Previous results summary

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Notation

The two main equations in Slud'spaper:

$$\rho(t) = \left[\left\{f(t)/\phi(t)\right\} - 1\right]\left[\left\{S(t)/S_X(t)\right\}\right]^{-1}$$

where,

- $f(t,s)$ is the joint distribtuion of surivival time and censor time.
- $f(t) = \int f(t,s)ds$
- $S(t) = \int_t^\infty f(t)dt$
- $S_X(t) = P(T > t, C > t)$
- $\phi(t) = \int_t^\infty f(t,s)ds = \int_t^\infty f(s|t)f(t)ds = f(t)\int_t^\infty f(s|t)ds = f(t)P(C > t|T = t)$. It can be treated as the derivation of $\Psi(t)$, where

$$\begin{aligned}\Psi(t) = &\int_0^t \psi(t)dt = \int_0^t \int_t^\infty f(s,t)dsdt \\ =&P(T < t, C > t) = P(T < t < C) \\ =&P(min(T,C) < t, C > t) \\ =&P(X < t, I = 1) \text{ time before t and not censor}\end{aligned},$$

$\rho(t)$ shows the proportion of death hazard at time t, conditioning on censored before t or after t, that is:

$$
\begin{aligned}
\rho(t) =& \lim_{\delta \to 0} \frac{P(t < T < t+\delta | T > t, C \leq t)}{P(t < T < t+\delta | T > t, C < t)} \\
=& \lim_{\delta \to 0} \frac{\frac{P(t<T<t+\delta, C \leq t)}{P(T>t, C\leq t)}}{\frac{P(t<T<t+\delta, C>t)}{P(T>t, C>t)}} \\
=& \lim_{\delta \to 0} \frac{\frac{P(t<T<t+\delta, C \leq t)}{P(t<T<t+\delta, C>t)}}{\frac{P(T>t, C\leq t)}{P(T>t, C>t)}} \\
=& \lim_{\delta \to 0} \frac{\frac{P(t<T<t+\delta, C \leq t)}{P(t<T<t+\delta, C>t)} + 1 - 1}{\frac{P(T>t, C\leq t)}{P(T>t, C>t)} + 1 - 1} \\
=& \lim_{\delta \to 0} \frac{\frac{P(t<T<t+\delta)}{P(t<T<t+\delta, C>t)} - 1}{\frac{P(T>t)}{P(T>t, C>t)} - 1} \\
=& \lim_{\delta \to 0} \frac{1/P(C > t | t < T < t+\delta) - 1}{S(t)/S_x(t) - 1} \\
=& \frac{1/P(C > t | T = t) - 1}{S(t)/S_x(t) - 1} \\
=& \Big[\Big\{ f(t)/\phi(t) \Big\} - 1 \Big] \Big[\Big\{ S(t)/S_X(t) - 1 \Big\} \Big]^{-1}
\end{aligned}
$$

The property of $\rho(t)$

Since

- $f(t)/\phi(t) = \frac{f(t)}{f(t)P(C>t|T=t)} = \frac{1}{P(C>t|T=t)} \geq 1$
- $S(t)/S_X(t) = \frac{P(T>t)}{P(T>t, C>t)} \geq 1$

Therefore, $\rho(t) \in [0, \infty]$

- When $\rho(t) = 0$, $f(t)/\phi(t) = 1$, that is $P(C > t | T = t) = 1$, which means that there is no censoring.
- When $\rho(t) = 1$, $1/P(C > t | T = t) = P(T > t)/P(T > t, C > t)$, which is $P(C > t | T = t) = P(C > t | T > t)$. That is, the C and T are independent. When $\rho(t) = 1$ the survival time and the censor time are independent.
- When $\rho(t) > 1$, we have a positive dependence between censor and death. The larger the ρ is, the larger the dependence is.

