MSMS-306 : Lifetime Data Analysis
Assignment: 02

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1. Question : Derive the estimators of the model parameters under the right censoring scheme when lifetimes are assumed to be coming from a Pareto distribution.

   Ans: The probability density function of a lifetime T having pareto distribution with shape $\alpha > 0$ and minimum value $t_m > 0$ is given by

$$f(t) = \begin{cases} \dfrac{\alpha t_{\mathrm{m}}^{\alpha}}{t^{\alpha+1}}, & t \geq t_{\mathrm{m}} \\ 0, & t < t_{\mathrm{m}} \end{cases}$$

   The corresponding Cumulative Distribution Function and the Survival Function are

$$\begin{aligned} F(t) = P(T \leq t) = \int_{t_{\mathrm{m}}}^{t} f(x)\,dx &= \int_{t_{\mathrm{m}}}^{t} \frac{\alpha t_{\mathrm{m}}^{\alpha}}{x^{\alpha+1}}\,dx \\ &= \alpha t_{\mathrm{m}}^{\alpha} \int_{t_{\mathrm{m}}}^{t} x^{-(\alpha+1)}\,dx \\ &= \alpha t_{\mathrm{m}}^{\alpha} \left[\frac{x^{-\alpha}}{-\alpha}\right]_{t_{\mathrm{m}}}^{t} \\ &= -t_{\mathrm{m}}^{\alpha}\left(t^{-\alpha} - t_{\mathrm{m}}^{-\alpha}\right) \\ &= 1 - \left(\frac{t_{\mathrm{m}}}{t}\right)^{\alpha}, \quad t \geq t_{\mathrm{m}} \end{aligned}$$

$$S(t) = \left(\frac{t_{\mathrm{m}}}{t}\right)^{\alpha}, \quad t \geq t_{\mathrm{m}}$$

   ☞ <u>Type I Censoring</u> : Let the observations are censored after time C.

$$\text{Define } \forall i, \delta_i = \begin{cases} 1, & i-\text{th observation is not censored} \\ 0, & i-\text{th observation is censored} \end{cases}$$

Then the likelihood function is given by

$$
\begin{aligned}
L(\alpha, t_{\mathrm{m}}) &= \prod_{i=1}^{n} f(t_i)^{\delta_i} \cdot S(C)^{1-\delta_i} \\
&= \prod_{i \in A} f(t_i) \cdot \prod_{i \in A^c} S(C) \quad \text{where } A = \{i | \delta_i = 1\} \\
&= \left[\left(\frac{t_{\mathrm{m}}}{C} \right)^{\alpha} \right]^{n-r} \cdot \prod_{i \in A} \frac{\alpha t_{\mathrm{m}}^{\alpha}}{t_i^{\alpha+1}}, \quad \text{on assuming the number of uncensored patients to be } r \\
&= \alpha^r t_{\mathrm{m}}^{\alpha n} \cdot \left(\prod_{i \in A} \frac{1}{t_i^{\alpha+1}} \right) \cdot \frac{1}{C^{\alpha(n-r)}}
\end{aligned}
$$

The log-likelihood function is given by

$$
\ell(\alpha, t_{\mathrm{m}}) = \log L(\alpha, t_{\mathrm{m}}) = r \log \alpha + \alpha n \log t_{\mathrm{m}} - (\alpha + 1) \sum_{i \in A} \log t_i - \alpha(n - r) \log C
$$

The partial derivative of $\ell(\alpha, t_{\mathrm{m}})$ w.r.t. α is as follows.

$$
\frac{\partial \ell}{\partial \alpha} = \frac{\partial}{\partial \alpha} (r \log \alpha) + \frac{\partial}{\partial \alpha} (\alpha n \log t_{\mathrm{m}}) - \frac{\partial}{\partial \alpha} \left((\alpha + 1) \sum_{i \in A} \log t_i \right) - \frac{\partial}{\partial \alpha} (\alpha(n - r) \log C)
$$

$$
= \frac{r}{\alpha} + n \log t_{\mathrm{m}} - \sum_{i \in A} \log t_i - (n - r) \log C
$$

We set the partial derivative to 0 and solve for α.

$$
\frac{r}{\alpha} + n \log t_{\mathrm{m}} - \sum_{i \in A} \log t_i - (n - r) \log C = 0
$$

$$
\Rightarrow \frac{r}{\alpha} = \sum_{i \in A} \log t_i + (n - r) \log C - n \log t_{\mathrm{m}}
$$

$$
\therefore \hat{\alpha}_{MLE} = \frac{r}{\sum\limits_{i \in A} \log t_i + (n - r) \log C - n \log \hat{t}_{\mathrm{m}}}
$$

$$
\text{and } \hat{t}_{\mathrm{m}} = t_{(1)} \quad \text{as } \hat{t}_{\mathrm{m}} \le t_i \ \forall i
$$

