Example 5, n =500

Mimic Zhiliang Ying's paper and to make things easy, let's look at the Joint CDF which is:

$$S(T \geq x, C \geq y) = \begin{cases} e^{-\theta x} e^{-(e^{\theta y}-1)\big((x-y)+1\big)} & x \geq y \\ e^{-\theta x} e^{-(e^{\theta y}-1)} & x < y \end{cases}$$

For $x > y$ scenario

$$\frac{\partial}{\partial x} e^{-(e^{\theta y}-1)\big((x-y)+1\big)} = -(e^{\theta y} + \theta - 1) e^{-\theta x - (e^{\theta y}-1)\big((x-y)+1\big)}$$

$$-\frac{\partial}{\partial y}(e^{\theta y} + \theta - 1) e^{-\theta x - (e^{\theta y}-1)\big((x-y)+1\big)}$$

$$= \Big\{ \big[(\theta x - \theta y) + (\theta - 1)\big] e^{2\theta y} + \big[(\theta^2 - \theta)(x-y) + (\theta-1)(\theta-2)\big] e^{\theta y} + (\theta - 1) \Big\} e^{-\theta x - (e^{\theta y}-1)\big((x-y)+1\big)}$$

which is bigger than 0 for sure when $\theta > 2$

Therefore, the pdfs are

$$f_{T,C}(x,y) = \begin{cases} \Big\{ \big[(\theta x - \theta y) + (\theta - 1)\big] e^{2\theta y} + \big[(\theta^2 - \theta)(x-y) + (\theta-1)(\theta-2)\big] e^{\theta y} + (\theta - 1) \Big\} \\ \times e^{-\theta x - (e^{\theta y}-1)\big((x-y)+1\big)} & x \geq y \\ \\ \theta^2 e^{-e^{\theta y} + \theta y - \theta x + 1} & x < y \end{cases}$$

Besides, the functions $S(t), S_z(t), f(t), f_z(t)$ are the same, therefore, $m(t)$ is the same as Dr. Ying's example.

$$S_T(t) = P(T > t) = P(T > t, C > 0) = e^{-\theta t} e^{-(e^{\theta 0}-1)\big((t-0)+1\big)} = e^{-\theta t}$$

$$f_T(t) = \frac{\partial}{\partial t}(1 - S_T(t)) = \frac{\partial}{\partial t}(1 - e^{-\theta t}) = \theta e^{-\theta t}$$

$$S_Z(t) = P(T > t, C > t) = e^{-\theta t} e^{-(e^{\theta t}-1)} = e^{-e^{\theta t} - \theta t + 1}$$

$$f_Z(t) = \frac{\partial}{\partial t}(1 - S_Z(t)) = 1 - e^{-e^{\theta t} - \theta t + 1} = \theta(1 + e^{\theta t}) e^{-e^{\theta t} - \theta t + 1}$$

$$\psi(t) = \int_t^\infty f(t,c)dc = \int_t^\infty \theta^2 e^{-e^{\theta c} + \theta c - \theta t + 1} dc = \theta e^{-e^{\theta t} - \theta t + 1}$$

Therefore, the $m()$ function is:

$$m(t) = \frac{\lambda_F(t)}{\lambda_H(t)} = \frac{f_T(t)}{S_T(t)} / \frac{f_Z(t)}{S_Z(t)} = \frac{\theta e^{-\theta t}}{e^{-\theta t}} / \frac{\theta(1 + e^{\theta t}) e^{-e^{\theta t} - \theta t + 1}}{e^{-e^{\theta t} - \theta t + 1}} = \frac{1}{1 + e^{\theta t}}$$

And for the $\rho()$ function,

$$\begin{aligned} \rho = & \frac{f(t)/\psi(t) - 1}{S(t)/S_x(t) - 1} \\ = & \frac{\theta e^{-\theta t} / (\theta e^{-e^{\theta t} - \theta t + 1}) - 1}{e^{-\theta t} / e^{-e^{\theta t} - \theta t + 1} - 1} \\ = & 1 \end{aligned}$$

Results

Simulation settings

The survival time and the censor time were jointly simulated from the joint distribution above. The parameter θ is chosen as 1, 1.5, 2, and 5.

