Homework assignment 1 - Qubits and Quantum gates

Semiconducting & Superconducting Quantum Computers

Due: Sept. 15th, 2022

NOTE: You can get a maximum of 50 points but the total available points with the +1 problem is 60 points, i.e., you don't have to solve everything to reach the maximum 50 points.

Problem 1 - The Pauli matrices and uncertainty principle for qubits (10 pts / no collaboration)

In class, we have introduced the four Pauli matrices, which form the basis for the vector space of 2×2 Hermitian matrices:

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

a) Show that these matrices are Hermitian and unitary. Find the eigenvalues and eigenvectors of each of them. Feel free to use the Python Qutip package for the calculations.

b) Show that the Pauli matrices satisfy the following commutation relation: $[\sigma_x, \sigma_y] = 2i\sigma_z$, and find similar expressions for $[\sigma_y, \sigma_z]$, and $[\sigma_z, \sigma_x]$.

c) The Heisenberg uncertainty principle is one of the most well-known results of quantum mechanics. It states that it is not possible to accurately measure both the position and the momentum of a particle at the same time, which mathematically means that the product of the standard deviation of position and the standard deviation of the momentum is greater than zero: $\Delta p \cdot \Delta x \geq \hbar/2$. Here the standard deviation for an operator $\mathcal{O}$ is defined as $\Delta\mathcal{O} = \sqrt{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}$. The momentum-position uncertainty is just one example for the general Heisenberg uncertainty principle, which states for two observables $\mathcal{O}$ and $\mathcal{P}$ that

$$\Delta\mathcal{O} \cdot \Delta\mathcal{P} \geq \left| \frac{\langle [\mathcal{O}, \mathcal{P}] \rangle}{2} \right|.$$

Using the commutation relationships derived above, write down the uncertainty principle for the σ_x and σ_y operators of a qubit, and find the uncertainty for measuring this two quantities for a qubit prepared in the $|0\rangle$ state.

Problem 2 - Free evolution of a qubit (7 pts / no collaboration)

A qubit with non-degenerate energy levels can be described by the Hamiltonian

$$H = -\frac{1}{2}\hbar\omega_0\sigma_z.$$

a) Similar to what we derived in class, find the eigenenergies of the Hamiltonian and show that the eigenstates indeed correspond to $|0\rangle$ and $|1\rangle$ states, and the qubit energy is ω_0.

b) Express the time-evolution of the state vector $|\psi(t)\rangle$ when the qubit is prepared in the $|0\rangle$ or $|1\rangle$. What is the probability of finding the qubit in the $|0\rangle$ or $|1\rangle$ states, i.e. $|\langle 0|\psi(t)\rangle|^2$ and $|\langle 1|\psi(t)\rangle|^2$?

c) How about the time-evolution of the qubit state $|\psi(t)\rangle$ if we prepared the state in the $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ state at $t = 0$? Plot the probability of finding the qubit in the $|+\rangle$ state as a function of time, $|\langle +|\psi(t)\rangle|^2$.

Problem 3 - Measurement of single and two qubits (10 pts / collaboration is encouraged)

a) Consider the qubit state

$$|\psi\rangle = \frac{i}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle.$$

i) What is the probability of measuring the qubit in the $|0\rangle$ state?

ii) What is the probability of measuring the qubit in the $|+\rangle$ state?

iii) What is the probability of measuring the qubit in the $|0\rangle$ state after we have measured the qubit in the $|+\rangle$ basis?

b) Assume that a memory register of a 2-qubit quantum computer is in the following state:

$$|\psi\rangle = \frac{1}{6}|00\rangle - \frac{1}{2\sqrt{3}}|01\rangle + \frac{\sqrt{2}}{3}|10\rangle - \sqrt{\frac{2}{3}}|11\rangle$$

i) Write down the state as a vector with four elements.

ii) What is the probability of finding the *first* qubit in the $|0\rangle$ state?

iii) What is the probability of finding *both* qubits in the $|0\rangle$ state?

Problem 4 - Equivalent gates (8 pts / no collaboration)

Many times a single- and a two-qubit gate can be expressed by the product of other gates. In this problem, we investigate a few examples for this. (Using the predefined gates in QuTip is encouraged.)

a) Show that for the single-qubit gates $X = HZH$.

b) Show that the Hadamard gate can be expressed by two rotational gates (around the x and z axis) and a global phase factor

$$H = e^{i\phi} \cdot R_x(\theta = \pi) \cdot R_y(\theta = \pi/2), \tag{1}$$

where

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

What is the value of the global phase?

c) Show that the two-qubit entangling CNOT and CZ gates are related through the following relationship $\text{CNOT} = [I \otimes R_y(\pi/2)] \cdot \text{CZ} \cdot [I \otimes R_y(-\pi/2)]$.

Problem 5 - Swapping qubits for entanglement (15 pts / collaboration is encouraged)

When designing two-qubit gates, it is important to realize schemes that transform unentangled product states into entangled states. An interesting set of gates are the SWAP-type gates that exchange the excitations between two qubits. The simplest version among them is the SWAP gate, which can be described by the following unitary matrix in the qubit eigenbasis

$$\text{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

While the SWAP gate does not lead to entangled states, its modification, the iSWAP gate is a perfect entangler. During the iSWAP gate the exchange of excitation is accompanied by a complex phase i, and this small modification enhances the entangling power of the gate. The unitary of the gate is

$$\text{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

a) Starting from the initial product state $|+-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, show that $\text{SWAP}|+-\rangle$ is a separable state (concurrence is zero), while $\text{iSWAP}|+-\rangle$ is a maximally entangled state (concurrence is one). Find the two states $|\psi\rangle$ and $|\phi\rangle$ that satisfy $\text{SWAP}|+-\rangle = |\psi\rangle \otimes |\phi\rangle$.

b) An entangling two-qubit does not always result in an entangled state starting from product state. For example, show that $\text{iSWAP}|10\rangle$ is still a separable state.

c) Now, we can look at a general separable state and investigate the degree of entanglement of the iSWAP gate depending on the initial states. We pick an arbitrary state on the Bloch sphere of the first qubit [at angles (θ_1, ϕ_1)] and similarly on the Bloch sphere of the second qubit [at angles (θ_2, ϕ_2)], and create the following product state

$$|\psi\rangle = \begin{pmatrix} \cos(\theta_1/2) \\ e^{i\phi_1}\sin(\theta_1/2) \end{pmatrix} \otimes \begin{pmatrix} \cos(\theta_2/2) \\ e^{i\phi_2}\sin(\theta_2/2) \end{pmatrix}.$$

Calculate the concurrence of iSWAP$|\psi\rangle$, $c(\text{iSWAP}|\psi\rangle)$. Show that if you calculate the same for the SWAP operator, $c(\text{SWAP}|\psi\rangle)= 0$ for any initial state.

d) Obtain the value of the entangling power of the iSWAP gate by averaging out the concurrence c over the two Bloch spheres. This can be done by evaluating the following spherical integral (the $\sin\theta_{1/2}$ appears to ensure that the density of points on the sphere is uniform)

$$\text{EP(iSWAP)} = \langle c^2(\text{iSWAP}|\psi\rangle)\rangle_{|\psi\rangle}/2 = \frac{1}{\Omega^2}\int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_1 \sin\theta_2 \times \\ \times c^2(\text{iSWAP}|\psi\rangle)/2,$$

where the normalization factor Ω is the area of a Bloch sphere, $\Omega = 4\pi$.

Problem +1 - Entanglement genesis by parity measurement (10 pts / collaboration is encouraged)

The parity of a two-qubit system is defined by the eigenvalues of the $\sigma_z \otimes \sigma_z$ operator. Explicitly, if both qubits are in the same state, $|00\rangle$ or $|11\rangle$, the system has even parity, while if the two qubits are in the opposite states, $|01\rangle$ or $|10\rangle$, the system has odd parity. Consider a locally prepared product state $|\psi\rangle = |++\rangle$, where $|+\rangle = (|0\rangle+|1\rangle)/\sqrt{2}$. Show that by measuring the parity of this system, the originally unentangled wavefunction will collapse into a fully entangled state.