For the term $\frac{P(t<T<t+dt, C\leq t)}{P(t<T<t+dt, C>t)}$,

$$\begin{aligned}
\frac{P(t<T<t+dt, C\leq t)}{P(t<T<t+dt, C>t)} =& \frac{P(t<T<t+dt) - P(t<T<t+dt, C>t)}{P(t<T<t+dt, C>t)} \\
=& \frac{P(T>t) - P(T>t+dt) - P(T>t, C>t) + P(T>t+dt, C>t)}{P(T>t, C>t) - P(T>t+dt, C>t)} \\
=& \frac{P(T>t) - P(T>t+dt) - P(T>t)P(C>t) + P(T>t+dt)P(C>t)}{P(T>t)P(C>t) - P(T>t+dt)P(C>t)} \\
=& \frac{\big[P(T>t) - P(T>t+dt)\big]\big[1 - P(C>t)\big]}{\big[P(T>t) - P(T>t+dt)\big]P(C>t)}
\end{aligned}$$

Therefore,when the conditions are true,

$$\Rightarrow \rho(t) = \lim_{dt\to 0} \frac{P(t<T<t+dt|T>t, C\leq t)}{P(t<T<t+dt|T>t, C>t)} = \lim_{dt\to 0} \frac{\big[P(T>t) - P(T>t+dt)\big]\big[1 - P(C>t)\big]}{\big[P(T>t) - P(T>t+dt)\big]P(C>t)} \frac{P(C>t)}{1-P(C>t)} = 1$$