Sample size: 500

Iteration time: 500

Table 1: Mean absolute difference between estimated and true S()

	With true m()				With estimated m()			
Quantile	KM	Exp m()	Dikta 1	Dikta 2	KM	Exp m()	Dikta 1	Dikta 2
theta = 1								
t0.1	0.00992	0.01044	0.01051	0.01055	0.00992	0.01058	0.01066	0.01070
t0.25	0.01399	0.02018	0.02046	0.02062	0.01399	0.02162	0.02190	0.02206
t0.5	0.02562	0.02891	0.02977	0.03030	0.02562	0.03509	0.03599	0.03654
t0.75	0.05239	0.02606	0.03253	0.03672	0.05239	0.03863	0.04548	0.04991
t0.9	0.07029	0.06051	0.09993	0.09988	0.07029	0.04441	0.09981	0.09976
theta = 1.5								
t0.1	0.01018	0.01666	0.01678	0.01683	0.01018	0.01721	0.01732	0.01738
t0.25	0.01610	0.03935	0.03969	0.03988	0.01610	0.04305	0.04340	0.04359
t0.5	0.04131	0.04206	0.04303	0.04364	0.04131	0.05601	0.05701	0.05762
t0.75	0.06403	0.02911	0.03489	0.03966	0.06403	0.05629	0.06278	0.06762
t0.9	0.06371	0.05706	0.10000	0.10000	0.06371	0.02589	0.10000	0.10000
theta = 2								
t0.1	0.01054	0.01571	0.01582	0.01588	0.01054	0.01618	0.01629	0.01635
t0.25	0.01562	0.03483	0.03517	0.03536	0.01562	0.03809	0.03843	0.03862
t0.5	0.04020	0.03610	0.03701	0.03758	0.04020	0.04830	0.04926	0.04984
t0.75	0.04648	0.02224	0.02651	0.02989	0.04648	0.04340	0.04898	0.05300
t0.9	0.07095	0.06153	0.09996	0.09993	0.07095	0.03195	0.09990	0.09988
theta = 5								
t0.1	0.01021	0.02920	0.02933	0.02940	0.01021	0.03205	0.03217	0.03224
t0.25	0.02689	0.09228	0.09269	0.09293	0.02689	0.11146	0.11190	0.11213
t0.5	0.22492	0.19632	0.19827	0.19956	0.22492	0.25869	0.26066	0.26176
t0.75	0.22701	0.08810	0.09980	0.10726	0.22701	0.16783	0.17647	0.18087
t0.9	0.08807	0.04907	0.04312	0.03318	0.08807	0.03332	0.04463	0.05010

Table 2: MSE

	With true m()				With estimated m()			
Quantile	KM	Exp m()	Dikta 1	Dikta 2	KM	Exp m()	Dikta 1	Dikta 2
theta = 1								
t0.1	0.00015	0.00017	0.00017	0.00018	0.00015	0.00018	0.00018	0.00018
t0.25	0.00030	0.00056	0.00057	0.00058	0.00030	0.00062	0.00064	0.00065
t0.5	0.00099	0.00112	0.00118	0.00121	0.00099	0.00157	0.00164	0.00169
t0.75	0.00429	0.00111	0.00211	0.00252	0.00429	0.00222	0.00332	0.00384
t0.9	0.00686	0.00374	0.00999	0.00998	0.00686	0.00250	0.00997	0.00997
theta = 1.5								
t0.1	0.00017	0.00037	0.00037	0.00038	0.00017	0.00039	0.00039	0.00040