$$
\hat{S}_p(t) = \frac{1}{N} \Big\{ n(t) + \sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \Big\}
$$

where,

- N is the total subjects in the trial

- X_i is the time, $X_i = min(T_i, C_i)$, and X_i is ordered from 1 to N: $X_1 \leq X_2 \leq ... \leq X_N$
- d the total number of death
- $X_{(i)}$ is the death time, and ordered as $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(d)}$
- $n(t)$ is the number of subjects who are still alive at time t
- $d(t)$ is the number of total death people at time t
- n_i is the number of people who survived after the ith death time ($X_j \geq X_{(i)}$)
- c_i is the number of censer between the ith death time $X_{(i)}$ and the $(i+1)$th death time $X_{(i+1)}$

Derivation of $S_{p2}(t)$

To make it is easy to distinguish Slud's equation and the new derivated equation, let's call Slud's one as $S_{p1}(t)$ and the new one $S_{p2}(t)$.

$$\hat{S}_{p1}(t) = \frac{1}{N}\Big\{n(t) + \sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1}\Big\}$$

$$\hat{S}_{p2}(t) = \frac{1}{N}\Big\{n(t) + c_{d(t)-1} + \sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho(X_i)}{n_i})\Big\}$$

To get $\hat{S}_{p2}(t)$, let's begin with:

$$P(T > X_{(j)}) = P(T > X_{(j)}, C < X_{(j-1)}) + P(T > X_{(j)}, C > X_{(j-1)})$$
$$P(T > X_{(j)}) = P(T > X_{(j)}, C > X_{(j)}) + P(T > X_{(j)}, C < X_{(j)})$$

Therefore,

$$P(T > X_{(j)}, C < X_{(j-1)}) + P(T > X_{(j)}, C > X_{(j-1)}) = P(T > X_{(j)}, C > X_{(j)}) + P(T > X_{(j)}, C < X_{(j)})$$

Besides, the term

$$\begin{aligned}
P(T > X_{(j)}, C < X_{(j-1)}) =&P(T > X_{(j)}, T > X_{(j-1)}, C < X_{(j-1)})\\
=&P(T > X_{(j)} | T > X_{(j-1)}, C < X_{(j-1)}) \times P(T > X_{(j-1)}, C < X_{(j-1)})\\
=&(1 - P(T < X_{(j)} | T > X_{(j-1)}, C < X_{(j-1)})) \times P(T > X_{(j-1)}, C < X_{(j-1)})
\end{aligned}$$

The term

$$\begin{aligned}
P(T > X_{(j)}, C > X_{(j-1)}) =&P(T > X_{(j)}, T > X_{(j-1)}, C > X_{(j-1)})\\
=&P(T > X_{(j)} | T > X_{(j-1)} C > X_{(j-1)}) P(T > X_{(j-1)} C > X_{(j-1)})\\
=&P(T > X_{(j)} | T > X_{(j-1)}, C > X_{(j-1)}) \times P(W > X_{(j-1)})\\
\approx&\frac{n_{j-1} - 1}{n_{j-1}} \times \frac{n_{j-1}}{N} = \frac{n_{j-1} - 1}{N}
\end{aligned}$$

Therefore,

$$\begin{aligned}
P(T > X_{(j)}, C < X_{(j)}) =& P(T > X_{(j)}, C < X_{(j-1)}) + P(T > X_{(j)}, C > X_{(j-1)}) - P(T > X_{(j)}, C > X_{(j)}) \\
=&(1 - P(T < X_{(j)}|T > X_{(j-1)}, C < X_{(j-1)})) \times P(T > X_{(j-1)}, C < X_{(j-1)}) \\
&+ P(T > X_{(j)}|T > X_{(j-1)}, C > X_{(j-1)}) \times P(W > X_{(j-1)}) - P(T > X_{(j)}, C > X_{(j)})
\end{aligned}$$

As $X_{(j)} - X_{(j-1)} \to 0$,

- $\rho_i = \frac{P(T<X_{(j)}|T>X_{(j-1)},C<X_{(j-1)})}{P(T<X_{(j)}|T>X_{(j-1)},C>X_{(j-1)})}$, then $P(T < X_{(j)}|T > X_{(j-1)}, C < X_{(j-1)}) = \rho_i P(T < X_{(j)}|T > X_{(j-1)}, C > X_{(j-1)})$
- The $P(T < X_{(j)}|T > X_{(j-1)}, C > X_{(j-1)})$ can be estimated as $\frac{1}{n_{j-1}}$
- $P(T > X_{(j-1)}, C > X_{(j-1)}) \approx P(W > X_{(j-1)}) = \frac{n_{j-1}}{N}$
- $P(T > X_{(j)}, C > X_{(j)}) \approx P(W > X_{(j)}) = \frac{n_j}{N}$
- $P(T > X_{(j)}|T > X_{(j-1)}, C > X_{(j-1)}) \approx \frac{n_{j-1}-1}{n_{j-1}}$

