. Due to the boundary conditions which force the endpoints of the mass-spring chain to be fixed in position, we notice that following property holds at all times:

$$x_{(0)} = x_{(N+1)} = 0 \tag{4}$$

To clarify, there are only N masses in our chain, so $x_{(0)}$ and $x_{(N+1)}$ do not actually refer to the displacement of any physical masses. They are simply being used to label the displacement of the end-points of our mass-spring chain. As equation (4) implies, the end points never move.We have now completely defined the system of differential equations which govern the dynamics of our mass-spring chain. It should be noted that, in general, coupled differential equations can be extremely challenging to solve analytically. However, in our situation, the differential equations are both linear and homogeneous. Furthermore, they are coupled in a way that is somewhat unique, thereby allowing us to find the general solution in a fairly elegant manner.

First, we can try to re-write the system of differential equations in matrix form. We begin by defining the column vector $\vec{x}$ whose *ith* entry is simply the displacement of $m_{(i)}$ from its equilibrium position (this is just $x_{(i)}$). This can be described by equation (5).

$$\vec{x} = (x_{(1)}, x_{(2)}, x_{(3)} \ldots . x_{(N-1)}, x_{(N)})^T \tag{5}$$

By taking two time derivatives, this implies equation (6).

$$\ddot{\vec{x}} = (\ddot{x}_{(1)}, \ddot{x}_{(2)}, \ddot{x}_{(3)} \ldots . \ddot{x}_{(N-1)}, \ddot{x}_{(N)})^T \tag{6}$$

Next, we define the *mass matrix M* which is an $N \times N$ diagonal matrix as given by equation (7).

$$M = \text{diag}(m_{(1)}, m_{(2)}, m_{(3)} \ldots . m_{(N-1)}, m_{(N)}) \tag{7}$$

We assume that every mass in our chain is positive and non-zero (otherwise there is no physical meaning). This automatically implies that M always has an inverse and this is given by equation (8).

$$M^{-1} = \text{diag}\left(\frac{1}{m_{(1)}}, \frac{1}{m_{(2)}}, \frac{1}{m_{(3)}} \ldots . \frac{1}{m_{(N-1)}}, \frac{1}{m_{(N)}}\right) \tag{8}$$

Finally, we can define the *stiffness matrix K* which is an $N \times N$ symmetric matrix The entries of this matrix are given by the following relations:

$$\begin{cases} K_{(i),(j)} = k_{(i),(i+1)} + k_{(i),(i-1)} & \text{if: } j = i \\ K_{(i),(j)} = -k_{(i),(i+1)} & \text{if: } j = i+1 \\ K_{(i),(j)} = -k_{(i),(i-1)} & \text{if: } j = i-1 \\ K_{(i),(j)} = 0 & \text{if: } j = \text{anything else} \end{cases}$$

Now, we can conveniently re-write our system of N differential equations as a single matrix equation as given in equation (9).

$$M\ddot{\vec{x}} = -K\vec{x} \tag{9}$$

By carrying out the matrix multiplication explicitly, it is straightforward to verify that this represents the same system that is defined by equations (3) and (4). When written in this alternative form we notice that this equation looks remarkably similar to that of a simple harmonic oscillator! Of course, there are clear differences: M and K are matrices and no longer just scalars, and our displacement coordinate is no longer a number, but it is actually a vector. Nevertheless, we can try to solve this

equation! We begin by trying an Ansatz that is similar to the one used for a simple harmonic oscillator, but this time we need to use a vector version of the Ansatz as shown in equation (10).

$$\vec{x} = \vec{a} \cdot cos(\omega t + \phi) \tag{10}$$

In equation (10), ω is the frequency of oscillation, ϕ is the phase angle, and the vector $\vec{a}$ contains the amplitude of each mass in the chain. As visible, this solution assumes that every mass in the chain oscillates with the same frequency and phase angle, but allows for different amplitudes. There is no clear reason as to why one might assume such a solution even exists, but as we will demonstrate, this is justified simply because it works in practice! Differentiating equation (10) twice with respect to time and plugging the result into equation (9) yields equation (11).

$$-M\omega^2\vec{x} = -K\vec{x} \tag{11}$$

Since M always has an inverse, we can rearrange things. Further, since $\vec{x}$ has the form given by equation (10) , we can cancel out the $cos(\omega t + \phi)$ term on both sides (thereby completely eliminating time-dependence) to obtain equation (12). As we will see, this is a very important result.

$$M^{-1}K\vec{a} = \omega^2\vec{a} \tag{12}$$

In particular, we shall denote the $N \times N$ matrix which is $M^{-1}K$ by the symbol T. This finally gives us equation (13). We notice the mathematical form of this equation is rather simple and elegant.

$$T\vec{a} = \omega^2\vec{a} \tag{13}$$

It is easy to see that the elements of T are given by the following set of formulae:

$$\begin{cases} T_{(i),(j)} = \frac{k_{(i),(i+1)}+k_{(i),(i-1)}}{m_{(i)}} & \text{if: } j = i \\ T_{(i),(j)} = -\frac{k_{(i),(i+1)}}{m_{(i)}} & \text{if: } j = i+1 \\ T_{(i),(j)} = -\frac{k_{(i),(i-1)}}{m_{(i)}} & \text{if: } j = i-1 \\ T_{(i),(j)} = 0 & \text{if: } j = \text{anything else} \end{cases}$$

As visible from equation (13), what we have is an eigenvalue problem and this is where all the linear algebra comes in! We have managed to reduce the complicated system of differential equations that govern the dynamics of our mass-spring chain to an eigenvalue problem. We can clearly see that ω^2 is an eigenvalue of T, and $\vec{a}$ is an eigenvector. To recall, we wish to solve a system of N coupled linear homogeneous differential equations. Due to the homogeneity property, we can use the *superposition principle* to our advantage. If we manage to somehow find N linearly independent solutions to our system, then, the general solution can be constructed by superposing them. Since T is an $N \times N$ matrix it has N eigenvalues, and if T is diagonalizable, then, T has N linearly independent eigenvectors. This is a very important feature because it guarantees that we can find the general solution when T is diagonalizable. We should also note that since we are dealing with 2nd order differential equations, and there are N of them, the general solution needs to have 2N undetermined constants that can be fixed by the specifying the initial conditions of the problem. Therefore, if T is diagonalizable, the general solution must have the form given by equation (14).

$$\vec{x} = \sum_{l=1}^{N} (a_l\vec{u}_l \cdot cos(\omega_l t + \phi_l)) \tag{14}$$

Here, for each value of l, I have written $\vec{a}$ as $a\vec{u}$ where $\vec{u}$ represents a unit eigenvector and a is just an arbitrary constant. So what we have is a linear combination of the N independent solutions. The set of a_l and ϕ_l terms are the 2N undetermined constants which can be found if we are given the initial conditions. We clarify that the ω_l and $\vec{u}_l$ terms have nothing to do with the initial conditions. They can be found simply by calculating the eigenvalues and eigenvectors of the T matrix (provided it is diagonalizable). This is an important point and it will help us introduce the concept of a *normal mode*. We note that M^{-1} and K are real symmetric matrices and hence they are diagonalizable as stated by the spectral theorem. However, $T = M^{-1}K$ is not necessarily symmetric. Nevertheless, it so happens that T is actually diagonalizable because it falls under the category of what is known as a self-adjoint matrix and the more general version of the spectral theorem guarantees that self-adjoint matrices are diagonalizable. Although many sources indicate otherwise, it should be noted that a self-adjoint matrix is *not* the same as a Hermitian matrix, but it is actually defined by an inner product which depends on the particular problem one is dealing with. At the end of the day, for our purposes, all this means is that we can *always* find the general solution to our system of differential equations! From equation (14) we can see that the general solution is a sum of N independent solutions. Each one of these is what is known as a *normal mode solution* and it corresponds to motion where each mass in the chain oscillates in simple harmonic motion at the same angular frequency ω_l which is known as the corresponding *normal frequency*. It should be noted that the amplitude of oscillation of each mass need not be identical. What is interesting is that any possible motion of our mass-spring chain can be written as a linear combination of motion at a set of discrete frequencies! A *normal mode solution* is simply given by equation (15).

