An Introduction to Linear Algebra

- **Vector**: A vector is an object that has both *magnitude* and *direction*. It is often denoted as $\vec{v}$ and can be represented in the following forms:

$$\langle v_1, v_2, v_3, \ldots, v_n \rangle \quad \text{or} \quad \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}.$$

  A matrix with only one row is a row vector. A matrix with only one column is a column vector.

- **Magnitude**: The magnitude of a vector is the length of the vector. It is often denoted as $||\vec{v}||$. We define the magnitude of a vector as

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2 + \cdots + v_n^2}$$

- **Scalar Multiplication**: We can multiply a vector by a scalar α (a normal number) as such, assuming $\alpha \in \mathcal{R}$:

$$\alpha \langle v_1, v_2, \ldots, v_n \rangle = \langle \alpha v_1, \alpha v_2, \ldots, \alpha v_n \rangle.$$

- **Vector Multiplication**: There are two main ways to multiply vectors: the *dot product* and the *cross product*. We will only discuss the dot product. The dot product takes two vectors and produces a scalar. We can define the dot product as:

$$\langle v_1, v_2, \ldots, v_n \rangle \cdot \langle w_1, w_2, \ldots, w_n \rangle = v_1 \cdot w_1 + v_2 \cdot w_2 + \cdots + v_n \cdot w_n$$

- **Matrix**: A matrix is a rectangular array of numbers - it is a way to store numbers in rows and columns. We say that the *size* of a matrix is $m \times n$, (read “m by n”) where m is the number of rows the matrix has and n is the number of columns. A matrix, denoted here as A, has the general form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

  where $a_{i,j} \in \mathcal{R}$, simply meaning that the entries of the matrix are real numbers.

  - A matrix is **square** if $m = n$.
  - A column vector is a $m \times 1$ matrix while a row vector is a $1 \times n$ matrix.
  - Two matrices are equal, $A = B$ if and only if every entry of matrix A and matrix B are equal.

- **Matrix Addition**: We can add two matrices if and only if they are the same size. Matrix addition is done entry by entry. If A and B are both $m \times n$ matrices, then their sum $A+B=C$ can be defined as:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

- **Scalar Multiplication**: Again, a scalar is just a normal number. When we multiply a matrix by a scalar, similarly to when we multiply a scalar to a vector, we multiply each entry by that scalar. Therefore:

$$\alpha \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \alpha \cdot a_{11} & \alpha \cdot a_{12} & \cdots & \alpha \cdot a_{1n} \\ \alpha \cdot a_{21} & \alpha \cdot a_{22} & \cdots & \alpha \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha \cdot a_{m1} & \alpha \cdot a_{m2} & \cdots & \alpha \cdot a_{mn} \end{bmatrix}$$

- **Matrix Multiplication**: We can multiply two matrices, or a vector and a matrix, if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix. If we want to compute the product AB of two matrices A size $m_A \times n_A$ and B size $m_B \times n_B$, then we can only do so if $n_A = m_B$. An example with two 2×2 matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11})(b_{11}) + (a_{12})(b_{21}) & (a_{11})(b_{12}) + (a_{12})(b_{22}) \\ \\ (a_{21})(b_{11}) + (a_{22})(b_{21}) & (a_{21})(b_{12}) + (a_{22})(b_{22}) \end{bmatrix}$$

  It is important to note that it is often the case that $AB \neq BA$. In other words, the order of multiplication does matter when dealing with matrices.

- **Systems of Linear Equations**: We can represent a system of linear equations using matrices and vectors. A general linear system of m equations (rows) and n unknown variables (columns) has the form:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m \end{aligned}$$

  We have all of the coefficients, the numbers multiplying the variables, $(a_{11}, a_{12}, \ldots, a_{mn})$, all of the variables $(x_1, x_2, \ldots, x_n)$, and all of the right-hand sides of the equations $(b_1, b_2, \ldots, b_n)$. We can thus write this system in the form:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If we define the coefficient matrix as A, the vector containing the variables as $\vec{x}$, and the vector containing the right-hand sides of the equations as $\vec{b}$, we can rewrite this system in the form:

$$A\vec{x} = \vec{b}.$$

- **Reduced Augmented Matrices**: We can write an *augmented* matrix by adding the right-hand side vector to the coefficient matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{array}\right]$$

to form an $m \times (n+1)$ matrix. This allows for a more convenient representation of the system when computing matrix reduction operations. We will not go over the actual matrix reduction process, but it involves a series of *elementary row operations* that result in the matrix looking a certain way. We will go over a few forms that a reduced augmented matrix might take and what information they provide.

  - **A Unique Solution**: A system of linear equations will have *one unique solution* if the reduced augmented matrix has a form similar to:

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n \end{array}\right].$$

    Notably, all rows have one nonzero entry followed by an entry in the right-hand side vector that may or may not be equal to zero.

  - **No Solutions**: A system of linear equations has *no solutions* if the reduced augmented matrix has a form similar to:

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_n \neq 0 \end{array}\right]$$

    that is, if the reduced augmented matrix has a row of all zeros as the entries followed by a nonzero entry to the right of the bar (as the value of the right-hand side of that equation). We cannot obtain a nonzero number by the addition of all zeros, so the system has no solutions.

- **Infinitely Many Solutions**: A system of linear equations has *infinitely many solutions* if the reduced augmented matrix has a form similar to:

$$\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1 \\ 0 & 1 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right].$$

  If there is a row entirely full of zeros, as in this case, this indicates that the number of unknown variables is greater than the number of equations and therefore the system will have infinitely many solutions.