Therefore, as $X_{(j)} - X_{(j-1)} \to 0$,

$$\begin{aligned}
P(T > X_{(j)}, C < X_{(j)}) =&(1 - P(T < X_{(j)}|T > X_{(j-1)}, C < X_{(j-1)})) \times P(T > X_{(j-1)}, C < X_{(j-1)}) \\
&+ P(T > X_{(j)}|T > X_{(j-1)}, C > X_{(j-1)}) \times P(W > X_{(j-1)}) - P(T > X_{(j)}, C > X_{(j)}) \\
\approx&(1 - \frac{\rho_{j-1}}{n_{j-1}}) \times P(T > X_{(j-1)}, C < X_{(j-1)}) + \frac{n_{j-1} - 1}{n_{j-1}} \frac{n_{j-1}}{N} - \frac{n_j}{N} \\
=&(1 - \frac{\rho_{j-1}}{n_{j-1}}) \times P(T > X_{(j-1)}, C < X_{(j-1)}) + \frac{c_{j-1}}{N}
\end{aligned}$$

Let $Y_j = P(T > X_{(j)}, C < X_{(j)})$, $A_j = 1 - \frac{\rho_j}{n_j}$, $B_j = \frac{c_j}{N}$ to make it is easier to see.

Since $Y_0 = P(T > X_{(0)}, C < X_{(0)})$, we can treat it as something that will never happen and probability $= 0$. Therefore,

- $Y_1 = A_0Y_0 + B_0 = B_0$, since $Y_0 = 0$. Begin with 0 since $k = 0$ in the equation 0
- $Y_2 = A_1Y_1 + B_1 = A_1B_0 + B_1$
- $Y_3 = A_2Y_2 + B_2 = A_2A_1B_0 + A_2B_1 + B_2$
- ...
- Therefore the equation is:

$$\begin{aligned}
Y_n =& B_0 \prod_{i=1}^{n-1} A_i + B_1 \prod_{i=2}^{n-1} A_i + B_2 \prod_{i=3}^{n-1} A_i + ...B_{n-2} \prod_{i=n-1}^{n-1} A_i + B_{n-1} \\
=&[\sum_{k=0}^{n-2} B_k \prod_{i=k+1}^{n-1} A_i] + B_{n-1} \\
=&[\sum_{k=0}^{n-2} \frac{c_k}{N} \prod_{i=k+1}^{n-1} (1 - \frac{\rho(X_i)}{n_i})] + \frac{c_{n-1}}{N}
\end{aligned}$$

Therefor, the $\hat{S}_{p2}(t)$ equation is:

$$\hat{S}_{p2}(t) = \frac{1}{N}\Big\{n(t) + c_{d(t)-1} + \sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho(X_i)}{n_i})\Big\}$$

The second version of $\hat{S}_{p1}(t)$

We know that:

1. $c_k = n_k - n_{k+1} - 1$

2. $$\begin{aligned} N =& n(t) + d(t) + \sum_{i=0} d(t) - 1 c_k \\ =& n(t) + d(t) + \sum_{i=0} d(t) - 1(n_k - n_{k+1} - 1) \\ =& n(t) + d(t) + n_0 - n_{d(t)} - d(t) \\ =& n(t) + N + 1 - n_{d(t)} \end{aligned}$$

