A generic hybrid construction of CPA encryption based on secure PRF

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A Graduate Course in Applied Cryptography version 0.5, Dan Boneh and Victor Shoup, pp. 180 – 185, 5.4.1

Construction

- $\varepsilon = (E, D)$ be a cipher, defined over (K, M, C).
- F be a PRF defined over (K', X, K); so,the output space of F is the key space of ε.
- We define a new cipher $\varepsilon' = (E', D')$, defined over $(K', M, X \times C)$. As follows:
  - For $k' \in K'$ and $m \in M$,

$$E'(k', m) := x \overset{R}{\leftarrow} X, k \leftarrow F(k', x), c \overset{R}{\leftarrow} E(k, m), \text{ output } (x, c);$$

  - For $k' \in K'$ and $c' = (x, c) \in X \times C$,

$$D'(k', c') := k \leftarrow F(k', x), m \leftarrow D(k, c), \text{output } m.$$

Clearly,ε' is a probabilistic cipher.

Then ,ε' is CPA encryption based on secure PRF.

Theorem and proof idea

- **Theorem. If $\boldsymbol{F}$ is a secure $\boldsymbol{PRF}$,$\boldsymbol{\varepsilon}$ is a semantically secure cipher , and $\boldsymbol{N} \coloneqq |\boldsymbol{X}|$ is super-poly, then the cipher $\boldsymbol{\varepsilon'}$described above is a $\boldsymbol{CPA}$ secure cipher.**
  - For a CPA adversary A that attacks ε', and it makes at most Q queries to its challenger, there exists a PRF adversary B_F to attacks F, and an SS adversary B_ε that attacks ε, where both B_F and B_ε are elementary wrappers around A, such that

$$CPAadv[A, \varepsilon'] \leq \frac{Q^2}{N} + 2 \cdot PRFadv[B_F, F] + Q \cdot SSadv\ [B_\varepsilon, \varepsilon]$$

- Its bit guessing version:

$$CPAadv^*[A, \varepsilon'] \leq \frac{Q^2}{2N} + PRFadv[B_F, F] + Q \cdot SSadv[B_\varepsilon, \varepsilon]$$

Basic strategy of the proof

- Define several games: Game 0, Game 1, Game 2, and Game 3. Each of these games is played between *A* and a different challenger. In each of these games, b denotes the random bit chosen by the challenger, while b'denotes the bit output by *A*. Also, for $j = 0, \ldots, 3$,we define W_j to be the event that $b' = b$ in Game j .
- The proof idea is to make sure for $j = 1, \ldots, 3$:

$$\left|Pr\left[W_j\right] - Pr\left[W_{j-1}\right]\right| = negligible$$

Game 0

Game 0 plays between A and the challenger in the bit-guessing version of $CPA\ security$ attack game . The challenger runs:

$b \xleftarrow{R} \{0,1\}$

$k' \xleftarrow{R} K'$

$for\ i \leftarrow 1\ to\ Q\ do$

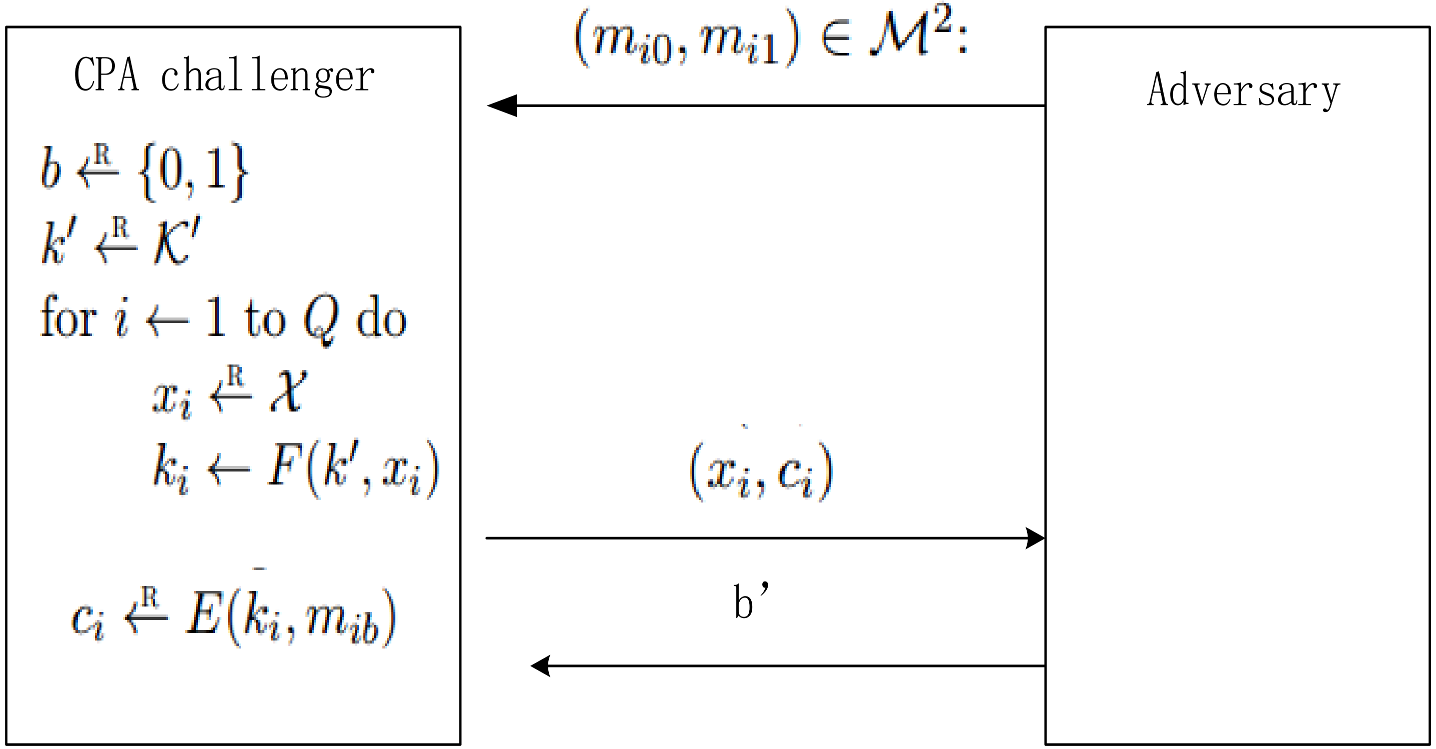
$\qquad x_i \xleftarrow{R} X$

$\qquad \boxed{k_i \leftarrow F(k', x_i)}$

upon receiving the ith query $(m_{i0}, m_{i1}) \in M^2$;

$\qquad c_i \xleftarrow{R} E(k_i, m_{ib})$

$\qquad$ send (x_i, c_i) to the adversary.



$$CPAadv^*[A, \varepsilon'] = |Pr[W_0] - 1/2|$$

Game 1

Game 1 plays "PRF card," replacing $F(k',\cdot)$ by a truly random function $f \in Funs[X, K]$. The challenger runs:

$b \xleftarrow{R} \{0,1\}$

$f \xleftarrow{R} Funs[X, K]$.

$for\ i \leftarrow 1\ to\ Q\ do$

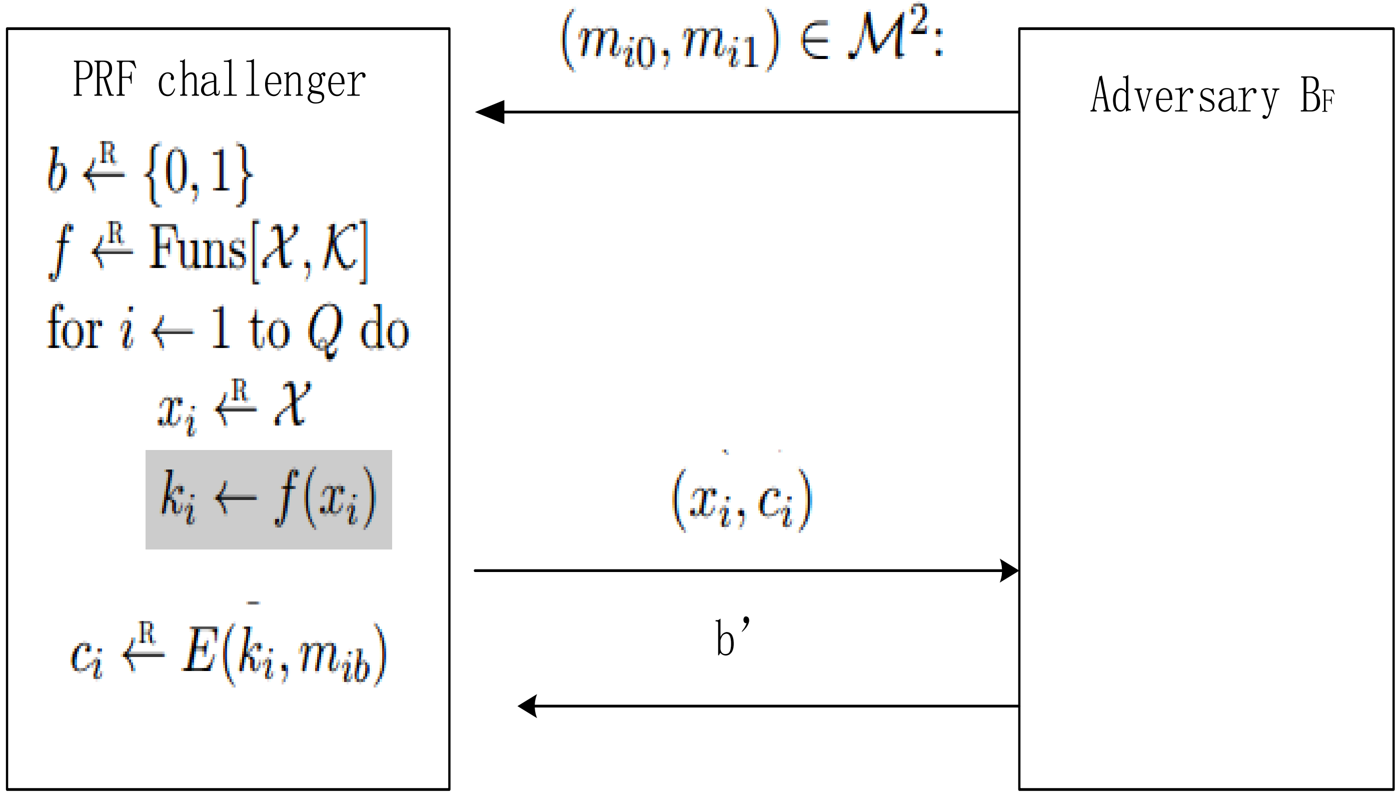
$\quad x_i \xleftarrow{R} X$

$\quad \boxed{k_i \leftarrow f(x_i)}$

upon receiving the ith query $(m_{i0}, m_{i1}) \in M^2$;

$\quad c_i \xleftarrow{R} E(k_i, m_{ib})$

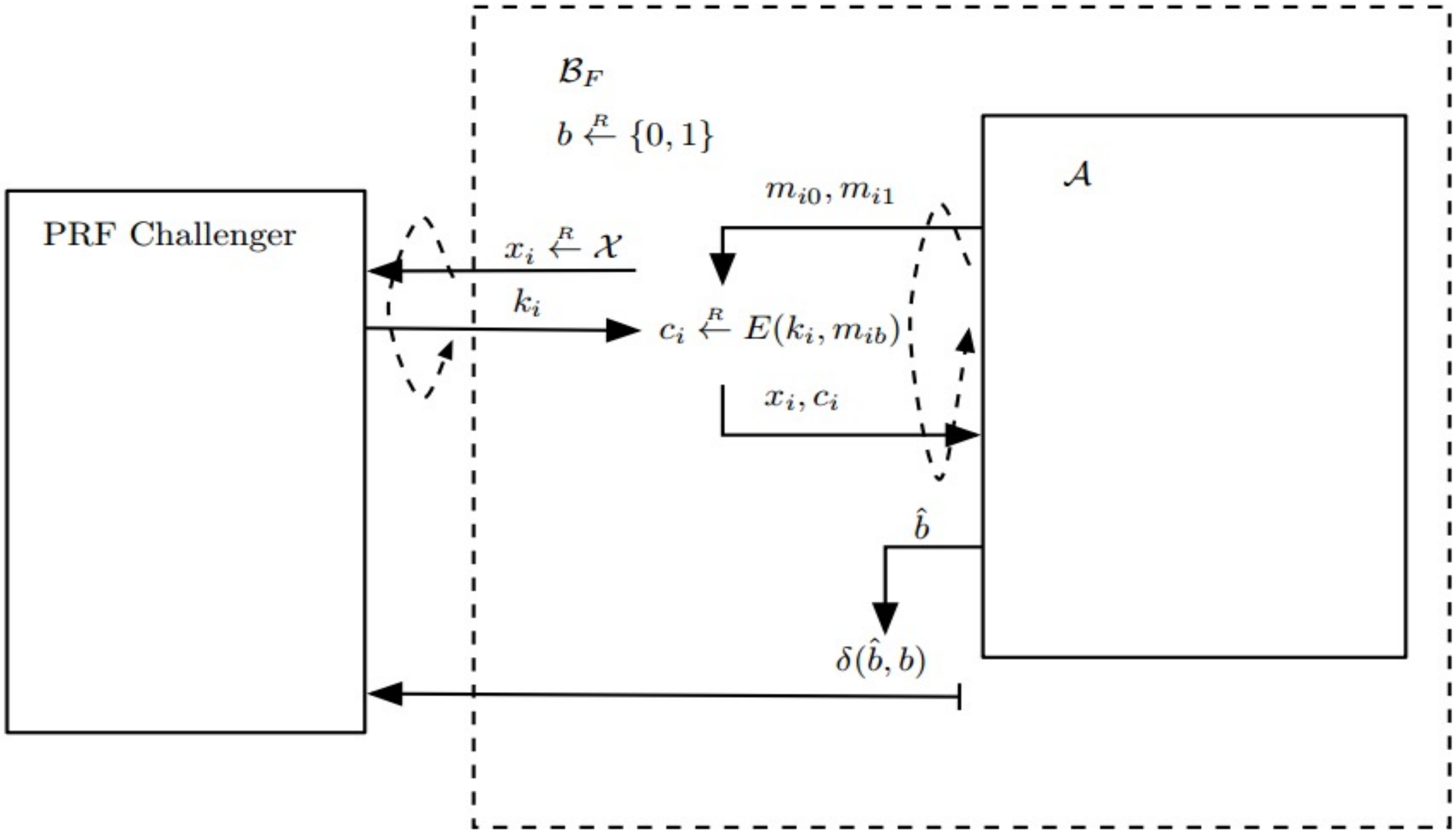
send (x_i, c_i) to the adversary.



$$|\Pr[W_1] - \Pr[W_0]| = PRFadv[B_F, F]$$

B_F is an efficient PRF adversary; assuming that F is a secure PRF,then $PRFadv[B_F, F]$ is negligible.

- For Game 0 and Game 1, let adversary B_F plays the role of challenger to A.



  - Eventually, A halts and outputs a bit b', at which time adversary B_F halts and outputs 1 if $b' = b$, and outputs 0 otherwise.

Game 2

Game 2 implements the random function f. Challenger keeps track of the inputs to f, and detect if the same input is used twice. Challenger runs:

$b \xleftarrow{R} \{0,1\}$

$for\ i \leftarrow 1\ to\ Q\ do$

$\quad x_i \xleftarrow{R} X$

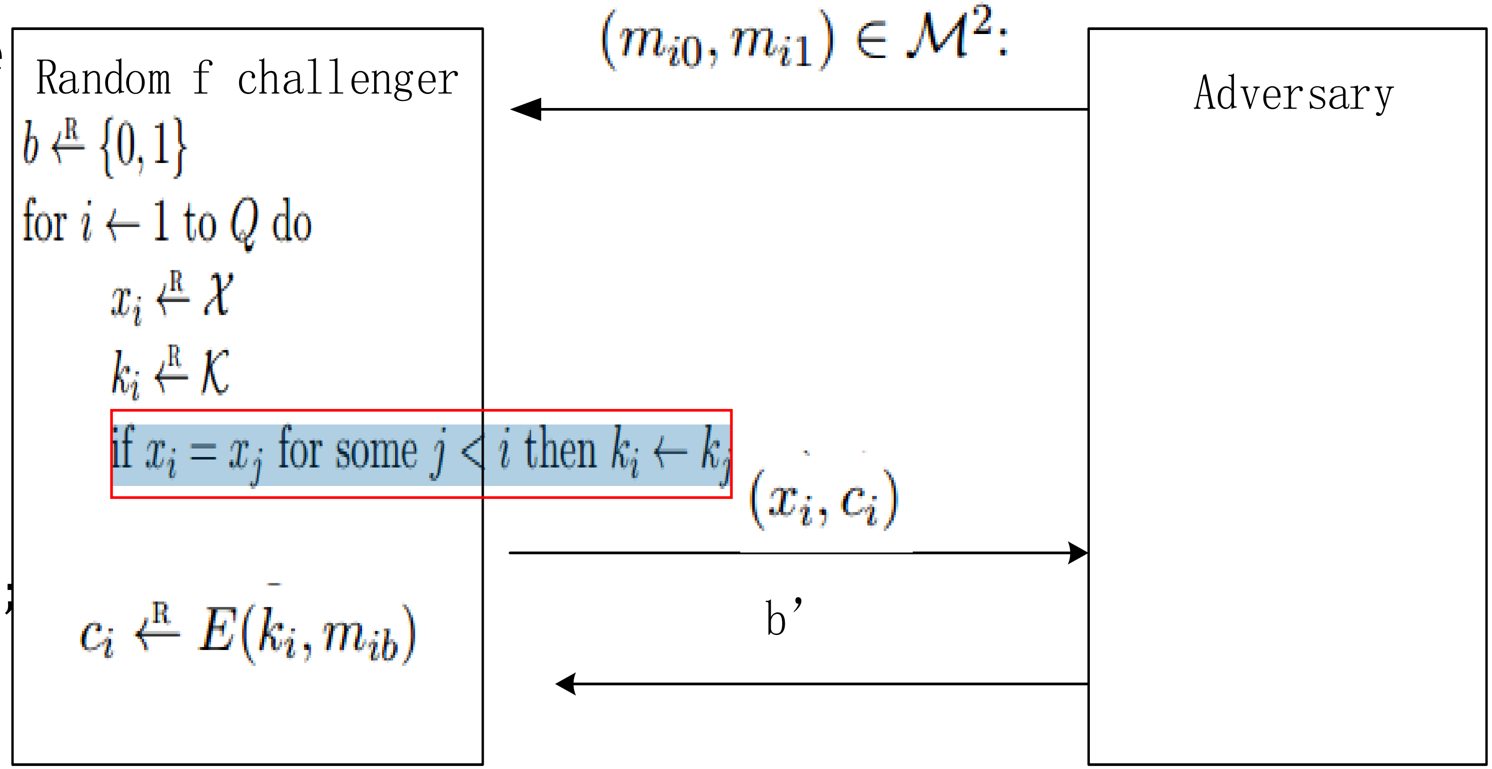
$\quad k_i \xleftarrow{R} K$

$\quad if\, x_i = x_j\ for\ some\ j < i\ then\ k_i \leftarrow k_j$

upon receiving the ith query $(m_{i0}, m_{i1}) \in M^2$:

$\qquad c_i \xleftarrow{R} E(k_i, m_{ib})$

$\qquad$send (x_i, c_i) to the adversary.



f is a faithful implementation of the random function ,then $\Pr[W_1] = \Pr[W_2]$

Game 3

Game3, dropping the highlight line in the previous game2:

$b \xleftarrow{R} \{0,1\}$
$for\ i \leftarrow 1\ to\ Q\ do$
$\quad x_i \xleftarrow{R} X$
$\quad k_i \xleftarrow{R} K$
upon receiving the ith query
$\quad (m_{i0}, m_{i1}) \in M^2;$
$\quad c_i \xleftarrow{R} E(k_i, m_{ib})$
$\quad$ send (x_i, c_i) to the adversary.

- Define Z to be the event that $x_i = x_j$for some $i \neq j$. Games 2 and 3 proceed identically unless Z occurs; particularly, $W_2 \land \bar{Z}$occurs if and only if $W_3 \land \bar{Z}$ occurs. Applying the Difference Lemma, we have

