Example

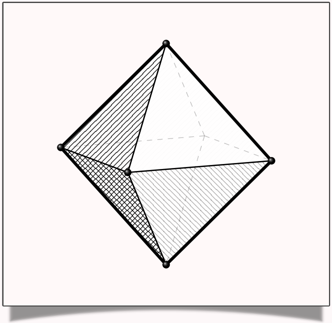
- Suppose that we have the following problem

$$
\begin{aligned}
\max\ & 2x + y - 5z \\
\text{s.t. } & x + y + z \leq 1 \\
& x + y - z \leq 1 \\
& x - y + z \leq 1 \\
& x - y - z \leq 1 \\
& -x + y + z \leq 1 \\
& -x + y - z \leq 1 \\
& -x - y + z \leq 1 \\
& -x - y - z \leq 1
\end{aligned}
$$

Example

- Suppose that we have the following problem

$$
\begin{aligned}
\max\ & 2x + y - 5z \\
\text{s.t. } & x + y + z \le 1 \\
& x + y - z \le 1 \\
& x - y + z \le 1 \\
& x - y - z \le 1 \\
& -x + y + z \le 1 \\
& -x + y - z \le 1 \\
& -x - y + z \le 1 \\
& -x - y - z \le 1
\end{aligned}
$$



feasible region

Fixing the Objective Value

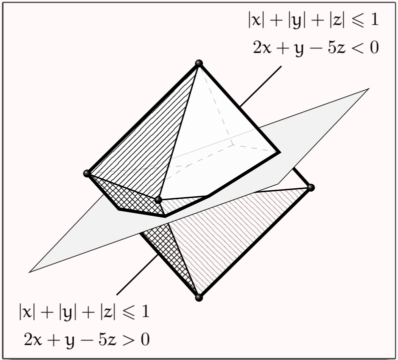
- Consider the hyperplane $F = \{(x, y, z) \mid 2x + y - 5z = c_0\}$

Fixing the Objective Value

- Consider the hyperplane $F = \{(x, y, z) \mid 2x + y - 5z = c_0\}$
  - its normal is $(2, 1, -5)$, so it faces down

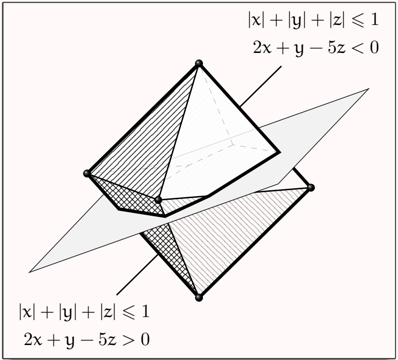
Fixing the Objective Value

- Consider the hyperplane $F = \{(x, y, z) \mid 2x + y - 5z = c_0\}$
  - its normal is $(2, 1, -5)$, so it faces down



Fixing the Objective Value

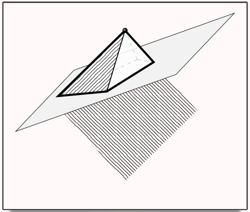
- Consider the hyperplane $F = \{(x, y, z) \mid 2x + y - 5z = c_0\}$
  - its normal is $(2, 1, -5)$, so it faces down



- What happens if we increase c_0?

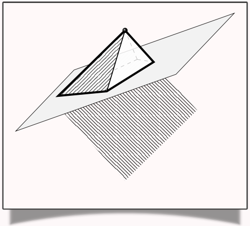
Finding the Optimum

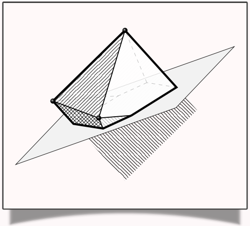
- Increasing $c_0 \implies$ moving plane towards optimum



Finding the Optimum

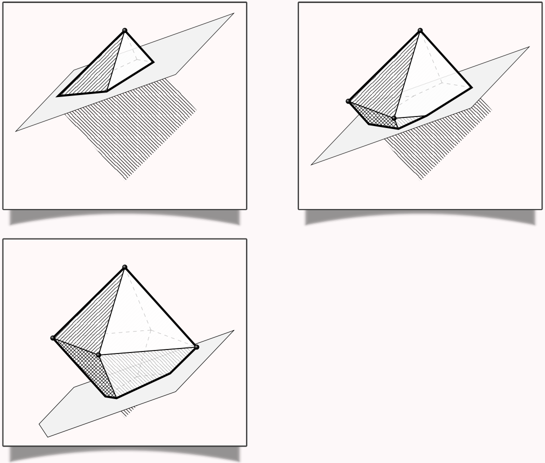
- Increasing $c_0 \implies$ moving plane towards optimum





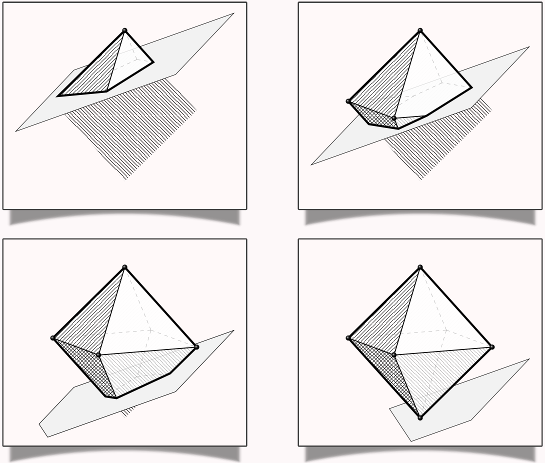
Finding the Optimum

- Increasing $c_0 \implies$ moving plane towards optimum



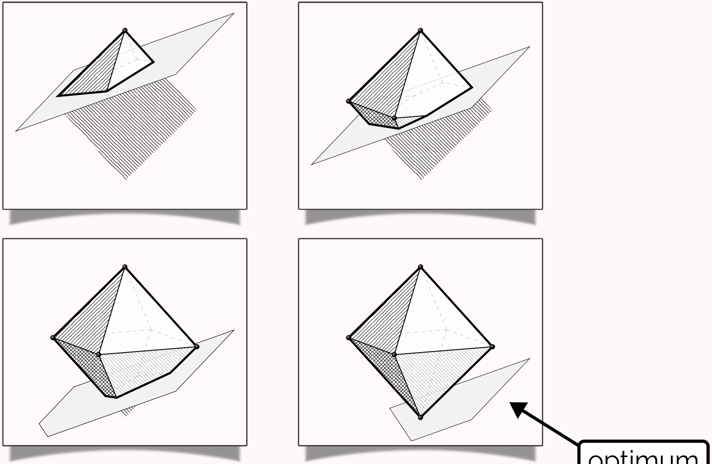
Finding the Optimum

- Increasing $c_0 \implies$ moving plane towards optimum



Finding the Optimum

- Increasing $c_0 \implies$ moving plane towards optimum



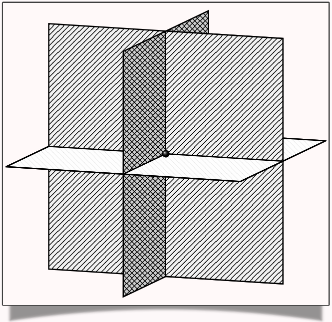
- Optimal region is either a vertex or contains one

Finding a Vertex

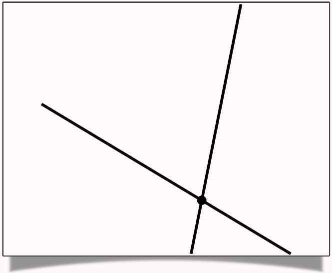
- A vertex in $\mathbb{R}^d$ is the intersection of d linearly independent hyperplanes

Finding a Vertex

- A vertex in $\mathbb{R}^d$ is the intersection of d linearly independent hyperplanes



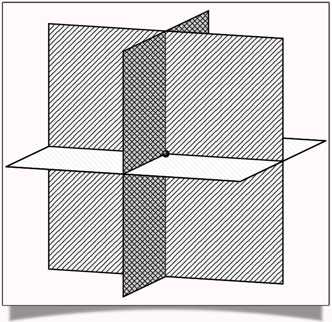
$\mathbb{R}^3$



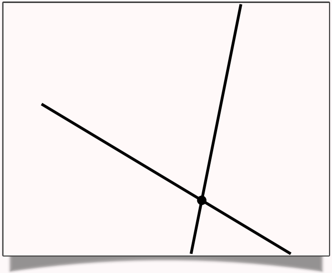
$\mathbb{R}^2$

Finding a Vertex

- A vertex in $\mathbb{R}^d$ is the intersection of d linearly independent hyperplanes



$\mathbb{R}^3$



$\mathbb{R}^2$

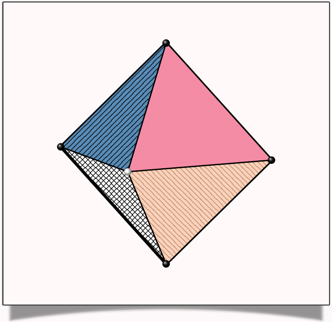
- *In other words, a solution to a linear system*

Example

$x + y + z = 1$

$x + y - z = 1$

$x - y + z = 1$

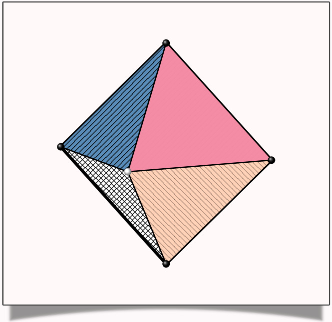


Example

$x + y + z = 1$

$x + y - z = 1$

$x - y + z = 1$



- The associated vertex is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Duality

The Primal Problem

$$\max_{\boldsymbol{x} \in \mathbb{R}^{n \times 1}} \{ \boldsymbol{c}\boldsymbol{x} \mid A\boldsymbol{x} \leq \boldsymbol{b} \}$$

Duality

The Primal Problem

$$\max_{\boldsymbol{x} \in \mathbb{R}^{n \times 1}} \{ \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \}$$

The Dual Problem

$$\min_{\boldsymbol{\pi} \in \mathbb{R}^{1 \times m}} \{ \boldsymbol{\pi}\boldsymbol{b} \mid \boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{c}, \boldsymbol{\pi} \geq \boldsymbol{0} \}$$

Duality

The Primal Problem

$$\max_{\boldsymbol{x} \in \mathbb{R}^{n \times 1}} \{ \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \}$$

The Dual Problem

$$\min_{\boldsymbol{\pi} \in \mathbb{R}^{1 \times m}} \{ \boldsymbol{\pi}\boldsymbol{b} \mid \boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{c}, \boldsymbol{\pi} \geq \boldsymbol{0} \}$$

- Both problems are deeply connected

Duality

The Primal Problem

$$\max_{\boldsymbol{x} \in \mathbb{R}^{n\times 1}} \{\, \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \,\}$$

The Dual Problem

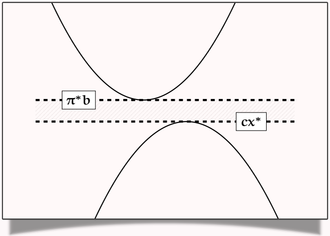
$$\min_{\boldsymbol{\pi} \in \mathbb{R}^{1\times m}} \{\, \boldsymbol{\pi}\boldsymbol{b} \mid \boldsymbol{\pi}\boldsymbol{A} = \boldsymbol{c}, \boldsymbol{\pi} \geq \boldsymbol{0} \,\}$$

- Both problems are deeply connected
  - In fact, the dual provides a certificate of optimality for primal solutions

Weak Duality

Theorem (Weak Duality)

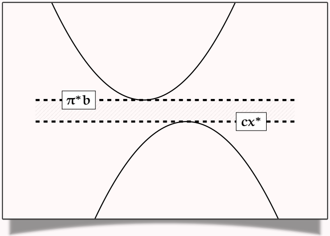
For any primal feasible $\boldsymbol{x}$ and any dual feasible $\boldsymbol{\pi}$, $\boldsymbol{cx} \leq \boldsymbol{\pi b}$



Weak Duality

Theorem (Weak Duality)

For any primal feasible $\boldsymbol{x}$ and any dual feasible $\boldsymbol{\pi}$, $\boldsymbol{cx} \leq \boldsymbol{\pi b}$



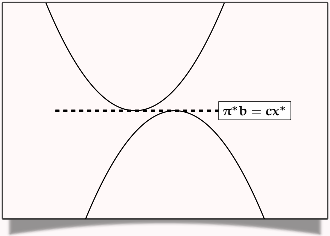
- **Proof.** $Ax \leq b \implies \underbrace{(\pi A)}_{\boldsymbol{c}} x \leq \pi b.$ $\square$

Strong Duality

Theorem (Strong Duality)

$$\max_{\boldsymbol{x} \in \mathbb{R}^{n\times 1}} \{ \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \} = \min_{\pi \in \mathbb{R}^{1\times m}} \{ \pi\boldsymbol{b} \mid \pi\boldsymbol{A} = \boldsymbol{c}, \pi \ge \boldsymbol{0} \}$$

as long as both problems are feasible

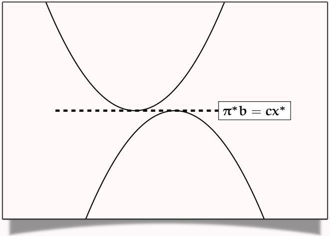


Strong Duality

Theorem (Strong Duality)

$$\max_{\boldsymbol{x} \in \mathbb{R}^{n \times 1}} \{\, \boldsymbol{cx} \mid \boldsymbol{Ax} \leq \boldsymbol{b} \,\} = \min_{\boldsymbol{\pi} \in \mathbb{R}^{1 \times m}} \{\, \boldsymbol{\pi b} \mid \boldsymbol{\pi A} = \boldsymbol{c}, \boldsymbol{\pi} \geq \boldsymbol{0} \,\}$$

as long as both problems are feasible



- Proof is a little more elaborate
  - Requires Farkas' Lemma

Strong Duality

Theorem (Farkas' Lemma)

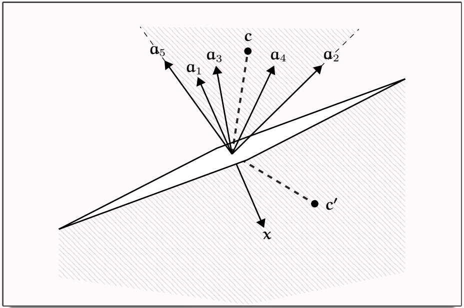
$$\exists \pi \geq 0, \pi A = c \iff \nexists x, Ax \leq 0, cx > 0$$

Strong Duality

Theorem (Farkas' Lemma)

$$\exists \pi \geq 0, \pi A = c \iff \nexists x, Ax \leq 0, cx > 0$$

- Provides conditions under which a system of inequalities is solvable



Strong Duality

- **Proof of Strong Duality.** Suppose the minimum of the dual problem is equal to $\delta \in \mathbb{R}$

Strong Duality

- **Proof of Strong Duality.** Suppose the minimum of the dual problem is equal to $\delta \in \mathbb{R}$

- The following system is obviously infeasible for any $\varepsilon > 0$

$$
\begin{aligned}
\boldsymbol{\pi A} &= \boldsymbol{c} \\
\boldsymbol{\pi b} + s &= \delta - \varepsilon \\
\boldsymbol{\pi} &\geq \boldsymbol{0} \\
s &\geq 0
\end{aligned}
$$

Strong Duality

- **Proof of Strong Duality.** Suppose the minimum of the dual problem is equal to $\delta \in \mathbb{R}$

- The following system is obviously infeasible for any $\varepsilon > 0$

$$\begin{aligned} \pi \boldsymbol{A} &= \boldsymbol{c} \\ \pi \boldsymbol{b} + s &= \delta - \varepsilon \\ \pi &\geq \boldsymbol{0} \\ s &\geq 0 \end{aligned}$$

$$(\pi, s)\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{bmatrix} = (\boldsymbol{c}, \delta - \varepsilon)$$

$$(\pi, s) \geq \boldsymbol{0}$$

Strong Duality

- **Proof of Strong Duality.** Suppose the minimum of the dual problem is equal to $\delta \in \mathbb{R}$

- The following system is obviously infeasible for any $\varepsilon > 0$

$$\begin{aligned} \boldsymbol{\pi} \boldsymbol{A} &= \boldsymbol{c} \\ \boldsymbol{\pi} \boldsymbol{b} + s &= \delta - \varepsilon \\ \boldsymbol{\pi} &\geq \boldsymbol{0} \\ s &\geq 0 \end{aligned}$$

$$(\boldsymbol{\pi}, s) \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{bmatrix} = (\boldsymbol{c}, \delta - \varepsilon)$$

$$(\boldsymbol{\pi}, s) \geq \boldsymbol{0}$$

- As such, there is $\boldsymbol{x}$ and λ s.t.

Strong Duality

- **Proof of Strong Duality.** Suppose the minimum of the dual problem is equal to $\delta \in \mathbb{R}$

- The following system is obviously infeasible for any $\varepsilon > 0$

$$\begin{aligned} \pi \boldsymbol{A} &= \boldsymbol{c} \\ \pi \boldsymbol{b} + s &= \delta - \varepsilon \\ \pi &\geq \boldsymbol{0} \\ s &\geq 0 \end{aligned}$$

$$(\pi, s) \begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{bmatrix} = (\boldsymbol{c}, \delta - \varepsilon)$$
$$(\pi, s) \geq \boldsymbol{0}$$

- As such, there is $\boldsymbol{x}$ and λ s.t.

$$\begin{aligned} \boldsymbol{A}\boldsymbol{x} + \lambda \boldsymbol{b} &\leq \boldsymbol{0} \\ \lambda &\leq 0 \\ \boldsymbol{c}\boldsymbol{x} + \lambda(\delta - \varepsilon) &> 0 \end{aligned}$$

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{bmatrix} \begin{pmatrix} \boldsymbol{x} \\ \lambda \end{pmatrix} \leq \boldsymbol{0}$$
$$(\boldsymbol{c}, \delta - \varepsilon) \begin{pmatrix} \boldsymbol{x} \\ \lambda \end{pmatrix} > 0$$

Strong Duality

- If $\lambda = 0$, then $\boldsymbol{Ax} \leq \boldsymbol{0}$ and $\boldsymbol{cx} > 0$

Strong Duality

- If $\lambda = 0$, then $\boldsymbol{Ax} \leq \mathbf{0}$ and $\boldsymbol{cx} > 0$
  - The dual problem is infeasible. A contradiction.

Strong Duality

- If $\lambda = 0$, then $\boldsymbol{Ax} \leq \mathbf{0}$ and $\boldsymbol{cx} > 0$
  - The dual problem is infeasible. A contradiction.
- If $\lambda < 0$, then $\boldsymbol{Ax} \leq -\lambda \boldsymbol{b}$ and $\boldsymbol{cx} > -\lambda(\delta - \varepsilon)$

Strong Duality

- If $\lambda = 0$, then $\boldsymbol{Ax} \leq \boldsymbol{0}$ and $\boldsymbol{cx} > 0$
  - The dual problem is infeasible. A contradiction.
- If $\lambda < 0$, then $\boldsymbol{Ax} \leq -\lambda \boldsymbol{b}$ and $\boldsymbol{cx} > -\lambda(\delta - \varepsilon)$
  - Multiplying both sides by $-\lambda^{-1}$, we have that $\boldsymbol{A}(-\lambda^{-1}\boldsymbol{x}) \leq \boldsymbol{b}$ and $\boldsymbol{c}(-\lambda^{-1}\boldsymbol{x}) > \delta - \varepsilon$

Strong Duality

- If $\lambda = 0$, then $\boldsymbol{Ax} \leq \boldsymbol{0}$ and $\boldsymbol{cx} > 0$
  - The dual problem is infeasible. A contradiction.
- If $\lambda < 0$, then $\boldsymbol{Ax} \leq -\lambda \boldsymbol{b}$ and $\boldsymbol{cx} > -\lambda(\delta - \varepsilon)$
  - Multiplying both sides by $-\lambda^{-1}$, we have that $\boldsymbol{A}(-\lambda^{-1}\boldsymbol{x}) \leq \boldsymbol{b}$ and $\boldsymbol{c}(-\lambda^{-1}\boldsymbol{x}) > \delta - \varepsilon$
  - Thus $\boldsymbol{y} = -\lambda^{-1}\boldsymbol{x}$ is a primal solution whose objective value is greater than $\delta - \varepsilon$

Strong Duality

- If $\lambda = 0$, then $\boldsymbol{Ax} \leq \boldsymbol{0}$ and $\boldsymbol{cx} > 0$
  - The dual problem is infeasible. A contradiction.
- If $\lambda < 0$, then $\boldsymbol{Ax} \leq -\lambda \boldsymbol{b}$ and $\boldsymbol{cx} > -\lambda(\delta - \varepsilon)$
  - Multiplying both sides by $-\lambda^{-1}$, we have that $\boldsymbol{A}(-\lambda^{-1}\boldsymbol{x}) \leq \boldsymbol{b}$ and $\boldsymbol{c}(-\lambda^{-1}\boldsymbol{x}) > \delta - \varepsilon$
  - Thus $\boldsymbol{y} = -\lambda^{-1}\boldsymbol{x}$ is a primal solution whose objective value is greater than $\delta - \varepsilon$
- From Weak Duality, $\delta - \varepsilon < \boldsymbol{cy} \leq \delta$

Strong Duality

- If $\lambda = 0$, then $\boldsymbol{Ax} \leq \boldsymbol{0}$ and $\boldsymbol{cx} > 0$
  - The dual problem is infeasible. A contradiction.
- If $\lambda < 0$, then $\boldsymbol{Ax} \leq -\lambda \boldsymbol{b}$ and $\boldsymbol{cx} > -\lambda(\delta - \varepsilon)$
  - Multiplying both sides by $-\lambda^{-1}$, we have that $\boldsymbol{A}(-\lambda^{-1}\boldsymbol{x}) \leq \boldsymbol{b}$ and $\boldsymbol{c}(-\lambda^{-1}\boldsymbol{x}) > \delta - \varepsilon$
  - Thus $\boldsymbol{y} = -\lambda^{-1}\boldsymbol{x}$ is a primal solution whose objective value is greater than $\delta - \varepsilon$
- From Weak Duality, $\delta - \varepsilon < \boldsymbol{cy} \leq \delta$
- Since ε was arbitrary, and it can be as close as possible to 0, we have that $\boldsymbol{cy} = \delta$. $\square$

Complementary Slackness

Theorem (Complementary Slackness)

$$\boldsymbol{x} \text{ and } \boldsymbol{\pi} \text{ optimal} \iff \boldsymbol{\pi}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) = 0$$

Complementary Slackness

Theorem (Complementary Slackness)

$$\boldsymbol{x} \text{ and } \boldsymbol{\pi} \text{ optimal} \iff \boldsymbol{\pi}(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}) = 0$$

- **Proof.** Note that $\boldsymbol{c}\boldsymbol{x} = \boldsymbol{\pi}\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{\pi}\boldsymbol{b}$

Complementary Slackness

Theorem (Complementary Slackness)

$$\boldsymbol{x} \text{ and } \pi \text{ optimal} \iff \pi(\boldsymbol{b} - A\boldsymbol{x}) = 0$$

- **Proof.** Note that $\boldsymbol{c}\boldsymbol{x} = \pi A\boldsymbol{x} \leq \pi \boldsymbol{b}$
- Besides, $\boldsymbol{c}\boldsymbol{x} = \pi \boldsymbol{b}$ (that is, $\boldsymbol{x}$ and π are optimal) if and only if $\pi A\boldsymbol{x} = \pi \boldsymbol{b}$. $\square$

Finding an Optimal Solution

- Suppose we want to find $\max\{\, \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \,\}$

Finding an Optimal Solution

- Suppose we want to find $\max\{ \boldsymbol{cx} \mid \boldsymbol{Ax} \leq \boldsymbol{b} \}$
  - We only need to check the vertices

Finding an Optimal Solution

- Suppose we want to find $\max\{ \boldsymbol{cx} \mid \boldsymbol{Ax} \leq \boldsymbol{b} \}$
  - We only need to check the vertices
  - In other words, choose a subsystem $\boldsymbol{A_B x} \leq \boldsymbol{b_B}$ and force equality, i.e., find $\boldsymbol{x}$ s.t. $\boldsymbol{A_B x} = \boldsymbol{b}$

Finding an Optimal Solution

- Suppose we want to find $\max\{\, \boldsymbol{cx} \mid \boldsymbol{Ax} \leq \boldsymbol{b} \,\}$
  - We only need to check the vertices
  - In other words, choose a subsystem $\boldsymbol{A_B x} \leq \boldsymbol{b_B}$ and force equality, i.e., find $\boldsymbol{x}$ s.t. $\boldsymbol{A_B x} = \boldsymbol{b}$

Primal solution

$$\boldsymbol{x} = A_{\boldsymbol{B}}^{-1} \boldsymbol{b_B}$$

Finding an Optimal Solution

- Suppose we want to find $\max\{\, \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \,\}$
  - We only need to check the vertices
  - In other words, choose a subsystem $\boldsymbol{A}_B\boldsymbol{x} \leq \boldsymbol{b}_B$ and force equality, i.e., find $\boldsymbol{x}$ s.t. $\boldsymbol{A}_B\boldsymbol{x} = \boldsymbol{b}$

Primal solution

$$\boldsymbol{x} = A_B^{-1}\boldsymbol{b}_B$$

Dual solution

$$(\pi_B, \pi_N) = (A_B^{-1}\boldsymbol{c}, \boldsymbol{0})$$

Finding an Optimal Solution

- Suppose we want to find $\max\{\,\boldsymbol{cx} \mid \boldsymbol{Ax} \le \boldsymbol{b}\,\}$
  - We only need to check the vertices
  - In other words, choose a subsystem $\boldsymbol{A_B x} \le \boldsymbol{b_B}$ and force equality, i.e., find $\boldsymbol{x}$ s.t. $\boldsymbol{A_B x} = \boldsymbol{b}$

Primal solution

$$\boldsymbol{x} = A_B^{-1}\boldsymbol{b_B}$$

Dual solution

$$(\pi_B, \pi_N) = (A_B^{-1}\boldsymbol{c}, \boldsymbol{0})$$

- From Complementary Slackness, $\boldsymbol{x}$ is optimal as long as $\boldsymbol{Ax} \le \boldsymbol{b}$ and $\boldsymbol{\pi} \ge \boldsymbol{0}$

Example

$$
\begin{aligned}
\max\, & 2x + y - 5z \\
\text{s.t. } & x + y + z \le 1 \\
& x + y - z \le 1 \\
& x - y + z \le 1 \\
& x - y - z \le 1 \\
& -x + y + z \le 1 \\
& -x + y - z \le 1 \\
& -x - y + z \le 1 \\
& -x - y - z \le 1
\end{aligned}
$$

$$
\begin{aligned}
\min\, & \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \pi_6 + \pi_7 + \pi_8 \\
\text{s.t. } & \pi_1 + \pi_2 + \pi_3 + \pi_4 - \pi_5 - \pi_6 - \pi_7 - \pi_8 = 2 \\
& \pi_1 + \pi_2 - \pi_3 - \pi_4 + \pi_5 + \pi_6 - \pi_7 - \pi_8 = 1 \\
& \pi_1 - \pi_2 + \pi_3 - \pi_4 + \pi_5 - \pi_6 + \pi_7 - \pi_8 = -5 \\
& \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8 \ge 0
\end{aligned}
$$

