Calculate the Kendall's tau for Tsiatis Copula

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Copula

In the Tsiatis's paper, the related functions are:

Function	Description	Expression
$P(T < t, C < c)$	Joint CDF	$1 + exp(-\lambda t - \mu c - \theta tc) - exp(-\lambda t) - exp(-\mu c)$
$f(t, c)$	Joint PDF	$(\lambda\mu - \theta + \lambda\theta t + \mu\theta c + \theta^2 tc)exp(-\lambda t - \mu c - \theta ct)$
$f_t(t)$	Marginal PDF of T	$\lambda exp(-\lambda t)$
$S_t(t)$	Survival function of T	$exp(-\lambda t)$
$f_c(c)$	Marginal PDF of C	$\mu exp(-\mu c)$
$S_c(c)$	$P_c(C > c)$	$exp(-\mu c)$
$S_x(t)$	$P(T > t, C > t)$	$exp(-\lambda t - \mu t - \theta t^2)$
$\psi(t)$	$\int_t^\infty f(t, c)dc$	$(\lambda + \theta t)exp(-\lambda t - \mu t - \theta t^2)$

Therefore,

$$F(t, c) = C(F(t), G(c))F(t)G(c)$$

Where

- F(t,c) is the joint pdf of survival time and censor time
- F(t) is the marginal function of survival time
- G(c) is the marginal function of censor time

That is:

$$\begin{aligned}
C(U, V) =& \frac{1 + exp(-\lambda t - \mu c - \theta tc) - exp(-\lambda t) - exp(-\mu c)}{(1 - exp(-\lambda t))(1 - exp(-\mu t))} \\
=& \frac{1 + (1 - U)(1 - V)exp(-\frac{\theta}{\mu\lambda}ln(1 - U)ln(1 - V)) - (1 - U) - (1 - V)}{UV} \\
=& \frac{U + V - 1 + (1 - U)(1 - V)exp(-\frac{\theta}{\mu\lambda}ln(1 - U)ln(1 - V))}{UV}
\end{aligned}$$

The pdf of the copula distribution is:

$$c(F(t), G(c)) = \frac{f(t, c)}{f_t(t)g(c)} = \frac{(\lambda\mu - \theta + \lambda\theta t + \mu\theta c + \theta^2 tc)exp(-\lambda t - \mu c - \theta ct)}{\lambda exp(-\lambda t)\mu exp(-\mu c)}$$

Therefore, $c(U, V) = \frac{1}{\lambda\mu}\Big[(\lambda - \frac{\theta}{\mu}ln(1 - V))(\mu - \frac{\theta}{\lambda}ln(1 - U)) - \theta\Big]exp(-\frac{\theta}{\mu\lambda}ln(1 - U)ln(1 - V))$

According to the relationship between the copula function and Kendall's τ:

$$\tau = 4E(C(u, v)) - 1$$

$$
\begin{aligned}
E(C(u,v)) &= \int_0^1 \int_0^1 C(u,v)dC(u,v) \\
&= \int_0^1 \int_0^1 C(u,v)c(u,v)dudv \\
&= \int_0^1 \int_0^1 \frac{u+v-1+(1-u)(1-v)exp(-\frac{\theta}{\mu\lambda}ln(1-u)ln(1-v)}{uv} \\
& \frac{1}{\lambda\mu}\Big[(\lambda - \frac{\theta}{\mu}ln(1-v))(\mu - \frac{\theta}{\lambda}ln(1-u)) - \theta\Big]exp(-\frac{\theta}{\mu\lambda}ln(1-u)ln(1-v))dudv
\end{aligned}
$$

However, no antiderivative could be found since the formula is too complicated.

Survival Copula

The joint survival function is:

$$P(T > t, C > c) = S(t,c) = exp(-\lambda t - \mu c - \theta tc)$$

The survival function is:

$$P(T > t) = S_t(t) = exp(-\lambda t)$$

The censor function is:

$$P(C > c) = S_c(c) = exp(-\mu c)$$

Therefore, the survival copula is:

$$C_s(S_t(t), S_c(c)) = S(t,c)/(S_t(t)S_c(c)) = \frac{exp(-\lambda t - \mu c - \theta tc)}{exp(-\lambda t)exp(-\mu c)} = exp(-\theta tc)$$

That is:

$$C_s(U,V) = exp(-\frac{\theta}{\lambda\mu}ln(U)ln(V))$$

The survival copula is also a copula

This is because

$$
\begin{aligned}
S(t,c) =& P(T > t, C > c) \\
=& P(T > t) - P(T > t, C < c) \\
=& P(T > t) - (P(C < c) - P(T < t, C < c)) \\
=& 1 - F_t(t) - F_c(C) + F(t,c) \\
=& S_t(t) + S_c(c) + C(1 - S_t(t), 1 - S_c(c)) \\
=& C_s(S_t(t), S_c(t))
\end{aligned}
$$

That is

$$C_s(u,v) = u + v - 1 + C(1-u, 1-v)$$

The margins of C_s are uniform:

- $C_s(u,1) = u + C(1-u,0) = u$, $C_s(1,v) = v + C(0,1-v) = v$

We may also proof that $C_s(t,c) \geq 0$ and it is a non-negative function.

