Supplemental Sheet for Polynomials and Rational Functions

This is a worksheet to supplement the material learned in class - it will go over some things you may see in the future that we do not have time to go over during lecture. The completion of this assignment is not required: it will not be turned in nor will it be graded, but you may find it useful to work through these problems.

1 Polynomials

Theorems concerning polynomials:

- **Synthetic Division**: Synthetic division can be used when dividing a polynomial by a binomial i.e, dividing some expression of the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0$ by an expression of the form $x - k$. You <u>cannot</u> use synthetic division if the divisor is not a binomial of degree one with a leading coefficient of one.

$$\begin{array}{r|cccccc}
k & a_n & a_{n-1} & \cdots & a_2 & a_1 & a_0 \\
+ & \downarrow & (k \cdot a_n) & \cdots & \cdots & \cdots & \cdots \\
\hline
 & a_n & (a_{n-1} + k \cdot a_n) & \cdots & \cdots & \cdots & \text{remainder}
\end{array}$$

  We set up the division as seen in the table above, with the coefficients of the polynomial inside the division box and k outside. Then, we can compute as follows:

  1. Pull down the a_n for the first entry beneath the box.
  2. Multiply k by next entry below the box (first will be a_n, then $(a_{n-1} + k \cdot a_n)$ etc.)
  3. Add the two rows within the division box, write sum in entry below the box (we first have $(a_{n-1} + k \cdot a_n)$, next would be $(a_{n-2} + k(a_{n-1} + k \cdot a_n)$ etc.)
  4. Continue this process until we reach the final sum, $a_0 + \ldots$ in the final, rightmost entry below the division box. If this quantity is equal to 0, we have no remainder. Otherwise, we must include this quantity as the remainder.
  5. The entries below the division box are the coefficients of our answer, a polynomial of degree $n-1$ i.e., our answer will be of the form: $a_nx^{n-1} + (a_{n-1} + k \cdot a_n)x^{n-2} + \cdots + \frac{remainder}{x-k}$.

- **Remainder Theorem**: When a polynomial, $p(x)$ is divided by some $x-k$, then the remainder of that division is equal to the value of the polynomial at the point $x = k$. So, if $\frac{p(x)}{x-k}$ has a remainder, then the remainder $= p(k)$. If we use synthetic division, this theorem can make evaluating complicated polynomials at certain points much easier (and able to be done without a calculator)!

- **Rational Root Theorem**: For a polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x^1 + a_0 = 0$, the rational roots theorem states that the only possible rational solutions to the equation (the only possible rational roots of the polynomial) will be of the most simplified form of $\frac{p}{q}$, where p is an integer factor of the constant term a_0 and q is an integer factor of the leading coefficient a_n. We can formulate a list of all of the possible rational roots of a polynomial and test them. For example, if the polynomial $3x^3 + x + 4$ has any rational roots, they will be one of the following numbers: ± 1, $\pm\frac{1}{3}$,± 4, $\pm\frac{4}{3}$, ± 2, or $\pm\frac{2}{3}$. Note that some polynomials will have no rational roots.

- **Conjugate Zeros Theorem**: If a polynomial $p(x)$ has a complex root $a - ib$, then the complex conjugate of the root, $a + ib$, is also a root of the same polynomial.

With this, answer the following:

1. Compute $\frac{3x^3+4x^2+5}{x+3}$ using synthetic division.

   If we rewrite $x + 3$ as $x - (-3)$, we see that $k = -3$. We also have $a_3 = 3$, $a_2 = 4$, $a_1 = 0$, and $a_0 = 5$. Following the steps above, we have

$$\begin{array}{r|rrrr} -3 & 3 & 4 & 0 & 5 \\ + & \downarrow & -9 & 15 & -45 \\ \hline & 3 & -5 & 15 & -40 \end{array}$$

   which gives us a new polynomial with coefficients $b_2 = 3$, $b_1 = -5$, $b_0 = 15$, and a remainder of -40. Writing the remainder over $x + k$, we have our final solution of

$$\frac{3x^3 + 4x^2 + 5}{x + 3} = 3x^2 - 5x + 15 + \frac{(-40)}{x + 3}.$$

2. $p(x) = 7x^4 + 48x^3 - 120x^2 - 6$. Evaluate $p(2)$ using the remainder theorem.

