Introduction to Cryptography

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The slide is based on the Online Cryptography Course by Dan Boneh
Also refer to https://crypto.stanford.edu/~dabo/courses/OnlineCrypto/

Cryptography is everywhere

Secure communication:

- web traffic: HTTPS
- wireless traffic: 802.11i WPA2 (and WEP), GSM, Bluetooth

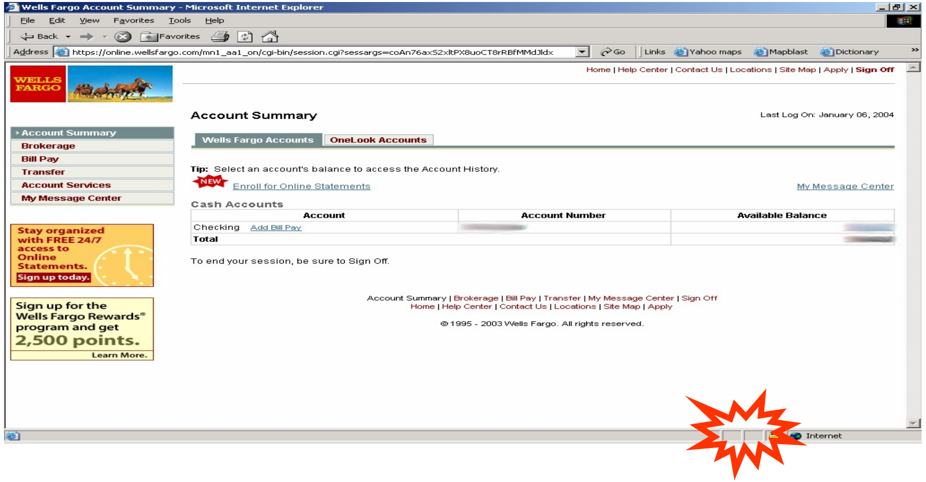
Encrypting files on disk: EFS, TrueCrypt

Content protection (e.g. DVD, Blu-ray): CSS, AACS

User authentication

… and much much more

Secure communication



HTTPS

no eavesdropping
no tampering

Secure Sockets Layer / TLS

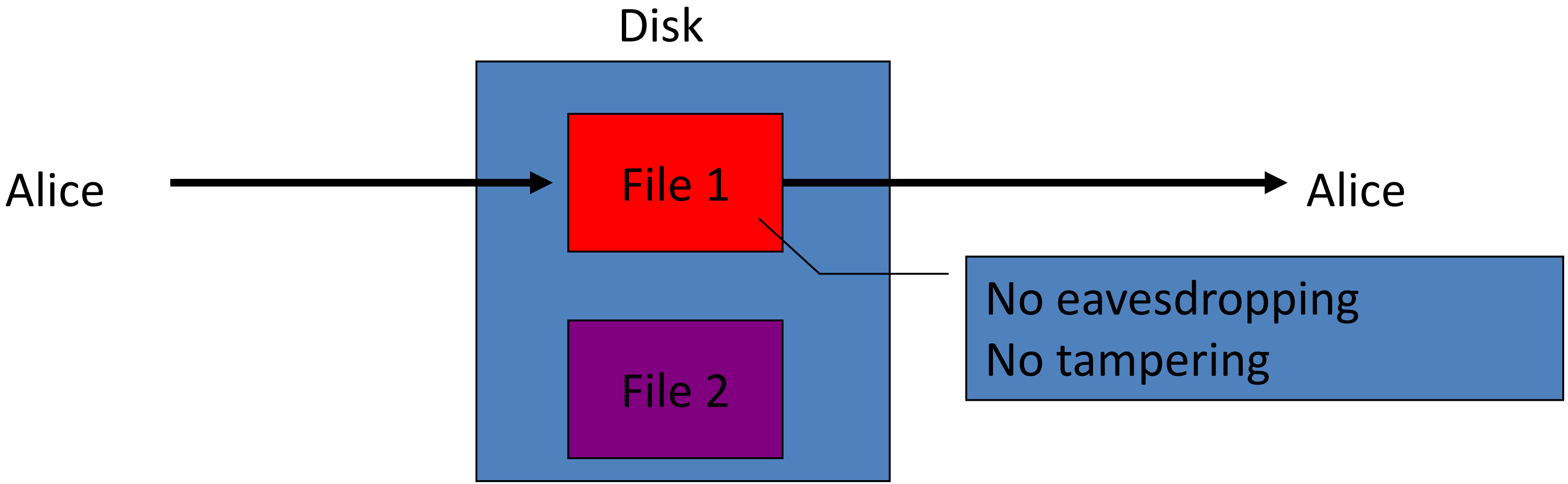
<u>Two main parts</u>

1. Handshake Protocol: **Establish shared secret key using public-key cryptography** (2nd part of course)

2. Record Layer: **Transmit data using shared secret key**

   Ensure confidentiality and integrity (1st part of course)

