Untitled

2020-02-16

More details for the data generation.

example 1

CDF:

$$S(x,y) = \begin{cases} e^{-\theta x} e^{-(e^{\theta y}-1)(x-y+1)} & x \geq y \\ e^{-\theta x} e^{-(e^{\theta y}-1)} & x < y \end{cases}$$

PDF:

$$f(x,y) = \begin{cases} \Big[(\theta(x-y+1)-1)e^{2\theta y} + (\theta^2(x-y+1) - \theta(x-y+1) - 2(\theta+1))e^{\theta y} + \theta - 1\Big] \\ \times e^{-\theta x} e^{-(e^{\theta y}-1)(x-y+1)} & x \geq y \\ \theta^2 e^{(-e^{\theta y} + \theta y - \theta x + 1)} & x < y \end{cases}$$

Conditional CDF:

$$F_{C|T}(x,y) = \begin{cases} 1 - \frac{1}{\theta}(e^{\theta y} + \theta - 1)e^{-(e^{\theta y}-1)(x-y+1)} & x \geq y \\ e^{1-e^{\theta x}} - e^{1-e^{\theta y}} & x < y \end{cases}$$

$$F_{C|T}(x,y) = \int_0^y f(y|x)dy = \int_0^y \frac{f(x,y)}{f(x)}dy = \int_0^y \frac{\partial F(x,y)}{\partial x \partial y f(x)}dy = \frac{1}{f(x)}\frac{\partial F(x,y)}{\partial x}|_0^u$$

$$F_{C|T}(x,y) = \begin{cases} 1 - \frac{1}{\theta_1}(2(e^{\theta_2 y} - 1)(x-y) + \theta_1)e^{-(e^{\theta_2 y}-1)(x-y+1)} & x \geq y \\ e^{1-e^{\theta_2 x}} - e^{1-e^{\theta_1 x}} & x < y \end{cases}$$

example 2

CDF:

$$S(x,y) = \begin{cases} e^{-\theta x} e^{-(e^{2\theta y}-1)(x-y+1)} & x \geq y \\ e^{-\theta x} e^{-(e^{2\theta y}-1)} & x < y \end{cases}$$

PDF:

$$f(x,y) = \begin{cases} \Big[(2\theta(x-y+1)-1)e^{4\theta y} + 2\theta(\theta-1)(x-y)e^{2\theta y} + (2\theta-1)(\theta-2)e^{2\theta y} + \theta - 1\Big] \\ \times e^{-\theta x} e^{-(e^{2\theta y}-1)(x-y+1)} & x \geq y \\ 2\theta^2 e^{(-e^{2\theta y} + 2\theta y - \theta x + 1)} & x < y \end{cases}$$

Conditional CDF:

$$F_{C|T}(x,y) = \begin{cases} 1 - \frac{1}{\theta}(e^{2\theta y} + \theta - 1)e^{-(e^{2\theta y}-1)(x-y+1)} & x \geq y \\ e^{1-e^{2\theta x}} - e^{1-e^{2\theta y}} & x < y \end{cases}$$

example 3

$$F_{C|T}(x,y) = \begin{cases} 1 - \frac{1}{\theta_1}(e^{\theta_2 y} + \theta_1 - 1)e^{-(e^{\theta_2 y}-1)(x-y+1)} & x \geq y \\ e^{1-e^{\theta_2 x}} - e^{1-e^{\theta_2 y}} & x < y \end{cases}$$

all different

e$^{(e}$(by) - by -1)

example

e^(-ax) e$^{(-(e}$y - 1)(x-y + 1)) ((x-y)e^(2y) + (a-1)(x-y)e^y + a-1)

Consider joint distribution of T and C,

$$S_{T,C}(t,s;X) = \begin{cases} S_T(t;X)K(t,s;X) & t \geq s \\ S_T(t;X)S_C(s;X) & t < s \end{cases}$$

where $F_T(t;X) = 1 - S_T(t;X)$ and $F_C(s;X) = 1 - S_C(s;X)$ are CDF functions. The X is the baseline vectors, which serves as parameters for the joint distribution. Besides, $K(t,s;X)$ is a joint function of T and C, where

