MSMS-306 : Lifetime Data Analysis
Assignment : 01

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1. Question : Establish the exponential distribution as a suitable lifetime model by analyzing its key characteristics with graphical illustrations. Also, compare it with other standard lifetime models, highlighting its advantages and limitations.

   Ans: Lifetime data / survival time data measure the time to a certain event, such as failure, death etc. These times are subject to random variations, and like any random variables, form a probability distribution.

   Time(T) is a non-negative continuous ($T \geq 0$) quantity and the exponential distribution also has similar properties. That makes Exponential Distribution one of the simplest distributions to model lifetime data.

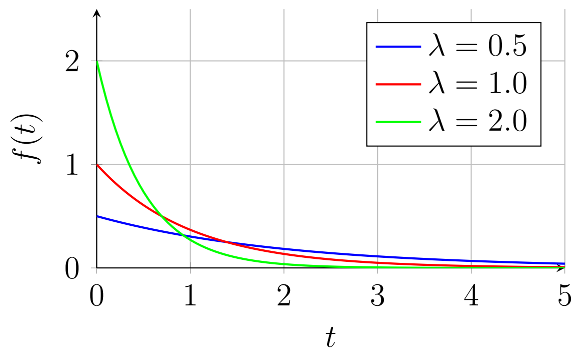
   The distribution of survival times is usually characterized by 3 functions as follows :

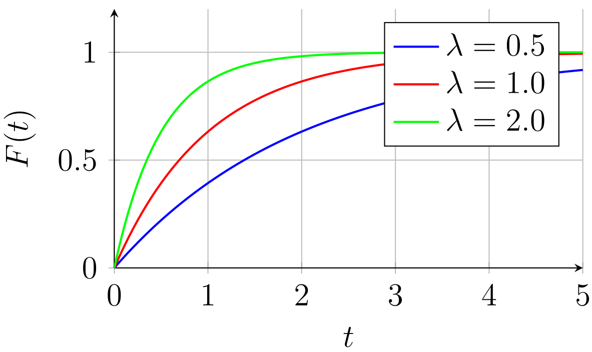
   - the Probability Density Function
   - the Survival Function
   - the Hazard Function

   There is also the Cumulative Hazard Function.

   ☞ Probability Density Function : If the lifetime T follows the exponential distribution with rate $\lambda \in \mathbb{R}^+$, the probability density function $f : (-\infty, \infty) \to [0, \infty)$ is given by

$$f(t) = \begin{cases} \lambda e^{-\lambda t}, & t > 0 \\ 0, & \text{otherwise.} \end{cases}$$





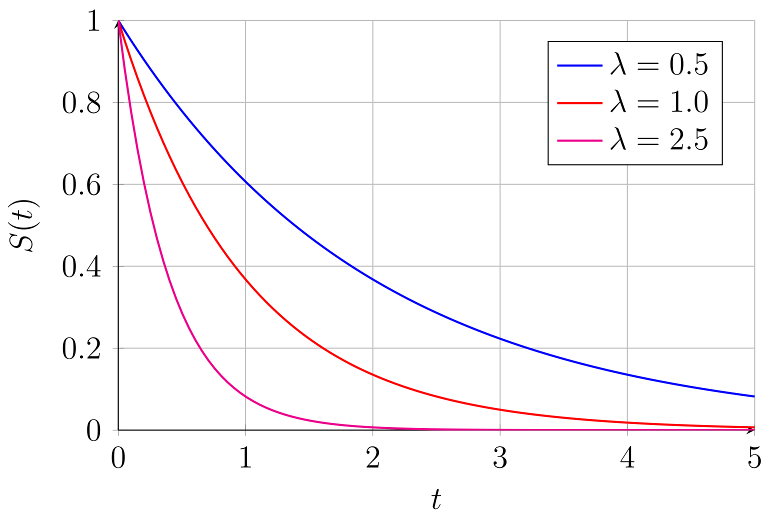
☞ <u>Survival Function</u> : From definition, with $F(\cdot)$ being the CDF of T, Survival function $S : [0, \infty) \to [0, 1]$ is given by

$$S(t) = P[T > t] = 1 - F(t) \ \forall t \geq 0.$$

We calculate

$$\begin{aligned} \forall t \geq 0,\ F(t) &= \int_0^t f(x)dx \\ &= \int_0^t \lambda e^{-\lambda x} dx = \lambda \left[-\frac{e^{-\lambda x}}{\lambda} \right]_0^t \\ &= 1 - e^{-\lambda t} \end{aligned}$$

$\therefore\ S(t) = e^{-\lambda t} \ \forall t \geq 0$



☞ <u>Hazard Function</u> : The Hazard function $h : [0, \infty) \to [0, \infty)$ is defined as

$$h(t) = \lim_{\Delta t \to 0} \frac{P\left[\begin{array}{c} \text{an individual fails in the time interval } (t, t + \Delta t) \\ \text{given the individual has survived to } t \end{array} \right]}{\Delta t} \ \forall t \geq 0.$$

