Engineering
Mathematics and Computing

Task 1: Coursework Assessment

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Git Repo : https://github.com/sakx7/mathcompuni

Chapter 1

Part A: Mathematics

Notes:

1.1 Q1

Use the quotient rule to differentiate the function $y = \frac{\ln 3x}{2x}$

The quotient rule states that if

$$y = \frac{u}{v} \quad \text{then} \quad y' = \frac{u'v - uv'}{v^2}$$

Here:

$$u = \ln 3x \qquad v = 2x$$

First, find the derivatives:

$$u = \ln(3x)$$

Let:

$$\delta = 3x \qquad u = \ln(\delta)$$

$$\frac{du}{d\delta} = \frac{1}{\delta} \qquad \frac{d\delta}{dx} = 3$$

Using the chain rule, we can find:

$$\frac{du}{dx} = \frac{du}{d\delta} \cdot \frac{d\delta}{dx}$$

$$\frac{du}{dx} = \frac{3}{\delta} = \frac{3}{3x} = \frac{1}{x}$$

General rule for composite/nested functions is solved via chain rule it's:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

$$v = 2x$$

$$v' = \frac{d}{dx}(2x) = 2$$

Now we can apply the quotient rule:

$$y' = \frac{\left(\frac{1}{x}\right)(2x) - (\ln 3x)(2)}{(2x)^2} = \frac{2 - 2\ln 3x}{4x^2}$$

$$\boxed{y' = \frac{1 - \ln 3x}{2x^2}}$$

1.2 Q2

Find the angle between the vectors $2i - 11j - 10k$ and $5i + 8j + 7k$

The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by:

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

Calculate the dot product:

$$\mathbf{a}\cdot\mathbf{b} = \underbrace{(2)(5)}_{i} + \underbrace{(-11)(8)}_{j} + \underbrace{(-10)(7)}_{k} = -148$$

Calculate the magnitudes:

$$\|\mathbf{a}\| = \sqrt{2^2 + (-11)^2 + (-10)^2} = \sqrt{225} = 15$$

$$\|\mathbf{b}\| = \sqrt{5^2 + 8^2 + 7^2} = \sqrt{138}$$

Find $\cos\theta$:

$$\cos\theta = \frac{-148}{15 \times \sqrt{138}}$$

Thus, the angle θ is:

$$\boxed{\theta = \cos^{-1}\left(\frac{-148}{15 \times \sqrt{138}}\right) \approx 147.1^\circ}$$

1.3 Q3

Find the rate of change of $y = \ln(16t^2 + 19)$ at the specified point $t = 9$

Differentiate y with respect to t:

Note prior (q1) that the general rule for composite/nested functions is:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Here for $y = \ln(16t^2 + 19)$

$$\frac{dy}{dt} = \frac{1}{16t^2 + 19} \cdot \frac{d}{dt}(16t^2 + 19)$$

$$\frac{d}{dt}(16t^2 + 19) = 32t$$

$$\frac{dy}{dt} = \frac{1}{16t^2 + 19} \cdot 32t = \frac{32t}{16t^2 + 19}$$

Evaluate at $t = 9$:

$$\boxed{\left.\frac{dy}{dt}\right|_{t=9} = \frac{32(9)}{16(9)^2 + 19} = \frac{288}{1315} \approx 0.219}$$

1.4 Q4

Express $\cos t - 8\sin t$ in the form $A\cos(\omega t + \alpha)$, where $\alpha \geq 0$

you see the thing is the left side, $\cos t - 8\sin t$, is standard, but the right side introduces a different frequency, ω, in $A\cos(\omega t + \alpha)$.

Since the left side has a frequency of 1, im just gonna assume $\omega = 1$ for simplicity. If ω were different, you would need to rewrite the left side in terms of ωt, but this isn't specified in the problem.

The angle subtraction formula is:

$$A\cos(t - \phi) = A\cos(\phi)\cos(t) + A\sin(\phi)\sin(t)$$

By comparing coefficients from both sides, we have:

$$a = A\cos(\phi)$$
$$b = A\sin(\phi)$$

To find A we use $A = \sqrt{a^2 + b^2}$

This arises from squaring both equations $a = A\cos(\phi)$ and $b = A\sin(\phi)$:

$$a^2 + b^2 = (A\cos(\phi))^2 + (A\sin(\phi))^2 = A^2(\cos^2(\phi) + \sin^2(\phi)) = A^2$$

To find the phase shift ϕ, we use $\tan\phi = \frac{b}{a}$

This comes from the definitions of sine and cosine:

$$\tan\phi = \frac{A\sin(\phi)}{A\cos(\phi)} = \frac{b}{a}$$

In so we derive and make use of:

$$A\cos(t - \phi) = a\cos(t) + b\sin(t)$$

Where $A = \sqrt{a^2 + b^2}$ and $\tan\phi = \frac{b}{a}$

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for $b = -8$ and $a = 2$

$$A = \sqrt{1^2 + (-8)^2} = \sqrt{65}$$

$$\tan\phi = \frac{-8}{1} = -8 \qquad \phi = \arctan(-8) \approx -82.87°$$

In so plugging in gives

$$\begin{aligned}\cos t - 8\sin t &= A\cos(t - \phi)\\ &= \sqrt{65}\cos(t - \arctan(-8))\\ &= \sqrt{65}\cos(t - (-82.87°))\end{aligned}$$

$$\boxed{\cos t - 8\sin t \approx 8.06\cos(t + 82.87°)}$$

1.5 Q5

Solve the following system of three linear equations using Cramer's rule

$$\begin{cases} 11v_1 - v_2 + v_3 = 31.4 \\ v_1 + \frac{v_2}{2} - v_3 = 1.9 \\ -9v_1 + 11v_3 = -12 \end{cases}$$

The system can be written in matrix form $A\mathbf{v} = \mathbf{b}$, specifically because of the variable distributions where:

$$A = \begin{bmatrix} 11 & -1 & 1 \\ 1 & \frac{1}{2} & -1 \\ -9 & 0 & 11 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 31.4 \\ 1.9 \\ -12 \end{bmatrix}$$

Calculate the determinant $\det(A)$.

$$\det(A) = 67$$

Now solve for each variable using Cramer's rule:

$$A_1 = \begin{bmatrix} 31.4 & -1 & 1 \\ 1.9 & \frac{1}{2} & -1 \\ -12 & 0 & 11 \end{bmatrix} \quad A_2 = \begin{bmatrix} 11 & 31.4 & 1 \\ 1 & 1.9 & -1 \\ -9 & -12 & 11 \end{bmatrix} \quad A_3 = \begin{bmatrix} 11 & -1 & 31.4 \\ 1 & \frac{1}{2} & 1.9 \\ -9 & 0 & -12 \end{bmatrix}$$

Calculate the determinants:

$$\det(A_1) \approx 187.6 \quad \det(A_2) \approx 40.2 \quad \det(A_3) \approx 80.4$$

$\mathbf{v}$ can be found as:

$$v_1 = \frac{\det(A_1)}{\det(A)}, \quad v_2 = \frac{\det(A_2)}{\det(A)}, \quad v_3 = \frac{\det(A_3)}{\det(A)}$$

$$\boxed{\begin{aligned} v_1 &= \tfrac{187.6}{67} = 2.8 \\ v_2 &= \tfrac{40.2}{67} = 0.6 \\ v_3 &= \tfrac{80.4}{67} = 1.2 \end{aligned}}$$

There are a few ways to calculate the determinants, but especially in this working out i chose not show the method i used since it's not the main focus of the question.

If you're curious about how I did it, the Sarrus method for simplicity.

1.6 Q6

Transpose $z = d + a\sqrt{y}$ to make y the subject.

Starting with:

$$z = d + a\sqrt{y}$$

$$z - d = a\sqrt{y}$$

$$\frac{z-d}{a} = \sqrt{y}$$

$$\left(\frac{z-d}{a}\right)^2 = y$$

Thus, the expression for y is:

$$\boxed{y = \frac{(z-d)^2}{a^2}}$$

i generally consider this the most concise and general expression for y.
im aware you could expand $(z-d)^2$:

$$y = \frac{z^2 - 2zd + d^2}{a^2}$$

and additionally take into account of both possible factorisations:

$$y = \frac{z^2 + d(d - 2z)}{a^2}$$

$$y = \frac{d^2 + z(z - 2d)}{a^2}$$

1.7 Q7

Find a vector that is perpendicular to both of the vectors

$$\boldsymbol{a} = 4\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$$

$$\boldsymbol{b} = 3\boldsymbol{i} + 4\boldsymbol{j} - 6\boldsymbol{k}$$

Hence find a unit vector that is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

The unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{r}}{||\vec{r}||}$$

Here we want to define $\vec{r}$ as a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, to do that we take there cross product $\mathbf{a} \times \mathbf{b}$

$$\vec{r} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 5 \\ 3 & 4 & -6 \end{vmatrix} = -38\mathbf{i} + 39\mathbf{j} + 7\mathbf{k}$$

Find the magnitude:

$$||\vec{r}|| = \|\mathbf{a} \times \mathbf{b}\| = \sqrt{(-38)^2 + 39^2 + 7^2} = \sqrt{3014}$$

The unit vector is:

$$\boxed{\frac{-38\mathbf{i} + 39\mathbf{j} + 7\mathbf{k}}{\sqrt{3014}}}$$

1.8 Q8

If $M = \begin{pmatrix} 7 & 9 \\ 1 & -2 \end{pmatrix}$ and $N = \begin{pmatrix} 2 & 1 \\ -2 & 6 \end{pmatrix}$ find MN and NM

Calculate MN:

$$MN = \begin{bmatrix} 7 & 9 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \cdot 2 + 9 \cdot (-2) & 7 \cdot 1 + 9 \cdot 6 \\ 1 \cdot 2 + (-2) \cdot (-2) & 1 \cdot 1 + (-2) \cdot 6 \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} -4 & 61 \\ 6 & -11 \end{bmatrix}}$$

Calculate NM:

$$NM = \begin{bmatrix} 2 & 1 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 7 & 9 \\ 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 7 + 1 \cdot 1 & 2 \cdot 9 + 1 \cdot (-2) \\ -2 \cdot 7 + 6 \cdot 1 & -2 \cdot 9 + 6 \cdot (-2) \end{bmatrix}$$

$$\boxed{= \begin{bmatrix} 15 & 16 \\ -8 & -30 \end{bmatrix}}$$

1.9 Q9

If $y = x^4 - 4x^3 - 90x^2$, find the values of x for which $y'' = 0$

First, find the first derivative:

$$y' = \frac{d}{dx}(x^4 - 4x^3 - 90x^2) = 4x^3 - 12x^2 - 180x$$

Find the second derivative:

$$y'' = \frac{d}{dx}(4x^3 - 12x^2 - 180x) = 12x^2 - 24x - 180$$

Set $y'' = 0$:

$$12x^2 - 24x - 180 = 0$$

Divide by 12:

$$x^2 - 2x - 15 = 0$$

Factor:

$$(x - 5)(x + 3) = 0$$

$$\boxed{x = 5,\ x = -3}$$

1.10 Q10

Transpose $b = g + t(a - 3)$ to make a the subject

Starting with:

$$b = g + t(a - 3)$$

this is easily rearranged to isolate a:

$$b - g = t(a - 3)$$

$$\frac{b - g}{t} = a - 3$$

$$\boxed{a = \frac{b - g}{t} + 3}$$

Chapter 2

Part B: Computing

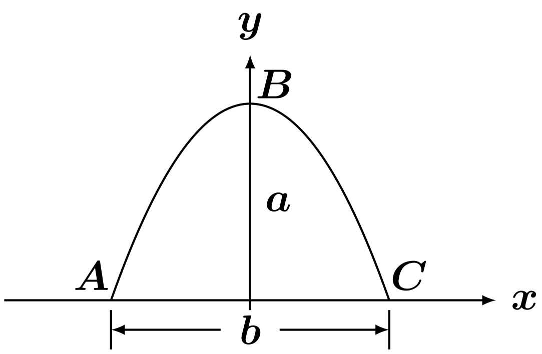
Notes:

- All graphs and images were created using TikZ PGF.
- Code implementations are original and written in MATLAB and to strictly follow guidelines with some python also.
  - Relying exclusively on MATLAB is short-sighted, especially when open-source alternatives like Python offer more flexibility and long-term accessibility. Student licenses for MATLAB are temporary, leaving costly fees after graduation.
  - While MATLAB is easy to learn, this short-term benefit can lead to dependency on a single tool. Python may take longer to master but provides skills that are more transferable and valuable in a variety of fields.
  - Courses focusing on MATLAB may neglect teaching transferable programming skills. Python, with its broad applications and extensive libraries, is a better long-term choice for students in a world dominated by open-source tools.
  - Despite the fact that using MATLAB is required to complete this module, I'm not sure I'm geared up to go face-first into something that contradicts the objectives of the module as a whole.
- Please consider the answers as a whole.

2.1 Q1

The arc length of a segment of a parabola ABC of an ellipse with semi-minor axes a and b is given approximately by:

$$L_{ABC} = \frac{1}{2}\sqrt{b^2 + 16a^2} + \frac{b^2}{8a}\ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$



Write a universal, user-friendly code, test your programme and determine L_{ABC} if $a = 11$ cm and $b = 9$ cm.

Since the goal is utilize Matlab, this question is fairly easy this version should suffice if a widely used and intuitive code is required.

matlab scripts/q1_easy.m

```matlab
%make a function to call for the calculation
function L_ABC = calculate_arc_length(a, b)
    %write in ascci math using matlab built in functions
    L_ABC = (1/2) * sqrt(b^2 + 16 * a^2) + (b^2 / (8 * a)) * log((4 * a + sqrt(b^2 +
        16 * a^2)) / b);
end

function arc_length_parabola()
    %prompt the user for input values
    a = input('Enter the value of a (height): ');
    b = input('Enter the value of b (width): ');

    %calculate the arc length by calling the function
    L_ABC = calculate_arc_length(a, b);

    %display the result by printing
    fprintf('The arc length L_ABC is: %.2f\n', L_ABC);
end
%realistically not much error handling is inputed but i dont think that is the gaol
    here
```

2.1.1 Advanced version

We are tasked with analyzing a parabola where:

- a represents the height of the parabola
- b represents the width of the parabola

My goal is to plot the parabola, first by formulating an equation in terms of these factors. Initially, we begin with a simple parabola in the general shape (x^2) similar to shown in the graph. Our goal is to select a reasonable function, with variables associated with the x scale and y location of the parabola:

$$y = -(\beta x)^2 + \gamma$$

Given our problem statement:

- γ represents the height, so $\gamma = a$
- The width is related to the roots of the equation when $y = 0$

To find β, we solve:

$$0 = -(\beta x)^2 + \gamma$$

$$\beta = \frac{2\sqrt{\gamma}}{b}$$

since we know γ:

$$\beta = \frac{2\sqrt{a}}{b}$$

Note: This requires $b \neq 0$ and $\sqrt{a} \neq 0$ (i.e., $a > 0$).
Substituting γ and β into our original equation:

$$\boxed{y = -\left(\frac{2\sqrt{a}}{b}x\right)^2 + a}$$

This is our final parabola equation in terms of a and b. The arc length of the parabola from $x = -\frac{b}{2}$ to $x = \frac{b}{2}$ is given by:

$$L_{ABC} = \frac{1}{2}\sqrt{b^2 + 16a^2} + \frac{b^2}{8a} \ln\left(\frac{4a + \sqrt{b^2 + 16a^2}}{b}\right)$$

When implementing this in a program:

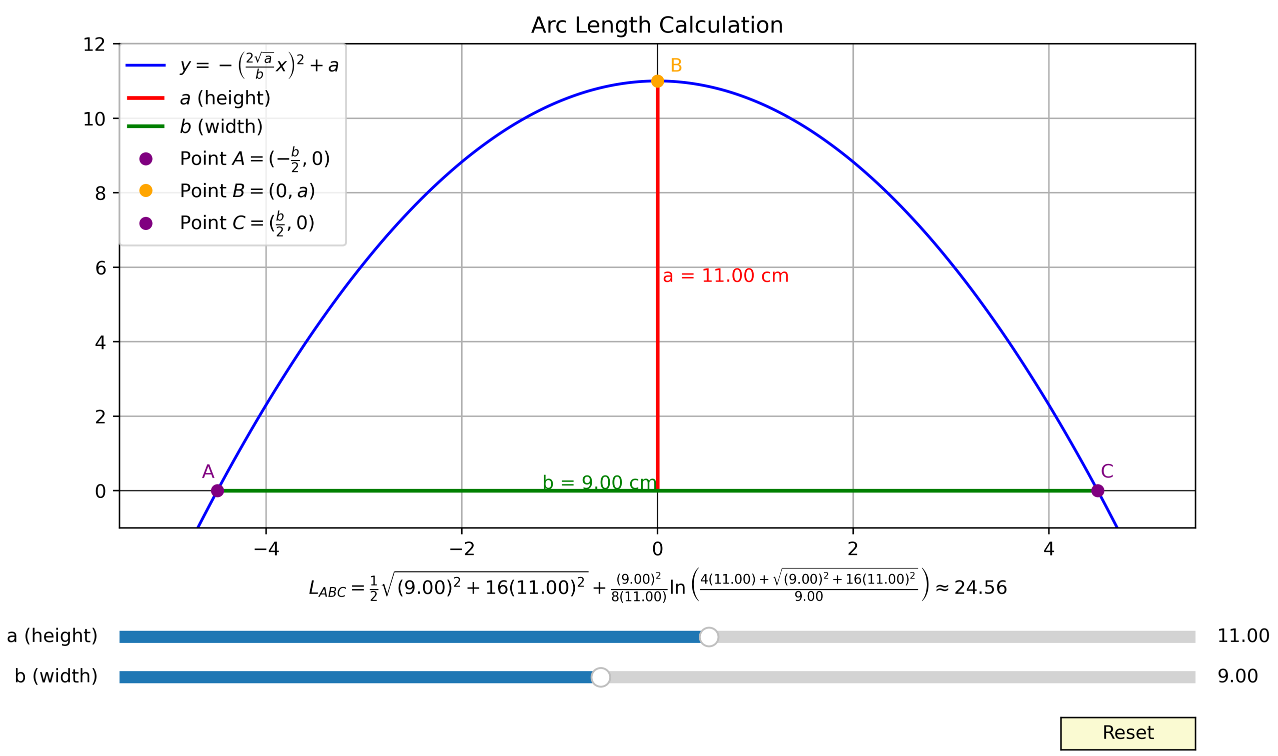
- Ensure that when there are no limit parameters for the inputs a and b, the conditions $a > 0$ and $b \neq 0$ are appropriately handled.
- Update the plot of $y = -\left(\frac{2\sqrt{a}}{b}x\right)^2 + a$ so that the sliders for a and b directly control the height and width, respectively.
- Calculate the arc length of the parabola using a function with inputs a and b, and display the result in the plot.

The rest is just nitpicky presentation, all just preference.

scripts/q1.py

```python
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.widgets import Slider, Button

def arc_length_parabola(a, b):
    L_ABC=(1/2)*np.sqrt(b**2+16*a**2)+(b**2/(8*a))*np.log((4*a+np.sqrt(b**2+16*a**2))/b)
    return L_ABC

fig, ax = plt.subplots(figsize=(10, 6))
plt.subplots_adjust(left=0.1, bottom=0.3, right=0.9, top=0.9)
initial_a = 11.0
initial_b = 9.0

ax_slider_a = plt.axes([0.1, 0.15, 0.8, 0.03])
ax_slider_b = plt.axes([0.1, 0.1, 0.8, 0.03])

slider_a = Slider(ax_slider_a, 'a (height)', 0.1, 20, valinit=initial_a, valfmt='%0.2f')
slider_b = Slider(ax_slider_b, 'b (width)', 0.1, 20, valinit=initial_b, valfmt='%0.2f')

x = np.linspace(-10, 10, 1000)

def update(val):
    a = slider_a.val
    b = slider_b.val
    L_ABC = arc_length_parabola(a, b)

    ax.clear()
    # should add if a > 0 and b != 0: but im not going to since i set the slider limits accordingly
    y = -((2*np.sqrt(a))*x/b)**2 + a
    ax.plot(x, y, label=r'$y = -\left(\frac{2\sqrt{a}}{b}x\right)^2 + a$', color='blue')

    ax.vlines(0, 0, a, color='red',linewidth=2 , zorder=10, label=r'$a$ (height)')
    ax.hlines(0, -b/2, b/2, color='green',linewidth=2, zorder=10, label=r'$b$ (width)')

    ax.text(0.05, a/2, f'a = {a:.2f} cm', color='red', fontsize=10, verticalalignment='bottom')
    ax.text(0, 0.05, f'b = {b:.2f} cm', color='green', fontsize=10, horizontalalignment='right')

    A_x = -b/2
    C_x = b/2

    ax.plot(A_x, 0, 'o', zorder=10, color='purple', label=r'Point $A = (-\frac{b}{2}, 0)$')
    ax.plot(0, a, 'o',zorder=10,  color='orange', label=r'Point $B = (0, a)$')
    ax.plot(C_x, 0, 'o',zorder=10, color='purple', label=r'Point $C = (\frac{b}{2}, 0)$')

    ax.annotate('A', (A_x, 0), textcoords="offset points", xytext=(-5,7), ha='center', color='purple')
    ax.annotate('B', (0, a), textcoords="offset points", xytext=(10,5), ha='center', color='orange')
    ax.annotate('C', (C_x, 0), textcoords="offset points", xytext=(5,7), ha='center', color='purple')
```

```
    ax.set_title(f'Arc Length Calculation')
    ax.set_xlabel(r'$L_{ABC} = \frac{1}{2}\sqrt{' + f'({b:.2f})^2 + 16({a:.2f})^2' + r'} + \frac{' + f'({b:.2f})^2' + r'}{8(' + f'{a:.2f}' + r')}\ln\left(\frac{4(' + f'{a:.2f}' + r') + \sqrt{' + f'({b:.2f})^2 + 16({a:.2f})^2' + r'}}{' + f'{b:.2f}' + r'}\right) \approx ' + f'{L_ABC:.2f}$')

    ax.set_ylim(-1, a + 1)
    ax.set_xlim(-b/2 - 1, b/2 + 1)

    ax.legend(bbox_to_anchor=(0, 1), loc='upper left', borderaxespad=0.)

    ax.grid(True)
    ax.axhline(y=0, color='k', linestyle='-', linewidth=0.5)
    ax.axvline(x=0, color='k', linestyle='-', linewidth=0.5)
    plt.draw()

slider_a.on_changed(update)
slider_b.on_changed(update)

resetax = plt.axes([0.8, 0.025, 0.1, 0.04])
button = Button(resetax, 'Reset', color='lightgoldenrodyellow', hovercolor='0.975')

def reset(event):
    slider_a.reset()
    slider_b.reset()
    update(None)

button.on_clicked(reset)

update(None)

plt.savefig('figures/Figure_1.png',dpi=400)
plt.show()
```



2.2 Q1

The voltage difference V_{ab} between points a and b in the Wheatstone bridge circuit is given by the formula:

$$V_{ab} = V\left(\frac{R_1R_3 - R_2R_4}{(R_1 + R_2)(R_3 + R_4)}\right)$$

Write a universal, user-friendly program that calculates the voltage difference V_{ab}. Test your program using the following values:

- $V = 14$ volts
- $R_1 = 120.6\ \Omega$
- $R_2 = 119.3\ \Omega$
- $R_3 = 121.2\ \Omega$
- $R_4 = 118.8\ \Omega$