21-259 Calculus in 3D

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Midterm 2 Edition

Chapter 0

Disclaimer

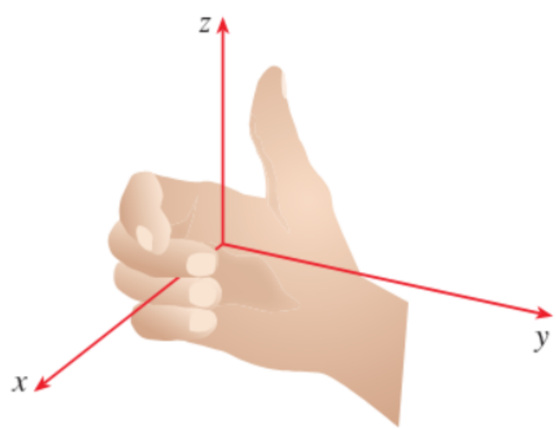
This is not a textbook. Do not use this text to learn the concepts. This is a revision guide or notes used to refresh your memory of the theorems. The best way to learn the content is to read the book, work through the guided exercises and do some practice problems.
Good luck,
SH

Chapter 12

Vectors and the Geometry of Space

12.1 Three-Dimensional Coordinate System

Right-hand rule for determining the coordinate axis:



Projection of a point (x, y, z) onto

- $x - y$ plane: (x, y, 0)
- $x - z$ plane: (x, 0, z)
- $y - z$ plane: (0, y, z)

Distance formula in 3 dimensions: Between two points (x_1, y_1, z_1) and (x_2, y_2, z_2), the distance is given by:

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere with center at (h, k, l) and radius r is:

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

12.2 Vectors

I will skip the basics, like vector addition definition, etc.
Components of a vector (for notation):

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$,

$$\overrightarrow{AB} = \mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Magnitude of a vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Unit vectors:

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle$$

Algebraic Properties of Vectors in $\mathbb{R}^n$

$\forall$ $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalars c and d:

1. **Commutativity of addition:** $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. **Associativity of addition:** $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
3. **Additive identity:** $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. **Additive inverse:** $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. **Distributivity of scalar multiplication over addition (vectors):** $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
6. **Distributivity of scalar multiplication over addition (scalars):** $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
7. **Associativity of scalar multiplication:** $(cd)\mathbf{u} = c(d\mathbf{u})$
8. **Scalar identity:** $1\mathbf{u} = \mathbf{u}$

12.3 The Dot Product

Definition 1. If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of the Dot Product

$\forall$ vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and scalar c:

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Theorem 1. If θ is the angle between two vectors $\mathbf{a}, \mathbf{b}$ then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Corollary 1. Two vectors $\mathbf{a}, \mathbf{b}$ are orthogonal iff $\mathbf{a} \cdot \mathbf{b} = 0$

The **direction angles** of a non-zero vector $\mathbf{a}$ are the angles α, β, γ that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively.
The cosines of these angles are called **direction cosines**:

$$\cos\alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|}, \quad \cos\beta = \frac{a_2}{|\mathbf{a}|}, \quad \cos\gamma = \frac{a_3}{|\mathbf{a}|}$$

Some nice properties of direction cosines:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\frac{\mathbf{a}}{|\mathbf{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Projections:
The **projection of b onto a** means we drop a perpendicular from the tip of **b** onto the line spanned by **a**. The resulting vector is called the vector projection of **b** onto **a**.

The **scalar projection of b onto a** is the length of this projection, given by

$$\text{comp}_{\mathbf{a}}\mathbf{b} = |\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|},$$

where θ is the angle between **a** and **b**.

The **vector projection** of **b** onto **a** is obtained by multiplying the unit vector in the direction of **a** by the scalar projection:

$$\text{proj}_{\mathbf{a}}\mathbf{b} = (\text{comp}_{\mathbf{a}}\mathbf{b})\,\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

12.4 The Cross Product

Definition 2. For $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, the **cross product** of **a** an **b** is:

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Theorem 2. The vector $\mathbf{a}\times\mathbf{b}$ is orthogonal to both **a** and **b**

Theorem 3. The length of the cross product is given by:

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

where $\theta$is the angle between **a** and **b**.

