Regression

A Learning Problem

- *A computer program* is said to *learn* from experience **T** with respect to some task **F** and some performance measure **P**, *if its performance on F, as measured by P, improves with experience T*.

Regression

- Regression is the determination of a function $y=h_\theta(\mathbf{x})$ fitting a set of points $\mathbf{T}=\{(\mathbf{x}^{(i)}, y^{(i)}), i=1,\ldots m, \mathbf{x}^{(i)} \in R^n, y^{(i)} \in R\}$.

- The function depends on a number of *unknown* parameters $\theta=[\theta_1, \theta_2\ldots., \theta_n]^T$

Regression framework

Learning phase

- Given a training set $\mathbf{T}=\{(\mathbf{x}^{(i)}, y^{(i)}), i=1,\ldots m, y^{(i)} \in R\}$
- Given some hypotheses regarding $h_\theta(\mathbf{x})$
- Estimate parameters $\theta=[\theta_1, \theta_2\ldots., \theta_n]^T$ of $h_\theta(\mathbf{x})$ by using $\mathbf{T}$

Estimation phase

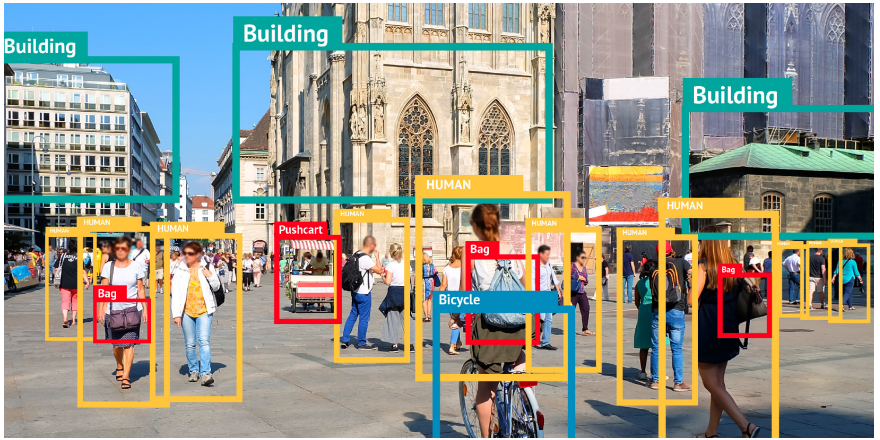
- Use the estimated parameters $\theta=[\theta_1, \theta_2\ldots., \theta_n]^T$ to determine $h_\theta(\mathbf{x})$ at new $\mathbf{x}$

Examples

- Image Based Localization
- Object Detection
- Pose Estimation

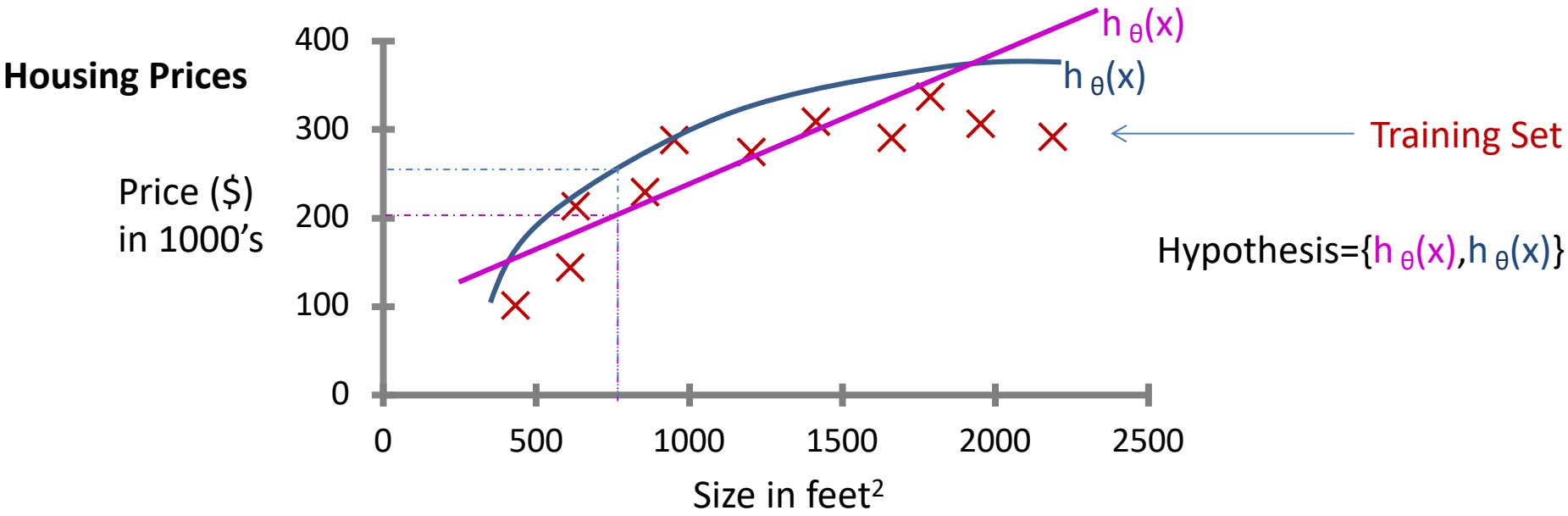
:

:



Linear Regression with one Variable

Supervised Learning Problem: Regression



Supervised Learning

"right answers" are given for each example in the data

Regression: Predict continuous valued output (price)

Training set of housing prices (Portland, OR)

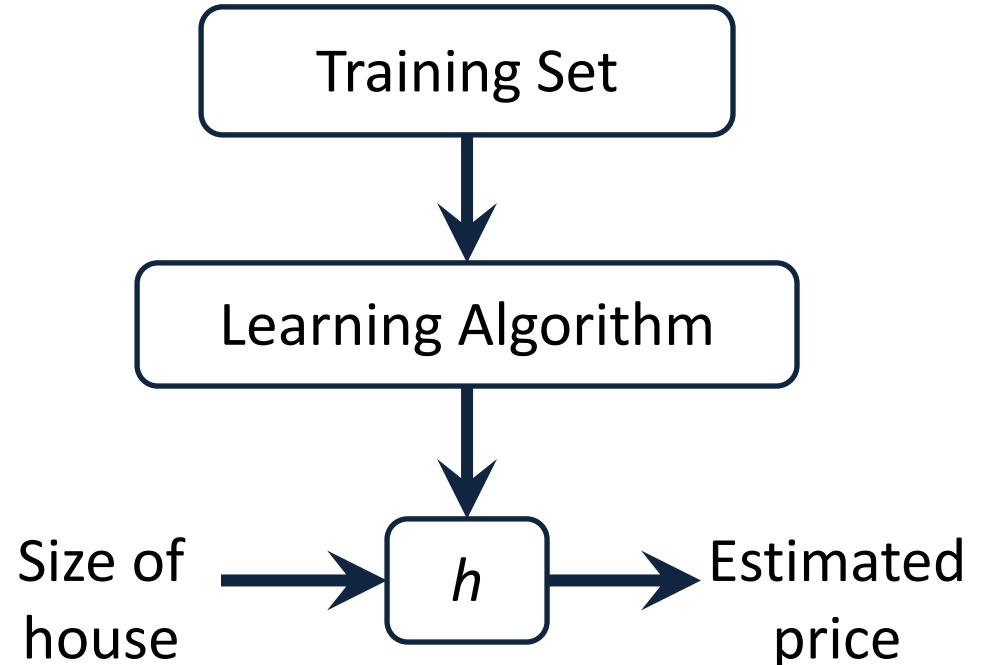
Size in feet² (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178
…	…

m

Notation:

- m = Number of training examples
- x's = "input" variable / features
- y's = "output" variable / "target" variable
- (x,y) = one training example
- $(x^{(i)},y^{(i)})$ = i^{th} training example

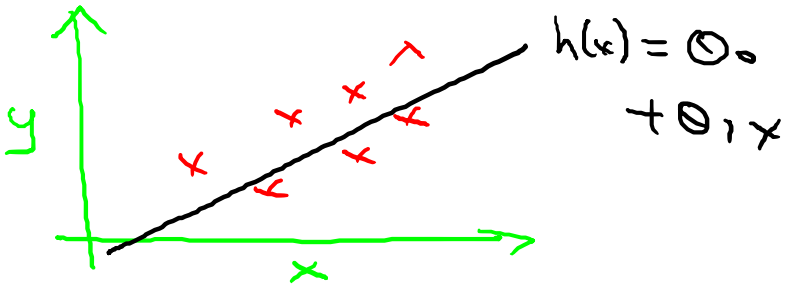
What is $y^{(3)}$?



