Homework assignment 5
Bosonic superconducting qubits

Semiconducting & Superconducting Quantum Computers
Due: Nov 20th, 2022

Problem 1 - Fast or slow gates? (15 pts / no collaboration)

One of the main disadvantages of the transmon qubit is that its small anharmonicity limits the logical gate speed. When the gate is too fast, the higher lying states become populated, leading to leakage error. When the gate is too slow, leakage errors are negligible but decoherence during the gate limits the gate performance. In this problem, we investigate this trade-off, and estimate the optimal gate time that maximizes the gate performance.

The gates in the transmon are based on microwave drives. If the qubit frequency is $\omega_{01}/2\pi$, and a resonant microwave signal with amplitude Ω_{01} excites the qubit, $V(t) = \hbar\Omega_{01} \times \cos(\omega_{01}t)$, the driven system is described by the Rabi Hamiltonian (in the rotating frame of the drive and after applying a rotating wave approximation):

$$H = \frac{\hbar\Omega_{01}}{2}\left[|0\rangle\langle 1| + |1\rangle\langle 0|\right]. \tag{1}$$

It is important to notice that the drives are not applied at all the time because we have to turn them off when we are done with a gate or want to do a different operation on the qubit. Thus, the voltage excitations are modulated by envelope functions, which, in most cases, have Gaussian shapes; for example,

$$V(t) = \hbar\Omega_{01} \times \cos(\omega_{01}t) \times e^{-t^2/2t_{\text{gate}}^2}. \tag{2}$$

In this case, the excitation is on around $t = 0$ for a characteristic gate time t_{gate}, while it can be considered to be off at other times. This modification causes that the drive will have frequency components different from the qubit frequency $\omega_{01}/2\pi$, as we show below.

a) First, as a warm up, show that a continuous drive has a Fourier component only at the qubit frequency (consider only positive frequencies here):

$$\hbar\Omega(\omega) = \int_{-\infty}^{\infty} V(t)e^{-i\omega t} = \int_{-\infty}^{\infty} \hbar\Omega_{01} \times \cos(\omega_{01}t) \times e^{-i\omega t} \propto \frac{\hbar\Omega_{01}}{2} \times \delta(\omega - \omega_{01}). \tag{3}$$

Now, show that when the drive is turned on and off (modulated by a Gaussian shape as explained above), the frequency components are broadened, and the Fourier transform of the signal is a Gaussian centered around ω_{01},

$$\hbar\Omega(\omega) = \int_{-\infty}^{\infty} V(t)e^{-i\omega t} = \int_{-\infty}^{\infty} \hbar\Omega_{01} \times \cos(\omega_{01}t) \times e^{-t^2/2t_{\text{gate}}^2} \times e^{-i\omega t} \propto \frac{\hbar\Omega_{01}}{2} \times e^{-(\omega-\omega_{01})^2 t_{\text{gate}}^2/2}. \tag{4}$$

Draw the shape of the pulse and its Fourier transform with axes properly labeled.

Note that for this signal, the frequency component with highest amplitude is still at the qubit frequency, $\hbar\Omega(\omega = \omega_{01}) = \hbar\Omega_{01}/2$. This component leads to the population transfer between the qubit $|0\rangle$ and $|1\rangle$ states. However, there is another component ($\hbar\Omega_{12}/2$) that is on-resonant with the $|1\rangle$ to $|2\rangle$ transition, with frequency of $\omega_{12} = \omega_{01} - E_C/\hbar$, such that $\hbar\Omega(\omega = \omega_{12}) = \hbar\Omega_{12}/2$ (E_C is the anharmonicity of the transmon, as we have shown in the previous problem set). This component can induce transitions between the $|1\rangle$ and $|2\rangle$ states, and move the population of the qubit into state $|2\rangle$. What is the amplitude ratio of this spurious frequency component compared to the component corresponding to the qubit transition, i. e., $\lambda = \Omega_{12}/\Omega_{01}$? How does it depend on the gate time?

