QUESTIONS

1. A random sample of size 9 is drawn from the distribution with PDF

$$f_\theta(x) \propto \frac{x^2}{\theta^3};\ -3\theta < x < \theta;\ \theta > 0 \text{ and}$$
$$f_\theta(x) = 0;\ \text{otherwise}$$

   and the observations are found to be 10, –30, 14, –45, –34, 7, 12, 11, –13.

   Find the maximum likelihood estimate of θ. Also find (with justification) the maximum likelihood estimate of the variance for the above distribution.

2. The time a client waits to be served by the mortgage specialist at a bank has probability density function

$$f(x) = \frac{1}{2\theta^3} x^2 e^{-x/\theta};\ x > 0;\ \theta > 0.$$

   The waiting times of 15 clients are found to be 6, 12, 15, 14, 12, 10, 8, 9, 10, 9, 8, 7, 10, 7 and 3 minutes. Calculate the values of the maximum likelihood estimate and the method of moments estimate of θ.

3. Consider the life-time of an electric bulb which is exponentially distributed with mean 3θ. The life-time of 20 bulbs are found to be 0.11, 2.28, 6.33, 0.67, 3.68, 1.46, 4.17, 2.96, 4.93, 9.49, 1.02, 0.60, 3.85, 1.32, 0.24, 2.38, 0.41, 4.98, 6.91, 2.14 years. Obtain

   (i) MLE of θ.

   (ii) unbiased estimate of θ.

   (iii) MLE of $P[X_5 > 6]$.

4. A random sample of size 20 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

   and the observations are found to be 0.10, 0.34, 0.77, 0.57, 1.63, 0.06, 1.14, 0.40, 0.07, 1.17, 0.17, 0.17, 0.16, 0.01, 0.79, 0.06, 0.38, 0.28, 0.17, 1.30.

   Obtain method of moments estimate of θ.

5. A random sample of size 20 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

   and the observations are found to be 2.13, 3.30, 3.78, 1.99, 2.91, 1.68, 1.18, 2.97, 3.89, 4.71, 2.05, 1.56, 0.48, 2.26, 3.27, 2.16, 2.71, 4.46, 1.78, 4.47.

   Obtain method of moments estimates of α and β.

6. An urn contains 6 marbels of which θ are white and the others black. In order to test the null hypothesis $H_0 : \theta = 3$ against the alternative $H_1 : \theta = 4$, two marbels are drawn at random (without replacement) and H_0 is rejected if both the marbels are white; otherwise H_0 is accepted. Find the probabilities of committing type I and type II errors.

7. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}x} \cdot \exp\{-\frac{1}{8}(\ln x - \mu)^2\}, & x > 0, \ -\infty < \mu < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Let the random sample be X_1, X_2, ..., X_{10}. To test $H_0 : \mu = 1$ against $H_1 : \mu > 1$, we reject H_0 if $\frac{1}{10}\sum_{i=1}^{10} \ln X_i > 1$.

Find P(type I error). Also find P(type II error) and power of the test when $\mu = 2$. Also draw a sketch of the power curve for $\mu = -3, -2, -1, 0, 1, 2, 3$ and comment on your findings.

8. Let X be a random variable with PMF under H_0 and H_1 given by

x	1	2	3	4	5	6
$f_0(x)$	0.01	0.01	0.01	0.01	0.01	0.95
$f_1(x)$	0.05	0.04	0.03	0.02	0.01	0.85

Find the most powerful test of level $\alpha = 0.03$. Also find the power of the test.

9. Let X be a random variable with PMF under H_0 and H_1 given by

x	1	2	3	4	5
$f_0(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$f_1(x)$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$

Find the most powerful test of level $\alpha = 0.5$. Also find the power of the test.

10. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}x} \cdot \exp\{-\frac{1}{8}(\ln x - \mu)^2\}, & x > 0, \ -\infty < \mu < \infty \\ 0, & \text{otherwise.} \end{cases}$$

and the observations are found to be 1.71, 1.18, 3.50, 0.82, 2.01, 0.60, 0.60, 1.68, 0.31, 1.10. Find the uniformly most powerful test of level $\alpha = 0.05$ for testing $H_0 : \mu = 1$ against $H_1 : \mu > 1$ and draw your conclusions.

11. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \theta\,(1-x)^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

and the observations be X_1, X_2, ..., X_{10}.

(i) Find the uniformly most powerful test of level $\alpha = 0.05$ for testing $H_0 : \theta = 1$ against $H_1 : \theta < 1$.

(ii) Find the uniformly most powerful test of level $\alpha = 0.05$ for testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$.

(iii) Obtain the power functions for both the cases. Hence, explain why UMP test for tesing $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ does not exist.

12. Let X be a random variable with PMF

$$f(x) = \begin{cases} \dfrac{2 + 4\alpha_1 + \alpha_2}{6}, & \text{if } x = 1, \\[2ex] \dfrac{2 - 2\alpha_1 + \alpha_2}{6}, & \text{if } x = 2, \\[2ex] \dfrac{1 - \alpha_1 - \alpha_2}{3}, & \text{if } x = 3. \end{cases}$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are unknown parameters such that $\alpha_1 + \alpha_2 \leq 1$. For testing the null hypothesis $H_0 : \alpha_1 + \alpha_2 = 1$ against the alternative hypothesis $H_1 : \alpha_1 = \alpha_2 = 0$, suppose the critical region is $C = \{2, 3\}$. Check whether the said critical region is unbiased.

13. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

and the observations be X_1, X_2, ..., X_{10}. Apply the method of likelihood ratio testing to develop a critical region with size $\alpha = 0.05$ for testing the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta = 2$.

14. Let X_1, X_2, ..., X_{10} be a random sample of size 10 drawn from a $N(\mu, 4)$ population. Use the method of likelihood ratio testing to develop a critical region with size $\alpha = 0.05$ for testing the null hypothesis $H_0 : \mu = 1$ against the alternative hypothesis $H_1 : \mu \neq 1$.