MSMS 106

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Practical 07

Gauss-Legendre Integration Methods

Question 1 :

Write an R program to approximate the integral of $f(x)$ over the finite interval $[a, b]$ using Gauss-Legendre one point formula.

Gauss-Legendre one point formula is given by

$$\int_{-1}^{1} f(x)\, dx \approx 2 \cdot f(0).$$

We transform the given interval $[a, b]$ to $[-1, 1]$ by the transformation $x = \frac{b-a}{2} \cdot t + \frac{b+a}{2}$.

```
transformation <- function(t, a, b){
  return( ( (b - a) / 2 ) * t + (b + a) / 2)
}
```

```
one_point <- function(func, a, b){

  integration <- 2 * func(transformation(0, a, b))

  result <- ( (b - a) / 2 ) * integration

  return(result)
}
```

Question 2 :

Write an R program to approximate the integral of $f(x)$ over the finite interval $[a, b]$ using Gauss-Legendre two point formula.

Gauss-Legendre two point formula is given by

$$\int_{-1}^{1} f(x)\, dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

We transform the given interval $[a, b]$ to $[-1, 1]$ by the transformation $x = \frac{b-a}{2} \cdot t + \frac{b+a}{2}$.

```
two_point <- function(func, a, b){

  integration <- func(transformation(-(1 / sqrt(3)), a, b)) +
                 func(transformation(1 / sqrt(3), a, b))

  result <- ( (b - a) / 2 ) * integration

  return(result)
}
```

✎ Question 3 :

Write an R program to approximate the integral of $f(x)$ over the finite interval $[a, b]$ using Gauss-Legendre three point formula.

➜ Gauss-Legendre three point formula is given by

$$\int_{-1}^{1} f(x)\, dx \approx \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right].$$

We transform the given interval $[a, b]$ to $[-1, 1]$ by the transformation $x = \dfrac{b-a}{2} \cdot t + \dfrac{b+a}{2}$.

```
three_point <- function(func, a, b){

  temp <- 5 * func(transformation(-sqrt(3/5), a, b)) +
                 8 * func(transformation(0, a, b)) +
                 5 * func(transformation(sqrt(3/5), a, b))

  integration <- temp / 9

  result <- ( (b - a) / 2 ) * integration

  return(result)
}
```

☞ Evaluate the integral $\displaystyle\int_{1}^{2} \frac{2x}{1+x^4}\, dx$ using the Gauss-Legendre 1-point, 2-point and 3-point quadrature rules.

```
f1 <- function(x) 2*x / (1 + x^4)

one_point(f1, 1, 2)

## [1] 0.4948454
```

```
two_point(f1, 1, 2)

## [1] 0.5433755
```

```r
three_point(f1, 1, 2)

## [1] 0.5405911
```

☞ Evaluate the integrals

(i) $I = \int_0^2 \frac{1}{3+4x}\, dx$ (ii) $I = \int_0^2 \frac{1}{x^2+2x+10}\, dx$

(a) by Gauss-Legendre two-point and three-point formulas;

(b) Write I as $I_1 + I_2$ where $I_1 = \int_0^1 f(x)\, dx$ and $I_2 = \int_1^2 f(x)\, dx$. Then evaluate each of the integrals by Gauss-Legendre two-point and three-point formulas.

```r
f2 <- function(x) 1 / (3 + 4*x)

two_point(f2, 0, 2)

## [1] 0.3206107
```

```r
two_point(f2, 0, 1) + two_point(f2, 1, 2)

## [1] 0.3242383
```

```r
three_point(f2, 0, 2)

## [1] 0.3243897
```

```r
three_point(f2, 0, 1) + three_point(f2, 1, 2)

## [1] 0.3247954
```

```r
f3 <- function(x) 1 / (x^2 + 2*x + 10)

two_point(f3, 0, 2)

## [1] 0.1546392
```

```r
two_point(f3, 0, 1) + two_point(f3, 1, 2)

## [1] 0.154554
```

```r
three_point(f3, 0, 2)
```

```
## [1] 0.154548
```

```r
three_point(f3, 0, 1) + three_point(f3, 1, 2)
```

```
## [1] 0.1545492
```

☞ Evaluate the integral

$$I = \int_{2}^{3} \frac{\cos 2x}{1 + \sin x}\, dx$$

by Gauss-Legendre two-point and three-point integration rules.

```r
f4 <- function(x) cos(2*x) / (1 + sin(x))

two_point(f4, 2, 3)
```

```
## [1] 0.2035084
```

```r
three_point(f4, 2, 3)
```

```
## [1] 0.2027139
```

☞ Obtain an approximate value of

$$I = \int_{-1}^{1} \sqrt{1 - x^2}\, \cos x\, dx$$

by Gauss-Legendre three-point formula.

```r
f5 <- function(x) sqrt(1 - x^2) * cos(x)

three_point(f5, -1, 1)
```

```
## [1] 1.391131
```

☞ Evaluate the integral

$$I = \int_{0}^{1} \frac{1}{1 + x}\, dx$$

by using Gauss-Legendre three-point formula.

```r
f6 <- function(x) 1 / (1 + x)

two_point(f6, 0, 1)
```

```
## [1] 0.6923077
```

```r
three_point(f6, 0, 1)
```

```
## [1] 0.6931217
```

☞ Obtain the approximate value of

$$I = \int_{-1}^{1} e^{-x^2} \cos x \; dx$$

by Gauss-Legendre three-point formula.

```r
f7 <- function(x) exp(-x^2) * cos(x)

three_point(f7, -1, 1)
```

```
## [1] 1.324708
```

☞ Evaluate the integral

$$I = \int_{1}^{2} \frac{1}{1+x^3} \; dx$$

by Gauss-Legendre three-point formula.

```r
f8 <- function(x) 1 / (1 + x^3)

three_point(f8, 1, 2)
```

```
## [1] 0.254387
```

☞ Evaluate the integral

$$I = \int_{0}^{2} \frac{x^2+2x+1}{1+(x+1)^4} \; dx$$

by Gauss-Legendre three-point formula.

```r
f9 <- function(x) (x^2 + 2*x + 1) / (1 + (x + 1)^4)

three_point(f9, 0, 2)
```

```
## [1] 0.5364222
```

☞ Apply Gauss-Legendre two-point formula to evaluate $\int_{-1}^{1} \frac{1}{1+x^2} \; dx$.

```r
f10 <- function(x) 1 / (1 + x^2)

two_point(f10, -1, 1)
```

```
## [1] 1.5
```

☞ Use Gauss-Legendre three-point formula to evaluate

(i) $\displaystyle\int_1^2 \frac{1}{x}\,dx$

(ii) $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

(iii) $\displaystyle\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$

(iv) $\displaystyle\int_{0.2}^{1.5} e^{-x^2}\,dx$

```
f11 <- function(x) 1 / x

three_point(f11, 1, 2)

## [1] 0.6931217
```

```
three_point(f10, 0, 1)

## [1] 0.785267
```

```
f12 <- function(x) 1 / sqrt(1 + x^4)

three_point(f12, 0, 1)

## [1] 0.9271835
```

```
f13 <- function(x) exp(-x^2)

three_point(f13, 0.2, 1.5)

## [1] 0.6586021
```