MSMS 301 - Time Series

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1 Moving Average Process

Suppose $\{Z_t\}$ is a purely random process with mean 0 and variance σ^2. A process $\{X_t\}$ derived as a weighted sum of **present and past q** white noises is said to be a Moving Average Process of order q (abbreviated to $MA(q)$).

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \ldots + \beta_q Z_{t-q} \tag{1}$$

where all β_i's are constants. The Z's are usually scaled so that $\beta_0 = 1$.

It immediately follows

- $E(X_t) = 0 \ \forall t$ as $E(Z_t) = 0 \ \forall t$
- $Var(X_t) = \sigma^2 \sum_{i=0}^{q} \beta_i^2 \ \forall t$ as $Z_t \overset{ind}{\sim}$ variance $= \sigma^2 \ \forall t$.

1.1 MA(1) Process

With $Z_t \sim$ White Noise $(0, \sigma^2)$, the first order moving average process $\{X_t\}$ is defined as

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1}; \ \beta_0 = 1 \tag{2}$$

$$\text{or } X_t = Z_t + \theta Z_{t-1} \tag{3}$$

where $\theta \in \mathbb{R}$ (as the process is finite, θ is free to be any real constant).

Clearly $E(X_t) = 0 \ \forall t$ and $Var(X_t) = \sigma^2(1 + \theta^2)$.

Now

$$\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov(Z_t + \theta Z_{t-1}, Z_{t+h} + \theta Z_{t+h-1}) \\
&= cov(Z_t, Z_{t+h}) + \theta \cdot cov(Z_t, Z_{t+h-1}) + \theta \cdot cov(Z_{t-1}, Z_{t+h}) + \theta^2 \cdot cov(Z_{t-1}, Z_{t+h-1}) \\
&= \begin{cases} \sigma^2(1+\theta^2), \ h = 0 \\ \theta\sigma^2, \ h = \pm 1 \\ 0, \ h = \pm 2, \pm 3, \pm 4, \ldots \end{cases}
\end{aligned}$$

Then

$$\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} 1, \ h = 0 \\ \dfrac{\theta}{1+\theta^2}, \ h = \pm 1 \\ 0, \ h = \pm 2, \pm 3, \pm 4, \ldots \end{cases}
\end{aligned}$$

✎ So, for MA(1) process, the autocorrelation function vanishes after lag 1. This is an identifier for MA(1) process.

1.2 MA(q) Process

$MA(q)$ process as in (1) can be written as $X_t = \sum\limits_{i=0}^{q} \beta_i Z_{t-i}$.

Then

$$
\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{j=0}^{q} \beta_j Z_{t+h-j}\right) \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{s=-h}^{q-h} \beta_{s+h} Z_{t-s}\right) \quad [s = j-h] \\
&= cov\left(\sum_{i=0}^{q} \beta_i Z_{t-i}, \sum_{i=-h}^{q-h} \beta_{i+h} Z_{t-i}\right) \quad [\text{runner } s \text{ is just a dummy variable}] \\
&= \sum_{i=0}^{q-h} \beta_i \beta_{i+h} \cdot cov(Z_{t-i}, Z_{t-i}) \\
&= \sigma^2 \sum_{i=0}^{q-h} \beta_i \beta_{i+h} \\
&= \begin{cases} \sigma^2 \sum\limits_{i=0}^{q-h} \beta_i \beta_{i+h}, \ h = 0, 1, \ldots, q \\ 0, \ h > q \\ \gamma(-h), \ h < 0 \end{cases}
\end{aligned}
$$

Consequently

$$
\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} \dfrac{\sum\limits_{i=0}^{q-h} \beta_i \beta_{i+h}}{\sum\limits_{i=0}^{q} \beta_i^2}, \ h = 0, 1, \ldots, q \\ 0, \ h > q \\ \rho(-h), \ h < 0 \end{cases}
\end{aligned}
$$

✎ It can be seen from the above expression that ACF becomes 0 at lag $k > q$. It shows that ACF of MA process cuts off at lag q which is a special characteristics of MA(q) process.

@ 03.08.2025, Sunday (Yesss !! On a goddaamn Sunday !! 🥲)

Random Cosine Curve : Consider a stochastic process $\{X_t\}$ given by

$$X_t = \cos\left(2\pi\left(\frac{t}{12} + \Phi\right)\right). \ t = 0, \pm 1, \pm 2, \ldots$$

where $\Phi \sim U(0, 1)$. Find $E(X_t)$ and $Var(X_t)$.

2 Autoregressive Process

2.1 AR(1) Process

The first-order autoregressive process $\{X_t\}$ is defined as

$$X_t = \alpha X_{t-1} + Z_t \quad t = 0, \pm 1, \pm 2, \ldots \tag{4}$$

where Z_t's are White Noise with mean 0 and variance σ^2; $\alpha \in \mathbb{R}$, constant.

Notice (4) reduces to a random walk model for $\alpha = 1$.

We may write (4) as $X_t = \alpha B X_t + Z_t$ where B is a backshift operator with $BX_t = X_{t-1}$.

