Example of Independence -2, 3

2019-10-22

The pairwise example in Slud's paper

The joint distribution is:

$$f(t,s) = \begin{cases} f_1(t)f_C(s) & (t \le s) \\ f_C(s)\dfrac{S_1(s)}{S_2(s)}f_2(t) & (t > s) \end{cases}$$

Let

- $f_1(t) = \exp(-t)$, $S_1(s) = \exp(-x)$
- $f_C(s) = \exp(-s)$, $S_C(s) = \exp(-s)$
- $f_2(t) = \rho\exp(-\rho t)$, $S_2(s) =]exp(-\rho t)$
- $\rho(t) = \frac{h_2(t)}{h_1(t)} = \rho$, which is a constant.

Then

$$f(t,s) = \begin{cases} \exp(-t-s) & (t \le s) \\ \rho\exp(-\rho t + (\rho-2)s) & (t > s) \end{cases}$$

And

$$f(t) = \frac{2\rho-2}{\rho-2}\exp(-2t) - \frac{\rho}{\rho-2}\exp(-\rho t)$$

$$S(t) = \frac{\rho-1}{\rho-2}exp(-2t) - \frac{1}{\rho-2}\exp(-\rho t)$$

$$\psi(t) = \exp(-2t)$$

$$S_x(x) = P(X = T \wedge C > x) = P(T > x, C > x) = P(T > C > x) + P(C > T > x)$$

$$\begin{aligned} S_x(x) =& P(X = T \wedge C > x) = P(T > x, C > x) = P(T > C > x) + P(C > T > x) \\ =& \int_x^\infty \int_x^t f(t,s)dsdt + \int_x^\infty \int_x^s f(t,s)dtds \\ =& \int_x^\infty \int_x^t \rho\exp(-\rho t + (\rho-2)s)dsdt + \int_x^\infty \int_x^t \exp(-t-s)dtds \\ =& \int_x^\infty \rho\Big(\frac{\exp(-2t)}{\rho-2} - \frac{\exp(\rho x - 2x - \rho t)}{\rho-2}\Big)dt + \int_x^\infty \exp -x - s - \exp(-2s)ds \\ =& \frac{\rho}{\rho-2}\frac{\rho-2}{2\rho}\exp(-2x) + \frac{\exp(-2x)}{2} \\ =& \exp(-2x) \end{aligned}$$

Therefore,

$$S_H(t) = S_x(t) = \exp(-2t), \lambda_H(t) = 2 \text{ ,(consistent to previous notation))}$$

Then the $m()$ function is

$$m(t) = \frac{\lambda_F(t)}{\lambda_H(t)} = \frac{\frac{\frac{2\rho-2}{\rho-2}\exp(-2t) - \frac{\rho}{\rho-2}\exp(-\rho t)}{\frac{\rho-1}{\rho-2}\exp(-2t) - \frac{1}{\rho-2}\exp(-\rho t)}}{2} = \frac{1}{2}\frac{(2\rho-2)\exp(-2t) - \rho\exp(-\rho t)}{(\rho-1)\exp(-2t) - \exp(-\rho t)}$$

And from the above formula, we can know that when $\rho = 1$, $m(t) = 1$.

Simulation

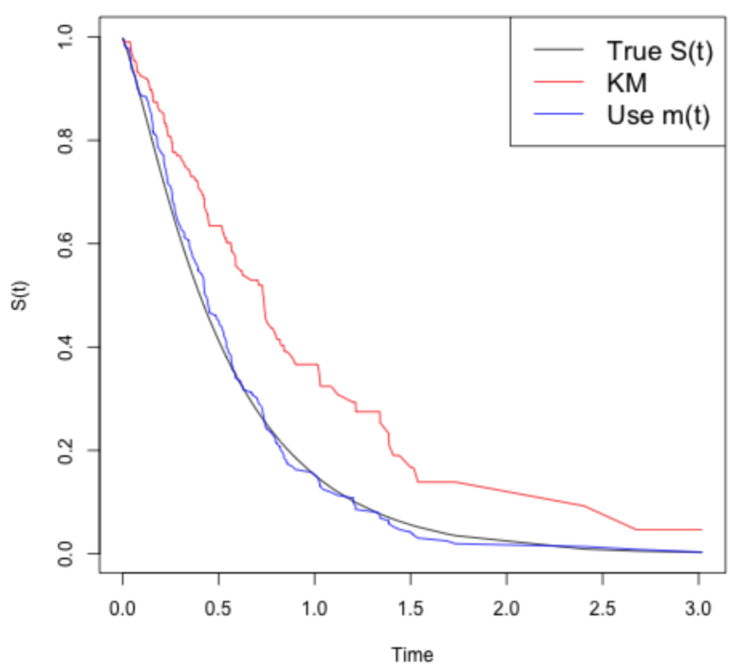
Let's take $\rho = 10$ as an example, then

$$f(t,s) = \begin{cases} \exp(-t-s) & (t \le s) \\ 10\exp(-10t + 8s) & (t > s) \end{cases}$$

The censoring percentage is 50 %.

The mean absolute difference between true $S(t)$ and Kaplan meier estimator is 0.146

The mean absolute difference between true $S(t)$ and use $\hat{\lambda}_F(t) = m(t)\hat{\lambda}_H$ is 0.021 (the $m(t)$ is used as true value).



The example in Dr. Ying's paper

In Zhiliang Ying's paper, the Joint CDF is:

$$S(T \geq x, U \geq y) = \begin{cases} e^{-x} e^{-(e^y - 1)\left((x-y)^2+1\right)} & x \geq y \\ e^{-x} e^{-(e^y-1)} & x < y \end{cases}$$

The corresponding marginal distributions:

- $P(T > x) = P(T > x, U > 0) = e^{-x} e^{-(e^0-1)\left((x-0)^2+1\right)} = e^{-x}$
- $F_T(x) = 1 - e^{-x}$, $f_T(x) = e^{-x}$
- $P(U > x) = P(U > x, T > 0) = e^{-0} e^{-(e^y-1)} = e^{-(e^y-1)}$
- $F_U(x) = 1 - e^{-(e^y-1)}$, $f_U(x) = e^{1+y-e^y}$

And the distribution of $X = T \wedge U$ is

$$P(X > x) = P(T > x, U > x) = e^{-x} e^{-(e^x-1)}$$

Therefore,

$$F_X(x) = 1 - e^{1-x-e^x}, f_X(x) = (1+e^x)e^{1-x-e^x}$$

The $m()$ function is:

$$m(t) = \frac{\lambda_F(t)}{\lambda_H(t)} = \frac{f_T(t)}{S_T(t)} / \frac{f_X(t)}{S_X(t)} = \frac{e^{-t}}{e^{-t}} / \frac{(1+e^t)e^{1-t-e^t}}{e^{1-t-e^t}} = \frac{1}{1+e^t}$$

The censoring percentage Since

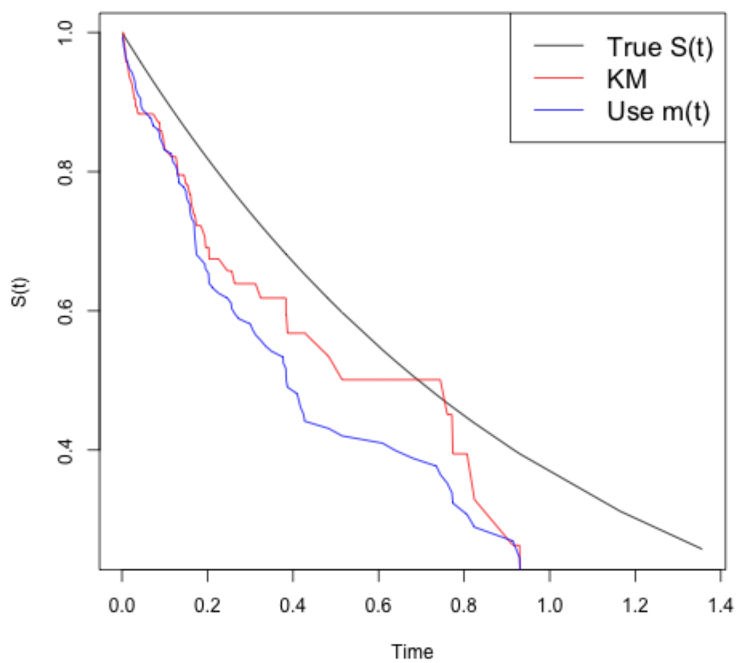
$$\begin{aligned} P(T < x < U) =& P(T < x, U > x) = P(U > x) - P(T > x, U > x) \\ =& \exp(-(\exp(x)-1)) - \exp(-x)\exp(-\exp(x)+1) \\ =& (1-\exp(-x))\exp(-(\exp(x)-1)) \end{aligned}$$

Then we can calculate $P(T < U)$ as:

$$\begin{aligned} P(T < U) =& \int_0^\infty P(T < x < U) dx \\ =& \int_0^\infty (1-\exp(-x))\exp(-(\exp(x)-1)) dx \\ =& [-e(\Gamma(0, e^x)) - \Gamma(-1, e^x)]|_0^\infty \\ \approx& 0.2 \end{aligned}$$

The censoring percentage is 1 - 0.2 = 0.8.

However, $S(T > x, U > y)$, $(x < y)$ doesn't mean that $T < U$. I met some problem in calculating the joint pdf, and the result did not look good, may be because the data isn't generated in the correct way.


True S(t)
KM
Use m(t)
S(t)
1.0
0.8
0.6
0.4
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Time