C_s is 2-nondecreasing:

If $0 \leq u_1 \leq u_2 < \infty$, $0 \leq v_1 \leq v_2 < \infty$

$$\begin{aligned}
&C_s(v_1,v_2) + C_s(u_1,u_2) - C_s(u_1,v_2) - C_s(u_2,v_1)\\
&= C(1-v_1,1-v_2) + C(1-u_1,1-u_2) - C_s(1-u_1,1-v_2) - C_s(1-u_2,1-v_1)\\
&\geq 0
\end{aligned}$$

Therefore, survival copula is a copula has a more simple formula than the copula.

According to the relationship between the copula function and Kendall's τ:

$$\tau = 4E(C_s(u,v)) - 1$$

The kendall's tau for the Tsiatis Copula should be:

$$E(C_s(u,v)) = \int_0^1 \int_0^1 C_s(u,v)dC_s(u,v)$$

$$\begin{aligned}
c_s(u,v) =& dC_s(u,v) = \frac{\partial C_s(u,v)}{\partial u \partial v} = \frac{\partial exp(-\frac{\theta}{\lambda\mu}ln(u)ln(v))}{\partial u \partial v}\\
&= \frac{\partial\Big[-\frac{\theta logv}{\lambda\mu u}exp(-\frac{\theta}{\lambda\mu}ln(u)ln(v))\Big]}{\partial v}\\
&= -\frac{\theta}{\lambda\mu uv}exp(-\frac{\theta}{\lambda\mu}ln(u)ln(v)) + \frac{\theta^2 ln(u)ln(v)}{\lambda^2\mu^2 uv}exp(-\frac{\theta}{\lambda\mu}ln(u)ln(v))
\end{aligned}$$

Therefore,

$$\begin{aligned}
E(C_s(u,v)) &= \int_0^1 \int_0^1 C_s(u,v)dC(u,v)\\
&= \int_0^1 \int_0^1 exp(-\frac{\theta}{\lambda\mu}ln(u)ln(v))\Big[-\frac{\theta}{\lambda\mu uv}exp(-\frac{\theta}{\lambda\mu}ln(u)ln(v))\\
&+ \frac{\theta^2 ln(u)ln(v)}{\lambda^2\mu^2 uv}exp(-\frac{\theta}{\lambda\mu}ln(u)ln(v))\Big]dudv\\
&= \int_0^1 \int_0^1 -\frac{\theta}{\lambda\mu uv}exp(-\frac{2\theta}{\lambda\mu}ln(u)ln(v)) + \frac{\theta^2 ln(u)ln(v)}{\lambda^2\mu^2 uv}exp(-\frac{2\theta}{\lambda\mu}ln(u)ln(v))dudv\\
&= \int_0^1 \int_{-\infty}^0 -\frac{\theta}{\lambda\mu v}exp(-\frac{2\theta}{\lambda\mu}Xln(v)) + \frac{\theta^2 Xln(v)}{\lambda^2\mu^2 v}exp(-\frac{2\theta}{\lambda\mu}Xln(v))dXdv \text{ ,(where } X = ln(u))
\end{aligned}$$

For $\int_{-\infty}^0 -\frac{\theta}{\lambda\mu v}exp(-\frac{2\theta}{\lambda\mu}Xln(v)) + \frac{\theta^2 Xln(v)}{\lambda^2\mu^2 v}exp(-\frac{2\theta}{\lambda\mu}Xln(v))dX$, Let $A = \frac{\theta}{\mu\lambda}$ to make it look more clear.

$$
\begin{aligned}
&\int_{-\infty}^{0} -\frac{A}{v}exp(-2Aln(v)X) + \frac{A^2ln(v)X}{v}exp(-2Aln(v)X)dX \\
=&\frac{1}{2vln(v)}exp(-2Aln(v)X)|_{-\infty}^{0} + \Big(-\frac{A}{2v}exp(-2Aln(v)X)X|_{-\infty}^{0} + \int_{-\infty}^{0}\frac{A}{2v}exp(-2Aln(v)X)dx\Big) \\
=&\frac{1}{2vln(v)}exp(-2Aln(v)X)|_{-\infty}^{0} + \Big[-\frac{A}{2v}exp(-2Aln(v)X)X|_{-\infty}^{0} + \Big(-\frac{1}{4vln(v)}exp(-2Aln(v)X)|_{-\infty}^{0}\Big)\Big] \\
=&\frac{1}{2vln(v)} + (0 - \frac{1}{4vln(v)}) \\
=&\frac{1}{4vln(v)}
\end{aligned}
$$

And then:

$$
\begin{aligned}
\int_{0}^{1}\frac{1}{4vln(v)}dv =& \int_{-\infty}^{0}\frac{1}{4Y}dY, \text{ (where } Y = ln(v)) \\
=&\frac{1}{4}ln(|Y|)|_{-\infty}^{0}
\end{aligned}
$$

The intergral is divergent.

The Kendall's Tau is not avaiable for this copula?