How do we represent *h* in Regression?

$h_\theta(x) = \theta_0 + \theta_1 x$

Shorthand: $h(x)$



-Linear regression with one variable. (x)

-Univariate linear regression.

↳ one variable

Training Set

Size in feet² (x)	Price ($) in 1000's (y)
2104	460
1416	232
1534	315
852	178
…	…

Hypothesis - Model: $h_\theta(x) = \theta_0 + \theta_1 x$

θ_i's: Parameters

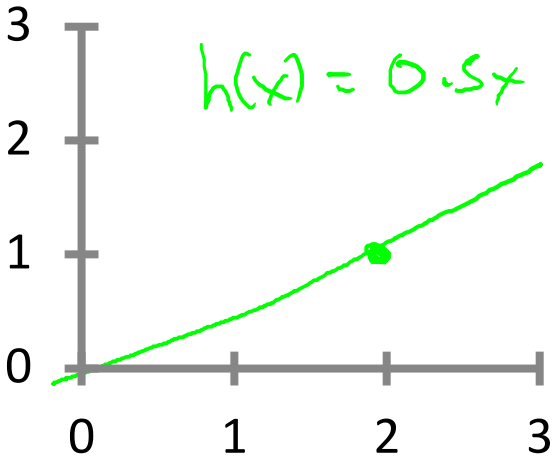
How to choose θ_i's ?

By choosing different parameters we have different hypotesis

$$h_\theta(x) = \theta_0 + \theta_1 x$$

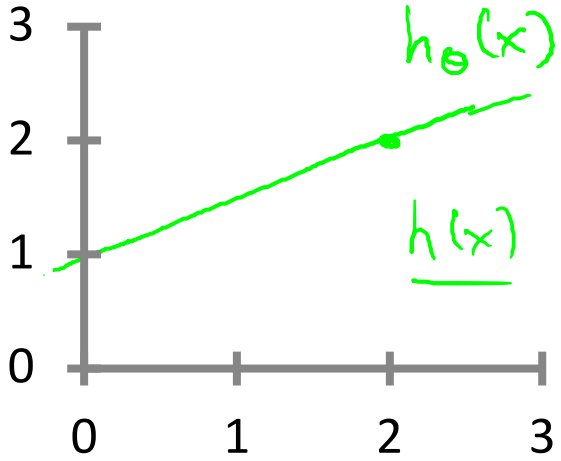
→ $\theta_0 = 1.5$

→ $\theta_1 = 0$



→ $\theta_0 = 0$

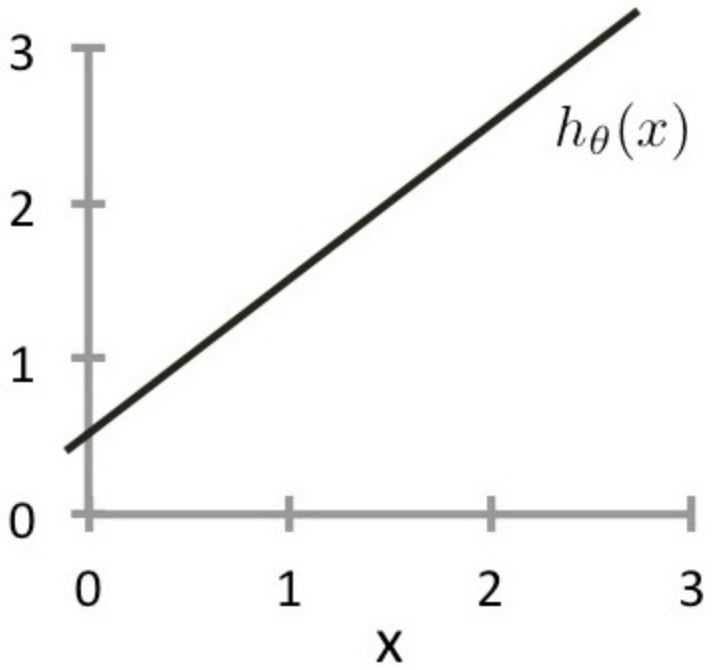
→ $\theta_1 = 0.5$



→ $\theta_0 = 1$

→ $\theta_1 = 0.5$

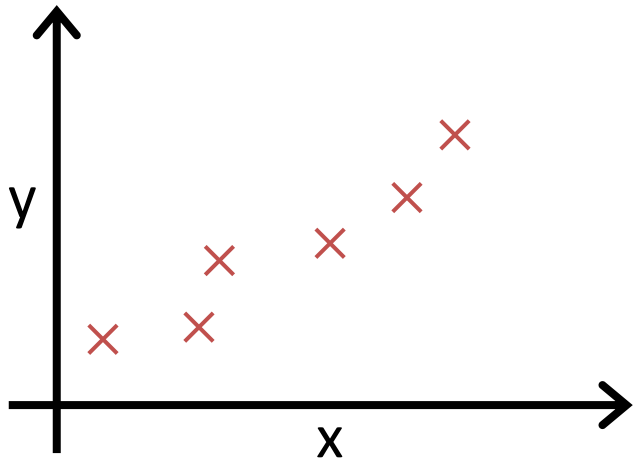
Consider the plot below of $h_\theta(x) = \theta_0 + \theta_1 x$. What are θ_0 and θ_1?



- $\theta_0 = 0, \theta_1 = 1$
- $\theta_0 = 0.5, \theta_1 = 1$
- $\theta_0 = 1, \theta_1 = 0.5$
- $\theta_0 = 1, \theta_1 = 1$

The Cost Function

The Cost Function J



$(x^{(i)}, y^{(i)})$

#training examples

$$\text{minimize}_{\theta_0,\theta_1} \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

$$h_\theta(x^{(i)}) = \theta_0 + \theta_1 x^{(i)}$$

$$J(\theta_0,\theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

Idea: Choose θ_0, θ_1 so that $h_\theta(x)$ is close to y for our training examples (x, y)

x, y

$$\text{Minimize}_{\theta_0,\theta_1} J(\theta_0,\theta_1)$$

Cost function

Squared error function

What cost function is doing and why we want to use it?