Therefore, $n(t) = n_{d(t)} - 1$

Therefore,

$$\begin{aligned} \hat{S}_p(t) =& \frac{1}{N}\Big\{ n(t) + \sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \Big\} \\ =& \frac{1}{N}\Big\{ n_{d(t)} - 1 + \sum_{k=0}^{d(t)-1} (n_k - n_{k+1} - 1) \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \Big\} \\ =& \frac{1}{N}\Big\{ n_{d(t)} - 1 + \sum_{k=0}^{d(t)-1} (n_k - 1) \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} - \sum_{k=0}^{d(t)-1} n_{k+1} \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \Big\} \\ =& \frac{1}{N}\Big\{ n_{d(t)} - 1 + \sum_{k=1}^{d(t)-1} (n_k - 1) \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} + (n_0 - 1) \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} - \sum_{k=0}^{d(t)-1} n_{k+1} \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \Big\} \\ =& \frac{1}{N}\Big\{ n_{d(t)} - 1 + \sum_{k=1}^{d(t)-1} (n_k - 1) \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} + N \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} - \sum_{k=0}^{d(t)-1} n_{k+1} \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \Big\} \end{aligned}$$

Besides,

$$\sum_{k=0}^{d(t)-1} n_{k+1} \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} = \sum_{k=1}^{d(t)} n_k \prod_{i=k}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1}$$

And

$$\begin{aligned} \sum_{k=1}^{d(t)-1} (n_k - 1) \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} &= \sum_{k=1}^{d(t)-1} (n_k + \rho_k - 1) \frac{n_k - 1}{n_k + \rho_k - 1} \prod_{i=k+1}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \\ &= \sum_{k=1}^{d(t)-1} (n_k + \rho_k - 1) \prod_{i=k}^{d(t)} \frac{n_i - 1}{n_i + \rho_i - 1} \end{aligned}$$

$$n_{d(t)} - 1 = (n_{d(t)} + \rho_{d(t)} - 1) \frac{n_{d(t)} - 1}{n_{d(t)} + \rho_{d(t)} - 1}$$

Therefore,

$$\frac{1}{N}\Big\{n_{d(t)}-1+\sum_{k=1}^{d(t)-1}(n_k-1)\prod_{i=k+1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}+N\prod_{i=k+1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}-\sum_{k=0}^{d(t)-1}n_{k+1}\prod_{i=k+1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}\Big\}$$

$$=\frac{1}{N}\Big\{(n_{d(t)}+\rho_{d(t)}-1)\frac{n_{d(t)}-1}{n_{d(t)}+\rho_{d(t)}-1}$$

$$+\sum_{k=1}^{d(t)-1}(n_k+\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}+N\prod_{i=k+1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}-\sum_{k=1}^{d(t)}n_k\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}\Big\}$$

$$=\frac{1}{N}\Big\{\sum_{k=1}^{d(t)}(n_k+\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}+N\prod_{i=k+1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}-\sum_{k=1}^{d(t)}n_k\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}\Big\}$$

$$=\frac{1}{N}\Big\{\sum_{k=1}^{d(t)}(\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}+N\prod_{i=k+1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}\Big\}$$

$$=\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}+\frac{1}{N}\sum_{k=1}^{d(t)}(\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}$$

Therefore, the another version of $S_{p1}(t)$ is:

$$\hat{S}_{p1}(t)=\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}+\frac{1}{N}\sum_{k=1}^{d(t)}(\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}$$

The second version of $\hat{S}_{p2}(t)$

$$\begin{aligned}\hat{S}_{p2}(t)=&\frac{1}{N}\Big\{n(t)+c_{d(t)-1}+\sum_{k=0}^{d(t)-2}c_k\prod_{i=k+1}^{d(t)-1}(1-\frac{\rho(X_i)}{n_i})\Big\}\\ =&\frac{1}{N}\Big\{n_{d(t)}-1+c_{d(t)-1}+\sum_{k=0}^{d(t)-2}(n_k-n_{k+1}-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}\Big\}\\ =&\frac{1}{N}\Big\{n_{d(t)}-1+c_{d(t)-1}+\sum_{k=0}^{d(t)-2}(n_k-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}-\sum_{k=0}^{d(t)-2}n_{k+1}\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}\Big\}\end{aligned}$$

And

1. $$\begin{aligned}\sum_{k=0}^{d(t)-2}n_{k+1}\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}&=\sum_{k=1}^{d(t)-1}n_k\prod_{i=k}^{d(t)-1}\frac{n_i-\rho_i}{n_i}\\ &=\sum_{k=1}^{d(t)-2}n_k\frac{n_k-\rho_k}{n_k}\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}+(n_{d(t)-1}-\rho_{d(t)-1})\\ &=\sum_{k=1}^{d(t)-2}(n_k-\rho_k)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}+(n_{d(t)-1}-\rho_{d(t)-1})\end{aligned}$$