Corollary 2. Two nonzeros vectors **a** and **b** are parallel iff

$$\mathbf{a}\times\mathbf{b} = \mathbf{0}$$

Area of parallelogram determined by **a** and **b** is $|\mathbf{a}\times\mathbf{b}|$

Area of triangle determined by **a** and **b** is $\frac{1}{2}|\mathbf{a}\times\mathbf{b}|$

IMPORTANT: cross product is NOT commutative, and associative law for multiplication does not usually hold.

Properties of the Cross Product

$\forall$ vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and scalar c:

1. $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
2. $(c\mathbf{a})\times\mathbf{b} = c(\mathbf{a}\times\mathbf{b}) = \mathbf{a}\times(c\mathbf{b})$
3. $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c}$
4. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c}$
5. $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$
6. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

Triple products:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is:

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

Useful: if the volume is 0, then the vectors must lie in the same plane, i.e. **coplanar**

12.5 Equations of Lines and Planes

12.5.1 Equation of Lines

The equation of a line that goes through point with position vector $\mathbf{r}_0$, and is parallel to vector $\mathbf{v}$ is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

If $\mathbf{v} = \langle a, b, c\rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ then

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle$$

.

Parametric form:

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc$$

Direction numbers of a line L are the numbers a, b, c from above.

Symmetric form: (eliminating t)

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

IMPORTANT: If a, b or c is zero, then we cannot divide by them. For example, if $a = 0$ then the equation becomes:

$$x = x_0, \quad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

In general, the symmetric equations between two points $P_0(x_0, y_0, z_0), P_1(x_1, y_1, z_1)$ are

$$\frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0} = \frac{z - z_0}{z_1 - z_0}$$

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation:

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

12.5.2 Equation of Planes

Let:

- $\mathbf{n} = \langle a, b, c\rangle$: vector orthogonal to plane
- $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$: a position vector for a point in the plane
- $\mathbf{r} = \langle x, y, z\rangle$: a position vector for an arbitrary point in the plane.

Then the **vector equation of the plane** is:

$$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r}_0) = 0 \quad \text{or} \quad \mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r}_0$$

The **scalar equation** is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

The **linear equation** is:

$$ax + by + cz - (ax_0 + by_0 + cz_0) = 0$$

Two planes are parallel if their normal vectors are parallel.
If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors.
The **distance** D from point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Chapter 13

Vector Function

Not coming in midterm 1 :)

Chapter 14

Partial Derivatives

14.1 Functions of Several Variables

Not coming in midterm 1 :)

14.2 Limits and Continuity

Not coming in midterm 1 :)

14.3 Partial Derivatives

If f is a function of two variables, its **partial derivatives** are the functions f_x, f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

and

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Also written:

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x}$$

Rules for finding the partial derivatives of $z = f(x,y)$

- To find $\frac{\partial z}{\partial x}$ regard y as a constant and differentiate $f(x,y)$ with respect to x
- To find $\frac{\partial z}{\partial y}$ regard x as a constant and differentiate $f(x,y)$ with respect to y

Interpretation of partial derivatives:
The partial derivatives represent the instantaneous rate of change of the function's output with respect to one variable, while all other variables are held constant.

- $f_x(x,y)$ is the rate at which f changes with respect to x when y is fixed.
- $f_y(x,y)$ is the rate at which f changes with respect to y when x is fixed.

Higher Derivatives: we can take second partial derivatives of f. Here is the notation:

$$(f_x)_x = f_{xx} = \frac{\partial^2 f}{\partial x^2}$$
$$(f_x)_y = f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$
$$(f_y)_x = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$
$$(f_y)_y = f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

Theorem 4. Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If f_{xy} and f_{yx} are both continuous on D, then:

$$f_{xy}(a, b) = f_{yx}(a, b)$$

14.4 Tangent Planes and Linear Approximations

The equation of a **tangent plane** to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = \frac{\partial z}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial z}{\partial y}(x_0, y_0)(y - y_0)$$

if f has continuous partial derivatives.

