Look into $S_{p1}(t)$, $S_{p2}(t)$ and S_{km}

2019-07-28

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1. Ratio between $S_{p1}(t)$ and $S_{km}(t)$

The Kaplan-Meier equation:

$$\hat{S}_{km}(t) = \prod_{i=k}^{d(t)} \frac{n_i - 1}{n_i}$$

The Slud's equation:

$$\begin{aligned}
\hat{S}_{p1}(t) =& \frac{1}{N}\Big\{n(t) + \sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1}\Big\} \\
=& \prod_{i=1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} + \frac{1}{N}\sum_{k=1}^{d(t)} (\rho_k - 1) \prod_{i=k}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1}
\end{aligned}$$

where,

- N is the total subjects in the trial
- X_i is the time, $X_i = min(T_i, C_i)$, and X_i is ordered from 1 to N: $X_1 \leq X_2 \leq ... \leq X_N$
- d the total number of death
- $X_{(i)}$ is the death time, and ordered as $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(d)}$
- $n(t)$ is the number of subjects who are still alive at time t
- $d(t)$ is the number of total death people at time t
- n_i is the number of people who survived after the ith death time ($X_j \geq X_{(i)}$)
- c_i is the number of censer between the ith death time $X_{(i)}$ and the $(i+1)$th death time $X_{(i+1)}$

Let's look at the ration of $\hat{S}_{p1}(t)$ and $\hat{S}_{km}(t)$.

Let

- $\epsilon_{max} = max_{[i\in[1,d(t)]}(\delta_i)$, $\epsilon_{min} = min_{[i\in[1,d(t)]}(\delta_i)$
- $\epsilon_1 = min_{[i\in[1,d(t)]}(\frac{n_i}{\delta_i})$,
- $\epsilon_2 = max_{[i\in[1,d(t)]}(\frac{n_i}{\delta_i})$
- $\epsilon_3 = min_{[i\in[1,d(t)]}(|\frac{n_i}{\delta_i}|)$
- $\epsilon_4 = max_{[i\in[1,d(t)]}(|\frac{n_i}{\delta_i}|)$

For the ratio:

$$\begin{aligned}\frac{S_{p1}(t)}{S_{KM}(t)} =& \frac{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1} + \frac{1}{N}\sum_{k=1}^{d(t)}(\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \\ =& \frac{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} + \frac{\frac{1}{N}\sum_{k=1}^{d(t)}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}}\end{aligned}$$

Term 1 $= \frac{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}}$

Term 2 $= \frac{\frac{1}{N}\sum_{k=1}^{d(t)}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}}$

1.1 $\rho_i = 1$

If $\rho_i = 1$, clearly, $S_{p1}(t) = S_{km}(t)$

1.2 $\rho_i = 0$

When $\rho_i = 0$, which means that $f(t) = \psi(t) \to \int_0^\infty(t,s)ds = \int_t^\infty f(t,s)ds \to \int_0^t f(t,s)ds = 0$. That is, there is no censoring.

$S_{p1}(t) = \prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1} + \frac{1}{N}\sum_{k=1}^{d(t)}(\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1} = 1 - \frac{d(t)}{N}$

$S_{km}(t) = \prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}$.

And if there is no censor, $n_i = N - i$, $\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i} = \prod_{i=1}^{d(t)}\frac{N-i-1}{N-i} = \frac{N-1-d(t)}{N-1} = 1 - \frac{d(t)}{N-1}$.

Therefore, $S_{p1}(t) \approx S_{km}(t)$.

1.3 $\rho_i = 1 + \delta_i$, $\delta_i \in [-M, +M]$, $M \leq \infty$ and $\frac{\delta_i}{n_i} \to 0$

Previous, we set $|\delta_i| \to 0$. However, we do not need δ_i to have such a restrict condition.

Let's give δ_i a looser restriction and explore what condition we need.

