CSCI-PHYS 3090 - Quantum Computing - Spring 2022
Homework #11

Due Monday, April 25, at the start of class

Homework is graded for clarity of explanation as much as for mere "correctness" of the final answer. You will earn partial credit much easier if your writing is legible and organized. Please scan and submit online, making sure that your scan can be easily read!

Problem 1: Combined QFT (40 points)

The goal of this problem is to construct a quantum circuit for the QFT on a larger Hilbert space using two QFTs on smaller spaces (and another operation). That is, for some computational basis state $|x\rangle$ in a pq-dimensional Hilbert space, we wish to take

$$|x\rangle \to \frac{1}{\sqrt{pq}} \sum_{y=0}^{pq-1} e^{2\pi i xy/pq} |y\rangle$$

for relatively prime p and q using a QFT mod p and a QFT mod q.

(a) Show that the sets $A := \{p \cdot 0 \mod q, \, p \cdot 1 \mod q, \ldots, p \cdot (q-1) \mod q\}$ and $B := \{0, 1, \ldots, q-1\}$ are equal. That is, construct a bijection $f : A \to B$. You must prove that f is indeed a 1–1 mapping. [Hint: Use the facts that p and q are relatively prime. This is in fact not true if they are not coprime; think of 2 and 4 as a counterexample.]

[Hint: You are being asked to show that the map $x \to p \cdot x$ is injective, which means that $p \cdot x = p \cdot y \mod q$ is true if and only if $x = y \mod q$. After you show this it is sufficient to check that A and B have an equal number of elements.]

Given some $0 \leq x < pq$, we can decompose x into $x = x_1 p + x_2$ where $0 \leq x_1 < q$ and $0 \leq x_2 < p$. Similarly for $0 \leq y < pq - 1$ we can write $y = y_1 q + y_2$ for $0 \leq y_2 < q$ and $0 \leq y_1 < p$.

Now we can rewrite our computational basis state:

$$|x\rangle = |x_1 p + x_2\rangle = |\widetilde{x_1}\rangle \, |x_2\rangle$$

where $\widetilde{x_1} = x_1 p \mod q$.

(b) Fill in the blanks:

(i) $|x\rangle$ is an _____-dimensional vector.

(ii) $|\widetilde{x_1}\rangle$ is an _____-dimensional vector.

(iii) $|x_2\rangle$ is an _____-dimensional vector.

(c) Show that $e^{2\pi ixy/pq} = e^{2\pi ix_1y_2/q}e^{2\pi ix_2y_1/p}e^{2\pi ix_2y_2/pq}$.

(d) Consider the function $U_a |\widetilde{x_1}\rangle = \frac{1}{\sqrt{q}} \sum_{y_2=0}^{q-1} e^{2\pi i\widetilde{x_1}y_2/q} |y_2\rangle$. What is U_a?

(e) Consider the function $U_b |x_2\rangle = \frac{1}{\sqrt{p}} \sum_{y_1=0}^{p-1} e^{2\pi ix_2y_1/p} |y_1\rangle$. What is U_b?

Assume that we have a phase operator:

$$\phi_{pq} |y_2\rangle |x_2\rangle = e^{2\pi ix_2y_2/pq} |y_2\rangle |x_2\rangle$$

It will be useful to extend the above functions to the full pq-dimensional Hilbert space:

$$\widetilde{U_a} := U_a \otimes I_p, \qquad \widetilde{U_b} := I_q \otimes U_b$$

.

(f) Combine the above arguments and operators to construct $U = QFT_{pq}$:

$$U|x\rangle = \frac{1}{\sqrt{pq}} \sum_{y=0}^{pq-1} e^{2\pi ixy/pq}|y\rangle$$

You may need to re-index your summations when composing the operators.

Problem 2: Quantum phase estimation (20 points)

Consider U to be a unitary operator with eigenvector $|\psi\rangle$ such that $U|\psi\rangle = e^{2\pi i\theta}|\psi\rangle$. This means:

$$U^{2^j} |\psi\rangle = U^{2^j-1}U |\psi\rangle = U^{2^j-1}e^{2\pi i\theta} |\psi\rangle = \cdots = e^{2\pi i 2^j\theta} |\psi\rangle$$

As we showed in class, quantum phase estimation involves a circuit in which applying sequentially n-controlled-unitary operations $\mathrm{C-U}^{2^j}$, with $0 \le j \le n-1$, to the uniform superposition in a 'counting' register, gives;

$$|\psi_2\rangle = \frac{1}{2^{n/2}}\left(|0\rangle + e^{2\pi i\theta 2^{n-1}} |1\rangle\right) \otimes \ldots \otimes \left(|0\rangle + e^{2\pi i\theta 2^1} |1\rangle\right) \otimes \left(|0\rangle + e^{2\pi i\theta 2^0} |1\rangle\right) \otimes |\psi\rangle$$

(a) Show that if k denotes the integer representation of n-bit binary numbers, this expression can be simplified to

$$|\psi_2\rangle = \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i\theta k} |k\rangle \otimes |\psi\rangle$$

(b) If we inverse Fourier transform $|\psi_2\rangle$, we arrive at the state

$$|\psi_3\rangle = \frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{2\pi i k(x-2^n\theta)/2^n}\,|x\rangle\otimes|\psi\rangle$$

Explain why it is that by measuring x, we have a high likelihood of finding θ.

Problem 3: Quantum phase estimation qiskit circuit example (40 points)

Fill out problem 3 on the attached Jupyter notebook.