The Tipping Point: Analysis of Conditions

Reference Paper: *The Tipping Point: A Mathematical Model for the Profit Driven Abandonment of Restaurant Tipping*, Sarah M. Clifton et al (2018)

Abstract

The Tipping Point: A Mathematical Model for the Profit Driven Abandonment of Restaurant Tipping by Sarah M. Clifton et al introduces a dynamical model that represents tipping behavior. The dynamical system models the behavior of restaurant operations based on diners, cooks, and waiters. In this analysis, different conditions will be explored to determine how no tipping and equal wage impact restaurant operations. This paper will explore the model based on the changes in the ratio of quality to service and the fraction of diners, waiters, and cooks.

APPM 3010 Chaos in Dynamical Systems

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Introduction

Compared to other countries, tipping culture is prominent in the United States. The act and meaning of tipping is to show gratitude and appreciation. To balance the minimum wage between tipped and non-tipped jobs, the minimum wage for tipped employees is much lower. *The Tipping Point: A Mathematical Model for the Profit-Driven Abandonment of Restaurant Tipping* by Clifton et al. (2018) explores the behavior of tipping culture in the United States then translates it into three ordinary differential equations that model the dynamics of restaurants. The three ordinary differential equations represent perspectives of the cook, waiters, and diners. It proposes that tipping will be forbidden in restaurants when it reaches a certain critical threshold because the restaurant profitability will be higher when compared to tipped cases. Although the base pay for cooks and waiters will drop, the overall diner value and satisfaction will increase because of the drop in menu price. Diners will perceive that the quality of the restaurant will decrease; however, it will balance out with the decrease in menu price. In this paper, constant conditions will be explored and compared with graphs generated using MATLAB. The first condition will be when the tip rate is zero, the fraction of diners, cooks, and waiters are equal, and the base wage pay for both waiters and diners is equal. For the computational analysis, initial conditions related to the number of waiters, diners, and cooks will be tested. Furthermore, the ratio of quality to service will also be tested to compare the dynamics.

Mathematical Formulations

Fraction of Diners

$$\frac{dD}{dt} = (1-D)\frac{v_1}{v_1+v_2} - D\frac{v_2}{v_1+v_2} \tag{1}$$

$$v_1(W,C) = \frac{W + rr_{CW}C}{m_1(1+r_1)} \tag{2}$$

$$v_2(W,C) = \frac{(1-W) + rr_{CW}(1-C)}{m_2(1-r_2)} \tag{3}$$

$v_1(W,C)$ and $v_2(W,C)$ represent the restaurant quality of two competing restaurants
D variable is the fraction of diners
W variable is the fraction of waiters

Fraction of Waiters

$$\frac{dW}{dt} = (1-W)\frac{b_{W1}+g_1}{b_{W1}+g_1+b_{W2}+g2} - W\frac{b_{W2}+g_2}{b_{W1}+g_1+b_{W2}+g2} \tag{4}$$

$$g_1(D,W) = \frac{m_1 r_{DW} D T_1}{W} \tag{5}$$

$$g_2(D,W) = \frac{m_2 r_{DW}(1-D)T_2}{W} \tag{6}$$

$g_1(D,W)$ and $g_2(D,W)$ represent the hourly gratuity and b_{W1} and b_{W2} represent base pay at two competing restaurants.
W is the fraction of waiters
D is the fraction of diners

Fraction of Cooks

$$\frac{dC}{dt} = (1-C)\frac{b_{C1}}{b_{C1}+b_{C2}} - C\frac{b_{C2}}{b_{C1}+b_{C2}} \tag{7}$$

b_{C1} and b_{C2} represent base pay for cooks.
C is the fraction of cooks

Hourly Profit

$$P(D,W,C) = m_1 r_{DW} D - b_{W1}W - b_{C1} r_{CW} C \tag{8}$$

Variables

r is the ratio of food to service importance for customers

$$r_{DW} = \frac{N_D}{N_W} \tag{9}$$

$$r_{CW} = \frac{N_C}{N_W} \tag{10}$$

Equation (9) represents the ratio of diners to waiters. Equation (10) represents the ratio of cooks to waiters. N_C, N_D, and N_W represent the number of cooks, number of diners, and number of waiters in the system.