- 1. cannot be factored as a production of a function that only contain T and a function that only contain C
- 2. $K(t,s;X) \geq 0$ when $t, s \geq 0$
- 3. $K(t,0;X) = 1$
- 4. $K(s,s;X) = S_C(s;X)$
- 5. $K(t,s;X) = 0$ as $t, s \to \infty$

Then, the marginal distribution for event time is

$$\begin{aligned} P(T > t; X) &= \int_t^\infty f_T(t;X)dt \\ &= \int_t^\infty \Big\{ \int_0^\infty f_{T,C}(t,s;X)ds \Big\} dt \\ &= P(T > t, C > 0; X) = S_T(t;X)K(t,0;X) \\ &= S_T(t;X) \end{aligned}$$

The marginal distribution for the censoring time is

$$\begin{aligned}
P(C > s; X) &= \int_s^\infty f_C(s; X) ds \\
&= \int_s^\infty \Big\{ \int_0^\infty f_{T,C}(t, s; X) dt \Big\} ds \\
&= P(T > 0, C > s; X) = S_T(0; X) S_C(s; X) \\
&= S_C(s; X)
\end{aligned}$$

The distribution for the observed time $Z = T \wedge C$ is

$$S_Z(t; X) = P(T > t, C > t; X) = S_T(t; X) S_C(t; X)$$

with pdf:

$$f_Z(t; X) = -\frac{\partial [S_T(t; X) S_C(t; X)]}{\partial t} = f_T(t; X) S_C(t; X) + S_T(t; X) f_C(t; X)$$

Suppose the hazard for event is

$$\lambda_T(t; X) = \frac{f_T(t; X)}{S_T(t; X)}$$

The hazard for the censoring is

$$\lambda_C(t; X) = \frac{f_C(t; X)}{S_C(t; X)}$$

Then the $m(t; X)$ is

$$m(t; X) = \frac{\frac{f_T(t;X)}{S_T(t;X)}}{\frac{f_Z(t;X)}{S_Z(t;X)}} = \frac{S_T(t; X) S_C(t; X) \frac{f_T(t;X)}{S_T(t;X)}}{f_T(t; X) S_C(t; X) + S_T(t; X) f_C(t; X)} = \frac{\lambda_T(t; X)}{\lambda_T(t; X) + \lambda_C(t; X)}$$

Suppose the event time is from a cox PH model, then $S_T(t; X) = S_0(t)^{\exp(\beta X)}$ and $\lambda_T(t; X) = \lambda_0(t) \exp(\beta X)$. Then

$$m(t; X) = \frac{\lambda_T(t; X)}{\lambda_T(t; X) + \lambda_C(t; X)} = \frac{1}{1 + \frac{\lambda_C(t;X)}{\lambda_0(t)} \exp(-\beta X)} = \frac{1}{1 + \exp\Big(\ln(\lambda_C(t; X)) - \ln(\lambda_0(t)) - \beta X \Big)}$$

which can be treated as a logistic regression, with independent variables, transformed X, the baseline covariates and transformed t, the observed time.

To make the model more clear, let's look at two examples with the above model structrue.

Example 1

Let construct a joint distribution as following

$$S_{T,C}(t, s; \theta_1, \theta_2) = \begin{cases} e^{-\theta_1 t} e^{-(e^{\theta_2 s} - 1)((t-s)^2 + 1)} & \text{when } t \geq s \\ e^{-\theta_1 t} e^{-(e^{\theta_2 s} - 1)} & \text{when } t < s \end{cases}$$

where

$$S_T(t;X) = e^{-\theta_1 t}, S_C(s;X) = e^{-(e^{\theta_2 s}-1)}; K(t,s;X) = e^{-(e^{\theta_2 s}-1)((t-s)^2+1)}$$

and θ_1, θ_2 are parameters associated with X. Suppose x_1 has four levels 1,2,3,4 and x_2 has two levels 0,1, which is an indicator for gender,

$$x_2 = \begin{cases} 1 & \text{female} \\ 0 & \text{male} \end{cases}$$

Let $\theta_1 = \beta x_1, \theta_2 = \beta x_1 x_2$. Therefore,

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\beta x_1 t} e^{-(e^{(\beta x_1 x_2)s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\beta x_1 t} e^{-(e^{(\beta x_1 x_2)s}-1)} & \text{when } t < s \end{cases}$$

