Application of Conformal Mapping to Solve a Non-Traditional Boundary Value Problem in Electrostatics

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Abstract

Conformal mapping is invaluable for solving problems in engineering and physics that can be expressed in terms of functions of a complex variable yet exhibit inconvenient geometries. By choosing an appropriate mapping, one can transform the inconvenient geometry into a much more convenient one. Our project will cover the mathematical formulation and theory behind conformal mapping and we will demonstrate its application to solve a non-traditional boundary value problem in electrostatics.

1 Introduction

Boundary value problems (BVPs) are a common family of problems undergraduate physics students solve in electromagnetism. Given there are no electric charges in the region of interest, and given the voltages on the the boundaries, one can theoretically solve the Laplace equation $\nabla^2 V = 0$ to find the electric potential (and electric field) in the region. Due to the limited mathematical tools students are usually equipped with at that stage of their academic journey, the BVPs are usually restricted to a simpler set of problems in which they can exploit high symmetry. A well-known BVP is finding the potential field between the inner metal cable and the outer metal casing of a coaxial cable. Assuming the cable and the casing (centre core and metallic shield in Fig. 1) are concentric (centered with respect to one another), it is a relatively simple BVP one can solve analytically. If we decided to shift the inner cable slightly, such that the cable and the casing are non-concentric, this becomes a very hard problem to solve. The complication is due to a non-radially-symmetric potential field.

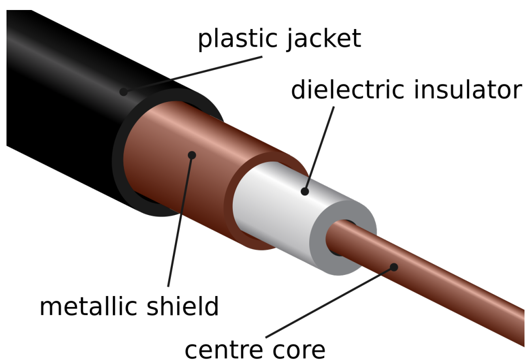


Figure 1: Diagram of a coaxial cable.

If the physics students happen to later take a course in complex variables and applications (like two of us), they learn that there is a method called *conformal mapping* that they can use to make this non-concentric coaxial cable problem much easier to solve. In fact, using conformal mapping, we can map the non-concentric problem to the well-known concentric one, solve for the potential there, and map it back to get the solution for the potential in our original region. At this point you might ask: How do we know if such a mapping exists, and if it does exist, how can we construct the right one.

The objective of this paper is to exactly answer these questions.

With the non-concentric coax cable problem in mind, we will introduce the reader in Sec. 2 with the tools required to solve such problem. In Sec. 2.1, we will introduce the theory behind analytic and conformal mapping, and cover concepts such as mapping composition and more specific transformations we will need using examples. In Sec. 2.2, we will construct a general mapping of annuli which may or may not be concentric. In Sec. 2.3, we will reintroduce the the electromagnetism problem we are solving more formally, and walk through examples to show how we solve the problem. Finally, in Sec. 3, we will conclude our work and have a brief discussion about possible directions for future work.

2 Project Description

2.1 Analytic and Conformal Mapping

Before we move on to conformal transformations, we need to give a brief mention about *analytic maps*. This derivation mainly follows [4]. Complex analysis provides powerful tools when it comes to solving the Laplace equation on different domains, thus we want to explore changes of variables defined by complex functions. The analytic function

$$\zeta = g(z) \quad \text{or} \quad \xi + i\eta = p(x, y) + iq(x, y) \tag{1}$$

is a mapping, $g : \Omega \to D$, taking a point $z \in \Omega \subset \mathbb{C}$ to a point $\zeta \in D = g(\Omega) \subset \mathbb{C}$. In many cases, we will choose D to be the unit disk. In order to unambiguously relate functions on Ω to functions on D, we require the mapping in Eq. (1) to be a bijection. Thus, the inverse $z = g^{-1}(\zeta)$ is a well-defined map from D back to Ω. The derivative of the inverse function yields

$$\frac{d}{d\zeta} g^{-1}(\zeta) = \frac{1}{g'(z)} \tag{2}$$

at $\zeta = g(z)$, implying that the derivative of g must be nonzero everywhere on Ω.

Example 2.1.1. An important example is the analytic map

$$\zeta = \frac{z-1}{z+1} = \frac{x^2 + y^2 - 1}{(x+1)^2 + y^2} + i\frac{2y}{(x+1)^2 + y^2}. \tag{3}$$

The mapping is one-to-one with analytic inverse

$$z = \frac{1+\zeta}{1-\zeta} = \frac{1-\xi^2-\eta^2}{(1-\xi)^2+\eta^2} + i\frac{2\eta}{(1-\xi)^2+\eta^2}, \tag{4}$$

provided $z \neq -1$ and $\zeta \neq 1$. This analytic map maps the right half plane $R = \{x = \text{Re } z > 0\}$ to the unit disk $D = \{|\zeta| < 1\}$.

