Exponential Distribution as a Lifetime Model

Barsha—Arnab—Ritik—Rudransh—Sayan—Sujil—Ananda

Contents

Introduction

Lifetime data / survival time data measure the time to a certain event, such as failure, death etc. These times are subject to random variations, and like any random variables, form a probability distribution.

Introduction

Lifetime data / survival time data measure the time to a certain event, such as failure, death etc. These times are subject to random variations, and like any random variables, form a probability distribution.

Time(T) is a non-negative continuous ($T \geq 0$) quantity and one of the simplest distributions to model lifetime data is the Exponential Distribution.

Introduction

The distribution of survival times is usually characterized by 3 functions as follows :

Introduction

The distribution of survival times is usually characterized by 3 functions as follows :

- the Probability Density Function

Introduction

The distribution of survival times is usually characterized by 3 functions as follows :

- the Probability Density Function
- the Survival Function

Introduction

The distribution of survival times is usually characterized by 3 functions as follows :

- the Probability Density Function
- the Survival Function
- the Hazard Function

Introduction

The distribution of survival times is usually characterized by 3 functions as follows :

- the Probability Density Function
- the Survival Function
- the Hazard Function

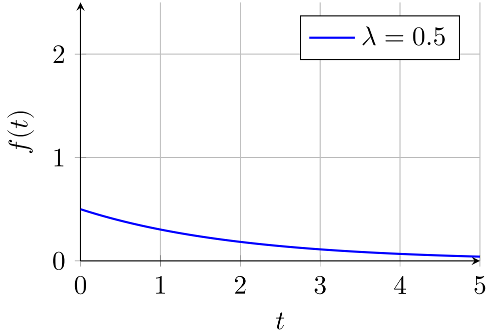
⁙ the Cumulative Hazard Function

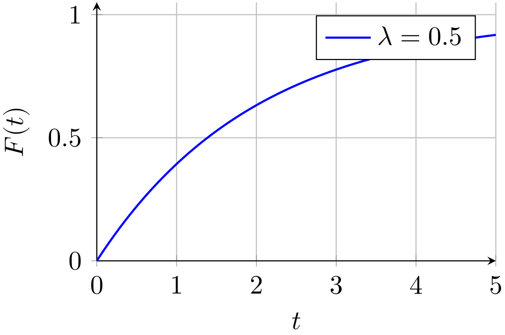
Probability Density Function

If the lifetime T follows the exponential distribution with <u>rate $\lambda \in \mathbb{R}^+$</u>, the probability density function $f : (-\infty, \infty) \to [0, \infty)$ is given by

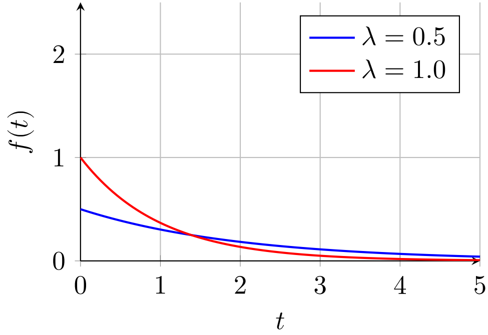
$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & \text{otherwise.} \end{cases}$$

