new assumption and rho

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Calculate the $\rho(t)$ function when the following assumption is true.

$$\text{Condition A: } \lim_{dt \to 0} \left\{ P(C > t, T \geq t + dt) - P(C > t)P(T \geq t + dt) \right\} = 0$$

We know the $\rho(t)$ function is

$$\text{Condition B: } \rho(t) = \lim_{dt \to 0} \frac{P(t < T < t + dt | T > t, C \leq t)}{P(t < T < t + dt | T > t, C > t)}$$

However, this two conditions are not equivalent.

$$\text{Condition B } \subseteq \text{Condition A}$$

Our new assumption is looser than $\rho = 1$ in terms of independent relationship between death time and censor time.

Part 1

New assumption is true $\not\to \rho(t) = 1$.

Counter example:

$$S_{T,C}(x, y) = (1 - x)(1 - y)(1 + \frac{C}{8}xy(x - y)(x + y - 1))$$
$$S_T(x) = 1 - x, S_C(y) = 1 - y$$

where $(x, y) \in [0, 1] \times [0, 1], C \in [-4, 4]$. It satifies the condition A, since:

$$\begin{aligned}
P(T > x + y, C > x) =& (1 - x - y)(1 - x)(1 + \frac{C}{8}xy(x + y)(2x + y - 1)) \\
=& [(1 - x)^2 - (1 - x)y][1 + \frac{C}{8}\left\{(2x^3 - x^2)y + (3x^2 - x)y^2 + xy^3\right\}] \\
=& (1 - x)^2 - (1 - x)y \\
&+ \frac{C}{8}\left\{(1 - x)^2(2x^3 - x^2)y + (1 - x)^2(3x^2 - x)y^2 + x(1 - x)^2y^3\right\} \\
&- \frac{C}{8}\left\{(1 - x)(2x^3 - x^2)y^2 + (1 - x)(3x^2 - x)y^3 + x(1 - x)y^4\right\} \\
=& (1 - x)^2 + \frac{C}{8}\left[(1 - x)^2(2x^3 - x^2) - (1 - x)\right]y \\
&+ \frac{C}{8}\left[(1 - x)^2(3x^2 - x) - (1 - x)(2x^3 - x^2)\right]y^2 \\
&+ \frac{C}{8}\left[x(1 - x)^2 - (1 - x)(3x^2 - x)\right]y^3 - \frac{C}{8}x(1 - x)y^4 \\
=& (1 - x)^2 + A_1y + A_2y^2 + A_3y^3 + A_4y^4
\end{aligned}$$

where

- $A_1 = \frac{C}{8}[(1 - x)^2(2x^3 - x^2) - (1 - x)]$
- $A_2 = \frac{C}{8}[(1 - x)^2(3x^2 - x) - (1 - x)(2x^3 - x^2)]$
- $A_3 = \frac{C}{8}[x(1 - x)^2 - (1 - x)(3x^2 - x)]$

- $A_4 = -\frac{C}{8}[x(1-x)]$

And when $y \to 0$,

$$\lim_{y\to 0} P(T > x+y, C > x) = \lim_{y\to 0} \left\{(1-x)^2 + A_1 y + A_2 y^2 + A_3 y^3 + A_4 y^4\right\} = (1-x)^2 = P(T > t)P(C > t)$$

For $\rho(t)$ calculation,

$$\begin{aligned}
\rho(t) &= \lim_{dt\to 0} \frac{P(t < T < t+dt | T > t, C \le t)}{P(t < T < t+dt | T > t, C > t)} \\
&= \lim_{dt\to 0} \frac{\frac{P(t<T<t+dt, C\le t)}{P(T>t, C\le t)}}{\frac{P(t<T<t+dt, C>t)}{P(T>t, C>t)}} = \lim_{dt\to 0} \frac{P(t < T < t+dt, C \le t)}{P(t < T < t+dt, C > t)} \frac{P(T > t, C > t)}{P(T > t, C \le t)}
\end{aligned}$$

For $\frac{P(T>t,C>t)}{P(T>t,C\le t)}$, under our assumption,

$$\frac{P(T > t, C > t)}{P(T > t, C \le t)} = \frac{P(T > t)P(C > t)}{P(T > t) - P(T > t, C > t)} = \frac{P(T > t)P(C > t)}{P(T > t) - P(T > t)P(C > t)} = \frac{P(C > t)}{1 - P(C > t)}$$