$$\vec{x}_l = (a_l\vec{u}_l \cdot cos(\omega_l t + \phi_l)) \tag{15}$$

Here, ω_l is the square root of the lth eigenvalue of T, and $\vec{u}_l$ is the corresponding unit eigenvector. As we will see through multiple examples, The "shape" the mechanical mode corresponding to ω_l is contained in this eigenvector and can be visualized rather easily. As their name implies, in general, normal mode solutions are orthogonal to each other because the eigenvectors of T are orthogonal. However, one must be cautious as this is not always with respect to the standard dot product, but rather, it is with respect to an inner product that depends on the precise structure of the T matrix. In the special case that the T matrix is symmetric, its eigenvectors are orthogonal under the standard dot product (as given by the spectral theorem). This actually has an interesting consequence. If we drive our mass spring chain with a sinusoidal force at one of the normal frequencies, then we only excite the normal mode corresponding to that frequency, and the motion of the chain has no contribution from any of the other normal modes. With this, we conclude our mathematical formulation and move to look at specific examples and applications of normal modes. To become familiar with all of this new mathematics, we will start by studying the simplest possible case of a linear mass-spring chain where we have just two masses connected by three springs before moving to more complex problems.

2 Numerical Examples and Results

2.1 Short Mass-Spring Chains

In this section, we will study short mass-spring chains that comprise of no more than 5 masses. To start, we will look at the simplest non-trivial example which is a system with 2 identical masses and 3

identical springs. We will choose our masses to have a mass of 1 kg and our springs to have a spring constant of 100 N/m. We begin by constructing our T matrix of known constants before doing any computation. This is shown in equation (16) below.

$$T = \begin{bmatrix} \left(\frac{k_1+k_2}{m_1}\right) & \left(\frac{-k_2}{m_2}\right) \\ \left(\frac{-k_1}{m_1}\right) & \left(\frac{k_2+k_3}{m_2}\right) \end{bmatrix} = \begin{bmatrix} 200 & -100 \\ -100 & 200 \end{bmatrix} \tag{16}$$

Using *Matlab*, we solved for the of amplitude vectors ($\vec{a}$) and their corresponding angular frequencies (ω) by computing the eigenvectors and the positive square roots of the eigenvalues of T as suggested by equation (13). With this, we found the normal modes of our system. As expected we obtained a *symmetric* and an *anti-symmetric mode* and this is shown in equations (17) and (18) respectively.

$$\vec{x_1}(t) = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} cos(10t + \phi_1) \tag{17}$$

$$\vec{x_2}(t) = a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} cos(10\sqrt{3}t + \phi_2) \tag{18}$$

In the symmetric mode given by equation (17), the two masses move in phase with the same amplitude, while in the anti-symmetric mode given by (18) they move perfectly out of phase with the same amplitude. It is worth mentioning that molecules such as CO_2 actually display the anti-symmetric vibrational mode and this can be detected via infrared spectroscopy! With the normal modes in hand, we can construct the general solution using equation (14) to obtain equation (19) as shown below.

$$\vec{x}(t) = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} cos(10t + \phi_1) + a_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} cos(10\sqrt{3}t + \phi_2) \tag{19}$$

From here, we can choose a set of initial conditions and plug them into the general solution. This will give a system of 4 linear equations in the 4 unknowns a_1, a_2, ϕ_1 and ϕ_2. Solving for these unknowns will allow us to determine the motion of each mass in our system at all future times. To start off, we chose $\vec{x}(0) = [0.5, -0.5]^T$ and $\dot{\vec{x}}(0) = [1, 1]^T$. Then, we used *Matlab* to numerically solve for the unknown constants. After this, we plotted the solutions to show how the position of each mass in our chain varies with time. This is shown in figure (2) below. To clarify, although the masses move parallel to the direction of the chain, we plot the motion as if they move perpendicular to the chain as this helps us visualize the situation more easily.

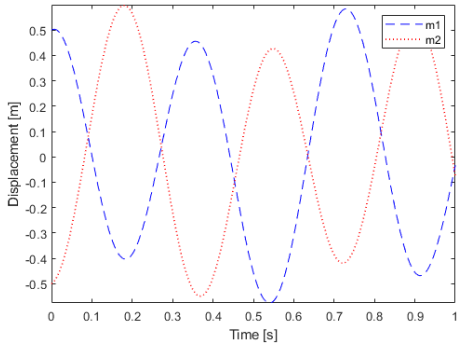


Figure 2: Solution for the motion for m_1 and m_2

Since the motion of the entire system is a linear combination of the normal modes, we can decompose the motion to see the contribution from each normal mode (this is analogous to decomposing a vector into its components by using a suitable basis). The results are shown in figure (3) below.

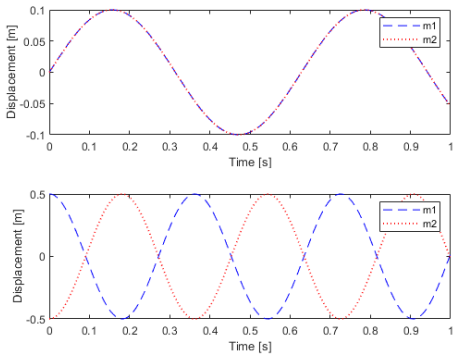


Figure 3: Contribution to Motion From Symmetric (top) and Anti-Symmetric (bottom) Modes

As we can see, the anti-symmetric mode contributes more because the overall amplitude of motion is larger, and the symmetric mode has a relatively small contribution. In fact, this is evident in figure (2) since the motion is quite similar to the anti-symmetric mode. However, it is not precisely the same and this is because there is a non-zero contribution from the symmetric mode. Overall, the general motion of each mass is the sum of motion at two frequencies, and is therefore non-trivial as seen in figure (2).

Next, we look at a slightly more complicated situation to try and observe the phenomenon of *beats*. We let $m_1 = 3$ kg, $m_2 = 1$ kg, $k_1 = k_3 = 100$ N/m, and $k_2 = 150$ N/m. Then we let the initial conditions be $\vec{x}(0) = [0, 0]^T$ and $\dot{\vec{x}}(0) = [2, -2]^T$ and used *Matlab* to find the unknown constants. Here, we can see how knowledge of the system's normal modes can offer meaningful insight to otherwise chaotic motion. The details are shown in figure (4) below. Part a) shows the motion of the two masses, and Parts b) and c) show the contributions from the two normal modes of the system.

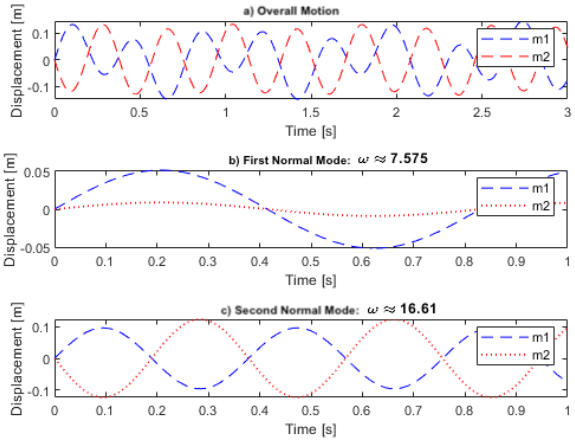


Figure 4: **a)** Overall Motion **b)** First Mode ($\omega = 7.575$) **c)** Second Mode ($\omega = 16.61$)

As visible above, when compared to the simple case we studied earlier where the masses and springs were identical, the overall motion of each mass and the normal modes are more complicated. This is

evident because the two masses no longer have the same amplitude in the normal modes. Further, if we plot the motion of the masses individually over long time periods, we see the interesting phenomenon of *beats* which was absent in the earlier example. This is shown in figure (5). We can see beats in the motion of m_1. The phenomenon of beats has to do with what is known as *weak-coupling* between the two masses and can be derived analytically. However, for the sake of simplicity, we will stick to a numerical example. Essentially it arises due to the non-homogeneous chain in this problem.

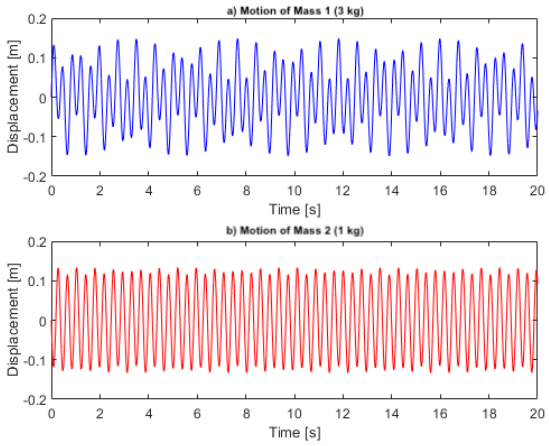


Figure 5: The Phenomenon of Beats

Finally, before concluding this section, we did a similar experiment for a mass-spring chain with 5 masses. For this, we went back to the homogeneous mass-spring chain and let every mass be 1 kg and let every spring have a spring constant of 100 N/m. After the usual steps, we used the initial conditions $\vec{x}(0) = [1, 1, 0, -1, -1]^T$ and $\dot{\vec{x}}(0) = [0, 0, 0, 0, 0]^T$ to find the unknown constants numerically on *Matlab*. As before, we plotted the motion of each mass and how it changes with time (shown in figure (6). Interestingly, because of the initial conditions chosen, m_3 never moves, and out of the 4 remaining masses, pairs of two move completely in phase with each other. Furthermore, each pair moves completely out of phase with the other. In essence, we can actually think of this as two smaller sub-chains with m_3 acting like a wall! To conclude, through multiple examples, we have seen that mass-spring chains can display complex and interesting dynamics that depend upon the nature of the chain and also the initial conditions of the system.

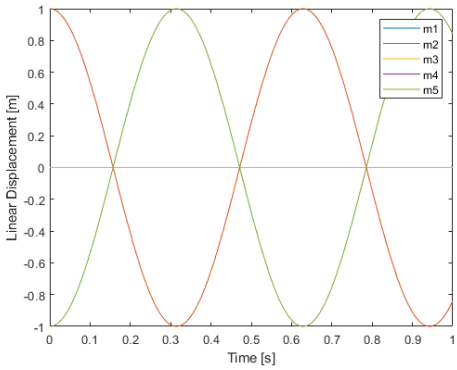


Figure 6: Overall Motion of the 5 Masses

2.2 N $\longrightarrow \infty$ and Waves

We now pivot to study mass-spring chains with a very large number of masses. This type of system is very interesting because it acts as a bridge between systems of discrete masses and those where we have a continuous mass distribution. It is important to emphasize that real-world systems, at the macroscopic level, are most often comprised of bodies with a continuous mass distribution. It is therefore of great importance to understand the dynamics of these systems. Although we will not go too much into the details, it is worth mentioning that the study of the normal modes of continuous systems encompasses a field known as *continuum mechanics* and it is here that the famous *wave equation* from physics shows up! The wave equation is a partial differential equation, and it can often be solved by a technique that is known as the *separation of variables*. This can give rise to solutions in the form of *Fourier Series*, and this has very fundamental connections to linear algebra on function spaces. To get a feel for all of this, we will just use a brute-force approach to find and plot the *shapes* of the normal modes for homogeneous mass spring chain. We will let all the masses have a mass of $m = 1$ and all the springs have a spring constant of $k = 1$ (arbitrary units). Then, we will let $N = 500$ to allow for the wave nature of the normal modes to show up. To find the normal mode shapes for this system, we first constructed the T matrix and then found its eigenvectors. This was done on *Mathematica*. To visualize the mode shapes, we plotted the entries of the eigenvectors against their index using a list plot feature. It is important to clarify that the overall amplitude of the modes in the plots are totally arbitrary (since we can always scale an eigenvector and it still remains an eigenvector). The important thing is to look at how much masses move *relative* to each other in each mechanical mode. In figure (7) below, we can see the mode shapes of multiple mechanical modes.

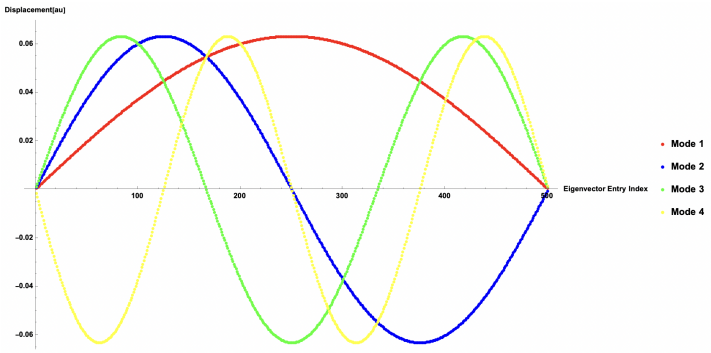


Figure 7: Mechanical Modes of a Long Homogeneous Mass-Spring Chain

These are the modes of a string which is clamped at both ends! In fact, the modes we are seeing here are called *standing waves*. If we assume the system is continuous and let u be the displacement of the string at the horizontal position x (the displacement is parallel to the string but we are plotting it this way for visualization) it is not difficult to show that the mode shapes are given by the equation (20).

$$u(x) = sin\left(\frac{k\pi x}{L}\right) \tag{20}$$

Here, L is the length of our chain (in this case $L = 500$) and k is any natural number (1, 2, 3 and so on). It is easy to verify that this matches with the mode shapes we are seeing above. And with this, we close our discussion of waves and move to look at musical instruments!

2.3 Normal Modes of Vibrating Instruments

To relate normal modes to music, we specifically look at the normal modes of vibrating instruments such as string instruments. To be more specific, when a wave is confined to a space it is then filled up with vibrations which allows for the creation of sound. This is a phenomenon called *standing waves*, which is produced when, say, the strings of a guitar are plucked. In regards to normal modes, a standing wave is simply a continuous form of a normal mode. The most general form of this kind of system would be the superposition of its normal modes. Having multiple modes with motion will cause for the waves to combine and create a sum of the waves that have crashed into each other, affecting the overall sound that would be generated from an instrument.

The normal modes of any vibrating instrument (strings for instance) are called "harmonics" or "overtones". To understand this better, we reduce this into a set of harmonic parts for greater analysis, taking apart the wave before adding the components in their superposition form. Normal modes are important as a result as they allows us to reduce waves into the form of a simple harmonic oscillator as a *basis* for any kind of wave.

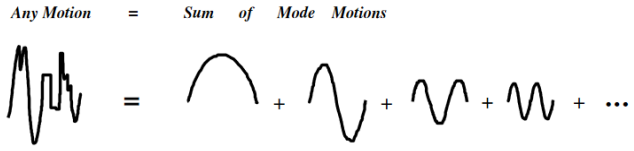


Figure 8: Sum of Mode Motions

If we look at the string of a guitar for example, when plucking it, there would appear to be only one fundamental mode going back and forth from its highest point to its lowest in amplitude based on the strength of the pluck at say for example a 100 Hz frequency. But in reality the string is vibrating within the whole spectrum of normal mode shapes and frequencies during the entire time of its motion.

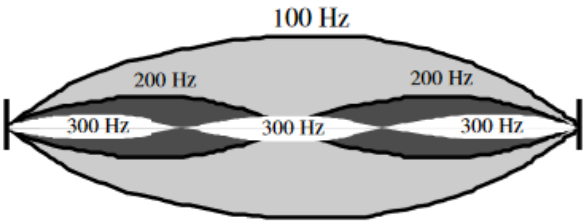


Figure 9: Decomposed Vibration of Wave

The resulting tone one would get from the guitar is the fundamental tone, which has a frequency f, that is combined with a series of higher and fainter overtones which have frequencies of 2f, 3f, 4f and so on. This overall tone is different for every instrument even if the same note were to be played at the same pitch! We know that this string pluck is a sum of its harmonic parts due to *Fourier's Theorem*, which states that a periodic function (or vibration in this case) can be decomposed into a sum of a series of its sine or cosine parts.

