B. Stat UGA 2010

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15. For any real number x, let $\tan^{-1}(x)$ denote the unique real number θ in $(-\pi/2, \pi/2)$ such that $\tan\theta = x$. Then

$$\lim_{n\to\infty} \sum_{m=1}^{n} \tan^{-1} \frac{1}{1+m+m^2}$$

(A) is equal to $\pi/2$;

(B) is equal to $\pi/4$;

(C) does not exist;

(D) none of the above.

Answer ::

Now,

$$\begin{aligned}
\tan^{-1}\frac{1}{1+m+m^2} &= \tan^{-1}\frac{(m+1)-m}{1+m(m+1)} \\
&= \tan^{-1}(m+1) - \tan^{-1} m
\end{aligned}$$

$$\begin{aligned}
\therefore \sum_{m=1}^{n} \tan^{-1}\frac{1}{1+m+m^2} &= \sum_{m=1}^{n} \left[\tan^{-1}(m+1) - \tan^{-1} m\right] \\
&= \tan^{-1}(n+1) - \tan^{-1} 1
\end{aligned}$$

$$\begin{aligned}
\therefore \lim_{n\to\infty} \sum_{m=1}^{n} \tan^{-1}\frac{1}{1+m+m^2} &= \lim_{n\to\infty} \left[\tan^{-1}(n+1) - \tan^{-1} 1\right] \\
&= \tan^{-1}(\infty) - \tan^{-1} \\
&= \frac{\pi}{2} - \frac{\pi}{4} \\
&= \frac{\pi}{4}
\end{aligned}$$

(B) is equal to $\frac{\pi}{4}$

22. Suppose that α and β are two distinct numbers in the interval $(0, \pi)$. If

$$\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$$

then the value of $\sin 3\alpha + \sin 3\beta$ is

(A) 0;

(B) $2\sin\dfrac{3(\alpha+\beta)}{2}$;

(C) $2\cos\dfrac{3(\alpha-\beta)}{2}$

(D) $\cos\dfrac{3(\alpha-\beta)}{2}$.

Answer ::

Given that, $\sin\alpha + \sin\beta = \sqrt{3}(\cos\alpha - \cos\beta)$

$$\therefore 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = \sqrt{3}\cdot 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

$$or, \cos\frac{\beta-\alpha}{2} = \sqrt{3}\cdot\sin\frac{\beta-\alpha}{2}$$

$$or, \tan\frac{\beta-\alpha}{2} = \frac{1}{\sqrt{3}}$$

$$or, \tan\frac{\beta-\alpha}{2} = \tan\frac{\pi}{6}$$

$$\therefore \frac{\beta-\alpha}{2} = \frac{\pi}{6} \Rightarrow \beta - \alpha = \frac{\pi}{3}$$

Now,

$$\begin{aligned}
\sin 3\alpha + \sin 3\beta &= 2\sin\frac{3(\alpha+\beta)}{2}\cdot\cos\frac{3(\alpha-\beta)}{2} \\
&= \quad 2\sin\frac{3(\alpha+\beta)}{2}\cdot\cos\left(\frac{3}{2}\cdot\frac{\pi}{3}\right)\ [\because \cos a = \cos -a] \\
&= 2\sin\frac{3(\alpha+\beta)}{2}\cdot\cos\frac{\pi}{2} \\
&= 0 \quad \left[\because \cos\frac{\pi}{2} = 0\right]
\end{aligned}$$

(A) 0