Linear Regression Problem (one variable)

Hypothesis - Model:

$$h_\theta(x) = \theta_0 + \theta_1 x$$

Parameters:

$$\theta_0, \theta_1$$

Cost Function:

$$J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)} \right)^2$$

Goal:

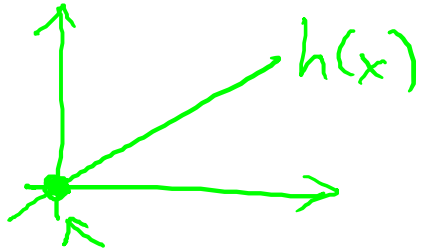
$$\underset{\theta_0, \theta_1}{\text{minimize}}\ J(\theta_0, \theta_1)$$

Simplified

$$h_\theta(x) = \theta_1 x$$

$\theta_0 = 0$

θ_1

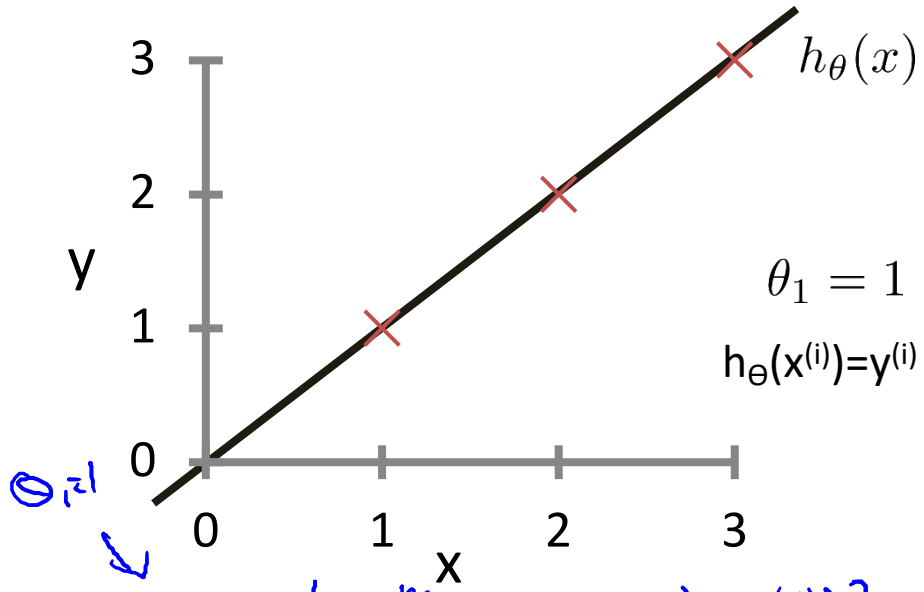


$$J(\theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)} \right)^2$$

$$\underset{\theta_1}{\text{minimize}}\ J(\theta_1)$$

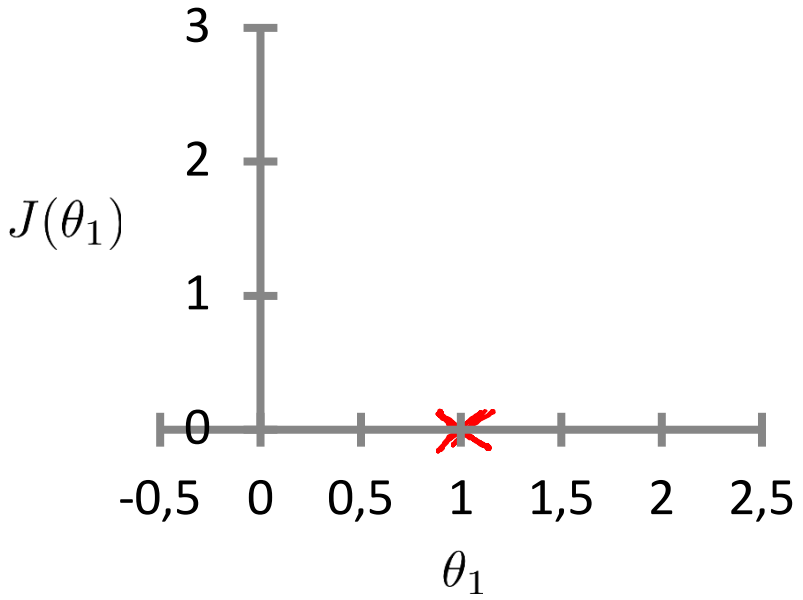
$$h_\theta(x) = \theta_1 x$$

(for fixed θ_1, this is a function of x)



$$J(\theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

(function of the parameter θ_1)



$\theta_1 = 1$

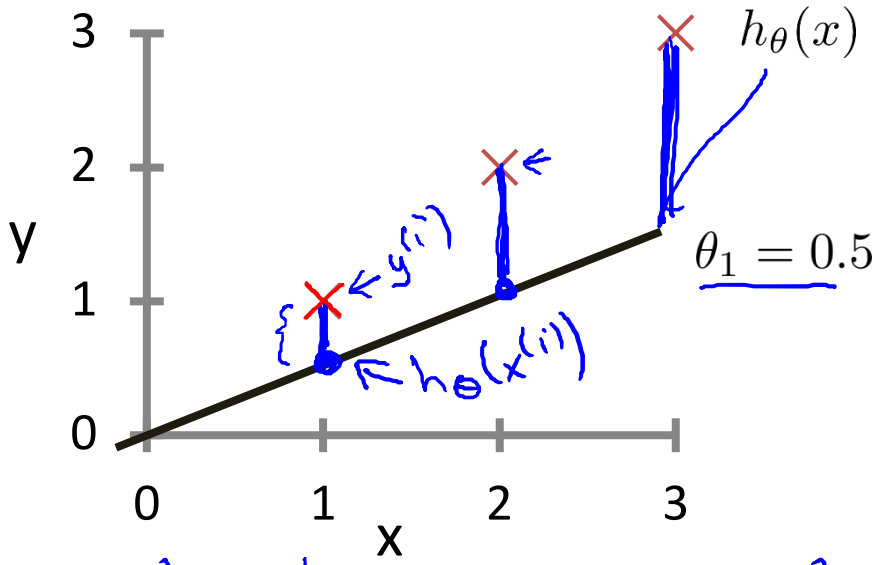
$J(\theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$

$= \frac{1}{2m}\sum_{i=1}^{m}\left(\theta_1 x^{(i)} - y^{(i)}\right)^2 = \frac{1}{2m}\left(0^2 + 0^2 + 0^2\right) = 0$

$J(1) = 0$

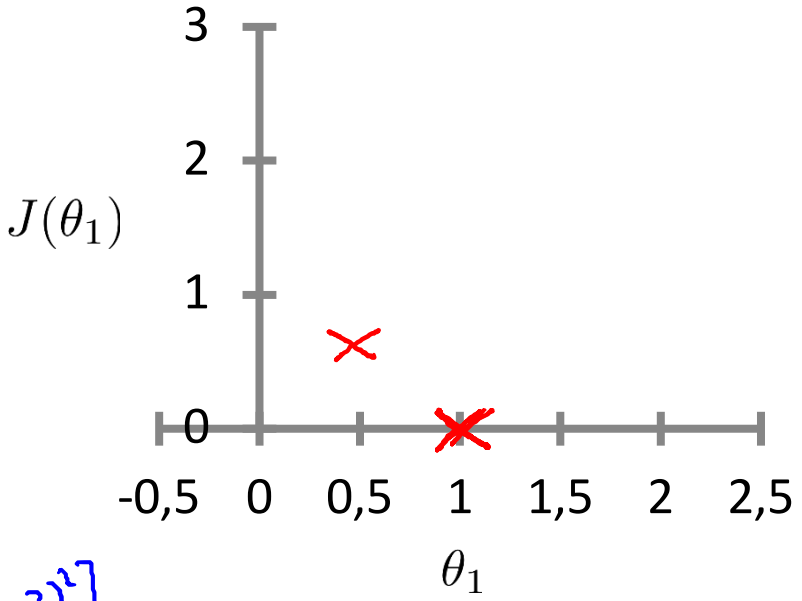
$$h_\theta(x) = \theta_1 x$$

(for fixed θ_1, this is a function of x)



$$J(\theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$$

(function of the parameter θ_1)