2. $$\sum_{k=0}^{d(t)-2}(n_k-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}=\sum_{k=1}^{d(t)-2}(n_k-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}+(n_0-1)\prod_{i=1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}$$

Therefore,

$$
\begin{aligned}
S_{p2}(t) =&\frac{1}{N}\Big\{n(t) + c_{d(t)-1} + \sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1}(1-\frac{\rho_i}{n_i})\Big\}\\
=&\frac{1}{N}\Big\{n_{d(t)} - 1 + c_{d(t)-1} + \sum_{k=1}^{d(t)-2}(n_k-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i}\\
&+(n_0-1)\prod_{i=1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} - \sum_{k=1}^{d(t)-2}(n_k-\rho_k)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} - (n_{d(t)-1}-\rho_{d(t)-1})\Big\}\\
=&\frac{1}{N}\Big\{n_{d(t)} - 1 + c_{d(t)-1} + \sum_{k=1}^{d(t)-2}(\rho_i-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} + N\prod_{i=1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} - (n_{d(t)-1}-\rho_{d(t)-1})\Big\}\\
=&\prod_{i=1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} + \frac{1}{N}\Big\{\sum_{k=1}^{d(t)-2}(\rho_i-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} + n_{d(t)} - n_{d(t)-1} + n_{d(t)-1} - n_{d(t)} - 1 + \rho_{d(t)-1} - 1\Big\}\\
=&\prod_{i=1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} + \frac{1}{N}\Big\{\sum_{k=1}^{d(t)-2}(\rho_i-1)\prod_{i=k+1}^{d(t)-1}\frac{n_i-\rho_i}{n_i} + \rho_{d(t)-1} - 2\Big\}
\end{aligned}
$$

Ratio of $S_{p1}(t)$ and $S_{kM}(t)$

Let's show when $\rho_i \to 0$, $S_{p1}(t) \to S_{kM}(t)$, which we can illustrate by $\frac{S_{p1}(t)}{S_{kM}(t)} \to 1$

Let

- Each $\rho_i = 1 + \delta_i$, where $\delta_i \in [-\epsilon, \epsilon]$, $\epsilon \to 0$
- $\epsilon_{max} = max_{[i\in[1,d(t)]}(\delta_i)$, $\epsilon_{min} = min_{[i\in[1,d(t)]}(\delta_i)$
- $\epsilon_1 = min_{[i\in[1,d(t)]}(\frac{n_i}{\delta_i})$,
- $\epsilon_2 = max_{[i\in[1,d(t)]}(\frac{n_i}{\delta_i})$
- $\epsilon_3 = min_{[i\in[1,d(t)]}(|\frac{n_i}{\delta_i}|)$
- $\epsilon_4 = max_{[i\in[1,d(t)]}(|\frac{n_i}{\delta_i}|)$

For the ratio:

$$
\begin{aligned}
\frac{S_{p1}(t)}{S_{KM}(t)} =&\frac{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1} + \frac{1}{N}\sum_{k=1}^{d(t)}(\rho_k-1)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\rho_i-1}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}}\\
=&\frac{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} + \frac{\frac{1}{N}\sum_{k=1}^{d(t)}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}}
\end{aligned}
$$

The first term equals to:

$$
\frac{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} = \prod_{i=1}^{d(t)}\frac{n_i}{n_i+\delta_i} = \prod_{i=1}^{d(t)}(1-\frac{1}{\frac{n_i}{\delta_i}+1})
$$

