MSMS 301 - Time Series

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Contents

@ 01.08.2025, Friday

1 Moving Average Process

Suppose $\{Z_t\}$ is a purely random process with mean 0 and variance σ^2. A process $\{X_t\}$ derived as a weighted sum of **present and past q** white noises is said to be a Moving Average Process of order q (abbreviated to $MA(q)$).

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \ldots + \beta_q Z_{t-q} \tag{1}$$

where all β_i's are constants. The Z's are usually scaled so that $\beta_0 = 1$.

It immediately follows

- $E(X_t) = 0 \ \forall t$ as $E(Z_t) = 0 \ \forall t$
- $Var(X_t) = \sigma^2 \sum_{i=0}^{q} \beta_i^2 \ \forall t$ as $Z_t \overset{ind}{\sim}$ variance $= \sigma^2 \ \forall t$.

1.1 MA(1) Process

With $Z_t \sim$ White Noise $(0, \sigma^2)$, the first order moving average process $\{X_t\}$ is defined as

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1}; \ \beta_0 = 1 \tag{2}$$

$$\text{or } X_t = Z_t + \theta Z_{t-1} \tag{3}$$

where $\theta \in \mathbb{R}$ (as the process is finite, θ is free to be any real constant).

Clearly $E(X_t) = 0 \ \forall t$ and $Var(X_t) = \sigma^2(1 + \theta^2)$.

Now

$$\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}) \\
&= cov(Z_t, Z_{t+h}) + \theta \cdot cov(Z_t, Z_{t+h-1}) + \theta \cdot cov(Z_{t-1}, Z_{t+h}) + \theta^2 \cdot cov(Z_{t-1}, Z_{t+h-1}) \\
&= \begin{cases} \sigma^2(1+\theta^2), \ h = 0 \\ \theta\sigma^2, \ h = \pm 1 \\ 0, \ h = \pm 2, \pm 3, \pm 4, \ldots \end{cases}
\end{aligned}$$

Then

$$\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} 1, \ h = 0 \\ \dfrac{\theta}{1+\theta^2}, \ h = \pm 1 \\ 0, \ h = \pm 2, \pm 3, \pm 4, \ldots \end{cases}
\end{aligned}$$

✎ So, for MA(1) process, the autocorrelation function vanishes after lag 1. This is an identifier for MA(1) process.

1.2 MA(q) Process

$MA(q)$ process as in (1) can be written as $X_t = \sum_{i=0}^{q} \beta_i Z_{t-i}$.

Then

$$\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{j=0}^{q} \beta_j Z_{t+h-j}\right) \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{s=-h}^{q-h} \beta_{s+h} Z_{t-s}\right) \quad [s = j-h] \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{i=-h}^{q-h} \beta_{i+h} Z_{t-i}\right) \quad [\text{runner } s \text{ is just a dummy variable}] \\
&= \sum_{i=0}^{q-h} \beta_i \beta_{i+h} \cdot cov(Z_{t-i}, Z_{t-i}) \\
&= \sigma^2 \sum_{i=0}^{q-h} \beta_i \beta_{i+h} \\
&= \begin{cases} \sigma^2 \sum_{i=0}^{q-h} \beta_i \beta_{i+h}, \; h = 0, 1, \ldots, q \\ 0, \; h > q \\ \gamma(-h), \; h < 0 \end{cases}
\end{aligned}$$

Consequently

$$\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} \dfrac{\sum_{i=0}^{q-h} \beta_i \beta_{i+h}}{\sum_{i=0}^{q} \beta_i^2}, \; h = 0, 1, \ldots, q \\ 0, \; h > q \\ \rho(-h), \; h < 0 \end{cases}
\end{aligned}$$

✎ It can be seen from the above expression that ACF becomes 0 at lag $k > q$. It shows that ACF of MA process cuts off at lag q which is a special characteristics of MA(q) process.

@ 03.08.2025, Sunday (Yesss !! On a goddaamn Sunday !! 😢)

💡 <u>Random Cosine Curve</u> : Consider a stochastic process $\{X_t\}$ given by

$$X_t = \cos\left(2\pi\left(\frac{t}{12} + \Phi\right)\right). \; t = 0, \pm 1, \pm 2, \ldots$$

where $\Phi \sim U(0,1)$. Find $E(X_t)$ and $Var(X_t)$.

