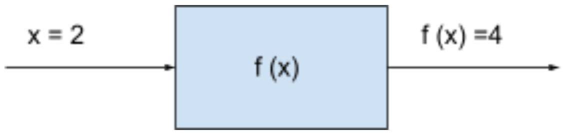
Lecture 2 + 3

Functions

Definition: Let's say we have set Y and set X. When we examine the properties of each set, we can find that there is element Y corresponding to each element X. We can call the elements in X as the domain and set Y as the codomain of the function.

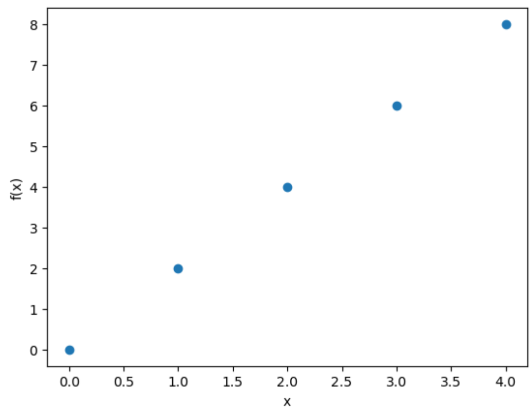


What would be the function above?

Maybe $f(x) = 2x$ or $f(x) = x^2$.

Let's look at this table below of the function $f(x) = 2x$

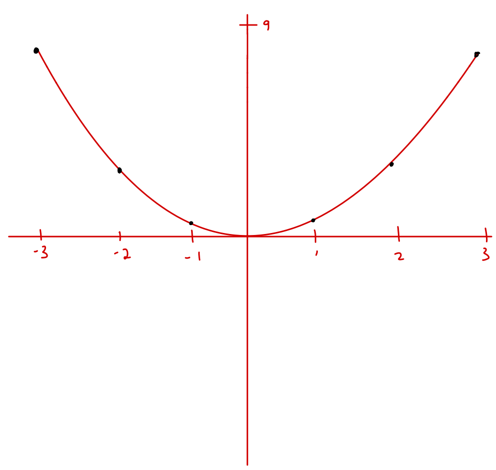
x	$f(x)$
0	0
1	2
2	4
3	6
4	8



Problems

1) Fill in the table below and draw the graph. Let $f(x) = x^2$

x	$f(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



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2) Let $f(x) = x^2 + 3x + 7$ and $g(x) = \frac{x+2}{x^2}$. Find $f(1)$, $f(x+3)$, and $g(10)$.

$f(1) = (1)^2 + 3(1) + 7 = 1 + 3 + 7 = 11$

$f(x+3) = (x+3)^2 + 3(x+3) + 7 = x^2 + 6x + 9 + 3x + 9 + 7 = x^2 + 9x + 15$

$g(10) = \frac{10+2}{(10)^2} = \frac{12}{100}$

3) Let $f(x) = x$ and $g(x) = x^2 + 1$. Find the composition $(f \circ g)(-1)$ and $(f \circ g)(10)$.

$(f \circ g)(-1) = f(g(-1))$

$f(g(x)) = (x^2+1)$ So $f(g(-1)) = 2$

$f(g(x)) = x^2+1$ So $f(g(10)) = 10^2 + 1 = 101$

If $f(x) = x^2$ then $f(g(x)) = (x^2+1)^2$

Domain and Range

Definition: Domain of a function is the set of values that we can plug into the function.

Definition: Range is the set of values that the function gives after plugging in the x values.

Problems

1) Find the domain of function $f(x) = \frac{6x-2}{3x-4}$.

Denominator can never be zero so find $3x-4=0$

$3x-4=0$

$3x=4$

$x=\frac{4}{3} \rightarrow$ @ $x=\frac{4}{3}$, $3x-4=0$ So DNE

Domain is $x > \frac{4}{3}$ $x < \frac{4}{3}$

$(-\infty, \frac{4}{3})$ $(\frac{4}{3}, \infty)$

2) Find the domain of the function $f(x) = \sqrt{5 - x}$.

Cannot have $\sqrt{-x}$ so find $\sqrt{5-x} = 0$

$\sqrt{5-x} = 0$

$(\sqrt{5-x})^2 = (0)^2$

$5 - x = 0$

$x = 5 \longrightarrow$ if $x = 6$ then $\sqrt{5-6} = \sqrt{-1}$

if $x = 4$ then $\sqrt{5-4} = \sqrt{1} = 1$ $\rightarrow$ So domain $x \le 5$ or $(-\infty, 5]$

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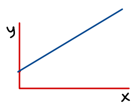
Inverses of Functions

Definition: Inverse is defined as $y = f^{-1}(x)$ if and only if $f(y) = x$. A function only has an inverse if the function is bijective (one to one correspondence).

