MSMS 301 - Time Series

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Contents

1 Some Important Functions

Let $\{X_t\}$ be any process. Then we have

(1) Mean Function : $\mu(t) = E(X_t)$

(2) Variance Functions : $\sigma^2(t) = E(X_t^2) - (E(X_t))^2$

(3) Covariance Function : $\gamma(t,s) = cov(X_t, X_s) = E(X_t X_s) - E(X_t)E(X_s)$

(4) Correlation Function : $\rho(t,s) = \dfrac{\gamma(t,s)}{\sqrt{\sigma^2(t) \cdot \sigma^2(s)}}$

2 Stationarity

Stationarity is a phenomenon of any stochastic process in which probabilistic nature of the random variables remain constant over time.

2.1 Strong/Strict Stationary

A stochastic process (or a time series to be specific) $\{X_t\}$ is said to be strictly stationary if the joint distribution of $\{X_{t_1}, X_{t_2}, \ldots, X_{t_k}\}$ is same as the joint distribution of $\{X_{t_k+h}, X_{t_k+h}, \ldots, X_{t_k+h}\}$ for all integers $k \geq 0$ and h *i.e.*

$$\{X_{t_1}, X_{t_2}, \ldots, X_{t_k}\} \overset{d}{\equiv} \{X_{t_k+h}, X_{t_k+h}, \ldots, X_{t_k+h}\}.$$

2.2 Weak Stationary

Strong Stationary is a much tough condition and is very rare to observe in practice. So we introduce a less strict set of conditions, called Weak Stationary Conditions.

A time series is said to be Weakly Stationary of order(r) if the moments of the process upto order r depend only on the time differences.

A time series $\{X_t\}$ is said to be Second Order Stationary if its mean is constant and autocovariance function depends only on the lag, provided all the required means exist *i.e.*

(i) $\forall t \; E(X_t) = \mu$.

(ii) $\forall t \; \gamma(h) = cov(X_t, X_{t+h})$ depends only on lag h, not depends on t.

Strong Stationary $\Rightarrow$ Weak Stationary, but the converse is not necessarily true. However, for a Gaussian Process, Weak Stationary does readily imply Strong Stationary as a Gaussian Process is completely determined by its mean and covariance.

Let $\{Z_t\} \sim$ White Noise $(0, \sigma^2)$. Get $E(X_t)$, $Var(X_t)$, $cov(X_t, X_x)$ and $corr(X_t, X_s)$ for the process

$$X_t = \frac{1}{3}(Z_{t-1} + Z_t + Z_{t+1})$$

. Thus comment on the stationarity of the process $\{X_t\}$.

$$E(X_t) = \frac{1}{3}(E[Z_{t-1}] + E[Z_t] + E[Z_{t+1}]) = 0, \text{ independent of } t.$$

$Var(X_t) = \frac{1}{9}(Var[Z_{t-1}] + Var[Z_t] + Var[Z_{t+1}]) = \frac{1}{9} \cdot 3\sigma^2 = \frac{\sigma^2}{3}$, independent of t.

Now

$$
\begin{aligned}
cov(X_t, X_s) &= cov\left(\frac{1}{3}\left(Z_{t-1} + Z_t + Z_{t+1}\right), \frac{1}{3}(Z_{s-1} + Z_s + Z_{s+1})\right) \\
&= \frac{1}{9}\left[cov(Z_{t-1}, Z_{s-1}) + cov(Z_{t-1}, Z_s) + cov(Z_{t-1}, Z_{s+1})\right] \\
&\quad + \frac{1}{9}\left[cov(Z_t, Z_{s-1}) + cov(Z_t, Z_s) + cov(Z_t, Z_{s+1})\right] \\
&\quad + \frac{1}{9}\left[cov(Z_{t+1}, Z_{s-1}) + cov(Z_{t+1}, Z_s) + cov(Z_{t+1}, Z_{s+1})\right] \\
&= \begin{cases} \frac{1}{3}\sigma^2, & |t-s| = 0 \\ \frac{2}{9}\sigma^2, & |t-s| = 1 \\ \frac{1}{9}\sigma^2, & |t-s| = 2 \\ 0, & |t-s| \geq 3 \end{cases}
\end{aligned}
$$

$\therefore$ So the covariances are independent of t and depending only on lag $|t-s|$.