2.4 Periodic Structures and Bandgaps

We now turn to investigate periodic structures and how they can lead to the formation of *bandgaps*. To start, we can consider the distribution of normal mode frequencies for a homogeneous mass-spring chain where all the masses have a mass of $m = 1$, and all the springs have a spring constant of $k = 1$. In addition we shall choose the number of masses in our chain to be large and let $N = 100$. We defined a piecewise function on *Mathematica* to create the T matrix that corresponds to this problem. Then, we used a sparse matrix plotting function to help visualize its structure as show below:

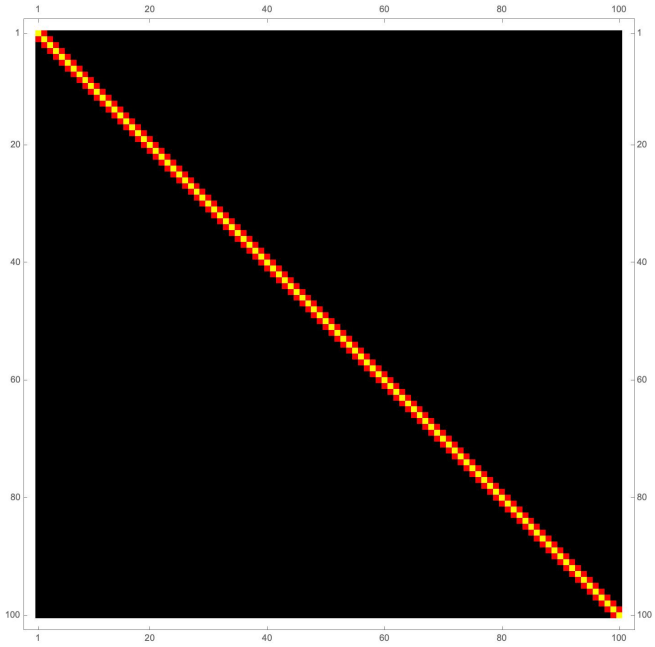


Figure 10: T Matrix for a Homogeneous Chain (black = 0, red = -1, yellow = 2)

The color key for figure (10) is given in the caption and it indicates the value of the entry stored at each position in the T matrix. Next, we used *Mathematica* to find the positive square roots of the eigenvalues of this matrix to obtain the normal mode frequencies, and then plotted the frequencies in ascending order against their index. This is show in figure (11). To clarify, the mode index goes from 1 to 100 since the T matrix has 100 eigenvalues (because $N = 100$). We notice from the graph that there is a fairly *even* distribution of normal mode frequencies between the maximum (about 2 au) and the minimum (about 0.1 au). Physically, this means that our homogeneous mass-spring chain can oscillate at any of the frequencies shown by the data points on the graph. Now, let us try an experiment. Instead of allowing every spring to have the same value for the spring constant, we can create a *periodic structure*. We shall do this by making the spring constants alternate between two different values which we will denote as k_a and k_b. So the first spring has spring constant k_a, for the second spring it is k_b, for the third it is k_a, for the fourth it is k_b, and so on. For ease of analysis, we will keep the sum of the spring constants fixed. In the homogeneous mass-spring chain that we just discussed, $k_a = k_b = 1$ and their sum is $k_a + k_b = 2$ (this can be seen on the diagonal of the T matrix). We will stick with the same setup here. To start we can make the spring constants slightly different: let $k_a = 1.1$ and let $k_b = 0.9$. The structure of the T matrix is shown in figure (12) and the color key is indicated in the caption.

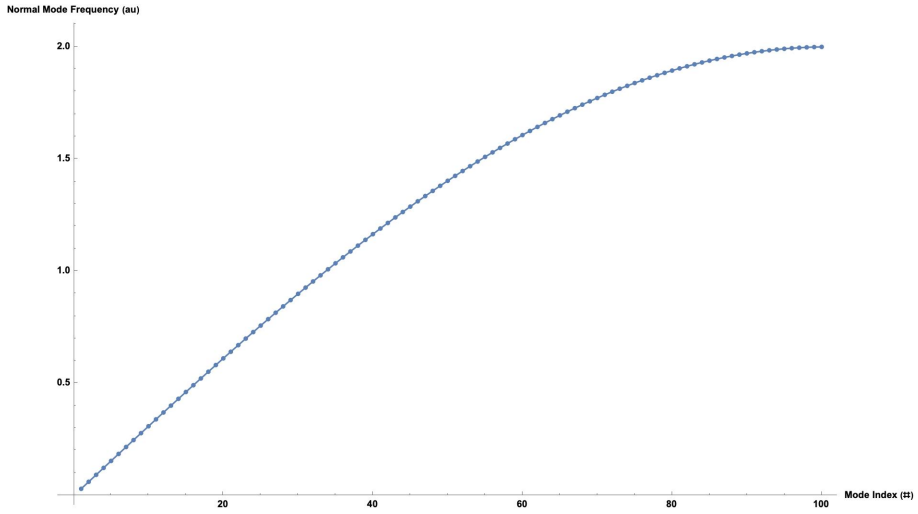


Figure 11: Normal Mode Frequencies vs Mode Index for a Homogeneous Chain

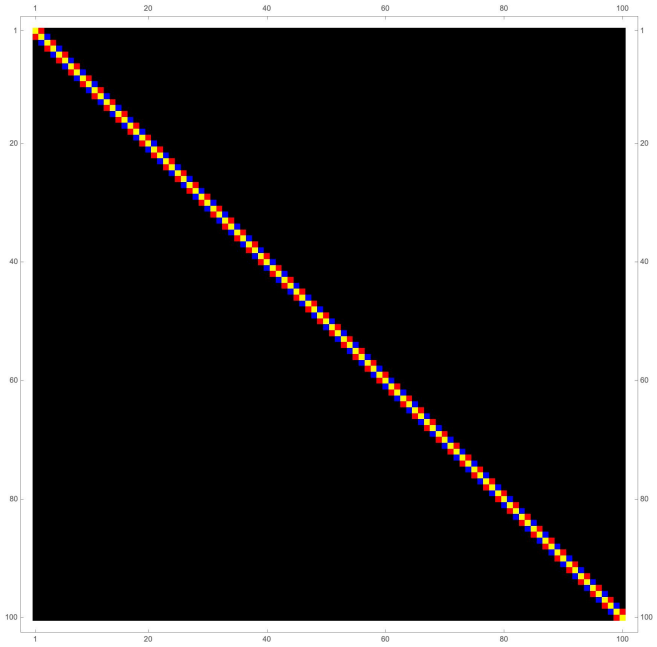


Figure 12: T Matrix for a Periodic Chain (black = 0, red = -1.1, blue = -0.9, yellow = 2)

For the mass-spring chain that corresponds to the T matrix in figure (12), the plot of the normal mode frequencies against their index is shown in figure (13). Overall, things look pretty similar to the homogeneous mass-spring chain. However, we notice that there is a small "gap" in the graph. There are no normal modes that have frequencies of vibration between 1.5 au and 1.3 au (approximately). But there are normal modes with frequencies both above and below this range. We did not see this behaviour in the homogeneous mass-spring chain! We can try to see what happens if we make the values of $k_a = 1.5$ and $k_b = 0.5$ such that the difference between them (in size) is larger than before. The results that we obtained are depicted in figure (14).