Numerical Results

Constant Conditions: No Tips, Equal Wage, and Equal $\mathbf{N}_D, N_W, N_C$

Refer below for the ODE modeled with the conditions listed above.
Fraction of Diners

$$\frac{dD}{dt} = \frac{1}{3} - \frac{2}{3}D \tag{11}$$

Fraction of Waiters

$$\frac{dW}{dt} = -W + \frac{1}{2} \tag{12}$$

Fraction of Cooks

$$\frac{dC}{dt} = -C + \frac{1}{2} \tag{13}$$

The following conditions were utilized to generate the set of ordinary differential equations listed above: the number of cooks, number of waiters, and number of diners are equal, tip rates are set to zero, the ratio of cook to waiter is one, menu prices are 15, and the ratio of food quality to service is 20. Furthermore, the base pay for cooks and waiters is equal and was set based on living wage standards. According to MIT, the living wage for a single adult in Boulder county is around \$18.53.

Based on the linear ordinary differential equations, the fixed point is 0.5.The fixed point is classified as stable. When eliminating the tip rate for waiters, the differential equations are nearly similar to the model for the cooks and diners. As a result, the dynamics of these three equations will be similar. Since the model doesn't have any influence based on gratuity and the initial conditions are constant, the fraction of waiters, cooks, and diners will be equal over time.When solving the ordinary differential equations using separation of variables, the time dependent solution for the fraction of diners is $D(t) = \frac{-e^{\frac{-2t+\frac{3}{2}ln(\frac{1}{3})}{3}}}{2} + \frac{1}{2}$. The solution for the fraction of waiters and cooks are $W(t) = -e^{-t+ln(\frac{1}{6})} + \frac{1}{2}$ and $C(t) = -e^{-t+ln(\frac{1}{6})} + \frac{1}{2}$. Figure one generates the fraction of waiters, cooks, and diners as time increases. Although the fraction of diners is lower than the fraction of waiters and cooks initially, it will start to approach the fixed point as time passes. Since the fixed point is stable, the number of waiters, cooks, and diners will stay around the same value. Since the fixed point is stable, cooks and waiters will want to continuously work at the restaurant. This means that the base pay for both cooks and waiters will remain constant, thus, causing the expenses to remain constant. When plotting the fraction of waiters and cooks to restaurant profitability, it is important to note that the profit is negative throughout time. The restaurant will continuously increase in negative profit. As the fixed point analysis has indicated, the fraction of cooks and waiters will remain constant at 0.5. In this case, N_D, N_W, and N_C are all 10. Since the number of diners and employees are equal, the restaurant will have negative profit because it will have to pay more expenses compared to the flow of cash (profit). The incoming cash flow will be used to pay waiters and cooks in equal wages. Furthermore, since the number of waiters and cooks are equal, it will further increase the expenses causing it to accumulate which is reflected as negative cash. The restaurant-quality is related to the profit because it takes into consideration the base pay and the number of workers. As the quality of the restaurant increase, the profit is going to decrease further as a negative value. Despite changes to other variables, the profit will always be negative indicating that optimization of the fraction of cooks and waiters is key in maintaining a good balance between cash flow and expenses. This optimization will produce desirable profit values which will positively impact quality and restaurant operations.

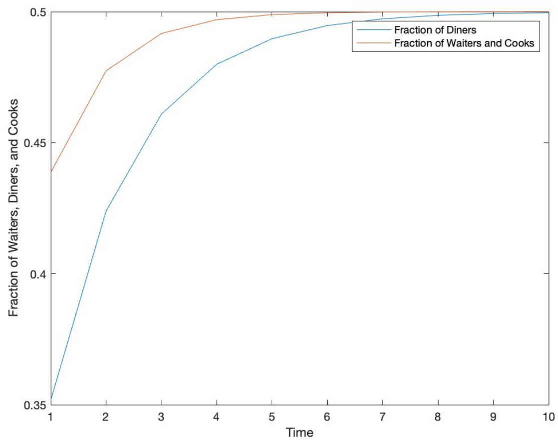


Figure 1: Time Dependent Solution (Diners, Waiters, and Cooks)

Computational Analysis (MATLAB)

In order to perform graphical analysis, the MATLAB code from the research paper was utilized with different conditions. Supplementary material was also used as reference to compare results with the original research paper.