Problems

1) Find the inverse of the function $f(x) = 5x + 1$. Does it exist?

is 5x-1 bijective? →



$y = 5x+1$ → 1) Replace y with x and x with y

2) $x = 5y+1$ now solve for y

3) $x - 1 = 5y$

$\boxed{y = \frac{x-1}{5}} \rightarrow y = f^{-1}(x)$

Polynomial and Rational Functions

Polynomial is a type of function that consists of variables and coefficients with varying powers. We can write a polynomial in the form below.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_2 x^2 + a_1 x + a_0$

Note that a_n is the coefficient value and n represents power. We can also write it in summation notation.

$f(x) = \sum_{k=0}^{n} a_k x^k$

Rational Functions

Rational functions is a function that is defined as a rational fraction. We can write it in the form below.

$f(x) = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$

Note that $P(x)$ and $Q(x)$ are functions.

Polynomial and rational functions have multiple real life applications in physics and biology.

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<u>Factor Theorem</u>

A polynomial $f(x)$ has a factor $(x - a)$ if and only if $f(a) = 0$.

Problems

1) Determine if $(x - 2)$ is a factor of $f(x) = 2x^3 - 5x^2 + 3x + 6$

X-2 is a factor So X-2=0, X=2 is a Root if f(2)=0

$f(2) = 2(2)^3 - 5(2)^2 + 3(2) + 6$

$= 16 - 20 + 6 + 6$

$= -4 + 12 = 8$

So $f(2) \neq 0$ meaning 2 is not a Root and x-2 is not a factor

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<u>Graphing Rational Functions</u>

Vertical Asymptotes occur when the denominator is zero.

Horizontal Asymptote is the ratio of the leading coefficient if the powers are the same.

Problem

1) Given the rational function $f(x) = \frac{3x^2-4x-1}{x^2+2x-3}$. Find the intercepts and asymptotes. Graph the results. Hint: the x- intercept is found by setting the numerator to zero. The y- intercept is found by evaluating $f(0)$.

Leading Power is 3 and is Same for numerator and Denominator So Horizontal asymptote $= \frac{3}{1} = 3$

Vertical asymptote: $x^2+2x-3=0$ → factor to find Roots $f(x) = (x-1)(x+3)$

Root is $x=1$, $x=-3$

So vertical asymptote is $x=1$ and $x=-3$.

X intercept: $3x^2-4x-1=0$ → $\frac{4 \pm \sqrt{4^2-4(3)(-1)}}{6} = \frac{4 \pm \sqrt{28}}{6}$ meaning $x = \frac{4+\sqrt{28}}{6}$ and $x = \frac{4-\sqrt{28}}{6}$

Y intercept: $\frac{3(0)^2-4(0)-1}{(0)^2+2(0)-3} = \frac{1}{3}$ So Y intercept is $y = 1/3$

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Rational Function with Holes

This occurs where the value of the function is not defined. When the numerator and denominator of a rational function have a common factor, we can cancel the factors. The canceled factor will create a hole in the graph where it is not defined.

Problems

1) Given the rational function $f(x) = \frac{x^2-9}{x-3}$. Determine if a hole exists and state where.

factor $\rightarrow$ $f(x) = \frac{(x+3)(x-3)}{(x-3)}$ (x-3) both in num + Denom So $x-3=0$, $x=3$ is a hole

2) Given the rational function $f(x) = \frac{x^2-4x-12}{x+3}$. Determine if a hole exists and state where.

factor $\rightarrow$ $\frac{(x-6)(x+2)}{(x+3)}$ no hole exists.

Rational Function with Slant Asymptote

This occurs when the degree of the numerator is one more than the denominator. When the degree of the numerator is greater than the denominator, then no horizontal asymptote exists. Instead, a slant asymptote exists

Problem

1) Given the rational function $f(x) = \frac{2x^3+5x^2-4x+1}{x+1}$. Determine the vertical asymptote and horizontal asymptote.

VA $\rightarrow$ $x+1=0$ So $x=-1$

HA $\rightarrow$ No HA but Slant Asymptote.

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<u>Properties of Polynomials</u>

We are able to characterize the end behavior of a polynomial based on the degree and leading coefficient.
<u>Note:</u> We are able to write a polynomial in the form $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$ where a_n is the leading coefficient and n is the highest degree.

Degree	**Leading Coefficient**	**Graph End Behavior**	**End Behavior**
Even	Positive	Rise to the Right Rise to the Left Ex) $f(x) = x^2$	$\lim_{x \to +\infty} f(x) = +\infty$ $\lim_{x \to -\infty} f(x) = +\infty$
Even	Negative	Falls to the Right Falls to the Left Ex) $f(x) = -x^2$	$\lim_{x \to +\infty} f(x) = -\infty$ $\lim_{x \to -\infty} f(x) = -\infty$
Odd	Positive	Rise to the Right Falls to the Left Ex) $f(x) = x^3$	$\lim_{x \to +\infty} f(x) = +\infty$ $\lim_{x \to -\infty} f(x) = -\infty$
Odd	Negative	Falls to the Right Rise to the Left Ex) $f(x) = -x^3$	$\lim_{x \to +\infty} f(x) = -\infty$ $\lim_{x \to -\infty} f(x) = +\infty$

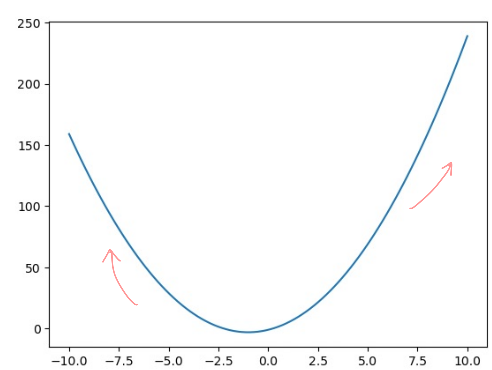
Maximum Number of Zeros (Roots) depends on the highest degree and will have at most n real zeros and $n - 1$ *local extrema*.

Problems

1) What is the left and right behavior of the function $f(x) = 2x^2 + 4x - 1$

$n=2$ and Coefficient is Even and Positive

Rise to the Right + Rise to the left.



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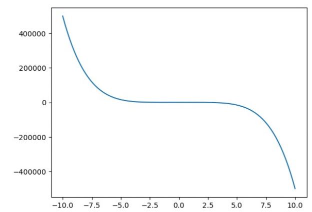
2) What is the left and right behavior of the function $f(x) = -5x^5 + 2x^2 + 1$

$n=5$ and Coefficent is negative

falls to the Right + Rise to the left $\rightarrow$



Polynomial Long Division

Example

Divide $f(x) = 4x^2 + 2x + 1$ by $x + 1$

$$
\begin{array}{r l}
 & 4x - 2 \\
x+1 \,\big) & 4x^2 + 2x + 1 \\
- & 4x^2 + 4x \\
\hline
 & -2x + 1 \\
- & -2x - 2 \\
\hline
 & 0 \quad 3
\end{array}
$$

answer: $4x - 2 + \frac{3}{x+1}$

Problems

1) Divide $f(x) = x^3 + 4x^2 + 1$ by $x + 5$

$$
\begin{array}{r l}
 & x^2 - x + 5 \\
x+5 \,\big) & x^3 + 4x^2 + 0x + 1 \\
- & x^3 + 5x^2 \\
\hline
 & 0 \quad -x^2 + 0x \\
- & -x^2 - 5x \\
\hline
 & 0 \quad 5x + 1 \\
- & 5x + 25 \\
\hline
 & -24
\end{array}
$$

$x^2(x+5) = x^3 + 5x^2$

$-x(x+5) = -x^2 - 5x$

$x^2 - x + 5 - \frac{24}{x+5}$

2) Given the rational function $f(x) = \frac{2x^3+5x^2-4x+1}{x+1}$. Find the slant asymptote.

$$
\begin{array}{r l}
 & 2x^2 + 3x - 7 \\
x+1 \,\big) & 2x^3 + 5x^2 - 4x + 1 \\
- & 2x^3 + 2x^2 \\
\hline
 & 3x^2 - 4x \\
- & 3x^2 + 3x \\
\hline
 & -7x + 1 \\
- & -7x - 7 \\
\hline
 & 8
\end{array}
$$

Slant $\rightarrow$ $2x^2 + 3x - 7 + \frac{8}{x+1}$

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Finding Zeros Of Polynomials

Find the roots of the polynomial using polynomial division.

Example

Find all of the zeros of $f(x) = x^3 - 4x^2 + x + 6$ given that 2 is one of the roots of this equation.

$$
\begin{array}{r|l}
 & x^2 - 2x - 3 \\
\hline
x - 2 & x^3 - 4x^2 + x + 6 \\
 & -\ (x^3 - 2x^2) \\
\hline
 & -2x^2 + x \\
 & -\ (-2x^2 + 4x) \\
\hline
 & 0 - 3x + 6 \\
 & -\ (-3x + 6) \\
\hline
 & 0
\end{array}
$$

$x^2 - 2x - 3$ factored is $(x-3)(x+1)$

So $(x-3)(x+1)(x-2)$ are factors meaning $x = 3, -1, 2$ are Roots.