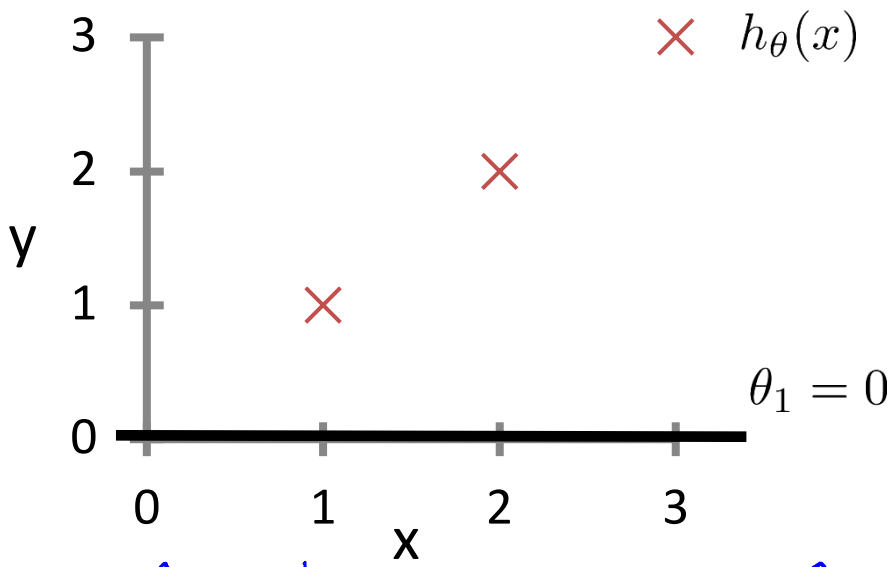


$J(0.5) = \frac{1}{2m}\left[(0.5-1)^2 + (1-2)^2 + (1.5-3)^2\right]$

$= \frac{1}{2\times 3}(3.5) = \frac{3.5}{6}$

$h_\theta(x) = \theta_1 x$

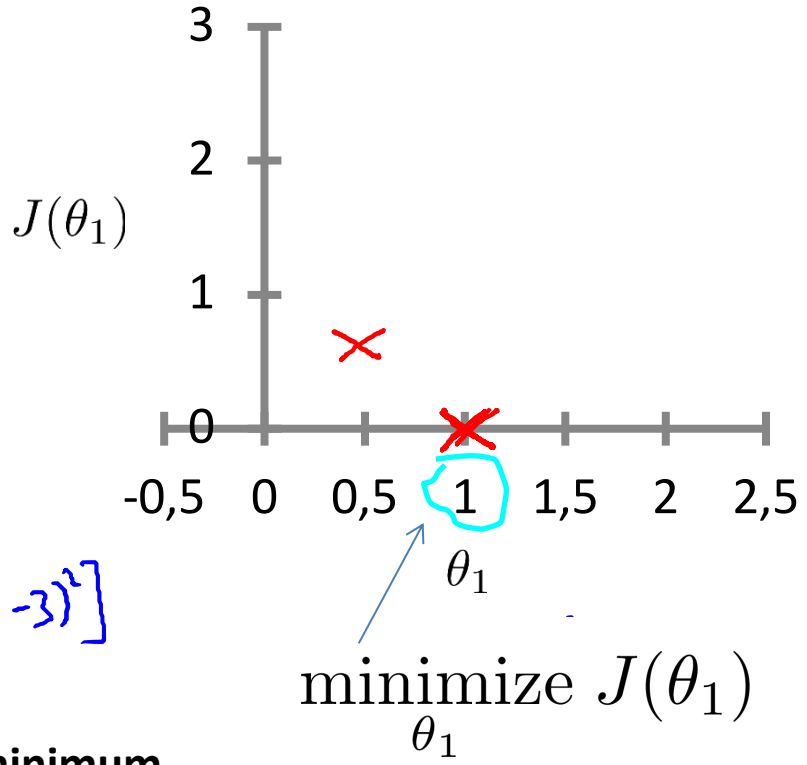
(for fixed θ_1, this is a function of x)



$J(0) = \frac{1}{2m}[(\quad -1)^2 + (-2)^2 + (\quad -3)^2]$

$J(\theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta(x^{(i)}) - y^{(i)}\right)^2$

(function of the parameter θ_1)



We need a procedure to automatically find the minimum

Original Problem

Hypothesis: $h_\theta(x) = \theta_0 + \theta_1 x$

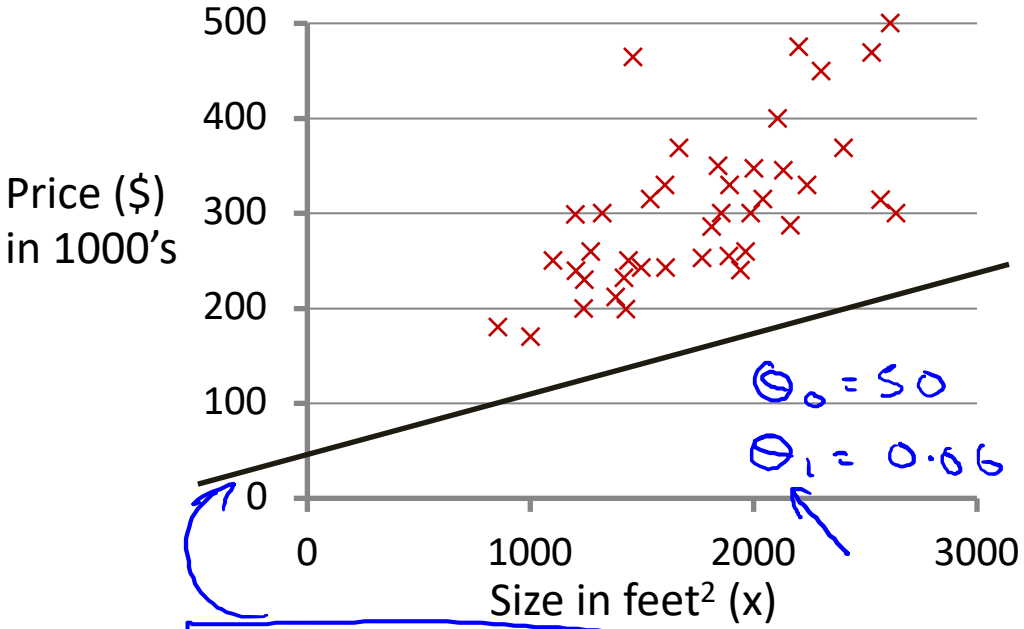
Parameters: θ_0, θ_1

Cost Function: $J(\theta_0, \theta_1) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_\theta(x^{(i)}) - y^{(i)} \right)^2$

Goal: $\underset{\theta_0, \theta_1}{\text{minimize}}\ J(\theta_0, \theta_1)$

$h_\theta(x)$

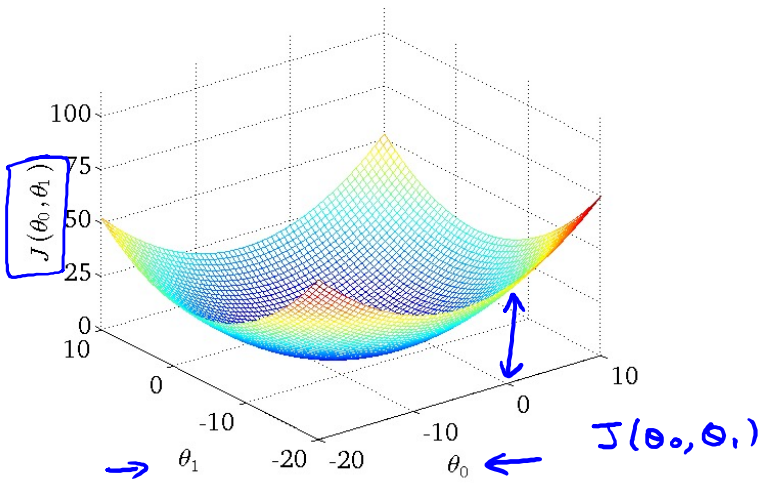
(for fixed θ_0, θ_1, this is a function of x)



$$h_\theta(x) = 50 + 0.06x$$

$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)

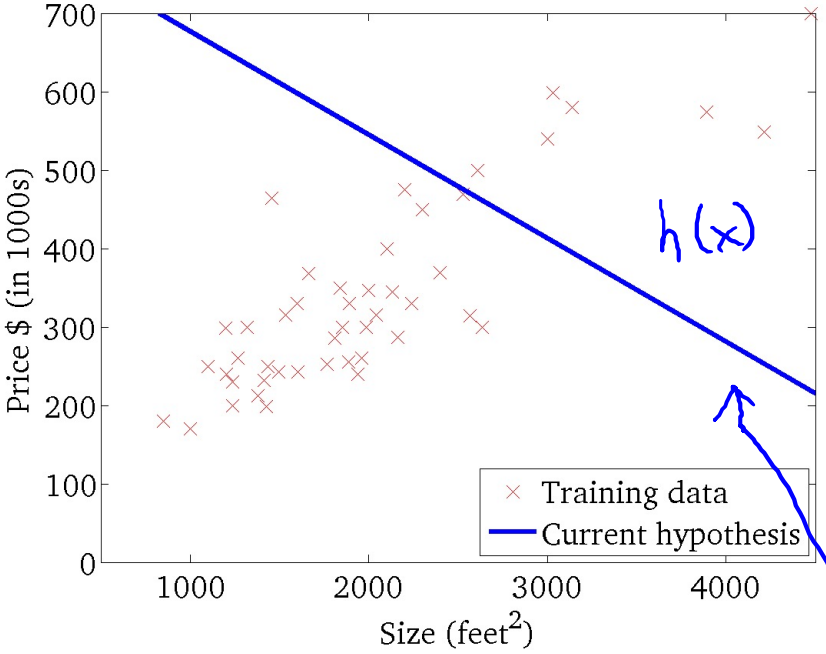


Note that we have just one global minimum.

Contour Plots….

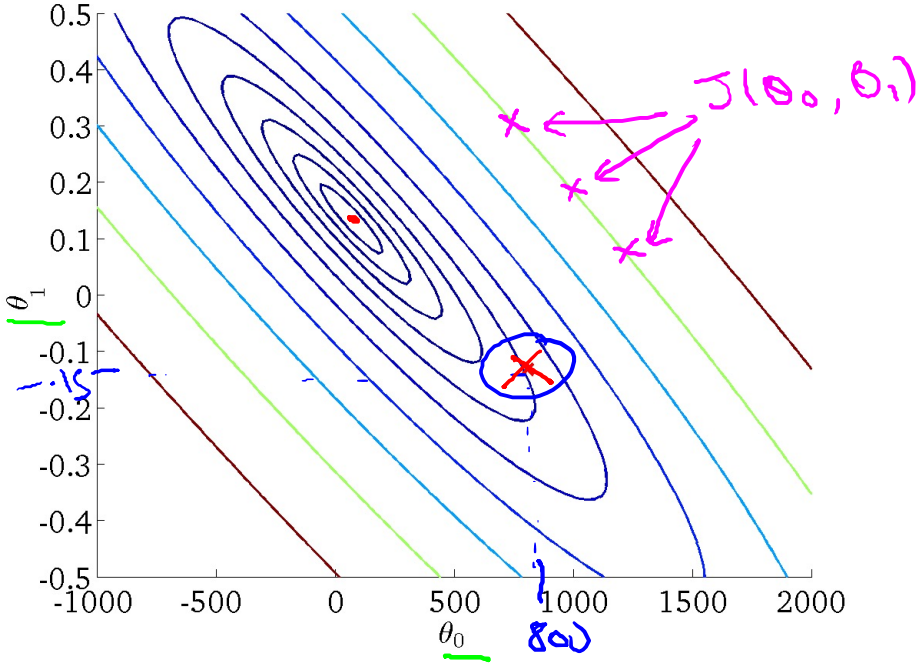
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



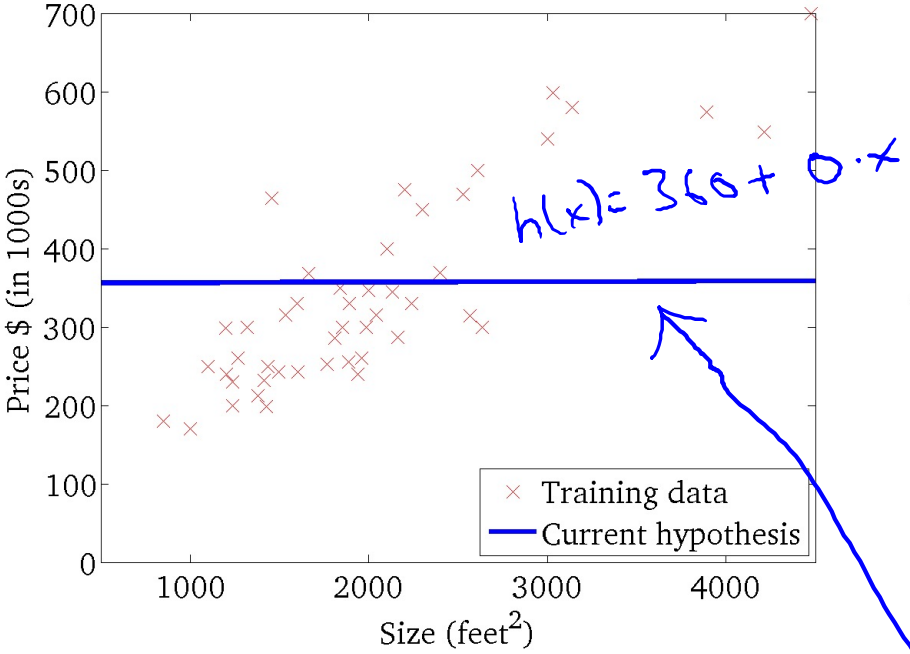
$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)



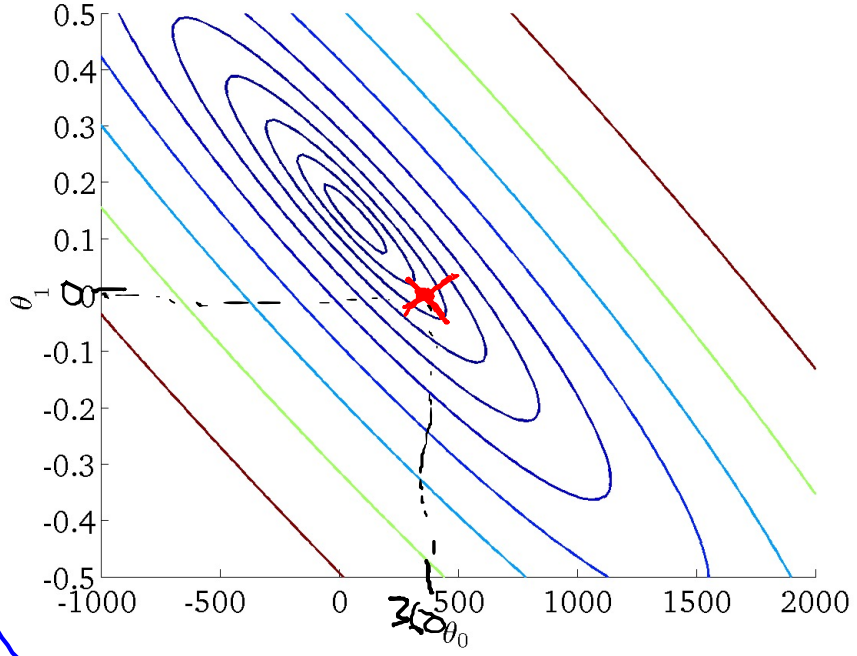
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)

