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$i \succ \bullet : o \bullet q b'' \} f \quad (x) \succ \hat{r} s \$: p \quad f(x+2) = f(x) \succ \neg h' |$

$$D = fz = x + iy \ 2 \ C \ j \ x; y \ 2 \ R; |y| \leq 5 \ g$$

$\succ \% \% B \grave{u} \ddot{p} r s \in \ p V'' q b'' \} \backslash \backslash p | \quad x Y w : p K'' \} \backslash w q V | \quad f(x)$

$x \tilde{N}'' \ae \Re f : 2 \% D \acute{o} p$

$$f(x) = \sum_{k=1}^{\infty} a_k e^{ikx}; \quad a_k = \frac{1}{2} \int_0^{2\pi} f(x) e^{-ikx} dx$$

U R “q m } \check{Z} < w \ddot{o} M t t Q ‘ }

(i) $\acute{o} \acute{E} \ \emptyset \acute{I} w : \ 0; 2; 2 + i; i \succ \backslash w q p A \ae p p V'' \tilde{O} M \ w \& \grave{I} t \ I h^* s$
 $u \ddot{u} \succ \beta Q'' \backslash q t' " | \acute{U}^{\text{TM}} w T : \quad k \ t \ 0' |$

$$a_k = \frac{e^k}{2} \int_0^{2\pi} f(x + i) e^{-ikx} dx$$

$\succ \hat{O} d \} \ddot{p} h |$

$$a_k = \frac{e^k}{2} \int_0^{2\pi} f(x - i) e^{-ikx} dx$$

$\succ \hat{O} d \}$

(ii) $L = \max \{ |f(z)| \mid z \in D \ g \ q \ b'' \} \acute{U}^{\text{TM}} w T : \quad k \ t \ 0' | a_{-k} \leq L e^{-|k|} \succ \hat{O} d \}$

(iii) $c > 1 \succ : q' |$

$$f(x) = \frac{1}{\cos x - c}$$

$q \ b'' \} \acute{U}^{\text{TM}} w Y w \hat{I} : \quad < \log(c + \sqrt{c^2 - 1}) \ t \ 0' | K'' \quad M > 0 \ U \ O' | b,$
 $ow T : \ k \ t \ 0' \quad a_k \leq M e^{-|k|} \ U R \text{ “} q m \backslash q \succ \hat{O} d \}$

An English Translation:

Applied Mathematics



Let i denote the imaginary unit. Let $f(x)$ be a real analytic function satisfying $f(x+2) = f(x)$ and having an analytic continuation on an open set including

$$D = \{z = x + iy \mid x \in \mathbb{R}; y \in \mathbb{R}; |y| \leq \delta\}$$

where $\delta > 0$ is a constant. Then the Fourier series of $f(x)$ converges to $f(x)$ and

$$f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}; \quad a_k = \frac{1}{2} \int_0^2 f(x) e^{-ikx} dx:$$

Answer the following questions.

- (i) Considering the contour integration along the rectangular path connecting the points $0; 2; 2 + i$ and i in this order on the complex plane, show that for any integer k ,

$$a_k = \frac{e^k}{2} \int_0^2 f(x + i) e^{ikx} dx:$$

Moreover show that

$$a_k = \frac{e^k}{2} \int_0^2 f(x - i) e^{ikx} dx:$$

- (ii) Let $L = \max_{|z| \leq 2} |f(z)|$. Show that for any integer k , $|a_k| \leq L e^{-|k|}$.

- (iii) Let $c > 1$ be a constant and let

$$f(x) = \frac{1}{\cos x - c}:$$

Show that for any positive real number $\epsilon < \log(c + \sqrt{c^2 - 1})$, there is a constant $M > 0$ such that for all integer k , $|a_k| \leq M e^{-|k|\epsilon}$ holds.