For term 1:

$$\frac{\prod_{i=1}^{d(t)} \frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)} \frac{n_i-1}{n_i}} = \prod_{i=1}^{d(t)} \frac{n_i}{n_i+\delta_i} = \prod_{i=1}^{d(t)} (1 - \frac{1}{\frac{n_i}{\delta_i}+1})$$

And

$$\prod_{i=1}^{d(t)} (1 - \frac{1}{1+\epsilon_1}) \leq \prod_{i=1}^{d(t)} (1 - \frac{1}{\frac{n_i}{\delta_i}+1}) \leq \prod_{i=1}^{d(t)} (1 - \frac{1}{1+\epsilon_2})$$

$$(1 - \frac{1}{1+\epsilon_1})^{d(t)} \leq \prod_{i=1}^{d(t)} (1 - \frac{1}{\frac{n_i}{\delta_i}+1}) \leq (1 - \frac{1}{1+\epsilon_2})^{d(t)}$$

$$(1 - \frac{1}{1+\epsilon_1})^{(1+\epsilon_1)\frac{d(t)}{1+\epsilon_1}} \leq \prod_{i=1}^{d(t)} (1 - \frac{1}{\frac{n_i}{\delta_i}+1}) \leq (1 - \frac{1}{1+\epsilon_2})^{(1+\epsilon_2)\frac{d(t)}{1+\epsilon_2}}$$

where $\epsilon_1 = min_{[i\in[1,d(t)]}(\frac{n_i}{\delta_i})$, $\epsilon_2 = max_{[i\in[1,d(t)]}(\frac{n_i}{\delta_i})$

Since $\epsilon_1 = min_{i\in[1,d(t)]}(\frac{n_i}{\delta_i})$, $\epsilon_2 = max_{i\in[1,d(t)]}(\frac{n_i}{\delta_i})$, and $\frac{\delta_i}{n_i} \to 0$, $1+\epsilon_1 \to \infty$ and $1+\epsilon_2 \to \infty$. (if all $\delta_i > 0$, $1+\epsilon_1 \to +\infty$, $1+\epsilon_2 \to +\infty$; if all $\delta_i < 0$, $1+\epsilon_1 \to -\infty$, $1+\epsilon_2 \to -\infty$; otherwise, $1+\epsilon_1 \to -\infty$, $1+\epsilon_2 \to +\infty$. however, the sign of ∞ will not affect the result)

Recall:

$$\lim_{n\to\infty}(1+\frac{a}{n})^{bn} = e^{ab}$$

Therefore,

$$\lim_{\epsilon_1\to\infty}(1 - \frac{1}{1+\epsilon_1})^{(1+\epsilon_1)\frac{d(t)}{1+\epsilon_1}} = \lim_{\epsilon_1\to\infty}[\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_1}}$$

$$\lim_{\epsilon_2\to\infty}(1 - \frac{1}{1+\epsilon_2})^{(1+\epsilon_2)\frac{d(t)}{1+\epsilon_2}} = \lim_{\epsilon_2\to\infty}[\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_2}}$$

Therefore,

$$\lim_{\epsilon_1\to\infty}[\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_1}} \leq \lim_{\infty} \text{ term1 } \leq \lim_{\epsilon_2\to\infty}[\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_2}}$$

For the term 2:

$$\frac{1}{N}\sum_{k=1}^{d(t)}(D_{min})\prod_{i=k}^{d(t)}\frac{\frac{n_i-1}{n_i+\delta_i}}{\frac{n_i-1}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \leq \frac{\frac{1}{N}\sum_{k=1}^{d(t)}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \leq \frac{1}{N}\sum_{k=1}^{d(t)}(D_{max})\prod_{i=k}^{d(t)}\frac{\frac{n_i-1}{n_i+\delta_i}}{\frac{n_i-1}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}$$

where D_{min} is the smallest δ_i and D_{max} is the biggest δ_i.

Here, since D_{max}, D_{min} may goes to $+\infty, -\infty$, we may discuss different scenarios.

Scenario 1

$\rho_i = 1 + \delta_i$, $\delta_i \in [-M, +M]$ and $0 < M < \infty$.

In this scenario, $D_{min} < D_{max} < \infty$.

Let's look at the right part first:

$$
\begin{aligned}
\frac{1}{N}\sum_{k=1}^{d(t)}(D_{max})\prod_{i=k}^{d(t)}\frac{\frac{n_i-1}{n_i+\delta_i}}{\frac{n_i-1}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} =&\frac{1}{N}\sum_{k=1}^{d(t)}(D_{max})\prod_{i=k}^{d(t)}\frac{1}{1+\frac{\delta_i}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}\\
\le&\frac{1}{N}\sum_{k=1}^{d(t)}(D_{max})\prod_{i=k}^{d(t)}\frac{1}{1-1/\epsilon_3}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1},\text{ where }\epsilon_3=min_{[i\in[1,d(t)]}(|\frac{n_i}{\delta_i}|)\\
=&\frac{D_{max}}{N}\sum_{k=1}^{d(t)}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)-k+1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}\\
=&\frac{D_{max}}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k=1}^{d(t)}\Big[1-1/\epsilon_3\Big]^{k-1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}\\
\le&\frac{D_{max}}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k=1}^{d(t)}\Big[1-1/\epsilon_3\Big]^{k-1}\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]^{k-1}\\
=&\frac{D_{max}}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\frac{1-A^{d(t)}}{1-A}
\end{aligned}
$$

where $A=\Big[1-1/\epsilon_3\Big]\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]$.