2 Autoregressive Process

2.1 AR(1) Process

The first-order autoregressive process $\{X_t\}$ is defined as

$$X_t = \alpha X_{t-1} + Z_t \;\; t = 0, \pm 1, \pm 2, \dots \tag{4}$$

where Z_t's are White Noise with mean 0 and variance σ^2; $\alpha \in \mathbb{R}$, constant.

Notice (4) reduces to a random walk model for $\alpha = 1$.

We may write (4) as $X_t = \alpha B X_t + Z_t$ where B is a backshift operator with $BX_t = X_{t-1}$.

Then $(1 - \alpha B)X_t = Z_t$ so that

$$\begin{aligned} X_t &= (1 - \alpha B)^{-1} Z_t \\ &= (1 + \alpha B + \alpha^2 B^2 + \alpha^3 B^3 + \dots) Z_t \\ &= Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \alpha^3 Z_{t-3} + \dots \\ &= \sum_{j=0}^{\infty} \alpha^j Z_{t-j} \text{ provided the sum exists } i.e. \ |\alpha| < 1 \end{aligned} \tag{5}$$

Using (5), $E(X_t) = 0$, $t = 0, \pm 1, \pm 2, \dots$

And $Var(X_t) = \sigma^2 \sum_{j=0}^{\infty} \alpha^{2j} = \dfrac{\sigma^2}{1 - \alpha^2}$, provided $|\alpha| < 1$.

Then

$$\begin{aligned} \gamma(h) &= cov(X_t, X_{t+h}) \\ &= cov\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}, \sum_{j=0}^{\infty} \alpha^j Z_{t+h-j}\right) \\ &= cov\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}, \sum_{k=-h}^{\infty} \alpha^{k+h} Z_{t-k}\right), \text{ taking } k = j - h \\ &= \sigma^2 \sum_{j=0}^{\infty} \alpha^j \alpha^{j+h} \\ &= \sigma^2 \alpha^h \sum_{j=0}^{\infty} \alpha^{2j} \\ &= \frac{\sigma^2 \alpha^h}{1 - \alpha^2} \text{ provided } |\alpha| < 1, h \geq 0 \end{aligned}$$

Note that $\gamma(h)$ does not depend on t. So AR(1) model is weak stationary only if $|\alpha| < 1$.

Consequently

$$\begin{aligned}\rho(h) &= \frac{\gamma(h)}{\gamma(0)}\\ &= \begin{cases}1, \ h=0\\ \alpha^h, \ h=1,2,\dots\\ \rho(-h), \ h=-1,-2,\dots\end{cases}\\ &= \alpha^{|h|}, \ h=\pm1,\pm2,\dots\end{aligned}$$

Multiplying X_{t-k} on both sides of (4) and taking expectation we get

$$\begin{aligned}E(X_t\cdot X_{t-h}) &= \alpha E(X_{t-1}\cdot X_{t-h}) + E(Z_t\cdot X_{t-h}),\\ \Rightarrow \gamma(h) &= \alpha\,\gamma(h-1)\end{aligned} \tag{6}$$

Remember Z_t, X_{t-h} are independent and $E(X_t)=0\ \forall t$. Also, using (5), $cov(Z_t, X_{t-h})=0$.

Following (6) we have

$$\begin{aligned}\gamma(1) &= \alpha\,\gamma(0)\\ \gamma(2) &= \alpha^2\,\gamma(0)\\ &\vdots\\ \gamma(h) &= \alpha^h\gamma(0)\end{aligned} \tag{7}$$

On dividing both sides of (7) by $\gamma(0)$, we get

$$\rho(h) = \alpha^{|h|}, \ h=0,\pm1,\pm2,\dots$$

Find ACVF & ACF and draw the ACF plot for the following $AR(1)$ processes.

(i) $X_t = 0.35\,X_{t-1} + Z_t$

(ii) $X_t = 0.85\,X_{t-1} + Z_t$

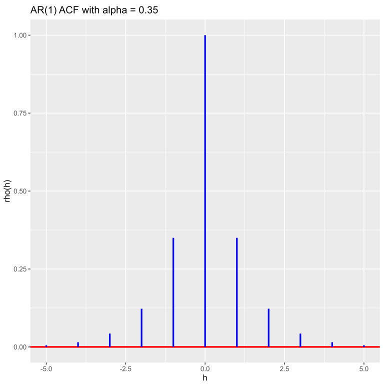
(iii) $X_t = -0.35\,X_{t-1} + Z_t$

The ACVFs and ACFs are as follows.

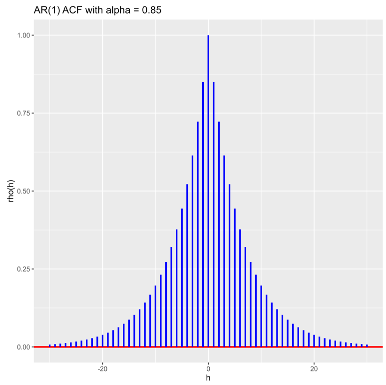
(i) $\gamma(h) = 0.8775\cdot\sigma^2\,0.35^{|h|}$, $\rho(h) = 0.35^{|h|}\ \forall h=0,\pm1,\pm2,\dots$

(ii) $\gamma(h) = 0.2775\cdot\sigma^2\,0.85^{|h|}$, $\rho(h) = 0.85^{|h|}\ \forall h=0,\pm1,\pm2,\dots$

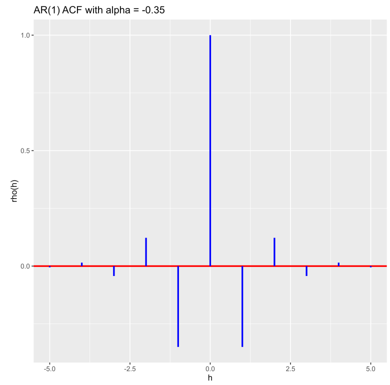
(iii) $\gamma(h) = 0.8775\cdot\sigma^2\,(-0.35)^{|h|}$, $\rho(h) = (-0.35)^{|h|}\ \forall h=0,\pm1,\pm2,\dots$



(i)



(ii)



(iii)

2.2 AR(2) Process

The second-order autoregressive process $\{X_t\}$ is defined as

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t \;\; t = 0, \pm 1, \pm 2, \ldots \tag{8}$$

where Z_t's are White Noise with mean 0 and variance σ^2; $\alpha_1, \alpha_2 \in \mathbb{R}$, constants.

The AR(2) process in (8) can be written in terms of backshift operator B as follows.

$$\begin{aligned} X_t &= \alpha_1 B X_t + \alpha_2 B^2 X_t + Z_t \\ \left(1 - \alpha_1 B - \alpha_2 B^2\right) X_t &= Z_t \\ X_t &= \left(1 - \alpha_1 B - \alpha_2 B^2\right)^{-1} Z_t \\ &= (\alpha(B))^{-1} Z_t \\ &= \psi(B) Z_t \end{aligned}$$

where $\alpha(z) = 1 - \alpha_1 z - \alpha_2 z^2$ and $\psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \ldots = \sum\limits_{j=0}^{\infty} \psi_j z^j$ with $\psi_0 = 1$.

$$\begin{aligned} \therefore X_t &= \sum_{j=0}^{\infty} \psi_j B^j Z_t \\ &= \sum_{j=0}^{\infty} \psi_j Z_{t-j} \text{ with } \psi_0 = 1 \end{aligned} \tag{9}$$

Using (9), $E(X_t) = 0$, $t = 0, \pm 1, \pm 2, \ldots$

And $\gamma(0) = Var(X_t) = \sigma^2 \sum\limits_{j=0}^{\infty} \psi_j^2$, provided the sum exists.

Then

$$\begin{aligned}
\forall h > 0,\ \gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov\left(\sum_{j=0}^{\infty} \psi_j Z_{t-j}, \sum_{j=0}^{\infty} \psi_j Z_{t+h-j}\right) \\
&= cov\left(\sum_{j=0}^{\infty} \psi_j Z_{t-j}, \sum_{k=-h}^{\infty} \psi_{k+h} Z_{t-k}\right),\ \text{taking } k = j - h \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \text{ provided the sum exists}
\end{aligned}$$

Thus

$$\gamma(h) = \begin{cases} \sigma^2 \sum\limits_{j=0}^{\infty} \psi_j \psi_{j+h},\ h = 1, 2, 3, \ldots \\ \sigma^2 \sum\limits_{j=0}^{\infty} \psi_j^2,\ h = 0 \\ \rho(-h),\ h = -1, -2, -3, \ldots \text{ provided all the sums exist} \end{cases}$$

Consequently

$$\rho(h) = \begin{cases} \dfrac{\sum\limits_{j=0}^{\infty} \psi_j \psi_{j+h}}{\sum\limits_{j=0}^{\infty} \psi_j^2},\ h = 0, 1, 2, 3, \ldots \\ \rho(-h),\ h = -1, -2, -3, \ldots \text{ provided all the sums exist} \end{cases}$$