b) Now, we can model our drive such that it has two relevant frequency components, which can induce transitions between the $|0\rangle - |1\rangle$, and $|1\rangle - |2\rangle$ states. The two-tone Rabi Hamiltonian takes the following form:

$$H = \frac{\hbar\Omega_{01}}{2}\left[|0\rangle\langle 1| + |1\rangle\langle 0|\right] + \frac{\hbar\Omega_{12}}{2}\left[|1\rangle\langle 2| + |2\rangle\langle 1|\right], \tag{5}$$

which in the basis of $\{|0\rangle, |1\rangle, |2\rangle\}$ states is

$$H = \frac{\hbar\Omega_{01}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & \lambda & 0 \end{pmatrix}.$$

Solve analytically for the time evolution of the state in the form of $|\psi(t)\rangle = c_0(t)\cdot|0\rangle + c_1(t)\cdot|1\rangle + c_2(t)\cdot|2\rangle$, assuming that at $t=0$, the qubit is in the $|0\rangle$ state $[c_0(t=0)=1,\ c_1(t=0)=0,\ c_2(t=0)=0]$.

c) If the calculation went well, you will find that the first excited state population is

$$P_1(t) = |c_1(t)|^2 = \frac{1}{1+\lambda^2}\sin^2\left(\frac{1}{2}\Omega_{01}\sqrt{1+\lambda^2}\cdot t\right). \tag{6}$$

Thus, the gate scheme never reaches full population transfer between the $|0\rangle - |1\rangle$ states, because $P_1(t) < 1$ for all t. What time would you set the gate length to achieve the best possible X gate (for fixed λ and Ω_{01}), and what fidelity can you achieve in this case?

d) In part a), you have shown that slower gates lead to smaller λ. Now, you can see that if λ is smaller, the better the gate is because more population ends up in the first excited state [$P_1(t_{\text{gate}})$ is larger]. However, slowing down the gate will have the negative consequence that dephasing during the gate will lower the excited state population. Here, we just build a very rough model, and assume that only energy relaxation plays a role, and in such a simple way that it lowers the excited state population with a factor of e^{-t_{gate}/T_1}, i.e., $P_1(t_{\text{gate}}) = |c_1(t_{\text{gate}})|^2 \times e^{-t_{\text{gate}}/T_1} = \frac{1}{1+\lambda^2}\times e^{-t_{\text{gate}}/T_1}$. Explain by words why this is a reasonable model. Plot this function vs. gate time up to a 50 ns ($E_C/\hbar = 2\pi\cdot$ 0.2 GHz, $T_1 = 1$ μs). What is the optimal gate time for this system?

Problem 2 - Coherent states of a harmonic oscillator and the two-legged cat qubit (15 pts / no collaboration)

Although there is no separate two-level subsystem in the spectrum of the quantum harmonic oscillator, it can still host a qubit. In fact, the equidistant energy spectrum offers a rich set of codewords to encode a qubit. One of these encodings use the coherent states of the oscillator. Coherent states are the eigenstates of the annihilation operator and share several similarities to the states of a classical harmonic oscillator; thus, coherent states are considered to be quasi-classical states.

In this problem, we focus on the quantum LC oscillator, whose Hamiltonian takes the form of

$$\hat{H} = \frac{1}{2C}\hat{Q}^2 + \frac{1}{2L}\hat{\Phi}^2, \tag{7}$$

where the charge and flux operators satisfy the usual commutation relation $[\hat{\Phi}, \hat{Q}] = i\hbar$. As before, we introduce the creation ($a^\dagger$) and annihilation (a) operators with commutation relations $[a, a^\dagger] = 1$,

$$\begin{aligned} \hat{Q} &= Q_{\text{ZPF}} \times i(a^\dagger - a), \\ \hat{\Phi} &= \Phi_{\text{ZPF}} \times (a^\dagger + a), \end{aligned} \tag{8}$$

where the zero-point fluctuations are $Q_{\text{ZPF}} = \sqrt{\frac{\hbar}{2Z}}$ and $\Phi_{\text{ZPF}} = \sqrt{\frac{\hbar Z}{2}}$ with impedance $Z = \sqrt{\frac{L}{C}}$. With these definitions, the Hamiltonian is

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}), \tag{9}$$

where the transition frequency is $\omega = \frac{1}{\sqrt{LC}}$.

