MSMS 206 : Practical 06 - Supplementary Derivations

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$$f(t;\alpha,\lambda) = \frac{\alpha}{\lambda}\left(\frac{t}{\lambda}\right)^{\alpha-1} e^{-(t/\lambda)^\alpha}, \quad t > 0.$$

For a sample of size n, the likelihood function is given by

$$\begin{aligned} L(\alpha,\lambda) &= \prod_{i=1}^{n} \frac{\alpha}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\alpha-1} e^{-(x_i/\lambda)^\alpha} \\ &= \left(\frac{\alpha}{\lambda}\right)^n \left(\prod_{i=1}^{n}\frac{x_i}{\lambda}\right)^{\alpha-1} \exp\left\{-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^\alpha\right\}. \end{aligned}$$

The log-likelihood function is given by

$$l(\alpha,\lambda) = n\log\alpha - n\log\lambda + (\alpha-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^\alpha.$$

Now,

$$\frac{\partial}{\partial\alpha} l(\alpha,\lambda) = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^\alpha \log\left(\frac{x_i}{\lambda}\right) = u(\alpha,\lambda), \text{ say}$$

and

$$\begin{aligned} \frac{\partial}{\partial\lambda} l(\alpha,\lambda) &= -\frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{\lambda}{x_i}\cdot\left(-\frac{x_i}{\lambda^2}\right) + \sum_{i=1}^{n}\frac{\alpha\cdot x_i^\alpha}{\lambda^{\alpha+1}} \\ &= -\frac{n}{\lambda} - (\alpha-1)\frac{n}{\lambda} + \sum_{i=1}^{n}\frac{\alpha\cdot x_i^\alpha}{\lambda^{\alpha+1}} \\ &= -\frac{n\alpha}{\lambda} + \sum_{i=1}^{n}\frac{\alpha\cdot x_i^\alpha}{\lambda^{\alpha+1}} = v(\alpha,\lambda), \text{ say.} \end{aligned}$$

Now,

$$u_\alpha(\alpha,\lambda) = \frac{\partial}{\partial\alpha}u(\alpha,\lambda) = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{\alpha}\left[\log\left(\frac{x_i}{\lambda}\right)\right]^2;$$

$$\begin{aligned} u_\lambda(\alpha,\lambda) = \frac{\partial}{\partial\lambda}u(\alpha,\lambda) &= \sum_{i=1}^{n}\frac{\lambda}{x_i}\cdot\left(-\frac{x_i}{\lambda^2}\right) - \sum_{i=1}^{n}\left[\frac{(-\alpha)x_i^{\alpha}}{\lambda^{\alpha+1}}\log\left(\frac{x_i}{\lambda}\right) + \left(\frac{x_i}{\lambda}\right)^{\alpha}\frac{\lambda}{x_i}\cdot\left(-\frac{x_i}{\lambda^2}\right)\right] \\ &= -\frac{n}{\lambda} + \sum_{i=1}^{n}\left[\frac{x_i^{\alpha}}{\lambda^{\alpha+1}}\left(\alpha\log\left(\frac{x_i}{\lambda}\right)+1\right)\right]. \end{aligned}$$

Also,

$$\begin{aligned} v_\alpha(\alpha,\lambda) = \frac{\partial}{\partial\alpha}v(\alpha,\lambda) &= -\frac{n}{\lambda} + \sum_{i=1}^{n}\left[\frac{x_i^{\alpha}}{\lambda^{\alpha+1}} + \frac{\alpha}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\alpha}\log\left(\frac{x_i}{\lambda}\right)\right] \\ &= -\frac{n}{\lambda} + \sum_{i=1}^{n}\left[\frac{x_i^{\alpha}}{\lambda^{\alpha+1}}\left(\alpha\log\left(\frac{x_i}{\lambda}\right)+1\right)\right]; \end{aligned}$$

$$v_\lambda(\alpha,\lambda) = \frac{\partial}{\partial\lambda}v(\alpha,\lambda) = \frac{n\alpha}{\lambda^2} - \sum_{i=1}^{n}\frac{\alpha(\alpha+1)x_i^{\alpha}}{\lambda^{\alpha+2}}.$$