MSMS 308 : Practical 06

Ananda Biswas

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Question

Obtain a sufficiently large sample from an exponential distribution under a Type II censoring scheme. After generating the sample, calculate the maximum likelihood (ML) estimate of the distribution parameter. Also, evaluate the performance of the estimate by computing the bias, variance, and mean squared error (MSE) for different sample sizes.

Theory

Let $T_1, \ldots, T_n \overset{iid}{\sim} \text{Exp}(\lambda)$ with density $f(t)$ and survival $S(t)$. Under Type II censoring we observe the first r failure times $0 < t_{(1)} \leq \cdots \leq t_{(r)}$; the remaining $n-r$ units are right-censored at the common time $t_{(r)}$.

The likelihood function is

$$
\begin{aligned}
L(\lambda) &= \frac{n!}{(n-r)!} \cdot \left[\prod_{i=1}^{r} f\big(t_{(i)}\big)\right] \cdot \Big[S\big(t_{(r)}\big)\Big]^{n-r}. \\
&= \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left(\lambda e^{-\lambda t_{(i)}}\right) \cdot \left(e^{-\lambda t_{(r)}}\right)^{n-r} \\
&= \frac{n!}{(n-r)!} \lambda^r \exp\left(-\lambda \sum_{i=1}^{r} t_{(i)}\right) \exp\big(-\lambda (n-r)\, t_{(r)}\big) \\
&= \frac{n!}{(n-r)!} \lambda^r \exp\left\{-\lambda \left[\sum_{i=1}^{r} t_{(i)} + (n-r)\, t_{(r)}\right]\right\}
\end{aligned}
$$

The log-likelihood function is

$$
\ell(\lambda) = \log\left(\frac{n!}{(n-r)!}\right) + r \log \lambda - \lambda \left[\sum_{i=1}^{r} t_{(i)} + (n-r)\, t_{(r)}\right].
$$

Now we differentiate $\ell(\lambda)$ w.r.t. λ and set the derivative to zero.

$$
\frac{\partial \ell}{\partial \lambda} = \frac{r}{\lambda} - \left[\sum_{i=1}^{r} t_{(i)} + (n-r)\, t_{(r)}\right] = 0
$$

$$
\Rightarrow \frac{r}{\lambda} = \sum_{i=1}^{r} t_{(i)} + (n-r)\, t_{(r)}
$$

$$\therefore \widehat{\lambda}_{MLE} = \frac{r}{\sum_{i=1}^{r} t_{(i)} + (n-r)\, t_{(r)}}$$

R Program

Let $n = 100$ and $r = 75$.

```
n <- 100; r <- 75
```

```
x <- sort(rexp(n, rate = 2))
t_values <- x[1:r]
```

```
lambda_hat <- r / (sum(t_values) + (n - r) * max(t_values)); lambda_hat

## [1] 2.247462
```

$\therefore \widehat{\lambda}_{MLE} = 2.2474623.$

Now we shall compare the estimates for different sample sizes by evaluating their bias, variance and MSE.

```
bias = variance = MSE = c()

n <- c(50, 100, 150, 200, 250, 300)

# suppose we censor after having first 75% observations
r <- round(n * 0.75)
```

```
true_lambda <- 1
```

```
for (i in 1:length(r)) {

  lambda_hat <- c()

  for (j in 1:1000) {
    x <- sort(rexp(n[i], rate = true_lambda))

    s <- x[1:r[i]]

    lambda_hat[j] <- r[i] / (sum(s) + (n[i] - r[i]) * max(s))
  }

  bias[i] <- mean(lambda_hat) - true_lambda

  variance[i] <- mean( (lambda_hat - mean(lambda_hat))^2 )
}

MSE <- bias^2 + variance
```

```r
df <- data.frame(Sample.Size = n,
                 Observed = r, Censored = n - r,
                 Bias = bias, Variance = variance, MSE = MSE)
```

```r
library(stargazer)
```

Bias, variance and MSE of the estimates for different sample sizes are as follows :

```r
stargazer(df, summary = FALSE, rownames = FALSE, digits = 5)
```

Table 1:

Sample.Size	Observed	Censored	Bias	Variance	MSE
50	38	12	0.03321	0.02758	0.02869
100	75	25	0.01595	0.01363	0.01389
150	112	38	0.00654	0.00936	0.00940
200	150	50	0.00144	0.00658	0.00659
250	188	62	0.00470	0.00527	0.00530
300	225	75	0.00800	0.00508	0.00514

✎ With increasing sample size, there is a steady decrease in bias, variance and mean squared error of the estimates.