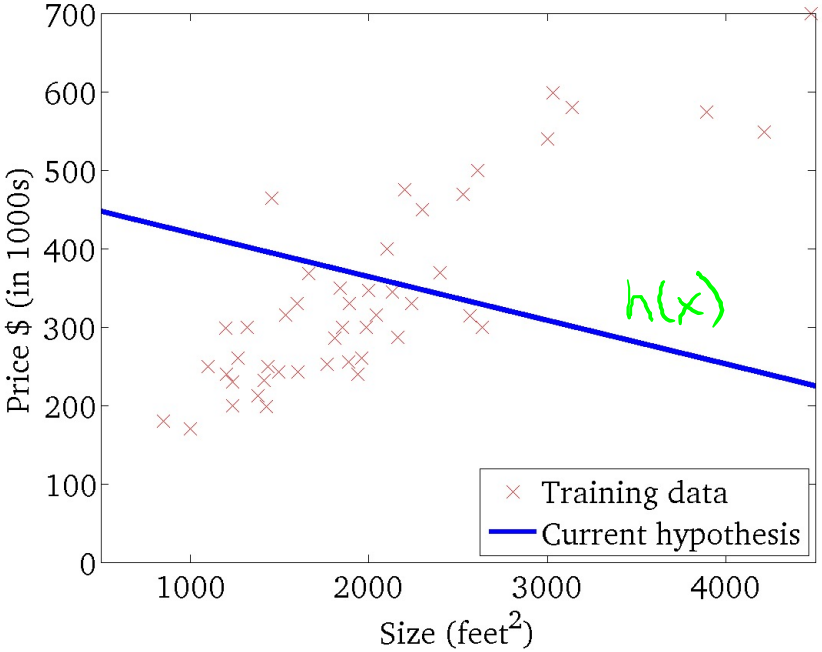


$\theta_0 = 360$

$\theta_1 = 0$

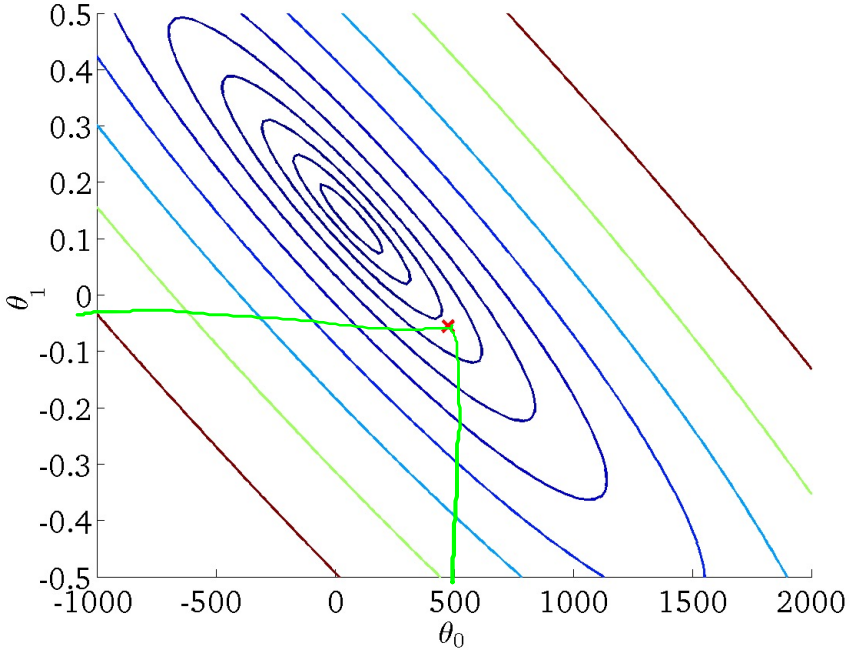
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



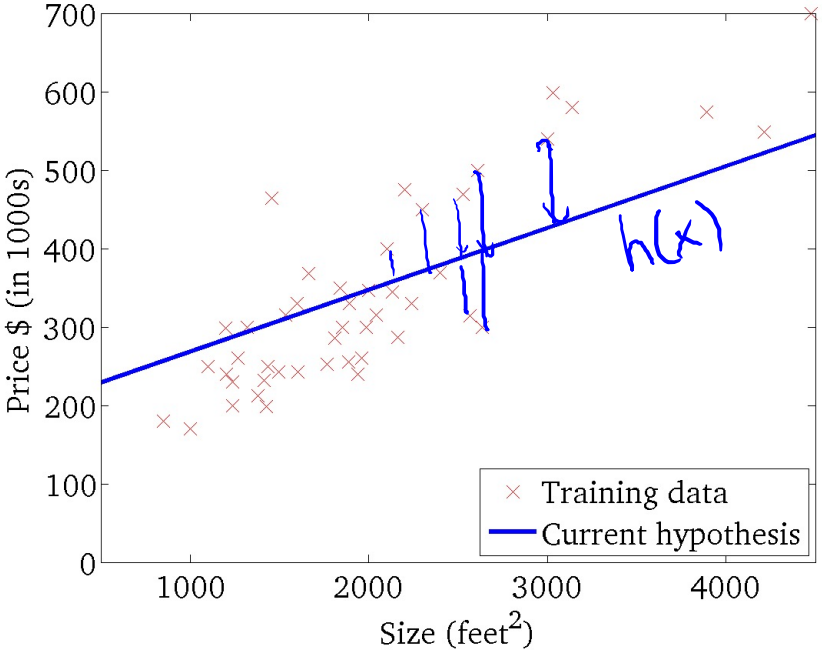
$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)



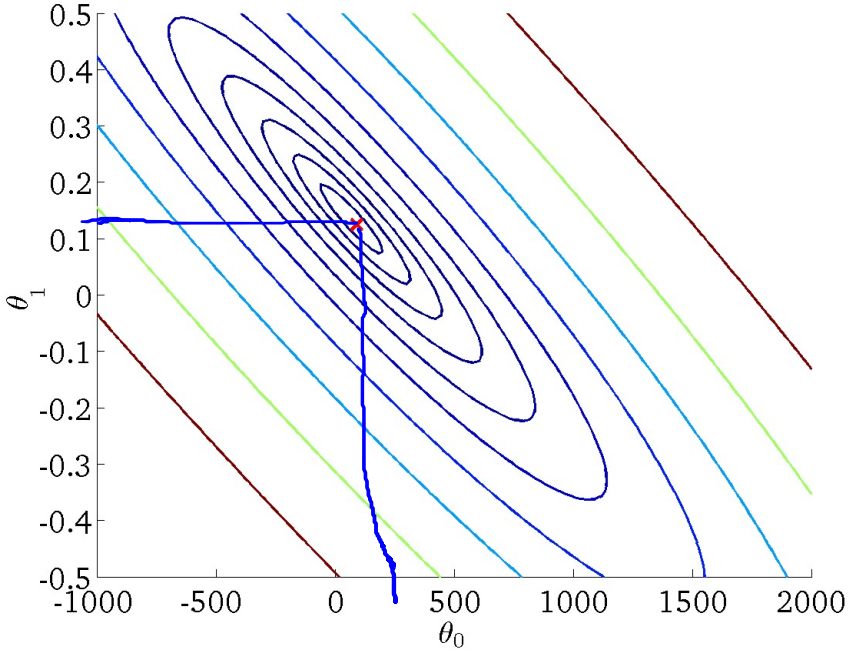
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



$J(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)



We want an algorithms that find automatically the best values for the parameters.

