Supplemental Sheet for Polynomials and Rational Functions

This is a worksheet to supplement the material learned in class - it will go over some things you may see in the future that we do not have time to go over during lecture. The completion of this assignment is not required: it will not be turned in nor will it be graded, but you may find it useful to work through these problems.

1 Polynomials

Theorems concerning polynomials:

- **Synthetic Division**: Synthetic division can be used when dividing a polynomial by a binomial i.e, dividing some expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$ by an expression of the form $x - k$. You cannot use synthetic division if the divisor is not a binomial of degree one with a leading coefficient of one.

$$\begin{array}{r|cccccc} k & a_n & a_{n-1} & \cdots & a_2 & a_1 & a_0 \\ + & \downarrow & (k \cdot a_n) & \cdots & \cdots & \cdots & \cdots \\ \hline & a_n & (a_{n-1} + k \cdot a_n) & \cdots & \cdots & \cdots & \text{remainder} \end{array}$$

  We set up the division as seen in the table above, with the coefficients of the polynomial inside the division box and k outside. Then, we can compute as follows:

  1. Pull down the a_n for the first entry beneath the box.
  2. Multiply k by next entry below the box (first will be a_n, then $(a_{n-1} + k \cdot a_n)$ etc.)
  3. Add the two rows within the division box, write sum in entry below the box (we first have $(a_{n-1} + k \cdot a_n)$, next would be $(a_{n-2} + k(a_{n-1} + k \cdot a_n)$ etc.)
  4. Continue this process until we reach the final sum, $a_0 + \ldots$ in the final, rightmost entry below the division box. If this quantity is equal to 0, we have no remainder. Otherwise, we must include this quantity as the remainder.
  5. The entries below the division box are the coefficients of our answer, a polynomial of degree $n-1$ i.e., our answer will be of the form: $a_n x^{n-1} + (a_{n-1} + k \cdot a_n)x^{n-2} + \cdots + \frac{remainder}{x-k}$.

- **Remainder Theorem**: When a polynomial, $p(x)$ is divided by some $x-k$, then the remainder of that division is equal to the value of the polynomial at the point $x = k$. So, if $\frac{p(x)}{x-k}$ has a remainder, then the remainder $= p(k)$. If we use synthetic division, this theorem can make evaluating complicated polynomials at certain points much easier (and able to be done without a calculator)!

- **Rational Root Theorem**: For a polynomial equation $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0 = 0$, the rational roots theorem states that the only possible rational solutions to the equation (the only possible rational roots of the polynomial) will be of the most simplified form of $\frac{p}{q}$, where p is an integer factor of the constant term a_0 and q is an integer factor of the leading coefficient a_n. We can formulate a list of all of the possible rational roots of a polynomial and test them. For example, if the polynomial $3x^3 + x + 4$ has any rational roots, they will be one of the following numbers: ± 1, $\pm\frac{1}{3}$,± 4, $\pm\frac{4}{3}$, ± 2, or $\pm\frac{2}{3}$. Note that some polynomials will have no rational roots.

- **Conjugate Zeros Theorem**: If a polynomial $p(x)$ has a complex root $a - ib$, then the complex conjugate of the root, $a + ib$, is also a root of the same polynomial.

With this, answer the following:

1. Compute $\frac{3x^3+4x^2+5}{x+3}$ using synthetic division.
2. $p(x) = 7x^4 + 48x^3 - 120x^2 - 6$. Evaluate $p(2)$ using the remainder theorem.
3. Find all of the roots of the following polynomials:
   (a) $3x^3 - 4x^2 - 8x - 1$
   (b) $x^3 - x^2 + 25x - 25$ given that one of the roots is $-5i$

2 Functions

Function rules:

- **Relations versus Functions**: A relation may show the relationship between an input and an output, but a **function** is a more specific classification of a relation. While a relation can have multiple outputs for one input, a **function** must have exactly one output for a given input.

- **Odd and Even Functions**: We classify a function $f(x)$ as **odd** if $f(-x) = -f(x)$.We classify a function as **even** if $f(-x) = f(x)$. Odd functions show symmetry across the x-axis, while even functions show symmetry across the y-axis. For example, x^3 is an odd function, while x^2 is an even function.

- **Bijective Functions**: We say that a function is **bijective** when it is both **one-to-one** and **onto**. A function is **one-to-one** if every element in the function's domain (each x value) is mapped to exactly one element in the function's range (each y value). In other words, if $f(a) = f(b)$ and the function is one-to-one, then $a = b$. A function is **onto** if every element in the function's range (each y value) can be produced by an element in the function's domain (an x value).

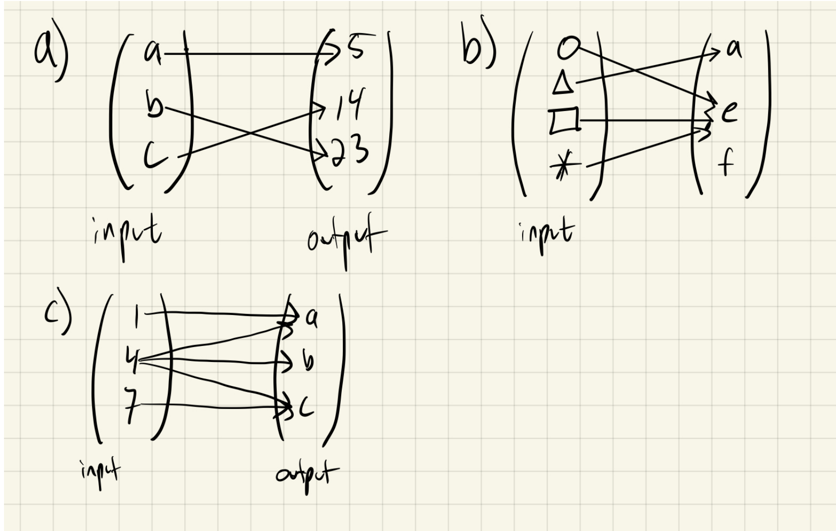
1. Classify the following as one-to-one, onto, neither, or bijective:

   (a) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = e^x$ (b) $f : \mathbf{R} \to \mathcal{R}$, $f(x) = x^3 + 5x^2 + x + 1$ (c) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = x^3$ (d) $f : \mathcal{R} \to \mathcal{R}$, $f(x) = x^2$

2. Classify the following functions as even, odd, or neither:

   (a) $\cos(x)$ (b) $\sin(x)$ (c) $|x|$ (d) $x + 7$

3. Classify the following as relations or functions:



3 More Inequalities

Some things to keep in mind:

- **Square Roots**: When dealing with an inequality similar to the form $x^2 > r$, we cannot simply write $x > \pm\sqrt{r}$. We simplify as such:
  1. $x^2 > r \to x > \sqrt{r}$ OR $x < -\sqrt{r}$
  2. $x^2 < r \to -\sqrt{r} < x < \sqrt{r}$
  3. $x^2 \geq r \to x \geq \sqrt{r}$ OR $x \leq -\sqrt{r}$
  4. $x^2 \leq r \to -\sqrt{r} \leq x \leq \sqrt{r}$
- **Interval Notation**: When we solve an inequality, we can write our solution in interval notation instead of using $>, <, \geq, \leq$. Say a, b are positive constants and $a < b$. If we have a $<$ or $>$ we use parenthesis (and when we have a $\leq$ or $\geq$ we use a bracket [. When we write two solutions, like $x < 1$ OR $x > 4$, we can unify these with the symbol $\cup$. Here are some examples of how we can change notations:
  1. $x > a \to x \in (a, \infty)$
  2. $x < a \to x \in (-\infty, a)$
  3. $a < x \leq b \to x \in (a, b]$
  4. $x \leq -a$ OR $x \geq b \to (-\infty, -a] \cup [b, \infty)$

With this, answer the following:

1. Simplify $x^2 \geq 16$. Write your answer in interval notation.
2. Write the following in interval notation: $-43 < x \leq 7$ OR $9 \leq x \leq 18$ OR $x > 89$.