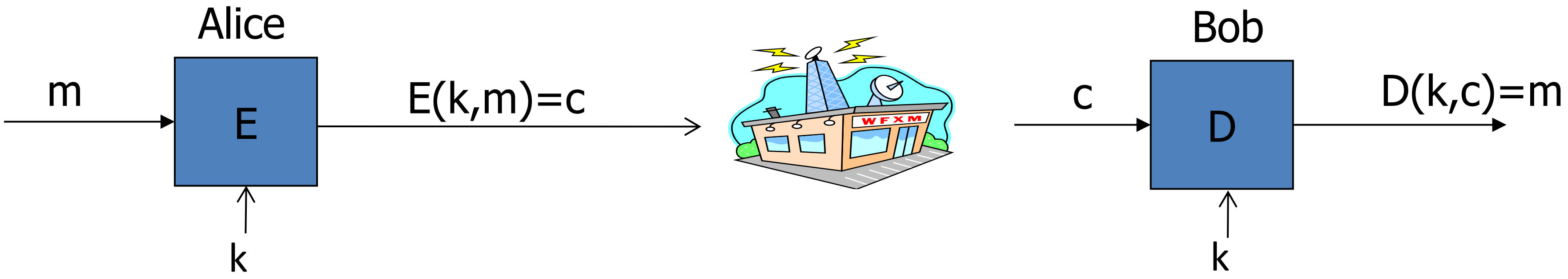
Protected files on disk



Analogous to secure communication:

Alice today sends a message to Alice tomorrow

Building block: sym. encryption



E, D: cipher   k: secret key (e.g. 128 bits)

m, c: plaintext, ciphertext

Encryption algorithm is publicly known

- Never use a proprietary cipher

Use Cases

Single use key: (one time key)

- Key is only used to encrypt one message
  - encrypted email: new key generated for every email

Multi use key: (many time key)

- Key used to encrypt multiple messages
  - encrypted files: same key used to encrypt many files
- Need more machinery than for one-time key

Things to remember

Cryptography is:

- A tremendous tool
- The basis for many security mechanisms

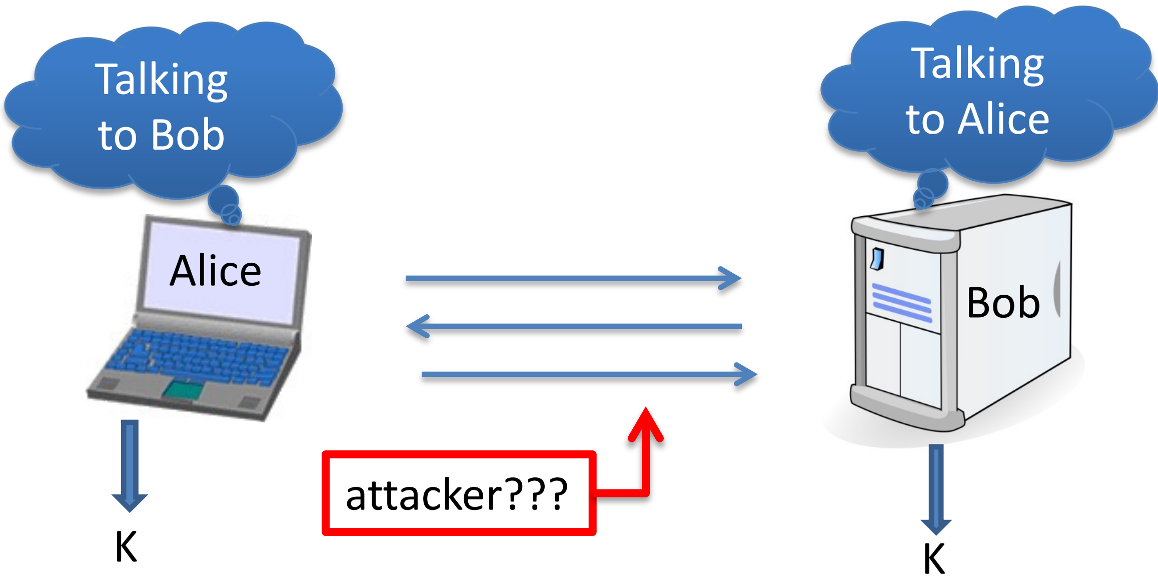
Cryptography is not:

- The solution to all security problems
- Reliable unless implemented and used properly
- Something you should try to invent yourself
  - many many examples of broken ad-hoc designs

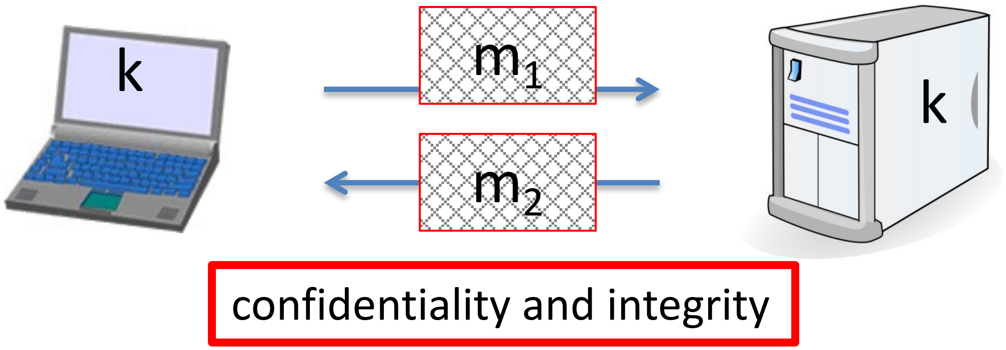
End of Segment

Crypto core

Secret key establishment:



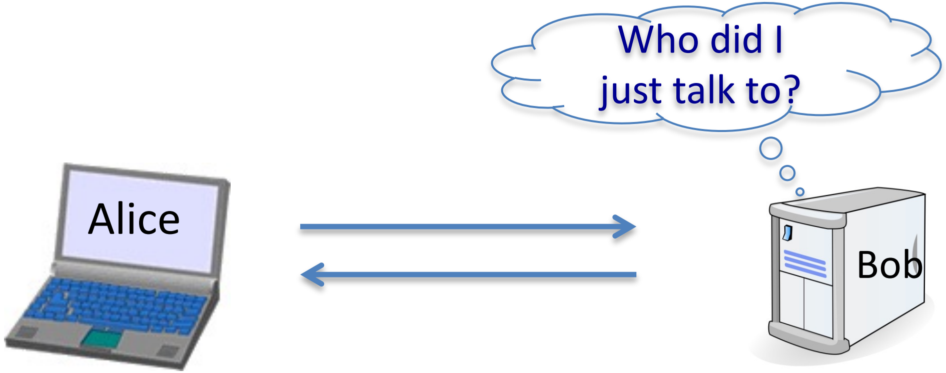
Secure communication:



But crypto can do much more

- Digital signatures

- Anonymous communication

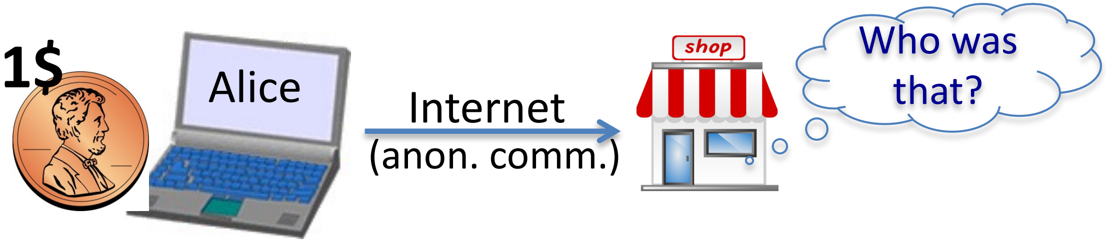


But crypto can do much more

- Digital signatures

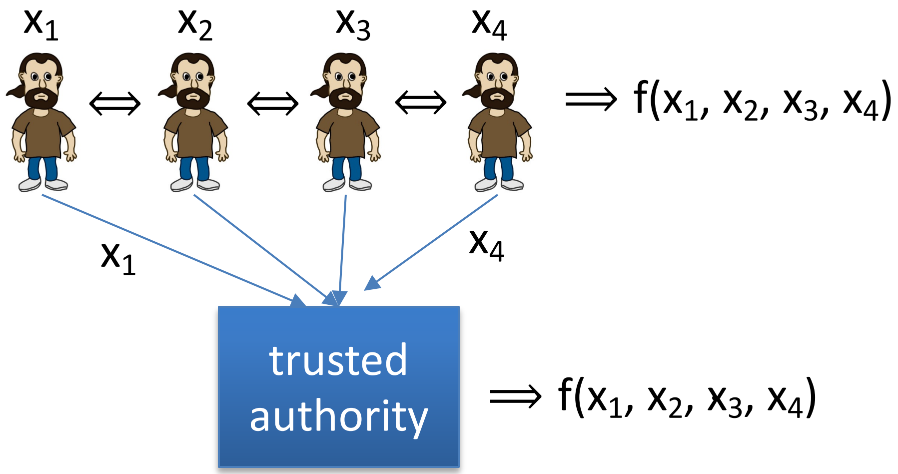
- Anonymous communication

- Anonymous **digital** cash
  - Can I spend a "digital coin" without anyone knowing who I am?
  - How to prevent double spending?



Protocols

- Elections
- Private auctions



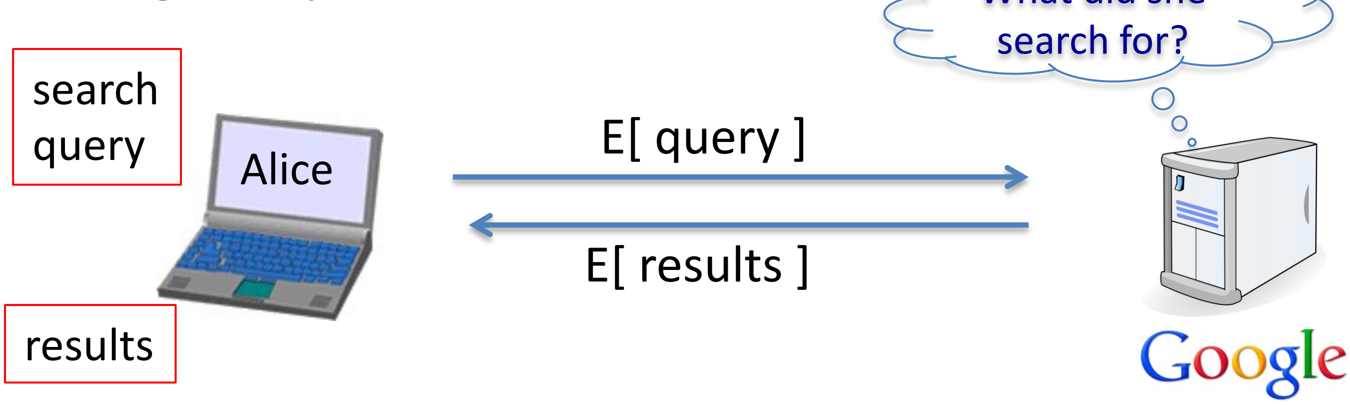
Goal: compute $f(x_1, x_2, x_3, x_4)$

"Thm:" anything that can done with trusted auth. can also be done without

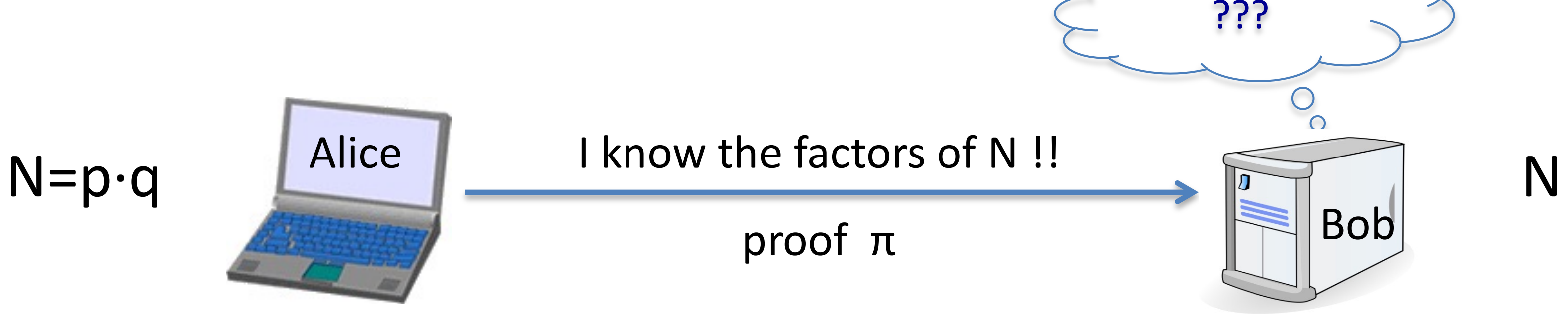
- Secure multi-party computation

Crypto magic

- Privately outsourcing computation



- Zero knowledge (proof of knowledge)