The Linear Approximation of f at (a, b) is:

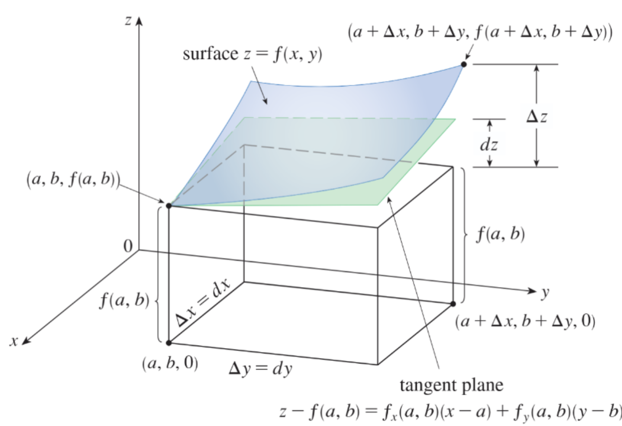
$$f(x, y) \approx f(a, b) + \frac{\partial z}{\partial x}(a, b)(x - a) + \frac{\partial z}{\partial y}(a, b)(y - b)$$

Remark: The section in the book about differentiability with epsilon has been omitted for brevity.

Theorem 5. If partial derivatives of f exist near (a, b) and are continuous at (a, b) then f is differentiable at (a, b)

Total differential dz is defined as

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$



14.5 Chain Rule

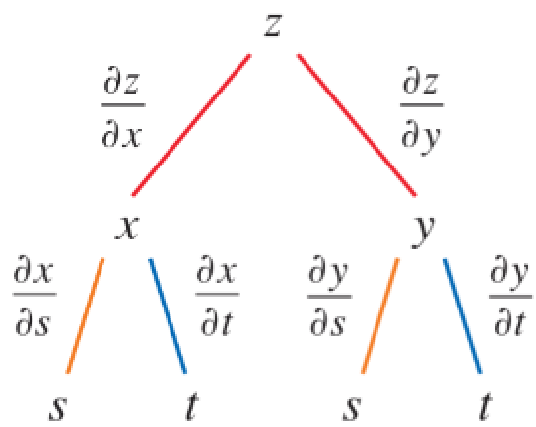
Basic chain rule reminder: $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$

Chain Rule Case 1: If $z = f(x, y)$ is a differentiable function of x and y which are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Chain Rule Case 2: If $z = f(x, y)$ is a differentiable function of x and y which are both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$



Chain Rule General Case: If

- u is a differentiable function of n variables $x_1, \ldots, x_n$
- Each x_i is a differentiable function of m variables $t_1, \ldots, t_m$

Then u is a function of $t_1, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for all $i = 1, \ldots, m$

Implicit Differentiation

Formula for Implicit Differentiation (2D)

Given an equation $F(x, y) = 0$, the derivative of y with respect to x is:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

provided that $F_y \neq 0$.

Formulas for Implicit Differentiation (3D)

Given an equation $F(x, y, z) = 0$, the partial derivatives of z are:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

provided that $F_z \neq 0$.

14.6 Directional Derivatives and the Gradient Vector

Definition 3. If f is a differentiable function of x, y, the **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Definition 4. If f is a function of x, y, then the **gradient** of f is the vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle$$

With these definitions, we can write the directional derivative cleanly as:

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Equivalently:

$$D_{\mathbf{u}} f(x, y) = |\nabla f(x, y)| \cos \theta$$

where θ os the angle between $\nabla f(x, y)$ and $\mathbf{u}$
This generalized to three variable case:

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Theorem 6. If f is differentiable function of 2 or 3 variables, the max value of the directional derivative is $|\nabla f(\mathbf{x})|$ and it happens when $\mathbf{u}$ has the same direction as the vector $\nabla f(\mathbf{x})$

Theorem 7. If f is differentiable function of 2 or 3 variables, the min value of the directional derivative is $-|\nabla f(\mathbf{x})|$ and it happens when $\mathbf{u}$ has the same direction as the vector $-\nabla f(\mathbf{x})$

Equation of tangent plane to level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Equation of normal line to level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ is:

$$\frac{x - x_0}{F_x} = \frac{y - y_0}{F_y} = \frac{z - z_0}{F_z}$$

$\nabla f(\mathbf{x})$ is perpedicular to the level curve or level surface of f through $\mathbf{x}$

14.7 Maximum and Minimum Values

Definition 5. A point (a, b) is called a **critical point** of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 8. If f has a local max or min at (a, b) and first order partial derivatives exist there then $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Second Derivative Test:
Let $D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$

- If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then (a, b) is a local min
- If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then (a, b) is a local max
- If $D(a, b) < 0$, then (a, b) is a saddle point
- If $D(a, b) = 0$ then the test gives no information

If you are crazy then you can use the following to remember the formula for D:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

Definition 6. $f(a, b)$ is **absolute maximum** of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D
$f(a, b)$ is **absolute minimum** of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D

Theorem 9. Extreme Value Theorem: A continuous function f will attain both an absolute minimum and an absolute maximum somewhere on a bounded, closed set in $\mathbb{R}^2$

To find where this happens:

1. Find the values of f at the critical points of f in D
2. Find the extreme values of f on the boundary of D
3. The largest from these two is absolute max, the smallest is absolute min

14.8 Lagrange Multipliers

The Method of Lagrange Multipliers
To find the minimum or maximum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$:

- Find all values of x, y, z, λ such that
$$\nabla f(x, y, z) = \lambda \, \nabla g(x, y, z)$$
and
$$g(x, y, z) = k.$$
- Evaluate f at all points obtained above. The largest value gives the maximum; the smallest gives the minimum.

The equations in the first step can be written componentwise as:

$$f_x = \lambda g_x, \qquad f_y = \lambda g_y, \qquad f_z = \lambda g_z, \qquad g(x, y, z) = k.$$

Two Constraints
If we want to find extrema subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$, we solve:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z),$$

together with
$$g(x, y, z) = k, \qquad h(x, y, z) = c.$$

Equivalently, this system is:
$$\begin{aligned} f_x &= \lambda g_x + \mu h_x, \\ f_y &= \lambda g_y + \mu h_y, \\ f_z &= \lambda g_z + \mu h_z, \\ g(x, y, z) &= k, \\ h(x, y, z) &= c. \end{aligned}$$

Chapter 15

Multiple Integrals

15.1 Double Integrals over Rectangles

Volumes and Double Integrals
If $f(x,y) \geq 0$, then the volume V of solid that lies above the rectangle R and below surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

Midpoint Rule for Double Integrals:

$$\iint_D f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x_i}, \bar{y_j})\Delta A$$

where $\bar{x_i}$ is the midpoint of $[x_{i-1}, x_i]$, $\bar{y_j}$ is the midpoint of $[y_{j-1}, y_j]$
Fubini's Theorem.: If f is continuous on a rectangle $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$ then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

If $f(x,y)$ can be written in the form $f(x,y) = g(x)h(y)$ then

$$\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

where $R = [a,b] \times [c,d]$
Average value of a function f on a rectangle R is:

$$f_{\text{average}} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$$

Interesting: if $f(x,y) \geq 0$, then $\text{Area}(R) \times f_{\text{average}} = \iint_R f(x,y)\,dA$
This says that the box with base R and height f_{average} has the same volume as the solid that lies under the graph of f.

15.2 Double Integrals over General Regions

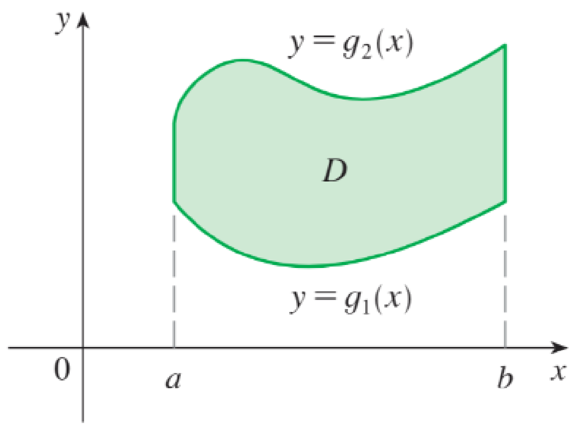
A plane region D is **type 1** if it lies between the graphs of two continuous functions of x, i.e.