   Using the remainder theorem, we want to compute the remainder of $\frac{7x^4+48x^3-120x^2-6}{x-2}$ so that we have a $k = 2$. We use synthetic division for the calculation:

$$\begin{array}{r|rrrrr} 2 & 7 & 48 & -120 & 0 & -6 \\ + & \downarrow & 14 & 124 & 8 & 16 \\ \hline & 7 & 62 & 4 & 8 & 10 \end{array}$$

   As we can see, we are left with a remainder of 10 and thus, by the remainder theorem

$$p(2) = 10$$

3. Find all of the roots of the following polynomials:

   (a) $3x^3 - 4x^2 - 8x - 1$

   Using the rational roots theorem, we know that if any rational roots of this polynomial exist, they will be one of the factors of $p = -1$ divided by $q = 3$. The only factors of -1 is itself and positive one, so any possible rational roots will be either $\pm\frac{1}{1}$ or $\pm\frac{1}{3}$, so $1, -1, \frac{1}{3}$, or $-\frac{1}{3}$. We can test each one using synthetic division and we find that $x = -1$ is the only one that produces no remainder:

$$\begin{array}{r|rrrr} -1 & 3 & -4 & -8 & -1 \\ + & \downarrow & -3 & 7 & 1 \\ \hline & 3 & -7 & -1 & 0 \end{array}$$

This shows us that if we divide the original polynomial by $(x+1)$, we are left with the polynomial $3x^2-7x-1$. We cannot directly factor this polynomial, so we use the quadratic formula to find the remaining two roots:

$$\frac{7 \pm \sqrt{(-7)^2-(4)(3)(-1)}}{2(3)} = \frac{7 \pm \sqrt{61}}{6}.$$

Thus, the three roots of the original polynomial $3x^3-4x^2-8x-1$ are $\{-1, \frac{7+\sqrt{61}}{6}, \frac{7-\sqrt{61}}{6}\}$.

(b) $x^3-x^2+25x-25$ given that one of the roots is $-5i$

By the conjugate zeros theorem, we know that if $5i$ is a root, then so is $-5i$. So, we know that $(x-5i)$ and $(x+5i)$ are factors of $x^3-x^2+25x-25$ and thus so is the product $(x-5i)\cdot(x+5i)=x^2+25$. So, we can divide: $\frac{x^3-x^2+25x-25}{x^2+25}$. We use long division:

$$\begin{array}{r|l} & x-1 \\ \hline x^2+25 & x^3-x^2+25x-25 \\ & -\quad(x^3+25x) \\ \hline & 0-x^2-25 \\ & -\quad(-x^2-25) \\ \hline & 0 \end{array}$$

We see that $(x-1)$ is the result, and thus we can write $x^3-x^2+25x-25=(x-5i)(x+5i)(x-1)$ and therefore the roots of the given polynomial are $\{5i, -5i, 1\}$.

2 Functions

Function rules:

- **Relations versus Functions**: A relation may show the relationship between an input and an output, but a **function** is a more specific classification of a relation. While a relation can have multiple outputs for one input, a **function** must have exactly one output for a given input.

- **Odd and Even Functions**: We classify a function $f(x)$ as **odd** if $f(-x)=-f(x)$.We classify a function as **even** if $f(-x)=f(x)$. ~~Odd functions show symmetry across the x-axis~~ Odd functions are symmetric about the origin, while even functions show symmetry across the y-axis. For example, x^3 is an odd function, while x^2 is an even function.

- **Bijective Functions**: We say that a function is **bijective** when it is both **one-to-one** and **onto**. A function is **one-to-one** if every element in the function's domain (each x value) is mapped to exactly one element in the function's range (each y value). In other words, if $f(a)=f(b)$ and the function is one-to-one, then $a=b$. A function is **onto** if every element in the function's range (each y value) can be produced by an element in the function's domain (an x value).

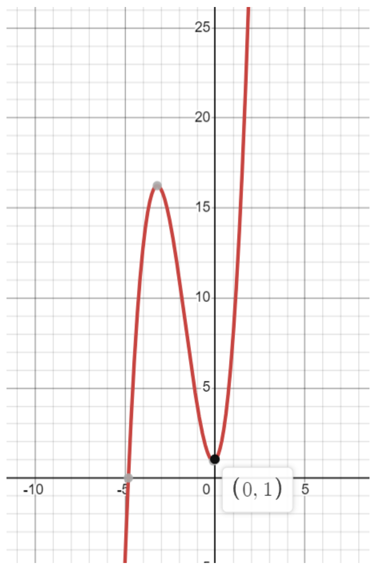
1. Classify the following as one-to-one, onto, neither, or bijective:

(a) $f : \mathcal{R} \rightarrow \mathcal{R}$, $f(x) = e^x$

For every x in the set of real numbers, we get exactly one output and each output (possible value of e^x) is produced by exactly one input. Therefore, the function is one-to-one. However, we cannot produce any negative outputs from e^x, i.e., $e^x > 0$, but the range is all real numbers, even negatives. Because we cannot produce the entire range, the function is not onto and therefore not bijective either.

(b) $f : \mathbf{R} \rightarrow \mathcal{R}$, $f(x) = x^3 + 5x^2 + x + 1$

If we look at the graph of this function, we can see that it will be able to produce



every real number as an output. Therefore, it is onto, as our given range is all real numbers. However, from the graph we can see that some outputs can be produced by multiple inputs. If we have a horizontal line at $y = 10$, we can see that the function crosses the line three times at different x coordinates. Therefore, the function is not one-to-one and thus it is not bijective.

(c) $f : \mathcal{R} \rightarrow \mathcal{R}$, $f(x) = x^3$

This function has every output produced by exactly one input, and can produce an output for every real number. Therefore, it is bijective.

(d) $f : \mathcal{R} \rightarrow \mathcal{R}$, $f(x) = x^2$

Every output of this function is not produced by exactly one input, in fact, every output can be produced by two inputs (for example, $(2)^2 = 4$ and $(-2)^2 = 4$, so the output 4 can be produced by both 2 and -2) so this function is not one-to-one. We also know that x^2 is always positive. But, the range is all real numbers, including negative ones. Because the function cannot produce negative numbers, it is not onto. Therefore, the function is neither one-to-one nor onto, nor is it bijective.

2. Classify the following functions as even, odd, or neither:

   (a) $\cos(x)$

   We can look at a graph of $\cos(x)$ to see that it is symmetric across the y-axis. Also, $\cos(-x) = \cos(x)$ an example being $\cos(2\pi) = 1$ and $\cos(-2\pi) = 1$. So, this function is even.

   (b) $\sin(x)$

   We can look at a graph of $\sin(x)$ to see that it is symmetric about the origin. $\sin(-x) = -\sin(x)$, an example being $\sin(\frac{\pi}{2}) = 1$ and $\sin(-\frac{\pi}{2}) = -1$. So, $\sin(x)$ is an odd function.

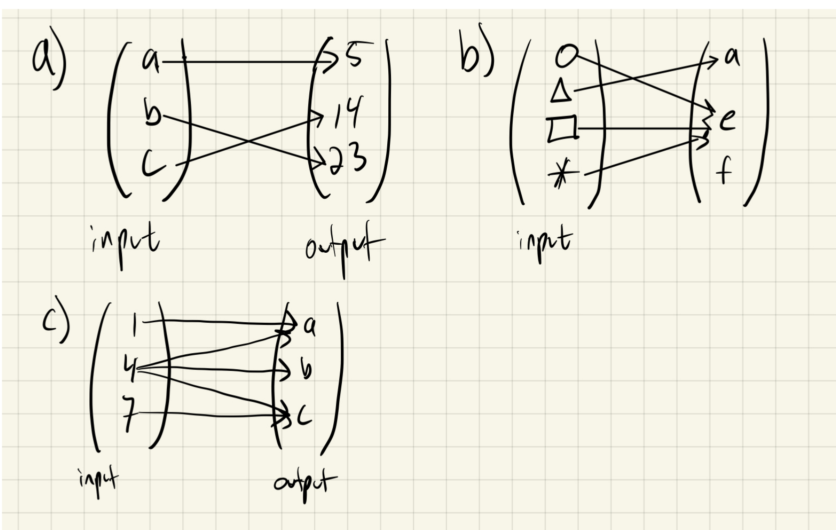
   (c) $|x|$

   We know by the definition of the absolute value function that $|-x| = x = |x|$, so this is an even function.

   (d) $x + 7$

   We can test this with one $\pm x$ value. Let's choose $x = 3$ and $x = -3$, though any pair would work. We can see that when $x = 3$, we get $f(3) = 3 + 7 = 10$ and for $x = -3$, we get $f(-3) = -3 + 7 = 4$. Clearly, $4 \neq -10$ or 10, which would be the expected values if the function was odd or even, respectively. Therefore, this function is neither even nor odd.

3. Classify the following as relations or functions:



a) This is a function as every input maps to exactly one output.

b) This is a function as every input maps to exactly one output. Note that this is still a function even though multiple inputs map to the same output and even though one of the outputs, f, does not get mapped to by any input (this means that it is not a one-to-one function nor an onto function, but it's a function nonetheless!)

c) This is not a function, only a relation, as the second input, 4, maps to three different outputs. For it to be a function, each input must map to exactly one output.

3 More Inequalities

Some things to keep in mind:

- **Square Roots**: When dealing with an inequality similar to the form $x^2 > r$, we cannot simply write $x > \pm\sqrt{r}$. We simplify as such:
  1. $x^2 > r \to x > \sqrt{r}$ OR $x < -\sqrt{r}$
  2. $x^2 < r \to -\sqrt{r} < x < \sqrt{r}$
  3. $x^2 \geq r \to x \geq \sqrt{r}$ OR $x \leq -\sqrt{r}$
  4. $x^2 \leq r \to -\sqrt{r} \leq x \leq \sqrt{r}$
- **Interval Notation**: When we solve an inequality, we can write our solution in interval notation instead of using $>, <, \geq, \leq$. Say a, b are positive constants and $a < b$. If we have a $<$ or $>$ we use parenthesis (and when we have a $\leq$ or $\geq$ we use a bracket [. When we write two solutions, like $x < 1$ OR $x > 4$, we can unify these with the symbol $\cup$. Here are some examples of how we can change notations:
  1. $x > a \to x \in (a, \infty)$
  2. $x < a \to x \in (-\infty, a)$
  3. $a < x \leq b \to x \in (a, b]$
  4. $x \leq -a$ OR $x \geq b \to (-\infty, -a] \cup [b, \infty)$

With this, answer the following:

1. Simplify $x^2 \geq 16$. Write your answer in interval notation.

   We take the square root of both sides. Note that $\sqrt{16} = \pm 4$. So, by rule 3 above, we have $x \geq 4$ OR $x \leq -4$. We can write this in interval notation as $(\infty, -4] \cup [4, \infty)$.

2. Write the following in interval notation: $-43 < x \leq 7$ OR $9 \leq x \leq 18$ OR $x > 89$.

$$(-43, 7] \cup [9, 18,] \cup (89, \infty)$$