Homework assignment 3 - Quantum errors and quantum error correction

Semiconducting & Superconducting Quantum Computers
Due: Oct. 19th, 2022

Problem 1 - Dissipative Rabi system
(20 pts / collaboration is encouraged)

Consider a qubit that is under a continuous external Rabi drive and also coupled to a dissipative environment. The drive is on-resonant with the qubit transition frequency, and the Rabi Hamiltonian in the frame of the drive takes the usual form

$$H_{\mathrm{Rabi}} = \frac{1}{2}\hbar\Omega\sigma_x, \tag{1}$$

where Ω is the Rabi frequency, which is proportional to the drive amplitude.

a) If there is no coupling between the qubit and the environment, and the qubit starts in the $|0\rangle$ state, the qubit will have a trajectory on the surface of the Bloch sphere. Show this by writing down the time dependence of the density matrix, $\rho(t)$ based on the von Neumann equation for time evolution

$$\dot{\rho} = -\frac{i}{\hbar}\left[H_{\mathrm{Rabi}}, \rho\right], \tag{2}$$

and express the position of the state on the Bloch sphere $(r_x(t), r_y(t), r_z(t))$, where

$$\rho(t) = \frac{1}{2}\left[\sigma_0 + r_x(t)\sigma_x + r_y(t)\sigma_y + r_z(t)\sigma_z\right]. \tag{3}$$

b) However, if there is a coupling between the qubit and the environment, the qubit state will move inside the Bloch sphere. We show this by using the Lindblad master equation, but instead of solving the general time dependence, we will focus only the final state of the qubit after a long time passed. We also assume that the environment introduces only dissipation with a rate of γ_1. To solve the time evolution of the system, we use the Lindblad master equation as discussed in class (take a look at the online notes):

$$\dot{\rho} = -\frac{i}{\hbar}\left[H_{\mathrm{Rabi}}, \rho\right] + \gamma_1\left[\sigma_-\rho\sigma_+ - \frac{1}{2}\{\sigma_+\sigma_-, \rho\}\right]. \tag{4}$$

Write down the differential equations for the time-evolution of the elements of the density matrix ρ_{ij}, where

$$\rho = \begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix}.$$

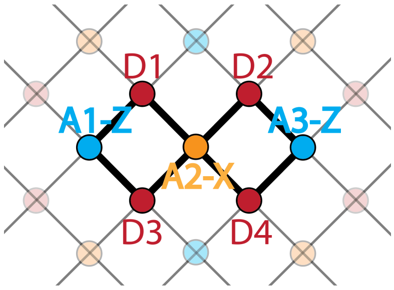
c) Find the steady state solution of the problem ($\dot{\rho} = 0$). Hint: keep in mind the general properties of the density matrix $\rho_{10} = \rho_{01}^*$, and $\rho_{00} + \rho_{11} = 1$.

d) Determine the position of the steady state inside the Bloch sphere (r_x, r_y, r_z), where

$$\rho = \frac{1}{2}\left[\sigma_0 + r_x\sigma_x + r_y\sigma_y + r_z\sigma_z\right]. \tag{5}$$

Problem 2 - A small surface code
(15 pts / no collaboration)

In superconducting qubits, the surface code is one of the most promising ways to perform error correction. In this code, each data qubit is connected to two X- and two Z-ancillary measurement qubits that can detect if an X error or a Z error occurred at a given position. Thus, the data and ancillary qubits form a rectangular lattice. Here, we investigate the smallest possible surface code (see the Figure) that consists of four data qubits (D1, D2, D3, D4), and three ancillary qubits, among which two can measure the Z parity (A1-Z, A3-Z) and one can measure the X parity (A2-X). This code can detect one error but can not correct it because the error syndromes are indistinguishable.