And:

- $D_{max}<\infty$, $\frac{D_{max}}{N}\to 0$
- $\lim_{\epsilon_3\to\infty}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}=\lim_{\epsilon_3\to\infty}(1+\frac{1}{\epsilon_3-1})^{d(t)}=e^{\frac{d(t)}{\epsilon_3+1}}$
- Since $\epsilon_3\to\infty$, $n_{k-1}\to\infty$, $A=\Big[1-1/\epsilon_3\Big]\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]=\frac{\frac{\epsilon_3-1}{n_{k-1}-1}}{\frac{\epsilon_3}{n_{k-1}}}\to 1$, and $A<1$. Therefore, $1-A^{d(t)}\to 0$ and $1-A\to 0$. According to L'Hôpital's rule,

$$\lim_{A\to 1}\frac{1-A^{d(t)}}{1-A}=\lim_{A\to 1}d(t)A^{d(t)-1}\approx\lim_{A\to 1}d(t)A^{d(t)}$$

  And

$$
\begin{aligned}
A=&\Big[1-1/\epsilon_3\Big]\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]\\
=&1+\frac{1}{n_{k-1}+1}-\frac{1}{\epsilon_3}-\frac{1}{\epsilon_3(n_{k-1}+1)}\\
=&[1+\frac{1}{n_{k-1}+1}-\frac{1}{\epsilon_3}-\frac{1}{\epsilon_3(n_{k-1}+1)}]^{1/(\frac{1}{n_{k-1}+1}-\frac{1}{\epsilon_3}-\frac{1}{\epsilon_3(n_{k-1}+1)})\times(\frac{1}{n_{k-1}-1}-\frac{1}{\epsilon_3}-\frac{1}{\epsilon_3(n_{k-1}+1)})}\\
&\lim_{n_{k-1},\epsilon_3\to\infty}A=e^{(\frac{1}{n_{k-1}-1}-\frac{1}{\epsilon_3}-\frac{1}{\epsilon_3(n_{k-1}-1)})}=e^{\frac{\epsilon_3-n_{k-1}}{\epsilon_3(n_{k-1}-1)}}
\end{aligned}
$$

Therefore, limitation of right partof term 2 inequation is:

$$
\begin{aligned}
&\frac{D_{max}}{N}\times e^{\frac{d(t)}{\epsilon_3+1}}\times d(t)[e^{\frac{\epsilon_3-n_{k-1}}{\epsilon_3(n_{k-1}-1)}}]^{d(t)}\\
=&\frac{d(t)D_{max}}{N}e^{d(t)\times(\frac{1}{\epsilon_3+1}+\frac{\epsilon_3-n_{k-1}}{\epsilon_3(n_{k-1}-1)})}
\end{aligned}
$$

Therefore,

$$\lim \frac{S_{p1}(t)}{S_{KM}(t)} \leq \lim \Big\{ [\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_2}} + \frac{d(t)D_{max}}{N} e^{d(t)\times(\frac{1}{\epsilon_3+1}+\frac{\epsilon_3-n_{k-1}}{\epsilon_3(n_{k-1}-1)})} \Big\}$$

The left part is similar.

Therefore, if $\frac{d(t)}{\epsilon_i} \to 0$,

- $[\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_2}} \to 1$
- $d(t) \times \frac{1}{\epsilon_3+1} \to 0$, $d(t) \times (\frac{\epsilon_3-n_{k-1}}{\epsilon_3(n_{k-1}-1)}) \to 0$,

That is, $e^{d(t)\times(\frac{1}{\epsilon_3+1}+\frac{\epsilon_3-n_{k-1}}{\epsilon_3(n_{k-1}-1)})} \to 1$, or it is less than ∞,

Then we need $\frac{d(t)D_{max}}{N} \to 0$. If $d(t)$ and N have same order, we need $D_{max} \to 0$, that is, $\rho_i \to 1$

This means that, we need the majority of ρ_i goes to 1.

Scenario 2

There are finity amound of δ_i, whose absolute value is large.

Let's divide subjects into two sets: E_1, E_2.

