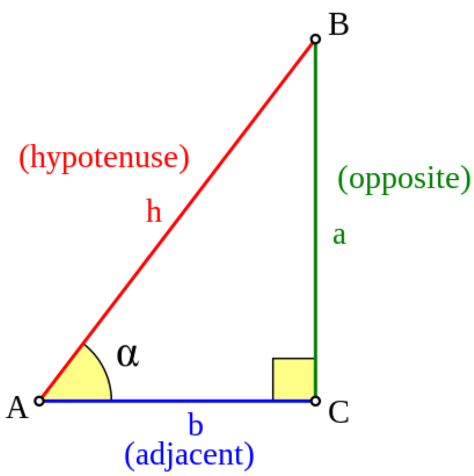
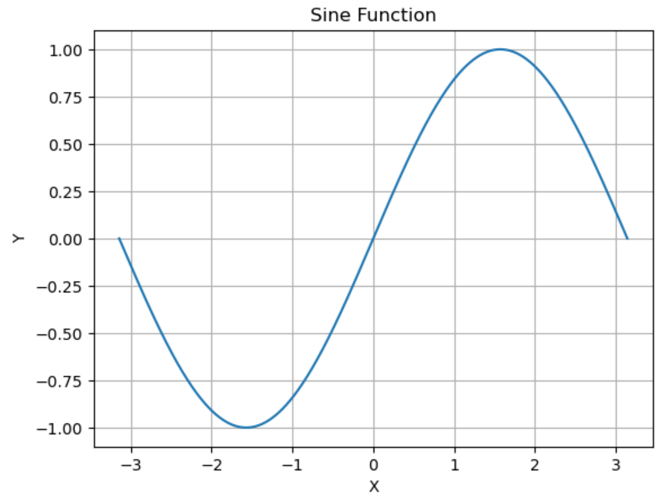
Lecture 6 + 7

Sine Function

Definition: The sine function is defined as the ratio of the opposite side of a right triangle over the hypotenuse. We can write this as $sin(\alpha) = \frac{opposite}{hypotenuse}$. We can express the function as radians or degrees.





The graph to the right is the sine function. The graph of sine starts at $y = 0$ and then peaks at $y = 1$. It then crosses the x - axis and reaches $y = -1$.

Properties of the Sine Function

1) The sine function is a periodic function meaning it oscillates and repeats the same pattern.
2) The period of the sine function is 2π.
3) The sine function is an odd function.
4) The general expression is $f(x) = a * sin(bx - c) + d$ where $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ is the period, $\frac{c}{b}$ is the phase shift, and d is the vertical shift.
5) The inverse of the sine function exists and we write it as $f(x) = sin^{-1}(x)$.

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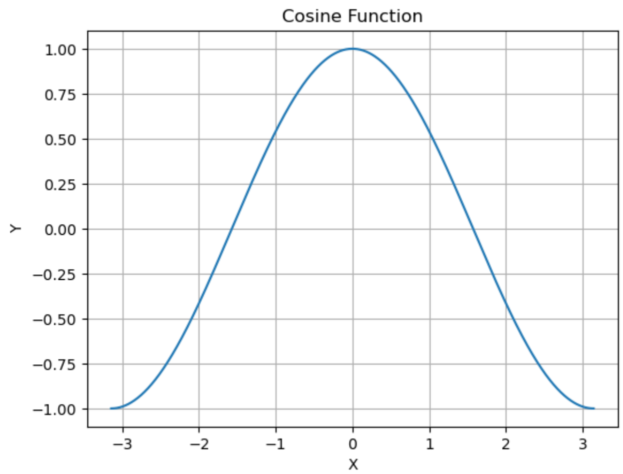
Cosine Function

Definition: The cosine function is defined as the ratio of the adjacent side of the right triangle over the hypotenuse. We are able to write this as $cos(\alpha) = \frac{adjacent}{hypotenuse}$.



Here is a graph of the cosine function. The graph starts at $y = 1$ and then goes down to $y = 0$. It then crosses the x - axis and reaches $y = -1$.

Properties of the Cosine Function

1) The cosine function is periodic.
2) It has a period of 2π.
3) The function is an even function.
4) The inverse of the cosine function exists and we are able to write it as $f(x) = cos^{-1}(x)$.

HW#2 → 07/07

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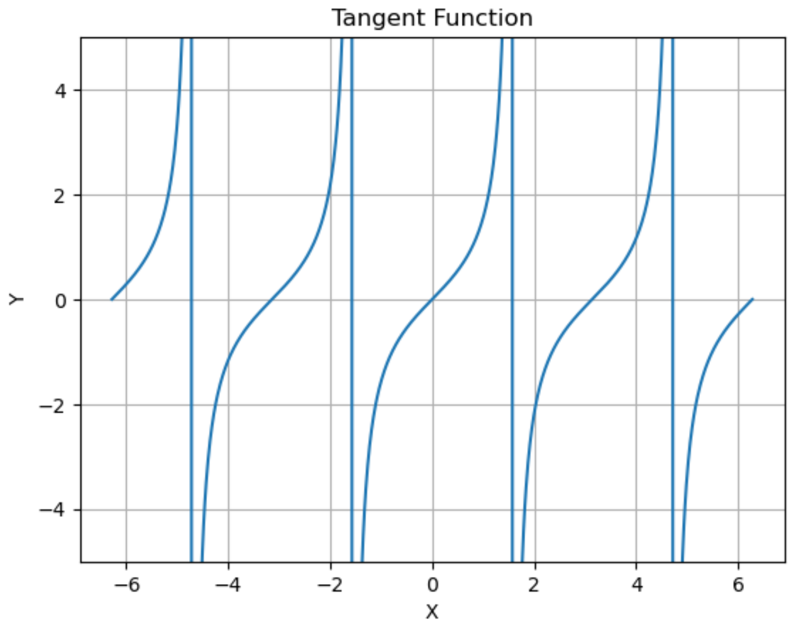
Tangent Function

Definition: The tangent function is defined as the ratio of the opposite length over the adjacent length. We can write this as $tan(\alpha) = \frac{opposite}{adjacent}$.



Here is a graph of the tangent function. Note that the tangent function has an infinite number of vertical asymptotes.

Properties of the Tangent Function

1) The period of the tangent function is π.
2) We can also write the tangent function as $tan(\theta) = \frac{sin(\theta)}{cos(\theta)}$.
3) The tangent function is an odd function.

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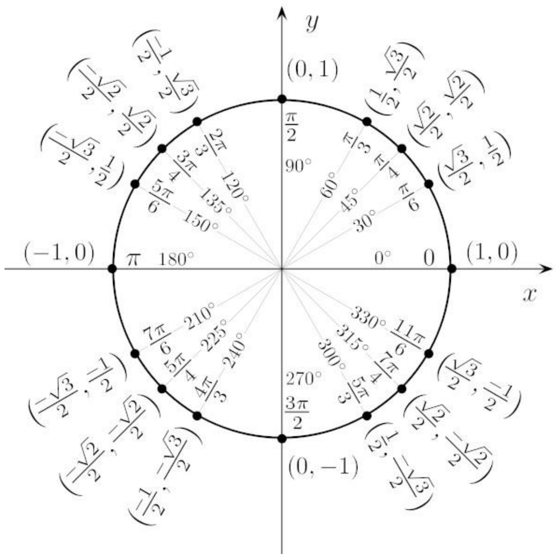
Unit Circle

Definition: The unit circle is a circle of unit radius. It is centered at the origin of $(0, 0)$. This circle is commonly used in trigonometry because it is easier to use with trigonometric functions and angles.



Here is the unit circle with degrees, radians, and the corresponding (x, y) values.

- The quadrant on the top right is quadrant 1 where sine, cosine, and tangent are positive.
- The quadrant on the top left is quadrant 2 where sine is positive, cosine is negative, and tangent is negative.
- The quadrant on the bottom left is quadrant 3 where tangent is positive, cosine is negative, and sine is negative.
- The quadrant on the bottom right is quadrant 4 where cosine is positive, sine is negative, and tangent is negative.

Problems

1) Determine the coordinates for the angle $\frac{\pi}{3}$ on the unit circle.

$\frac{\pi}{3} = 60° \rightarrow \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

2) Calculate the coordinates for the angle $\frac{-\pi}{6}$ on the unit circle.

$-\frac{\pi}{6} + \frac{12\pi}{6} = \frac{11\pi}{6} = 330° \rightarrow \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

Lecture 6 + 7

Converting Between Degrees and Radians

Conversion Formula: $1^{\circ} = \frac{\pi}{180} radians$ (Degrees → Radians) and $1\ radian = (\frac{180}{\pi})^{\circ}$ (Radians → Degrees)

Example

Convert 120° to radians.

$120^{\circ}(\frac{\pi}{180^{\circ}}) \rightarrow \frac{2\pi}{3}$

Problems

1) Convert $-\frac{5\pi}{4}$ radians to degrees.

$-\frac{5\pi}{4}(\frac{180^{\circ}}{\pi}) \rightarrow -\frac{900}{4}$ or -225

2) Convert 60° to radians.

$60^{\circ}(\frac{\pi}{180^{\circ}}) \rightarrow \frac{\pi}{3}$

Half Angle Identities

Half angle identities allow the user to find the exact value of a trigonometric function.

$$sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-cos(\theta)}{2}}$$

$$cos(\frac{\theta}{2}) = \pm\sqrt{\frac{1+cos(\theta)}{2}}$$

$$tan(\frac{\theta}{2}) = \pm\sqrt{\frac{1-cos(\theta)}{1+cos(\theta)}}$$

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Example

Use the half angle identity to find the exact value of $sin(\frac{7\pi}{8})$.

$\sin(\frac{7\pi}{8}) = \sin\left(\frac{\frac{7\pi}{4}}{2}\right)$

Note: x is cosine and y is sin

$\sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$

$= \sqrt{\frac{1-\cos(\frac{7\pi}{4})}{2}}$ using unit circle $\cos(\frac{7\pi}{4}) = \frac{\sqrt{2}}{2}$ so $\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} \rightarrow \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \underline{\sqrt{\frac{2-\sqrt{2}}{4}}}$

$\frac{7\pi}{8}\left(\frac{180^\circ}{\pi}\right) = 157.5^\circ$ Positive Sin

Problem

1) Use the half angle identity to find the value of $sin(\frac{\theta}{2})$ if $sin(\theta) = \frac{3}{5}$ and θ is in the first quadrant.

$\sin(\frac{\theta}{2}) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$

$\sin^2(\theta) + \cos^2(\theta) = 1$

$\sin(\theta) = \frac{3}{5}$ so $\sin^2(\theta) = (\frac{3}{5})^2 = \frac{9}{25}$

$\cos^2(\theta) = 1 - \frac{9}{25} \Rightarrow \cos^2(\theta) = \frac{16}{25}$

$\cos(\theta) = \sqrt{\frac{16}{25}}$ or $\frac{4}{5}$ Since it is in 1st quadrant

$\underline{\sin(\frac{\theta}{2}) = \sqrt{\frac{1-\frac{4}{5}}{2}} = \sqrt{\frac{\frac{1}{5}}{2}} = \sqrt{\frac{1}{10}}}$

Inverse Trigonometric Functions

The inverse of trigonometric functions are also called arc functions. We can write them as Arcsin, Arccosine, and Arctangent.

Here is a list of inverse trigonometric formulas.

1) $sin^{-1}(x)$
2) $cos^{-1}(x)$
3) $tan^{-1}(x)$
4) $cot^{-1}(x)$
5) $sec^{-1}(x)$
6) $csc^{-1}(x)$

Example

Solve for x given the expression $sin^{-1}(x) = \frac{\pi}{6}$.

$x = \sin(\frac{\pi}{6})$ So $\underline{x = \frac{1}{2}}$

Lecture 6 + 7

Problems

1) Find the exact value of $tan(cos^{-1}(\frac{\sqrt{3}}{2}))$

$\cos^{-1}(\frac{\sqrt{3}}{2}) = \theta$

$\frac{\sqrt{3}}{2} = \cos(\theta)$

$\theta = 30° \rightarrow \tan(30°) = \frac{\sin(30°)}{\cos(30°)} = \frac{1}{2}(\frac{2}{\sqrt{3}}) = \frac{1}{\sqrt{3}}$

2) Find the exact value of $sin^{-1}(cos(\frac{4\pi}{3}))$

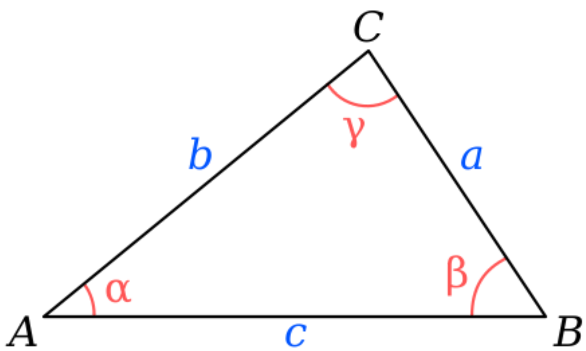
$\cos(\frac{4\pi}{3}) = \frac{-1}{2}$ So $\sin^{-1}(\frac{-1}{2}) = -30$

$-\frac{1}{2} = \sin(\theta)$

$\theta = -30$

Law of Cosines

Definition: The law of cosines relates the length of the side of a triangle to the cosine of the target angle.



Given this triangle, we are able to find the length and the angles using the equation

$c^2 = a^2 + b^2 - 2abcos(\gamma)$.

The law of cosine is utilized when the question provides three sides or two sides and an angle.

Example

In a triangle, side a has a length of 7, side b has length 9, and angle between sides a and b is 60°. Find the length of side c.

$a = 7$

$b = 9$

$\theta = 60°$

$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

$c^2 = 7^2 + 9^2 - 2(7)(9)\cos(60°)$

$c^2 = 49 + 81 - 2(7)(9)(\frac{1}{2})$

$c^2 = 49 + 81 - 63$

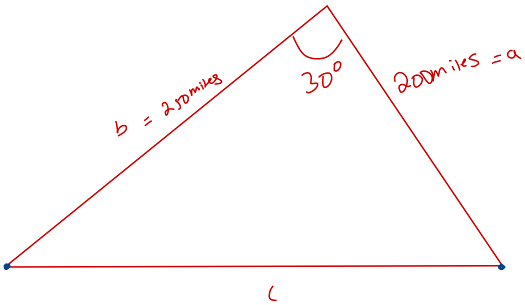
$c^2 = 67$

$c = \sqrt{67}$

Lecture 6 + 7

Problems

1) Two ships leave a harbor at the same time. Ship A sails due north for 200 miles, while Ship B sails 30 degrees east of north for 250 miles. What is the distance between the two ships?



$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$

$c^2 = 200^2 + 250^2 - 2(200)(250)\cos(30^\circ)$

$c^2 = 15897.46$

$c = \sqrt{15897.46}$

$c = 126.09$ miles

Law of Sines

Definition: The law of sine relates the length of the sides of any triangle to the sines of the angles. We are able to write the law of sines as the equation below.

$$\frac{sin(A)}{a} = \frac{sin(B)}{b} = \frac{sin(C)}{c}$$

The law of sine is utilized when the question provides two angles and one side or two sides and an angle.

Example

In a triangle, side a has length of 8 and angle A has a measure of 40 degrees. Side b has length 10. Find the measure of angle B and the length of side c.

$a = 8$ $b = 10$

$A = 40^\circ$

$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} \rightarrow \frac{\sin(40^\circ)}{8} = \frac{\sin(B)}{10}$

So $\sin(B) = \frac{10\sin(40^\circ)}{8}$

$B = \sin^{-1}\left(\frac{10\sin(40^\circ)}{8}\right) = 53.46^\circ$

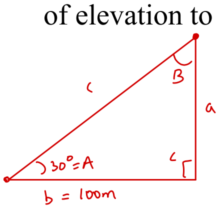
$\rightarrow$ $180 - A - B = C$ so $180 - 40 - 53.46 = 86.54^\circ$

$\frac{\sin(40)}{8} = \frac{\sin(86.54)}{c}$

$c = \frac{8\sin(86.54)}{\sin(40^\circ)} = 12.42$

Problem

1) An observer stands 100 meters away from the base of a tower. The observer measures the angle of elevation to the top of the tower as 30°. How tall is the tower?



$180^\circ - 30^\circ - 90^\circ = 60^\circ$

$\frac{\sin(30^\circ)}{a} = \frac{\sin(60^\circ)}{100\text{ m}}$

$a = \frac{100\text{m}\sin(30^\circ)}{\sin(60^\circ)} = 57.74\text{m}$

Lecture 6 + 7

Sum and Difference Identities

The sum and difference identities allow us to find the trigonometric values given the angles or radians. Here is a list of sum and difference identities.

1) $sin(a + b) = sin(a)cos(b) + cos(a)sin(b)$
2) $sin(a - b) = sin(a)cos(b) - cos(a)sin(b)$
3) $cos(a + b) = cos(a)cos(b) - sin(a)sin(b)$
4) $cos(a - b) = cos(a)cos(b) + sin(a)sin(b)$
5) $tan(a + b) = \frac{tan(a)+tan(b)}{1-tan(a)tan(b)}$
6) $tan(a - b) = \frac{tan(a)-tan(b)}{1+tan(a)tan(b)}$

Example

Use the sum and difference identity to find the exact value of $sin(75^{\circ})$.

$sin(45^\circ + 30^\circ) = sin(45^\circ)cos(30^\circ) + cos(45^\circ)sin(30^\circ)$

$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$

$= \frac{\sqrt{2}\sqrt{3}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

Problems

1) Use the sum and difference identity to find the exact value of $tan(105^{\circ})$.

$60^\circ + 45^\circ = 105^\circ$

$tan(60^\circ + 45^\circ) = \frac{tan(60^\circ) + tan(45^\circ)}{1 - tan(60^\circ)\,tan(45^\circ)}$

$tan(60^\circ) = \frac{sin(60^\circ)}{cos(60^\circ)} = \left(\frac{\sqrt{3}}{2}\left(\frac{2}{1}\right)\right) = \sqrt{3}$

$tan(45^\circ) = \frac{sin(45^\circ)}{cos(45^\circ)} = \frac{\sqrt{2}}{2}\left(\frac{2}{\sqrt{2}}\right) = 1$

$= \frac{\sqrt{3} + 1}{1 - \sqrt{3}}$

2) Use the sum and difference identity to find the exact value of $sin(165^{\circ})$.

$135^\circ + 30^\circ = 165^\circ$

$sin(135^\circ + 30^\circ) = sin(135^\circ)cos(30^\circ) + cos(135^\circ)sin(30^\circ)$

$= \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right) + -\frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)$

$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

Solving Trigonometric Equations

Example

Solve the equation $2cos(x) - 1 = 0$ for $0 \leq x \leq 2\pi$.

$2cos(x) = 1$

$cos(x) = \frac{1}{2}$

$x = cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$

Lecture 6 + 7

Problems

1) Solve the equation $sin(2x) = cos(x)$ for $0 \leq x \leq 2\pi$.

Using Double angle identity $\rightarrow$ $\sin(2x) = 2\sin(x)\cos(x)$

$2\sin(x)\cos(x) = \cos(x)$

$2\sin(x) = 1$

$\sin(x) = \frac{1}{2}$

$x = \sin^{-1}(\frac{1}{2}) = 30°$

2) Solve the equation $tan(x) = \sqrt{3}$ for $0 \leq x \leq \pi$.

$\tan(x) = \sqrt{3}$

$x = \tan^{-1}(\sqrt{3}) = 60°$

Lecture 6 + 7

Trigonometry Function Identities

Trigonometry Function Identities

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Reciprocal Indentities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\csc^2\theta - \cot^2\theta = 1$$

Even/Odd Indentities

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \cot(-\theta) = -\cot\theta$$

$$\csc(-\theta) = -\csc\theta \qquad \sec(-\theta) = \sec\theta$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta \qquad \cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$$

$$\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta \qquad \sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$$

$$\frac{\pi}{2} \text{ radians} = 90°$$

Sum/Difference Indentities

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$$

$$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Double Angle Identities

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Half Angle Indentities

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Sum to Product of Two Angles

$$\sin\theta + \sin\phi = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\sin\theta - \sin\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\cos\theta - \cos\phi = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

Product to Sum of Two Angles

$$\sin\theta\sin\phi = \frac{[\cos(\theta - \phi) - \cos(\theta + \phi)]}{2}$$

$$\cos\theta\cos\phi = \frac{[\cos(\theta - \phi) + \cos(\theta + \phi)]}{2}$$

$$\sin\theta\cos\phi = \frac{[\sin(\theta + \phi) + \sin(\theta - \phi)]}{2}$$

$$\cos\theta\sin\phi = \frac{[\sin(\theta + \phi) - \sin(\theta - \phi)]}{2}$$

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