That is, for females:

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\beta x_1 t} e^{-(e^{(\beta x_1)s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\beta x_1 t} e^{-(e^{(\beta x_1)s}-1)} & \text{when } t < s \end{cases}$$

For males:

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\beta x_1 t} & \text{when } t \geq s \\ e^{-\beta x_1 t} & \text{when } t < s \end{cases} = e^{-\beta x_1 t}$$

There is no censorship in males.

For the event time distribution,

$$S_T(t;x_1,x_2) = e^{-\beta x_1 t} = [e^{-t}]^{\beta x_1} = [e^{-t}]^{\exp(\ln(\beta x_1))} = [e^{-t}]^{\exp(\ln(\beta)+\ln(x_1))}$$

Therefore, $S_T(t;x_1,x_2)$ is a cox PH model. And

$$f_T(t;x_1,x_2) = \beta x_1 e^{-\beta x_1 t},$$

$$\lambda_T(t;x_1,x_2) = \frac{f_T(t)}{S_T(t)} = \frac{\beta x_1 e^{-\beta x_1 t}}{e^{-\beta x_1 t}} = \beta x_1 = \lambda_{T0}(t)\exp(\ln(\beta)+\ln(x_1)), \quad \lambda_{T0}(t) = 1$$

The censoring time is

$$S_C(t;x_1,x_2) = e^{-(e^{(\beta x_1 x_2)s}-1)}$$

$$f_C(s;x_1,x_2) = \Big(\beta x_1 x_2\Big) e^{-e^{(\beta x_1 x_2)s}+(\beta x_1 x_2)s+1}$$

$$\lambda_C(s;x_1,x_2) = \frac{f_C(s;x_1,x_2)}{S_C(s;x_1,x_2)} = \Big(\beta x_1 x_2\Big) e^{(\beta x_1 x_2)s}$$

The associated $m()$ function is

$$\begin{aligned} m(t;x_1,x_2) =& \frac{\lambda_T(t;x_1,x_2)}{\lambda_T(t;x_1,x_2)+\lambda_C(t;x_1,x_2)} \\ =& \frac{\beta x_1}{\beta x_1 + \Big(\beta x_1 x_2\Big) e^{(\beta x_1 x_2)t}} \\ =& \frac{1}{1+x_2 e^{(\beta x_1 x_2)t}} \\ =& \begin{cases} \frac{1}{1+e^{(\beta x_1)t}} & x_2 = 1, \text{female} \\ \frac{1}{1+0} = 1 & x_2 = 0, \text{male} \end{cases} \end{aligned}$$

Example 2

To avoid θ_1, θ_2 generating 0, we reset the range for x_2, where

$$x_2 = \begin{cases} 2 & \text{female} \\ 1 & \text{male} \end{cases}$$

The other functions are the same. That is

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\beta x_1 t} e^{-(e^{(\beta x_1 x_2) s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\beta x_1 t} e^{-(e^{(\beta x_1 x_2) s}-1)} & \text{when } t < s \end{cases}$$

For females:

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\beta x_1 t} e^{-(e^{(2\beta x_1) s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\beta x_1 t} e^{-(e^{(2\beta x_1) s}-1)} & \text{when } t < s \end{cases}$$

For males:

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\beta x_1 t} e^{-(e^{(\beta x_1) s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\beta x_1 t} e^{-(e^{(\beta x_1) s}-1)} & \text{when } t < s \end{cases}$$