- When $i \in E_1$, $\delta_i \in (-d_1, d_1)$, where $d_1 \to 0$
- When $i \in E_2$, $\delta_i \in (-d_2, d_2)$, where $0 < d_2 < \infty$. The size of set E_2 is finite, say, $|E_2| = l$.

Then the $\frac{n_i}{\delta_i} \to \infty$ is still true, therefore, the derivative of term 1 will not change.

$$\lim_{\epsilon_1 \to \infty} [\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_1}} \leq \lim_{\infty} \text{ term1 } \leq \lim_{\epsilon_2 \to \infty} [\frac{1}{e}]^{\frac{d(t)}{1+\epsilon_2}}$$

Regrad to term 2:

$$\begin{aligned} &\frac{\frac{1}{N}\sum_{k=1}^{d(t)}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} = \\ &\frac{\frac{1}{N}\sum_{k\in E_1}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} + \frac{\frac{1}{N}\sum_{k\in E_2}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \end{aligned}$$

- For the points in E_1, the previous derivation is still true

$$\begin{aligned} \frac{\frac{1}{N}\sum_{k\in E_1}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \leq &\frac{\frac{1}{N}\sum_{k\in E_1}(d_1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \\ = &\frac{d_1}{N}\sum_{k\in E_1}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)-k+1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \\ \leq &\frac{d_1}{N}\sum_{k=1}^{d(t)}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)-k+1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \end{aligned}$$

which throught the above derivation goes to 0 as d_1 goes to 0.

- For the points in E_2

$$\begin{aligned}
\frac{\frac{1}{N}\sum_{k\in E_2}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \le & \frac{\frac{1}{N}\sum_{k\in E_2}(d_2)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \\
= & \frac{d_2}{N}\sum_{k\in E_2}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)-k+1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \\
= & \frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_2}\Big[1-1/\epsilon_3\Big]^{k-1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \\
\le & \frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_2}\Big[1-1/\epsilon_3\Big]^{k-1}\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]^{k-1} \\
= & \frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_2}A^{k-1}
\end{aligned}$$

where $A=\Big[1-1/\epsilon_3\Big]\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]$.

Previously, we know that A goes to 1 from the right. As k is larger, A^k is smaller. Therefore,

$$\begin{aligned}
\frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_2}A^{k-1} \le & \frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\Big[|E_2|A^{k_0-1}\Big], \text{ where } k_0 \text{ is the smallest value in } E_2 \\
= & \frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\Big[lA^{k_0-1}\Big]
\end{aligned}$$

within it,

- A^{k_0-1} is a constant, since $0<A<1$, $k_0<d_2<\infty$.
- $\lim_{\epsilon_3\to\infty}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}=\lim_{\epsilon_3\to\infty}(1+\frac{1}{\epsilon_3-1})^{d(t)}=e^{\frac{d(t)}{\epsilon_3+1}}$, when $\frac{d(t)}{\epsilon_3}<\infty$, it is a constant.
- $d_2\times l<\infty$ is a constant.

Therefore, $\frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\Big[lA^{k_0-1}\Big]=\frac{C}{N}\to 0$ as $N\to\infty$, where $C=d_2\times l\times A^{k_0-1}\times e^{\frac{d(t)}{\epsilon_3+1}}$.

Therefore, similarly, intotal, when $\frac{d(t)}{\epsilon_3}\to 0$, term1 goes to 1 and term 2 goes to 0. S_{p1} is close to S_{km}

Scenario 3

Let's divide subjects into three sets: E_1, E_2, E_3

- When $i\in E_1$, $\delta_i\in(-d_1,d_1)$, where $d_1\to 0$
- When $i\in E_2$, $\delta_i\in(-d_2,d_2)$, where $0<d_2<\infty$. The size of set E_2 is finite, say, $|E_2|=l$.
- When $i\in E_3$, $\delta_i\to\infty$, where $0<d_2<\infty$, and the order is smaller than n_i, that is, $\frac{\delta_i}{n_i}\to 0$. The size of set E_3 is finite, say, $|E_3|=l_3$.

Then the $\frac{n_i}{\delta_i}\to\infty$ is still true, therefore, the derivative of term 1 will not change.

