MSMS 206 : Practical 04

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➜ Question

Consider a lifetime variable that follows a **Weibull distribution** with shape parameter α and scale parameter λ. The probability density function of this distribution is given by:

$$f(t;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{t}{\lambda}\right)^{\alpha-1} e^{-(t/\lambda)^{\alpha}}, \quad t>0.$$

The objective is to evaluate the performance of **maximum likelihood estimation (MLE)** for different sample sizes. First, generate random samples of sizes $n = 60, 80, 100, 120$ and 140 from this Weibull distribution. For each sample, estimate the parameters α and λ using MLE.

Next, compute the **standard errors** of the ML estimates and evaluate their accuracy by estimating the **bias** and the **mean squared error (MSE)**. Also evaluate the distribution of MLE of each parameter. Discuss the asymptotic properties of MLE, particularly its consistency and efficiency, based on the observed results.

➜ Build-up for obtaining MLE

For a sample of size n, the likelihood function is given by

$$\begin{aligned} L(\alpha,\lambda) &= \prod_{i=1}^{n} \frac{\alpha}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\alpha-1} e^{-(x_i/\lambda)^{\alpha}} \\ &= \left(\frac{\alpha}{\lambda}\right)^{n}\left(\prod_{i=1}^{n}\frac{x_i}{\lambda}\right)^{\alpha-1} \exp\left\{-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha}\right\}. \end{aligned}$$

The log-likelihood function is given by

$$l(\alpha,\lambda) = n\log\alpha - n\log\lambda + (\alpha-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha}.$$

Now,

$$\frac{\partial}{\partial\alpha} l(\alpha,\lambda) = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right) = u(\alpha,\lambda), \text{ say and}$$

$$\frac{\partial}{\partial\lambda}l(\alpha,\lambda) = -\frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{\lambda}{x_i}\cdot\left(-\frac{x_i}{\lambda^2}\right) + \sum_{i=1}^{n}\frac{\alpha\cdot x_i^{\alpha}}{\lambda^{\alpha+1}}$$

$$= -\frac{n}{\lambda} - (\alpha-1)\frac{n}{\lambda} + \sum_{i=1}^{n}\frac{\alpha\cdot x_i^{\alpha}}{\lambda^{\alpha+1}}$$

$$= -\frac{n\alpha}{\lambda} + \sum_{i=1}^{n}\frac{\alpha\cdot x_i^{\alpha}}{\lambda^{\alpha+1}} = v(\alpha,\lambda), \text{ say.}$$

Solutions of the likelihood equations $u(\alpha,\lambda) = 0$ and $v(\alpha,\lambda) = 0$ can not be obtained in closed form. So we opt for numerical approach to get approximate solutions.

By Newton-Raphson method for system of non-linear equations, approximate solution $(\alpha_{k+1}, \lambda_{k+1})$ after $(k+1)$ iterations is given by

$$\begin{bmatrix}\alpha_{k+1}\\ \lambda_{k+1}\end{bmatrix} = \begin{bmatrix}\alpha_{k}\\ \lambda_{k}\end{bmatrix} - \begin{bmatrix}u_\alpha(\alpha_k,\lambda_k) & u_\lambda(\alpha_k,\lambda_k)\\ v_\alpha(\alpha_k,\lambda_k) & v_\lambda(\alpha_k,\lambda_k)\end{bmatrix}^{-1}\cdot\begin{bmatrix}u(\alpha_k,\lambda_k)\\ v(\alpha_k,\lambda_k)\end{bmatrix}, \quad k = 0,1,2,\ldots$$

with notations having their usual meanings.

Now,

$$u_\alpha(\alpha,\lambda) = \frac{\partial}{\partial\alpha}u(\alpha,\lambda) = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha}\left[\log\left(\frac{x_i}{\lambda}\right)\right]^2;$$

$$u_\lambda(\alpha,\lambda) = \frac{\partial}{\partial\lambda}u(\alpha,\lambda) = \sum_{i=1}^{n}\frac{\lambda}{x_i}\cdot\left(-\frac{x_i}{\lambda^2}\right) - \sum_{i=1}^{n}\left[\frac{(-\alpha)x_i^{\alpha}}{\lambda^{\alpha+1}}\log\left(\frac{x_i}{\lambda}\right) + \left(\frac{x_i}{\lambda}\right)^{\alpha}\frac{\lambda}{x_i}\cdot\left(-\frac{x_i}{\lambda^2}\right)\right]$$

$$= -\frac{n}{\lambda} + \sum_{i=1}^{n}\left[\frac{x_i^{\alpha}}{\lambda^{\alpha+1}}\left(\alpha\log\left(\frac{x_i}{\lambda}\right)+1\right)\right].$$

Also,

$$v_\alpha(\alpha,\lambda) = \frac{\partial}{\partial\alpha}v(\alpha,\lambda) = -\frac{n}{\lambda} + \sum_{i=1}^{n}\left[\frac{x_i^{\alpha}}{\lambda^{\alpha+1}} + \frac{\alpha^2}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right)\right];$$

$$v_\lambda(\alpha,\lambda) = \frac{\partial}{\partial\lambda}v(\alpha,\lambda) = \frac{n\alpha}{\lambda^2} - \sum_{i=1}^{n}\frac{\alpha(\alpha+1)x_i^{\alpha}}{\lambda^{\alpha+2}}.$$

➔ Conclusion

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