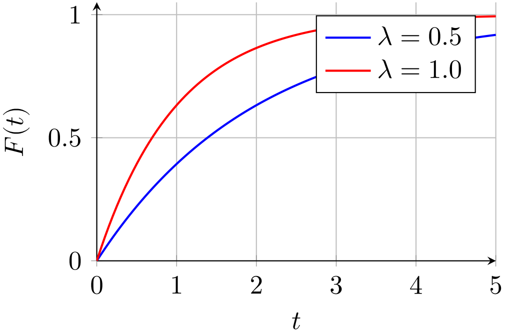
Exponential PDF and CDF



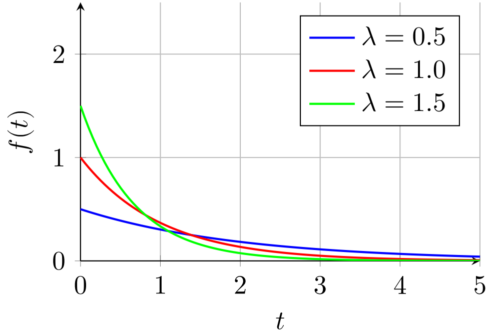


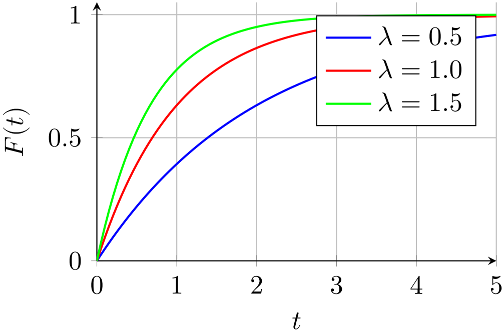
Exponential PDF and CDF



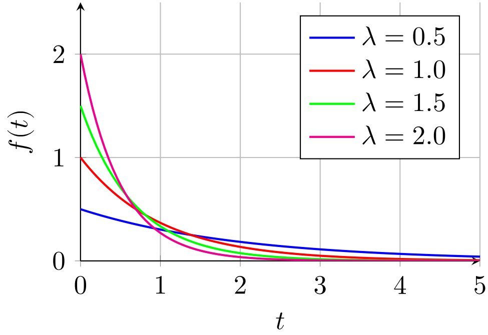


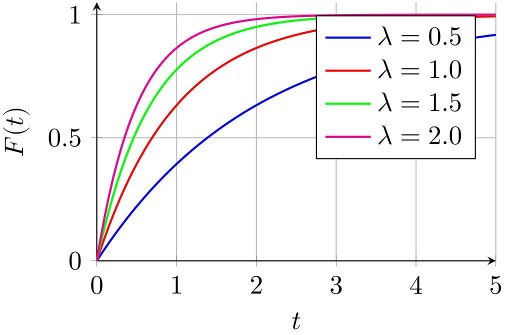
Exponential PDF and CDF





Exponential PDF and CDF





Survival Function

From definition, with $F(\cdot)$ being the CDF of T, Survival function $S : [0, \infty) \to [0, 1]$ is given by

$$S(t) = P[T > t] = 1 - F(t) \ \forall t \geq 0.$$

Survival Function

From definition, with $F(\cdot)$ being the CDF of T, Survival function $S : [0, \infty) \to [0, 1]$ is given by

$$S(t) = P[T > t] = 1 - F(t) \ \forall t \geq 0.$$

We calculate

Survival Function

From definition, with $F(\cdot)$ being the CDF of T, Survival function $S : [0, \infty) \to [0, 1]$ is given by

$$S(t) = P[T > t] = 1 - F(t) \ \forall t \geq 0.$$

We calculate

$$\forall t \geq 0, \ \ F(t) = \int_0^t f(x)dx$$

Survival Function

From definition, with $F(\cdot)$ being the CDF of T, Survival function $S : [0, \infty) \to [0, 1]$ is given by

$$S(t) = P[T > t] = 1 - F(t) \ \forall t \geq 0.$$

We calculate

$$\begin{aligned} \forall t \geq 0, \ \ F(t) &= \int_0^t f(x)dx \\ &= \int_0^t \lambda e^{-\lambda x} dx = \lambda \left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^t \\ &= 1 - e^{-\lambda t} \end{aligned}$$

Survival Function

From definition, with $F(\cdot)$ being the CDF of T, Survival function $S : [0, \infty) \to [0, 1]$ is given by