Do you know any naive algorithm for this?

Gradient Descent

(can be used not only for linear regression – useful to minimize functions)

Problem Setup

Gradient Descend it is useful also to more general function

Have some arbitrary function $J(\theta_0, \theta_1)$

Want an algorithm for

$$\min_{\theta_0, \theta_1} J(\theta_0, \theta_1)$$

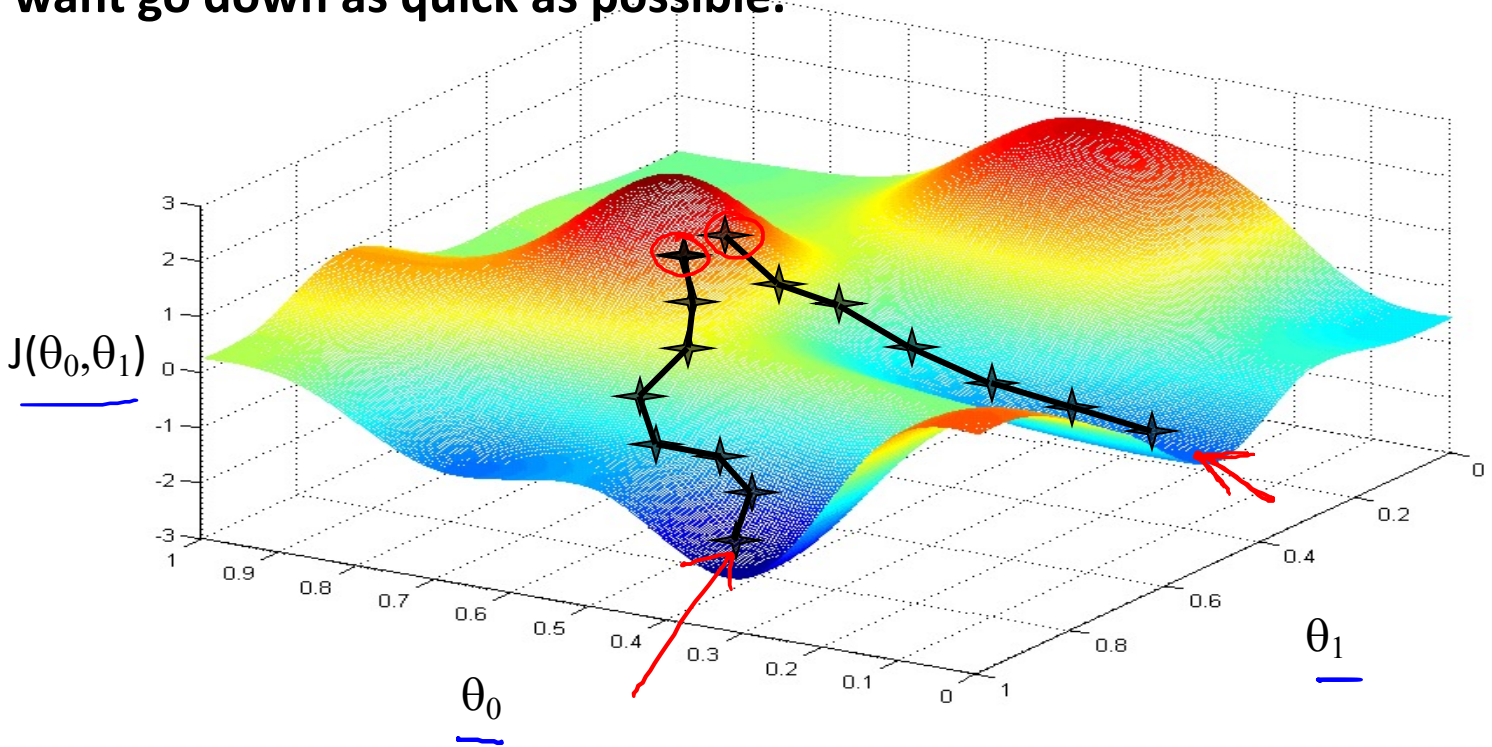
Outline of the Gradient Descent algorithm:

- Start with some θ_0, θ_1
- Keep changing θ_0, θ_1 to reduce $J(\theta_0, \theta_1)$

  until we hopefully end up at a minimum

Suppose you stand in that point, look around 360°
and choose a direction to perform a step down.
You want go down as quick as possible.

General $J(\theta_1,\theta_2)$ function



Starting from a different point a different minimum is reached.

Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1) \quad (\text{for } j = 0 \text{ and } j = 1)$$

}

learning rate > 0

derivative

:= denote assignment operator

Learning rate controls how big is the step

Derivative term give the direction of the step
(we will derive later – do not be scared! ;-)

Gradient descent algorithm

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1) \qquad (\text{for } j = 0 \text{ and } j = 1)$$

}

Correct: Simultaneous update

$$\text{temp0} := \theta_0 - \alpha \frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$$
$$\text{temp1} := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$$
$$\theta_0 := \text{temp0}$$
$$\theta_1 := \text{temp1}$$

Incorrect:

$$\text{temp0} := \theta_0 - \alpha \frac{\partial}{\partial \theta_0} J(\theta_0, \theta_1)$$
$$\theta_0 := \text{temp0}$$
$$\text{temp1} := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_0, \theta_1)$$
$$\theta_1 := \text{temp1}$$

We have to understand what the gradient descend algorithm is doing and the role of the learning rate and of the derivative term

repeat until convergence {

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta_0, \theta_1)$$

(simultaneously update $j = 0$ and $j = 1$)

}

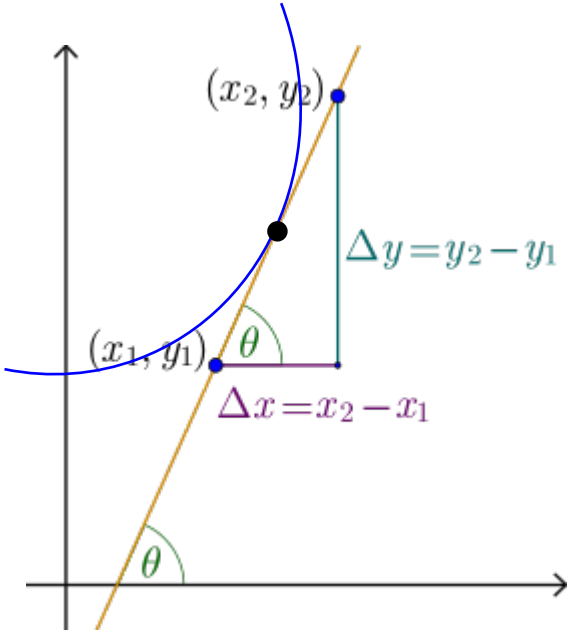
learning rate > 0

derivative

To figure out this let consider the simplified case:

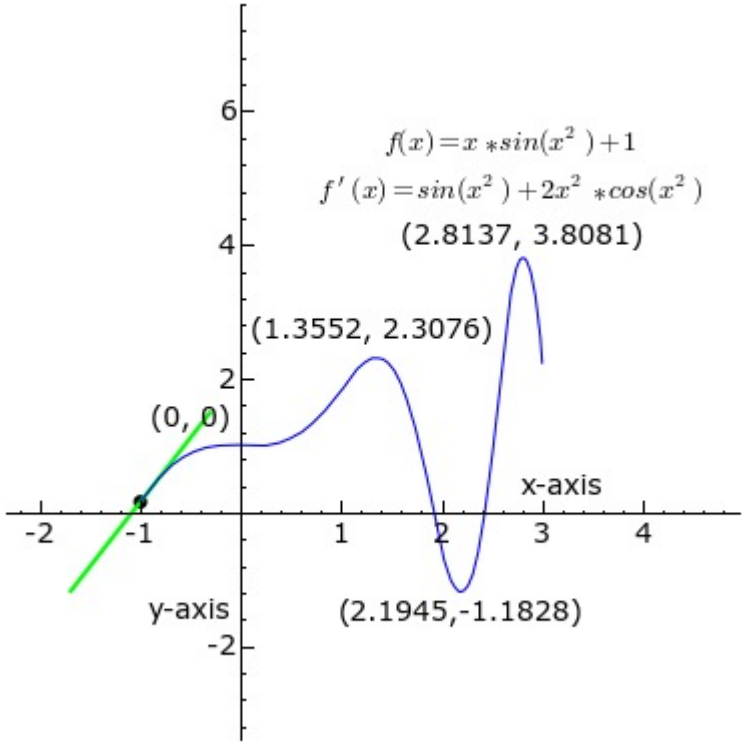
$$\min_{\theta_1} J(\theta_1) \qquad \theta_1 \in \mathbb{R}$$

Gradient Descent Intuition (simplified case $J(\theta_1)$) – What is the derivative



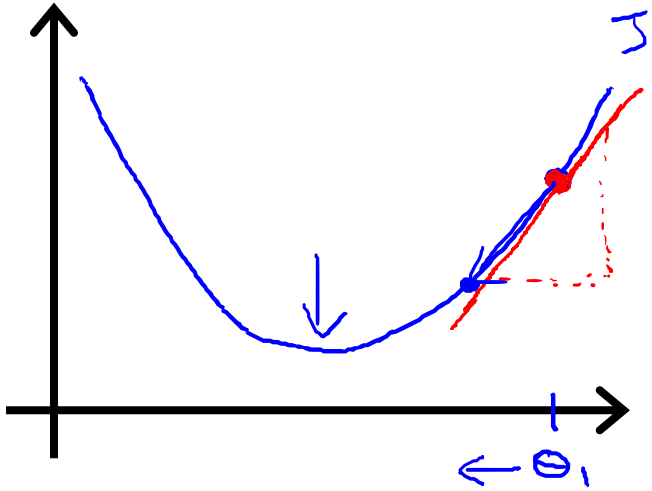
Coefficiente angolare

$$m = \frac{\Delta y}{\Delta x} = \tan(\theta)$$



At each point, the line is always tangent to the curve. Its **slope is the derivative**; the positivity, negativity and zeroes of the derivative are marked by green, red and black respectively.

Gradient Descent Intuition (simplified case)



$(\theta_1 \in \mathbb{R})$

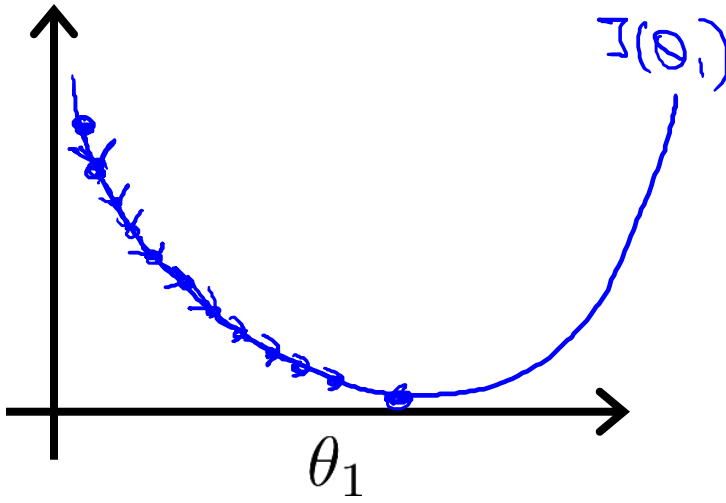
$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

$\frac{d}{d\theta_1} J(\theta_1) \geq 0$

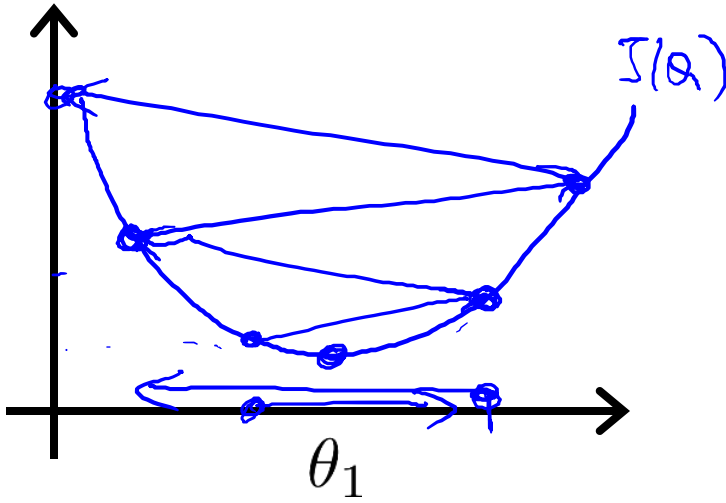
$$\theta_1 := \theta_1 - \alpha \cdot (\text{positive number})$$

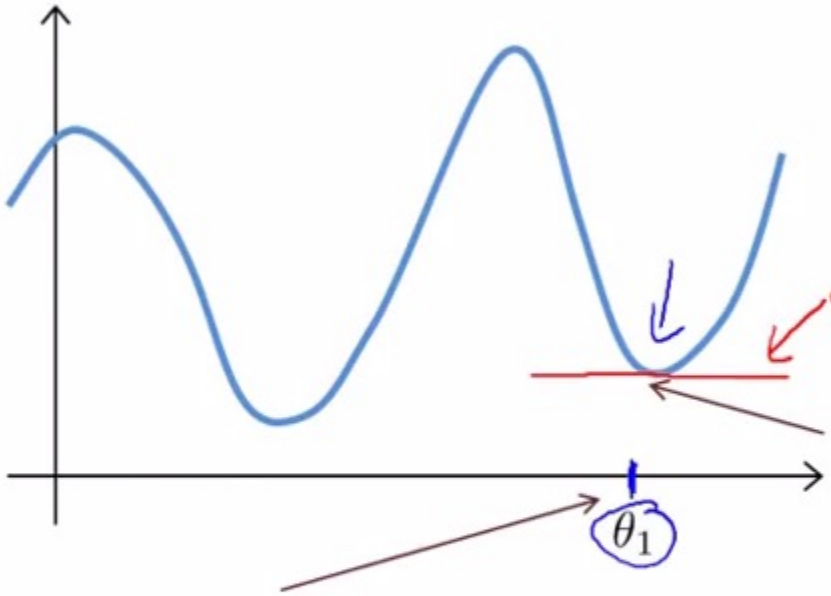
$$\theta_1 := \theta_1 - \alpha \frac{\partial}{\partial \theta_1} J(\theta_1)$$

If α is too small, gradient descent can be slow.



If α is too large, gradient descent can overshoot the minimum. It may fail to converge, or even diverge.





$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

$$\theta_1 := \theta_1 - \alpha \cdot 0$$

$$= 0$$

Gradient descent can converge to a local minimum, even with the learning rate α fixed.

$$\theta_1 := \theta_1 - \alpha \frac{d}{d\theta_1} J(\theta_1)$$

As we approach a local minimum, gradient descent will automatically take smaller steps. So, no need to decrease α over time.