$\Rightarrow$ The process $\{X_t\}$ is weak stationary.

Then

$$
\begin{aligned}
corr(X_t, X_s) &= \frac{cov(X_t, X_s)}{\sqrt{Var(X_t) \cdot Var(X_s)}} \\
&= \frac{cov(X_t, X_s)}{\frac{\sigma^2}{3}} \\
&= \begin{cases} 1, & |t-s| = 0 \\ \frac{2}{3}, & |t-s| = 1 \\ \frac{1}{3}, & |t-s| = 2 \\ 0, & |t-s| \geq 3 \end{cases}
\end{aligned}
$$

3 Random Walk

@ 01.08.2025, Friday

For a process of the form $X_t = Z_{t-1} + 2Z_t + Z_{t+1},\ t = 0, 1, 2, \ldots$ where $Z_t \sim$ White Noise $(0, \sigma^2)$, determine the ACVF and ACF as a function of lag h. Also plot the ACF as a function of h.

From the given, $E(X_t) = E(Z_{t-1}) + 2E(Z_t) + E(Z_{t+1}) = 0\ \forall t$.

The ACVF is given by

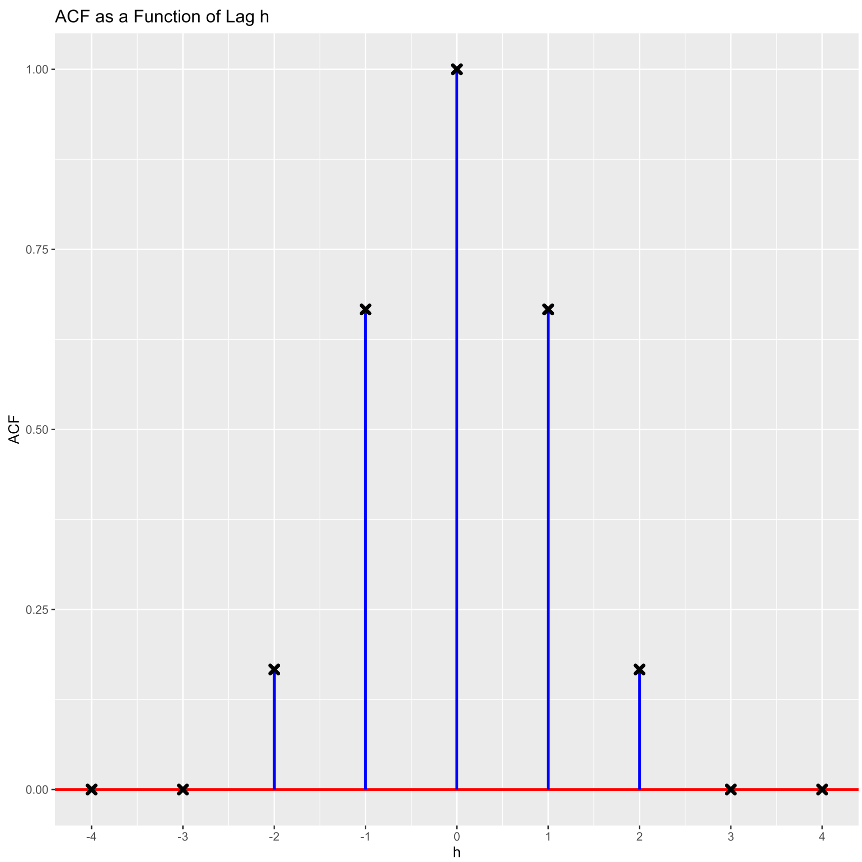
$$
\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov(Z_{t-1} + 2Z_t + Z_{t+1}, Z_{t+h-1} + 2Z_{t+h} + Z_{t+h+1}) \\
&= cov(Z_{t-1}, Z_{t+h-1}) + 2 \cdot cov(Z_{t-1}, Z_{t+h}) + cov(Z_{t-1}, Z_{t+h+1}) \\
&\quad + 2 \cdot cov(Z_t, Z_{t+h-1}) + 4 \cdot cov(Z_t, Z_{t+h}) + 2 \cdot cov(Z_t, Z_{t+h+1}) \\
&\quad + cov(Z_{t+1}, Z_{t+h-1}) + 2 \cdot cov(Z_{t+1}, Z_{t+h}) + cov(Z_{t+1}, Z_{t+h+1})
\end{aligned}
$$