t0.25	0.00040	0.00177	0.00180	0.00181	0.00040	0.00208	0.00212	0.00213
t0.5	0.00222	0.00209	0.00218	0.00223	0.00222	0.00352	0.00364	0.00371
t0.75	0.00562	0.00128	0.00187	0.00237	0.00562	0.00395	0.00492	0.00566
t0.9	0.00563	0.00332	0.01000	0.01000	0.00563	0.00101	0.01000	0.01000
theta = 2								
t0.1	0.00017	0.00034	0.00034	0.00034	0.00017	0.00035	0.00036	0.00036
t0.25	0.00038	0.00143	0.00145	0.00146	0.00038	0.00166	0.00169	0.00171
t0.5	0.00216	0.00168	0.00175	0.00180	0.00216	0.00275	0.00285	0.00291
t0.75	0.00341	0.00080	0.00128	0.00157	0.00341	0.00263	0.00338	0.00387
t0.9	0.00684	0.00386	0.00999	0.00999	0.00684	0.00145	0.00999	0.00998
theta = 5								
t0.1	0.00017	0.00096	0.00097	0.00097	0.00017	0.00114	0.00115	0.00116
t0.25	0.00100	0.00877	0.00885	0.00889	0.00100	0.01271	0.01280	0.01286
t0.5	0.05201	0.03893	0.03971	0.04023	0.05201	0.06719	0.06822	0.06879
t0.75	0.05167	0.00812	0.01095	0.01255	0.05167	0.02849	0.03160	0.03319
t0.9	0.00778	0.00258	0.00218	0.00153	0.00778	0.00131	0.00233	0.00281

Table 3: Standard deviations of the estimated S()

	With true m()				With estimated m()			
Quantile	KM	Exp m()	Dikta 1	Dikta 2	KM	Exp m()	Dikta 1	Dikta 2
theta = 1								
t0.1	0.01245	0.01020	0.01021	0.01022	0.01245	0.01023	0.01024	0.01025
t0.25	0.01697	0.01421	0.01423	0.01424	0.01697	0.01425	0.01428	0.01429
t0.5	0.02247	0.01848	0.01855	0.01860	0.02247	0.01939	0.01945	0.01949
t0.75	0.04784	0.02601	0.03581	0.03709	0.04784	0.03144	0.03867	0.03963
t0.9	0.08103	0.00883	0.01146	0.01066	0.08103	0.02328	0.00962	0.00884
theta = 1.5								
t0.1	0.01305	0.01033	0.01034	0.01035	0.01305	0.01039	0.01040	0.01040
t0.25	0.01760	0.01516	0.01518	0.01520	0.01760	0.01543	0.01545	0.01546
t0.5	0.02376	0.01829	0.01836	0.01841	0.02376	0.01972	0.01979	0.01983
t0.75	0.04315	0.02450	0.02828	0.03036	0.04315	0.02930	0.03209	0.03355
t0.9	0.07412	0.00798	0.00000	0.00000	0.07412	0.02122	0.00000	0.00000
theta = 2								
t0.1	0.01324	0.01015	0.01016	0.01017	0.01324	0.01019	0.01020	0.01021
t0.25	0.01743	0.01464	0.01465	0.01467	0.01743	0.01464	0.01465	0.01467
t0.5	0.02475	0.01994	0.02001	0.02006	0.02475	0.02064	0.02071	0.02075
t0.75	0.04287	0.02400	0.02892	0.03023	0.04287	0.02989	0.03311	0.03402
t0.9	0.07804	0.00872	0.00805	0.00747	0.07804	0.02281	0.00680	0.00623
theta = 5								
t0.1	0.01282	0.01051	0.01051	0.01052	0.01282	0.01075	0.01075	0.01076
t0.25	0.01867	0.01593	0.01597	0.01599	0.01867	0.01679	0.01683	0.01684
t0.5	0.03767	0.01977	0.02005	0.02020	0.03767	0.01631	0.01652	0.01664
t0.75	0.01188	0.01906	0.03150	0.03228	0.01188	0.01795	0.02145	0.02177
t0.9	0.00518	0.01321	0.03445	0.03264	0.00518	0.01472	0.01843	0.01748

The estimate of $m()$

Table 4: mean absolute difference between hat m() and true m()

1	1.5	2	5
0.0106151	0.0166431	0.0155151	0.0575267

The row name shows the θ value

Table 5: standard deviation of estimated m()

1	1.5	2	5
0.0080496	0.0098042	0.0095389	0.0097455

The row name shows the θ value

Table 6: estimated theta from logitic regression

1	1.5	2	5
0.9093124	1.209593	1.652898	0.7815693

The survival function plots: