Example of Independence-1

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Distribution description

We have a joint pdf function $f_{T1,T2}(t_1, t_2)$, which

$$f_{T1,T2}(x, y) = 1 + h(t_1, t_2)(t_1 - t_2),\ (t_1, t_2) \in [0, 1] \times [0, 1] \tag{1}$$

where

$$h(t_1, t_2) = C_0(t_1 - \frac{1}{2})(t_2 - \frac{1}{2})(t_1 + t_2 - 1) \tag{2}$$

Then we could calculate $P(T_1 > t_1, T_2 > t_2)$ and its associated marginal distributions and the $m()$ function. That is:

$$\begin{aligned} S_{T_1,T_2} =& P(T_1 > t_1, T_2 > t_2) = \int_{t_2}^{1}\int_{t_1}^{1} f_{T_1,T_2}(x, y)dxdy \\ =& (1 - t_1)(1 - t_2)(1 + \frac{C_0}{8} t_1 t_2 (t_1 - t_2)(t_1 + t_2 - 1)) \end{aligned} \tag{3}$$

CDF/PDF validation

To make equation (1) a valid PDF function and equation (3) a valid CDF fucnction, we need:

- 1. $f_{T_1,T_2}(t_1, t_2) \geq 0$
- 2. $S(0, 0) = 1$
- 3. $S(1, 1) = 0$
- 4. $S(t_1, t_2) > 0$, which is equivalent to show that $f_{t_1,t_2} > 0$
- 5. $S(t_1, t_2)$ is non-increasing.

It is easy to show that

$$S(0, 0) = (1 - 0)(1 - 0)(1 + \frac{C_0}{8} \times 0 \times 0 \times (0 + 0 - 1)) = 1$$

$$S(1, 1) = (1 - 1)(1 - 1)(1 + \frac{C_0}{8} \times 1 \times 1 \times (1 + 1 - 1)) = 0$$

Therefore, (2) and (3) are satisfied.

Show that $f_{T_1,T_2}(t_1,t_2) \geq 0$. To make it easier, we may change t_1, t_2 to x, y, where $x = t_1 - \frac{1}{2}$ and $y = t_2 - \frac{1}{2}$, then,

$$f(x,y) = 1 + C_0 xy(x+y), \ x, y \in [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$$

We may find the min and max value of $f(x,y)$.

$$\frac{\partial f(x,y)}{\partial x} = C_0(2xy + y^2), \ \frac{\partial^2 f(x,y)}{\partial^2 x} = 2C_0 y \rightarrow \begin{cases} C_0 > 0, y > 0 & \text{convex function, } x = -\frac{y}{2} \text{ is the min} \\ C_0 < 0, y < 0 & \text{convex function, } x = -\frac{y}{2} \text{ is the min} \\ C_0 > 0, y > 0 & \text{concave function, } x = -\frac{y}{2} \text{ is the max} \\ C_0 < 0, y > 0 & \text{concave function, } x = -\frac{y}{2} \text{ is the max} \end{cases}$$

And

$$f(-\frac{y}{2}, y) = 1 - C_0 \frac{y^3}{4}, \tag{4}$$

$$f(\frac{1}{2}, y) = 1 + C_0(\frac{y^2}{2} + \frac{y}{4}) \tag{5}$$

$$f(-\frac{1}{2}, y) = 1 + C_0(-\frac{y^2}{2} + \frac{y}{4}) \tag{6}$$

The extreme value for function (4) is

$$1 - C_0 \times \frac{1}{4}(\frac{1}{2})^3 = 1 - \frac{C_0}{32}, \text{ or}$$

$$1 - C_0 \times \frac{1}{4}(-\frac{1}{2})^3 = 1 + \frac{C_0}{32}$$

The extreme value for function (5) is

$$1 + C_0((\frac{1}{2})^2 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}) = 1 + \frac{C_0}{4}, \text{ or}$$

$$1 + C_0((-\frac{1}{2})^2 \times \frac{1}{2} - \frac{1}{2} \times \frac{1}{4}) = 1$$

The extreme value for function (5) is

$$1 + C_0(-(\frac{1}{2})^2 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{4}) = 1, , \text{ or}$$

$$1 + C_0(-(-\frac{1}{2})^2 \times \frac{1}{2} - \frac{1}{2} \times \frac{1}{4}) = 1 - \frac{C_0}{4}$$

Therefore, we need

$$1 - \frac{C_0}{32} \geq 0, 1 + \frac{C_0}{32} \geq 0, 1 - \frac{C_0}{4} \geq 0, 1 + \frac{C_0}{4} \geq 0 \rightarrow C_0 \in [-4, 4]$$

That is, to satisfy condition (1) and condition (4), we need $C_0 \in [-4, 4]$

The marginal function for the survival time and censoring time are all uniform distributions:

$$
\begin{aligned}
f_{t_1}(x) &= \int_0^1 f_{t_1,t_2}(x,y)dy \\
&= \Big\{y - \frac{C_0}{4}(x - \frac{1}{2})(y^4 - 2y^3 + (-2x^2 + 2x + 1)y^2 + (2x^2 - 2x)y)\Big\}\Big|_0^1 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
f_{t_2}(y) &= \int_0^1 f_{t_1,t_2}(x,y)dx \\
&= \Big\{\frac{C_0}{4}(y - \frac{1}{2})\Big[x^4 - 2x^3 + (-2y^2 + 2y + 1)x^2 + (2y^2 - 2y)x\Big] + x\Big\}\Big|_0^1 \\
&= 1
\end{aligned}
$$

That is,

$$
f_{T_1}(t_1) = I_{[0,1]}(t_1),\ \ f_{T_2}(t_2) = I_{[0,1]}(t_2)
$$
$$
P(T_1 > t_1) = 1 - t_1,\ \ P(T_2 > t_2) = 1 - t_2
$$

Therefore, the condition (5) is satisfied.

And hazard rate function λ_F for the survival time is:

- $S_F(t) = 1 - t$, $\Lambda_F(t) = -\log(1-t)$, $\lambda_F(t) = \frac{1}{1-t}$

The hazard rate function λ_H for the observed time is:

- $S_H(t) = P(Z > t) = (1-t)^2$, $\Lambda_H(t) = -2\log(1-t), \lambda_H(t) = \frac{2}{1-t}$

Then

$$
m(t) = \frac{\lambda_F(t)}{\lambda_H(t)} = 0.5
$$

And the associated $\rho(t)$ is

$$
\begin{aligned}
\psi(t) &= \int_t^1 f(t,s)ds = 1 + C_0(t_1 - \frac{1}{2})(t_2 - \frac{1}{2})(t_1 + t_2 - 1)(t_1 - t_2) \\
&= \frac{1}{8}\Big((1-t)\Big(C_0(t-1)t^2(2t-1) + 8\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\rho(t) &= \frac{f(t)/\psi(t) - 1}{S(t)/S_x(t) - 1} = \frac{1/\psi(t) - 1}{\frac{1-t}{(1-t)^2} - 1} \\
&= \frac{1-t}{t}\frac{2C_0t^5 - 5C_0t^4 + 4C_0t^3 - C_0t^2 + 8t}{\Big((1-t)\Big(C_0(t-1)t^2(2t-1) + 8\Big)\Big)} \\
&= \frac{2C_0t^4 - 5C_0t^3 + 4C_0t^2 - C_0t + 8}{C_0t^2(t-1)(2t-1) + 8}
\end{aligned}
$$

However,

When $C_0 = 0$, $\rho(t) = 1$, and T and U are independent. However, $m(t) \neq 1$ since $m(t) = 0.5$ is always true.

Is that means that when T and U are independent, $m(t) = 1$ is not alwauys true?

The results comparison