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$G \succ : B \cup V \mid \neg B \cup E \quad T'R''om \dot{E} A \dot{A}^2 \neg \dot{a} \tilde{N} q' \mid \pi \neg \quad e_2 E \quad t x \hat{i} : \langle w$
 $O^{\wedge} w(e) \cup \zeta \rangle^{\bullet} o M'' \} : w \ae \ddot{u} B \cup \quad X j \quad V \quad t Q_X' \quad X \quad q V n X \quad w \quad w \neg w B \cup \succ$
 $E(X) \quad q G b \} \neg w \ae \ddot{u} B \cup \quad S j \quad E \quad t 0' o \quad w(S), \quad w(e) \mid w_{\max}(S), \quad \max_{e_2 S} w(e) \quad q$
 $\check{S}'' \} \check{Z} < w \check{\delta} M t t Q' \}$

(i) $(X; F); X \neq V \succ G \quad w \ae \ddot{u} \ae q' \mid G \quad w 7 - \ae t x \ae \quad (X; F) \succ \% \langle w U \quad O b$
 $"q > b" \} \quad a_F = uv_2 E(X) \succ E(X) \quad w \pi p O^{\wedge} 7 - w \neg q b'' \} \setminus w q V \quad G$
 $w 7 - \ae t x \quad (X \mid fu; v \quad g; F \mid fa_F g) \succ \% \langle w U \quad O b'' \setminus q \succ \hat{A} \hat{i} d' \}$

(ii) $7 - \ae \succ \{ \check{S}'' \acute{O} \ae \ddot{U} O \succ G \setminus' \mid f w Y p Q \succ \hat{A} \hat{i} d' \}$

(iii) $(V; T) \succ G \quad w 7 - \ae q b'' \} \setminus w q V \quad G \quad w \acute{U}^{\text{TM}} w \P \neg \ae(V; T) \quad t 0' o \quad w_{\max}(T) \quad 5$
 $w_{\max}(T) \quad U R " q m \setminus q \succ \hat{A} \hat{i} d' \}$

An English Translation:

Graph Theory



Let G be a simple and connected undirected graph with a vertex set V and an edge set E such that each edge $e \in E$ is weighted by a real value $w(e)$. For a subset $X \subseteq V$ of vertices, let $E(X)$ denote the set of edges between X and $V \setminus X$. For a subset $S \subseteq E$ of edges, define $w(S) = \sum_{e \in S} w(e)$ and $w_{\max}(S) = \max_{e \in S} w(e)$. Answer the following questions.

- (i) Let $(X; F)$; $X \subseteq V$ be a subtree of G and assume that one of the minimum spanning trees of G contains the tree $(X; F)$. Let $a_F = uv \in E(X)$ be an edge with the minimum weight among the edges in $E(X)$. Prove that one of the minimum spanning trees of G contains $(X \cup \{u, v\}; F \cup \{a_F\})$.
- (ii) Describe Prim's method for computing a minimum spanning tree and prove its correctness.
- (iii) Let $(V; T)$ be a minimum spanning tree of G . Prove that $w_{\max}(T) \leq w_{\max}(T')$ holds for every spanning tree $(V; T')$ of G .

$$i \in \{1, 2, \dots, n\} \quad \text{and} \quad x_i \geq 0$$



Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{n \times n}$ be given. Consider the problem

$$\begin{aligned} P(x): \quad & \text{Minimize} \quad \sum_{i=1}^n (z^i)^T z^i + y^T y + x^T C x \\ & \text{subject to} \quad y \sum_{i=1}^n x_i z^i = A x - b \end{aligned}$$

where $z^i \in \mathbb{R}^m$, $i = 1, \dots, n$, and $x \in \mathbb{R}^n$. Let $f(x)$ denote the optimal value of $P(x)$. Consider the problem

(i) Find the Karush-Kuhn-Tucker conditions for the problem

(ii) Find the dual problem of the problem

(iii) Find the primal problem of the problem

$$\begin{aligned} P1: \quad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad x \in \mathbb{R}^n \end{aligned}$$

Let $x \in \mathbb{R}^n$ be a feasible point for $P1$. Let $\lambda \in \mathbb{R}^m$ be a Lagrange multiplier. Consider the problem

$$(x)^T x \leq \frac{b^T b}{\min(C)}$$

$$h(x) = \min(C) x^T C x - \{ (x)^T b \}$$

(iv) Find the dual problem of the problem

$$\begin{aligned} P2: \quad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad x \geq x_0 \end{aligned}$$

