B. Stat - B. Math UGA 2012

Ananda Biswas

3. Consider the functions $f_1(x) = x$, $f_2(x) = 2 + \log_e x$, $x > 0$ (where e is the base of natural logarithm). The graphs of the functions intersect

   (A) once in $(0, 1)$ and never in $(1, \infty)$
   (B) once in $(0, 1)$ and once in (e^2, ∞)
   (C) once in $(0, 1)$ and once in (e, e^2)
   (D) more than twice in $(0, \infty)$.

   Answer ::

   For $x = e^{-2} < 1$, $f_1(e^{-2}) = e^{-2}$ and $f_2(e^{-2}) = 2 - 2 = 0$. $\therefore f_1(e^{-2}) > f_2(e^{-2})$. $\ldots\ldots(i)$

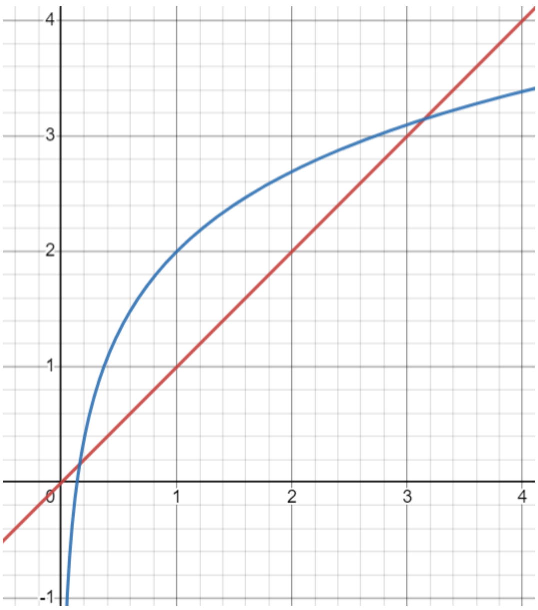
   Agian for $x = 1$, $f_1(1) = 1$ and $f_2(1) = 2$. $\therefore f_1(1) < f_2(1)$. $\ldots\ldots(ii)$

   As f_1 and f_2 are always continuous $\forall x > 0$, from (i) and (ii) we can conclude that, f_1 and f_2 must intersect once in $(e^{-2}, 1)$. $0 < e^{-2} < 1$. $\therefore f_1$ and f_2 intersect once in $(0, 1)$.

   Again, $f_1(e) = e$ and $f_2(e) = 2 + 1 = 3$. $2 < e < 3$. $\therefore f_1(e) < f_2(e)$. $\ldots\ldots(iii)$

   Also, $f_1(e^2) = e^2$ and $f_2(e^2) = 2 + 2 = 4$. $4 < e^2 < 9$. $\therefore f_1(e^2) > f_2(e^2)$. $\ldots\ldots(iv)$

   From (iii) and (iv) we can conclude that, f_1 and f_2 must intersect once in (e, e^2).



(C) once in $(0, 1)$ and once in (e, e^2)

7. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

   Then

(A) f is not continuous

(B) f is differentiable but f' is not continuous

(C) f is continuous but $f'(0)$ does not exist

(D) f is differentiable and f' is continuous.

Answer ::

From the given information, $f(x)$ is continuous everywhere, except at $x = 0$ where the continuity is questionable. So let's check the continuity of $f(x)$ at $x = 0$.

We have $f(0) = 0$; $\lim_{x\to 0^-} f(x) = 0$. Now, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0} e^{-\frac{1}{x}} = e^{-\infty} = 0$

$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$ $\therefore$ $f(x)$ is continuous at $x = 0$ and hence $f(x)$ is continuous everywhere.

Again from the given information, $f(x)$ is differentiable everywhere, except at $x = 0$ where the differentiability is questionable. So now let's check the differentiability of $f(x)$ at $x = 0$.

Now, $\lim_{x\to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \dfrac{0}{x} = 0.$

And, $\lim_{x\to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \dfrac{e^{-\frac{1}{x}}}{x} = \lim_{h\to\infty} he^{-h} = \lim_{h\to\infty} \dfrac{h}{e^h} = \dfrac{\text{tends to}\,\infty}{\text{tends to}\,e^{\infty}} = 0.$ $\left[\, x = \dfrac{1}{h} \,\right]$

$\therefore Lf'(0) = Rf'(0)$. $\therefore$ $f(x)$ is differentiable at $x = 0$ and hence $f(x)$ is differentiable everywhere.

Now,

$$f'(x) = \begin{cases} \dfrac{e^{-\frac{1}{x}}}{x^2}, & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Clearly, $f'(0) = 0$; $\lim_{x\to 0^-} f'(x) = 0$.

Now, $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0} \dfrac{e^{-\frac{1}{x}}}{x^2} = \lim_{h\to\infty} h^2 e^{-h} = \lim_{h\to\infty} \dfrac{h^2}{e^h} = \dfrac{\text{tends to}\,\infty}{\text{tends to}\,e^{\infty}} = 0$

So, f' is continuous everywhere.

(D) f is differentiable and f' is continuous.

19. What is the limit of

$$\left(1 + \frac{1}{n^2 + n}\right)^{n^2 + \sqrt{n}}$$

as $n \to \infty$?