Example

Primal solution

$$x + y + z = 1$$
$$x + y - z = 1$$
$$x - y + z = 1$$

Dual solution

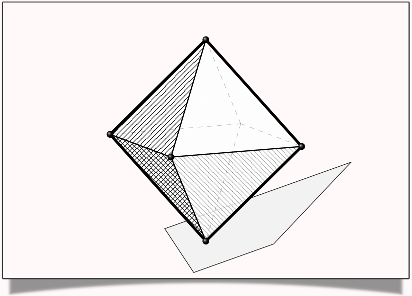
$$\pi_1 + \pi_2 + \pi_3 = 2$$
$$\pi_1 + \pi_2 - \pi_3 = 1$$
$$\pi_1 - \pi_2 + \pi_3 = -5$$
$$\pi_4, \ldots, \pi_8 = 0$$

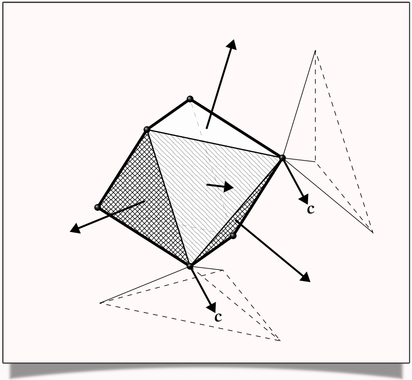
Geometrical Intuition

- $\boldsymbol{x}$ is optimal if $\boldsymbol{c}$ lies in the cone spanned by tight constraints

Geometrical Intuition

- x is optimal if c lies in the cone spanned by tight constraints





Standard Form

Theorem (Strong Duality)

$$\min_{\boldsymbol{x} \in \mathbb{R}^{n\times 1}} \{\, \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \,\} = \max_{\boldsymbol{\pi} \in \mathbb{R}^{1\times m}} \{\, \boldsymbol{\pi}\boldsymbol{b} \mid \boldsymbol{\pi}\boldsymbol{A} \leq \boldsymbol{c} \,\}$$

as long as both problems are feasible

Standard Form

Theorem (Strong Duality)

$$\min_{\boldsymbol{x} \in \mathbb{R}^{n\times 1}} \{\, \boldsymbol{c}\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \,\} = \max_{\boldsymbol{\pi} \in \mathbb{R}^{1\times m}} \{\, \boldsymbol{\pi}\boldsymbol{b} \mid \boldsymbol{\pi}\boldsymbol{A} \leq \boldsymbol{c} \,\}$$

as long as both problems are feasible

Primal solution

$$(\boldsymbol{x}_B, \boldsymbol{x}_N) = (\boldsymbol{A}_B^{-1}\boldsymbol{b}, \boldsymbol{0})$$

Standard Form

Theorem (Strong Duality)

$$\min_{x \in \mathbb{R}^{n\times 1}} \{\, cx \mid Ax = b, x \geq 0 \,\} = \max_{\pi \in \mathbb{R}^{1\times m}} \{\, \pi b \mid \pi A \leq c \,\}$$

as long as both problems are feasible

Primal solution

$$(x_B, x_N) = (A_B^{-1} b, 0)$$

Dual solution

$$\pi = A_B^{-1} c_B$$

Standard Form

Theorem (Strong Duality)

$$\min_{\boldsymbol{x}\in\mathbb{R}^{n\times 1}} \{\, \boldsymbol{cx} \mid \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \,\} = \max_{\boldsymbol{\pi}\in\mathbb{R}^{1\times m}} \{\, \boldsymbol{\pi b} \mid \boldsymbol{\pi A} \leq \boldsymbol{c} \,\}$$

as long as both problems are feasible

Primal solution

$$(\boldsymbol{x}_B, \boldsymbol{x}_N) = (\boldsymbol{A}_B^{-1}\boldsymbol{b}, \boldsymbol{0})$$

Dual solution

$$\boldsymbol{\pi} = \boldsymbol{A}_B^{-1}\boldsymbol{c}_B$$

- From Complementary Slackness, $\boldsymbol{x}$ is optimal as long as $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{\pi A} \leq \boldsymbol{c}$
  - The latter is equivalent to requiring positive reduced costs, that is, $\bar{\boldsymbol{c}} = \boldsymbol{c} - \boldsymbol{\pi A} = \boldsymbol{c} - (\boldsymbol{A}_B^{-1}\boldsymbol{c}_B)\boldsymbol{A} \geq \boldsymbol{0}$