$$|\Pr[W_3] - \Pr[W_2]| \leq \Pr[Z]$$

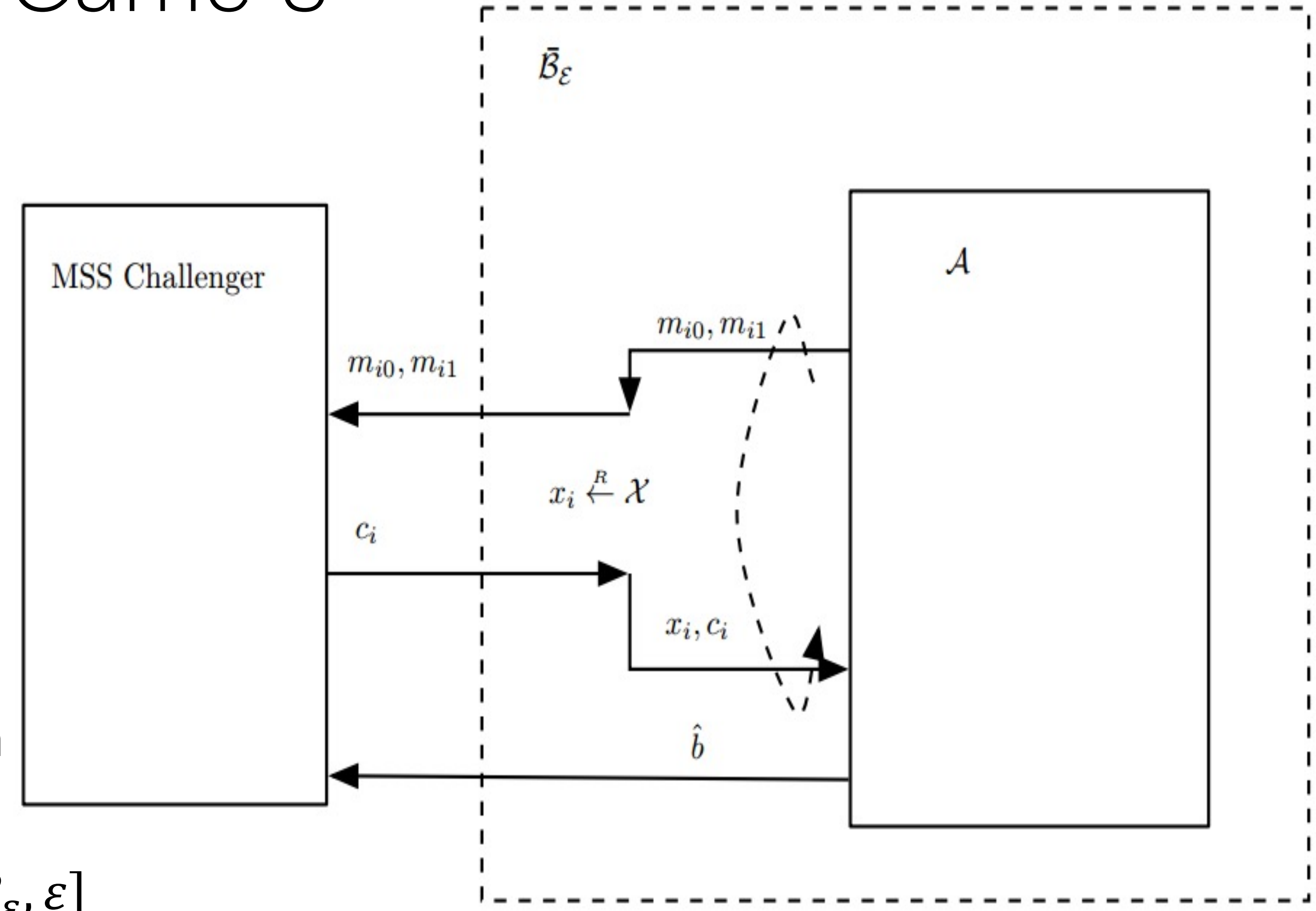
Because there are less than $\frac{Q^2}{2}$ such events , each event occurs with probability $\frac{1}{N}$.

So , $\Pr[Z] \leq \frac{Q^2}{2N}$

Game 3

$\bar{B}_\varepsilon$ is a multi-key semantic security adversary, playing multi-key semantic security attack with MSS challenger . It makes at most Q queries. Adversary $\bar{B}_\varepsilon$ plays the role of challenger to A.Then

$$\left|\Pr[W_3] - \frac{1}{2}\right| = MSSadv^*[\bar{B}_\varepsilon, \varepsilon]$$



Game3, independent encryption keys k_i are used to encrypt each message. So,

$$MSSadv^*[\bar{B}_\varepsilon, \varepsilon] = Q \cdot SSadv^*[B_\varepsilon, \varepsilon]$$

Putting together:

- $MSSadv^*[\bar{B}_\varepsilon, \varepsilon] = Q \cdot SSadv^*[B_\varepsilon, \varepsilon]$
- $CPAadv^*[A, \varepsilon'] = |Pr[W_0] - 1/2|$
- $|\Pr[W_1] - \Pr[W_0]| = PRFadv[B_F, F]$
- $\Pr[W_1] = \Pr[W_2]$
- $|\Pr[W_3] - \Pr[W_2]| \leq \Pr[Z]$
- $\Pr[Z] \leq \frac{Q^2}{2N}$
- $\left|\Pr[W_3] - \frac{1}{2}\right| = MSSadv^*[\bar{B}_\varepsilon, \varepsilon]$
- $\Rightarrow$ $CPAadv^*[A, \varepsilon'] \leq \frac{Q^2}{2N} + PRFadv[B_F, F] + Q \cdot SSadv^*[B_\varepsilon, \varepsilon]$