$$
= \begin{cases}
\sigma^2 + 0 + 0 + 0 + 4\sigma^2 + 0 + 0 + 0 + \sigma^2, \ h = 0 \\
0 + 0 + 0 + 2\sigma^2 + 0 + 0 + 0 + 2\sigma^2 + 0, \ h = 1 \\
0 + 2\sigma^2 + 0 + 0 + 0 + 2\sigma^2 + 0 + 0 + 0, \ h = -1 \\
0 + 0 + 0 + 0 + 0 + 0 + \sigma^2 + 0 + 0, \ h = 2 \\
0 + 0 + \sigma^2 + 0 + 0 + 0 + 0 + 0 + 0, \ h = 2 \\
0, \ h = \pm 3, \pm 4, \pm 5, \ldots
\end{cases}
$$

$$
= \begin{cases}
6\sigma^2, \ h = 0 \\
4\sigma^2, \ h = \pm 1 \\
\sigma^2, \ h = \pm 2 \\
0, \ h = \pm 3, \pm 4, \pm 5, \ldots
\end{cases}
$$

Consequently the ACF will be

$$
\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases}
1, \ h = 0 \\
\frac{2}{3}, \ h = \pm 1 \\
\frac{1}{6}, \ h = \pm 2 \\
0, \ h = \pm 3, \pm 4, \pm 5, \ldots
\end{cases}
\end{aligned}
$$



4 Moving Average Process

Suppose $\{Z_t\}$ is a purely random process with mean 0 and variance σ^2. A process $\{X_t\}$ derived as a weighted sum of **present and past q** white noises is said to be a Moving Average Process of order q (abbreviated to $MA(q)$).

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \ldots + \beta_q Z_{t-q} \tag{1}$$

where all β_i's are constants. The Z's are usually scaled so that $\beta_0 = 1$.

It immediately follows

- $E(X_t) = 0\ \forall t$ as $E(Z_t) = 0\ \forall t$
- $Var(X_t) = \sigma^2 \sum_{i=0}^{q} \beta_i^2\ \forall t$ as $Z_t \overset{ind}{\sim}$ variance $= \sigma^2\ \forall t$.

4.1 MA(1) Process

With $Z_t \sim$ White Noise $(0, \sigma^2)$, the first order moving average process $\{X_t\}$ is defined as

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1};\ \beta_0 = 1 \tag{2}$$

$$\text{or } X_t = Z_t + \theta Z_{t-1} \tag{3}$$

where $\theta \in \mathbb{R}$ (as the process is finite, θ is free to be any real constant).

Clearly $E(X_t) = 0\ \forall t$ and $Var(X_t) = \sigma^2(1 + \theta^2)$.

