Calculate the Kendall's tau for Tsiatis Copula

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A copula is the joint distribution of random variables $U_1, U_2, ..., U_p$, each of which is marginally uniformly distributed as $U(0, 1)$.

$$C(u_1, ..., u_p) = P(U1 \leq u_1, ..., U_p \leq u_p)$$

Therefore, the copula for distribution $F(t, s)$ is

$$F(t, s) = C[F_t(t), F_s(s)]$$

The pdf of copula distribution is:

$$c(u_1, u_2, ..., u_p) = \frac{\partial^p}{\partial u_1 \partial u_2 ... \partial u_p} C(u_1, u_2, ..., u_p)$$

That is,

$$c(F_1(x_1), ..F_p(x_p)) = \frac{f(x_1, x_2, ..., x_p)}{f_1(x_1) f_2(x_2) ... f_p(x_p)}$$

Therefore, in the Tsiatis's example, the copula is

$$C(F_t(t), F_c(c)) = 1 + exp(-\lambda t - \mu c - \theta tc) - exp(-\lambda t) - exp(-\mu c)$$

$$\begin{aligned} C(u, v) =& 1 + (1-u)(1-v) exp(-\frac{\theta}{\lambda\mu} ln(1-u) ln(1-v)) - (1-u) - (1-v) \\ =& u + v - 1 + (1-u)(1-v) exp(-\frac{\theta}{\lambda\mu} ln(1-u) ln(1-v)) \end{aligned}$$

The pdf of copula is

$$\begin{aligned} c(F_t(t), F_c(c)) =& \frac{f(t, c)}{f_t(t) f_c(c)} = \frac{(\lambda\mu - \theta + \lambda\theta t + \mu\theta c + \theta^2 tc) exp(-\lambda t - \mu c - \theta ct)}{\lambda exp(-\lambda t) \mu exp(-\mu c)} \\ =& \frac{(\lambda\mu - \theta + \lambda\theta t + \mu\theta c + \theta^2 tc)}{\lambda\mu} exp(-\theta tc) \\ =& \frac{1}{\lambda\mu} (\lambda\mu - \theta - \theta ln(1-u) - \theta ln(1-u) + \theta^2 ln(1-u) ln(1-v)) exp(-\frac{\theta}{\lambda\mu} ln(1-u) ln(1-v)) \end{aligned}$$

To calculate the Kendall's tau, we need $E(C(u, v))$

$$E(C(u, v)) = \int C(u.v) dC(u, v) = \int_0^1 \int_0^1 C(u.v) c(u, v) du dv$$

In this case, replace u, v with t, c may be easier to calculate, that is

$$\begin{aligned} & \int_0^\infty \int_0^\infty C(F_t(t).F_c(c)) c(F_t(t), F_c(c)) dF_t(t) dF_c(c) \\ =& F(t, c) \frac{f(t, s)}{f_t(t) f_c(c)} dF_t(t) dF_c(c) \\ =& F(t, c) \frac{f(t, s)}{f_t(t) f_c(c)} f_t(t) f_c(c) dt dc \end{aligned}$$

Here, to make it easy, let's set $\lambda = \mu = \theta = 1$, then

$$\begin{aligned}
E(C(u,v)) &= \int_0^\infty \int_0^\infty F(t,c)\frac{f(t,s)}{f_t(t)f_c(c)}f_t(t)f_c(c)dtdc \\
&= \int_0^\infty \int_0^\infty [1 + exp(-t-c-tc) - exp(-t) - exp(-c)] \\
&\quad (t+c+tc)exp(-tc)exp(-t)exp(-c)dtdc \\
&= \int_0^\infty \frac{e^{-2c}((4c^2+16c+12)e^c - 2c^3 - 11c^2 - 20c - 12)}{4c^3+20c^2+32c+16}dc
\end{aligned}$$

However, still, no antiderivative could be found since the formula