2.1.1 Conformal Transformations

This mathematical formulation of conformal transformations mainly follows [1]. Let γ be a curve in the complex plane z and let $f(z)$ be some analytic function of z. We construct a map from z to w such that $w = f(z)$. Under this transformation, the curve γ in the z-plane is mapped to a curve Γ in the w-plane. While the precise form of Γ depends on the form of γ, there exists a geometrical property of Γ that is independent of γ: Let z_0 be a point on the curve γ, and assume that $f'(z_0) \neq 0$. Under the transformation $w = f(z)$, the tangent to the curve γ at the point z_0 is rotated counterclockwise by $\Theta := \arg(f'(z_0))$ at $w_0 = f(z_0)$.

To show this, we assume that $f(z)$ is some analytic function of z, $\forall z \in D$, and consider a continuous curve γ that is contained in the region D. We call γ an *arc*, to emphasize that our analysis is local. Thus, the arc γ, can be defined parametrically,

$$\gamma \,:\, z(t) = x(t) + iy(t),\ \ t \in [a,b], \tag{5}$$

where $[a,b]$ is a real valued interval. If we let $w = u(x,y) + iv(x,y)$ where $u, v \in \mathbb{R}$, then the image of the arc γ is given by

$$\Gamma \,:\, w(t) = u(x(t), y(t)) + iv(x(t), y(t)),\ \ t \in [a,b]. \tag{6}$$

Since both x and y are continuous functions of t, then both u and v are continuous functions of t, establishing the continuity of Γ. Also, since γ is differentiable, so is Γ. However, an image of an non-intersecting arc is not necessarily non-intersecting itself, but one can avoid this if f is one-to-one. Now, if we define the derivative of z with respect to t as

$$\frac{dz(t)}{dt} = \frac{dx(t)}{dt} + i\frac{dy(t)}{dt},$$

and if we let $f(z)$ be analytic in the domain containing the open neighborhood of $z_0 := z(t_0)$, then the image Γ is $w(t) = f(z(t))$, and by the chain rule

$$\left.\frac{dw(t)}{dt}\right|_{t=t_0} = f'(z_0)\left.\frac{dz(t)}{dt}\right|_{t=t_0}. \tag{7}$$

If $f'(z_0) \neq 0$ and $z'(t_0)$, if follows that $w'(t_0) \neq 0$, and since $w'(t)$, $f'(z)$, and $z'(t)$ are complex valued functions, we can take the argument of both sides of Eq. (7) and get

$$\arg(w'(t_0)) = \arg(z'(t_0)) + \Theta, \tag{8}$$

where we used the fact that the argument of a product of complex numbers is the sum of the arguments of each. Again, we defined Θ as $\arg(f'(z_0))$. By this, we have shown that under a transformation $f(z)$, the direct tangent to any curve through $z_0 \in D$ is rotated by an angle Θ. The consequence of of Eq. (8) is that for points where $f'(z) \neq 0$, analytic transformations preserve angles. That is, if two curves on the z-plane intersect at the point z_0, the angles of both tangents at that point get rotated by Θ under a transformation. Thus, the angle between the tangents on the w-plane equals to the angle between the tangents on the z-plane - it is preserved under that transformation at $w_0 = f(z_0)$. A transformation with this property is called *conformal* and we state this as a theorem:

Theorem 2.1.1. *Assume that $f(z)$ is analytic and not constant $\forall z \in D$ of the complex z-plane. For any point z for which $f'(z) \neq 0$, this mapping is conformal, that is, it preserves the angles between two differentiable arcs.*

In addition to preserving angles, conformal mapping also magnifies distances near the image of z_0 by a factor of $|f'(z_0)|$. Since f is analytic, the following holds,

$$|f'(z_0)| = \lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}. \tag{9}$$

If we let $w = f(z)$, then

$$|w - w_0| \approx |f'(z_0)||z - z_0|. \tag{10}$$

2.1.2 Composition

One of the outstanding capabilities of conformal mapping is that it allows one to assemble a large repertoire of complicated domains by composing elementary mappings [4]. This relies on the fact that the composition of two analytic functions is also analytic.

Proposition 2.1.1. *If $w = f(z)$ is an analytic function of the complex variable $z = x + iy$, and $\zeta = g(w)$ is an analytic function of the complex variable $w = u + iv$, then the composition $\zeta := h(z) = g \circ f = g(f(z))$ is an analytic function of z.*

Proof. Suppose $f'(a)$, $g'(f(a))$ both exist for some $a \in \mathbb{C}$, then we want to show that $(g \circ f)'(a) = g'(f(a))f'(a)$. Since $f'(a)$ exists, f is continuous at a. Also, there exists an $r > 0$ and M such that

$$|g(w) - g(f(a))| \leq M|w - f(a)|\,,\ w \in D(f(a), r).$$

Suppose $f'(a) = 0$, then because f is continuous at a, $f(z) \in D(f(a), r)$ for z near a. For such $z \neq a$ we have

$$\left|\frac{g(f(z)) - g(f(a))}{z - a}\right| \leq M\left|\frac{f(z) - f(a)}{z - a}\right|.$$

The right hand side goes to zero, and thus we get the desired result.