A rigorous science

The three steps in cryptography:

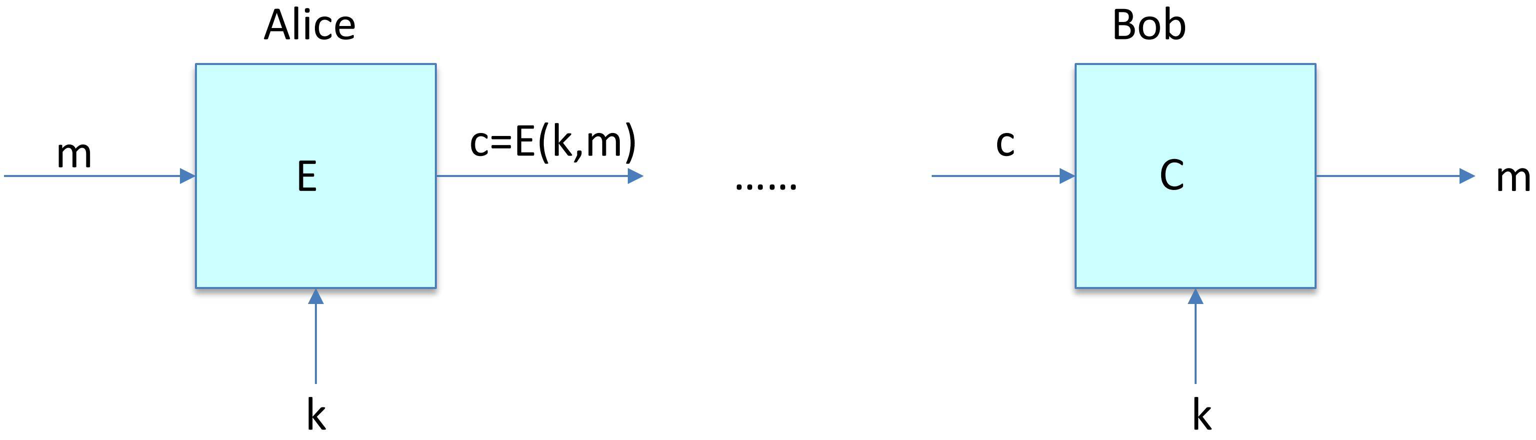
- Precisely specify threat model

- Propose a construction

- Prove that breaking construction under threat mode will solve an underlying hard problem

History

Symmetric Ciphers



Alice and Bob have the same key.

Few Historic Examples (all badly broken)

1. Substitution cipher

C:=E(k,"bcza")="wnac"

D(k,C)="bcza"

k :=

$a \rightarrow c$
$b \rightarrow w$
$c \rightarrow n$
…
…
$z \rightarrow a$

Caesar Cipher (no key)

Shift by 3:

$a \longrightarrow d$

$b \longrightarrow e$

$c \longrightarrow f$

…

…

$y \longrightarrow b$

$z \longrightarrow c$

What is the size of key space in the substitution cipher assuming 26 letters?

$|\mathcal{K}| = 26$

$|\mathcal{K}| = 26!$ $\quad\quad\quad\quad$ $26! \approx 2^{88}$

$|\mathcal{K}| = 2^{26}$

$|\mathcal{K}| = 26^2$

How to break a substitution cipher?

What is the most common letter in English text?

"X"

"L"

"E"

"H"

How to break a substitution cipher?

(1) Use frequency of English letters

“e”:12.7%, ”t”:9.1%， “a”:8.1%

(2) Use frequency of pairs of letters (digrams)

“he”, ”an”, “in”, “th”

⟹ CT only attack!