Regard to the term 2:

$$\frac{\frac{1}{N}\sum_{k=1}^{d(t)}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} =$$

$$\frac{\frac{1}{N}\sum_{k\in E_1}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} + \frac{\frac{1}{N}\sum_{k\in E_2}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} + \frac{\frac{1}{N}\sum_{k\in E_3}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}}$$

* When $i \in E_1$, the derivation is similar as scenario 1,

$$\frac{\frac{1}{N}\sum_{k\in E_1}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \to 0$$

* When $i \in E_2$, the derivation is similar as scenario 2,

$$\frac{\frac{1}{N}\sum_{k\in E_2}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \to 0$$

* When $i \in E_3$, the derivation is similar as scenario 3,

$$\begin{aligned}
\frac{\frac{1}{N}\sum_{k\in E_3}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \leq & \frac{\frac{1}{N}\sum_{k\in E_3}(d_3)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \\
= & \frac{d_3}{N}\sum_{k\in E_3}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)-k+1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \\
= & \frac{d_3}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_3}\Big[1-1/\epsilon_3\Big]^{k-1}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \\
\leq & \frac{d_3}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_3}\Big[1-1/\epsilon_3\Big]^{k-1}\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]^{k-1} \\
= & \frac{d_2}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_3}A^{k-1}
\end{aligned}$$

where $A = \Big[1-1/\epsilon_3\Big]\Big[\frac{n_{k-1}}{n_{k-1}-1}\Big]$. δ_i is bounded by d_3 and d_3 goes to ∞.

Previously, we know that A goes to 1 from the right. As k is larger, A^k is smaller. Therefore,

$$\begin{aligned}
\frac{d_3}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k\in E_3}A^{k-1} \leq & \frac{d_3}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\Big[|E_3|A^{k_0-1}\Big], \text{ where } k_0 \text{ is the smallest value in } E_3 \\
= & \frac{d_3}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\Big[l_3A^{k_0-1}\Big]
\end{aligned}$$

within it,

- A^{k_0-1} is a constant, since $0 < A < 1$, $k_0 < d_3 < \infty$.
- $\lim_{\epsilon_3\to\infty}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)} = \lim_{\epsilon_3\to\infty}(1+\frac{1}{\epsilon_3-1})^{d(t)} = e^{\frac{d(t)}{\epsilon_3+1}}$, when $\frac{d(t)}{\epsilon_3} < \infty$, it is a constant.
- $d_2 \times l < \infty$ is a constant.

Therefore, $\frac{d_3}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\Big[l_3A^{k_0-1}\Big] = \frac{d_3}{N}C \to 0$ as $N \to \infty$, where $C = l_3 \times A^{k_0-1} \times e^{\frac{d(t)}{\epsilon_3+1}}$. This is because $\frac{d_3}{N} \to 0$

Conclusion

In a conclusion, when for

- any i, $\frac{\rho_i - 1}{n_i} \to 0$,
- all $\rho_i \to 1$, except finite number of is, s.t. $|\rho_i| > 1$ ($|\rho_i|$ can be finite value or infinity)

The $S_p(t)/S_{km} \to 1$

2. Difference between $S_{p1}(t)$ and $S_{km}(t)$

Similar way as ratio, however, ratio is easier to show.

$$S_{p1}(t) - S_{km}(t) = S_{km}(t)\Big[\frac{S_{p1}(t)}{S_{km}(t)} - 1\Big]$$

As $\frac{S_{p1}(t)}{S_{km}(t)} \to 1$, $S_{p1}(t) - S_{km}(t) \to 0$.

3. Difference between $S_{p1}(t)$ and $S_{p2}(t)$

There was a wrong calculation in the difference between $S_{p1}(t)$ and $S_{p2}(t)$.

We find that there are less differneces between $S_{p1}(t)$ and $S_{p2}(t)$ from the simulations. Let's look at their distance.

$$\hat{S}_{p1}(t) = \frac{1}{N}\Big\{n(t) + \sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)} (1 - \frac{\rho_i}{n_i + \rho_i - 1})\Big\}$$

$$\hat{S}_{p2}(t) = \frac{1}{N}\Big\{n(t) + c_{d(t)-1} + \sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho(X_i)}{n_i})\Big\}$$

Within those two equations, the different parts are:

$$\begin{aligned}
\hat{S}_{p2}(t) - \hat{S}_{p1}(t) =& \frac{1}{N}c_{d(t)-1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1}(1 - \frac{\rho_i}{n_i}) - \frac{1}{N}\sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)}(1 - \frac{\rho_i}{n_i + \rho_i - 1}) \\
=& \frac{1}{N}c_{d(t)-1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1}(1 - \frac{\rho_i}{n_i}) - \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1}(1 - \frac{\rho_i}{n_i})(\frac{1 - \frac{\rho_i}{n_i + \rho_i - 1}}{1 - \frac{\rho_i}{n_i}}) \\
&- \frac{1}{N}c_{d(t)-1}(1 - \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1}) \\
=& \frac{1}{N}c_{d(t)-1}\frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1}(1 - \frac{\rho_i}{n_i})(1 - \prod_{i=k+1}^{d(t)-1}\frac{1 - \frac{\rho_i}{n_i + \rho_i - 1}}{1 - \frac{\rho_i}{n_i}}) \text{ equation (*)}
\end{aligned}$$

3.1 If all $\rho_i \leq 1$:

Let $c_{max} = max(c_k),\ k \in [0, d(t)-2]$.

$$equation(*) \leq \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} + \frac{c_{max}}{N} \sum_{k=0}^{d(t)-2} \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i}) \Big(1 - \prod_{i=k+1}^{d(t)-1} \frac{1 - \frac{\rho_i}{n_i + \rho_i - 1}}{1 - \frac{\rho_i}{n_i}}\Big)$$

And

$$\begin{aligned} \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i}) \leq & (1 - \frac{\rho_m}{n_m})^{d(t)-1-k} \\ = & [e^{-\rho_m}]^{\frac{d(t)-1-k}{n_m}},\ \text{as } n_m \to \infty \end{aligned}$$

where m is the index that achieve the min $\frac{\rho_i}{n_i},\ i \in [1, d(t)-1]$

$$\begin{aligned} 1 - \prod_{i=k+1}^{d(t)-1} \frac{1 - \frac{\rho_i}{n_i + \rho_i - 1}}{1 - \frac{\rho_i}{n_i}} = & 1 - \prod_{i=k+1}^{d(t)-1} \frac{(n_i - 1)n_i}{(n_i - 1)n_i + \rho_i - \rho_i^2} \\ = & 1 - \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i - \rho_i^2}{(n_i - 1)n_i + \rho_i - \rho_i^2}) \\ \leq & 1 - (1 - \frac{\rho_{d(t)-1} - \rho_{d(t)-1}^2}{(n_{d(t)-1} - 1)n_{d(t)-1} + \rho_{d(t)-1} - \rho_{d(t)-1}^2})^{d(t)-1-k} \end{aligned}$$

the inequality is because $\rho_i \in (0,1), 0 < \rho_i - \rho_i^2 < 1$, and $\frac{(n_i-1)n_i}{(n_i-1)n_i + \rho_i - \rho_i^2}$ is an monotone increasing function w.r.t n_i (the affect of ρ_i is too small comparing to ρ_i, we can just treat it as a small fixed value.)

$$\begin{aligned} & \lim_{n_i \to \infty} \Big[1 - (1 - \frac{\rho_{d(t)-1} - \rho_{d(t)-1}^2}{(n_{d(t)-1} - 1)n_{d(t)-1} + \rho_{d(t)-1} - \rho_{d(t)-1}^2})^{d(t)-1-k}\Big] \\ = & \lim_{n_i \to \infty} \Big[1 - (1 - \frac{\rho_{d(t)-1} - \rho_{d(t)-1}^2}{(n_{d(t)-1} - 1)n_{d(t)-1}})^{d(t)-1-k}\Big] \\ = & 1 - \Big(\frac{1}{e}\Big)^{\rho_{d(t)-1} - \rho_{d(t)-1}^2 \times \frac{d(t)-1-k}{(n_{d(t)-1} - 1)n_{d(t)-1}}} \end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{n_i\to\infty} equation(*) \leq & \lim_{n_i\to\infty} \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1}+\rho_{d(t)-1}-1} \\
& + \lim_{n_i\to\infty} \frac{c_{max}}{N} \sum_{k=0}^{d(t)-2} \Big(\frac{1}{e}\Big)^{\rho_m \times \frac{d(t)-1-k}{n_m}} \Big\{1-\Big(\frac{1}{e}\Big)^{\rho_{d(t)-1}-\rho^2_{d(t)-1}\times\frac{d(t)-1-k}{(n_{d(t)-1}-1)n_{d(t)-1}}}\Big\} \\
= & \lim_{n_i\to\infty} \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1}+\rho_{d(t)-1}-1} \\
& + \lim_{n_i\to\infty} \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} [\Big(\frac{1}{e}\Big)^{\frac{\rho_m}{n_m}}]^i \Big\{1-[\Big(\frac{1}{e}\Big)^{\frac{\rho_{d(t)-1}-\rho^2_{d(t)-1}}{n^2_{d(t)-1}-n_{d(t)-1}}}]^i\Big\} \\
= & \lim_{n_i\to\infty} \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1}+\rho_{d(t)-1}-1} \\
& + \lim_{n_i\to\infty} \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} A^i \times (1-B^i) \\
= & \lim_{n_i\to\infty} \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1}+\rho_{d(t)-1}-1} + \frac{c_{max}}{N}\Big[\frac{A}{1-A} - \frac{AB}{1-AB}\Big]
\end{aligned}$$

where $A = \left(\frac{1}{e}\right)^{\frac{\rho_m}{n_m}}, B = \left(\frac{1}{e}\right)^{\frac{\rho_{d(t)-1}-\rho^2_{d(t)-1}}{n^2_{d(t)-1}-n_{d(t)-1}}}$