If $f'(a) \neq 0$, then $f(z) \neq f(a)$ for some z in a punctured disc centered at a. For such $z \to a$, by the chain rule,

$$\lim_{z \to a} \frac{g(f(z)) - g(f(a))}{z - a} = \lim_{z \to a} \frac{g(f(z)) - g(f(a))}{f(z) - f(a)} \frac{f(z) - f(a)}{z - a},$$

obtaining $(g \circ f)'(a) = g'(f(a))f'(a)$. □

Thus, proposition 2.1.1 guarantees the existence of $\frac{d\zeta}{dz} = \frac{d\zeta}{dw}\frac{dw}{dz}$. This is enough to show that the composition of two conformal maps is also conformal, which is an immediate result of using the chain rule.

Example 2.1.2. Let us conformally map the lower half plane $L = \{\text{Im } z < 0\}$ to the unit disk $D = \{|\zeta|^2 < 1\}$. Since we already know that the transformation in Eq. (3) maps the right half plane $R = \{\text{Re } w > 0\}$ to $D = g(R)$, we can use the fact that a multiplication by $-i = e^{-i\pi/2}$, with $z = h(w) = -iw$, rotates the complex plane by 90° clockwise and so it maps R to $L = h(R)$. Its inverse $h^{-1}(z) = iz$ will therefore map L to $R = h^{-1}(L)$. Consequently, to map the lower half plane to the unit disk, we compose the two maps, leading to the conformal map $\zeta : L \to D$,

$$\zeta = g \circ h^{-1}(z) = \frac{iz - 1}{iz + 1}. \tag{11}$$

2.1.3 Bilinear Transformations

An important class of conformal mapping we will later use in this project is bilinear mapping, given by the following transformation

$$w = f(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0, \tag{12}$$

where $a, b, c, d \in \mathbb{C}$. Bilinear transformations have a number of remarkable properties, which are global and are valid for any z including $z = \infty$. While all the properties are listed in [1] (pages 351-356), we will provide here the ones that will help us solve the electrostatics problems later in the paper.

Property 1. Conformality: Bilinear transformations are conformal. This can be seen be examining the derivative:

$$f'(z) = \frac{ad - bc}{(cz + d)^2}, \tag{13}$$

As long as $c \neq 0$, and $z \neq \frac{-d}{c}$, then $f'(z) \neq 0$ for all other z and there is a conformal mapping defined by Eq. (12).

Property 2. Mapping Property: Bilinear transformations map circles and lines into circles or lines. For our purposes we would like to show that a circle maps to a circle. The expression for a circle in the complex plane with center z_0 and radius ρ is $(z - z_0)(\bar{z} - \bar{z_0}) = \rho^2$ or

$$z\bar{z} + \bar{B}z + B\bar{z} + c = 0; \;\; B = -z_0, \;\; c = |B|^2 - \rho^2. \tag{14}$$

Then, under the inversion transformation $w = \frac{1}{z}$, the circle is mapped to

$$cw\bar{w} + \bar{B}w + B\bar{w} = 0. \tag{15}$$

Thus for $c \neq 0$, equation (15) is also a circle of radius $\frac{|B|}{|c|}$ and centered at $\frac{-B}{c}$.

Property 3. Inverse Property: Bilinear transformations map inverse points (with respect to a circle) to inverse points. The points p and q are inverse with respect to the circle of radius ρ and center z_0, if z_0, p, q, lie, in that order, on the same line, and the distances $|z_0 - p|$ and $|z_0 - q|$ satisfy $|z_0 - p||z_0 - q| = \rho^2$. If the points indeed lie on the same line, it follows that $p = z_0 + r_1 e^{i\phi}$ and $q = z_0 + r_2 e^{i\phi}$, for some angle ϕ and $r_1, r_2 \in \mathbb{R}$. If the points are inverse, then $r_1 r_2 = (p - z_0)(\bar{q} - \bar{z}_0) = \rho^2$. Thus, the two inverse points are

$$p = z_0 + re^{i\phi}, \quad q = z_0 + \frac{\rho^2}{r} e^{i\phi}, \quad r \neq 0. \tag{16}$$

Using the above equation and $z = z_0 + \rho e^{i\theta}$, we have

$$\frac{z-p}{z-q} = \frac{\rho e^{i\theta} - re^{i\phi}}{\rho e^{i\theta} - \frac{\rho^2}{r} e^{i\phi}} = \frac{r}{\rho} \cdot \frac{re^{-i\theta} - \rho e^{-i\phi}}{re^{i\theta} - \rho e^{i\phi}} \cdot \left(-e^{i\phi} e^{i\theta}\right), \tag{17}$$

and taking the absolute value of both sides yields

$$\left| \frac{z-p}{z-q} \right| = k, \quad k \in (0, 1]. \tag{18}$$

If $k \neq 1$, the above equation represents a circle with center at z_0 and radius ρ given by

$$z_0 = \frac{p - k^2 q}{1 - k^2}, \quad \rho = \frac{k|p - q|}{1 - k^2}, \tag{19}$$

where p and q are inverse points with respect to the circle and p is inside the circle.

