15-382 Collective Intelligence

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First Edition

Disclaimer

This document aims to summarize the content of the slides for 15-382, including what the author considers important. As always, the definition of important is highly subjective so the author might have omitted something that was important to another person, or included something that is trivial to another.

Good luck,

SH

Dynamical Systems

Fingerprints of Complex Systems

- Multi-agent / multi-component
- Decentralized
- Local interactions
- Dynamic

Types of Abstract Models:

Agent-based Mechanistic implementation of the multi-component interactions.

Mathematical (white-box) Formally describe the relations among the relevant components.

Black-box Input-output pairs from the system are used to predict...

Statistical Describing patterns and correlations between variables.

Systems of ODEs

Here is a system of $n \geq 1$ Ordinary Differential Equations

$$\begin{cases} \frac{dx_1}{dt} = f_1(\mathbf{x}(t)) \\ \frac{dx_2}{dt} = f_2(\mathbf{x}(t)) \\ \vdots \\ \frac{dx_n}{dt} = f_n(\mathbf{x}(t)) \end{cases}$$

where $\mathbf{x}(t)$ is an n-dim vector.

A continuous-time Dynamical System is defined by a system of differential equations:

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}, t; \theta) \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t; \theta)$$

where $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ specifies how each component of the state evolves as t changes. It can depend on a set of given parameters θ

Some definitions:

- Initial conditions: where the system is at the beginning of the evolution: $\mathbf{x}(t_0)$
- Phase space: space of all possible states
- Trajectory: the curve traces by $\mathbf{x}(t)$ in the phase space starting from $\mathbf{x}(t_0)$
- Solution: is in the form $\mathbf{x}(t; t_0)$ that defines a family of time trajectories in the phase space. Once we fix t_0, we fix a unique trajectory

Vector fields and flows

How are solutions built? At any point, $\mathbf{f}$ assigns a vector that shows where the point is heading (direction of motion).

If we plot these arrows (vectors) in the phase space, we get can get an idea of how the system evolves.

Flow: $\Phi : \mathtt{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is the collection of all trajectories generated by all possible starting conditions.

$\Phi(t, \mathbf{x}_0) = \mathbf{x}(t; \mathbf{x}_0)$

A fundamental theorem guarantees that **two orbits corresponding to two different initial solutions never intersect with each other** , except at equilibrium

Basic Properties

An ODE is linear if

- $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ (Homogeneous)
- $\mathbf{f}(\mathbf{x}) = A\mathbf{x} + b$ (Affine)

Linear ODE enjoys closed form solutions, non-linear ODEs usually not

A system is autonomous if time doesn't appear in expression of $\mathbf{f}$.

Facts:

- Any n-order ODE can be rewritten as a system of 1st order ODEs in $\mathbb{R}^n$
- Any Non-Autonomous ODE can be rewritten as an autonomous one

So we will focus on 1st order, autonomous and linear ODEs

Solving!

General form of linear ODE:

$$\dot{\mathbf{x}} = A\mathbf{x}, \quad x \in \mathbb{R}^n$$

A solution is a function $\mathbf{x}(t)$ that satisfies the vector field A.

(Lots of derivation out is skipped, here's how to solve)

1. Solve $\det(A - \lambda I) = 0$ for λ
2. The roots λ_i are eigenvalues of A
3. For each λ_i, there exists a non-null eigenvector $\mathbf{u}_i$
4. Together they yield one solution: $\mathbf{x}(t) = \mathbf{u}_i e^{\lambda_i t}$
5. Each distinct eigen-pair gives ONE independent vector solution
6. The general solution is then the combination of these terms: $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{u}_n$ (at most n terms)

Important: the above is strictly true only if all eigenvalues are distinct

Matrix Exponential representation: $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ where $\mathbf{x}(0)$ is a generic initial condition

Exponentials and Asymtotic Behavior

Since the solution is a sum of exponentials, stuff is being pulled in the direction of the eigenvectors, weighted by their corresponding signed eigenvalues.

If the real part of $\lambda_i > 0$, mode i is unstable/diverging.

If the real part of $\lambda_i < 0$, mode i is stable/contracting.

At each point, the solution **mixes** the modes.

Equilibrium points

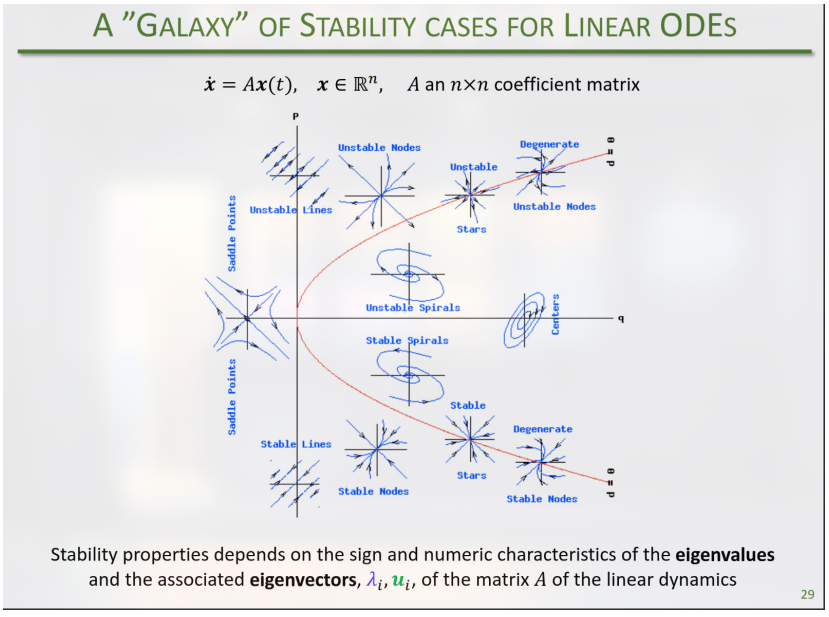
A state $\mathbf{x}_e$ is an equilibrium state of a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ if when at a time t_0 the sytem is at $\mathbf{x}_e$ then it stays there FOREVER

Why? Velocity of the field in $\mathbf{x}_e$ is null: $\mathbf{f}(\mathbf{x}_e) = 0$

For a linear ODE, the equilibrium points are the points of the **Null Space** (solutions to $A\mathbf{x} = 0$) Theres one trivial solution at $\mathbf{x} = \mathbf{0}$ if A is invertible, o.w. inifnitely many solutions

Taxonomy of equilibria

Equilibrium Type	System's Behavior	Trajectories
Equilibrium state	If at or arrives at $\mathbf{x_e}$, it stays at $\mathbf{x_e}$	Trajectory is constant: $\mathbf{x}(t) = \mathbf{x_e}$
Stable equilibrium (Lyapunov)	If started close to $\mathbf{x_e}$, stays close to $\mathbf{x_e}$ forever	Nearby trajectories remain in a neighborhood of $\mathbf{x_e}$
Asymptotically stable equilibrium	If started close to $\mathbf{x_e}$, stays close to $\mathbf{x_e}$ and moves toward $\mathbf{x_e}$ as $t \to \infty$	Nearby trajectories converge to $\mathbf{x_e}$
Unstable equilibrium	Even if started very close to $\mathbf{x_e}$, eventually diverges from $\mathbf{x_e}$	Nearby trajectories diverge from $\mathbf{x_e}$



Linear System Classification by Eigenvalues

(for the saddle case, keep in mind a product is negative only if exaclty one of the numbers is negative)

Eigenvalues	Critical Point	Stability
$r_1, r_2 > 0$	Node (real, distinct)	Unstable
$r_1, r_2 < 0$	Node (real, distinct)	Asymptotically stable
$r_1 r_2 < 0$	Saddle	Unstable
$r_1 = r_2 \neq 0$	Node / Improper node	Same as sign of r_1
$r_{1,2} = \lambda \pm i\mu$	Spiral (focus)	Same as sign of λ
$r_{1,2} = \pm i\mu$	Center	Neutrally stable

Perturbations

For Pure Imaginary Eigenvalue, small perturbations add a tiny real part to the eigenvalues:

- $\lambda > 0$ (positive real part) $\implies$ trajectories spiral outward (unstable spiral).
- $\lambda < 0$ (negative real part) $\implies$ trajectories spiral inward (stable spiral).

For Repeated Real Eigenvalues

- If the eigenvectors are linearly independent, the system stays a node, but may change to a saddle if the signs differents.
- If the eigenvectors are linearly dependent (a degenerate node), a small perturbation will typically turn it into either a spiral (if eigenvalues become complex) or a node (if eigenvalues become distinct real numbers).

Linearization around a Critical Point

Consider a nonlinear system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n$$

Let $\mathbf{x}_0$ be a critical point: $\mathbf{f}(\mathbf{x}_0) = 0$.

Step 1: Linear Approximation

$$\mathbf{f}(\mathbf{x}) \approx \mathbf{f}(\mathbf{x}_0) + J_{\mathbf{f}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

Since $\mathbf{f}(\mathbf{x}_0) = 0$, the linearized system is

$$\dot{\mathbf{x}} \approx J_{\mathbf{f}}(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0), \quad \mathbf{y} = \mathbf{x} - \mathbf{x}_0$$

Step 2: Jacobian Matrix

$$J_{\mathbf{f}}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_0}$$

Step 3: Eigenvalues, Eigenvectors, and General Solution

Compute eigenvalues λ_i and eigenvectors $\mathbf{v}_i$ of $J_{\mathbf{f}}(\mathbf{x}_0)$. The general solution of the linearized system is

$$\mathbf{y}(t) = \sum_{i=1}^{n} c_i e^{\lambda_i t} \mathbf{v}_i, \quad \mathbf{x}(t) = \mathbf{x}_0 + \mathbf{y}(t)$$

Step 4: Stability Analysis

Stability is determined by the eigenvalues of the Jacobian $J_{\mathbf{f}}(\mathbf{x}_0)$, just like in the linear case.

Global Behavior and Nullclines

- **Basin of Attraction:** The set of all initial conditions that eventually lead a trajectory to the same stable equilibrium point.
- **Separatrix:** The boundary between different basins of attraction.
- **Isoline:** The set of points where a function takes the same value.
- **Isocline:** The set of points where the vector field has the same **slope**.
- **Nullcline:** A specific type of isocline. It's the set of points where the slope is either zero or infinite (i.e., where one component of the vector field is zero).

Nullcline fun facts:

1. The x**-nullcline** is the curve where $\dot{x} = f(x, y) = 0$.
2. The y**-nullcline** is the curve where $\dot{y} = g(x, y) = 0$.
3. On the x-nullcline, the vector field has no horizontal component, so vectors can only point vertically (up or down).
4. On the y-nullcline, the vector field has no vertical component, so vectors can only point horizontally (left or right).
5. **Equilibrium points** are exactly at the intersection of the x-nullclines and y-nullclines (since $\dot{x} = 0$ and $\dot{y} = 0$ simultaneously).
6. Nullclines divide the phase space into regions. In each region, the sign of $\dot{x}$ and $\dot{y}$ is constant.

Limit cycles

A periodic orbit is just any trajectory that forms a closed loop.

A limit cycle is an isolated closed trajectory. This means neighboring trajectories are not closed; they either spiral into the limit cycle (a stable limit cycle) or spiral away from it (an unstable limit cycle). There's also mixed scenarios (half-stable)

Example: Van der Pol Oscillator

2D is Boring

Theorem 1. Any closed trajectory muyst encolse at least one equilibrium

Theorem 2. Poincare-Bendixson Theorem:

IF a trajectory is trapped in a closed, bounded region R,

AND this region R contains no equilibrium points,

THEN the trajectory must eventually approach a limit cycle.

In other words, 2D systems cannot have chaos

Chaos and Strange Attractors

3 main types of attractors: Fixed Points, Limit Cycles and Strange Attractors (the last appears only in 3D+)

Properties of strange attractors:

- Sensitive dependence on initial conditions: two initial conditions very close to each other become very far apart as time goes on (but remain confined in the set that defines the attractor)
- Fractal dimensions (e.g. 2.06). Non-integer

Example: Lorenz Attractor

1. For a parameter r=21, trajectories spiral into one of two stable fixed points.
2. For r=28, the fixed points become unstable. Trajectories are still bounded, but they never settle down. They move from one "wing" of the attractor to the other in an aperiodic, unpredictable way.

Definition 1. Chaos: **aperiodic long-term behavior** in a **deterministic** system that exhibits **sensitive dependence on initial conditions**

Iterated Maps

Intro

Discrete-time dynamical systems have a state which is only defined at integer steps.

The state is:

$$\mathbf{x}(n) = (x_1(n), \ldots, x_k(n))$$

where k is the dimension of the system and n is an integer step parameter

and the rule is:

$$\mathbf{x}(n) = (f)((\mathbf{x}(n-1), \ldots, \mathbf{x}(n-m)))$$

(so the system here depends on m previous states)

Can also be written nicely as $\mathbf{x}_n = \mathbf{f}(\mathbf{x}_{n-1}, \ldots, \mathbf{x}_{n-m})$

Next state is obtained by directly apply the map $\mathbf{f}$

1D Iterated Maps and Cobweb Plots

We focus on the simplest case: a 1D map $x_n = f(x_{n-1})$

- **Orbit:** sequence of points generated starting at x_0 and keep applying the map
- **Fixed point:** Point that maps to itself: $f(x^*) = x^*$