Variable	Meaning	Units	Range	Baseline
D	Fraction of diners at our restaurant	…	[0, 1]	…
W	Fraction of waiters at our restaurant	…	[0, 1]	…
C	Fraction of cooks at our restaurant	…	[0, 1]	…
t	Time (dimensionless)	…	$[0, \infty)$	…
r	Relative importance of food quality versus service quality, typically a value exceeding one	…	[4, 20]	12[a]
r_{CW}	Ratio of total cooks to waiters in the system	…	[0.5, 2]	1
r_{DW}	Ratio of total diners to waiters in the system	…	[1, 20]	12
m_1	Average menu cost per hour at our restaurant	$/h	[5, 20]	10
b_{W1}	Waiters' base pay per hour at our restaurant	$/h	[2.13[b], 25]	5.00[c]
b_{C1}	Cooks' base pay per hour at our restaurant	$/h	[7.25[b], 25]	10.40[c]
T_1	Average tip rate at our restaurant, determined by either social convention or mandated by restaurant owner	…	[0.01, 0.5]	0.19[d]
v_1	Meal value perceived by customers at our restaurant	1/$	…	…
g_1	Gratuity per hour at our restaurant	$/h	…	…

Figure 2: Variable Ranges for Simulation

No Tips, Equal Wage, and $\mathbf{N}_D = 10, N_W = 5, N_C = 3$

For this optimization, the ratio of food to service (R) was was set to 20 which means that the weight placed on the quality of food was greater than service. Furthermore, the same conditions were applied where the base pay for both waiters and cooks were set to $18.53. The MATLAB optimization requires a value for legal minimum wage. According to MIT, the minimum age for Boulder county is around $12. The initial number of diners, waiters, and cooks were set to $N_D = 10$, $N_W = 5$,$N_C = 3$. Based on initial observations, the dynamics of this model isn't sophisticated compared to the analysis done above. When inputting these values, the relationships between the variables are linear. The key difference between the model described in the paper and this optimization is that the restaurant operations tend to depend on the cooks or waiters. In this case, since the quality of food is more important to customers than service, the diner value, restaurant profitability, and wages depend on the cook. As seen in figure 3, the changes in these conditions will cause the fraction of waiters to decrease more than the fraction of cooks. Since diners desire quality of food, the fraction of cooks will nearly remain constant while the fraction of diners increases. It is also key to notice that the menu price, quality, and the cook pay are related. As seen in figure 4, as the menu price goes up, the cook pay increases. Furthermore, it is also observed that as the cook pay increases, the quality of the restaurant will increase. Since tipping is eliminated and wages between waiters and diners are equal, the restaurant will be want to increase cook pay since it is more profitable because diners desire better food quality. When comparing the importance of cook pay to waiter pay, waiter pay nearly remains constant as the menu price increases. It would be assumed that menu price wouldn't impact the waiter base pay because service doesn't impact the quality of food. Figure 5b represents the relationship between menu price and waiter pay. Although the graph has minimal oscillations as menu price increases, the range of waiter pay nearly remains constant. Since waiters do not receive gratuity, base wage and restaurant management can be leading factors to the decrease in fraction of waiters. In this case, service isn't desired compared to the importance of cooks. Therefore, the fraction of waiters would decrease if they find employment at a different place with better conditions.