$$D = \{(x,y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

In this case, we have:

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Illustration (think of drawing a vertical arrow across the region):



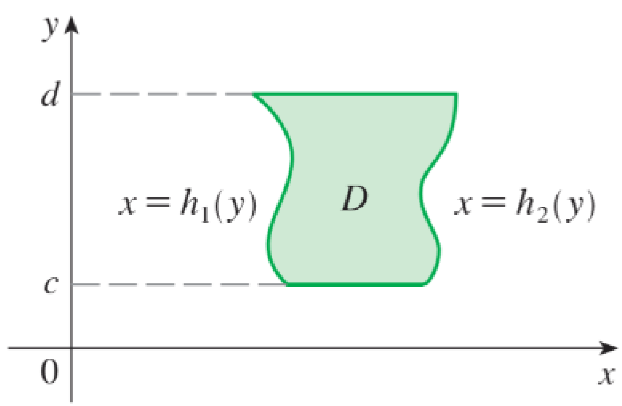
A plane region D is **type 2** if it lies between the graphs of two continuous functions of y, i.e.

$$D = \{(x, y)|c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

In this case, we have:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Illustration (think of drawing a horizontal arrow across the region):



Properties:

- $\iint_D [f(x,y) + g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$
- $\iint_D cf(x,y)\,dA = c\iint_R f(x,y)\,dA$
- If $f(x,y) \ge g(x,y)$ for all (x,y) in D, then $\iint_D f\,dA \ge \iint_D g\,dA$
- If $D = D_1 \cup D_2$ where D_1, D_2 don't overlap except at boundaries, then $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$
- $\iint_D 1\,dA = \text{Area}(D)$

Strategy: Switching the Order of Integration Sometimes an integral is impossible to evaluate in the given order. To solve this:

1. Sketch the region D based on the given bounds.
2. Interpret the region as the *other* Type (switch from Type I to II or vice versa).
3. Rewrite the integral with the new bounds and new order ($dx\,dy \leftrightarrow dy\,dx$).

15.3 Double Integrals in Polar Coordinates

Relationship between rectangular coordinates and polar coordinates:

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta$$

Double Integrals in Polar Coordinates
Polar Rectangle: $R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$

Theorem 10. If f is continuous on a polar rectangle R where $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$ and $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

More complicated region: $D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$ then the following is true:

$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Common Polar Bounds:

- **Full Circle** (radius a): $0 \le r \le a$, $0 \le \theta \le 2\pi$
- **Top Semicircle**: $0 \le r \le a$, $0 \le \theta \le \pi$
- **Right Semicircle**: $0 \le r \le a$, $-\pi/2 \le \theta \le \pi/2$
- **Annulus (Ring)** (radii a and b): $a \le r \le b$, $0 \le \theta \le 2\pi$

15.4 Applications of Double Integrals (Probability)

A function $f(x, y)$ is a **joint density function** for a pair of random variables (X, Y) if:

- $f(x, y) \ge 0$ for all (x, y)
- The total volume under the graph is 1:

$$\iint_{\mathbb{R}^2} f(x, y)\, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$$

The probability that (X, Y) lies in a region D is the volume under the surface over D:

$$P((X, Y) \in D) = \iint_D f(x, y)\, dA$$

- **Mean of X:**

$$E[X] = \iint_{\mathbb{R}^2} x f(x, y)\, dA$$

- **Mean of Y:**

$$E[Y] = \iint_{\mathbb{R}^2} y f(x, y)\, dA$$

Two random variables X and Y are **independent** if their joint density function is the product of their individual p.d.f.'s:

$$f(x, y) = f_1(x) f_2(y)$$

Where f_1 and f_2 are the single-variable density functions for X and Y.
Exponential Random Variables
If X and Y are independent exponential variables with means μ_1 and μ_2:

$$f(x, y) = \begin{cases} \frac{1}{\mu_1 \mu_2} e^{-x/\mu_1} e^{-y/\mu_2} & \text{if } x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The single variable form is $f(t) = \frac{1}{\mu} e^{-t/\mu}$ for $t \ge 0$.