MSMS 106 : Assignment 09

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➜ Objective

To find Maximum Likelihood Estimate of the parameters of the Exponential Distribution given as follows :

$$f_X(x) = \theta e^{-\theta x}\ I_{(0,\infty)}(x);\ \theta > 0$$

and to compare ML estimates for different sample sizes.

➜ Theory

For a sample of size n, the likelihood function is

$$L(\theta) = \theta^n \cdot exp\left\{-\theta \cdot \sum_{i=1}^{n} x_i\right\}.$$

The log-likelihood function is

$$l(\theta) = n \ln\theta - \theta \cdot \sum_{i=1}^{n} x_i.$$

The derivative of the log-likelihood w.r.t. θ is

$$\frac{d}{d\theta} l(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} x_i.$$

Setting it to 0 and solving for θ we get,

$$\hat{\theta} = \frac{n}{\sum\limits_{i=1}^{n} x_i} = \frac{1}{\bar{x}}. \tag{1}$$

• To compare MLEs from different sample sizes, we compare their MSEs.

For a fixed sample size n, $\text{MSE}(\hat{\theta}_{\text{MLE}}) = \frac{1}{k}\sum\limits_{i=1}^{k}(\hat{\theta}_i - \theta_0)^2$,
where $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \ldots, \hat{\theta}_k$ are MLEs from different samples of fixed size n and θ_0 is the true value of θ.

➜ R Program

The following function takes random sample as input and gives $\hat{\theta}$ as output.

```r
Exponential_MLE <- function(exp_sample) 1 / mean(exp_sample)
```

Now we calculate MLEs for different sample sizes.

```r
n <- c(100, 200, 500, 1000, 5000, 10000, 100000)
```

```r
estimated_theta <- c()
```

```r
for (i in 1:length(n)) {
  our_sample <- rexp(n[i], rate = 2)
  estimated_theta[i] <- Exponential_MLE(our_sample)
}
```

```r
Exponential_MLE_df1 <- data.frame(sample_size = n,
                                  theta_hat = estimated_theta)
```

```r
Exponential_MLE_df1

##   sample_size theta_hat
## 1       1e+02  2.357433
## 2       2e+02  1.929250
## 3       5e+02  1.933304
## 4       1e+03  1.984861
## 5       5e+03  1.991208
## 6       1e+04  2.012820
## 7       1e+05  2.003282
```

✎ As sample size increases, the estimate of parameter seems to converge at 2.

Now we shall compare the MSEs.

```r
MSE_theta_hat <- c()
```

```r
for(j in 1:length(n)){

  theta_hats <- c()

  for (i in 1:100) {
    a_sample <- rexp(n[j], rate = 2)
    theta_hats[i] <- Exponential_MLE(a_sample)
  }

  MSE_theta_hat[j] <- mean( (theta_hats - 2)^2 )
}
```

```r
Exponential_MLE_df2 <- data.frame(sample_size = n,
                                  MSE_theta_hat = MSE_theta_hat)
```

```r
Exponential_MLE_df2
```

```
##   sample_size MSE_theta_hat
## 1       1e+02  4.666483e-02
## 2       2e+02  2.078637e-02
## 3       5e+02  7.644937e-03
## 4       1e+03  4.285273e-03
## 5       5e+03  8.452473e-04
## 6       1e+04  3.824530e-04
## 7       1e+05  4.533569e-05
```

➔ Conclusion

✎ As sample size increases, MSE of the parameter decreases monotonically. This implies MLE gives better estimate as sample size increases. For larger and larger samples, the MLE will smoothly converge to the true value of the parameter.