And

$$
\prod_{i=1}^{d(t)}(1-\frac{1}{1+\epsilon_1}) \le \prod_{i=1}^{d(t)}(1-\frac{1}{\frac{n_i}{\delta_i}+1}) \le \prod_{i=1}^{d(t)}(1-\frac{1}{1+\epsilon_2})
$$

$$(1-\frac{1}{1+\epsilon_1})^{d(t)} \leq \prod_{i=1}^{d(t)}(1-\frac{1}{\frac{n_i}{\delta_i}+1}) \leq (1-\frac{1}{1+\epsilon_2})^{d(t)}$$

$$(1-\frac{1}{1+\epsilon_1})^{(1+\epsilon_1)\frac{d(t)}{1+\epsilon_1}} \leq \prod_{i=1}^{d(t)}(1-\frac{1}{\frac{n_i}{\delta_i}+1}) \leq (1-\frac{1}{1+\epsilon_2})^{(1+\epsilon_2)\frac{d(t)}{1+\epsilon_2}}$$

where $\epsilon_1 = min_{[i\in[1,d(t)]]}(\frac{n_i}{\delta_i})$, $\epsilon_2 = max_{[i\in[1,d(t)]]}(\frac{n_i}{\delta_i})$

Since $\epsilon_1 = min_{i\in[1,d(t)]]}(\frac{n_i}{\delta_i})$, $\epsilon_2 = max_{i\in[1,d(t)]]}(\frac{n_i}{\delta_i})$, and $\delta_i \to 0$, $1+\epsilon_1 \to \infty$ and $1+\epsilon_2 \to \infty$. (if all $\delta_i > 0$, $1+\epsilon_1 \to +\infty$, $1+\epsilon_2 \to +\infty$; if all $\delta_i < 0$, $1+\epsilon_1 \to -\infty$, $1+\epsilon_2 \to -\infty$; otherwise, $1+\epsilon_1 \to -\infty$, $1+\epsilon_2 \to +\infty$. however, the sign of ∞ will not affect the result)

Recall:

$$\lim_{n\to\infty}(1+\frac{a}{n})^{bn} = e^{ab}$$

Therefore,

$$\lim_{\epsilon_1\to\infty}(1-\frac{1}{1+\epsilon_1})^{(1+\epsilon_1)\frac{d(t)}{1+\epsilon_1}} = \lim_{\epsilon_1\to\infty}[e^{-d(t)}]^{\frac{1}{1+\epsilon_1}} = 1$$

$$\lim_{\epsilon_2\to\infty}(1-\frac{1}{1+\epsilon_2})^{(1+\epsilon_2)\frac{d(t)}{1+\epsilon_2}} = \lim_{\epsilon_2\to\infty}[e^{-d(t)}]^{\frac{1}{1+\epsilon_2}} = 1$$

Therefore, based on the squeeze theorem, $\lim_{\frac{n_i}{\delta_i}\to\infty}\prod_{i=1}^{d(t)}(1-\frac{1}{\frac{n_i}{\delta_i}+1}) = 1$. That is, $\lim_{\frac{n_i}{\delta_i}\to\infty}\frac{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} = 1$

For the term 2:

$$\frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{min})\prod_{i=k}^{d(t)}\frac{\frac{n_i-1}{n_i+\delta_i}}{\frac{n_i-1}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \leq \frac{\frac{1}{N}\sum_{k=1}^{d(t)}(\delta_i)\prod_{i=k}^{d(t)}\frac{n_i-1}{n_i+\delta_i}}{\prod_{i=1}^{d(t)}\frac{n_i-1}{n_i}} \leq \frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{max})\prod_{i=k}^{d(t)}\frac{\frac{n_i-1}{n_i+\delta_i}}{\frac{n_i-1}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}$$

Let's look at the right part first:

$$\begin{aligned}
\frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{max})\prod_{i=k}^{d(t)}\frac{\frac{n_i-1}{n_i+\delta_i}}{\frac{n_i-1}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} &= \frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{max})\prod_{i=k}^{d(t)}\frac{1}{1+\frac{\delta_i}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \\
&\leq \frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{max})\prod_{i=k}^{d(t)}\frac{1}{1-1/\epsilon_3}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1},\ \text{where } \epsilon_3 = min_{[i\in[1,d(t)]]}(|\frac{n_i}{\delta_i}|) \\
&= \frac{\epsilon_{max}}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k=1}^{d(t)}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}
\end{aligned}$$

