Purity calculation

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The new defined purity function:

$$g = \int log(f_1)f_1 - \int log(f_2)f_1 + \int log(f_2)f_2 - \int log(f_1)f_2$$

where,

$$f_1 = \frac{1}{\sqrt{((2\pi)^p|D_1|)}}exp(-\frac{1}{2}(z-\mu_1)'D_1^{-1}(z-\mu_1))$$

$$f_2 = \frac{1}{\sqrt{((2\pi)^p|D_2|)}}exp(-\frac{1}{2}(z-\mu_2)'D_2^{-1}(z-\mu_2))$$

We know that

$$\begin{aligned}\int f_1 log f_1 =& E_1(-\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_1|) - \frac{1}{2}(z-\mu_1)'D_1^{-1}(z-\mu_1))\\ =& -\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_1|) - \frac{1}{2}E_1[(z-\mu_1)'D_1^{-1}(z-\mu_1)]\end{aligned}$$

And

$$\begin{aligned}E_1[(z-\mu_1)'D_1^{-1}(z-\mu_1)] =& E_1[tr((z-\mu_1)'D_1^{-1}(z-\mu_1))]\\ =& E_1[tr(D_1^{-1}(z-\mu_1)'(z-\mu_1))]\\ =& tr(E_1[D_1^{-1}(z-\mu_1)'(z-\mu_1)])\\ =& tr(D_1^{-1}E_1[(z-\mu_1)'(z-\mu_1)])\\ =& tr(D_1^{-1}D_1) = tr(I_p) = p\end{aligned}$$

Therefore,

$$\int f_1 log f_1 = -\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_1|) - \frac{p}{2}$$

Similarly,

$$\int f_2 log f_2 = -\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_2|) - \frac{p}{2}$$

$$\begin{aligned}\int f_1 log f_2 =& E_1(-\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_2|) - \frac{1}{2}(z-\mu_2)'D_2^{-1}(z-\mu_2))\\ =& -\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_2|) - \frac{1}{2}E_1[(z-\mu_2)'D_2^{-1}(z-\mu_2)]\end{aligned}$$

And

$$\begin{aligned}E_1[(z-\mu_2)'D_2^{-1}(z-\mu_2)] =& E_1[(z-\mu_1+\mu_1-\mu_2)'D_2^{-1}(z-\mu_1+\mu_1-\mu_2)]\\ =& E_1[(z-\mu_1)'D_2^{-1}(z-\mu_1) + (\mu_1-\mu_2)'D_2^{-1}(z-\mu_1)\\ & + (z-\mu_1)'D_2^{-1}(\mu_1-\mu_2) + (\mu_1-\mu_2)'D_2^{-1}(\mu_1-\mu_2)]\\ =& E_1[(z-\mu_1)'D_2^{-1}(z-\mu_1)] + (\mu_1-\mu_2)'D_2^{-1}E_1(z-\mu_1) +\\ & E_1(z-\mu_1)')D_2^{-1}(\mu_1-\mu_2) + (\mu_1-\mu_2)'D_2^{-1}(\mu_1-\mu_2)\\ =& E_1[(z-\mu_1)'D_2^{-1}(z-\mu_1)] + 0 + 0 + (\mu_1-\mu_2)'D_2^{-1}(\mu_1-\mu_2)\end{aligned}$$

And

$$
\begin{aligned}
E_1[(z-\mu_1)'D_2^{-1}(z-\mu_1)] =& E_1[tr((z-\mu_1)'D_2^{-1}(z-\mu_1))] \\
=& E_1[tr(D_2^{-1}(z-\mu_1)'(z-\mu_1))] \\
=& tr(E_1[D_2^{-1}(z-\mu_1)'(z-\mu_1)]) \\
=& tr(D_2^{-1}E_1[(z-\mu_1)'(z-\mu_1)]) \\
=& tr(D_2^{-1}D_1)
\end{aligned}
$$

Therefore,

$$
\int f_1 log f_2 = -\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_2|) - \frac{1}{2}\big(tr(D_2^{-1}D_1) + (\mu_1-\mu_2)'D_2^{-1}(\mu_1-\mu_2)\big)
$$

Similarly,

$$
\int f_2 log f_1 = -\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_1|) - \frac{1}{2}\big(tr(D_1^{-1}D_2) + (\mu_1-\mu_2)'D_1^{-1}(\mu_1-\mu_2)\big)
$$

Therefore, the function is:

$$
\begin{aligned}
&\int log(f_1)f_1 - \int log(f_2)f_1 + \int log(f_2)f_2 - \int log(f_1)f_2 \\
&= \big(-\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_1|) - \frac{p}{2}\big) \\
&- \big(-\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_2|) - \frac{1}{2}\big(tr(D_2^{-1}D_1) + (\mu_1-\mu_2)'D_2^{-1}(\mu_1-\mu_2)\big)\big) \\
&+ \big(-\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_2|) - \frac{p}{2}\big) \\
&- \big(-\frac{p}{2}log(2\pi) - \frac{1}{2}log(|D_1|) - \frac{1}{2}\big(tr(D_1^{-1}D_2) + (\mu_1-\mu_2)'D_1^{-1}(\mu_1-\mu_2)\big)\big) \\
&= -p + \frac{1}{2}tr(D_2^{-1}D_1) + \frac{1}{2}tr(D_1^{-1}D_2) + \frac{1}{2}(\mu_1-\mu_2)'(D_1^{-1}+D_2^{-1})(\mu_1-\mu_2)
\end{aligned}
$$