As $n \to \infty, \rho \in (0,1)$, $A \to 1, B \to 1$, and $A < 1, B < 1$

$$\begin{aligned}
\lim_{A\to 1, B\to 1} \frac{A}{1-A} - \frac{AB}{1-AB} = & \lim_{A\to 1, B\to 1} \frac{A-AB}{1-A-AB+A^2B} \\
= & \lim_{A\to 1, B\to 1} \frac{1-B}{-1-B+2AB}, \text{ L'Hôpital's rule, derive r.t A} \\
= & \lim_{A\to 1, B\to 1} \frac{1}{-1+2A}, \text{ L'Hôpital's rule, derive r.t B} \\
= & 1
\end{aligned}$$

Therefore,

$$equation(*) \leq \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1}+\rho_{d(t)-1}-1} + \frac{c_{max}}{N}$$

3.2 If $\rho_i \to \infty$ and $\frac{n_i}{\rho_i} \to \infty$:

Since,

$$\hat{S}_{p2}(t) - \hat{S}_{p1}(t) = \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1}+\rho_{d(t)-1}-1} + \frac{1}{N} \sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1-\frac{\rho_i}{n_i})(1-\prod_{i=k+1}^{d(t)-1} \frac{1-\frac{\rho_i}{n_i+\rho_i-1}}{1-\frac{\rho_i}{n_i}})$$

And

$$1 - \prod_{i=k+1}^{d(t)-1} \frac{1-\frac{\rho_i}{n_i+\rho_i-1}}{1-\frac{\rho_i}{n_i}} \leq 0$$

Therefore, let's look at the absolute value of $\hat{S}_{p2}(t) - \hat{S}_{p1}(t)$

$$|\hat{S}_{p2}(t) - \hat{S}_{p1}(t)| = \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} + \frac{1}{N} \sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i})(-1 + \prod_{i=k+1}^{d(t)-1} \frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}})$$

where,

$$\begin{aligned}\lim_{N,n,\rho\to\infty} \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} &= \lim_{N,n,\rho\to\infty} \frac{1}{N} c_{d(t)-1} \frac{1}{\frac{n_{d(t)-1}}{\rho_{d(t)-1}} + 1 - \frac{1}{\rho_{d(t)-1}}} \\ &= 0\end{aligned}$$

since $\frac{n_{d(t)-1}}{\rho_{d(t)-1}} \to \infty$. And in the second term within the equation:

$$\frac{1}{N} \sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i})(-1 + \prod_{i=k+1}^{d(t)-1} \frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}})$$

Similar with pervious result:

$$\begin{aligned}\prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i}) &\leq (1 - \frac{\rho_m}{n_m})^{d(t)-1-k} \\ &= (1 - \frac{1}{\frac{n_m}{\rho_m}})^{\frac{n_m}{\rho_m}\frac{\rho_m}{n_m}(d(t)-1-k)} \\ &= [e^{-1}]^{\frac{\rho_m}{n_m}(d(t)-1-k)}, \text{ as } n_m \to \infty\end{aligned}$$

where m is the index that achieve the min $\frac{\rho_i}{n_i}$, $i \in [1, d(t)-1]$

And

$$\begin{aligned}\prod_{i=k+1}^{d(t)-1} \frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}} - 1 &= \prod_{i=k+1}^{d(t)-1} \frac{(n_i-1)n_i}{(n_i-1)n_i + \rho_i - \rho_i^2} - 1 \\ &= \prod_{i=k+1}^{d(t)-1} (1 + \frac{\rho_i^2 - \rho_i}{(n_i-1)n_i + \rho_i - \rho_i^2}) - 1 \\ &\leq (1 + \frac{\rho_l^2 - \rho_l}{(n_l-1)n_l + \rho_l - \rho_l^2})^{d(t)-1-k} - 1\end{aligned}$$

where l is the index that achieve the max $\frac{(n_i-1)n_i}{(n_i-1)n_i+\rho_i-\rho_i^2}$, $i \in [1, d(t)-1]$, and the limit:

$$\begin{aligned}&\lim_{n,\rho\to\infty} (1 + \frac{\rho_l^2 - \rho_l}{(n_l-1)n_l + \rho_l - \rho_l^2})^{d(t)-1-k} - 1 \\ &= e^{\frac{\rho_l^2-\rho_l}{(n_l-1)n_l+\rho_l-\rho_l^2} \times \left(d(t)-1-k\right)} - 1\end{aligned}$$

Therefore, the limit of the second term less than:

$$\lim_{n,\rho\to\infty} \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} \left\{ [e^{-1}]^{\frac{\rho_m}{n_m} i} \times \left[e^{\frac{\rho_l^2-\rho_l}{(n_l-1)n_l+\rho_l-\rho_l^2} \times i} - 1 \right] \right\}$$

Let $A = e^{-\frac{\rho_m}{n_m}}$, $B = e^{\frac{\rho_l^2 - \rho_l}{(n_l-1)n_l+\rho_l-\rho_l^2}}$, then $A \to 1$ and $B \to 1$, $A < 1, B > 1$. The limit equation becomes:

$$\begin{aligned}\lim_{n,\rho\to\infty} \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} A^i(B^i - 1) &= \lim_{n,\rho\to\infty} \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} (AB)^i - \lim_{n,\rho\to\infty} \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} A^i \\ &= \lim_{n,\rho\to\infty} \frac{c_{max}}{N} [\frac{AB}{1-AB} - \frac{A}{1-A}] \\ &= \lim_{n,\rho\to\infty} \frac{c_{max}}{N}\end{aligned}$$

Therefore, the results are similar as $|\rho| < 1$, $\lim |\hat{S}_{p1} - \hat{S}_{p2}| < \lim \frac{c_{max}}{N}$

3.3 If all $\rho_i \to \infty$, and $\frac{n_i}{\rho_i} \to H < \infty$:

$$\begin{aligned}\lim \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} &= \lim \frac{1}{N} c_{d(t)-1} \frac{1}{\frac{n_{d(t)-1}}{\rho_{d(t)-1}} + 1 - \frac{1}{\rho_{d(t)-1}}} \\ &= \lim \frac{1}{N} c_{d(t)-1} \frac{1}{H+1}\end{aligned}$$

In the second part $\lim \frac{c_{max}}{N} \sum_{k=0}^{d(t)-2} \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i})\Big(1 - \prod_{i=k+1}^{d(t)-1} \frac{1-\frac{\rho_i}{n_i+\rho_i-1}}{1-\frac{\rho_i}{n_i}}\Big)$, $1 - \frac{\rho_i}{n_i} < 1 - h$, where $h = min(\frac{\rho_i}{n_i})$.

$$\begin{aligned}\lim \prod_{i=k+1}^{d(t)-1} \frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}} &= \lim \prod_{i=k+1}^{d(t)-1} \frac{(n_i-1)n_i}{(n_i-1)n_i + \rho_i - \rho_i^2} \\ &= \lim \prod_{i=k+1}^{d(t)-1} \frac{n_i^2}{n_i^2 - \rho_i^2} \\ &\leq \lim \prod_{i=k+1}^{d(t)-1} \frac{1}{1-(h')^2}\end{aligned}$$

where $h' = max(\rho_i/n_i)$. Therefore,

$$\begin{aligned}\lim \frac{c_{max}}{N} \sum_{k=0}^{d(t)-2} \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i})\Big(1 - \prod_{i=k+1}^{d(t)-1} \frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}}\Big) &\leq \lim \frac{c_{max}}{N} \sum_{k=0}^{d(t)-2} \prod_{i=k+1}^{d(t)-1} (1-h) \prod_{i=k+1}^{d(t)-1} \frac{1}{1-(h')^2} \\ &= \lim \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} (1-h)^i \Big[\frac{1}{1-(h')^2}\Big]^i \\ &= \lim \frac{c_{max}}{N} \sum_{i=1}^{d(t)-1} \Big[\frac{1-h}{1-(h')^2}\Big]^i \\ &= \lim \frac{c_{max}}{N} \frac{1-h}{h-(h')^2}\end{aligned}$$

where $h = min(\frac{\rho_i}{n_i})$, $h' = max(\rho_i/n_i)$.

Therefore, in this scenario, the difference between $\hat{S}_{p1}$ and $\hat{S}_{p2}$ may be larger.