The eigenstates of the Hamiltonian $|n\rangle$ are called Fock states, where

$$H|n\rangle = \hbar\omega(n + \frac{1}{2})|n\rangle. \tag{10}$$

a) The Fock states are very different from the classical solutions of the harmonic oscillator; while in a classical harmonic oscillator the charge on the capacitor and the flux in the inductor oscillates as a function of time, the Fock states are more like stationary states. For example, show that $\langle\hat{\Phi}\rangle_n = \langle n|\hat{\Phi}|n\rangle = 0$ and $\langle\hat{Q}\rangle_n = \langle n|\hat{Q}|n\rangle = 0$.

b) Coherent states, on the other hand, have oscillating nature. These states are the combination of the Fock states with coefficients resembling the Poisson distribution

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{11}$$

Show that α describes the mean photon number of the state, i.e., $\langle \hat{n} \rangle_\alpha = \langle a^\dagger a \rangle_\alpha = |\alpha|^2$ and that $\langle \hat{\Phi}(t) \rangle_\alpha$ and $\langle \hat{Q}(t) \rangle_\alpha$ is an oscillating function of time similar to a classical harmonic oscillator. *Hint:* The time-evolution of a coherent state $|\alpha(t)\rangle = e^{-i\omega t/2}|\alpha_0 e^{-i\omega t}\rangle$, where $|\alpha(t=0)\rangle = |\alpha_0\rangle$.

c) Calculate the wavefunction of the coherent state in flux space $\psi_\alpha(\phi) = \langle \phi | \alpha \rangle$. Show that this wavefunction is just the ground state wavefunction displaced in flux space by $\langle \hat{\Phi} \rangle_\alpha$, i.e., $\psi_\alpha(\phi) \propto e^{-(\phi - \langle \hat{\Phi} \rangle_\alpha)^2/2Z\hbar}$. *Hint:* The annihilation operator can be expressed as $a = \frac{1}{\sqrt{2\hbar Z}} \left[\hat{\Phi} + Z\hbar \partial_\Phi \right]$.

d) Now, show that the different coherent states $|\alpha\rangle$ and $|\beta\rangle$ are quasi-orthogonal, i.e. $|\langle \beta | \alpha \rangle|^2 = e^{-|\alpha - \beta|^2}$. This enables us to define qubit *logical* states such that $|0_L\rangle = |\alpha\rangle$ and $|1_L\rangle = |-\alpha\rangle$, since the qubit basis states are almost orthogonal for large α, $|\langle 1_L | 0_L \rangle|^2 = \mathcal{O}\left(e^{-2|\alpha|^2}\right)$.

e) For the two-legged cat qubit, we follow exactly this definition for the qubit states, such that the $|0_L\rangle = |\alpha\rangle$ and $|1_L\rangle = |-\alpha\rangle$. In this case, the superposition qubit states are Schrödinger cat states because they are the superposition of quasi-classical states, $|+_L\rangle = \mathcal{N}_+(|\alpha\rangle + |-\alpha\rangle)$, and $|-_L\rangle = \mathcal{N}_-(|\alpha\rangle - |-\alpha\rangle)$. Find the normalization factors $\mathcal{N}_+$ and $\mathcal{N}_-$.

f) Estimate the bit-flip and phase-flip rates of this qubit due to photon loss. *Hint:* $1/T_1 \propto |\langle -\alpha | a | \alpha \rangle|^2$ and $1/T_\varphi \propto |\langle +_L | a | -_L \rangle|^2$. Which one is the dominatic loss channel?