Since n_i is a decreasing function,

$$\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \leq \prod_{i=1}^{k-1}\frac{n_{d(t)}}{n_{d(t)}-1} = [\frac{n_{d(t)}}{n_{d(t)}-1}]^{k-1}$$

Therefore,

$$\begin{aligned}
\frac{\epsilon_{max}}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k=1}^{d(t)}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} &\leq \frac{\epsilon_{max}}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}\sum_{k=1}^{d(t)}[\frac{n_{d(t)}}{n_{d(t)}-1}]^{k-1} \\
&= (n_{d(t)}-1)\frac{\epsilon_{max}}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}[(\frac{n_{d(t)}}{n_{d(t)}-1})^{d(t)}-1] \\
&= \frac{\epsilon_{max}(n_{d(t)}-1)}{N}\Big[\frac{n_{d(t)}}{(1-1/\epsilon_3)(n_{d(t)}-1)}\Big]^{d(t)} - \frac{\epsilon_{max}(n_{d(t)}-1)}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)}
\end{aligned}$$

If $\epsilon_{max} = o(\Big[\frac{n_{d(t)}}{(1-1/\epsilon_3)(n_{d(t)}-1)}\Big]^{-d(t)})$, as $n \to \infty$, $\frac{\epsilon_{max}(n_{d(t)-1})}{N}\Big[\frac{n_{d(t)}}{(1-1/\epsilon_3)(n_{d(t)}-1)}\Big]^{d(t)} \to 0$, $\frac{\epsilon_{max}(n_{d(t)-1})}{N}\Big[\frac{1}{1-1/\epsilon_3}\Big]^{d(t)} \to 0$. The above function goes to 0.

For the left part

$$
\begin{aligned}
\frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{min})\prod_{i=k}^{d(t)}\frac{\frac{n_i-1}{n_i+\delta_i}}{\frac{n_i-1}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} =& \frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{min})\prod_{i=k}^{d(t)}\frac{1}{1+\frac{\delta_i}{n_i}}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \\
\geq& \frac{1}{N}\sum_{k=1}^{d(t)}(\epsilon_{max})\prod_{i=k}^{d(t)}\frac{1}{1+1/\epsilon_4}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}, \text{ where } \epsilon_4 = max_{[i\in[1,d(t)]}(|\frac{n_i}{\delta_i}|) \\
=& \frac{\epsilon_{min}}{N}\Big[\frac{1}{1+1/\epsilon_4}\Big]^{d(t)}\sum_{k=1}^{d(t)}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1}
\end{aligned}
$$

Since n_i is a decreasing function,

$$
\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \geq \prod_{i=1}^{k-1}\frac{n_1}{n_1-1} = [\frac{n_1}{n_1-1}]^{k-1}
$$

Therefore,

$$
\begin{aligned}
\frac{\epsilon_{min}}{N}\Big[\frac{1}{1+1/\epsilon_4}\Big]^{d(t)}\sum_{k=1}^{d(t)}\prod_{i=1}^{k-1}\frac{n_i}{n_i-1} \geq& \frac{\epsilon_{min}}{N}\Big[\frac{1}{1+1/\epsilon_4}\Big]^{d(t)}\sum_{k=1}^{d(t)}[\frac{n_1}{n_1-1}]^{k-1} \\
=& (n_1-1)\frac{\epsilon_{min}}{N}\Big[\frac{1}{1+1/\epsilon_4}\Big]^{d(t)}[(\frac{n_1}{n_1-1})^{d(t)}-1] \\
=& \frac{\epsilon_{min}(n_1-1)}{N}\Big[\frac{n_1}{(1+1/\epsilon_4)(n_1-1)}\Big]^{d(t)} - \frac{\epsilon_{min}(n_1-1)}{N}\Big[\frac{1}{1+1/\epsilon_4}\Big]^{d(t)}
\end{aligned}
$$

If $\epsilon_{min} = o(\Big[\frac{n_1}{(1+1/\epsilon_4)(n_1-1)}\Big]^{-d(t)})$, as $n \to \infty$, the above function goes to 0.