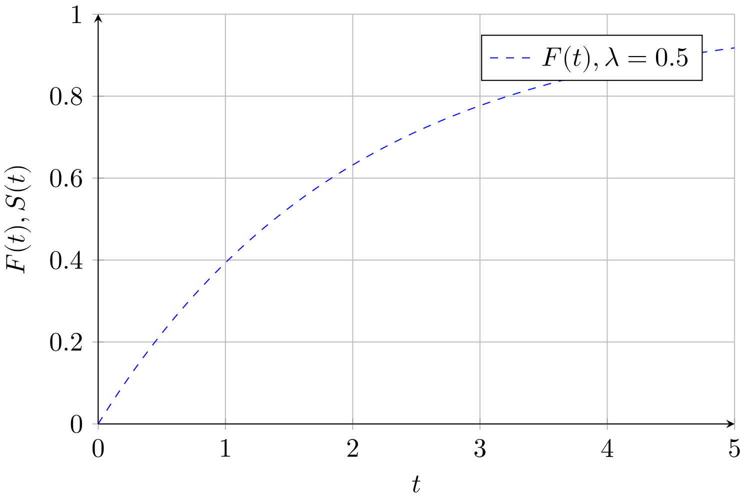
$$S(t) = P[T > t] = 1 - F(t) \ \forall t \geq 0.$$

We calculate

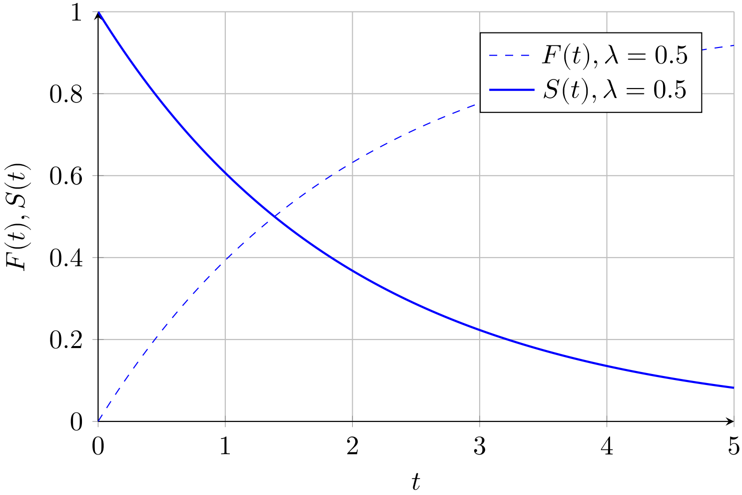
$$\begin{aligned}\forall t \geq 0, \ \ F(t) &= \int\limits_0^t f(x)dx \\ &= \int\limits_0^t \lambda e^{-\lambda x} dx = \lambda \left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^t \\ &= 1 - e^{-\lambda t}\end{aligned}$$

- $S(t) = e^{-\lambda t} \ \forall t \geq 0$

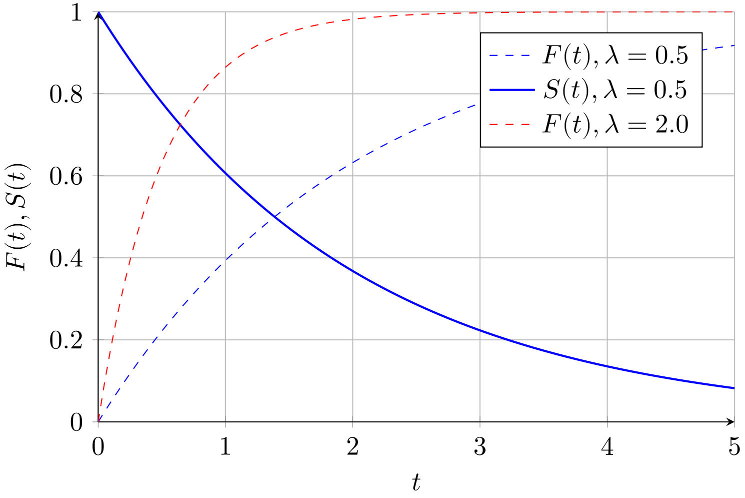
Exponential CDF and Survival Function



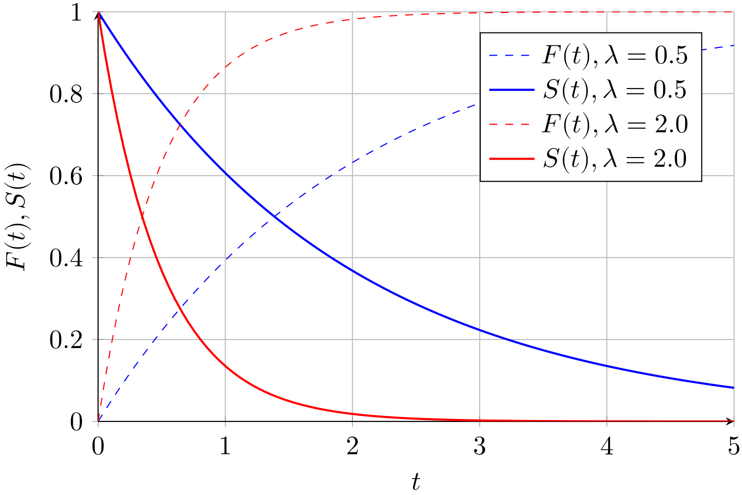
Exponential CDF and Survival Function



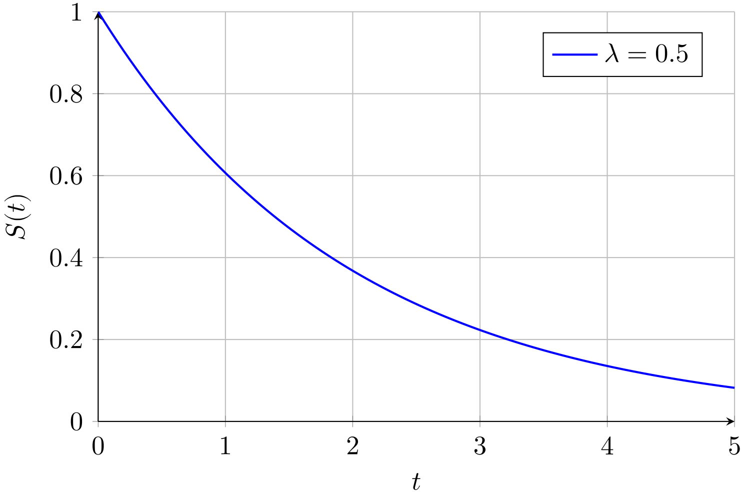
Exponential CDF and Survival Function



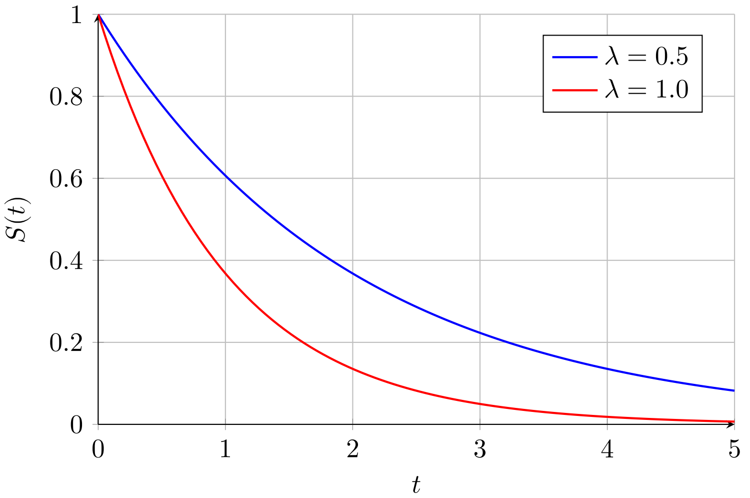
Exponential CDF and Survival Function



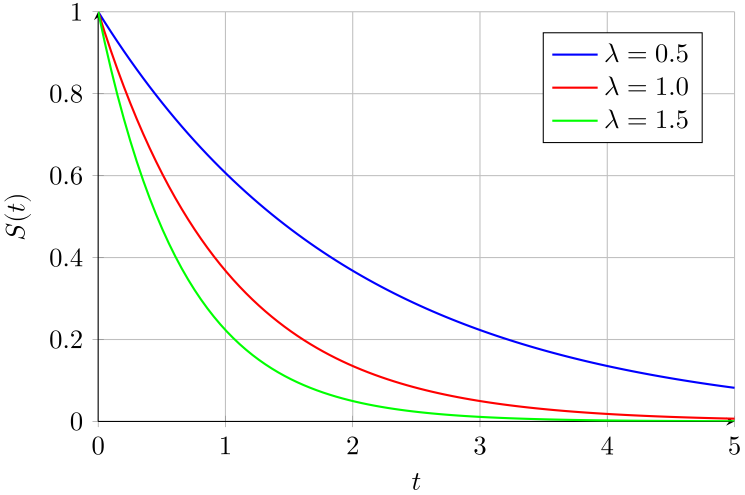
Exponential Survival Function



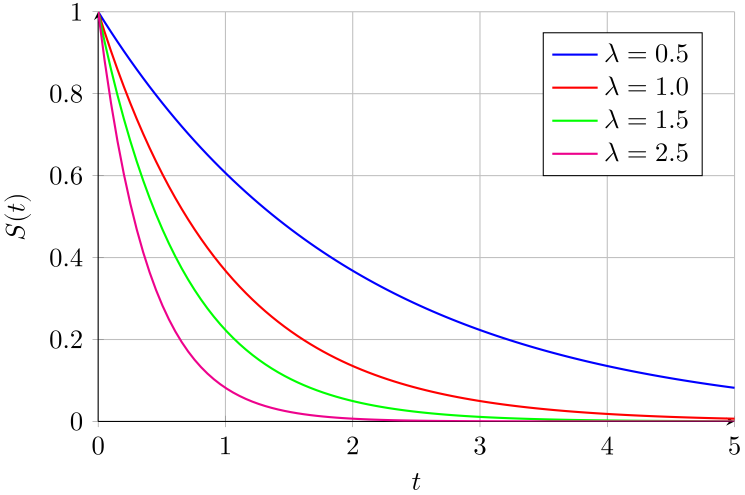
Exponential Survival Function



Exponential Survival Function



Exponential Survival Function



Hazard Function

The Hazard function $h : [0, \infty) \to [0, \infty)$ is defined as

$$h(t) = \lim_{\Delta t \to 0} \frac{P\begin{bmatrix} \text{an individual fails in the time interval } (t, t + \Delta t) \\ \text{given the individual has survived to } t \end{bmatrix}}{\Delta t} \quad \forall t \geq 0.$$

Hazard Function

The Hazard function $h : [0, \infty) \to [0, \infty)$ is defined as

$$h(t) = \lim_{\Delta t \to 0} \frac{P\left[\begin{array}{c}\text{an individual fails in the time interval } (t, t + \Delta t) \\ \text{given the individual has survived to } t\end{array}\right]}{\Delta t} \; \forall t \geq 0.$$

A simple derivation leads to

$$h(t) = \frac{f(t)}{S(t)} \; \forall t \geq 0.$$

Hazard Function

We have

- $f(t) = \lambda e^{-\lambda t} \cdot I_{(0,\infty)}(t),\ \lambda > 0.$

Hazard Function

We have

- $f(t) = \lambda e^{-\lambda t} \cdot I_{(0,\infty)}(t),\ \ \lambda > 0.$
- $S(t) = e^{-\lambda t}\ \forall t \geq 0.$

Hazard Function

We have

- $f(t) = \lambda e^{-\lambda t} \cdot I_{(0,\infty)}(t), \;\; \lambda > 0.$
- $S(t) = e^{-\lambda t} \;\; \forall t \geq 0.$

☞ $h(t) = \lambda \;\forall t \geq 0$, a constant, independent of t.