(A) e

(B) 1

(C) 0

(D) ∞.

Answer ::

The given limit is of the format 1^∞.

So, the result of the limit is of the form e^A where $A = \lim\limits_{n\to\infty}\left(\dfrac{n^2+\sqrt{n}}{n^2+n}\right)$.

Now, $A = \lim\limits_{n\to\infty}\left(\dfrac{1+\frac{1}{n\sqrt{n}}}{1+\frac{1}{n}}\right) = \left(\dfrac{1+0}{1+0}\right) = 1.$

$\therefore \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2+n}\right)^{n^2+\sqrt{n}} = e^1 = e.$

(A) e

20. Consider the function $f(x) = x^4 + x^2 + x - 1$, $x \in (-\infty, \infty)$. The function

(A) is zero at $x = -1$, but is increasing near $x = -1$

(B) has a zero in $(-\infty, -1)$

(C) has two zeros in $(-1, 0)$

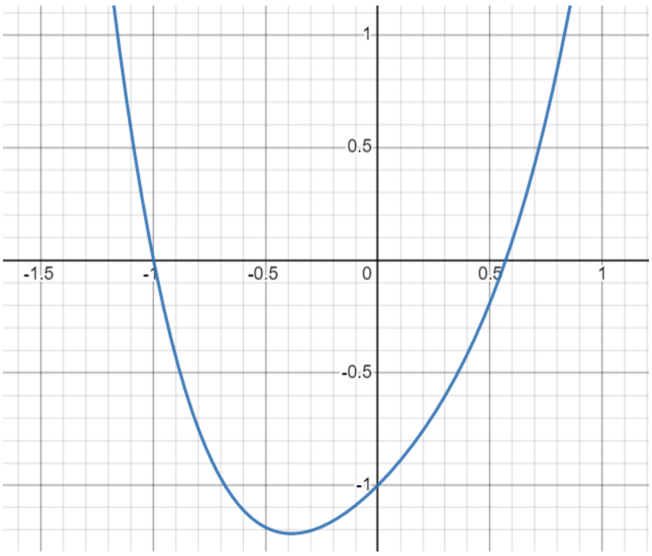
(D) has exactly one local minimum in $(-1, 0)$.

Answer ::

$f(-1) = 1+1-1-1 = 0$, $f(0) = 0+0+0-1 = -1$ and $f(1) = 1+1+1-1 = 2$

So there is a root of $f(x)$ in (0,1).

$f(x)$ is a polynomial. So $f(x)$ is continuous and differentiable everywhere. $f(x)$ is concave up and its graph looks like this.



Clearly, $\forall x \in (-\infty, -1)$, $f(x) > 0$.

So, option (B) is incorrect.

$\therefore f'(x) = 4x^3 + 2x + 1$. $\therefore f'(-1) = -4-2+1 = -5 < 0$. So $f(x)$ is decreasing at $x = -1$

So, option (A) is incorrect.

Again, $f'(1) = 4+2+1 = 7 > 0$. So $f(x)$ is increasing at $x = 1$

Also, $f'(0) = 0+0+1 = 1 > 0$. So $f(x)$ is increasing at $x = 0$

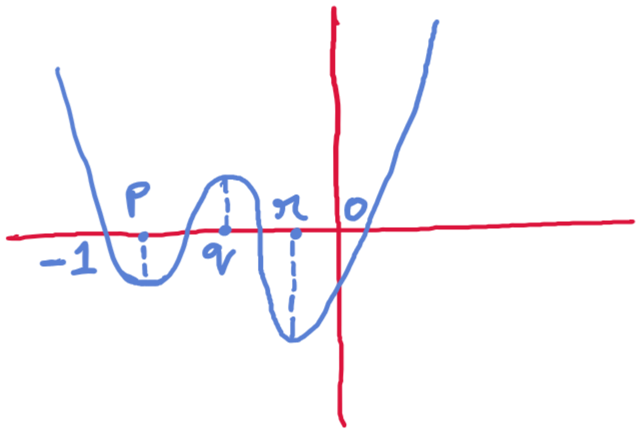
Observe that, $f'(-1) < 0$ and $f'(0) > 0$. So $\exists c \in (-1, 0)$ such that, $f'(c) = 0$. And at c, the change of sign must be from negative to positive.

So, $f(x)$ has one local minimum in $(-1, 0)$. Now we have to check whether this is the only minimum in $(-1, 0)$.

Now, $f''(x) = 12x^2 + 2$. So, $f''(x) > 0$ $\forall x \in (-1, 0)$. So, $f'(x)$ is always increasing in $(-1, 0)$. Therefore, $f'(x)$ can have maximum one root in between $(-1, 0)$. So, there exists only one $c \in (-1, 0)$ such that $f'(c) = 0$. Hence, $f(x)$ has exactly one local minimum in $(-1, 0)$.

So, option (D) is correct.

We shall also show that, $f(x)$ cannot have two zeros in $(-1, 0)$. If so, then the graph of $f(x)$ must look like this.



Then we would have 3 points p, q, r such that f' is zero. But we just showed that in $(-1, 0)$, f' cannot have more than one root. Consequently, in $(-1, 0)$, f cannot have any zero.

(D) has exactly one local minimum in $(-1, 0)$