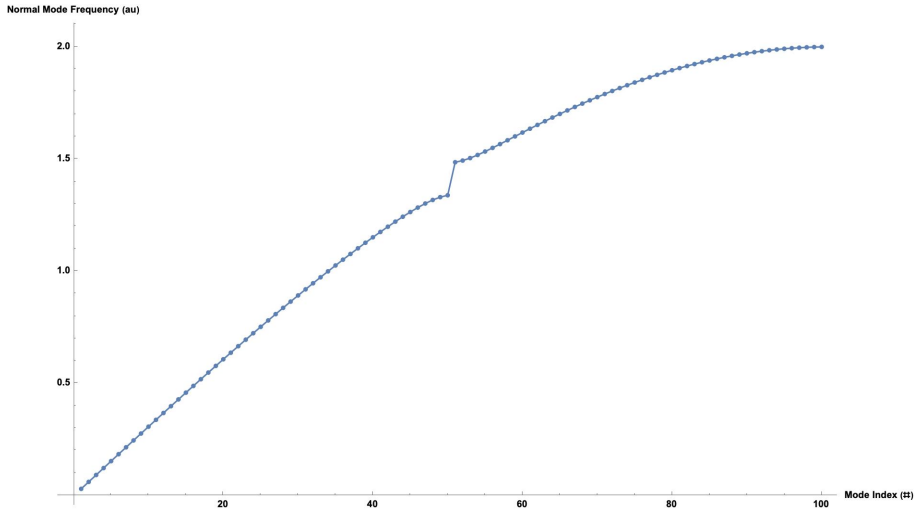


Figure 13: Normal Mode Frequencies vs Mode Index for a Periodic Chain ($k_a = 1.1$, $k_b = 0.9$)

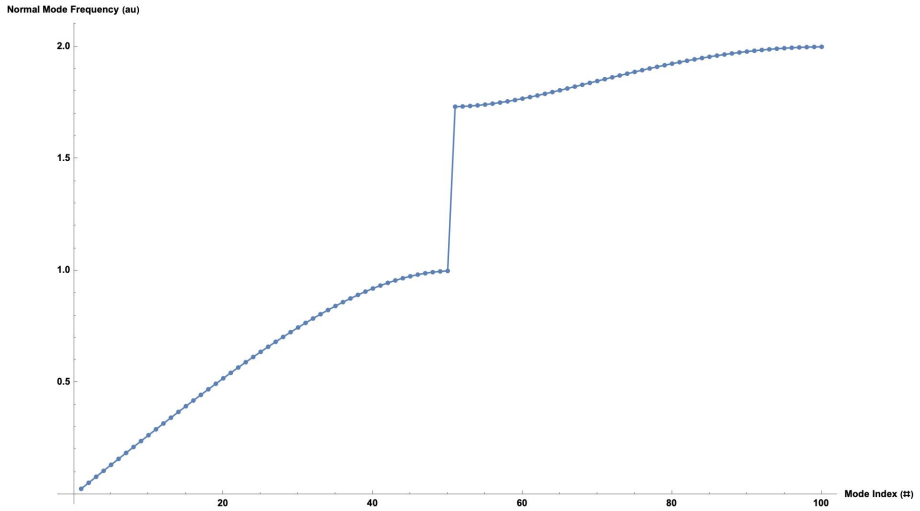


Figure 14: Normal Mode Frequencies vs Mode Index for a Periodic Chain ($k_a = 1.5$, $k_b = 0.5$)

We note that for this mass-spring chain, the basic structure of the T matrix is practically the same as what we saw in figure (12). Only the sizes of the entries are different, so we have omitted the sparse matrix plot for this as it shows us nothing new. Looking at figure (14), as evident, the "gap" is a wider than before! Going to the extreme case, we can set $k_a = 1.9$ and $k_b = 0.1$. The results are given in figure (15). As we can see, the same trend continues. We see that the larger the difference (in size) between the spring constants of neighbouring springs in the periodic chain, the wider the "gap". What we are seeing here is the formation of a *bandgap* due to the periodic nature of the chain. Mathematically, just by staring at the T matrix for periodic chains, it is difficult to get a good feeling for exactly *why* there are no longer any eigenvalues in some particular range. However, we can try to provide a physical explanation for why this occurs in terms of *waves*. The normal mode frequencies of vibration of our chain indicate the frequencies of mechanical waves that can propagate through the chain. One could also think of these waves as acoustic or sound waves.

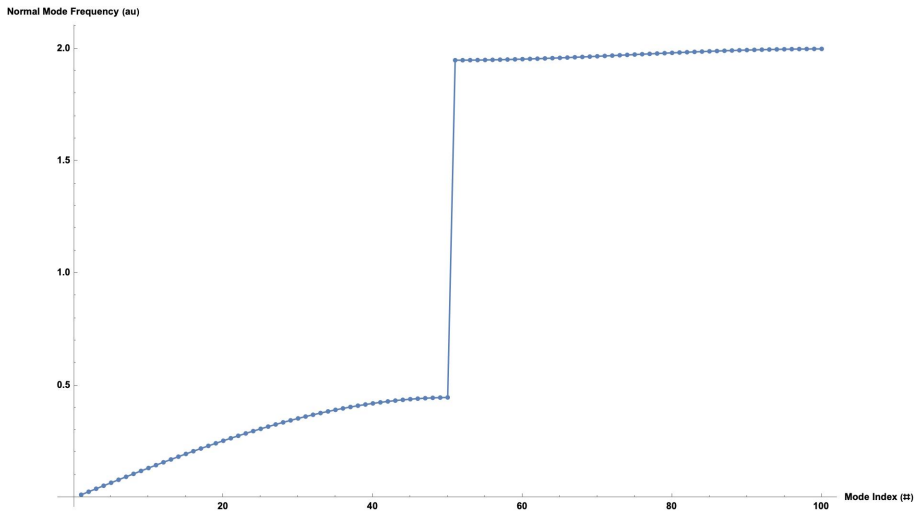


Figure 15: Normal Mode Frequencies vs Mode Index for a Periodic Chain ($k_a = 1.9$, $k_b = 0.1$)

As seen from our experiments, for periodic chains, due to the bandgap, there are certain frequencies at which the chain simply *cannot* vibrate. In other words, these frequencies of mechanical waves are not allowed to propagate through the chain. Why? In the limit where N is very large, our periodic mass-spring chain can be thought of as a continuous object composed of two physically different media that alternate between each other. When we send a mechanical wave through a periodic structure, at each media boundary, some fraction of the wave is reflected, and the rest is transmitted. The details depends on the frequency of wave we send in. Miraculously, for a certain range of frequencies of mechanical waves, all the reflected portions of the waves interfere constructively, and all the transmitted portions of the waves interfere destructively. In other words, the wave is not allowed to propagate through the structure, but it simply gets reflected back! And this is exactly what leads to the formation of a bandgap. Sometimes, one can cleverly engineer a "defect" in the periodic structure such that the "defect" vibrates at a frequency that lies in the bandgap. Then, the motion of the "defect" can be "trapped" in the bandgap. This has some truly remarkable applications.

The closest real-world example to what we have just studied is a *phononic* crystal. There are many ways to make a phononic crystal, but a common method uses thin-film membranes. These membranes can be patterned so that their film stress varies periodically, and this can lead to the formation of an acoustic bandgap, thereby giving rise to a phononic crystal. Phononic crystals are being used in fundamental research to help design high precision force sensors. They are also being tried as possible candidates for a quantum transducer that converts microwave photons to optical photons (a lot of this research is actually being done by our physics department at CU). This is very important for being able to effectively transport quantum information. The optical analogue of a phononic crystal is a *photonic* crystal which is made of a material that has a periodically varying dielectric constant. This leads to a photonic bandgap and light waves of certain frequencies cannot pass through the crystal! Apart from fundamental research applications, photonic crystals are used in thin-film optics and are also being used to made optical fibers since they can be used to "trap" light extremely efficiently. Finally, it is worth mentioning that semiconductor devices such as LEDs and transistors work in the way that they do because of *electronic* bandgaps which arise due to quantum mechanical effects. And with this, we conclude our discussion of periodic structures and bandgaps!

