Statistical inference I HW 3

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Question 1

Here, we would like to prove that if $\int_\Omega f dv = 0$ for an a.e. non-negative function f, then it must be the case that $f = 0$, a.e. We could show that if $f \neq 0$, a.e, then $\int_\Omega f dv > 0$.

Let's make up a set $A = \{w \in \Omega : f(w) > 0\}$, and let $A_n = \{w \in \Omega : f(w) > \frac{1}{n}\}$, where $n \in \mathbb{N}$. We can show that $A = \cup_{n=1}^{\infty} A_n$ by:

> For $\forall x \in A$, we know that $x > 0$, then there exists a n, s.t. $x > \frac{1}{n}$. Then $x \in A_n = \{w \in \Omega : f(w) > \frac{1}{n}\}$. Therefore, $x \in \cup_{n=1}^{\infty} A_n$. That is, $\forall x \in A, x \in \cup_{n=1}^{\infty} A_n$, $A \subseteq \cup_{n=1}^{\infty} A_n$.
>
> For $\forall x \in \cup_{n=1}^{\infty} A_n$, we can easily know that $x > 0$, therefore, $x \in A$. Then $\cup_{n=1}^{\infty} A_n \subseteq A$. Therefore, $A = \cup_{n=1}^{\infty} A_n$.

We could let function $f > 0$ on a given set A, whose measure is bigger than 0 (i.e. $v(A) > 0$). That is:

$$f() : \begin{cases} f(w) = 0 & \text{if } w \notin A \\ f(w) > 0 & \text{if } w \in A \end{cases}$$

Since $A = \cup_{n=1}^{\infty} A_n$ and $v(A) > 0$, we can find at least one A_i, s.t. $v(A_i) > 0$.

Then let the A_n as the set whose measure is bigger than 0, and when $w \in A_n$, $f(w) > \frac{1}{n}$. Then

$$\begin{aligned} \int_\Omega f dv =& \int_{w \notin A} f dv + \int_{w \in A} f dv \\ =& 0 + \int_{w \in A} f dv \\ \geq& \int_{w \in A_n} f dv \\ >& \int_{w \in A_n} \frac{1}{n} dv, \text{ (since } f(w) > \frac{1}{n} \text{ on } A_n) \\ =& \frac{1}{n} v(A_n) \\ >& 0 \end{aligned}$$

It is a contradiction. Therefore, if $\int_\Omega f dv = 0$ for an a.e. non-negative finction f, then it must be that $f = 0$, a.e.

Question 2

Let $f_n(x) = (1 + \frac{x}{n})^{-n} cos^2(\frac{x}{n})$, $g_n(x) = (1 + \frac{x}{n})^{-n}$, $f(x) = e^{-x}$, $g(x) = e^{-x}$.

It is easy to know that:

- $f_n(x) \leq g_n(x)$
- $f_n(x) \to f(x)$ as $n \to \infty$
- $g_n(x) \to g(x)$ as $n \to \infty$

And $\int g_n = \int (1+\frac{x}{n})^{-n} dx = -\frac{n}{n-1}(1+\frac{x}{n})^{-n+1}|_0^\infty = \frac{n}{n-1}$ (when $n > 1$).

Then $\lim_{n\to\infty} \int g_n = \lim_{n\to\infty} \frac{n}{n-1} = 1$

And $\int \lim_{n\to\infty} g_n = \int g = \int e^{-x} = 1$.

Therefore, $\lim_{n\to\infty} \int g_n \to 1 = \int g$, $\int g_n \to \int g$

Therefore, we know that

- $|f_n| \leq g_n$
- $f_n(x) \to f(x)$, $g_n(x) \to g(x)$, as $n \to \infty$
- $\int g_n \to \int g$

According to the conclusion we proved in question 3 (the extension of dominated convergence theorem), $\int f_n \to \int f$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty (1+\frac{x}{n})^{-n} cos^2(\frac{x}{n}) dx &= \int_0^\infty \lim_{n\to\infty} (1+\frac{x}{n})^{-n} cos^2(\frac{x}{n}) dx \\
&= \int_0^\infty e^{-x} \times 1 dx \\
&= 1
\end{aligned}$$

Question 3

We have the facts that:

Since $f_n \to f$, then $\limsup_{n\to\infty} f_n = f = \liminf_{n\to\infty} f_n$

Since $g_n \to g$, then $\limsup_{n\to\infty} g_n = g = \liminf_{n\to\infty} g_n$

Since $\int g_n \to g_n$, then $\limsup_{n\to\infty} \int g_n = \int g = \liminf_{n\to\infty} \int g_n$

And we have $|f_n| \leq g$, therefore, both $g_n + f_n \geq 0$, $g_n - f_n \geq 0$. Therefore,

$$\begin{aligned}
\int \liminf_{n\to\infty} (g_n - f_n) &\leq \liminf_{n\to\infty} \int (g_n - f_n) \text{ (by Fatou Lemma)} \\
&= \liminf_{n\to\infty} \int g_n - \limsup_{n\to\infty} \int f_n \\
&= \int g - \limsup_{n\to\infty} \int f_n \text{ , (since} \int g_n \to \int g)
\end{aligned}$$