Then $(1 - \alpha B)X_t = Z_t$ so that

$$\begin{aligned}
X_t &= (1-\alpha B)^{-1} Z_t \\
&= (1 + \alpha B + \alpha^2 B^2 + \alpha^3 B^3 + \ldots) Z_t \\
&= Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \alpha^3 Z_{t-3} + \ldots \\
&= \sum_{j=0}^{\infty} \alpha^j Z_{t-j} \text{ provided the sum exists } i.e. \ |\alpha| < 1
\end{aligned} \tag{5}$$

Using (5), $E(X_t) = 0$, $t = 0, \pm 1, \pm 2, \ldots$

And $Var(X_t) = \sigma^2 \sum_{j=0}^{\infty} \alpha^{2j} = \dfrac{\sigma^2}{1-\alpha^2}$, provided $|\alpha| < 1$.

Then

$$\begin{aligned}
\gamma(h) &= cov(X_t, X_{t+h}) \\
&= cov\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}, \sum_{j=0}^{\infty} \alpha^j Z_{t+h-j} \right) \\
&= cov\left(\sum_{j=0}^{\infty} \alpha^j Z_{t-j}, \sum_{k=-h}^{\infty} \alpha^{k+h} Z_{t-k} \right), \text{ taking } k = j - h \\
&= \sigma^2 \sum_{j=0}^{\infty} \alpha^j \alpha^{j+h} \\
&= \sigma^2 \alpha^h \sum_{j=0}^{\infty} \alpha^{2j} \\
&= \frac{\sigma^2 \alpha^h}{1-\alpha^2} \text{ provided } |\alpha| < 1, h \geq 0
\end{aligned}$$

Note that $\gamma(h)$ does not depend on t. So AR(1) model is <u>weak stationary</u> only if $|\alpha| < 1$.

Consequently

$$
\begin{aligned}
\rho(h) &= \frac{\gamma(h)}{\gamma(0)} \\
&= \begin{cases} 1, \ h = 0 \\ \alpha^h, \ h = 1, 2, \ldots \\ \rho(-h), \ h = -1, -2, \ldots \end{cases} \\
&= \alpha^{|h|}, \ h = \pm 1, \pm 2, \ldots
\end{aligned}
$$

Multiplying X_{t-k} on both sides of (4) and taking expectation we get

$$
\begin{aligned}
E(X_t \cdot X_{t-h}) &= \alpha E(X_{t-1} \cdot X_{t-h}) + E(Z_t \cdot X_{t-h}), \\
\Rightarrow \gamma(h) &= \alpha\, \gamma(h-1)
\end{aligned} \tag{6}
$$

Remember Z_t, X_{t-h} are independent and $E(X_t) = 0 \ \forall t$. Also, using (5), $cov(Z_t, X_{t-h}) = 0$.

Following (6) we have

$$
\begin{aligned}
\gamma(1) &= \alpha\, \gamma(0) \\
\gamma(2) &= \alpha^2\, \gamma(0) \\
&\vdots \\
\gamma(h) &= \alpha^h \gamma(0)
\end{aligned} \tag{7}
$$

On dividing both sides of (7) by $\gamma(0)$, we get

$$\rho(h) = \alpha^{|h|}, \ h = 0, \pm 1, \pm 2, \ldots$$

Find ACVF & ACF and draw the ACF plot for the following $AR(1)$ processes.

(i) $X_t = 0.35\, X_{t-1} + Z_t$

(ii) $X_t = 0.85\, X_{t-1} + Z_t$

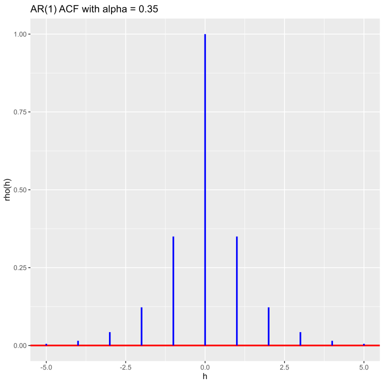
(iii) $X_t = -0.35\, X_{t-1} + Z_t$

The ACVFs and ACFs are as follows.

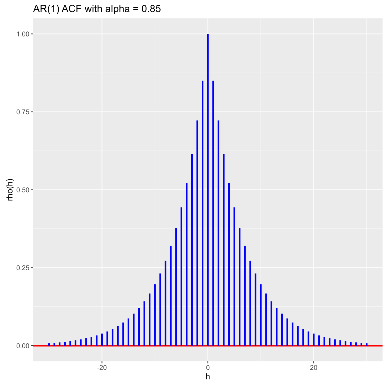
(i) $\gamma(h) = 0.8775 \cdot \sigma^2\, 0.35^{|h|}$, $\rho(h) = 0.35^{|h|} \ \forall h = 0, \pm 1, \pm 2, \ldots$

(ii) $\gamma(h) = 0.2775 \cdot \sigma^2\, 0.85^{|h|}$, $\rho(h) = 0.85^{|h|} \ \forall h = 0, \pm 1, \pm 2, \ldots$

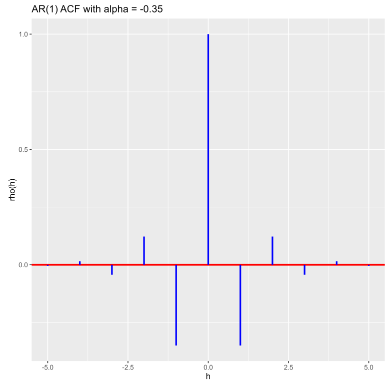
(iii) $\gamma(h) = 0.8775 \cdot \sigma^2\, (-0.35)^{|h|}$, $\rho(h) = (-0.35)^{|h|} \ \forall h = 0, \pm 1, \pm 2, \ldots$



(i)



(ii)



(iii)

2.2 AR(2) Process