Now

$$\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}) \\
&= cov(Z_t, Z_{t+h}) + \theta \cdot cov(Z_t, Z_{t+h-1}) + \theta \cdot cov(Z_{t-1}, Z_{t+h}) + \theta^2 \cdot cov(Z_{t-1}, Z_{t+h-1}) \\
&= \begin{cases} \sigma^2(1+\theta^2), \ h = 0 \\ \theta\sigma^2, \ h = \pm 1 \\ 0, \ h = \pm 2, \pm 3, \pm 4, \dots \end{cases}
\end{aligned}$$

Then

$$\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} 1, \ h = 0 \\ \dfrac{\theta}{1+\theta^2}, \ h = \pm 1 \\ 0, \ h = \pm 2, \pm 3, \pm 4, \dots \end{cases}
\end{aligned}$$

✎ So, for MA(1) process, the autocorrelation function vanishes after lag 1. This is an identifier for MA(1) process.

4.2 MA(q) Process

$MA(q)$ process as in (1) can be written as $X_t = \sum\limits_{i=0}^{q} \beta_i Z_{t-i}$.

Then

$$\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{j=0}^{q} \beta_j Z_{t+h-j} \right) \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{s=-h}^{q-h} \beta_{s+h} Z_{t-s} \right) \quad [s = j - h] \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{i=-h}^{q-h} \beta_{i+h} Z_{t-i} \right) \quad [\text{runner } s \text{ is just a dummy variable}] \\
&= \sum_{i=0}^{q-h} \beta_i \beta_{i+h} \cdot cov(Z_{t-i}, Z_{t-i}) \\
&= \sigma^2 \sum_{i=0}^{q-h} \beta_i \beta_{i+h} \\
&= \begin{cases} \sigma^2 \sum\limits_{i=0}^{q-h} \beta_i \beta_{i+h}, \ h = 0, 1, \dots, q \\ 0, \ h > q \\ \gamma(-h), \ h < 0 \end{cases}
\end{aligned}$$

Consequently

$$
\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} \dfrac{\sum\limits_{i=0}^{q-h} \beta_i \beta_{i+h}}{\sum\limits_{i=0}^{q} \beta_i^2}, \ h = 0, 1, \dots, q \\ 0, \ \ h > q \\ \rho(-h), \ \ h < 0 \end{cases}
\end{aligned}
$$

✎ It can be seen from the above expression that ACF becomes 0 at lag $k > q$. It shows that ACF of MA process cuts off at lag q which is a special characteristics of MA(q) process.

@ 03.08.2025, Sunday (Yesss !! On a goddaamn Sunday !! 😢)

💡 Random Cosine Curve : Consider a stochastic process $\{X_t\}$ given by

$$X_t = \cos\left(2\pi\left(\frac{t}{12} + \Phi\right)\right). \ t = 0, \pm 1, \pm 2, \dots$$

where $\Phi \sim U(0, 1)$. Find $E(X_t)$ and $Var(X_t)$.

5 Autoregressive Process

5.1 AR(1) Process

The first-order autoregressive process $\{X_t\}$ is defined as

$$X_t = \alpha X_{t-1} + Z_t \ \ t = 0, \pm 1, \pm 2, \dots \tag{4}$$

where Z_t's are White Noise with mean 0 and variance σ^2; $\alpha \in \mathbb{R}$, constant.

Notice (4) reduces to a random walk model for $\alpha = 1$.

We may write (4) as $X_t = \alpha B X_t + Z_t$ where B is a backshift operator with $BX_t = X_{t-1}$.

Then $(1 - \alpha B)X_t = Z_t$ so that

$$
\begin{aligned}
X_t &= (1 - \alpha B)^{-1} Z_t \\
&= (1 + \alpha B + \alpha^2 B^2 + \alpha^3 B^3 + \dots) Z_t \\
&= Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \alpha^3 Z_{t-3} + \dots \\
&= \sum_{j=0}^{\infty} \alpha^j Z_{t-j} \ \text{ provided the sum exists } i.e. \ |\alpha| < 1
\end{aligned} \tag{5}
$$

Using (5), $E(X_t) = 0, \ t = 0, \pm 1, \pm 2, \dots$

And $Var(X_t) = \sigma^2 \sum\limits_{j=0}^{\infty} \alpha^{2j} = \dfrac{\sigma^2}{1 - \alpha^2}$, provided $|\alpha| < 1$.

