Homework 11: Quantum Phase Estimation

Quantum phase estimation is one of the most important subroutines in quantum computation. It serves as a central building block for many quantum algorithms. The objective of the algorithm is the following:

Given a unitary operator U, the algorithm estimates θ in $U|\psi\rangle = e^{\mathbf{2\pi i\theta}}|\psi\rangle$. Here $|\psi\rangle$ is an eigenvector and $e^{\mathbf{2\pi i\theta}}$ is the corresponding eigenvalue. Since U is unitary, all of its eigenvalues have a norm of 1.

In [1]:
```python
### START DO NOT CHANGE ###
import numpy as np
from numpy import pi
from qiskit import QuantumCircuit, transpile, assemble, Aer, IBMQ, ClassicalRegister, QuantumRegister
from qiskit.providers.ibmq import least_busy
from qiskit.tools.monitor import job_monitor
from qiskit.visualization import plot_histogram, plot_bloch_multivector
import os.path
import matplotlib.pyplot as plt
import math

# Common variables
nqubits = 3
shots = 2048

def swap_registers(circuit, n):
    for qubit in range(n//2):
        circuit.swap(qubit, n-qubit-1)
    return circuit

def qft_rotations(circuit, n):
    """Performs qft on the first n qubits in circuit (without swaps)"""
    if n == 0:
        return circuit
    n -= 1
    circuit.h(n)
    for qubit in range(n):
        circuit.cp(pi/2**(n-qubit), qubit, n)
    # At the end of our function, we call the same function again on
    # the next qubits
    # NOTE: we reduced n by one earlier in the function
    qft_rotations(circuit, n)

def qft(circuit, n):
    """QFT on the first n qubits in circuit"""
    qft_rotations(circuit, n)
    swap_registers(circuit, n)
    return circuit.decompose() # .decompose() allows us to see the individual gates

def inverse_qft(circuit, n):
    """Does the inverse QFT on the first n qubits in circuit"""
    # First we create a QFT circuit of the correct size:
    qft_circ = qft(QuantumCircuit(n), n)
    # Then we take the inverse of this circuit
    invqft_circ = qft_circ.inverse()
    # And add it to the first n qubits in our existing circuit
    circuit.append(invqft_circ, circuit.qubits[:n])
    return circuit.decompose() # .decompose() allows us to see the individual gates

def qft_dagger(qc, n):
    """n-qubit QFTdagger the first n qubits in circ"""
    # Don't forget the Swaps!
    for qubit in range(n//2):
        qc.swap(qubit, n-qubit-1)
    for j in range(n):
        for m in range(j):
            qc.cp(-math.pi/float(2**(j-m)), m, j)
        qc.h(j)

def load_IBMQ():
    IBMQ.save_account('57ea5b774ba3ee6147f3fe472cf668743a6f248a4daea41769ebf98c286eff10c9c0ea88430fd2ac4680bee4f52df40b3c373d214390f8e77f2870479f6867d8', overwrite=True)
    IBMQ.load_account()
    provider = IBMQ.get_provider(hub='ibm-q')
    backend = least_busy(provider.backends(filters=lambda x: x.configuration().n_qubits >= nqubits
                                       and not x.configuration().simulator
                                       and x.status().operational==True))

    return provider, backend

def simulate(circuit):
    qc = circuit.copy()
    qobj = assemble(qc, shots=shots)
    job = sim.run(qobj)
    job_monitor(job)
    return job.result().get_counts()


def run_job(circuit):
    qc = circuit.copy()
    transpiled_qc = transpile(qc, backend, optimization_level=3)
    job = backend.run(transpiled_qc, shots=shots)
    job_monitor(job)

    return job.result().get_counts()

provider, backend = load_IBMQ()
sim = Aer.get_backend("aer_simulator")

### END DO NOT CHANGE ###
```

As an example, we will run QPE on the T-gate to estimate its phase. The T-gate is given by

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \tag{1}$$

Running QPE on T will give us θ where:

$$T|1\rangle = e^{2i\pi\theta}|1\rangle$$

We expect to find:

$$\theta = \frac{1}{8}$$

The Problem

In this notebook, you will estimate the phase of the rotation gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{8\pi i}{5}} \end{bmatrix}$$

using 3 and 5 counting qubits.

