Review for Pre-Calculus

SOLUTION SET

This is a review worksheet to re-familiarize yourself with concepts learned prior to pre-calculus that may be relevant within topics we will study over the course of the program. This assignment is not required: it will not be turned in nor will it be graded, but you may find it useful to work through some of these problems, particularly those in areas you may not remember as well.

1 Factoring

Recall the following methods of factoring/root finding:

- **Complete the Square**: When given a polynomial of the form $ax^2 + bx$, we can write this as $\left(ax + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.
- **Difference of Squares**: When given a polynomial of the form $a^2x^2 - b^2$, we can write this as $(ax - b)(ax + b)$.
- **Quadratic Formula**: A polynomial of the form $ax^2 + bx + c$ has roots of the form: $x_{1,2} = \frac{-(b)\pm\sqrt{b^2-4(a)(c)}}{2(a)}$.

With this, answer the following:

1. What is the discriminant? What can it tell us? (Think about the sign)

   The discriminant is a function of the coefficients of a quadratic equation. More specifically, when given an equation of the form $ax^2 + bx + c = 0$, it is equal to $(b)^2 - 4(a)(c)$, the part under the square root in the quadratic formula. It is often denoted with Δ. The discriminant can tell us how many valid solutions there are to a problem. If the discriminant is positive i.e., $\Delta = (b)^2 - 4(a)(c) > 0$, then we have two real solutions to our equation. If the discriminant is negative, $(b)^2 - 4(a)(c) < 0$, then we have two complex solutions and an i will be present in our solutions. If the discriminant is equal to zero, $(b)^2 - 4(a)(c) = 0$, then we have one real solution.

2. In what cases can we use the quadratic formula?

   We can use the quadratic formula anytime we wish to find the roots of a quadratic equation i.e., when finding the solutions to an equation of the form $ax^2 + bx + c = 0$. Note that while b and/or c can be equal to zero, we may not use the formula when $a = 0$ as this would require division by 0 and the original equation would not be quadratic.

3. Solve the following equations using different methods:

   (a) $x^2 + 6x + 9 = 0$

   This problem can be solved by factoring. We note that there are two factors of 9: $(1, 9)$ and $(3, 3)$. Since $(3, 3)$ also sums to 6, we use these and write the solution as

$(x+3)(x+3)=0 \rightarrow (x+3)^2=0$. This gives us two identical real roots found by solving $x_{1,2}+3=0 \rightarrow x_{1,2}=-3$. This problem can also be solved using the quadratic formula.

(b) $7x^2+3x+4=0$

Here we use the quadratic formula. Noting that $a=7, b=3$, and $c=4$, we can plug these into the formula: $x_{1,2}=\frac{-3\pm\sqrt{(3)^2-4(7)(4)}}{2(7)} \rightarrow x_{1,2}=\frac{-3\pm\sqrt{-75}}{14}$. Here, we will have two complex solutions as the discriminant, $\Delta=-75$ is negative. Since $\sqrt{-75}=i\sqrt{25(3)}=i5\sqrt{3}$, we can write our final solutions as $x_1=\frac{-3+5i\sqrt{3}}{14}$, $x_2=\frac{-3-5i\sqrt{3}}{14}$.

(c) $x^2+6x-16=0$

Here, we can complete the square to factor. We have $a=1, b=6$, and $c=-16$. To get the polynomial in the form given above, ax^2+bx, we rewrite the original equation as $x^2+6x=16$. We then seek the value of $\left(\frac{b}{2}\right)^2$, which in this case is $\left(\frac{6}{2}\right)^2=9$. We can then write our equation as $x^2+6x+9-9=16$ noting that by both adding and subtracting 9, the equation is essentially unchanged. We can then factor our equation into the form seen in the notes above: $(x+3)^2-9=16 \rightarrow (x+3)^2=25 \rightarrow (x+3)=\pm 5$ after taking the square roots of both sides in the final step. We then have the two equations for our roots: $x_1+3=5$, $x_2+3=-5$, yielding the solutions $x_1=2$, $x_2=-8$.

(d) $64x^2-56=0$

Here, we can use the quadratic formula, or we can treat the equation as a difference of squares, $a^2x^2-c^2$, as there is no bx term. 64 is a perfect square and we can write 56 as $\sqrt{56}^2$. We then have $64x^2-\sqrt{56}^2=0$ where $a=8$ and $c=\sqrt{56}$ which, following the notes above, we can write in the form $(ax-b)(ax+b) \rightarrow \left(8x-\sqrt{56}\right)\left(8x+\sqrt{56}\right)=0$. This gives us two equations for the roots $8x+\sqrt{56}=0$ and $8x-\sqrt{56}=0$ that yield the roots $x_1=-\frac{\sqrt{56}}{8}=-\frac{\sqrt{14}}{4}$ and $x_2=\frac{\sqrt{56}}{8}=\frac{\sqrt{14}}{4}$.

2 Radicals

Recall the following rules regarding simplifying radicals:

- No radicand contains a factor to a power greater than or equal to the index of the radical i.e., for $\sqrt[n]{b^m}$, then $m \ngeq n$.
- No power of the radicand and the index of the radical have a common factor other than 1 i.e., given the form above, n and m have no common factors.
- No radical appears in the denominator.
- No fraction appears within a radical.

1. How do you simplify an expression with a radical in the denominator?

   If we have an expression of the form $\frac{1}{\sqrt{b}}$, with any arbitrary expression multiplying the numerator, we can rationalize (simplify) it by multiplying this expression by $\frac{\sqrt{b}}{\sqrt{b}}$, noting that because it is a value divided by itself, it is equivalent to 1. Multiplying an expression by 1 will not change its value, but in this case it will change its form. So, we have $\frac{1}{\sqrt{b}} \cdot \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{b}}{\sqrt{b} \cdot \sqrt{b}}$ Using the square root properties found in the first lecture packet, we see that this is equal to $\frac{\sqrt{b}}{\sqrt{b^2}} = \frac{\sqrt{b}}{b}$. This expression no longer has a square root in the denominator, so it is rationalized (simplified).

2. Simplify the following expressions if they are not already fully simplified:
   (a) $\sqrt[8]{x^4}$

   $\sqrt[8]{x^4} = (x^4)^{\frac{1}{8}} = x^{\frac{4}{8}} = x^{\frac{1}{2}} = \sqrt{x}$

   (b) $\sqrt[9]{x^{13}}$

   $\sqrt[9]{x^{13}} = x^{\frac{9}{9}} \cdot x^{\frac{4}{9}} = x^1 \sqrt[9]{x^4}$

   (c) $\frac{\sqrt{\frac{5}{3}}}{\sqrt{2}}$

   $\frac{\sqrt{\frac{5}{3}}}{\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{3}\sqrt{2}} = \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5}\sqrt{6}}{\sqrt{6}\sqrt{6}} = \frac{\sqrt{30}}{6}$

3 Complex Numbers

A complex number has both a real and an imaginary part. If of the form $a + ib$, a is the real component and ib is the imaginary component of the complex number. Recall the following regarding the rules of imaginary and complex numbers:

- We define $i = \sqrt{-1}$.
- Therefore: $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$.
- The standard form of a complex number is expressed as $a + ib$.
- The complex conjugate of a complex number is found by negating the imaginary component of the complex number. The complex conjugate of $a + ib$ is $a - ib$.
- To rationalize expressions with a complex or imaginary number in the denominator, multiply both the denominator and the numerator of the expression by the complex conjugate of the denominator. If we have $\frac{1}{a+ib}$, we multiply the expression by $\frac{a-ib}{a-ib}$ to get $\frac{1}{a+ib} * \frac{a-ib}{a-ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$.

Answer the following questions regarding complex and imaginary numbers:

1. When do we encounter complex numbers?

   We encounter complex numbers when we deal with an expression containing the square root of a negative number i.e., when the discriminant $\Delta < 0$

2. What is i^6 equivalent to? (Hint: rewrite the expression with two exponents)

   Rewrite as $i^6 = (i^2)^3$ and using the definition above yields $(-1)^3 = -1$. Alternately, rewrite as $(i^3)^2$ and again using definitions above gives us $(-i)^2 = -1$

3. Write the following expressions in standard complex form:

   (a) $\frac{\sqrt{-49}}{2+3i}$

   We can write $\sqrt{-49}$ as $i\sqrt{49}$ which is equal to $7i$. We then multiply the expression by the complex conjugate of the denominator: $\frac{7i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{14i-21i^2}{4-3i+3i-9i^2} = \frac{14i+21}{9+4} = \frac{21}{11} + \frac{14}{11}i$

   (b) $83 + 16i - (3+4i)(1-2i)$

   Foiling the product term, we get $(83 + 16i - 3 - 6i + 4i - 8i^2 = 83 - 3 + 8) + i(16 - 6 + 4) = 88 + 14i$

4 Sets of Numbers

We introduce the definitions for various sets of real numbers to allow for familiarity with the language.

- We can think of the set of **real numbers** as containing all points on the number line. It is often denoted $\mathcal{R}$.
- The set of **integers** contains all positive and negative integers (whole numbers) as well as 0 and it is often denoted with $\mathcal{Z}$.
- The set of **natural numbers** contains all integers greater than and equal to 1. It is often denoted $\mathcal{N}$.
- The set of **rational numbers** is often denoted with $\mathcal{Q}$ and contains all numbers that can be expressed as a quotient of two integers. $\frac{1}{3}$, 5, and $\frac{37}{43}$ are rational numbers, while π and $\sqrt{2}$ are not.
- The set of **irrational numbers** contains all numbers that are not rational.

5 Word Problems

1. A ball is launched from the ground. Its trajectory can be modelled by the following equation: $f(x) = -x^2 + 8x + 5$ where $f(x)$ denotes the ball's position. At what x-coordinate does the ball land? How far does the ball travel? (Hint: for the total distance, consider both roots.)

   Extra hint for homework: At least one of the methods given in section 1 will work here. Considering the frame of reference of the roots when computing the total distance of the ball – does the first root (where the ball launches from) coincide with x = 0? If not, what must you do in your calculation?

2. Olivia owns a flower shop that has 2,000 dollars in the bank. Each arrangement sells for 20 dollars. The flower shop's bills total 3,000, and Olivia would like to have *more than* 1,500 dollars left over. How many arrangements does she need to sell to make this happen?

   We can model this situation with an inequality. Let x be the variable that represents the number of flower arrangements that Olivia sells. Define the following constants: p = price per flower arrangement, T = total money in the bank to start, B = total amount of bills to be paid, G = the goal amount of money remaining. Therefore: $T - B + px > G$. We have: $p = 20, T = 2000, B = 3000$ and $G = 1500$. This gives us $2000 - 3000 + 20x > 1500 \rightarrow -1000 + 20x > 1500 \rightarrow 20x > 2500 \rightarrow x > 125$. So, Olivia needs to sell more than 125 flower arrangements to have over 1500 dollars in the bank after paying her bills.

3. The width of a particular rectangular lawn is equal to 5 times its length. If the area must be *less than* 25 acres, what's the maximum value that the width can be?

   We start by defining a variable for length, l, a variable for width, w, and a variable for the area, A. We know that the width is 5 times the length, $w = 5l$ and that $A < 25$. Because the lawn is rectangular, we know that the area is equal to the product of the length and the width, $A = l \cdot w$. Substituting $5l$ in for w, we have $A < 25 \rightarrow l \cdot w < 25 \rightarrow l \cdot 5l < 25 \rightarrow 5l^2 < 25$. Solving for l, we get $l < \sqrt{5}$, where we only consider the positive root of 5 because length cannot be a negative value. Since $w = 5l$, we have $w < 5\sqrt{5}$, and therefore the width can be, at most, just under $5\sqrt{5} \approx 11.18$ acres.