MSMS 308 : Practical 07

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Exam Roll No. : 24419STC053

November 11, 2025

Question

Obtain maximum likelihood estimates of Weibull parameters for varying sample sizes and censoring proportions $p_c \in \{0.2, 0.3, 0.4\}$ under

(1) complete data (no censoring scheme),

(2) Type I censoring scheme,

(3) Type II censoring scheme.

and compare the estimates.

Theory and R Program

The probability density function of a lifetime T having Weibull distribution with shape $\alpha > 0$ and scale $\lambda > 0$ is given by

$$f(t) = \frac{\alpha}{\lambda}\left(\frac{t}{\lambda}\right)^{\alpha-1} \exp\left[-\left(\frac{t}{\lambda}\right)^{\alpha}\right], \qquad t \geq 0,$$

The Cumulative Distribution Function is given by

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\lambda}\right)^{\alpha}\right], \qquad t \geq 0,$$

The Survival Function is given by

$$S(t) = \exp\left[-\left(\frac{t}{\lambda}\right)^{\alpha}\right], \qquad t \geq 0.$$

1 MLE of α and λ with no censoring

Let $T_1, \ldots, T_n \overset{\text{iid}}{\sim} \text{Weibull}(\alpha, \lambda)$ with density $f(t)$ and survival $S(t)$.

The likelihood function is given by

$$\begin{aligned} L(\alpha, \lambda) &= \prod_{i=1}^{n} \frac{\alpha}{\lambda}\left(\frac{t_i}{\lambda}\right)^{\alpha-1} e^{-(t_i/\lambda)^{\alpha}} \\ &= \left(\frac{\alpha}{\lambda}\right)^{n} \left(\prod_{i=1}^{n} \frac{t_i}{\lambda}\right)^{\alpha-1} \exp\left\{-\sum_{i=1}^{n}\left(\frac{t_i}{\lambda}\right)^{\alpha}\right\}. \end{aligned}$$

The log-likelihood function is given by

$$l(\alpha,\lambda) = n\log\alpha - n\log\lambda + (\alpha-1)\sum_{i=1}^{n}\log\left(\frac{t_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{t_i}{\lambda}\right)^{\alpha}.$$

Now,

$$\frac{\partial}{\partial\alpha}l(\alpha,\lambda) = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(\frac{t_i}{\lambda}\right) - \sum_{i=1}^{n}\left(\frac{t_i}{\lambda}\right)^{\alpha}\log\left(\frac{t_i}{\lambda}\right) = u(\alpha,\lambda), \text{ say.} \tag{1}$$

and

$$\begin{aligned}\frac{\partial}{\partial\lambda}l(\alpha,\lambda) &= -\frac{n}{\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{\lambda}{t_i}\cdot\left(-\frac{t_i}{\lambda^2}\right) + \sum_{i=1}^{n}\frac{\alpha\cdot t_i^{\alpha}}{\lambda^{\alpha+1}} \\ &= -\frac{n}{\lambda} - (\alpha-1)\frac{n}{\lambda} + \sum_{i=1}^{n}\frac{\alpha\cdot t_i^{\alpha}}{\lambda^{\alpha+1}} \\ &= -\frac{n\alpha}{\lambda} + \sum_{i=1}^{n}\frac{\alpha\cdot t_i^{\alpha}}{\lambda^{\alpha+1}} = v(\alpha,\lambda), \text{ say.}\end{aligned}$$

Setting $v(\alpha,\lambda) = 0$ we get,

$$\begin{aligned}-\frac{n\alpha}{\lambda} + \sum_{i=1}^{n}\frac{\alpha\cdot t_i^{\alpha}}{\lambda^{\alpha+1}} &= 0 \\ \Rightarrow \frac{n}{\lambda} &= \sum_{i=1}^{n}\frac{t_i^{\alpha}}{\lambda^{\alpha+1}} \\ \Rightarrow \frac{n}{\lambda} &= \frac{1}{\lambda^{\alpha+1}}\sum_{i=1}^{n}t_i^{\alpha} \\ \Rightarrow \lambda^{\alpha} &= \frac{1}{n}\sum_{i=1}^{n}t_i^{\alpha} \\ \therefore \lambda &= \left(\frac{1}{n}\sum_{i=1}^{n}t_i^{\alpha}\right)^{\frac{1}{\alpha}}\end{aligned} \tag{2}$$

Setting $u(\alpha,\lambda) = 0$ does not yield any closed form solution. So for getting the ML estimate of α, we resort to numerical methods (here Newton-Raphson method).