3 Conclusions and Future Work

3.1 Conclusions

In our write-up, we studied the dynamics of 1-dimensional mass-spring chains. By employing linear algebra techniques, we introduced the concept of normal modes, and through this, we were able to gain an in-depth understanding into the behaviour of multi-particle systems. We studied short mass-spring chains to get a feel for normal modes, and then we moved to long mass-spring chains and discussed how they relate to waves. After this, we talked about waves and normal modes that arise in musical instruments. Finally, we investigated periodic structures and bandgaps, and discussed some of the fascinating real-world applications of bandgaps.

3.2 Future Work

In terms of future work, it would be interesting to extend our treatment of continuous systems and deal with the partial differential equations that arise from the wave equation. This would allow us to learn more about how linear algebra applies on function spaces. Other than this, we could also extend our study of waves and introduce concepts such as the *dispersion relation* as this could be used to derive the origin of bandgaps in continuous systems.

While we constrained our mass-spring chains to be 1-dimensional, it would be illuminating to explore the applications and appearances of normal modes in two and three dimensional structures. When using numerical techniques to find the behaviour of a given multi-particle system, adding dimensions will increase the strain of computation significantly, and it might be worth investigating more effective and scalable solution strategies.

Normal modes appear frequently when studying the vibrational modes of molecules and their infrared spectra, and it might be interesting to learn more about this.

4 References

[1] *Classical Mechanics* by John R Taylor, University Science Books, 2005

[2] *Applied Linear Algebra* by Olver and Shakiban, Springer, 2018

[3] *Analysis of Membrane Phononic Crystals with Wide Band Gaps and Low-Mass Defects* by Chris Reetz et al., Physical Review Applied, JILA and CU Boulder Physics Department, 2019

[4] *An Introduction to Computer Simulation Methods* by Gould, Tobochnik and Christian, Pearson Education, 2007

5 Appendix of Code

The code shown on the next few pages was written on *Mathematica* and used to obtain the results in sections 2.2 and 2.4 of the write-up.

```
ClearAll["Global`*"]
Clear["Global`*"]


(*Bandgap Simulation*)
f[i_, j_] :=
 Piecewise[{{2, Abs[i - j] == 0}, {-0.1, Abs[i - j] == 1 && Mod[(i + j), 4] == 1},
   {-1.9, Abs[i - j] == 1 && Mod[(i + j), 4] == 3}}]
matri = Table[f[i, j], {i, 100}, {j, 100}];
MatrixForm[matri];
vals = Reverse[Eigenvalues[matri]];
omegas = Sqrt[vals];
MatrixPlot[matri, ColorRules → {0 → Black, -1.9 → Red, -0.1 → Blue, 2 → Yellow}];

ListLinePlot[omegas, PlotMarkers → {Automatic, 6},
  AxesLabel → {"Mode Index (#)", "Normal Mode Frequency (au)"},
  LabelStyle → Directive[Black, Bold]];


ClearAll["Global`*"]
Clear["Global`*"]
```

```
(*Standing Waves Simulation*)
f[i_, j_] :=
 Piecewise[{{1, Abs[i - j] == 0}, {-0.5, Abs[i - j] == 1 && Mod[(i + j), 4] == 1},
   {-0.5, Abs[i - j] == 1 && Mod[(i + j), 4] == 3}}]
matri = Table[f[i, j], {i, 500}, {j, 500}];
MatrixForm[matri];
{vals, vecs} = Eigensystem[matri];
omegas = Sqrt[vals];
ListLinePlot[Reverse[omegas], PlotMarkers → {Automatic, 6}];
m1 =
  ListPlot[vecs[[1]], PlotStyle → Red, PlotLegends → LineLegend[{Red}, {"Mode 1"}],
   AxesLabel → {"Eigenvector Entry Index", "Displacement[au]"},
   LabelStyle → Directive[Bold, Black]];
m2 = ListPlot[vecs[[2]], PlotStyle → Blue,
   PlotLegends → LineLegend[{Blue}, {"Mode 2"}],
   AxesLabel → {"Eigenvector Entry Index", "Displacement[au]"},
   LabelStyle → Directive[Bold, Black]];
m3 = ListPlot[vecs[[3]], PlotStyle → Green,
   PlotLegends → LineLegend[{Green}, {"Mode 3"}],
   AxesLabel → {"Eigenvector Entry Index", "Displacement[au]"},
   LabelStyle → Directive[Bold, Black]];
m4 = ListPlot[vecs[[4]], PlotStyle → Yellow,
   PlotLegends → LineLegend[{Yellow}, {"Mode 4"}],
   AxesLabel → {"Eigenvector Entry Index", "Displacement[au]"},
   LabelStyle → Directive[Bold, Black]];
Show[m1, m2, m3, m4, PlotRange -> Full];
```

The code shown on the next few pages was written on *Matlab* and used to obtain the results in sections 2.1 of the write-up.

Table of Contents

auth: Matthew Brown

sec: APPM-3310-001

collaborators:

Sanju Kumar Keshava,

Helen Kim

One Dimensional, 2 Body System

```
close all
clear variables
clc

% bodies
n = 2;
m1 = 1; % [kg]
m2 = 1; % [kg]
m = [m1 m2]';

% springs
k1 = 100; % [N/m]
k2 = 100; % [N/m]
k3 = k1;

% construct 'T' matrix
T = [(k1+k2)/m1, -k2/m2;
    -k1/m1,   (k2+k3)/m2]

[A,Omega]=eig(T);

w = zeros(n,1);
for i = 1:n
    Omega(i,i) = sqrt(Omega(i,i));
    w(i) = Omega(i,i);
end

% correct normalization and negativity
A=-sqrt(2).*A
w
```

```
T =
```

```
   200  -100
  -100   200


A =

     1     1
     1    -1


w =

   10.0000
   17.3205
```

Solve with initial conditions

```
% parameters
C1 = [];
C2 = [];
PHI = [];
DELTA = [];

% initial conditions
x0 = [[0 0]' [1/2 -1/2]'];
v0 = [[0 0]' [1 1]'];

% general form of normal modes
syms c1 c2 phi delta t x(t) omega a n1(t) n2(t) xG(t) b
x1(t) = c1 * a * cos(omega*t + phi);  % mass 1
x2(t) = c2 * a * cos(omega*t + delta);% mass 2

% first normal mode
n1(t) = [subs(x1(t), [a omega],[A(1,1) w(1)]);
         subs(x1(t), [a omega],[A(2,1) w(1)])]

% second normal mode
n2(t) = [subs(x2(t), [a omega],[A(1,2) w(2)]);
         subs(x2(t), [a omega],[A(2,2) w(2)])];

% general solution by superpostion
xG(t) = n1(t) + n2(t)
dxG = diff(xG,t)

% solve for key parameters
for i = 1:n
    sol = solve([xG(0)==x0(:,i) dxG(0)==v0(:,i)],[c1 c2 phi
 delta],"Real",true);
    C1 = [C1 sol.c1];
    C2 = [C2 sol.c2];
    PHI = [PHI sol.phi];
    DELTA = [DELTA sol.delta];
```

```
end
```

```
n1(t) =

 c1*cos(phi + 10*t)
 c1*cos(phi + 10*t)
```