Indeed, if $k \neq 1$, then Eq. (18) yields

$$\begin{aligned}
&(z - p)(\bar{z} - \bar{p}) = k^2 (z - q)(\bar{z} - \bar{q}) \\
&(1 - k^2) z\bar{z} + (k^2 \bar{q} - \bar{p}) z + (k^2 q - p)\bar{z} + (p\bar{p} - k^2 q\bar{q}) = 0 \\
&z\bar{z} + \frac{k^2 \bar{q} - \bar{p}}{1 - k^2} z + \frac{k^2 q - p}{1 - k^2} \bar{z} + \frac{p\bar{p} - k^2 q\bar{q}}{1 - k^2} = 0,
\end{aligned}$$

which turns out to be Eq. (14), describing a circle with

$$B = \frac{k^2 q - p}{1 - k^2} = -z_0, \quad c = \frac{|p|^2 - k^2 |q|^2}{1 - k^2}.$$

Using Eq. (18) we can demonstrate that bilinear transformations map inverse points to inverse points. If we let $w = f(z) = az + b$ then we must show that the points $\tilde{p} = ap + b$ and $\tilde{q} = aq + b$ are inverse points with respect to the circle in the complex w-plane:

$$\left|\frac{w - \tilde{p}}{w - \tilde{q}}\right| = \left|\frac{az + b - ap - b}{az + b - aq - b}\right| = \left|\frac{z - p}{z - q}\right| = k,$$

showing that indeed $\tilde{p}$ and $\tilde{q}$ are inverse points in the w-plane. We can show a similar result if we take $w = 1/z$.

2.2 Conformal Mapping of Annuli

Consider the region bounded by two cylinders perpendicular to the z-plane, where their bases are the discs bounded by the two circles $|z| = R$ and $|z - a| = r,\ \ 0 < a < R - r,\ \ R, r, a \in \mathbb{R}$. Recall that Eq. (18) is the equation of a circle with respect to which the points p and q are inverse to one another. If we fix p and q, and let k vary, $|z - p| = k|z - q|$ describes a family of non-intersecting circles, which the two circles we are considering are a part of. Thus, given that p and q are inverse to one another, $pq = R^2$ for he first circle and $(p - a)(q - a) = r^2$ for the other. Solving for the two we find

$$q = \frac{R^2}{p} \text{ and } p = \frac{R^2 + a^2 - r^2 - A}{2a}, \tag{20}$$

where $A^2 := (R^2 + a^2 - r^2)^2 - 4a^2R^2$ and the sign of A is fixed by taking p inside and q outside both circles. Note that $p, q \in \mathbb{R}$. The bilinear transformation $w = \tilde{\kappa}(z - p)/(z - q)$ maps the above family of non-intersecting circles into a family of concentric circles. First, using Eq. (20),

$$w = \tilde{\kappa}\frac{z - p}{z - q} = \tilde{\kappa}\frac{z - p}{z - R^2/p} = -\tilde{\kappa}p\frac{z - p}{R^2 - pz},$$

we define $\kappa := -\tilde{\kappa}p$ such that

$$w = \kappa\frac{z - p}{R^2 - pz}. \tag{21}$$

As a convention, we choose to map the circle $|z| = R$ to the unit circle $|w| = 1$, thus we let $z = Re^{i\theta}$ and plug it to the above equation

$$w = \kappa\frac{Re^{i\theta} - p}{R^2 - pRe^{i\theta}} = \frac{\kappa}{Re^{i\theta}} \cdot \frac{Re^{i\theta} - p}{Re^{-i\theta} - p}.$$

Taking the absolute value of both sides,

$$|w| = \left|\frac{\kappa}{Re^{i\theta}}\right| \cdot \left|\frac{Re^{i\theta} - p}{Re^{-i\theta} - p}\right| = \frac{|\kappa|}{R} \cdot 1 = 1,$$

yielding $\kappa = R$. Thus, we get the bilinear map of the form

$$w = f(z) = R\frac{z - p}{R^2 - pz} = \left(\frac{-R}{p}\right)\left(\frac{z - p}{z - q}\right), \tag{22}$$

with p defined in Eq. (20). While we imposed that $|z| = R$ will be mapped $|w| = 1$, we want to find the correct mapping for the other circle $|z - a| = r$ onto $|w| = \rho_0$. From Eq. (18) we get $k_2 = |z - p|/|z - q|$ and using Eq. (19) for the circle $|z - a| = r$; $z_0 = a$, we get $k_2^2 = (p - a)/(q - a)$. Thus, taking the absolute value of Eq. (22) yields

$$|w| = \frac{|R|}{|p|}\frac{|z - p|}{|z - q|} = \frac{|R|}{|p|}k_2 = \frac{|R|}{|p|} \cdot \sqrt{\frac{p - a}{q - a}}, \tag{23}$$

and finally,

$$\rho_0 = \frac{|R|}{|p|} \cdot \sqrt{\frac{p - a}{q - a}}. \tag{24}$$

We have obtained everything we need to map the class of all non-concentric circles to concentric ones. This will be applied in the next section to solve the Laplace equation.