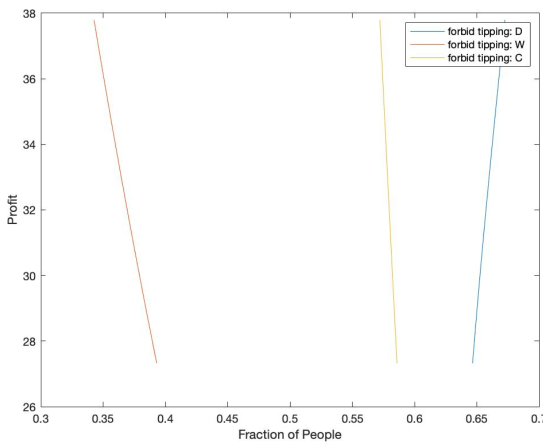
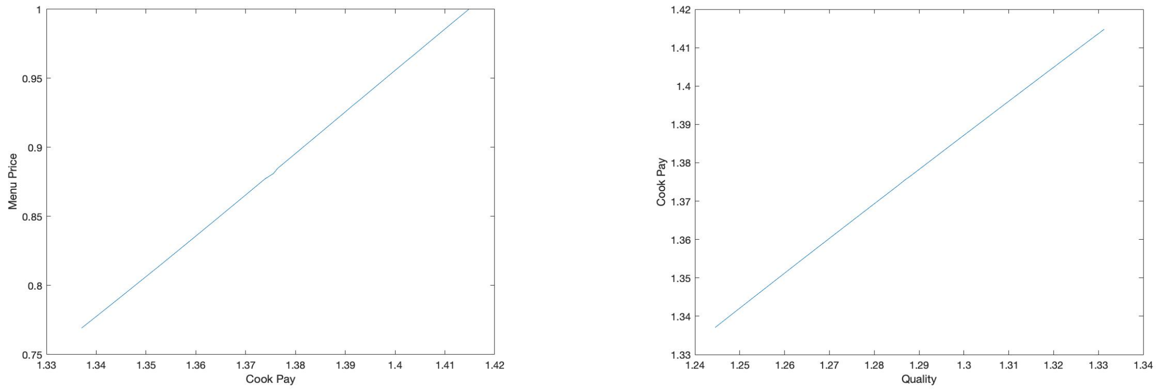


Figure 3: Fraction of People and Profit

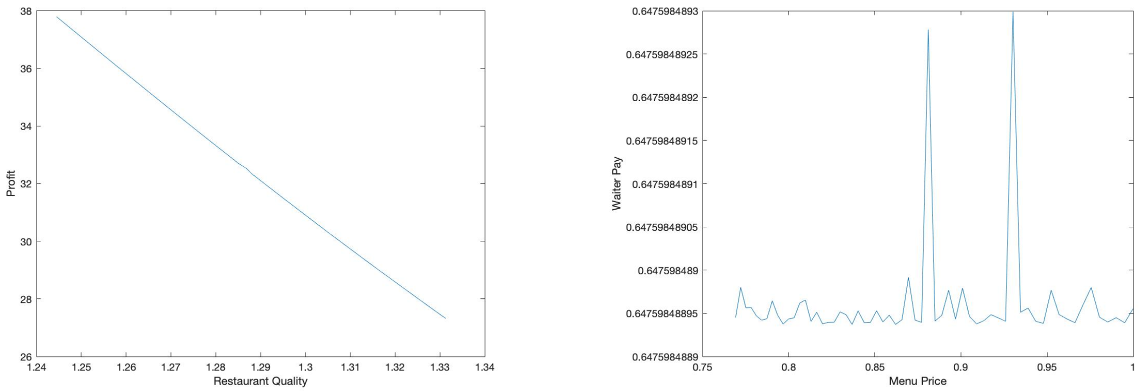
It is also important to notice that as the restaurant-quality increases, the profitability of the restaurant decreases. This can be caused by two different reasons. The first is that it is caused by an imbalance of cash flow and expenses. In other words, the restaurant will want to increase the base pay of cooks because it satisfies diners; however, this will also cause the expenses to increase. In return, it will cause the profit to decrease if other actions are not taken. If the restaurant lacks the ability to increase marketing tactics or other methods of balancing out expenses, it will cause the profit to decrease dramatically. The second reasoning is that the restaurant enters a different niche of restaurant quality. As mentioned above, diners desire good quality which means that the cook base pay and menu price will increase. When the menu price increases, the diner value will



(a) Cook Pay and Menu Price

(b) Cook Pay and Quality

Figure 4: Restaurant Operation Dependent on Cook



(a) Restaurant Quality and Profit

(b) Waiter Pay and Menu Price

Figure 5: Restaurant Quality Dependence

decrease because some diners will not be able to afford the price of each meal. If the restaurant continuously emphasizes the quality of food, it will eventually cause it to change its target market and niche of restaurant quality. The restaurant will transform into an upper-level dining restaurant with a niche target market. Limited populations will be able to afford the restaurant value. It should be noted that the dynamics of this situation isn't taken into consideration by this model because it lacks variables required to model different restaurant styles. Figure 4a represents the linear relationship between profit and restaurant quality.

No Tips, Equal Wage, and N$_D = 2, N_W = 5, N_C = 3$

For this simulation, the same conditions were applied; however, the number of diners, waiters, and cooks changed.The number of diners are two, number of waiters are five, and the number of cooks are three. The number of diners are less than the number of workers. Results for this simulation are similar to the first analysis when number of people are equal.

The quality of the restaurant takes into consideration the number of cooks and waiters and the weights placed on these groups. Furthermore, the profit also takes into consideration the fraction of people and the pay. When the quality of the restaurant increases, the profit decreases. Similar to the first simulation, the profit will be negative because the expenses are greater than the cash flow. The wages of cooks and waiters will remain nearly constant which means expenses will remain constant. In this case, the profit will be negative because there will be a limited flow of cash from diners. The number of diners is much less than the number of cooks and number of waiters. Although this might be a better situation for diners since they receive more attention and service, the restaurant will not be able to operate unless there is enough demand to equal out the number of employees. One lacking feature of this model is the inability to represent restaurant operations at different levels. For example, the operations of fine dining restaurants will be different from casual dining. Fine dining restaurants tend to have limited customers and niche markets. Operations of fine dining restaurants could have alternative cash flows besides the main flow from profit. The model assumes limited variables for restaurant operations which means it will not be able to properly model fine dining and other unique styles of restaurant operations.

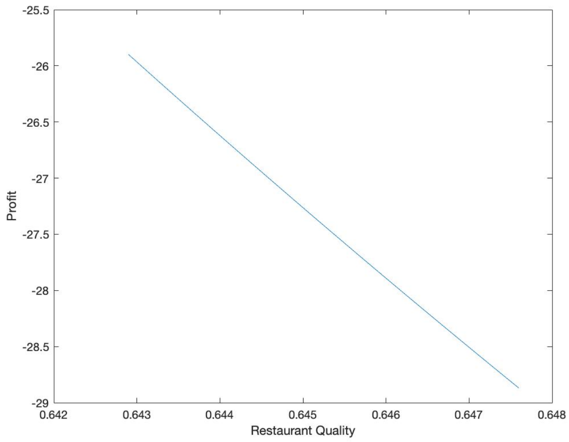
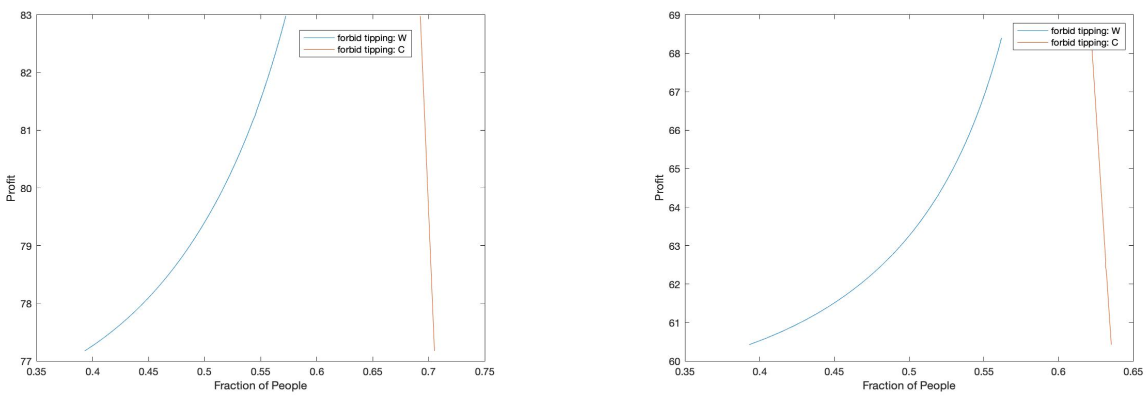


Figure 6: Restaurant Quality and Profit

No Tips, Equal Wage, $N_D = 10, N_W = 3, N_C = 1$

According to eposnow, there are ideal optimizations for the number of cooks and waiters. Usually, it is ideal to have one waiter for five tables and four cooks for 50 customers per hour. In this case, the number of diners N_D is 10. Using the optimization provided above, the restaurant will employ around three waiters and one cook. Utilizing this optimization produced the maximum amount of profit when compared to the simulation with $N_D = 10$, $N_W = 5$, and $N_C = 3$. As seen in figure 7a, the starting profit with the fraction of cooks and waiters is above 77 per hour which is greater than the starting profit for the simulation with $N_D = 10$, $N_W = 5$, and $N_C = 3$. In this plot, the profit reaches greater than 80 per hour when the fraction of cooks and waiters are further optimized. To further compare the results of employee optimization, another plot was generated with $N_D = 10$, $N_W = 2$, and $N_C = 2$. In this case, the number of cooks and waiters are equal; however, it is still less than the number of waiters used in previous simulations.



(a) Fraction of People and Profit Using Optimization (b) Fraction of People ($N_D = 10$,$N_W = 2 = N_C$) and Profit

Figure 7: Fraction of Employees Optimization and Profit

Figure 7b shows the profit and fraction of employees when the number of cooks and waiters are equal. In this case, the profit value per hour is smaller compared to the profit when the optimization rule is utilized ($N_D = 10, N_W = 3, N_C = 1$). The starting profit when the number of waiters and cooks is equal is around 60.5 per hour. It is important to note that the graph indicates an increase in profit when the fraction of employees is further optimized.

When the tip rate is zero and the wages are equal, optimization of employees is key in maximizing the profit value and operations of the restaurant. The simulation indicates that it is generally profitable when the number of cooks is less than the number of waiters. When considering the profit values for this simulation, it is important to note that it only takes into consideration the wage expenses. Profit is the difference between earned cash and expenses. For a more accurate reflection of profit, it would be key to indicate other expenses such as utilities and the purchase of raw materials. Although the model lacks real-world applications, it is generally accurate in predicting and visualizing trends related to restaurant operations, tips, and employee dynamics.

Discussion and Conclusion

When tipping is eliminated and wages are equal between waiters and cooks, restaurant operations depend heavily on the needs of the diners. Furthermore, profit depends on the optimization of employees. In some scenarios, restaurants would have to change business models to make them more attractive to diners. As seen in the first computational simulation, when the diners desire better food over service, cooks become more important in restaurant operations. Profit and menu prices depend on the fraction of cooks which eventually impacts the quality delivered to the diners. The R value becomes more critical in these situations.

In terms of employee dynamics, the results from this simulation differ from the results presented in the original paper. The original paper presents that the number of waiters decreases because of the elimination of tips. However, when the wages are set equal to cooks and tips are eliminated, the difference in the fraction of waiters is minimal. Waiters in this situation would be leaving because of a better wage offer from a competitor. The fraction of cooks also stay nearly constant at conditions utilized in this paper. When the number of employees (N_C and N_W) change, it is reflected in the profit values. As seen in the last two simulations, restaurants need to optimize the number of cooks and waiters for maximized profits. Employee optimizations are different depending on the type of restaurant. However, it is generally encouraged to have one waiter for five tables and four cooks for 50 customers per hour. Using this optimization, the simulation displayed maximum profit.

In conclusion, when tipping is eliminated and wages are equal, restaurant dynamics depend on diner needs. Furthermore, it is key for restaurants to optimize different aspects of operation. The model presented in the reference paper cannot accurately model different styles of restaurant dining. For future research, it would be interesting to analyze systems that can model the dynamics of different restaurants. For example, it would be interesting to see the difference between the models created for fine dining and fast-food restaurants. In addition, the next level of research would be to increase the accuracy of this model by considering different variables that contribute to operations. The current model only considers wage expenses; however, it would be important to consider additional expenses to create a more accurate model.

References

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Appendix

Part 1 Initial Conditions
Fraction of Diners: $D(t) = 10, t = 0$
Fraction of Waiters: $W(t) = 10, t = 0$
Fraction of Cooks: $C(t) = 10, t = 0$
MATLAB Code
NOTE: 033-optimization-restaurant-protocol.m file was used to generate data points. Another MATLAB code was generated to plot graphs with different variables. Please refer below for the detailed code for generating the graphs.

```
% RATIO COOK PAY AND RATIO QUALITY
figure
plot(ratio_quality_notip,ratio_cookpay_notip)
xlabel('Quality')
ylabel('Cook Pay')
% RATIO WAITER PAY AND QUALITY
figure
plot(ratio_quality_notip,ratio_waiterbasepay_notip)
xlabel('Quality')
ylabel('Waiter Pay')
%MODIFIED RESTAURANT PROFIT AND RESTAURANT QUALITY
figure
q1_1 = (equil_1(:,2)+R*equil_1(:,3)); q2_1 = (Nw-equil_1(:,2)+R*(Nc-equil_1(:,3)
    ));
q1_2 = (equil_2(:,2)+R*equil_2(:,3)); q2_2 = (Nw-equil_2(:,2)+R*(Nc-equil_2(:,3)
    ));
ratio_quality_tip = q1_1./q2_1; ratio_quality_notip = q1_2./q2_2;
pmax2
plot(ratio_quality_notip,pmax2)
xlabel('Restaurant Quality')
ylabel('Profit')
% MENU PRICE AND COOK PAY
figure
m1_1 = (m1*(1+tip_conventional')); m2_1 = (m2*(1+tip_conventional'));
m1_2 = (m1); m2_2 = (m2*(1+tip_conventional'));
ratio_menu_tip = m1_1./m2_1; ratio_menu_notip = m1_2./m2_2;
plot(ratio_cookpay_notip,ratio_menu_notip)
xlabel('Cook Pay')
ylabel('Menu Price')
% COOK PAY AND WAITER PAY
figure
plot(ratio_cookpay_notip,ratio_waiterbasepay_notip)
xlabel('Cook Pay')
ylabel('Waiter Pay')
% DINER VALUE AND MENU PRICE
figure
ratio_menu_notip % Menu Price
ratio_value_notip % Diner Value
plot(ratio_menu_notip,ratio_value_notip)
xlabel('Menu Price')
ylabel('Diner Value')
```

```
% DINER VALUE AND WAITER PAY
figure
plot(ratio_value_notip,ratio_waiterbasepay_notip)
xlabel('Diner Value')
ylabel('Waiter Pay')
%FRACTION OF Waiters AND PROFIT
[ equil_2(:,1)/Nd,  equil_2(:,2)/Nw, equil_2(:,3)/Nc] % Fraction of People
pmax2 % Profit
plot(equil_1(:,2)/Nw,pmax2)
xlabel('Fraction of Waiters')
ylabel('Profit')
%FRACTION OF WAITERS AND WAITER PAY
figure
plot(ratio_waiterbasepay_notip,[equil_2(:,2)/Nw])
xlabel('Waiter Pay')
ylabel('Fraction of Waiters')
% FRACTION OF COOK AND PAY
figure
plot(equil_2(:,3)/Nc,ratio_cookpay_notip)
xlabel('Fraction of Cooks')
ylabel('Cook Pay')
% FRACTION OF WAITERS AND PAY
figure
plot(equil_2(:,2)/Nw,ratio_waiterbasepay_notip)
xlabel('Fraction of Waiters')
ylabel('Waiter Pay')
% QUALITY AND FRACTION OF WAITERS
figure
plot(equil_2(:,2)/Nw,ratio_quality_notip)
xlabel('Fraction of Waiters')
ylabel('Quality')
% QUALITY AND FRACTION OF COOKS
figure
plot(equil_2(:,3)/Nc,ratio_quality_notip)
xlabel('Fraction of Cooks')
ylabel('Quality')
% MENU PRICE AND WAITER WAGE
figure
plot(ratio_menu_notip,ratio_waiterbasepay_notip)
xlabel('Menu Price')
ylabel('Waiter Pay')
% MENU PRICE AND COOK WAGE
figure
plot(ratio_menu_notip,ratio_cookpay_notip)
xlabel('Menu Price')
ylabel('Cook Pay')
%FRACTION OF DINERS AND PROFIT
plot([equil_2(:,1)/Nd],pmax2)
hold off
xlabel(' Pay')
ylabel('Profit')
legend('Cook Pay','Waiter Pay', 'Fraction of Diners')
%FRACTION OF DINERS AND PAY
plot([equil_2(:,1)/Nd],ratio_waiterpay_notip,[equil_2(:,1)/Nd],ratio_cookpay_notip)
xlabel('Fraction of Diners')
ylabel('Pay')
legend('Waiter Pay','Cook Pay')
% HAND CALCULATION: PART 1
figure
x=[1:1:10]
```

```
y=-(exp((-2*x+(3/2)*log(1/3))/3)/2)+1/2
y2=-(exp(-x+log(1/6)))+1/2
plot(x,y)
hold on
plot(x,y2)
hold off
xlabel('Time')
ylabel('Fraction of Waiters, Diners, and Cooks')
legend('Fraction of Diners', 'Fraction of Waiters and Cooks')
```