An Example

UKBYBIPOUZBCUFEEBORUKBYBHOBBRFESPVKBWFOFERVNBCVBZPRUBOFERVNBCVBPCYYFVUFO
FEIKNWFRFIKJNUPWRFIPOUNVNIPUBRNCUKBEFWWFDNCHXCYBOHOPYXPUBNCUBOYNRVNIWN
CPOJIOFHOPZRVFZIXUBORJRUBZRBCHNCBBONCHRJZSFWNVRJRUBZRPCYZPUKBZPUNVPWPCYVF
ZIXUPUNFCPWRVNBCVBRPYYNUNFCPWWJUKBYBIPOUZBCUIPOUNVNIPUBRNCHOPYXPUBNCUB
OYNRVNIWNCPOJIOFHOPZRNCRVNBCUNENVVFZIXUNCHPCYVFZIXUPUNFCPWZPUKBZPUNVR

B	36	➔ E
N	34	
U	33	➔ T
P	32	➔ A
C	26	

NC	11	➔ IN
PU	10	➔ AT
UB	10	
UN	9	

digrams

UKB	6	➔ THE
RVN	6	
FZI	4	

trigrams

2. Vigener cipher (16'th century, Rome)

k = $\boxed{\text{C R Y P T O}}$ C R Y P T O C R Y P T

(+ mod 26)

m = W H A T A N I C E D A Y T O D A Y

c = Z Z Z J U C | L U D T U N | W G C Q S

suppose most common = "H" ⟹ first letter of key = "H" – "E" = "C"

3. Data Encryption Standard (1974)

DES: # keys = 2^{56} , block size = 64 bits

Today: AES (2001), Salsa20 (2008) (and many others)

Discrete Probability

U: finite set (e.g. $U = \{0,1\}^n$)

Def: **Probability distribution** P over U is a function $P: U \longrightarrow [0,1]$

such that $\sum_{x \in U} P(x) = 1$

Examples:

1. Uniform distribution: for all $x \in U$: $P(x) = 1/|U|$
2. Point distribution at x_0: $P(x_0) = 1$, $\forall x \neq x_0$: $P(x) = 0$

Distribution vector: $\big(P(000), P(001), P(010), \ldots, P(111) \big)$

Events

- For a set $A \subseteq U$: $\Pr[A] = \sum_{x \in A} P(x) \in [0,1]$

note: $\Pr[U]=1$

- The set A is called an **event**

Example: $U = \{0,1\}^8$

- $A = \{$ all x in U such that $\text{lsb}_2(x)=11$ $\} \subseteq U$

for the uniform distribution on $\{0,1\}^8$: $\Pr[A] =$ 1/4

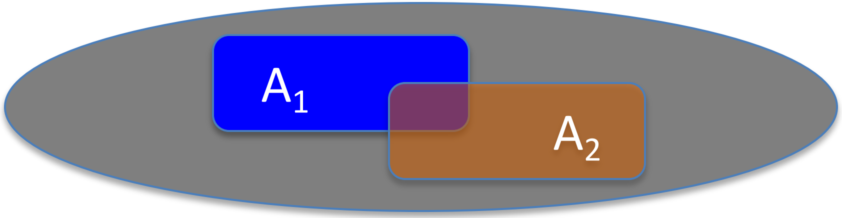
The union bound

- For events A_1 and A_2

$$\Pr[\,A_1 \cup A_2\,] \le \Pr[A_1] + \Pr[A_2]$$

If $A_1 \cap A_2 = \emptyset \Longrightarrow$

$$\Pr[\,A_1 \cup A_2\,] = \Pr[A_1] + \Pr[A_2]$$



Example:

$A_1 = \{$ all x in $\{0,1\}^n$ s.t $\mathrm{lsb}_2(x)=11$ $\}$; $A_2 = \{$ all x in $\{0,1\}^n$ s.t. $\mathrm{msb}_2(x)=11$ $\}$

$$\Pr[\,\mathrm{lsb}_2(x)=11 \text{ or } \mathrm{msb}_2(x)=11\,] = \Pr[A_1 \cup A_2] \le ¼+¼ = ½$$

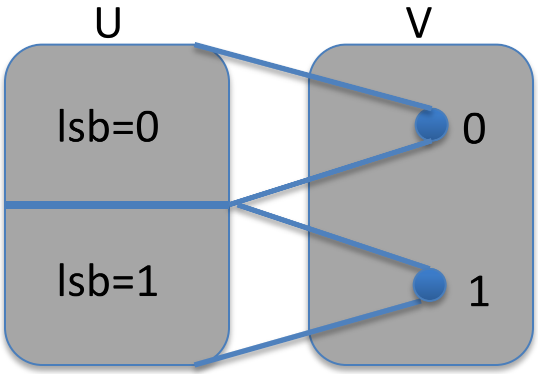
Random Variables

Def: a random variable X is a function $X: U \longrightarrow V$

Example: $X: \{0,1\}^n \longrightarrow \{0,1\}$; $X(y) = \text{lsb}(y) \in \{0,1\}$

For the uniform distribution on U:

$\Pr[X=0] = 1/2$, $\Pr[X=1] = 1/2$



More generally:

rand. var. X induces a distribution on V: $\Pr[X=v] := \Pr\left[X^{-1}(v)\right]$

The uniform random variable

Let U be some set, e.g. $U = \{0,1\}^n$

We write $r \xleftarrow{R} U$ to denote a **uniform random variable** over U

for all $a \in U$: $\Pr\big[\, r = a \,\big] = 1/|U|$

(formally, r is the identity function: $r(x)=x$ for all $x \in U$)

Let r be a uniform random variable on $\{0,1\}^2$

Define the random variable $X = r_1 + r_2$

Then $\Pr[X=2] = ¼$

Hint: $\Pr[X=2] = \Pr[\, r=11 \,]$

Randomized algorithms

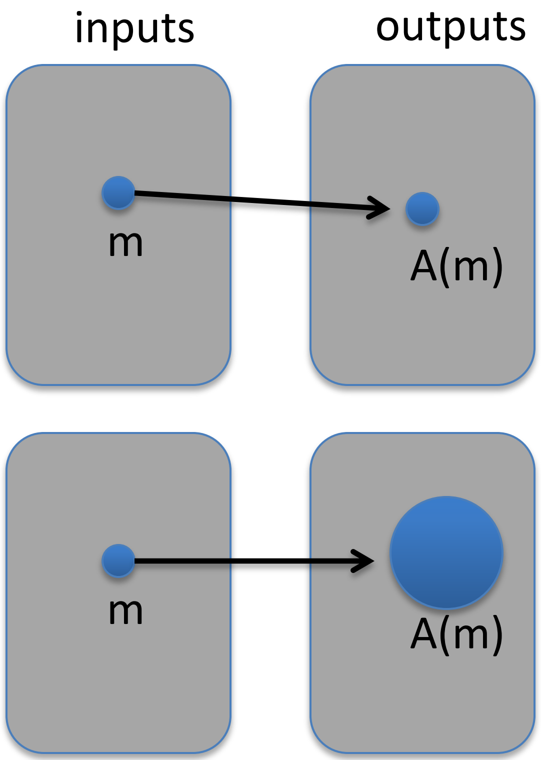
- Deterministic algorithm: $y \longleftarrow A(m)$

- Randomized algorithm

$$y \longleftarrow A(\,m\,;\,r\,) \quad \text{where} \quad r \xleftarrow{R} \{0,1\}^n$$

output is a random variable

$$y \xleftarrow{R} A(\,m\,)$$



Example: $A(m\,;\,k) = E(k,\,m)$, $y \xleftarrow{R} A(\,m\,)$

Independence

Def: events A and B are **independent** if $\Pr[\,A \text{ and } B\,] = \Pr[A] \cdot \Pr[B]$

random variables X,Y taking values in V are **independent** if

$\forall a,b \in V$: $\Pr[\,X=a \text{ and } Y=b] = \Pr[X=a] \cdot \Pr[Y=b]$

Example: $U = \{0,1\}^2 = \{00, 01, 10, 11\}$ and $r \xleftarrow{R} U$

Define r.v. X and Y as: $X = \text{lsb}(r)$, $Y = \text{msb}(r)$

$\Pr[\,X=0 \text{ and } Y=0\,] = \Pr[\,r=00\,] = 1/4 = \Pr[X=0] \cdot \Pr[Y=0]$

Review: XOR

XOR of two strings in $\{0,1\}^n$ is their bit-wise addition mod 2

X	Y	X $\oplus$ Y
0	0	0
0	1	1
1	0	1
1	1	0

$$
\begin{array}{r}
0\ 1\ 1\ 0\ 1\ 1\ 1 \\
\oplus \quad 1\ 0\ 1\ 1\ 0\ 1\ 0 \\
\hline
1\ 1\ 0\ \boxed{1\ 1\ 0\ 1}
\end{array}
$$

An important property of XOR

Thm: Y a rand. var. over $\{0,1\}^n$, X an indep. uniform var. on $\{0,1\}^n$

Then $Z := Y \oplus X$ is uniform var. on $\{0,1\}^n$

Y	Pr
0	P_0
1	P_1

X	Pr
0	½
1	½

Proof: (for n=1)

$\Pr[\ Z=0\ ] = \Pr[\ (X,Y)=(0,0) \text{ or } (X,Y)=(1,1)\ ]$

$= \Pr[\ (X,Y)=(0,0)\ ] + \Pr[\ (X,Y)=(1,1)\ ]$

$= P_0/2 + P_1/2 = 1/2$

X	Y	Pr
0	0	$P_0/2$
0	1	$P_1/2$
1	0	$P_0/2$
1	1	$P_1/2$

The birthday paradox

Let $r_1, \ldots, r_n \in U$ be indep. identically distributed random vars.