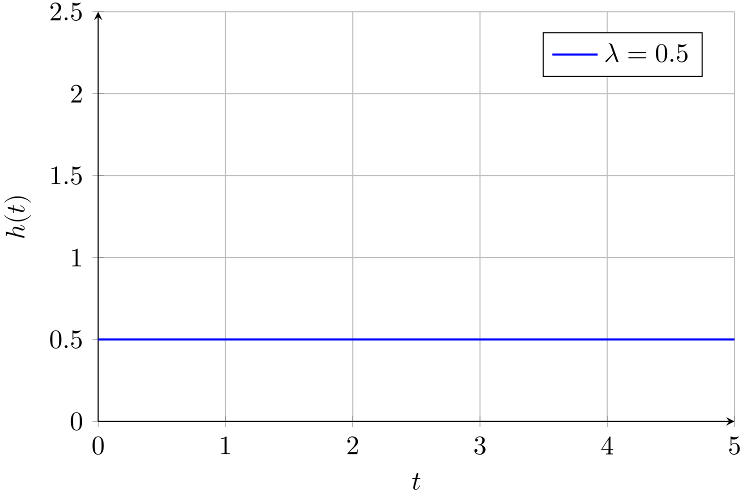
Hazard Function

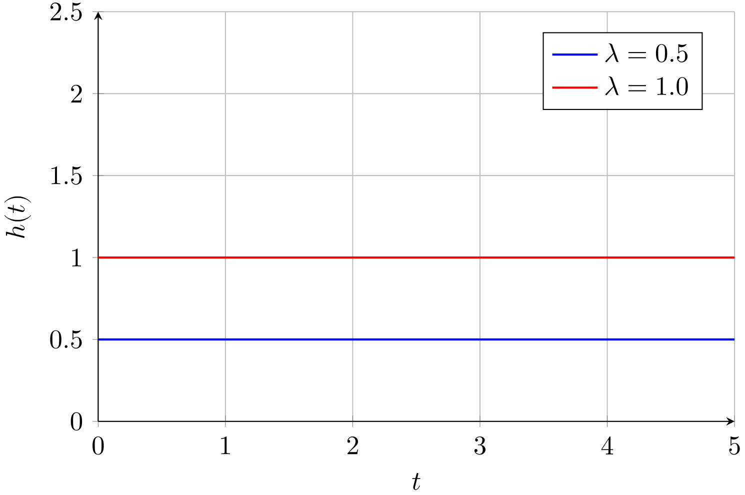
We have

- $f(t) = \lambda e^{-\lambda t} \cdot I_{(0,\infty)}(t), \;\; \lambda > 0.$
- $S(t) = e^{-\lambda t} \;\; \forall t \geq 0.$

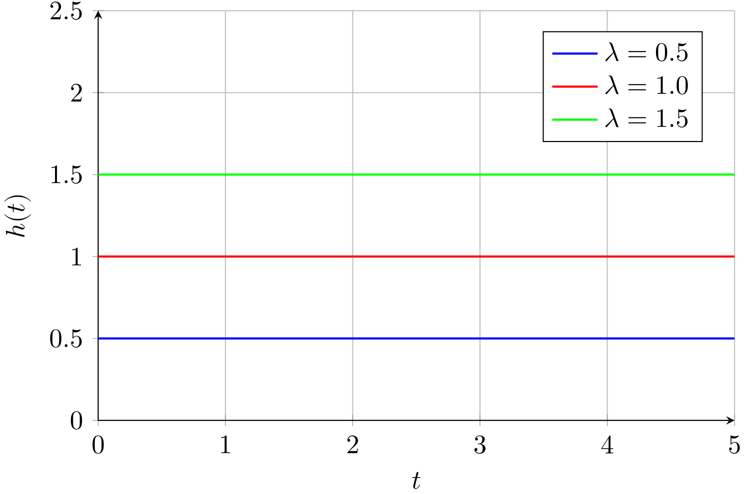
☞ $h(t) = \lambda \;\forall t \geq 0$, a constant, independent of t.