Above all, as $n \to \infty$ and $\epsilon_{max} = o(\Big[\frac{n_{d(t)}}{(1-1/\epsilon_3)(n_{d(t)}-1)}\Big]^{-d(t)})$, $\epsilon_{min} = o(\Big[\frac{n_1}{(1+1/\epsilon_4)(n_1-1)}\Big]^{-d(t)})$, then the term 2 goes to 0.

That is, combine the two parts,

$$
\lim_{\rho_i \to 1}\frac{S_{p1}(t)}{S_{kM}(t)} = 1 + 0 \to 1
$$

Difference between $S_{p1}(t)$ and $S_{p2}(t)$

We find that there are less differneces between $S_{p1}(t)$ and $S_{p2}(t)$ from the simulations. Let's look at their distance.

$$
\hat{S}_p(t) = \frac{1}{N}\Big\{n(t) + \sum_{k=0}^{d(t)-1}c_k\prod_{i=k+1}^{d(t)}(1-\frac{\rho_i}{n_i+\rho_i-1})\Big\}
$$

$$
\hat{S}_{p,corrected}(t) = \frac{1}{N}\Big\{n(t) + c_{d(t)-1} + \sum_{k=0}^{d(t)-2}c_k\prod_{i=k+1}^{d(t)-1}(1-\frac{\rho(X_i)}{n_i})\Big\}
$$

Within those two equations, the different parts are:

$$part1 = \frac{1}{N}\sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)} (1 - \frac{\rho_i}{n_i + \rho_i - 1})$$

$$part2 = \frac{1}{N} c_{d(t)-1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i})$$

$$\begin{aligned}
part2 - part1 =& \frac{1}{N} c_{d(t)-1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i}) - \frac{1}{N}\sum_{k=0}^{d(t)-1} c_k \prod_{i=k+1}^{d(t)} (1 - \frac{\rho_i}{n_i + \rho_i - 1}) \\
=& \frac{1}{N} c_{d(t)-1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i}) - \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i})(\frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}}) \\
&- \frac{1}{N} c_{d(t)-1}(1 - \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1}) \\
=& \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i}) \prod_{i=k+1}^{d(t)-1} (1 - \frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}}) \text{ equation (*)}
\end{aligned}$$

If $\rho_i \leq 1$ (which is our case), $\frac{\rho_i}{n_i+\rho_i-1} \geq \frac{\rho_i}{n_i}$

Then

$$\begin{aligned}
\text{equation (*)} \leq& \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} + \frac{1}{N}\sum_{k=0}^{d(t)-2} c_k \prod_{i=k+1}^{d(t)-1} (1 - \frac{\rho_i}{n_i}) A^{d(t)-1-k} \\
\leq& \frac{1}{N} c_{d(t)-1} \frac{\rho_{d(t)-1}}{n_{d(t)-1} + \rho_{d(t)-1} - 1} + (part2 - \frac{1}{N} c_{d(t)-1}) A \text{ equation (**)}
\end{aligned}$$

where A $= \max(1 - \frac{1-\frac{\rho_i}{n_i+\rho_i-1}}{1-\frac{\rho_i}{n_i}})$, $i \in [k+1, d(t) - 1]$.

Then we need to find A, or $B = min(\frac{1-\frac{\rho_i}{n_i+\rho_i-1}}{1-\frac{\rho_i}{n_i}})$.

$$\frac{1 - \frac{\rho_i}{n_i+\rho_i-1}}{1 - \frac{\rho_i}{n_i}} = \frac{(n_i - 1)n_i}{(n_i - 1)n_i + \rho_i - \rho_i^2}$$

With fixed n_i, when $\rho_i = 0.5$, B get the min value; with fixed ρ_i, B is a monotune increasing function w.r.t n_i ($n_i \geq 1$).

Therefore, $A = \frac{(n_{d(t)-1} - 1)n_{d(t)-1}}{(n_{d(t)-1} - 1)n_{d(t)-1} + \rho_{0.5} - \rho_{0.5}^2}$, $\rho_{0.5}$ is the ρ value that closed to 0.5.

And equation (**) $\approx$ part2 A, since $c_{d(t)-1}/N$ is usually small.

Ususally, part2 is small (less than 0.01) and A is small and less than 1. Therefore, the difference between Slud's equation and corrected Slud's equation is relatively small (usually less than 0.01).