Dr. Ying's Example, with parameter

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CDF with parameter

In Zhiliang Ying's paper, the Joint CDF is:

$$S(T \geq x, C \geq y) = \begin{cases} e^{-x} e^{-(e^y-1)\left((x-y)^2+1\right)} & x \geq y \\ e^{-x} e^{-(e^y-1)} & x < y \end{cases}$$

Let's add a parameter θ in the model:

$$S(T \geq x, C \geq y) = \begin{cases} e^{-\theta x} e^{-(e^{\theta y}-1)\left((x-y)^2+1\right)} & x \geq y \\ e^{-\theta x} e^{-(e^{\theta y}-1)} & x < y \end{cases}$$

PDF with parameter

Since

$$\begin{aligned} P(T \geq x, C \geq y) =& P(T \geq x) - P(T \geq x, C < y) \\ =& P(T \geq x) - (P(C < y) - P(C < y, T < x)) \\ =& P(T \geq x) + P(C \geq y) + P(C < y, T < x) - 1 \end{aligned}$$

Then

$$P(C < y, T < x) = 1 + P(T \geq x, C \geq y) - P(T \geq x) - P(C \geq y)$$

When $x \geq y$, the pdf is

$$\frac{\partial}{\partial x} P(C < y, T < x) = \theta e^{-\theta x} + \Big(- (e^{\theta y} - 1)(2x - 2y) - \theta \Big) e^{-\theta x} e^{-(e^{\theta y}-1)\left((x-y)^2+1\right)}$$

$$\begin{aligned} f_{T,C}(x,y) =& \frac{\partial}{\partial x \partial y} P(C < y, T < x) \\ =& ((2x-2y)(1-e^{\theta y}) - \theta)((2y-2x)(1-e^{\theta y}) - \theta(y^2 - 2xy + x^2 + 1)e^{\theta y})e^{(y^2-2xy+x^2+1)(1-e^{\theta y})-\theta x} \\ &+ (\theta(2y-2x)e^{\theta y} - 2(1-e^{\theta y}))e^{(y^2-2xy+x^2+1)(1-e^{\theta y})-\theta x} \end{aligned}$$

When $x < y$, the pdf is

$$\frac{\partial}{\partial x} P(C < y, T < x) = \theta e^{-\theta x} - \theta e^{-\theta x} e^{-(e^{\theta y}-1)}$$

$$\frac{\partial}{\partial x \partial y} P(C < y, T < x) = \theta^2 e^{-e^{\theta y} + \theta y - \theta x + 1}$$

Therefore, the total pdf is

$$f_{T,C}(x,y) = \begin{cases} ((2x-2y)(1-e^{\theta y}) - \theta)((2y-2x)(1-e^{\theta y}) - \theta(y^2 - 2xy + x^2 + 1)e^{\theta y})e^{(y^2-2xy+x^2+1)(1-e^{\theta y})-\theta x} \\ +(\theta(2y-2x)e^{\theta y} - 2(1-e^{\theta y}))e^{(y^2-2xy+x^2+1)(1-e^{\theta y})-\theta x} \\ \\ \theta^2 e^{-e^{\theta y} + \theta y - \theta x + 1} \end{cases}$$

$m()$ function, $\rho()$ function

Then

$$S_T(t) = P(T > t) = P(T > t, C > 0) = e^{-\theta t} e^{-(e^{\theta 0}-1)\left((t-0)^2+1\right)} = e^{-\theta t}$$

$$f_T(t) = \frac{\partial}{\partial t}(1 - S_T(t)) = \frac{\partial}{\partial t}(1 - e^{-\theta t}) = \theta e^{-\theta t}$$

$$S_x(t) = P(T > t, C > t) = e^{-\theta t} e^{-(e^{\theta t}-1)} = e^{-e^{\theta t}-\theta t+1}$$

$$f_x(t) = \frac{\partial}{\partial t}(1 - S_x(t)) = 1 - e^{-e^{\theta t}-\theta t+1} = \theta(1 + e^{\theta t})e^{-e^{\theta t}-\theta t+1}$$

$$\psi(t) = \int_t^{\infty} f(t, c)dc = \int_t^{\infty} \theta^2 e^{-e^{\theta c}+\theta c-\theta t+1} dc = \theta e^{-e^{\theta t}-\theta t+1}$$

Therefore, the $m()$ function is:

$$m(t) = \frac{\lambda_F(t)}{\lambda_H(t)} = \frac{f_T(t)}{S_T(t)} / \frac{f_X(t)}{S_X(t)} = \frac{\theta e^{-\theta t}}{e^{-\theta t}} / \frac{\theta(1 + e^{\theta t})e^{-e^{\theta t}-\theta t+1}}{e^{-e^{\theta t}-\theta t+1}} = \frac{1}{1 + e^{\theta t}}$$

And for the $\rho()$ function,

$$\begin{aligned} \rho =& \frac{f(t)/\psi(t) - 1}{S(t)/S_x(t) - 1} \\ =& \frac{\theta e^{-\theta t}/(\theta e^{-e^{\theta t}-\theta t+1}) - 1}{e^{-\theta t}/e^{-e^{\theta t}-\theta t+1} - 1} \\ =& 1 \end{aligned}$$

Simulation

Data generation

Censoring percentage

$$\begin{aligned} P(T < C) =& \int_0^{\infty} \int_0^{y} \theta^2 e^{-e^{\theta y}+\theta y-\theta x+1} dxdy \\ =& \int_0^{\infty} \theta(e^{\theta y} - 1)e^{-e^{\theta y}+1} \\ \approx& 0.4 \end{aligned}$$

Conditional distribution

Since $f_t(x) = \theta e^{-\theta x}$,

- when $X < Y$:

$$f_{c|t}(y) = \frac{f_{t,c}(x, y)}{f_t(x)} = \theta^2 e^{-e^{\theta y}+\theta y-\theta x+1}/(\theta e^{-\theta x}) = \theta e^{-e^{\theta y}+\theta y+1}$$

$$F_{c|t}(x) = \int_0^{x} f_{c|t}(y)dy = \int_0^{x} \theta e^{-e^{\theta y}+\theta y+1} dy = 1 - e^{1-e^{\theta x}}$$

$$F_{c|t}^{-1}(x) = \frac{1}{\theta}\ln(1 - \ln(1 - x))$$

Which also means that, when $x < y$, the simulation of T and C can be independent.

- when $X \geq Y$:

$$\begin{aligned}
f_{c|t}(y) =& \frac{f_{t,c}(x,y)}{f_t(x)} = \frac{f_{t,c}(x,y)}{\theta e^{-\theta x}} \\
=& \frac{1}{\theta}((2x - 2y)(1 - e^{\theta y}) - \theta)((2y - 2x)(1 - e^{\theta y}) - \theta(y^2 - 2xy + x^2 + 1)e^{\theta y})e^{(y^2-2xy+x^2+1)(1-e^{\theta y})} \\
&+ ((2y - 2x)e^{\theta y} - 2(1 - e^{\theta y}))e^{(y^2-2xy+x^2+1)(1-e^{\theta y})}
\end{aligned}$$

$$\begin{aligned}
F_{c|t}(c) =& \int_0^c f_{c|t}(y)dy \\
=& 1 - \left((2e^{c+1} - 2e)e^{c^2}x + (-2ce^{c+1} + 2ec + e)e^{c^2}e^{-e^c x^2 + x^2 + 2ce^c x - 2cx - c^2 e^c - e^c}\right)
\end{aligned}$$