1 Data Generation with Two-Dimensional θ

For a two-dimensional parameter vector $\theta = (\theta_1, \theta_2)$, proposing an elliptical relationship to generate data points (M, R). The elliptical relation is defined as follows:

$$\frac{M^2}{\theta_1} + \frac{R^2}{\theta_2} = 1 \tag{1}$$

Equation (1) represents an ellipse in the MR-plane, where θ_1 and θ_2 are the squares of the lengths of the semi-major and semi-minor axes, respectively. To generate data points (M, R) that follow this elliptical pattern, we can use the parametric equations of an ellipse.

$$M = \sqrt{\theta_1} \cdot \cos(p_i) \tag{2}$$

$$R = \sqrt{\theta_2} \cdot \sin(p_i) \tag{3}$$

where p_i is a parameter that varies and helps in traversing the ellipse to generate data points along its perimeter.

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n = 100 ; p_true = runif(n, 0, pi/2); theta1 = 4 ; theta2 = 1
sigma = 0.1
m = sqrt(theta1) * cos(p_true) + rnorm(n, mean=0, sd=sigma)
r = sqrt(theta2) * sin(p_true)  + rnorm(n, mean=0, sd=sigma)
plot(m, r)
```

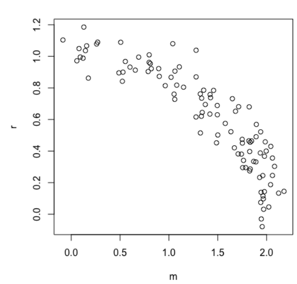

Figure 1: Data Generation

2 Bayesian Model Setup

$$
\begin{aligned}
(M_i, R_i) \mid p_i, \theta_1, \theta_2, \Sigma &\sim \mathcal{N}_2\left(\{\sqrt{\theta_1}\cos(p_i), \sqrt{\theta_2}\sin(p_i)\}, \Sigma\right), \\
p_i &\sim \text{Discrete\_uniform}(0, 2\pi), \\
\theta_1 &\sim \text{Discrete\_uniform}(0, 5), \\
\theta_2 &\sim \text{Discrete\_uniform}(0, 5), \\
\Sigma &\sim \text{Inverse\_Wishart}\left(\nu = 2, \Psi = \text{diag}(2)\right).
\end{aligned}
$$

The likelihood and the full conditionals take the form:

$$
\begin{aligned}
P(\mathbf{M}, \mathbf{R} | \mathbf{p}, \theta_1, \theta_2, \Sigma) &= \prod_{i=1}^{n} \mathcal{N}_2\left(\{M(p_i, \theta_1), R(p_i, \theta_2)\}, \Sigma\right), \\
P(\theta_1, \theta_2 \mid \mathbf{M}, \mathbf{R}, \Sigma, \mathbf{p}) &\propto P(\mathbf{M}, \mathbf{R} \mid \theta_1, \theta_2, \Sigma, \mathbf{p}) \cdot P(\theta_1) \cdot P(\theta_2), \\
P(\Sigma \mid \mathbf{M}, \mathbf{R}, \theta_1, \theta_2, \mathbf{p}) &\propto P(\mathbf{M}, \mathbf{R} \mid \theta_1, \theta_2, \Sigma, \mathbf{p}) \cdot P(\Sigma), \\
P(p_i \mid \mathbf{M}, \mathbf{R}, \theta_1, \theta_2, \Sigma) &\propto \mathcal{N}_2\left(\{M(p_i, \theta_1), R(p_i, \theta_2)\}, \Sigma\right) P(p_i).
\end{aligned}
$$