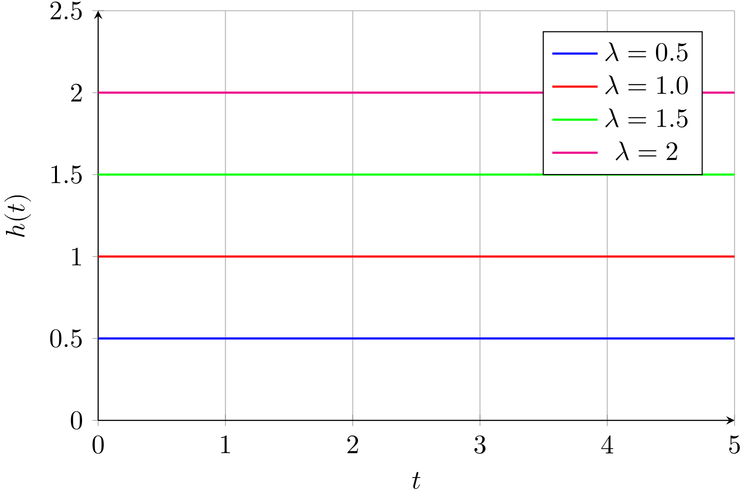
✎ A constant hazard rate is a necessary and sufficient condition for a continuous lifetime distribution to be exponential.


$\lambda = 0.5$
$h(t)$
t
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5



2.5
2
1.5
1
0.5
0
0
1
2
3
4
5
$h(t)$
t
$\lambda = 0.5$
$\lambda = 1.0$


$\lambda = 0.5$
$\lambda = 1.0$
$\lambda = 1.5$
$h(t)$
t
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5



2.5
2
1.5
1
0.5
0
0
1
2
3
4
5
$h(t)$
t
$\lambda = 0.5$
$\lambda = 1.0$
$\lambda = 1.5$
$\lambda = 2$