Problem 3 - Quantum error correction with the kitten code (20 pts / collaboration is encouraged)

The energy spectrum of the harmonic oscillator offers multiple ways to encode an error-correctable qubit. Roughly speaking, the key requirement for error correction is that (1) both qubit basis states ($|0_L\rangle, |1_L\rangle$) are the eigenstate of an operator with the same eigenvalue, and (2) when a certain error happens, the eigenvalue of this operator changes. Thus, when there is no error, a measurement of this operator does not change the qubit states, but when an error occurs, we can detect it. We will investigate this in the example of two bosonic encodings. In both cases, the operator is the parity operator $P = (-1)^{a^\dagger a}$, and the measurement determines whether there are an odd or an even number of photons in the oscillator.

a) The first code is the simplest kitten code, where we use the following code words for the logical qubit states:

$$
\begin{aligned}
|0_L\rangle &= \frac{|0\rangle + |4\rangle}{\sqrt{2}}, \\
|1_L\rangle &= |2\rangle.
\end{aligned}
\tag{12}
$$

Here, the states on the right hand side, $|n\rangle$ are the Fock states, the eigenstates of the harmonic oscillator. Show that both $|0_L\rangle$ and $|1_L\rangle$ states have the same average photon number, and the same photon number parity (even), so if we measure the parity of these states, we will not gain any information about the qubit states and their superposition remains uneffected.

b) When an error occurs (a photon loss), both states will change parity. The photon loss is described by the following event

$$
|\psi\rangle \to \frac{a|\psi\rangle}{\sqrt{\langle \psi | a^\dagger a | \psi \rangle}}
\tag{13}
$$

Important to note that the denominator is the square root of the photon number of the original state. Calculate the states after the photon loss, and show that they have odd parity. Thus, when we measure the parity, and detect a parity change, we can just add back a photon into the cavity and correct an error.

c) Assume that we did not correct for the error, and a second photon leaves the cavity. Calculate the resultant state, and show that now a superposition state is distorted, and the quantum information is lost. *Hint:* Assume that the original state is $|\psi^{(0)}\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$, and show that the α and/or β coefficients get distorted after two-photon-loss events, when the wavefunction becomes $|\psi^{(2)}\rangle = \tilde{\alpha}|0_{EE}\rangle + \tilde{\beta}|1_{EE}\rangle$, where $|0_{EE}\rangle$ and $|1_{EE}\rangle$ are the basis states after two photon loss. Thus, this kitten code is not protected against two-photon loss events.

d) A more complicated code, however, can lead to protection against two-photon loss as well. In this case, take the following code words

$$
\begin{aligned}
|0_L\rangle &= \frac{|0\rangle + \sqrt{3}|6\rangle}{2}, \\
|1_L\rangle &= \frac{\sqrt{3}|3\rangle + |9\rangle}{2}.
\end{aligned}
\tag{14}
$$

Again, assume that $|\psi^{(0)}\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$, and calculate the wavefunction after one ($|\psi^{(1)}\rangle$) and two photon losses ($|\psi^{(2)}\rangle$), and show that the α and β coefficients remain the same over these processes, and they get distorted after the third photon leaves the resonator.

Note, this code can be generalized to be protected to arbitrary number of photon loss, and it is called as the binomial code [Phys. Rev. X 6, 031006 (2016)].

Alternative Homework
(40 pts / no collaboration, instead of solving Problems 1-2-3)

1. Explain the idea behind the bosonic qubits, and how error correction can extend the lifetime of these particular qubits. (2 pages, single spaced).
2. Derive the circuit quantization of the fluxonium. Sketch the energy levels, and the qubit wavefunctions in flux bases, and explain the benefits of fluxonium over transmon (2 pages, single spaced).