(a) What are the eigenstates and eigenvalues of this operator? What is the phase θ?

|0> is an eigenstate with eigenvalue 1 and |1> is an eigenstate with eigenvalue $e^{\frac{8i\pi}{5}}$

Moving forward, we let $|\psi\rangle$ be the eigenstate of S that is not equal to $|0\rangle$.

Three Counting Qubits

We will first estimate the phase θ using 3 qubits.

(b) How many qubits total do we need for this process?

4 qubits are needed

(c)(i) State preparation: We provide a circuit that has the amount of qubits from answer (b). Your job is to prepare the eigenstate $|\psi\rangle$ on the last qubit.

In [2]:

```
### BEGIN DO NOT CHANGE ###
qpe = QuantumCircuit(4,3)
### END DO NOT CHANGE ###
### BEGIN YOUR CODE ###
qpe.x(3);
### BEGIN YOUR CODE ###
```

(c)(ii) Apply H-gates to all the counting qubits

In [3]:

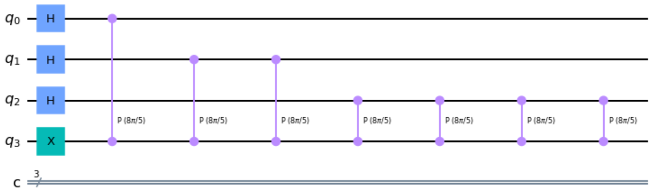
```
### BEGIN YOUR CODE ###
qpe.h(0);
qpe.h(1);
qpe.h(2);
### BEGIN YOUR CODE ###
```

(d) Now we need to add the controlled U operations. Most of the logic is in the provided method `controlled_U`, your task is to call this function with the correct angle.

In [4]:

```
### BEGIN DO NOT CHANGE ###
def controlled_U(circuit, angle, num_counting_qubits):
    repetitions = 1
    for counting_qubit in range(num_counting_qubits):
        for i in range(repetitions):
            circuit.cp(angle, counting_qubit, num_counting_qubits);
        repetitions *= 2

    return circuit
### END DO NOT CHANGE ###

### BEGIN YOUR CODE ###

controlled_U(qpe, (pi*(8/5)), 3)

### END YOUR CODE ###

qpe.draw('mpl')
```

Out[4]:

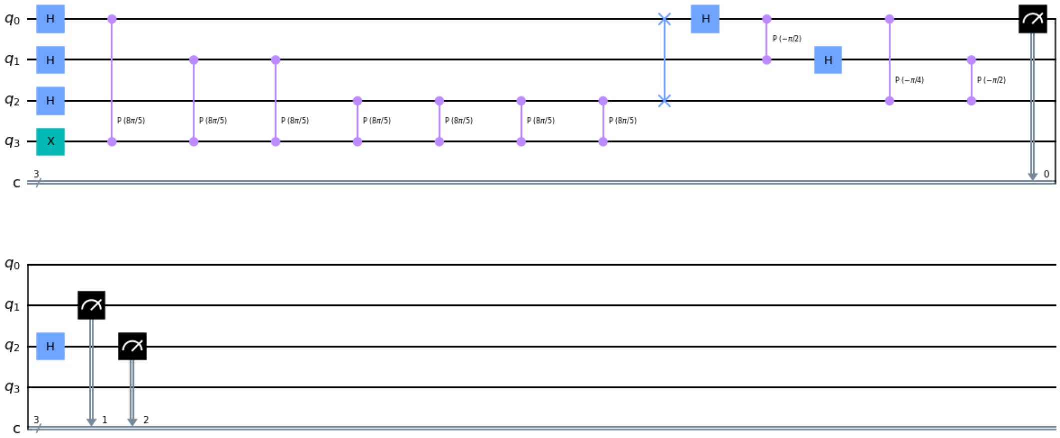


(e) Lastly, add the inverse QFT to the appropriate qubits using the `qft_dagger` method

In [5]:

```
### BEGIN YOUR CODE ###

qft_dagger(qpe, 3)

### BEGIN YOUR CODE ###
### BEGIN DO NO CHANGE ###
for n in range(3):
    qpe.measure(n,n)
qpe.draw('mpl')
### END DO NO CHANGE ###
```

