作业七

CarBO

目录

1

推导李代数 $\mathfrak{se}(3)$ 的指数映射。已知

$$\mathfrak{se}(3)=\left\{\xi=\begin{bmatrix}\rho\\ \phi\end{bmatrix}\in\mathbb{R}^6,\rho\in\mathbb{R}^3,\phi\in\mathfrak{so}(3),\xi^\wedge=\begin{bmatrix}\phi^\wedge & \rho\\ 0^T & 0\end{bmatrix}\in\mathbb{R}^{4\times 4}\right\}$$

1.1

证明

$$\exp(\xi^\wedge)=\begin{bmatrix}\sum\limits_{n=0}^{\infty}\frac{1}{n!}(\phi^\wedge)^n & \sum\limits_{n=0}^{\infty}\frac{1}{(n+1)!}(\phi^\wedge)^n\rho\\ 0^T & 1\end{bmatrix}$$

证：由指数无穷级数有

$$\begin{aligned}\exp(\xi^\wedge)&=\sum_{n=0}^{\infty}\frac{1}{n!}(\xi^\wedge)^n\\ &=I+\sum_{n=1}^{\infty}\frac{1}{n!}(\xi^\wedge)^n\end{aligned}$$

因为

$$\xi^\wedge=\begin{bmatrix}\phi^\wedge & \rho\\ 0^T & 0\end{bmatrix}$$

则有

$$(\xi^\wedge)^2 = \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} * \begin{bmatrix} \phi^\wedge & \rho \\ 0^T & 0 \end{bmatrix} = \begin{bmatrix} (\phi^\wedge)^2 & \phi^\wedge\rho \\ 0^T & 0 \end{bmatrix}$$

以此类推得

$$\sum_{n=1}^{\infty} \frac{1}{n!}(\xi^\wedge)^n = \begin{bmatrix} \sum\limits_{n=1}^{\infty} \frac{1}{n!}(\phi^\wedge)^n & \sum\limits_{n=1}^{\infty} \frac{1}{n!}(\phi^\wedge)^{n-1}\rho \\ 0^T & 0 \end{bmatrix}$$

故

$$\begin{aligned} \exp(\xi^\wedge) &= I + \sum_{n=1}^{\infty} \frac{1}{n!}(\xi^\wedge)^n \\ &= I + \begin{bmatrix} \sum\limits_{n=1}^{\infty} \frac{1}{n!}(\phi^\wedge)^n & \sum\limits_{n=1}^{\infty} \frac{1}{n!}(\phi^\wedge)^{n-1}\rho \\ 0^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} \sum\limits_{n=0}^{\infty} \frac{1}{n!}(\phi^\wedge)^n & \sum\limits_{n=0}^{\infty} \frac{1}{(n+1)!}(\phi^\wedge)^n\rho \\ 0^T & 1 \end{bmatrix} \end{aligned}$$

证毕。

1.2

令 $\phi = \theta a$, 那么

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\phi^\wedge)^n = \frac{\sin\theta}{\theta} I + \left(1 - \frac{\sin\theta}{\theta}\right) aa^T + \frac{1-\cos\theta}{\theta} a^\wedge \triangleq J$$

证：将 $\phi = \theta a$ 代入上式有

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!}(\theta a^\wedge)^n$$

因为 a 为单位向量，则有

$$\begin{aligned} (a^\wedge)^2 &= aa^T - I \\ (a^\wedge)^3 &= -a^\wedge \\ (a^\wedge)^4 &= -(a^\wedge)^2 \end{aligned}$$

因此

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\theta^n(a^\wedge)^n &= I+\frac{-a^\wedge}{\theta}(-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4+\cdots) \\
&\quad+\frac{-(aa^T-I)}{\theta}(-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5+\cdots) \\
&= I+\frac{-a^\wedge}{\theta}(-1+1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4+\cdots) \\
&\quad+\frac{-(aa^T-I)}{\theta}(-\theta+\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5+\cdots) \\
&= I+\frac{-a^\wedge}{\theta}(\cos\theta-1)+\frac{-(aa^T-I)}{\theta}(\sin\theta-\theta) \\
&= \frac{\sin\theta}{\theta}I+\left(1-\frac{\sin\theta}{\theta}\right)aa^T+\frac{1-\cos\theta}{\theta}a^\wedge
\end{aligned}
$$

证毕。