The logical qubit states of the code are

$$
\begin{aligned}
|0\rangle_L &= \frac{1}{\sqrt{2}}\left(|0000\rangle + |1111\rangle\right), \\
|1\rangle_L &= \frac{1}{\sqrt{2}}\left(|0101\rangle + |1010\rangle\right).
\end{aligned}
\tag{6}
$$

a) Show that these states are indeed the eigenstates of the stabilizers,

$$
\begin{aligned}
S_1 &= X_{D1}X_{D2}X_{D3}X_{D4}, \\
S_2 &= Z_{D1}Z_{D3}, \\
S_3 &= Z_{D2}Z_{D4},
\end{aligned}
\tag{7}
$$

thus, measuring these operators does not collapse a logical state.

b) Draw the quantum circuit that measures S_1 and S_2. *Hint:* We discussed similar quantum circuits in class.

c) Assume that an X error occurred on the D1 data qubit. Find the error syndrome outcome of the stabilizer measurements. (The eigenvalues of the stabilizers are called error syndromes.) Is the error detectable?

d) Show that if an X error occurs on the D3 data qubit, you get the same error syndromes, illustrating that the errors are not correctable (we don't know where the error occurred with this small code).

e) Show that the logical operators are the following gates

$$
\begin{aligned}
Z_L &= Z_{D1}Z_{D2} \\
X_L &= X_{D1}X_{D3},
\end{aligned}
\tag{8}
$$

and find the expression for Y_L.

f) Find another set of logical operators. *Hint:* the logical gates correspond to the product of single qubit gates along lines in the code that connects the boundaries of the plaquette.

Problem 3 - Build your own single-qubit simulator (15 pts / collaboration is encouraged)

In this problem, we will simulate a gate implementing a π-pulse in a noisy qubit. To be more realistic, we consider real superconducting qubit parameters. We assume that the qubit frequency is $\omega_0/2\pi = 5$ GHz, the qubit relaxation time is $T_1 = 5\ \mu$s, and pure dephasing time is $T_\phi = 1\ \mu$s. We assume that the drive is on for $t_{\text{gate}} = 100$ ns, and the phase of the drive has only in-phase component, i.e., it is a purely σ_x drive without a σ_y component.

In QuTip, the convention is $\hbar = 1$, so the Hamiltonian reads simply as

$$
H_{\text{Rabi}} = -\frac{1}{2}(\omega_0 - \omega_L)\sigma_z + \frac{1}{2}\Omega\sigma_x, \tag{9}
$$

where $\Omega/2\pi$ is the Rabi frequency, and $\omega_L/2\pi$ is the drive frequency. Furthermore the jump operators are defined in QuTip such that they include the corresponding rates, for example, the jump operator for relaxation is $L_- = \sqrt{\frac{1}{T_1}}\sigma_-$, and the jump operator for dephasing is $L_\phi = \sqrt{\frac{1}{2T_\phi}}\sigma_z$. (Note the factor of 2 for T_ϕ). With these definitions, the Lindblad master equation takes the following form:

$$
\dot{\rho} = -i\left[H_{\text{Rabi}}, \rho\right] + \left[L_-\rho L_+ - \frac{1}{2}\{L_+\L_-, \rho\}\right] + \left[L_\phi \rho L_\phi^\dagger - \frac{1}{2}\{L_\phi^\dagger \L_\phi, \rho\}\right]. \tag{10}
$$

QuTip is capable to solve the master equation, when the jump operators are passed to the *mesolve* function (see examples on Canvas). The best way to define these jump operators in QuTip is to include them in a list, for example:

```
c_ops = [np.sqrt(1/T_1) * sm, np.sqrt(1/(2*T_phi)) * sz]
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where sm and sz are the σ_- and σ_z operators.

a) Assume that the drive is on-resonant with the qubit frequency ($\omega_0 = \omega_L$). Find the amplitude of the Rabi drive $\Omega/2\pi$ in units of MHz, so that the pulse implements the best possible $R_x(\pi)$ pulse for this noisy qubit. Note that because of the decoherence processes, it is not possible to achieve a perfect gate.

b) Plot the position of the state during the evolution of the gate on (inside) the Bloch sphere.

c) What is the fidelity of the final state? This example shows that gate fidelities are often limited by the intrinsic decoherence properties of the qubit.

Alternative Homework
(40 pts / no collaboration, instead of solving Problems 1-2-3)

Explain the idea of quantum error correction in about 5 pages, discuss the case of bit-flip error correcting codes, surface codes, fault-tolerance and error threshold theorem.