Thm: when $n = 1.2 \times |U|^{1/2}$ then $\Pr[\,\exists i \neq j: \; r_i = r_j\,] \geq ½$

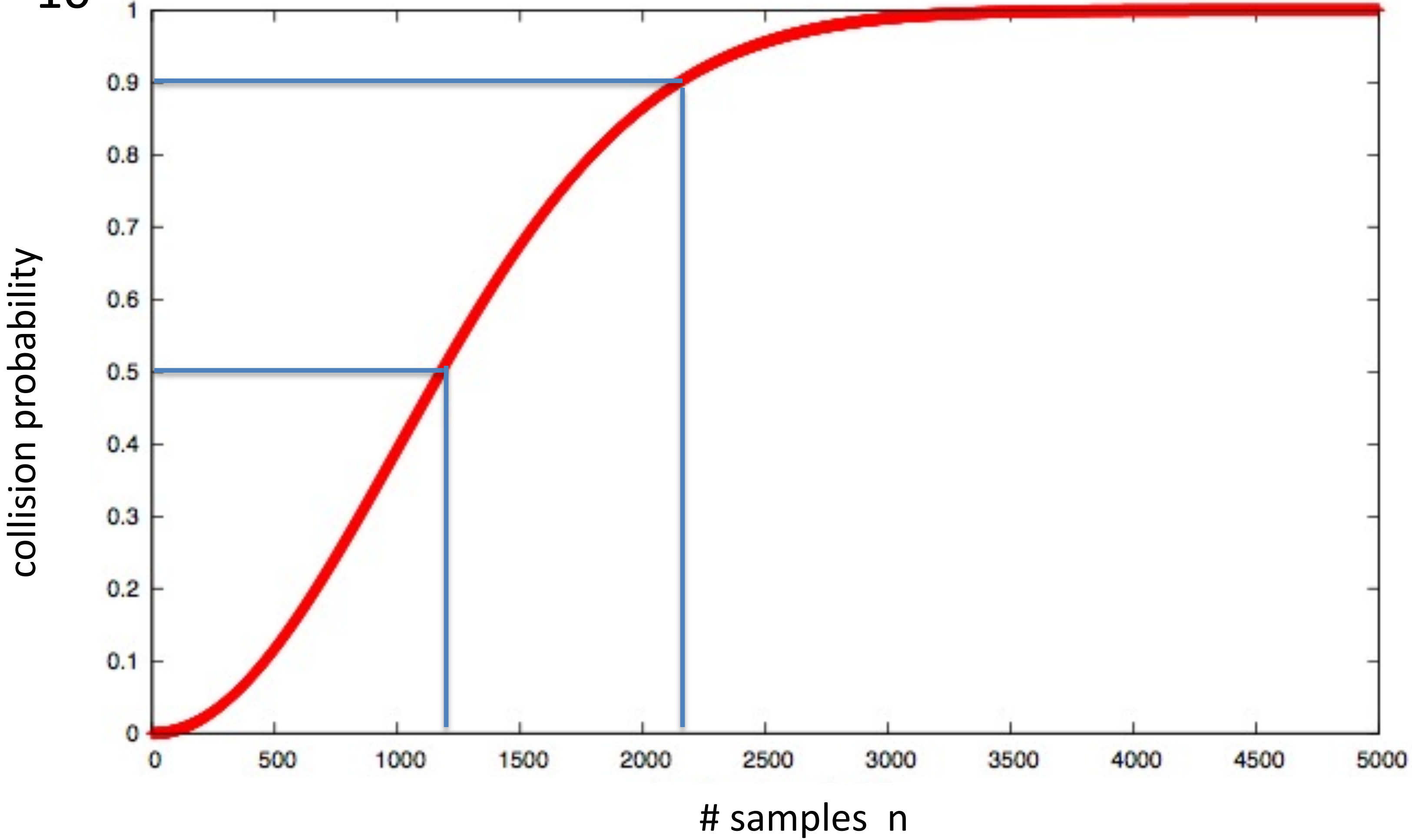
notation: $|U|$ is the size of U

Example: Let $U = \{0,1\}^{128}$

After sampling about 2^{64} random messages from U,

some two sampled messages will likely be the same

$|U|=10^6$



Dan Boneh