2.3 Application: Electrostatics

Having covered the theory behind conformal maps in a rigorous manner, we now move to study applications to problems in the field of electrostatics. The primary goal of electrostatics is to find the spatially dependent electric field $\vec{E}(x, y, z)$ due to a spatially dependent charge distribution $\rho(x, y, z)$ [3]. This can always be done numerically using Coulomb's law in vector integral form, however, we wish to find analytical solutions where possible. As it happens, but for a small class of problems, Coulomb's law in vector integral form cannot be used to find analytical solutions. Further, this method assumes the charge distribution $\rho(x, y, z)$ is known. An alternative approach involves finding the electric potential $V(x, y, z)$ due to the charge distribution, and this can be done using a modified version of Coulomb's law in scalar integral form. Then, the electric field can be found by taking the negative gradient of the electric potential: $\vec{E}(x, y, z) = -\vec{\nabla} V(x, y, z)$. Although this approach is easier to use and can be applied to a wider variety of problems, it still requires apriori knowledge of the charge distribution $\rho(x, y, z)$. It is often the case that we wish to find the electric field (by first finding the potential) in some physical domain knowing only the potential on the boundary of this domain. This type of problem is known as a boundary-value problem and it frequently requires one to solve partial differential equations.

Before we solve boundary value problems in electrostatics, we need to re-cast Coulomb's law into differential form to obtain Poisson's equation: $\nabla^2 V(x, y, z) = \rho(x, y, z)/\epsilon_0$. For the purpose of this project, we will focus on finding the potential in charge-free regions and restrict ourselves to 2-dimensional systems. Further, since the electric field can always be found by simply computing the negative gradient of the potential, we will not bother with calculating it. In this case Poisson's equation reduces to the 2-dimensional version of Laplace's equation: $\nabla^2 V(x, y) = 0$. This is a partial differential equation, and in simple cases it can be solved using standard techniques (such as the separation of variables). However, for inconvenient geometries with complicated boundary conditions, this becomes impossible, and we are often forced to use numerical methods. Rather surprisingly, conformal maps can allow us to solve some of these problems analytically. The overall idea involves re-writing the entire boundary value problem in a new coordinate system where an analytical solution can be found with ease [5]. Then, one is able to transform the solution back into the old coordinate system to obtain a solution to the original problem. Before we move to study a specific examples, we will go over some important technical details [2].

A function which satisfies Laplace's equation is known as a harmonic function and harmonic functions are analytic. Assume we want to solve a boundary value problem for Laplace's equation on some domain in a given coordinate system. We first wish to show that when we use a conformal map to move to a new coordinate system, we are still dealing with the same partial differential equation (Laplace's equation). In other words, we want to show that when a harmonic function undergoes a conformal transform, it is still harmonic.

Proposition 2.3.1. *If $U(\xi, \eta)$ is an harmonic function of ξ and η and there exists a conformal map $g(z) = \zeta$ where $\zeta = \xi + i\eta$ and $z = x + iy$ then, $V(x, y) = U(\xi(x, y), \eta(x, y))$ is also a harmonic function.*

Proof. Begin by using the chain rule for differentiation:

$$V_x = U_\xi \xi_x + U_\eta \eta_x \text{ and } V_y = U_\xi \xi_y + U_\eta \eta_y \tag{25}$$

$$V_{xx} = U_{\xi\xi}(\xi_x)^2 + 2U_{\eta\xi}(\xi_x)(\eta_x) + U_{\eta\eta}(\eta_x)^2 + U_\xi \xi_{xx} + U_\eta \eta_{xx} \tag{26}$$

$$V_{yy} = U_{\xi\xi}(\xi_y)^2 + 2U_{\eta\xi}(\xi_y)(\eta_y) + U_{\eta\eta}(\eta_y)^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy} \tag{27}$$

Now we invoke the Cauchy-Reimann conditions:

$$\xi_x = -\eta_y \text{ and } \xi_y = \eta_x \tag{28}$$

Then, using all this, we can show that

$$\nabla^2 V = |g'(z)|^2 \, \nabla^2 U \tag{29}$$

Thus, if $U(\xi, \eta)$ is an harmonic function of ξ and η then $\nabla^2 U = 0$ and from the above relation this implies that $\nabla^2 V = 0$, and this means $V(x, y)$ is a harmonic function of x and y. Note that since $g(z)$ is a conformal transform, it is analytic and therefore we can be sure that $g'(z)$ is finite.