Within this equation, $\mu_1 = \beta_1 + \Gamma_1\alpha' x$, $\mu_2 = \beta_2 + \Gamma_2\alpha' x$. Therefore,

$$
\begin{aligned}
(\mu_1-\mu_2)'(D_1^{-1}+D_2^{-1})(\mu_1-\mu_2) =& \big(\beta_1-\beta_2+(\Gamma_1-\Gamma_2)\alpha' x\big)'(D_1^{-1}+D_2^{-1})\big(\beta_1-\beta_2+(\Gamma_1-\Gamma_2)\alpha' x\big) \\
=& (\beta_1-\beta_2)'(D_1^{-1}+D_2^{-1})(\beta_1-\beta_2) + (\beta_1-\beta_2)'(D_1^{-1}+D_2^{-1})(\Gamma_1-\Gamma_2)\alpha' x \\
&+ \big(\Gamma_1-\Gamma_2)\alpha' x\big)'(D_1^{-1}+D_2^{-1})\big(\beta_1-\beta_2\big) + \big((\Gamma_1-\Gamma_2)\alpha' x\big)'(D_1^{-1}+D_2^{-1})\big((\Gamma_1-\Gamma_2)\alpha' x\big) \\
&= (\beta_1-\beta_2)'(D_1^{-1}+D_2^{-1})(\beta_1-\beta_2) \\
&+ \big[(\beta_1-\beta_2)'(D_1^{-1}+D_2^{-1})(\Gamma_1-\Gamma_2) + (\Gamma_1-\Gamma_2)'(D_1^{-1}+D_2^{-1})(\beta_1-\beta_2)\big]x'\alpha \\
&+ \alpha' x\big((\Gamma_1-\Gamma_2)\big)'(D_1^{-1}+D_2^{-1})\big((\Gamma_1-\Gamma_2)\big)x'\alpha
\end{aligned}
$$

Let

$$
g(\alpha) = \int log(f_1)f_1 - \int log(f_2)f_1 + \int log(f_2)f_2 - \int log(f_1)f_2
$$

Then ?

$$
\begin{aligned}
\frac{\partial(g(\alpha))}{\partial\alpha} =& \big[(\Gamma_1-\Gamma_2)'(D_1^{-1}+D_2^{-1})(\beta_1-\beta_2)x'\alpha \\
&+ \alpha' x\big((\Gamma_1-\Gamma_2)\big)'(D_1^{-1}+D_2^{-1})\big((\Gamma_1-\Gamma_2)\big)x'\alpha\big]' \\
=& \big[(\Gamma_1-\Gamma_2)'(D_1^{-1}+D_2^{-1})(\beta_1-\beta_2)x'\big] + \\
& \alpha'\big[x\big((\Gamma_1-\Gamma_2)\big)'(D_1^{-1}+D_2^{-1})\big((\Gamma_1-\Gamma_2)\big)x'\big] + \\
& \alpha'\big[x\big((\Gamma_1-\Gamma_2)\big)'(D_1^{-1}+D_2^{-1})\big((\Gamma_1-\Gamma_2)\big)x'\big]'
\end{aligned}
$$

I am sorry I am a little bit confused about the derivative of $\frac{\partial X'AX}{\partial X}$. I feel it should be $(A+A')X$ instead of the first eigenvector of matrix A.

Since:

https://atmos.washington.edu/~dennis/MatrixCalculus.pdf

Proposition 8 *For the special case in which the scalar α is given by the quadratic form*

$$\alpha = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} \tag{43}$$

where $\mathbf{x}$ is $n \times 1$, $\mathbf{A}$ is $n \times n$, and $\mathbf{A}$ does not depend on $\mathbf{x}$, then

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \mathbf{x}^{\mathsf{T}}\left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right) \tag{44}$$

Proof*: By definition*

$$\alpha = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}x_i x_j \tag{45}$$

Differentiating with respect to the kth element of $\mathbf{x}$ we have

$$\frac{\partial \alpha}{\partial x_k} = \sum_{j=1}^{n} a_{kj}x_j + \sum_{i=1}^{n} a_{ik}x_i \tag{46}$$

for all $k = 1, 2, \ldots, n$, and consequently,

$$\frac{\partial \alpha}{\partial \mathbf{x}} = \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} + \mathbf{x}^{\mathsf{T}}\mathbf{A} = \mathbf{x}^{\mathsf{T}}\left(\mathbf{A}^{\mathsf{T}} + \mathbf{A}\right) \tag{47}$$

q.e.d.

If the above derivative is correct, then

$$\begin{gathered}\frac{\partial(g(\alpha))}{\partial \alpha} \equiv 0 \rightarrow \\ \big[(\Gamma_1 - \Gamma_2)'(D_1^{-1} + D_2^{-1})(\beta_1 - \beta_2)x'\big] + \\ \alpha'\big[x\big((\Gamma_1 - \Gamma_2)\big)'(D_1^{-1} + D_2^{-1})\big((\Gamma_1 - \Gamma_2)\big)x'\big] + \\ \alpha'\big[x\big((\Gamma_1 - \Gamma_2)\big)'(D_1^{-1} + D_2^{-1})\big((\Gamma_1 - \Gamma_2)\big)x'\big]' = 0\end{gathered}$$

That is

$$\alpha = -(B+B')^{-1}A$$

where $A = (\Gamma_1 - \Gamma_2)'(D_1^{-1} + D_2^{-1})(\beta_1 - \beta_2)x'$, $B = x\big((\Gamma_1 - \Gamma_2)\big)'(D_1^{-1} + D_2^{-1})\big((\Gamma_1 - \Gamma_2)\big)x'$