Let $x \in \mathbb{R}^n$ be a feasible point for $P2$. Let $\lambda \in \mathbb{R}^m$ be a Lagrange multiplier. Consider the problem

An English Translation:

Operations Research



Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and $C \in \mathbb{R}^{n \times n}$. Consider the following nonlinear programming problem with parameter $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$:

$$\begin{aligned} P(x): \quad & \text{Minimize} \quad \sum_{i=1}^n (z^i)^T z^i + y^T y + x^T C x \\ & \text{subject to} \quad y \sum_{i=1}^n x_i z^i = Ax - b; \end{aligned}$$

where the decision variables are $z^i \in \mathbb{R}^m$ ($i = 1, \dots, n$), with T denoting transposition. Moreover, denote by $f(x)$ the optimal value of problem $P(x)$, assuming that it is well-defined for all x .

Answer the following questions.

- (i) Write out the Karush-Kuhn-Tucker conditions of $P(x)$.
- (ii) Prove that the objective function of problem $P(x)$ is convex with respect to $y; z^i \in \mathbb{R}^m$ ($i = 1, \dots, n$).
- (iii) Assume that C is symmetric positive definite and consider the following optimization problem:

$$\begin{aligned} P1: \quad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad x \in \mathbb{R}^n; \end{aligned}$$

Show that the following inequality holds when $x \in \mathbb{R}^n$ is a global optimal solution of problem P1:

$$(x^*)^T x \leq \frac{b^T b}{\lambda_{\min}(C)};$$

where $\lambda_{\min}(C)$ denotes the smallest eigenvalue of C .

- (iv) Assume that A is the $m \times n$ zero matrix and b is the m -dimensional zero vector. Consider the following optimization problem:

$$\begin{aligned} P2: \quad & \text{Minimize} \quad f(x) \\ & \text{subject to} \quad x^T x \leq \alpha; \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a positive constant. Show that $f(x^*) = f(x)$ holds, when both $(x^*, y^*); (x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ satisfy the Karush-Kuhn-Tucker conditions of problem P2.

An English Translation:

Modern Control Theory



A linear system is described by the state equation

$$\frac{d}{dt}x = Ax + Bu(t); \quad x(0) = x_0;$$

where $A \in \mathbb{R}^{n \times n}$; $B \in \mathbb{R}^{n \times m}$; $x_0 \in \mathbb{R}^n$. A matrix algebraic equation

$$A^T P + P A - P B B^T P + I = 0 \quad (1)$$

with respect to a symmetric matrix $P \in \mathbb{R}^{n \times n}$ is introduced. The transpose of a matrix A is denoted by A^T . The transpose and the norm of a vector x are denoted by x^T and $\|x\| = \sqrt{x^T x}$, respectively. Answer the following questions.

(i) Let $n = 2$, $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with $(a; b) \in \mathbb{R}^2$ such that $ab \neq 0$. Then, find the number of positive definite solution P to (1) for $(a; b)$ which makes this system uncontrollable.

(ii) Suppose that $B = 0$ and that a positive definite matrix P satisfies (1). Prove $\lim_{t \rightarrow \infty} \|x(t)\| = 0$ holds for any x_0 .

(iii) Suppose that P is a solution to (1). Prove that

$$\int_0^\infty (\|x(t)\|^2 + \|u(t)\|^2) dt = x_0^T P x_0 - x(\infty)^T P x(\infty) + \int_0^\infty \|u(t) + B^T P x(t)\|^2 dt$$

holds for any x_0 and $\infty > 0$.

(iv) Define $H = \begin{pmatrix} A & BB^T \\ I & A^T \end{pmatrix}$. Prove that for any eigenvalue λ of H , $\bar{\lambda}$ is also an eigenvalue of H .