That is

$$\int g - \int f = \int (g - f) = \int \liminf_{n\to\infty} (g_n - f_n) \leq \int g - \limsup_{n\to\infty} \int f_n$$

$$\int f \geq \limsup_{n\to\infty} \int f_n$$

Similarly,

$$\begin{aligned}\int \liminf_{n\to\infty}(g_n + f_n) &\le \liminf_{n\to\infty} \int (g_n + f_n) \text{ (by Fatou Lemma)} \\ &= \liminf_{n\to\infty} \int g_n + \liminf_{n\to\infty} \int f_n \\ &= \int g + \liminf_{n\to\infty} \int f_n \text{ , (since} \int g_n \to \int g)\end{aligned}$$

That is

$$\int g + \int f = \int (g+f) = \int \liminf_{n\to\infty}(g_n + f_n) \le \int g + \liminf_{n\to\infty} \int f_n$$

$$\int f \le \liminf_{n\to\infty} \int f_n$$

Therefore,

$$\limsup_{n\to\infty} \int f_n \le \int f \le \liminf_{n\to\infty} \int f_n$$

while

$$\liminf_{n\to\infty} \int f_n \le \limsup_{n\to\infty} \int f_n$$

Therefore, by the sandwich theorem,

$$\limsup_{n\to\infty} \int f_n = \int f = \liminf_{n\to\infty} \int f_n$$

That is, $\int f_n < \infty$, it is integrable and $\int f_n \to \int f$

Question 4

We know that the estimation of β_1 in a simple linear regression model is:

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

In the estimation of $\hat{\beta}_1^*$, let set it as a function of β_1

$$f(\beta_1) = (\beta_1 - \hat{\beta}_1)^2 + \lambda|\beta_1| = \begin{cases} (\beta_1 - \hat{\beta}_1)^2 + \lambda\beta_1 & \text{when } \beta_1 \ge 0 \\ (\beta_1 - \hat{\beta}_1)^2 - \lambda\beta_1 & \text{when } \beta_1 < 0 \end{cases}$$

Therefore,

$$\frac{\partial f(\beta_1)}{\partial \beta_1} = \begin{cases} 2\beta_1 - 2\hat{\beta}_1 + \lambda & \text{when } \beta_1 \ge 0 \\ 2\beta_1 - 2\hat{\beta}_1 - \lambda & \text{when } \beta_1 < 0 \end{cases} \tag{1}$$

To solve $\frac{\partial f(\beta_1)}{\partial \beta_1} = 0$, we can get that

$$\beta_1 = \begin{cases} \hat{\beta}_1 - \frac{\lambda}{2} & \text{when } \beta_1 \ge 0 \\ \hat{\beta}_1 + \frac{\lambda}{2} & \text{when } \beta_1 < 0 \end{cases} \tag{2}$$

Therefore, the value of β_1 depends on the value of $\hat{\beta}_1$ and λ.

For the part $\beta_1 \geq 0$

- If $\hat{\beta}_1 \geq \frac{\lambda}{2} > 0$, the minimum value is at the $\beta_1 - \frac{\lambda}{2}$, the $f()$ function meets the minimum value. which is $\lambda\hat{\beta}_1 - \frac{\lambda^2}{4}$
- If $\hat{\beta}_1 < \frac{\lambda}{2}$, the minimum value is at the $\hat{\beta}_1 = 0$, the $f()$ function meets the minimum value, which is $\hat{\beta}_1^2$

For the part $\beta_1 < 0$

- If $\hat{\beta}_1 \leq -\frac{\lambda}{2} < 0$, the minimum value is at the $\beta_1 + \frac{\lambda}{2}$, the $f()$ function meets the minimum value, which is $\lambda\hat{\beta}_1 + \frac{3\lambda^2}{4}$
- If $-\frac{\lambda}{2} < \hat{\beta}_1$, the minimum value is at the $\hat{\beta}_1 = 0$, the $f()$ function meets the minimum value, which is $\hat{\beta}_1^2$

That is

- If $\hat{\beta}_1 \geq \frac{\lambda}{2}$, since then $\hat{\beta}_1^2 > \lambda\hat{\beta}_1 - \frac{\lambda^2}{4}$, the min of $f()$ is at point $\hat{\beta}_1 - \frac{\lambda}{2}$, and the min value is $\lambda\hat{\beta}_1 - \frac{\lambda^2}{4}$
- If $\hat{\beta}_1 \leq -\frac{\lambda}{2}$, since then $\hat{\beta}_1^2 > \lambda\hat{\beta}_1 + \frac{3\lambda^2}{4}$, the min of $f()$ is at point $\hat{\beta}_1 + \frac{\lambda}{2}$, and the min value is $\lambda\hat{\beta}_1 + \frac{3\lambda^2}{4}$
- If $\hat{\beta}_1 \in (-\frac{\lambda}{2}, \frac{\lambda}{2})$, then the min value is $\hat{\beta}_1^2$

Therefore, That is,

$$\hat{\beta}_1^* = \begin{cases} \lambda\hat{\beta}_1 - \frac{\lambda^2}{4} & \hat{\beta}_1 \geq \frac{\lambda}{2} \\ \lambda\hat{\beta}_1 + \frac{3\lambda^2}{4} & \hat{\beta}_1 \leq -\frac{\lambda}{2} \\ \hat{\beta}_1^2 & \hat{\beta}_1 \in (-\frac{\lambda}{2}, \frac{\lambda}{2}) \end{cases}$$

where $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$