A simple derivation leads to

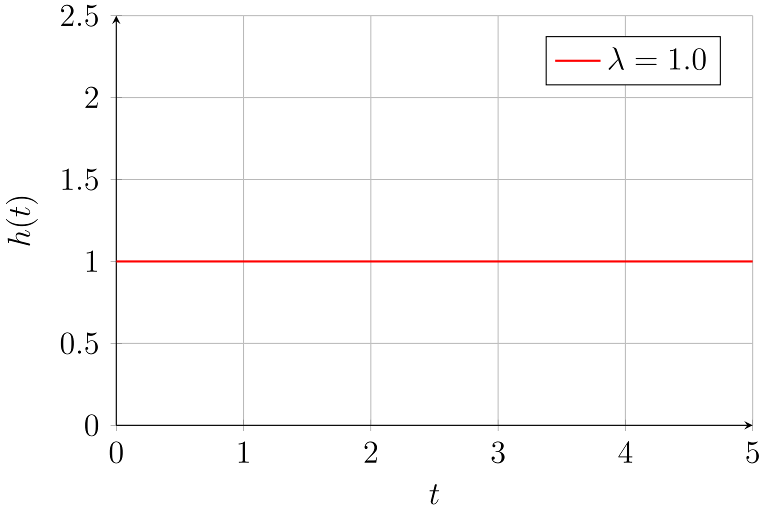
$$h(t) = \frac{f(t)}{S(t)} \ \forall t \geq 0.$$

Here we have

- $f(t) = \lambda e^{-\lambda t} \cdot I_{(0,\infty)}(t),\ \lambda > 0.$
- $S(t) = e^{-\lambda t} \ \forall t \geq 0.$

$\therefore\ h(t) = \lambda\ \forall t \geq 0$, a constant, independent of t.

✎ A constant hazard rate is a necessary and sufficient condition for a continuous lifetime distribution to be exponential.

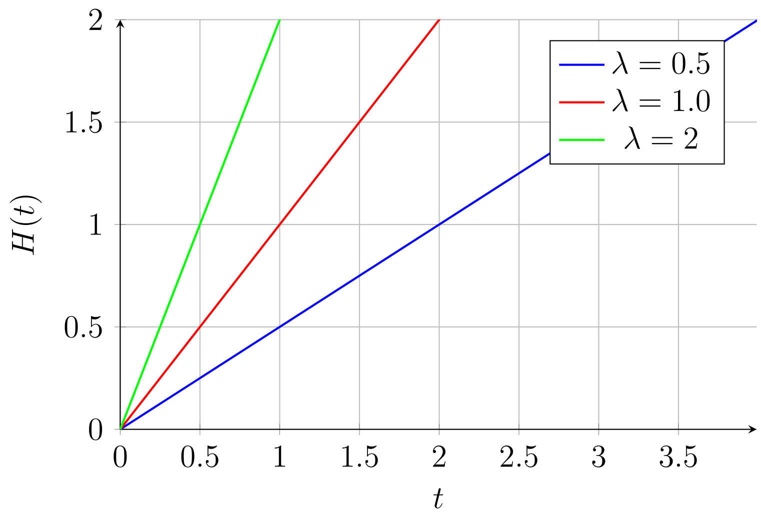


☞ Cumulative Hazard Function : The Cumulative Hazard Function $H : [0, \infty) \to [0, \infty)$ is defined by

$$H(t) = \int_0^t h(x)dx.$$

For $T \sim \text{Exp}(\text{rate} = \lambda)$,

$$H(t) = \lambda t\ \forall t \geq 0.$$



Comparison of Exponential Distribution with Other Standard Lifetime Models

Basis of Distinction	**Exponential**	**Weibull**	**Gamma**	**Lognormal**	**Pareto**
Hazard Function (Failure Rate)	Constant over time (no change)	Can increase or decrease based on shape	Usually increases with time	Increases first, then decreases (hump-shaped)	Decreases as time increases
Memoryless Property	Yes	No	No	No	No
Tail Behaviour (Extreme Events)	Light tails — extreme values are rare	Light tails — few extreme values	Moderate tails — some large values possible	Heavy tails — large values more common	Very heavy tails — high chance of extremes
Real-life Use Cases	Lifetimes of bulbs, radioactive decay	Mechanical systems, reliability engineering	Queues, survival analysis	Financial time durations, web session times	Insurance, wealth, income distribution
Mathematical Simplicity	Easiest	Moderate	Slightly complex	Complex	Moderate

Table 1: Comparison of exponential distribution with other standard lifetime models

Advantages of Exponential Distribution

Compared With	Advantage of Exponential Distribution
Weibull	Simpler to use and interpret. It has only one parameter (λ), making it easier to estimate. Suitable when the failure rate is constant over time.
Gamma	Computationally less intensive. Exponential is a special case of Gamma when shape = 1, so it works when only a single failure phase is present.
Lognormal	Easier to analyze mathematically. Exponential has memoryless property, unlike lognormal. Useful in simpler reliability models.
Pareto	Has lighter tails, making it better for modeling systems without extreme outliers. Also easier to fit and simulate.

Limitations of Exponential Distribution

Compared With	Limitation of Exponential Distribution
Weibull	Assumes constant failure rate, which is unrealistic in many aging systems. Weibull can model increasing or decreasing failure rates.
Gamma	Can't model multi-stage failure processes. Gamma handles systems that degrade over multiple phases more effectively.
Lognormal	Cannot model "typical" failure times with a peak. Lognormal captures bell-shaped distributions of lifetimes.
Pareto	Doesn't model rare/extreme events well. Pareto has heavy tails, useful for risk modeling and extreme lifetimes.