Methods of Estimation

1. A random sample of size 9 is drawn from the distribution with PDF

$$f_\theta(x) \propto \frac{x^2}{\theta^3};\ -3\theta < x < \theta;\ \theta > 0 \text{ and}$$
$$f_\theta(x) = 0;\ \text{otherwise}$$

and the observations are found to be 10, –30, 14, –45, –34, 7, 12, 11, –13.

Find the maximum likelihood estimate of θ. Also find (with justification) the maximum likelihood estimate of the variance for the above distribution.

2. The time a client waits to be served by the mortgage specialist at a bank has probability density function

$$f(x) = \frac{1}{2\theta^3} x^2 e^{-x/\theta};\ x > 0;\ \theta > 0.$$

The waiting times of 15 clients are found to be 6, 12, 15, 14, 12, 10, 8, 9, 10, 9, 8, 7, 10, 7 and 3 minutes. Calculate the values of the maximum likelihood estimate and the method of moments estimate of θ.

3. Consider the life-time of an electric bulb which is exponentially distributed with mean 3θ. The life-time of 20 bulbs are found to be 0.11, 2.28, 6.33, 0.67, 3.68, 1.46, 4.17, 2.96, 4.93, 9.49, 1.02, 0.60, 3.85, 1.32, 0.24, 2.38, 0.41, 4.98, 6.91, 2.14 years. Obtain

   (i) MLE of θ.

   (ii) unbiased estimate of θ.

   (iii) MLE of $P[X_5 > 6]$.

4. A random sample of size 20 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \theta e^{-\theta x}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

and the observations are found to be 0.10, 0.34, 0.77, 0.57, 1.63, 0.06, 1.14, 0.40, 0.07, 1.17, 0.17, 0.17, 0.16, 0.01, 0.79, 0.06, 0.38, 0.28, 0.17, 1.30.

Obtain method of moments estimate of θ.

5. A random sample of size 20 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \dfrac{\alpha^\beta}{\Gamma(\beta)} e^{-\alpha x} x^{\beta-1}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

and the observations are found to be 2.13, 3.30, 3.78, 1.99, 2.91, 1.68, 1.18, 2.97, 3.89, 4.71, 2.05, 1.56, 0.48, 2.26, 3.27, 2.16, 2.71, 4.46, 1.78, 4.47.

Obtain method of moments estimates of α and β.

Type-I, Type-II Errors & Power Curve

6. An urn contains 6 marbels of which θ are white and the others black. In order to test the null hypothesis $H_0 : \theta = 3$ against the alternative $H_1 : \theta = 4$, two marbels are drawn at random (without replacement) and H_0 is rejected if both the marbels are white; otherwise H_0 is accepted. Find the probabilities of committing type I and type II errors.

7. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}x} \cdot \exp\{-\frac{1}{8}(\ln x - \mu)^2\}, & x > 0,\ -\infty < \mu < \infty \\ 0, & \text{otherwise.} \end{cases}$$

Let the random sample be X_1, X_2, ..., X_{10}. To test $H_0 : \mu = 1$ against $H_1 : \mu > 1$, we reject H_0 if $\frac{1}{10}\sum_{i=1}^{10} \ln X_i > 1$.

Find P(type I error). Also find P(type II error) and power of the test when $\mu = 2$. Also draw a sketch of the power curve for $\mu = -3, -2, -1, 0, 1, 2, 3$ and comment on your findings.

Most Powerful Critical Region

8. Let X be a random variable with PMF under H_0 and H_1 given by

x	1	2	3	4	5	6
$f_0(x)$	0.01	0.01	0.01	0.01	0.01	0.95
$f_1(x)$	0.05	0.04	0.03	0.02	0.01	0.85

Find the most powerful test of level $\alpha = 0.03$. Also find the power of the test.

9. Let X be a random variable with PMF under H_0 and H_1 given by

x	1	2	3	4	5
$f_0(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$f_1(x)$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{6}$

Find the most powerful test of level $\alpha = 0.5$. Also find the power of the test.