Then

$$\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}, \sum_{j=0}^{\infty} \alpha^j Z_{t+h-j}\right) \\
&= cov\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}, \sum_{k=-h}^{\infty} \alpha^{k+h} Z_{t-k}\right), \text{ taking } k = j - h \\
&= \sigma^2 \sum_{j=0}^{\infty} \alpha^j \alpha^{j+h} \\
&= \sigma^2 \alpha^h \sum_{j=0}^{\infty} \alpha^{2j} \\
&= \frac{\sigma^2 \alpha^h}{1-\alpha^2} \text{ provided } |\alpha| < 1, h \geq 0
\end{aligned}$$

Note that $\gamma(h)$ does not depend on t. So AR(1) model is <u>weak stationary</u> only if $|\alpha| < 1$.

Consequently

$$\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} 1, \;\; h = 0 \\ \alpha^h, \;\; h = 1, 2, \ldots \\ \rho(-h), \;\; h = -1, -2, \ldots \end{cases} \\
&= \alpha^{|h|}, \;\; h = \pm 1, \pm 2, \ldots
\end{aligned}$$

Multiplying X_{t-k} on both sides of (4) and taking expectation we get

$$\begin{aligned}
E(X_t \cdot X_{t-h}) &= \alpha E(X_{t-1} \cdot X_{t-h}) + E(Z_t \cdot X_{t-h}), \\
\Rightarrow \gamma(h) &= \alpha\, \gamma(h-1) \qquad (6)
\end{aligned}$$

Remember Z_t, X_{t-h} are independent and $E(X_t) = 0 \; \forall t$. Also, using (5), $cov(Z_t, X_{t-h}) = 0$.

Following (6) we have

$$\begin{aligned}
\gamma(1) &= \alpha\, \gamma(0) \\
\gamma(2) &= \alpha^2\, \gamma(0) \\
&\vdots \\
\gamma(h) &= \alpha^h \gamma(0) \qquad (7)
\end{aligned}$$

On dividing both sides of (7) by $\gamma(0)$, we get

$$\rho(h) = \alpha^{|h|}, \;\; h = 0, \pm 1, \pm 2, \ldots$$

Find ACVF & ACF and draw the ACF plot for the following $AR(1)$ processes.

(i) $X_t = 0.35\, X_{t-1} + Z_t$

(ii) $X_t = 0.85\, X_{t-1} + Z_t$

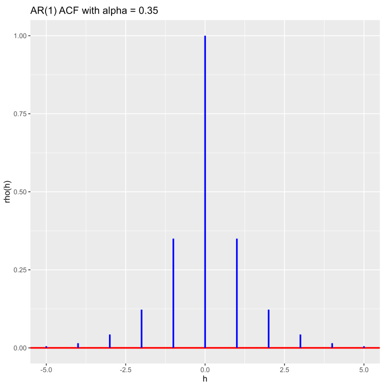
(iii) $X_t = -0.35\, X_{t-1} + Z_t$

The ACVFs and ACFs are as follows.

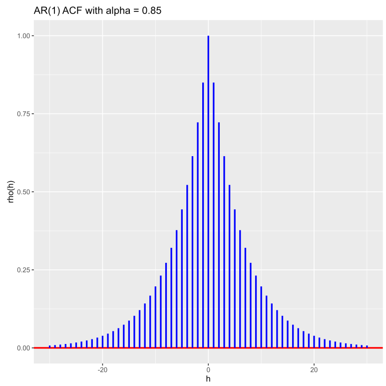
(i) $\gamma(h) = 0.8775 \cdot \sigma^2\, 0.35^{|h|}$, $\rho(h) = 0.35^{|h|}$ $\forall h = 0, \pm 1, \pm 2, \ldots$