Out[5]:



In [6]:

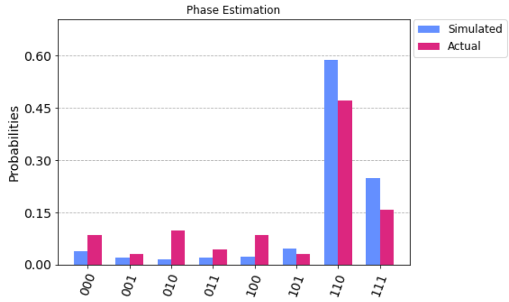
```
### BEGIN DO NO CHANGE ###
counts_simulated = simulate(qpe)
counts_real = run_job(qpe)
### END DO NO CHANGE ###
```

```
Job Status: job has successfully run
Job Status: job has successfully run
```

In [7]:

```
### BEGIN DO NO CHANGE ###
legend = ['Simulated', 'Actual']
plot_histogram([counts_simulated, counts_real], legend=legend, bar_labels=False, title='Phase Estimation')
### END DO NO CHANGE ###
```

Out[7]:



If $x_1x_2x_3$ is a bitstring that occurs with high probability and x is its decimal representation, then we expect that $\theta \approx x/2^3$. (The 3 corresponds to the 3 counting qubits)

(f) Interpret your results: which bitstring(s) have the highest probability? How does this correspond to the real θ?

Bitstring 110 which corresponds to 7 has the highest probability. We therefore expect $\theta \approx \frac{7}{8} = 0.875$ and this is pretty similar to the expected value of $\theta = \frac{4}{5} = 0.800$$

We will now repeat steps (b)-(f) from above using five counting qubits instead of three.

Five Counting Qubits

To get more precision we simply add more counting qubits. We are going to add two more counting qubits for 5 total.

(g) How many qubits total do we need for this process?

6 qubits are needed

(h)(i) State preparation: We construct a circuit that has the amount of qubits from answer (g). Your job is to prepare the eigenstate $|\psi\rangle$ on the last qubit.

In [8]:
```
### BEGIN DO NOT CHANGE ###
qpe5 = QuantumCircuit(6,5)
### END DO NOT CHANGE ###

### BEGIN YOUR CODE ###
qpe5.x(5);
### END YOUR CODE ###
```

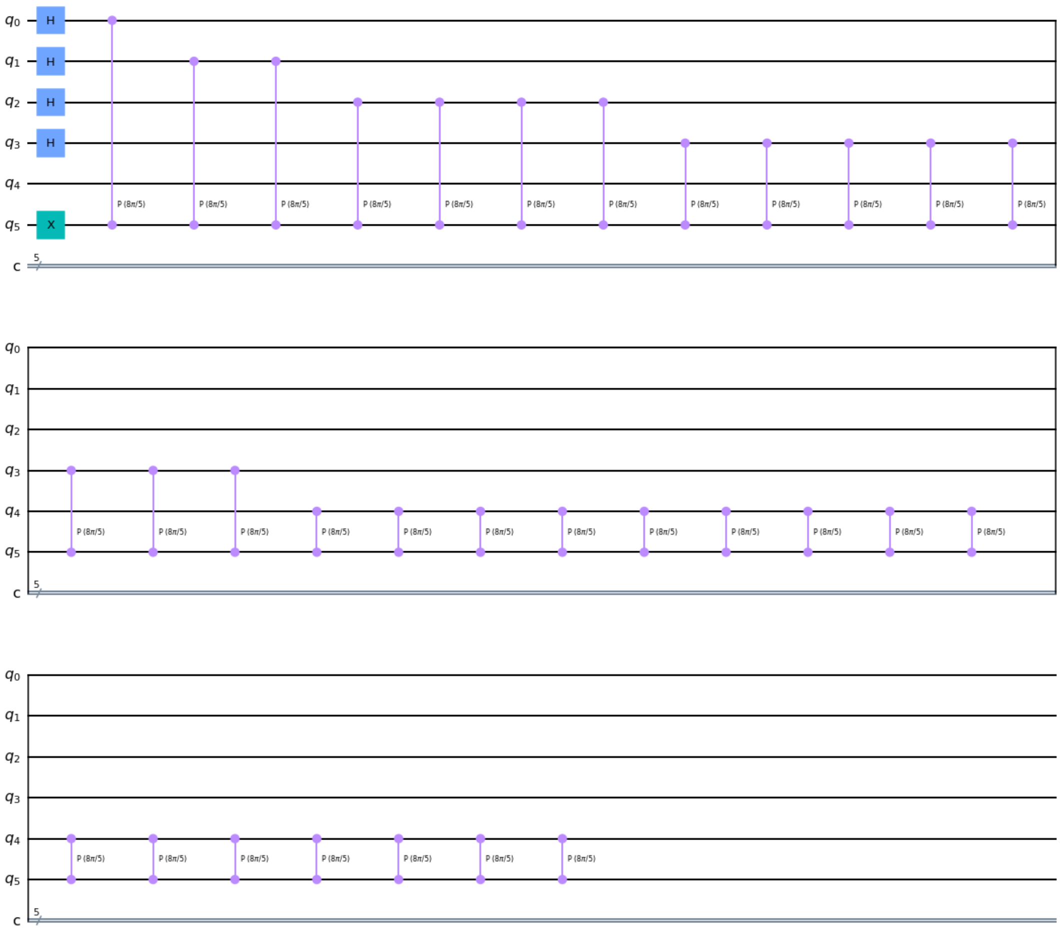
(h)(ii) Apply H-gates to all the counting qubits