Sawtooth diagram: Plotting the $(x, f(x))$ diagram. No time shown, all it tells you is if the system is at x now, the next state will be at $f(x)$

Cobweb Plot construction algorithm:

1. Draw $y = f(x)$ and $y = x$
2. Start at initial point x_0 on horizontal axis
3. Move **vertically** to $y = f(x)$ (this is point x_0, x_1)
4. Move **horizontally** to $y = x$ (this is point x_1, x_1)
5. Move **vertically** to $y = f(x)$ again (this is point x_1, x_2)
6. Keep going lil bro

The intersections of $y = f(x)$ and $y = x$ are fixed points. We can analyze stability of a fixed point by looking at the plot.

Stability of fixed points

Some simple derivation because its interesting:

Consider a point near a fixed point, $x_n = x^* + \epsilon_n$. Next point $x_{n+1} = f(x^* + \epsilon_n) = f(x^*) + f'(x^*)\epsilon_n + O(\epsilon_n^2)$

So a linear approximation shows that $\epsilon_{n+1} = f'(x^*)\epsilon_n$. Let $\lambda = f'(x^*)$

Solving: $\epsilon_n = \lambda^n \epsilon_0$. If you know some basics you can infer stability from this alone but I will write a table.

$\lambda = f'(x^*)$	**Stability Type**	**Behavior Near x^***
Stable (Attracting)		
$0 < \lambda < 1$	Monotonic	Converges from one side
$-1 < \lambda < 0$	Oscillatory	Zig-zag convergence (alternating sides)
$\lambda = 0$	Superstable	Very fast convergence
Unstable (Repelling)		
$\lambda > 1$	Monotonic	Diverges from one side
$\lambda < -1$	Oscillatory	Alternating divergence
Marginal		
$\lvert\lambda\rvert = 1$	Neutral	Linearization inconclusive

Logistic map and chaos

Logistic Map: $x_{n+1} = rx_n(1 - x_n)$ where $x_n \in [0, 1]$ is population, $r \in [0, 4]$ is growth rate

- $1 < r < 3$: The population converges to a single, stable fixed point $x^* = 1 - 1/r$.
- $r = 3$: The fixed point becomes unstable.
- $3 < r < 3.449\ldots$: The system no longer settles to one point. It settles into a stable period-2 cycle, oscillating between two values. This is a bifurcation.
- $r > 3.449\ldots$: The 2-cycle becomes unstable and splits into a stable period-4 cycle. This continues, creating an 8-cycle, 16-cycle, etc., in a period-doubling cascade.
- $r > r^\infty \approx 3.5699\ldots$: The cascade finishes, and the system enters the chaotic regime. The orbit becomes aperiodic, never settling down and seemingly random.

This behavior can be summarized in an **orbit diagram**.

- The x-axis is the parameter r.
- The y-axis plots the long-term attractor points for that r.

A **bifurcation** is a qualitative change in the long-term behavior of a system with a smooth variation of a parameter

A bunch of math deriving Lyapunov Exponents, but here is the final result:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'(x_k)|$$

The approximation for λ can be constructed numerically, by iterating the map!

- $\lambda < 0$: For stable fixed points and cycles
- $\lambda > 0$: For chaotic attractors
- $\lambda = 0$: This is the marginal case, which occurs at bifurcation points.
- λ is the same for all points in the basin of attraction of an attractor

Cellular Automata (CA)

CA properties

A **Cellular Automaton** is a multi-dimensional discrete-time dynamical system that is defined by the principle of locality. properties:

Systems's state is n-dimensional $\mathbf{x}(t) = (x_1(t), \ldots, x_n(t))$

Discrete time : updated at discrete time steps

State components arranged according to a given topology

Neighborhood defined based on topology : $N(x_i) = x_j : x_j$ is a neighbor of x_i

Neighborhood is the range for a cell to be influenced by other cells

1D Example: $N(x_i) = x_{i-1}, x_i, x_{i+1}$

2D Examples:

- Von Neumann: The cell and its 4 neighbors (up, down, left, right).
- Moore: The cell and its 8 surrounding neighbors (a 3x3 box).

Locality of updates : Each state component x_i evolves according to a rule that depends only on its own state and those of its neighbors in $N(x_i)$.

Local-state transition function, $F_i : S(N(x_i)) \to S_i$ where S_i s the set of values that state component x_i can take, and S is the state values from the cells in the neighborhood set.