And the associated $m()$ function is

$$\begin{aligned} m(t;x_1,x_2) =& \frac{\lambda_T(t;x_1,x_2)}{\lambda_T(t;x_1,x_2)+\lambda_C(t;x_1,x_2)} \\ =& \frac{\beta x_1}{\beta x_1 + \left(\beta x_1 x_2\right) e^{(\beta x_1 x_2) t}} \\ =& \frac{1}{1+x_2 e^{(\beta x_1 x_2) t}} \\ =& \frac{1}{1+e^{\ln(x_2)+(\beta x_1 x_2) t}} \\ =& \begin{cases} \frac{1}{1+e^{\ln(2)+(2\beta x_1) t}} & x_2 = 2, \text{female} \\ \frac{1}{1+e^{(\beta x_1) t}} & x_2 = 1, \text{male} \end{cases} \end{aligned}$$

Example 3

To mimic the cox PH model sitation, we set $\theta_1 = \exp(\beta x_1), \theta_2 = \exp(\beta x_1 x_2)$, where x_1 has four levels 1,2,3,4 and x_2 has two levels 0,1, which is an indicator for gender,

$$x_2 = \begin{cases} 1 & \text{female} \\ 0 & \text{male} \end{cases}, \quad \theta_2 = \begin{cases} \exp(\beta x_1) & \text{female} \\ 1 & \text{male} \end{cases}$$

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\left(e^{\beta x_1}\right) t} e^{-(e^{\left(e^{\beta x_1 x_2}\right) s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\left(e^{\beta x_1}\right) t} e^{-(e^{\left(e^{\beta x_1 x_2}\right) s}-1)} & \text{when } t < s \end{cases}$$

That is, for females:

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\left(e^{\beta x_1}\right)t} e^{-(e^{\left(e^{\beta x_1}\right)s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\left(e^{\beta x_1}\right)t} e^{-(e^{\left(e^{\beta x_1}\right)s}-1)} & \text{when } t < s \end{cases}$$

For males:

$$S_{T,C}(t,s;x_1,x_2) = \begin{cases} e^{-\left(e^{\beta x_1}\right)t} e^{-(e^{s}-1)((t-s)^2+1)} & \text{when } t \geq s \\ e^{-\left(e^{\beta x_1}\right)t} e^{-(e^{s}-1)} & \text{when } t < s \end{cases}$$

where

$$S_T(t;x_1,x_2) = e^{-\left(e^{\beta x_1}\right)t} = [e^{-t}]^{\left(e^{\beta x_1}\right)} = [e^{-t}]^{\exp(\beta x_1)}$$

where $S_0(t) = e^{-t}$. Therefore, $S_T(t;x_1,x_2)$ is a cox PH model. And

$$f_T(t;x_1,x_2) = \exp(\beta x_1) e^{-\exp(\beta x_1)t},$$

$$\lambda_T(t;x_1,x_2) = \frac{f_T(t)}{S_T(t)} = \exp(\beta x_1) = \lambda_{T0}(t)\exp(\beta x_1), \quad \lambda_{T0}(t) = 1$$

The censoring time is

$$S_C(t;x_1,x_2) = e^{-(e^{\exp(\beta x_1 x_2)s}-1)}$$

$$f_C(s;x_1,x_2) = \exp\left(\beta x_1 x_2\right) e^{-e^{\exp(\beta x_1 x_2)s}+\exp(\beta x_1 x_2)s+1}$$

$$\lambda_C(s;x_1,x_2) = \frac{f_C(s;x_1,x_2)}{S_C(s;x_1,x_2)} = \exp\left(\beta x_1 x_2\right) e^{\exp(\beta x_1 x_2)s}$$

The associated $m()$ function is

$$\begin{aligned} m(t;x_1,x_2) =& \frac{\lambda_T(t;x_1,x_2)}{\lambda_T(t;x_1,x_2)+\lambda_C(t;x_1,x_2)} \\ =& \frac{\exp(\beta x_1)}{\exp(\beta x_1)+\exp\left(\beta x_1 x_2\right) e^{\exp(\beta x_1 x_2)t}} \\ =& \frac{1}{1+\exp(-\beta x_1+\beta x_1 x_2) exp((\beta x_1 x_2)t)} \\ =& \frac{1}{1+\exp(-\beta x_1+\beta x_1 x_2+(\beta x_1 x_2)t)} \\ =& \begin{cases} \frac{1}{1+e^{(\beta x_1)t}} & x_2 = 1, \text{female} \\ \frac{1}{1+e^{-\beta x_1}} & x_2 = 0, \text{male} \end{cases} \end{aligned}$$

Joint CDF

We denote $T_i, i = 1, ..., N$ are the independent, identically, distributed (iid) lifetimes, whose corresponding cumulative distribution function (CDF) is F, probability distribution function (PDF) is f; the censoring time is defined as $C_i, i = 1, ..., N$. C_is are also iid, with CDF denoted as G and PDF denoted as g. We set the censors happen on the right and the observed time is $Z_i = T_i \wedge C_i$, whose CDF is H and PDF is h. The $\delta_i = I_{[T_i \leq C_i]}$ is the status indicator, which shows whether the event of the ith subject is censored ($\delta_i = 0$) or observed ($\delta_i = 1$).

$$S_{T,C}(t, s; \theta_1, \theta_2) = \begin{cases} e^{-\theta_1 t} e^{-(e^{\theta_2 s}-1)(t-s+1)} & \text{when } t \geq s \\ e^{-\theta_1 t} e^{-(e^{\theta_2 s}-1)} & \text{when } t < s \end{cases}$$

where

$$\theta_1 = \beta_1^T X_1 = \beta_{11}x_{11} + \beta_{12}x_{12} + ... + \beta_{1n_1}x_{1n_1},$$
$$\theta_2 = \beta_2^T X_2 = \beta_{21}x_{21} + \beta_{22}x_{22} + ... + \beta_{2n_2}x_{2n_2}$$

And its associated pdf is

$$f_{T,C}(t, s; \theta_1, \theta_2) = \begin{cases} \Big\{(\theta_2(t-s+1)-1)e^{2\theta_2 y} + \theta_2(\theta_1-1)(t-s)e^{\theta_2 s} + (\theta_2-1)(\theta_1-2)e^{\theta_2 s} + \theta_1 - 1\Big\} \\ \times e^{-\theta_1 t} e^{-(e^{\theta_2 t}-1)(t-s+1)} & \text{whe} \\ \theta_1\theta_2 e^{-e^{\theta_2 s}+\theta_2 s-\theta_1 t+1} & \text{whe} \end{cases}$$

The marginal CDF and PDF of survival time and censoring time are:

1- CDF: $S_T(t; \theta_1, \theta_2) = P(T > t) = P(T > t, C > 0) = e^{-\theta_1 t}$, PDF: $f_T(x) = \theta_1 e^{-\theta_1 t}$

1- CDF: $S_C(s; \theta_1, \theta_2) = P(C > s) = P(T > 0, C > s) = e^{-(e^{\theta_2 s}-1)}$, PDF: $f_C(x) = \theta_2 e^{-e^{\theta_2 s}+\theta_2 s+1}$

Then the conditional pdf of censoring time given the death time is:

- When $t \geq s$

$$f_{C|T=t}(s; \theta_1, \theta_2) = \frac{f_{T,C}(s, t; \theta_1, \theta_2)}{f_T(t; \theta_1, \theta_2)} =$$
$$\frac{1}{\theta_1}\Big\{(\theta_2(t-s+1)-1)e^{2\theta_2 y} + \theta_2(\theta_1-1)(t-s)e^{\theta_2 s} + (\theta_2-1)(\theta_1-2)e^{\theta_2 s} + \theta_1 - 1\Big\} e^{-(e^{\theta_2 s}-1)(t-s+1)}$$

$$F_{C|T=t}(s; \theta_1, \theta_2) = \int_0^s f_{C|T=t}(s; \theta_1, \theta_2)ds =$$

- When $t < s$

$$f_{C|T=t}(s; \theta_1, \theta_2) = \frac{f_{T,C}(s, t; \theta_1, \theta_2)}{f_T(t; \theta_1, \theta_2)} = \frac{\theta_1\theta_2 e^{-e^{\theta_2 s}+\theta_2 s-\theta_1 t+1}}{\theta_1 e^{-\theta_1 t}} = \theta_2 e^{-e^{\theta_2 s}+\theta_2 s+1}$$
$$F_{C|T=t}(s; \theta_1, \theta_2) = \int_t^s f_{C|T=t}(s; \theta_1, \theta_2)ds = e^{1-e^{\theta_2 t}} - e^{1-e^{\theta_2 s}}$$

parameter setting

For example, if we set

$$\theta_1 = \exp(\beta_0 + \beta_1 x_1) = \exp(1 + x_1), \theta_2 = 1$$

where $\beta_0 = \beta_1 = 1$

$$S_{T,C}(t, s; \theta_1, \theta_2) = \begin{cases} e^{-\theta_1 t} e^{-(e^s - 1)(t - s + 1)} & \text{when } t \geq s \\ e^{-\theta_1 t} e^{-(e^s - 1)} & \text{when } t < s \end{cases}$$

And its associated pdf is

$$f_{T,C}(t, s; \theta_1, \theta_2) = \begin{cases} \Big\{((t - s + 1) - 1)e^{2y} + (\theta_1 - 1)(t - s)e^s + \theta_1 - 1\Big\} \\ \times e^{-\theta_1 t} e^{-(e^s - 1)(t - s + 1)} \\ & \text{when } t \geq s \\ \theta_1 \theta_2 e^{-e^{\theta_2 s} + \theta_2 s - \theta_1 t + 1} & \text{when } t < s \end{cases}$$

(((x-y + 1) - 1)exp(2y) + (a-1)(x-y)exp(y) + a-1) exp(-(exp(t) -1)(x-y + 1))

The marginal CDF and PDF of survival time and censoring time are:

1- CDF: $S_T(t; \theta_1, \theta_2) = P(T > t) = P(T > t, C > 0) = e^{-\theta_1 t}$, PDF: $f_T(x) = \theta_1 e^{-\theta_1 t}$

1- CDF: $S_C(s; \theta_1, \theta_2) = P(C > s) = P(T > 0, C > s) = e^{-(e^s - 1)}$, PDF: $f_C(x) = e^{-e^s + s + 1}$

Then the conditional pdf of censoring time given the death time is:

- When $t \geq s$

$$f_{C|T=t}(s; \theta_1, \theta_2) = \frac{f_{T,C}(s, t; \theta_1, \theta_2)}{f_T(t; \theta_1, \theta_2)} =$$

$$\frac{1}{\theta_1} \Big\{(t - s)e^{2y} + (\theta_1 - 1)(t - s)e^s + \theta_1 - 1\Big\} e^{-(e^s - 1)(t - s + 1)}$$

$$F_{C|T=t}(s; \theta_1, \theta_2) = \int_0^s f_{C|T=t}(s; \theta_1, \theta_2) ds = 1 - \frac{1}{\theta_1}(e^s + \theta_1 - 1)e^{-(e^s - 1)(t - s + 1)}$$

- When $t < s$

$$f_{C|T=t}(s; \theta_1, \theta_2) = \frac{f_{T,C}(s, t; \theta_1, \theta_2)}{f_T(t; \theta_1, \theta_2)} = \frac{\theta_1 e^{-e^s + s - \theta_1 t + 1}}{\theta_1 e^{-\theta_1 t}} = e^{-e^s + s + 1}$$

$$F_{C|T=t}(s; \theta_1, \theta_2) = \int_t^s f_{C|T=t}(s; \theta_1, \theta_2) ds = e^{1 - e^t} - e^{1 - e^s}$$

$$m(t; \theta_1, \theta_2) = \frac{\theta_1}{\theta_1 + \theta_2 e^{\theta_2 t}} = \frac{1}{1 + \frac{\theta_2}{\theta_1} e^{\theta_2 t}} = \frac{1}{1 + e^{-\beta_0 - \beta_1 x_1} e^t} = \frac{1}{1 + e^{-\beta_0 - \beta_1 x_1 + t}}$$