□

So we have shown that a conformal map preserves the harmonic nature of functions. This means that, once we map our problem into a new coordinate system, we are still dealing with the same partial differential equation, but possibly with different boundary conditions on a different domain. We can now summarize all this in a compact manner. Assume we wish to solve the following problem:

PROBLEM 1: $\nabla^2 U = 0$ on ζ where $\zeta = \xi + i\eta$ and $\partial U = \Psi(\xi, \eta)$

If there is a suitable conformal map $g(z) = \zeta$, we can rewrite this as:

PROBLEM 2: $\nabla^2 V = 0$ on z where $z = x + iy$ and $\partial V = F(x, y)$

If the second problem is easy to solve analytically, once we find $V(x, y)$ we can use our conformal map to find $U(\xi, \eta)$ thereby solving the original boundary value problem. With all this said, we have covered the fundamentals of solving boundary value problems using conformal maps. We will now move to work with a specific example which has to do with finding the potential inside a 2-layer coaxial cable when the inner cable is not centered. This is described in detail below:

Example 2.3.1. Considering the following boundary value problem:

$$\textbf{PDE: } \nabla^2 V = 0 \text{ for } |z| < 1 \text{ and } \left|z - \frac{2}{5}\right| > \frac{2}{5}$$

$$\textbf{BCs: } V = b \text{ when } |z| = 1 \text{ and } V = a \text{ when } \left|z - \frac{2}{5}\right| = \frac{2}{5}$$

Essentially we want to solve Laplace's equation to find the potential inside an annulus with an off-centered hole given the potential on the boundaries. Fig. 2 below shows the shaded region where we wish to find the potential.

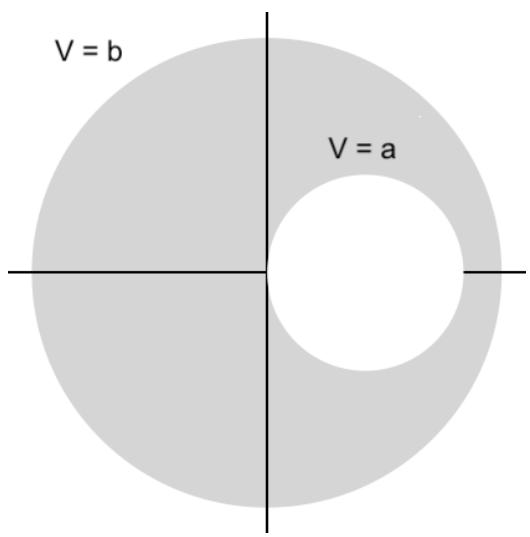


Figure 2: Boundary Value Problem in the z Plane

Due to the asymmetry, it is very difficult to solve this problem directly, but we can use a suitable conformal map to help us. To obtain the bilinear map that will map the two non-concentric circles to concentric ones, we plug $R = 1$, $r = 2/5$, and $a = 2/5$ to Eq. (20) to obtain p and q and by that we have the map in Eq. (22) and the radius of the inner circle in Eq. (24). Ultimately, this gives us the following set of results:

$$\zeta = \frac{1-2z}{z-2} \quad \text{and} \quad \rho_0 = \frac{1}{2}$$

With that, we can re-write our boundary value problem as follows:

$$\textbf{PDE: } \nabla^2 U = 0 \text{ for } \frac{1}{2} < |\zeta| < 1$$

$$\textbf{BCs: } U = b \text{ when } |\zeta| = 1 \text{ and } U = a \text{ when } |\zeta| = \frac{1}{2}$$

So we have now moved to the following domain visualized in Fig 3 below:

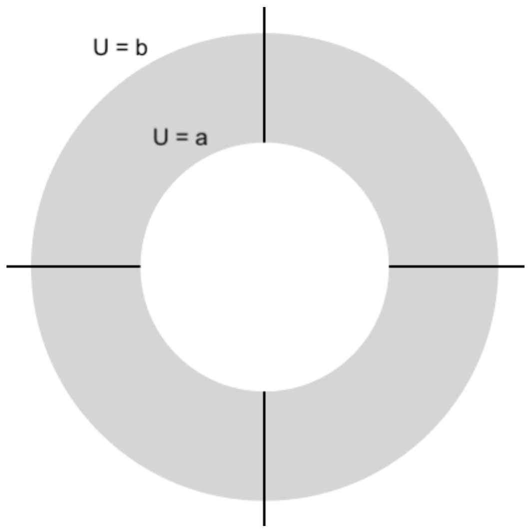


Figure 3: Boundary Value Problem in the ζ Plane

It so happens that we can solve the problem in this new domain very easily. The details of this are given below:

Since we have radial symmetry, we can claim that the potential $U(\xi, \eta) = U(\gamma)$ where $\gamma = |\zeta| = \sqrt{\xi^2 + \eta^2}$. Therefore we have that :

$$\nabla^2 U(\gamma) = 0$$

Using the definition of the laplacian in polar coordinates, and by noting that the potential does not depend on the phase angle, we can show that:

$$\frac{1}{\partial\gamma}\left(\gamma\cdot\frac{U(\gamma)}{\partial\gamma}\right)=0$$

Now, we integrate both sides with respect to γ and this implies:

$$\gamma\cdot\frac{U(\gamma)}{\partial\gamma}=C_1$$

Where C_1 is some constant. By rearranging the variables and integrating with respect to γ again we get:

$$U(\gamma)=C_1\cdot\ln(\gamma)+C_2$$

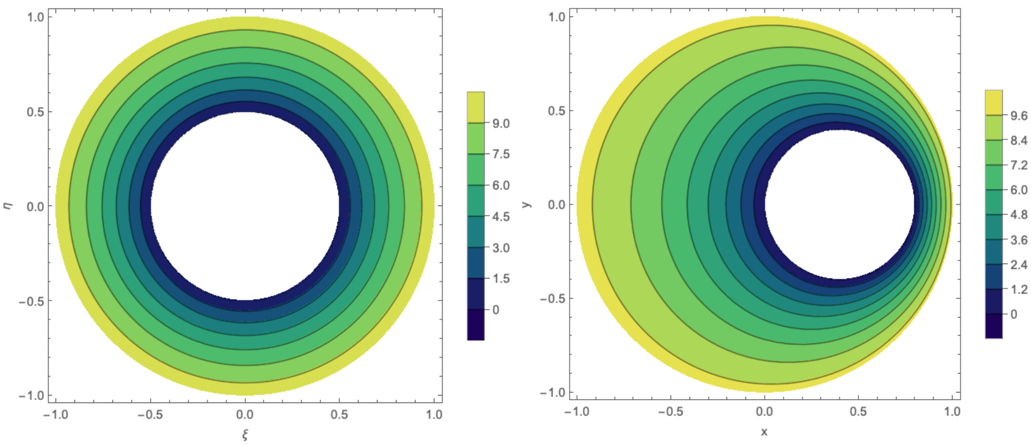
Where C_2 is once again some constant. Finally, we use the boundary condition to determine the unknown constants, and this gives:

$$U(\gamma)=\frac{b-a}{\ln(2)}\cdot\ln(\gamma)+b=U(\xi,\eta)=\frac{b-a}{2\ln(2)}\cdot\ln(\xi^2+\eta^2)+b$$

So we have successfully solved the boundary value problem in the ζ plane, and all that is left to do is to move back to the z plane. To do this we just run our function through the conformal map and this yields:

$$V(x,y)=\frac{b-a}{\ln(2)}\ln\left(\left|\frac{1-2z}{z-2}\right|\right)=\frac{b-a}{2\ln(2)}\cdot\ln\left(\frac{1-4x+4x^2+4y^2}{4-4x+x^2+y^2}\right)+b$$

With this, we have completed our solution to the original boundary value problem! Since we have the potential in our domain of interest, it is easy to find the electric field by taking the negative gradient of the potential if required. We can see the contour plots of the potential field on both ζ and z planes in Fig. 4 below, where we have set $a=0$ and $b=10$ (arbitrary units).



(a) Potential U on ζ Plane (b) Potential V on z Plane

Figure 4: (a) Solution to the problem on a symmetric domain. (b) Solution to the original problem on an asymmetric domain

Qualitatively, we can see that the solutions on both domains make sense. The boundary conditions appear to be met and we see a gradual growth of the potential from the inner boundary to the outer boundary. Further, there are no maximum or minimum points inside the domain, they are only on the boundaries (as expected for a valid solution to Laplace's equation).

Next, we consider a slightly more complicated variant of the same problem where the position of the inner cavity has been rotated about the origin.

Example 2.3.2. Consider the following boundary value problem:

$$\textbf{PDE: } \nabla^2 V = 0 \text{ for } |z| < 1 \text{ and } \left| z - \frac{2(1+i)}{5\sqrt{2}} \right| > \frac{2}{5}$$

$$\textbf{BCs: } V = b \text{ when } |z| = 1 \text{ and } V = a \text{ when } \left| z - \frac{2(1+i)}{5\sqrt{2}} \right| = \frac{2}{5}$$

This is essentially the same problem as the previous example, but rotated by $\pi/4$ radians counterclockwise. Fig. 5 shows the shaded region where we wish to find the potential. Using Prop. 2.1.1, we can easily compose a new map to help us solve this problem.

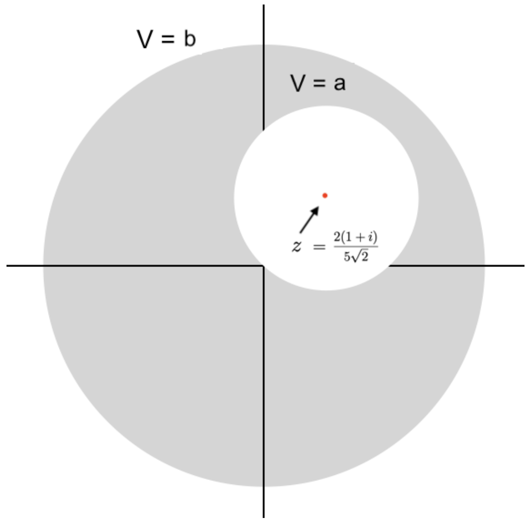


Figure 5: Boundary Value Problem in the z Plane

To rotate the z plane by $\pi/4$ radians counterclockwise, we use the following map:

$$z(\chi) = \chi e^{i\pi/4}$$

Thus, if we want to solve our problem using the previous example's result, we must use the inverse map:

$$\chi(z) = ze^{-i\pi/4}$$

We can obtain the desired transformation by composing χ and ζ from the previous example as follows

$$\psi(z) = \zeta \circ \chi\ (z) = \frac{1 - 2ze^{-i\pi/4}}{ze^{-i\pi/4} - 2} \tag{30}$$

Following similar steps to the previous example, we can solve the boundary value problem on the symmetric domain and map it back to z plane. The visualization of the resulting field is demonstrated in Fig. 6 below where we have again set $a = 0$ and $b = 10$ (arbitrary units):

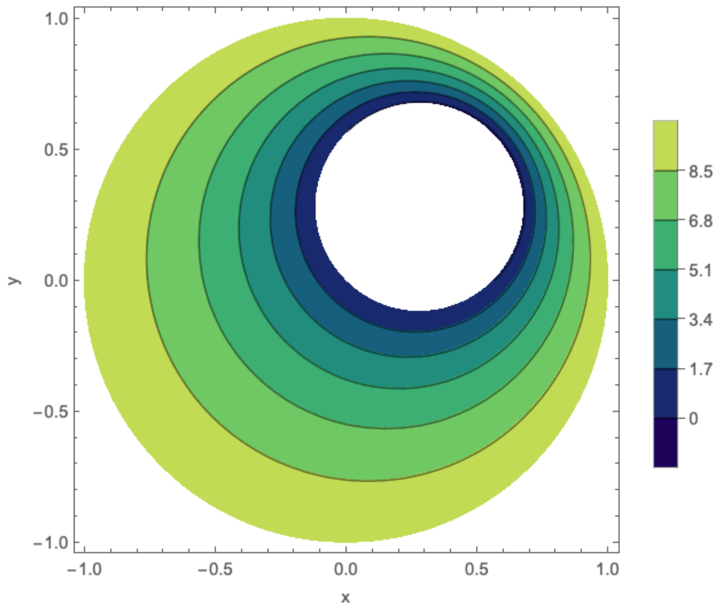


Figure 6: Contour plot of the potential field $V(x,y)$, due to a rotated cavity at $\pi/4$ radians above the horizontal axis

Once again, qualitatively, this solution makes sense. The boundary conditions are met, and the maximum and minimum points occur only on the boundary of the domain as one would expect for a valid solution to Laplace's equation.

For the same (inner and outer) radii and boundary conditions as above, if we were to shift the off-centered inner circle by θ radians instead, the resulting potential field on z can be shown to be:

$$V(x,y) = \frac{5}{\ln(2)} \ln\left(\frac{(2e^{i\theta}(x-iy)-1)(e^{i\theta}-2(x+iy))}{(e^{i\theta}(x-iy)-2)(2e^{i\theta}-x-iy)}\right) + 10. \tag{31}$$

So this gives us a more general solution. In fact, it is possible to solve the problem for variable inner and outer radii , however we have omitted this because the solution is very tedious.

Overall, looking back at this section, we have demonstrated the power of conformal maps and used them to solve a non-traditional boundary value problem for Laplace's equation. Although we have looked at one specific problem, it is worth mentioning that conformal maps can be used to solve a wide variety of boundary value problems.

3 Conclusion and Future Directions

3.1 Conclusion

In this project, we have demonstrated how we can solve the boundary value problem for the electric potential in a non-concentric auxiliary cable. At the typical level of undergraduate electromagnetism, this problem could be very hard to solve with standard techniques. With the help of conformal mapping, we were able to show how one could map the non-concentric case to the well-known concentric one, solve for the potential with relative ease, and finally map it back.

After walking the reader through elementary concepts and properties of conformal mapping, we were able to construct a framework that results in map that could be applied to any case of annuli and by that solve for the potential field in a non-concentric auxiliary cable problem, relying only on the initial dimensions and boundary conditions. We have demonstrated this through examples and provided diagrams to help visualize how the potential fields behave under these transformations.

3.2 Future Directions

In this report we have exclusively covered the application of conformal mapping to the homogeneous Laplace's equation, however conformal mapping can also be applied to the non homogeneous Poisson's equation defined by:

$$\nabla^2 V = \rho(x,y)/\epsilon_0 \text{ for } (x,y) \in \Omega$$

on some domain $\Omega \subset \mathbb{R}^2$. We use Poisson's equation in the presence of electric charges in the domain. This would allow us to solve a wider class of problems in the field of electrostatics.

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