when $P(T > t) \ne 0$

And we know that

$$\begin{aligned}
P(t < T < t+dt) &= dt \\
P(t < T < t+dt, C > t) &= P(T > t, C > t) - P(T > t+dt, C > t) \\
&= (1-t)^2 - (1-t)^2 - A_1 dt - A_2(dt)^2 - A_3(dt)^3 - A_4(dt)^4 \\
&= -A_1 dt - A_2(dt)^2 - A_3(dt)^3 - A_4(dt)^4
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{P(t < T < t+dt, C \le t)}{P(t < T < t+dt, C > t)} &= \frac{P(t < T < t+dt) - P(t < T < t+dt, C > t)}{P(t < T < t+dt, C > t)} \\
&= \frac{dt + A_1 dt + A_2(dt)^2 + A_3(dt)^3 + A_4(dt)^4}{-A_1 dt - A_2(dt)^2 - A_3(dt)^3 - A_4(dt)^4} \\
\lim_{dt\to 0} \frac{P(t < T < t+dt, C \le t)}{P(t < T < t+dt, C > t)} &= \lim_{dt\to 0} \frac{P(t < T < t+dt) - P(t < T < t+dt, C > t)}{P(t < T < t+dt, C > t)} \\
&= \lim_{dt\to 0} \frac{dt + A_1 dt + A_2(dt)^2 + A_3(dt)^3 + A_4(dt)^4}{-A_1 dt - A_2(dt)^2 - A_3(dt)^3 - A_4(dt)^4} \\
&= \lim_{dt\to 0} \frac{1 + A_1 + 2A_2(dt) + 3A_3(dt)^2 + 4A_4(dt)^3}{-A_1 - 2A_2(dt) - 3A_3(dt)^2 - 4A_4(dt)^3} \\
&= \frac{1 + A_1}{-A_1} = \frac{1 + \frac{C}{8}[(1-x)^2(2x^3 - x^2) - (1-x)]}{-\frac{C}{8}[(1-x)^2(2x^3 - x^2) - (1-x)]}
\end{aligned}$$

Therefore,

$$\rho(t) = \lim_{dt\to 0} \frac{P(t < T < t+dt, C \le t)}{P(t < T < t+dt, C > t)} \times \frac{P(C > t)}{1 - P(C > t)} \ne 1$$

Part 2

When $\rho(t) = 1$, our condition is true. Since

$$\begin{aligned}
\rho(t) &= \lim_{dt\to 0} \frac{P(t < T < t+dt | T > t, C \le t)}{P(t < T < t+dt | T > t, C > t)} \\
&= \lim_{dt\to 0} \frac{\frac{P(t<T<t+dt, C\le t)}{P(T>t, C\le t)}}{\frac{P(t<T<t+dt, C>t)}{P(T>t, C>t)}} \\
&= \lim_{dt\to 0} \frac{P(t < T < t+dt, C \le t)}{P(t < T < t+dt, C > t)} \frac{P(T > t, C > t)}{P(T > t, C \le t)} = 1
\end{aligned}$$

$\Longrightarrow$

$$\lim_{dt\to 0}\frac{P(T>t)-P(T>t+dt)-P(T>t,C>t)+P(T>t+dt,C>t)}{P(T>t,C>t)-P(T>t+dt,C>t)}\times\frac{P(T>t,C>t)}{P(T>t)-P(T>t,C>t)}=1$$

To make it looks more clean, let' replace the probablities with some other labels,

- $A = P(T > t)$
- $B = P(T > t + dt)$
- $C = P(T > t, C > t)$
- $D = P(T > t + dt, C > t)$

And $\lim_{dt\to 0}[A - B] = 0,\ \ \lim_{dt\to 0} P[C - D] = 0.$

Then the above function is

$$\begin{aligned}1 &= \lim_{dt\to 0}\frac{[A-B-(C-D)]C}{[C-D][A-C]}\\ &= \lim_{dt\to 0}\frac{AC-BC-C^2+CD}{AC-AD-C^2+CD}\\ &= \lim_{dt\to 0}\frac{(AC-C^2+CD)-AD+AD-BC}{(AC-C^2+CD)-AD}\\ &= \lim_{dt\to 0}\big\{1+\frac{AD-BC}{AC+CD-C^2-AD}\big\}\end{aligned}$$

Therefore,

$$\begin{aligned}&\lim_{dt\to 0}\frac{AD-BC}{AC+CD-C^2-AD}=0\\ &=\lim_{dt\to 0}\frac{AD-BC}{(C-D)(A-C)}\\ &=\lim_{dt\to 0}\frac{AD-BC}{C-D}\ \text{, since } A-C\neq 0\end{aligned}$$

Therefore, $AD - BC = o(C - D)$. Since $\lim_{dt\to 0} C - D = \lim_{dt\to 0} P(T > t, C > t) - P(T > t+dt, C > t) = 0$,

$$\lim_{dt\to 0} AD - BC = 0$$

$$\lim_{dt\to 0}\big\{P(T>t)P(T>t+dt,C>t)-P(T>t+dt)P(T>t,C>t)\big\}=0$$

$$\lim_{dt\to 0}\big\{\frac{P(T>t+dt,C>t)}{P(T>t+dt)}-\frac{P(T>t,C>t)}{P(T>t)}\big\}=0$$

$$\lim_{dt\to 0}\big\{P(C>t|T>t+dt)-P(C>t|T>t)\big\}=0$$

C and T are independent at the diagnoal neighborhood.