☞ <u>Type II Censoring</u> : Let $t_{(1)}, t_{(2)}, \ldots, t_{(r)}$ be the ordered observed lifetimes, and suppose the remaining $n - r$ units are right-censored at $t_{(r)}$. Then the likelihood function for the Pareto distribution is:

$$L(\alpha, t_{\mathrm{m}}) = \left[\prod_{i=1}^{r} f(t_i)\right] \cdot \left[S(t_{(r)})\right]^{n-r}$$

$$= \left[\prod_{i=1}^{r} \frac{\alpha t_{\mathrm{m}}^{\alpha}}{t_{(i)}^{\alpha+1}}\right] \cdot \left[\left(\frac{t_{\mathrm{m}}}{t_{(r)}}\right)^{\alpha}\right]^{n-r}$$

$$= \alpha^r t_{\mathrm{m}}^{\alpha r} \cdot \left(\prod_{i=1}^{r} \frac{1}{t_{(i)}^{\alpha+1}}\right) \cdot \left(\frac{t_{\mathrm{m}}}{t_{(r)}}\right)^{\alpha(n-r)}$$

$$= \alpha^r t_{\mathrm{m}}^{\alpha n} \cdot \left(\prod_{i=1}^{r} \frac{1}{t_{(i)}^{\alpha+1}}\right) \cdot \frac{1}{t_{(r)}^{\alpha(n-r)}}$$

The log-likelihood function is given by

$$\ell(\alpha, t_{\mathrm{m}}) = logL(\alpha, t_{\mathrm{m}}) = r\log\alpha + \alpha n \log t_{\mathrm{m}} - (\alpha+1)\sum_{i=1}^{r} \log t_{(i)} - \alpha(n-r)\log t_{(r)}$$

The partial derivative of $\ell(\alpha, t_{\mathrm{m}})$ w.r.t. α is as follows.

$$\frac{\partial \ell}{\partial \alpha} = \frac{\partial}{\partial \alpha}(r\log\alpha) + \frac{\partial}{\partial \alpha}(\alpha n \log t_{\mathrm{m}}) - \frac{\partial}{\partial \alpha}\left((\alpha+1)\sum_{i=1}^{r}\log t_{(i)}\right) - \frac{\partial}{\partial \alpha}\left(\alpha(n-r)\log t_{(r)}\right)$$

$$= \frac{r}{\alpha} + n\log t_{\mathrm{m}} - \sum_{i=1}^{r}\log t_{(i)} - (n-r)\log t_{(r)}$$

We set the partial derivative to 0 and solve for α.

$$\frac{r}{\alpha} + n\log t_{\mathrm{m}} - \sum_{i=1}^{r}\log t_{(i)} - (n-r)\log t_{(r)} = 0$$

$$\Rightarrow \frac{r}{\alpha} = \sum_{i=1}^{r}\log t_{(i)} + (n-r)\log t_{(r)} - n\log t_{\mathrm{m}}$$

$$\therefore \hat{\alpha}_{MLE} = \frac{r}{\sum_{i=1}^{r}\log t_{(i)} + (n-r)\log t_{(r)} - n\log \hat{t}_{\mathrm{m}}} \quad \text{and } \hat{t}_{\mathrm{m}} = t_{(1)} \quad \text{as } t_{\mathrm{m}} \le t_i \ \forall i$$

☞ Random Censoring : The likelihood function under random censoring is:

$$L(\alpha, t_{\mathrm{m}}) = \prod_{i=1}^{n} \left[f(t_i)^{\delta_i} \cdot S(t_i)^{1-\delta_i}\right]$$

$$= \prod_{i=1}^{n} \left[\left(\frac{\alpha t_{\mathrm{m}}^{\alpha}}{t_i^{\alpha+1}}\right)^{\delta_i} \cdot \left(\frac{t_{\mathrm{m}}}{t_i}\right)^{\alpha(1-\delta_i)}\right]$$

$$= \alpha^{\sum \delta_i} \cdot t_{\mathrm{m}}^{\alpha n} \cdot \prod_{i=1}^{n} \left(\frac{1}{t_i^{\delta_i(\alpha+1)+(1-\delta_i)\alpha}}\right)$$

The log-likelihood function is given by

$$\ell(\alpha, t_{\mathrm{m}}) = \log L(\alpha, t_{\mathrm{m}}) = \left(\sum_{i=1}^{n} \delta_i\right) \log \alpha + \alpha n \log t_{\mathrm{m}} - \sum_{i=1}^{n} [\delta_i(\alpha+1) + (1-\delta_i)\alpha] \log t_i$$

The partial derivative of $\ell(\alpha, t_{\mathrm{m}})$ w.r.t. α is as follows.

$$\frac{\ell}{\partial \alpha} = \left(\sum_{i=1}^{n} \delta_i\right) \frac{1}{\alpha} + n \log t_{\mathrm{m}} - \sum_{i=1}^{n} [\delta_i \cdot \log t_i + (1-\delta_i) \cdot \log t_i]$$

$$= \frac{1}{\alpha} \sum_{i=1}^{n} \delta_i + n \log t_{\mathrm{m}} - \sum_{i=1}^{n} \log t_i$$

We set the partial derivative to 0 and solve for α.

$$\frac{1}{\alpha} \sum_{i=1}^{n} \delta_i + n \log t_{\mathrm{m}} - \sum_{i=1}^{n} \log t_i = 0$$

$$\Rightarrow \frac{1}{\alpha} \sum_{i=1}^{n} \delta_i = \sum_{i=1}^{n} \log t_i - n \log t_{\mathrm{m}}$$

$$\therefore \hat{\alpha}_{MLE} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} \log t_i - n \log \hat{t}_{\mathrm{m}}} \quad \text{and } \hat{t}_{\mathrm{m}} = t_{(1)} \quad \text{as } t_{\mathrm{m}} \le t_i \ \forall i$$