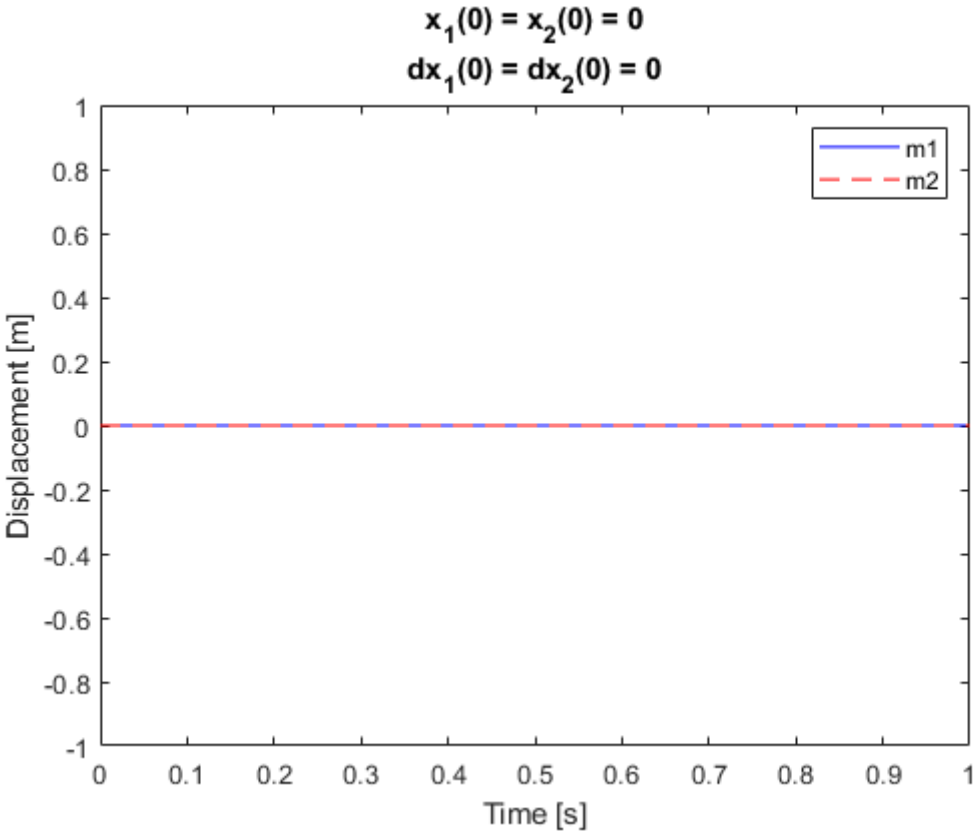
```
xG(t) =

 c2*cos(delta + 10*3^(1/2)*t) + c1*cos(phi + 10*t)
 c1*cos(phi + 10*t) - c2*cos(delta + 10*3^(1/2)*t)
```

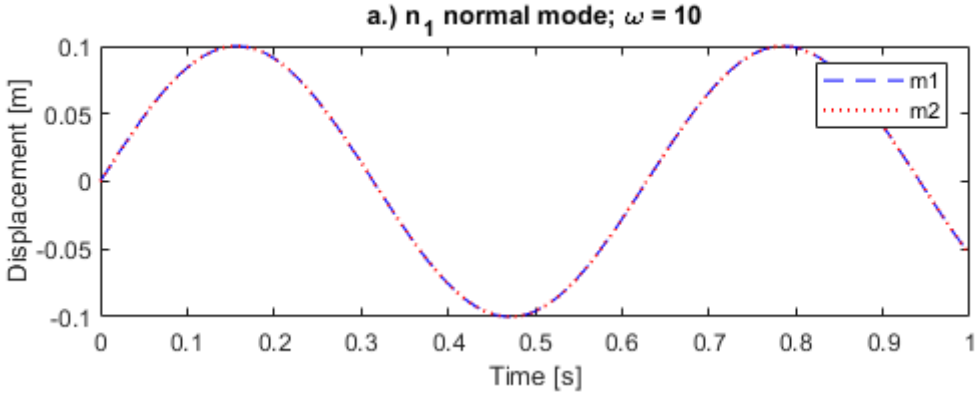
```
dxG(t) =

 - 10*c1*sin(phi + 10*t) - 10*3^(1/2)*c2*sin(delta + 10*3^(1/2)*t)
   10*3^(1/2)*c2*sin(delta + 10*3^(1/2)*t) - 10*c1*sin(phi + 10*t)
```

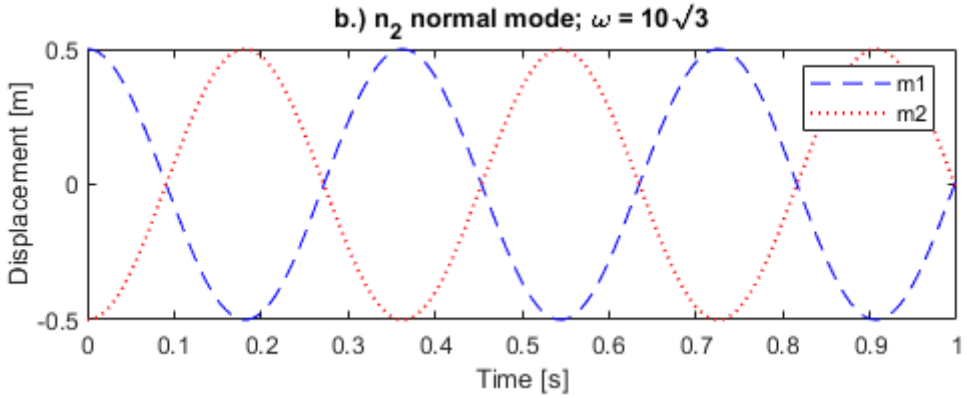
Plotting Solutions

```
% Trivial Solution: All bodies at rest at equilibrium position
i = 1;
j = 1;
soln1 = subs(xG(t),[c1 c2 phi delta],[C1(i,j) C2(i,j) PHI(i,j)
 DELTA(i,j)]);
figure
fplot(soln1(1,:),[0 1],'b')
hold on
fplot(soln1(2,:),[0 1],'--r')
titletext = {'x_1(0) = x_2(0) = 0','dx_1(0) = dx_2(0) = 0'};
title(titletext)
legend('m1','m2')
xlabel('Time [s]')
ylabel('Displacement [m]')

% Non-Trivial Solution
i = 1;
j = 2;
soln2 = subs(xG(t),[c1 c2 phi delta],[C1(i,j) C2(i,j) PHI(i,j)
 DELTA(i,j)]);
figure
fplot(soln2(1),[0 1],'--b')
hold on
fplot(soln2(2),[0 1],':r',"LineWidth",1.25)
hold off
legend('m1','m2')
titletext = {'x_1(0) = 1/2,  x_2(0) = -1/2','dx_1(0) = 1, dx_2(0) =
 1'};
title(titletext)
xlabel('Time [s]')
```

```
ylabel('Displacement [m]')

% plotting n2
j = 2;
soln2 = subs(n2(t),[c2 delta],[C2(i,j) DELTA(i,j)]);
subplot(2,1,2)
fplot(soln2(1),[0 1],'--b')
hold on
fplot(soln2(2),[0 1],':r',"LineWidth",1.25)
hold off
legend('m1','m2')
titletext = {'b.) n_2 normal mode; \omega = 10\surd{3}'};
title(titletext)
xlabel('Time [s]')
ylabel('Displacement [m]')

% n1
i = 1;
j = 2;
soln3 = subs(n1(t), [c1 phi],[C1(i,j) PHI(i,j)])
subplot(2,1,1)
fplot(soln3(1,:),[0 1],'--b')
hold on
fplot(soln3(2,:),[0 1],':r',"LineWidth",1.25)
hold off
legend('m1','m2')
titletext = {'a.) n_1 normal mode; \omega = 10'};
title(titletext)
xlabel('Time [s]')
ylabel('Displacement [m]')


soln3 =

 cos(10*t - pi/2)/10
 cos(10*t - pi/2)/10
```


$x_1(0) = x_2(0) = 0$
$dx_1(0) = dx_2(0) = 0$
m1
m2
Displacement [m]
Time [s]


a.) n_1 normal mode; $\omega = 10$
m1
m2
Displacement [m]
Time [s]


b.) n_2 normal mode; $\omega = 10\sqrt{3}$
m1
m2
Displacement [m]
Time [s]