In [9]:
```
### BEGIN YOUR CODE ###

for i in range(0,4,1):
    qpe5.h(i)

### END YOUR CODE ###
```

(i) Add the controlled-U gates

In [11]:
```
### BEGIN YOUR CODE ###

controlled_U(qpe5, (pi*(8/5)),5)

### END YOUR CODE ###

qpe5.draw('mpl')
```

Out[11]:

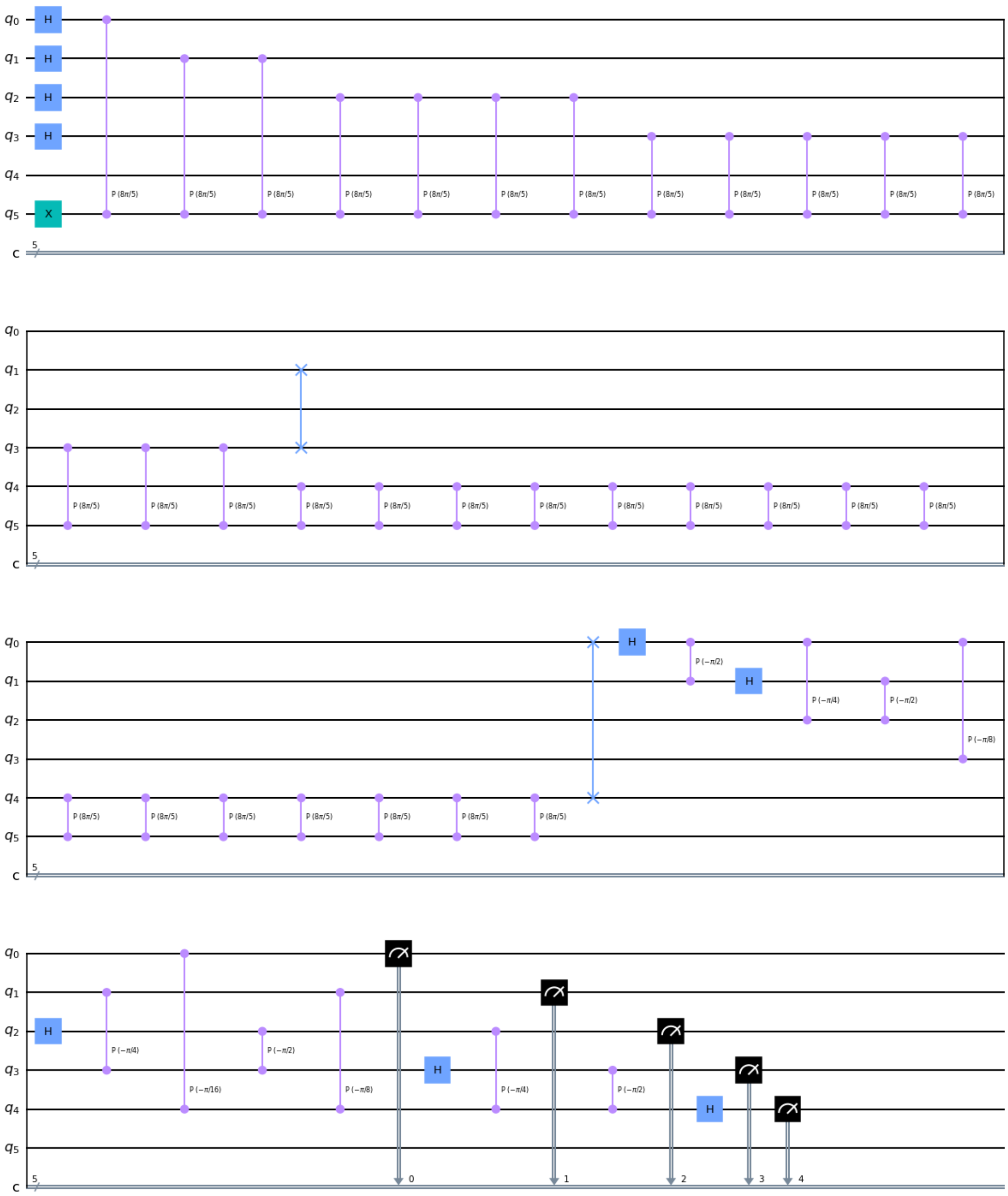


(j) Lastly, add the inverse QFT to the appropriate qubits using the `qft_dagger` method

In [13]:

```
### BEGIN YOUR CODE ###

qft_dagger(qpe5,5)

### END YOUR CODE ###

### BEGIN DO NO TOUCH ###
for n in range(5):
    qpe5.measure(n,n)
qpe5.draw('mpl')
### END DO NO TOUCH ###
```

Out[13]:

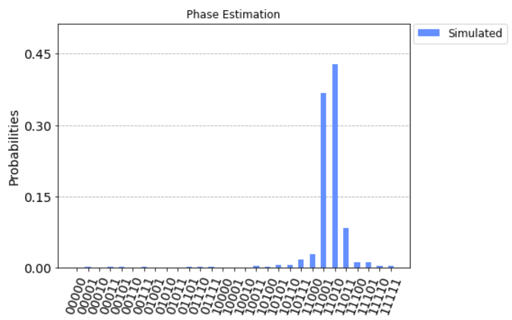


In [14]:
```
### BEGIN YOUR CODE ###
counts_simulated = simulate(qpe5)
### END YOUR CODE ###
```

```
Job Status: job has successfully run
```

In [15]:
```
### BEGIN YOUR CODE ###
legend = ['Simulated']
plot_histogram([counts_simulated], legend=legend, bar_labels=False, title='Phase Estimation')
### END YOUR CODE ###
```

Out[15]:



(k) Once again, interpret your results. How do these approximations compare to those in the 3-qubit case?

Bitstring 11010 which corresponds to 26 has the highest probability, we therefore expect $\theta \approx \frac{26}{32} = 0.8125$ and this is pretty close to the expected value of $\theta = \frac{4}{5} = 0.800$

As we can see the results are closer to the expectation, so there is greater accuracy and this is because we have more qubits.

References

[1] Michael A. Nielsen and Isaac L. Chuang. 2011. Quantum Computation and Quantum Information: 10th Anniversary Edition (10th ed.). Cambridge University Press, New York, NY, USA.

In [16]:
```
import qiskit.tools.jupyter
%qiskit_version_table
```

Version Information

Qiskit Software	Version
qiskit-terra	0.20.2
qiskit-aer	0.10.4
qiskit-ignis	0.7.1
qiskit-ibmq-provider	0.19.1
qiskit	0.36.2
System information	
Python version	3.9.7
Python compiler	Clang 10.0.0
Python build	default, Sep 16 2021 08:50:36
OS	Darwin
CPUs	8
Memory (Gb)	8.0
	Sat Sep 17 21:10:37 2022 MDT

In [ ]: