MSMS - 106

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Practical 07

✎ **Question :** Consider a random sample of size 20 from $Cauchy(\theta, 1)$ population.

5.637941, 4.942002, 4.861254, 3.469588, 5.009333, 7.702125, 5.473228, 3.613141, 3.444167, 4.509174, 5.171716, 3.680117, 2.365371, -4.959420, 5.030187, 4.815630, 4.564628, 4.224900, 4.426912, 4.471680

Obtain maximum likelihood estimate of θ by

(a) Newton-Raphson method,

(b) Fisher Scoring method.

MLE by Newton-Raphson method

The P.D.F. of a $Cauchy(\theta, 1)$ variate is $f(x) = \frac{1}{\pi} \cdot \frac{1}{1 + (x - \theta)^2}$; $x \in \mathbb{R}$; $\theta \in \mathbb{R}$.

$X_1, X_2, \ldots, X_{20} \overset{\text{iid}}{\sim} Cauchy(\theta, 1)$.

```
our_sample <- c(5.637941, 4.942002, 4.861254, 3.469588, 5.009333, 7.702125, 5.473228, 3.613141,
                3.444167, 4.509174, 5.171716, 3.680117, 2.365371, -4.959420, 5.030187, 4.815630,
                4.564628, 4.224900, 4.426912, 4.471680)
```

The likelihood function for θ is $L(\theta) = \prod_{i=1}^{20} \frac{1}{\pi} \cdot \frac{1}{1 + (x_i - \theta)^2}$; $x_i \in \mathbb{R}\ \forall i = 1(1)20$.

The log-likelihood function for θ is $l(\theta) = ln(L(\theta)) = -20\, ln(\pi) - \sum_{i=1}^{20} ln(1 + (x_i - \theta)^2)$.

Now, $l'(\theta) = \frac{\partial}{\partial \theta} l(\theta) = \sum_{i=1}^{20} \frac{2(x_i - \theta)}{1 + (x_i - \theta)^2} = \sum_{i=1}^{20} u(x_i, \theta)$ (say).

We shall solve the equation $l'(\theta) = 0$ for θ by Newton-Raphson method.

```
u <- function(x, theta) (x - theta)/(1 + (x - theta)^2)
```

```
l_dash <- function(theta){
  temp <- 0
  for (i in 1:length(our_sample)) {
    temp <- temp + u(our_sample[i], theta)
  }
  return(temp)
}
```

We start with an initial approximation θ_0 and successively calculate

$$\theta_{n+1} = \theta_n - \frac{l'(\theta_n)}{l''(\theta_n)} \text{ for } n = 0, 1, 2, \ldots.$$

Upon reaching desired accuracy or after doing a certain number of iterations, we report final θ_n as our approximate solution to $l'(\theta) = 0$.

Now, $l''(\theta) = \sum_{i=1}^{20} u'(x_i, \theta)$ where $u'(x_i, \theta) = \frac{\partial}{\partial\theta} u(x_i, \theta)$.

```
library(Deriv)
u_dash <- Deriv(u, "theta")
```

```
l_dash_dash <- function(theta){
  temp <- 0
  for (i in 1:length(our_sample)) {
    temp <- temp + u_dash(our_sample[i], theta)
  }
  return(temp)
}
```

```
newton_raphson <- function(func, func_dash, theta_0, iterations){

  i <- 1
  theta <- c(theta_0)

  while(i <= iterations){
    theta[i+1] <- theta[i] - func(theta[i]) / func_dash(theta[i])

    if(abs(func(theta[length(theta)])) < 0.001) break

    i <- i + 1
  }

  return(theta[length(theta)])
}
```

```
theta_hat_1 <- newton_raphson(func = l_dash, func_dash = l_dash_dash,
                              theta_0 = 2, iterations = 100)
theta_hat_1

## [1] 4.596972
```

We shall also check whether $l'''(\hat{\theta}) < 0$ or not.

```
u_dash_dash <- Deriv(u_dash, "theta")

l_dash_dash_dash <- function(theta){
```

```r
  temp <- 0
  for (i in 1:length(our_sample)) {
    temp <- temp + u_dash_dash(our_sample[i], theta)
  }
  return(temp)
}

l_dash_dash_dash(theta_hat_1) < 0

## [1] TRUE
```

So, by Newton-Raphson method $\hat{\theta}_{\text{MLE}} = 4.596972$.

➔ MLE by Fisher Scoring method

This is similar to Newton-Raphson method, except that we replace $l''(\theta)$ with $E_X[l''(\theta)]$.

Recall that $l(\theta)$ can also be expressed as $l(\theta; \mathbf{x})$.

Also, Fisher's Information about parameter θ in a sample of size n is $I^*(\theta) = -E_X[l''(\theta)]$.

So here, we start with an initial approximation θ_0 and successively calculate

$$\theta_{n+1} = \theta_n - \frac{l'(\theta_n)}{E_X[l''(\theta)]} = \theta_n + \frac{l'(\theta_n)}{I^*(\theta)} \text{ for } n = 0, 1, 2, \ldots.$$

Upon reaching desired accuracy or after doing a certain number of iterations, we report final θ_n as our approximate solution to $l'(\theta) = 0$.

```r
fisher_scoring <- function(func, fisher_info , theta_0, iterations){

  i <- 1
  theta <- c(theta_0)

  while(i <= iterations){
    theta[i+1] <- theta[i] + func(theta[i]) / fisher_info(theta[i])

    if(abs(func(theta[length(theta)])) < 0.001) break

    i <- i + 1
  }

  return(theta[length(theta)])
}
```

For a sample of size n from $Cauchy(\theta, 1)$ population, $I^*(\theta) = \frac{n}{2}$.

```
cauchy_fisher_info <- function(theta) length(our_sample) / 2
```

```
theta_hat_2 <- fisher_scoring(func = l_dash, fisher_info = cauchy_fisher_info,
                              theta_0 = 2, iterations = 100)
theta_hat_2

## [1] 4.596825
```

We shall also check whether $l'''(\hat{\theta}) < 0$ or not.

```
l_dash_dash_dash(theta_hat_2) < 0

## [1] TRUE
```

So, by Fisher Scoring method $\hat{\theta}_{\text{MLE}} = 4.596825$.