N-Body Mass Spring Chains

```
close all
clear variables
clc

% number of bodies
n = 5;

% mass of bodies
m = ones(n,1); % [kg]

% springs involved
k = ones(n+1,1);
k = 100*k; % [N/m]

% construct 'T' matrix
T = zeros(n);
Td = zeros(n,1); % diagonal of the matrix
ld = zeros(n-1,1); % lower diagonal
ud = ld; % upper diagnonal
for i = 1:n
    Td(i) = (k(i)+k(i+1))/m(i);
end
for i = 1:n-1
    ld(i) = k(i)/m(i);
end
for i = 2:n
    ud(i-1) = k(i)/m(i);
end
T = T+diag(Td)-diag(ld,-1)-diag(ud,+1)

% solve the system
[A,Omega] = eig(T);

w = zeros(n,1);
for i = 1:n
    Omega(i,i) = sqrt(Omega(i,i));
    w(i) = Omega(i,i);
end

% rescale A matrix
for i = 1:n
    set = A(:,i);
    scale = 1/max(set);
    A(:,i) = scale.*set;
end
A = -1.*A
w

syms x(t) c c1 c2 c3 c4 c5 phi phi1 phi2 phi3 phi4 phi5 t omega a N(t) nt(t)
Cs = sym([c1,c2,c3,c4,c5]);
PHIs = sym([phi1, phi2, phi3, phi4, phi5]);
```

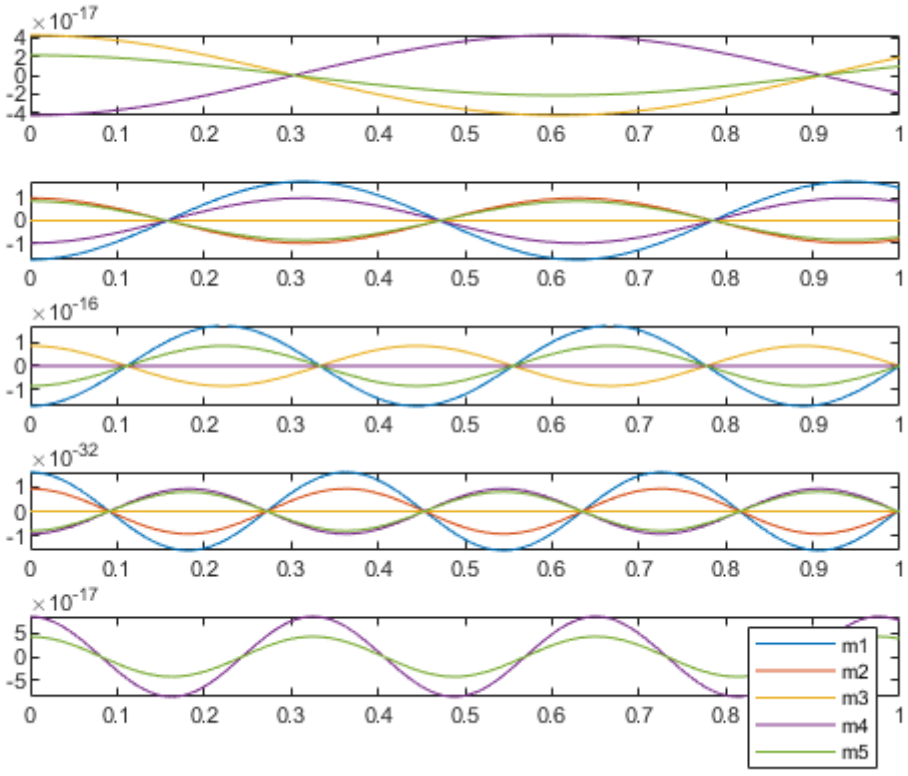
```
x(t) = c*a*cos(omega*t + phi);

% build normal mode solns
N = x(t).*ones(n);
for i = 1:n
    for j = 1:n
        N(i,j) = subs(x(t),[c phi omega a],[Cs(j) PHIs(j) w(j)
 A(i,j)]);
    end
end

% initial conditions
x0 = [1 1 0 -1 -1]';
v0 = [0 0 0 0 0]';

xG(t) = sum(N,2);
dxG = diff(xG,t);

sol = solve([xG(0)==x0, dxG(0)==v0],
[c1,c2,c3,c4,c5,phi1,phi2,phi3,phi4,phi5]);
C = [sol.c1(1) sol.c2(1) sol.c3(1) sol.c4(1) sol.c5(1)]';
PHI = [sol.phi1(1) sol.phi2(1) sol.phi3(1) sol.phi4(1) sol.phi5(1)];

% plot the various normal modes
for i = 1:n
    subplot(n,1,i)
    for j = 1:n
        nt(t) = subs(x(t),[c phi a omega],[C(i) PHI(i) A(i,j) w(i)]);
        fplot(nt(t),[0 1])
        hold on
    end
end
legend('m1','m2','m3','m4','m5')

% plot the general equation of motion for all 5 masses
figure
X = subs(xG(t),[c1,c2,c3,c4,c5],[C(1),C(2),C(3),C(4),C(5)]);
X = subs(X,[phi1,phi2,phi3,phi4,phi5],
[PHI(1),PHI(2),PHI(3),PHI(4),PHI(5)]);
for i = 1:n
    fplot(X(i),[0 1])
    hold on
end
legend('m1','m2','m3','m4','m5')
ylabel('Linear Displacement [m]')
xlabel('Time [s]')
```

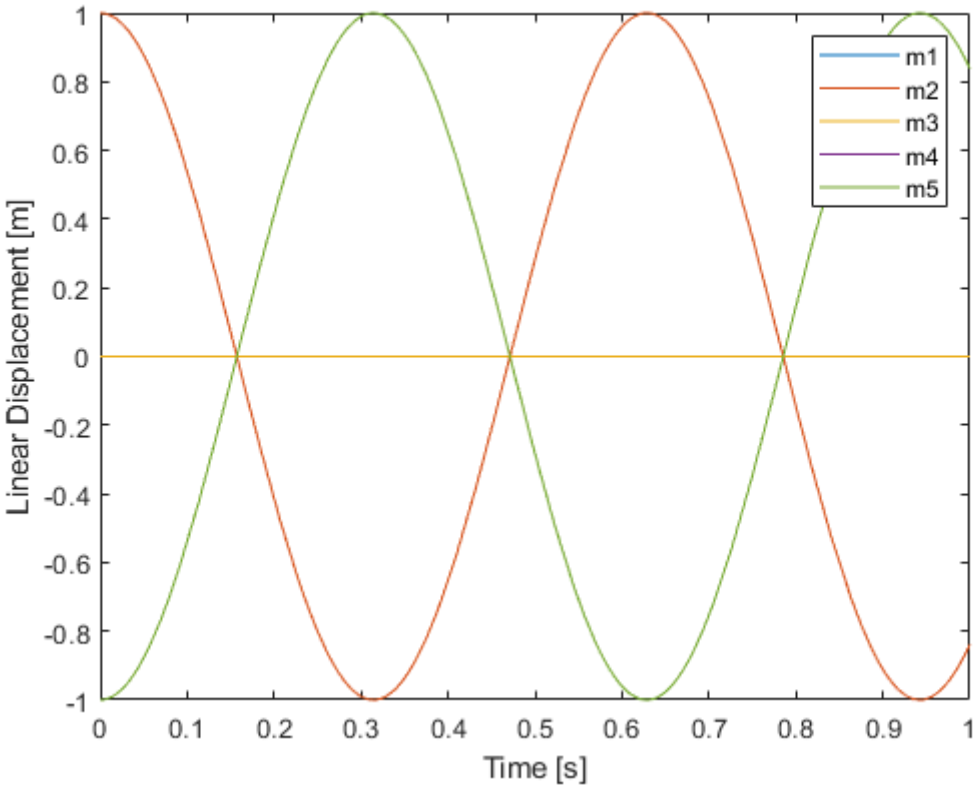
```
T =

   200  -100     0     0     0
  -100   200  -100     0     0
     0  -100   200  -100     0
     0     0  -100   200  -100
```

```
     0     0     0  -100   200
```

```
A =

   -1.0000    1.0000   -1.0000    1.0000   -0.5000
   -1.7321    1.0000   -0.0000   -1.0000    0.8660
   -2.0000   -0.0000    1.0000   -0.0000   -1.0000
   -1.7321   -1.0000    0.0000    1.0000    0.8660
   -1.0000   -1.0000   -1.0000   -1.0000   -0.5000
```

```
w =

    5.1764
   10.0000
   14.1421
   17.3205
   19.3185
```





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