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$$E_n = h \frac{1}{2} + n \quad n = 0; 1; 2; 3; \dots$$

s”ü[^]: (> 0) wü[^] % › ß Q”. \\ p h(> 0) x : p K “ , αÉç^a”èÕç
w V @ x Á X, % % w ü : Z x

$$Z = \sum_{n=0}^{\infty} \exp \frac{E_n}{kT}$$

p) Q ’ • ” q b ”. h i ` , k > 0 › Ø ç À Ú ĩ : , T › ^ 0 9 S q b ” . Ž < w ð M
t t Q ‘.

- (i) ü : Z › - % d ‘.
- (ii) αÉç^a” E w 8 4 ‹ hEi › { Š ‘.
- (iii) z ä C = $\frac{dhEi}{dT}$ › { Š ‘.
- (iv) z ä C w ÿ 9 Ã v (T ! 0) › { Š ‘.
- (v) z ä C w ô 9 Ã v (T ! 1) › { Š ‘.

An English Translation:

Physical Statistics



Consider an oscillator system of a frequency with the energy levels

$$E_n = h \left(\frac{1}{2} + n \right) \quad \text{for } n = 0; 1; 2; 3; \dots$$

where $h(> 0)$ is a constant and no energy level is degenerate. The distribution function Z of the system with the absolute temperature T is given by

$$Z = \sum_{n=0}^{\infty} \exp \left(-\frac{E_n}{kT} \right);$$

where $k(> 0)$ is the Boltzmann constant. Answer the following questions.

- (i) Compute the distribution function Z .
- (ii) Obtain the average energy $\langle E \rangle$.
- (iii) Obtain the specific heat $C = \frac{d\langle E \rangle}{dT}$.
- (iv) Obtain the specific heat C in the low temperature limit ($T \rightarrow 0$).
- (v) Obtain the specific heat C in the high temperature limit ($T \rightarrow \infty$).

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$a(t); b(t) \succ t w K'' \quad g \ddot{U} q` o \acute{I} w \hat{i} \bullet \ddot{U} M \quad \ddot{U} \succ \beta Q'' \}$

$$\frac{d^2x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = 0 \tag{1}$$

$\check{Z} < w \check{o} M t t Q' \}$

(i) $k = 1 \succ K'' T : q` o | x \quad = t^k U \ddot{U} (1) \quad w r p K'' h \check{S} w \quad a(t); b(t) \quad t \quad b'' \check{Z} A$
 $G \ddot{u} \acute{U} E \succ \{ \check{S}' \}$

$y \check{Z} < p x | K'' T : \quad k = 1 t 0` o \quad (i) \quad p \{ \check{S} h \acute{U} E U R " q m \langle w q` | \quad (t) \succ t^k q$
 $\emptyset \quad q s r q` o |$

$$p(t) = t \frac{d}{dt}(t) \quad k(t)$$

$q S X \}$

(ii) $a(t); b(t) \succ p(t) \succ ; M o^{-\sim} d \}$

(iii) $p(t) = t w q V a(t); b(t) \succ \check{S}' \}$

(iv) $\ddot{U} (1) w b, o w r U : p s M \quad \grave{o} \ddot{U} w q V | \quad a(t); b(t) \quad x \quad \grave{o} \ddot{U} p s M \backslash q \succ \hat{O} d \}$

An English Translation:

Mathematics for Dynamical Systems



Let $a(t)$ and $b(t)$ be rational functions of t . Consider the real ordinary differential equation

$$\frac{d^2x}{dt^2} + a(t) \frac{dx}{dt} + b(t)x = 0: \quad (1)$$

Answer the following questions.

- (i) Obtain a necessary and sufficient condition on $a(t)$ and $b(t)$ for $x = t^k$ to be a solution to Eq. (1) for each integer $k = 1$.

In the following, assume that the condition obtained in (i) holds for an integer $k = 1$, and let

$$p(t) = t \frac{d}{dt}(t^k(t));$$

where $t^k(t)$ is a solution which is linearly independent of t^k .

- (ii) Write down $a(t)$ and $b(t)$ in terms of $p(t)$.
- (iii) Determine $a(t)$ and $b(t)$ when $p(t) = t$.
- (iv) Show that $a(t)$ and $b(t)$ are not polynomials if all solutions to Eq. (1) are nonconstant polynomials.