(ii) $\gamma(h) = 0.2775 \cdot \sigma^2\, 0.85^{|h|}$, $\rho(h) = 0.85^{|h|}$ $\forall h = 0, \pm 1, \pm 2, \ldots$

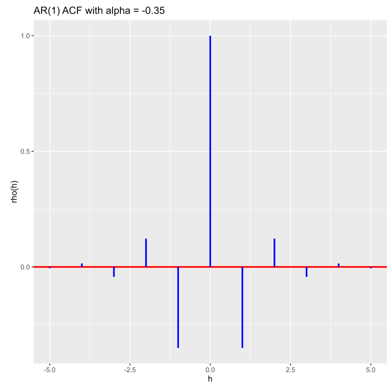
(iii) $\gamma(h) = 0.8775 \cdot \sigma^2\, (-0.35)^{|h|}$, $\rho(h) = (-0.35)^{|h|}$ $\forall h = 0, \pm 1, \pm 2, \ldots$



(i)



(ii)



(iii)

5.2 AR(2) Process

The second-order autoregressive process $\{X_t\}$ is defined as

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + Z_t \quad t = 0, \pm 1, \pm 2, \dots \tag{8}$$

where Z_t's are White Noise with mean 0 and variance σ^2; $\alpha_1, \alpha_2 \in \mathbb{R}$, constants.

The AR(2) process in (8) can be written in terms of backshift operator B as follows.

$$\begin{aligned} X_t &= \alpha_1 B X_t + \alpha_2 B^2 X_t + Z_t \\ \left(1 - \alpha_1 B - \alpha_2 B^2\right) X_t &= Z_t \\ X_t &= \left(1 - \alpha_1 B - \alpha_2 B^2\right)^{-1} Z_t \\ &= (\alpha(B))^{-1} Z_t \\ &= \psi(B) Z_t \end{aligned}$$

where $\alpha(z) = 1 - \alpha_1 z - \alpha_2 z^2$ and $\psi(z) = 1 + \psi_1 z + \psi_2 z^2 + \ldots = \sum_{j=0}^{\infty} \psi_j z^j$ with $\psi_0 = 1$.

$$\begin{aligned} \therefore X_t &= \sum_{j=0}^{\infty} \psi_j B^j Z_t \\ &= \sum_{j=0}^{\infty} \psi_j Z_{t-j} \text{ with } \psi_0 = 1 \end{aligned} \tag{9}$$

Using (9), $E(X_t) = 0$, $t = 0, \pm 1, \pm 2, \ldots$

And $\gamma(0) = Var(X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2$, provided the sum exists.

Then

$$
\begin{aligned}
\forall h > 0,\ \gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov\left(\sum_{j=0}^{\infty} \psi_j Z_{t-j}, \sum_{j=0}^{\infty} \psi_j Z_{t+h-j}\right) \\
&= cov\left(\sum_{j=0}^{\infty} \psi_j Z_{t-j}, \sum_{k=-h}^{\infty} \psi_{k+h} Z_{t-k}\right),\ \text{taking } k = j - h \\
&= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h} \text{ provided the sum exists}
\end{aligned}
$$

Thus

$$
\gamma(h) = \begin{cases} \sigma^2 \sum\limits_{j=0}^{\infty} \psi_j \psi_{j+h},\ h = 1, 2, 3, \ldots \\ \sigma^2 \sum\limits_{j=0}^{\infty} \psi_j^2,\ h = 0 \\ \rho(-h),\ h = -1, -2, -3, \ldots \text{ provided all the sums exist} \end{cases}
$$

Consequently

$$
\rho(h) = \begin{cases} \dfrac{\sum\limits_{j=0}^{\infty} \psi_j \psi_{j+h}}{\sum\limits_{j=0}^{\infty} \psi_j^2},\ h = 0, 1, 2, 3, \ldots \\ \rho(-h),\ h = -1, -2, -3, \ldots \text{ provided all the sums exist} \end{cases}
$$