MSMS 408 : Practical 02

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Question

Use Monte Carlo Integration to approximate the following integrals

(i) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\,dx$, (ii) $\displaystyle\int_0^{\infty}\int_0^{x} e^{-(x+y)}\,dy\,dx$

and check how the estimates converge to the true value as the sample size increases.

Theory

Suppose we have to compute

$$I = \int_{-\infty}^{\infty} f(x)dx \;\text{ or }\; I = \int_{a}^{\infty} f(x)dx \tag{1}$$

We rewrite

$$I = \int f(x)dx = \int \frac{f(x)}{p(x)}p(x)dx = E_X\left[\frac{f(x)}{p(x)}\right] \tag{2}$$

where $p(x)$ is a **suitable** probability density and $X \sim p$.

To approximate I, we generate random numbers $X_1, X_2, \ldots, X_n$ from $p(x)$ and calculate

$$I \approx \frac{1}{n}\sum_{i=1}^{n}\frac{f(X_i)}{p(X_i)}. \tag{3}$$

By law of large numbers, $\frac{1}{n}\sum_{i=1}^{n}\frac{f(X_i)}{p(X_i)} \to E_X\left[\frac{f(x)}{p(x)}\right]$ as $n \to \infty$.

R Program

```r
set.seed(22)
```

1 $\int\limits_{-\infty}^{\infty} e^{-x^2} dx$

Here $f(x) = e^{-x^2}$. We take $p(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$. Then $\frac{f(x)}{p(x)} = \sqrt{2\pi} \cdot \exp\left\{-\frac{x^2}{2}\right\}$.

```r
MCI.1 <- function(sample.size){

  x <- rnorm(sample.size)

  return(mean(exp(-x^2) / dnorm(x)))
}
```

```r
sample_sizes <- c(100, 500, 700, 1000, 3000, 5000, 7000, 9000, 10000)
```

```r
results.1 <- c()
```

```r
for (i in 1:length(sample_sizes)) {
  results.1[i] <- MCI.1(sample.size = sample_sizes[i])
}
```

```r
df1 <- data.frame(sample.size = sample_sizes,
                  approximate.integral.value = results.1)
```

```r
stargazer(df1, summary = FALSE, rownames = FALSE, label = "Table 1")
```

Table 1:

sample.size	approximate.integral.value
100	1.805
500	1.768
700	1.735
1,000	1.802
3,000	1.764
5,000	1.758
7,000	1.772
9,000	1.767
10,000	1.776

Note that, $\int\limits_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \approx 1.7725.$

2 $\int_0^{\infty}\int_0^{x} e^{-(x+y)}\,dy\,dx$

Let $X, Y \overset{iid}{\sim} \text{Exp}(1)$. Then $P(Y \leq X) = \int_0^{\infty}\int_0^{x} e^{-(x+y)}\,dy\,dx = 0.5$ as X and Y are independent.

```
MCI.2 <- function(sample.size){

  x <- rexp(sample.size); y <- rexp(sample.size)

  return(mean(y <= x))
}
```

```
results.2 <- c()
```

```
for (i in 1:length(sample_sizes)) {
  results.2[i] <- MCI.2(sample.size = sample_sizes[i])
}
```

```
df2 <- data.frame(sample.size = sample_sizes,
                  approximate.integral.value = results.2)
```

```
stargazer(df2, summary = FALSE, rownames = FALSE, label = "Table 2")
```

Table 2:

sample.size	approximate.integral.value
100	0.440
500	0.550
700	0.467
1,000	0.494
3,000	0.497
5,000	0.504
7,000	0.502
9,000	0.507
10,000	0.498

Conclusion

As sample size increases, the approximates gradually converge to the true integral value.