Uniformly Most Powerful Critical Region

10. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{2\pi}x} \cdot \exp\{-\frac{1}{8}(\ln x - \mu)^2\}, & x > 0,\ -\infty < \mu < \infty \\ 0, & \text{otherwise.} \end{cases}$$

and the observations are found to be 1.71, 1.18, 3.50, 0.82, 2.01, 0.60, 0.60, 1.68, 0.31, 1.10. Find the uniformly most powerful test of level $\alpha = 0.05$ for testing $H_0 : \mu = 1$ against $H_1 : \mu > 1$ and draw your conclusions.

11. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \theta\,(1-x)^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

and the observations be X_1, X_2, ..., X_{10}.

(i) Find the uniformly most powerful test of level $\alpha = 0.05$ for testing $H_0 : \theta = 1$ against $H_1 : \theta < 1$.

(ii) Find the uniformly most powerful test of level $\alpha = 0.05$ for testing $H_0 : \theta = 1$ against $H_1 : \theta > 1$.

(iii) Obtain the power functions for both the cases. Hence, explain why UMP test for tesing $H_0 : \theta = 1$ against $H_1 : \theta \neq 1$ does not exist.

Unbiased Critical Region

12. Let X be a random variable with PMF

$$f(x) = \begin{cases} \dfrac{2+4\alpha_1+\alpha_2}{6}, & \text{if } x = 1, \\[2ex] \dfrac{2-2\alpha_1+\alpha_2}{6}, & \text{if } x = 2, \\[2ex] \dfrac{1-\alpha_1-\alpha_2}{3}, & \text{if } x = 3. \end{cases}$$

where $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ are unknown parameters such that $\alpha_1 + \alpha_2 \leq 1$. For testing the null hypothesis $H_0 : \alpha_1 + \alpha_2 = 1$ against the alternative hypothesis $H_1 : \alpha_1 = \alpha_2 = 0$, suppose the critical region is $C = \{2, 3\}$. Check whether the said critical region is unbiased.

Likelihood Ratio Test

13. A random sample of size 10 is drawn from the distribution with PDF

$$f(x) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

and the observations be X_1, X_2, ..., X_{10}. Apply the method of likelihood ratio testing to develop a critical region with size $\alpha = 0.05$ for testing the null hypothesis $H_0 : \theta = 1$ against the alternative hypothesis $H_1 : \theta = 2$.

14. Let X_1, X_2, ..., X_{10} be a random sample of size 10 drawn from a $N(\mu, 4)$ population. Use the method of likelihood ratio testing to develop a critical region with size $\alpha = 0.05$ for testing the null hypothesis $H_0 : \mu = 1$ against the alternative hypothesis $H_1 : \mu \neq 1$.

Confidence Interval

15. A random sample of size 10 is drawn from $U(0, \theta)$ distribution and the observations are found to be 1.39, 1.69, 1.48, 0.99, 0.15, 1.12, 1.76, 1.79, 1.94, 0.18. Obtain a 95% confidence interval for θ.

16. A random sample of size 10 is drawn from a distribution with PDF

$$f(x) = \begin{cases} e^{-(x-\theta)}, & x \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

and the observations are found to be 3.29, 2.78, 3.11, 2.69, 4.44, 2.43, 3.64, 2.21, 2.43, 2.69. Obtain a 95% confidence interval for θ.

17. Suppose the body weights of 100 fathers and first-born sons are measured and the sample correlation coefficient is found to be $r = 0.38$. Obtain a 95% confidence interval for the population correlation coefficient ρ.

Large Sample Test based on Pearsonian χ^2

18. A six faced die is thrown 300 times and the results obtained are as follows:

Face	1	2	3	4	5	6
Frequency	31	52	46	40	54	77

Use the data to test whether the die is unbiased.

19. 1072 school boys are classified according to intelligence and at the same time their economic conditions are recorded. The results are shown in the following table.

Economic	Intelligence			
Condition	Excellent	Good	Mediocore	Dull
Good	48	199	181	82
Not good	81	185	190	106

Judge whether there is any association between intelligence and economic conditions.

20. There are two sections in a class having 120 and 100 pupils respectively. The following table gives their results in the half-yearly and annual examinations.

Section 1 — Half-yearly exam | Section 2 — Half-yearly exam

Failed	Passed				Passed	Failed
12	48	Passed	Annual Exam	Passed	21	8
52	8	Failed		Failed	6	65

(i) For each section, test if the annual exam results have any association with the results of the half-yearly exam.

(ii) Test whether the two sections may be regarded as random samples from the same population.