Cumulative Hazard Function

The Cumulative Hazard Function $H : [0, \infty) \to [0, \infty)$ is defined by

$$H(t) = \int_0^t h(x)dx.$$

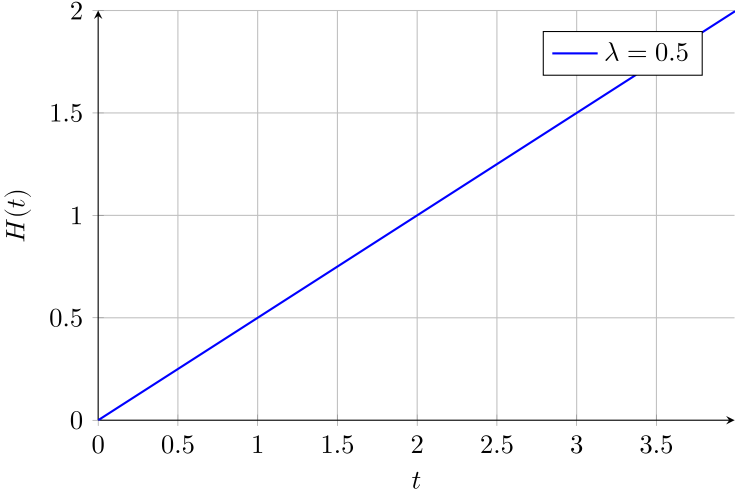
Cumulative Hazard Function

The Cumulative Hazard Function $H : [0, \infty) \to [0, \infty)$ is defined by

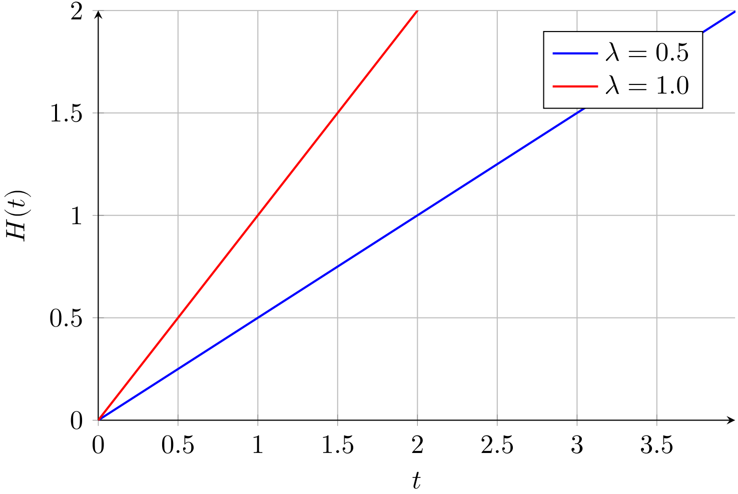
$$H(t) = \int_0^t h(x)dx.$$

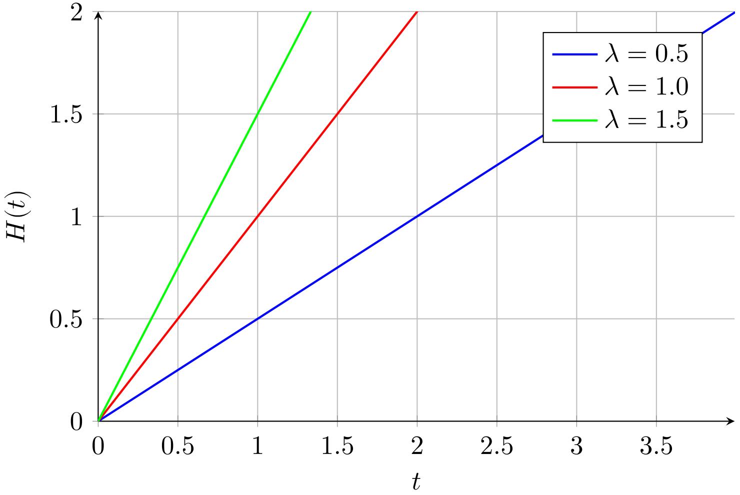
For $T \sim \text{Exp}(\text{rate} = \lambda)$,

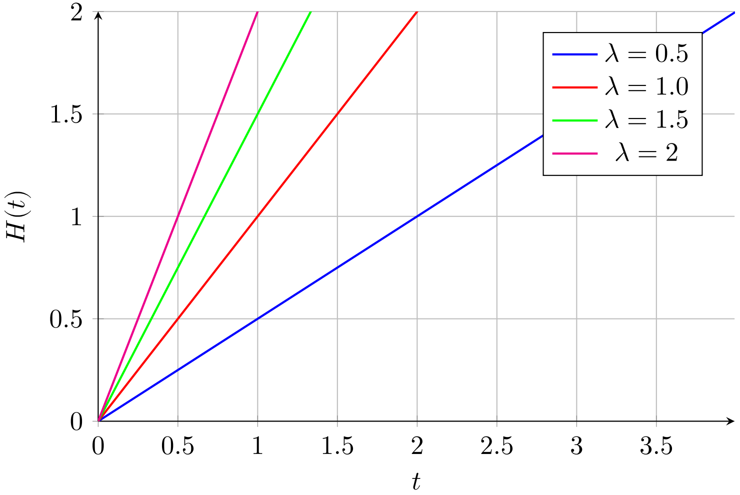
$$H(t) = \lambda t \ \forall t \geq 0.$$



$\lambda = 0.5$
$H(t)$
t
0
0.5
1
1.5
2
2.5
3
3.5


2
1.5
1
0.5
0
$H(t)$
0
0.5
1
1.5
2
2.5
3
3.5
t
$\lambda = 0.5$
$\lambda = 1.0$


$\lambda = 0.5$
$\lambda = 1.0$
$\lambda = 1.5$
$H(t)$
t
0
0.5
1
1.5
2
2.5
3
3.5



$H(t)$
t
0
0.5
1
1.5
2
2.5
3
3.5
$\lambda = 0.5$
$\lambda = 1.0$
$\lambda = 1.5$
$\lambda = 2$