Now,

$$u_{\alpha}(\alpha,\lambda) = \frac{\partial}{\partial\alpha}u(\alpha,\lambda) = -\frac{n}{\alpha^2} - \sum_{i=1}^{n}\left(\frac{t_i}{\lambda}\right)^{\alpha}\left[\log\left(\frac{t_i}{\lambda}\right)\right]^2; \tag{3}$$

At each iteration, with the present value of α we calculate λ by using (2); then we use the obtained value of λ in (1) and (3) to improve the estimate of α by Newton-Raphson method.

```
true_alpha <- 3; true_lambda <- 2
```

```r
sample_size <- c(50, 100, 150, 200, 250)
```

```r
estimate_lambda <- function(s, alpha) mean(s^alpha)^(1/alpha)
```

```r
u <- function(alpha, lambda, s){

  a <- length(s) / alpha

  b <- sum(log(s / lambda))

  c <- sum((s / lambda)^alpha * log(s / lambda))

  return(a + b - c)
}
```

```r
u_alpha <- function(alpha, lambda, s){

  a <- - length(s) / alpha^2

  b <- sum((s / lambda)^alpha * log(s / lambda)^2)

  return(a - b)
}
```

```r
estimate_alpha <- function(s, initial, epsilon = 0.0001, iterations = 100){

  alphas <- c(initial)

  for (i in 2:iterations) {
    l <- estimate_lambda(s, alphas[i-1])

    alphas[i] <- alphas[i-1] - u(alphas[i-1], l, s) / u_alpha(alphas[i-1], l, s)

    if(abs((alphas[i] - alphas[i-1])) < epsilon) break
  }

  return(alphas[length(alphas)])
}
```

```r
alpha_hat = lambda_hat = c()
```

```r
for(i in 1:length(sample_size)){

  x <- rweibull(sample_size[i], shape = true_alpha, scale = true_lambda)

  alpha_hat[i] <- estimate_alpha(x, 1)

  lambda_hat[i] <- estimate_lambda(x, alpha_hat[i])
}
```

Now we shall empirically calculate bias and MSE of the estimates for different sample sizes.

```
bias_and_MSE <- function(size){

  alpha_estimates = lambda_estimates = c()

  for (i in 1:100){

    x <- rweibull(size, shape = true_alpha, scale = true_lambda)

    alpha_estimates[i] <- estimate_alpha(x, 1)

    lambda_estimates[i] <- estimate_lambda(x, alpha_estimates[i])
  }

  alpha_bias <- mean(alpha_estimates) - true_alpha

  lambda_bias <- mean(lambda_estimates) - true_lambda

  alpha_MSE <- mean( (alpha_estimates - true_alpha)^2 )

  lambda_MSE <- mean( (lambda_estimates - true_lambda)^2 )

  return(c(alpha_bias, lambda_bias, alpha_MSE, lambda_MSE))
}
```

```
alpha_bias = lambda_bias = alpha_MSE = lambda_MSE = c()

for (i in 1:length(sample_size)) {

  temp <- bias_and_MSE(sample_size[i])

  alpha_bias[i] <- temp[1]

  lambda_bias[i] <- temp[2]

  alpha_MSE[i] <- temp[3]

  lambda_MSE[i] <- temp[4]
}
```

```
df1 <- data.frame(Sample.Size = sample_size,
                  alpha_hat = alpha_hat,
                  bias.alpha = alpha_bias,
                  MSE.alpha = alpha_MSE,
                  lambda_hat = lambda_hat,
                  bias.lambda = lambda_bias,
                  MSE.lambda = lambda_MSE)
```

```r
stargazer(df1, summary = FALSE, rownames = FALSE)
```

Table 1:

Sample.Size	alpha_hat	bias.alpha	MSE.alpha	lambda_hat	bias.lambda	MSE.lambda
50	2.545	0.069	0.104	1.922	0.006	0.009
100	3.496	0.022	0.043	1.934	-0.009	0.005
150	3.283	0.025	0.032	1.932	0.001	0.004
200	3.202	0.037	0.047	2.050	-0.002	0.002
250	2.890	0.0002	0.025	1.979	-0.001	0.002

2. MLE of α and λ with Right Censoring, Type I

Final Result and Conclusion