Initial Conditions : state at $t = 0$

Lattices and Boundaries:

Infinite/adaptive lattice: The grid grows as the pattern propagates

Finite lattice

- Hard boundary: fixed, edge cells have a fixed state
- Hard boundary: reflective, leftmost (rightmost) cell only diffuse right (left)
- Soft boundary: periodic boundary conditions, edges wrap around

Updating Schedules:

- Synchronous Updating: At time t, all cells read the states of their neighbors. Then, they all compute their next state. Finally, all cells update their state simultaneously to begin step $t + 1$.
- Asynchronous Updating: Cells update one at a time or in random groups. The order matters. A cell updating later in the step will see the already updated states of its neighbors who updated earlier.

Some Math

We assume they are homogeneous (same lattice, same N, and same rule F for all cells) and use synchronous updating.

Combinatorics Let:

- $k = |S|$ is the number of states per cell
- M is the number of cells
- r is the range ($floor(|N(a)|/2)$)

Then:

- Number of possible state configs: k^M
- Number of possible neighborhood configurations: $k^{|N|}$
- Number of possible evolution functions (rules): $k^{\left(k^{|N|}\right)}$ Why? This is the number of possible functions mapping from the set of neighborhood configurations (domain size $k^{|N|}$) to the set of possible next states (codomain size k). For each of the $k^{|N|}$ possible inputs, there are k possible outputs.

Wolfram Code

For Wolfram's 1D CA: $S = \{0, 1\}$ (so $k = 2$) and the neighborhood is $r = 1$ (i.e., $N(x_i) = \{x_{i-1}, x_i, x_{i+1}\}$, so $|N| = 3$).

Using our formula, the number of rules $= k^{\left(k^{|N|}\right)} = 2^{\left(2^3\right)} = 2^8 = 256$.

Four classes of Behavior and Chaos

Class	Behavior	Information Dynamics	Lyapunov Exponent
1	Evolves to a simple, stable, homogeneous state (all 0s or all 1s).	Small changes die; information is lost.	$\lambda \leq 0$
2	Evolves to simple periodic structures (stripes, oscillators).	Small changes may persist locally but do not spread.	$\lambda = 0$
3	Evolves to chaotic, aperiodic patterns (e.g., Rule 30).	Small changes spread out and affect distant regions.	$\lambda > 0$
4	Creates complex, localized, moving structures. (e.g., Rule 110 is **Turing complete**)	Small changes may or may not spread; irregular but structured dynamics.	$\lambda > 0$ (tends to 0)

Game of Life

- 2D Lattice of identical cells
- Moore neighborhood
- 2 states: dead or alive
- The 4 Rules:

  Loneliness A live cell with < 2 live neighbors dies.

  Overcrowding A live cell with > 3 live neighbors dies.

Survival A live cell with 2 or 3 live neighbors lives on.

Reproduction A dead cell with exactly 3 live neighbors becomes alive.

- Garden of Eden: A pattern that can only exist as initial pattern. In other words, no parent could possibly produce the pattern.

Networks

Basic Graph Theory, Terminology, Properties

Graph: two sets (V, E) of vertices/nodes and edges/arcs/links

Can be directed (interaction flows one way) or undirected (interaction flows both ways)

For Undirected graphs:

- Node degree k_i: number of edges adjacent to node i
- Handshake theorem: let L be the number of edges. Then $L = \frac{1}{2}\sum_{i=1}^{N} k_i$
- Average degree: $\langle k \rangle = \frac{2L}{N}$

For Directed Graphs:

- In-degree: number of edges going into node
- Out-degree: number of edges going out of the node
- Total degree $k_i = k_i^{\text{in}} + k_i^{\text{out}}$
- Average degree (either in or out): $\frac{L}{N}$

Degree Distribution: $p_k = \frac{N_k}{N}$ where N_k is number of nodes with degree k

In real networks degree distribution is highly heterogeneous

Complete graph: every possible edge is there. $L = \frac{N(N-1)}{2}$

In real networks, the number of edges is way less than that.

- **Walk:** A sequence of vertices (and corresponding edges) such that consecutive vertices are adjacent. Vertices and edges may repeat. For example, $(1 \to 2 \to 1)$ is a valid walk.
- **Path:** A walk in which all vertices (and therefore all edges) are distinct. For example, $(1 \to 2 \to 3)$ is a path, but $(1 \to 2 \to 1)$ is not.
- **Length:** The number of edges in a walk or path, or the sum of their weights if the graph is weighted.
- **Distance:** The length of the shortest path between two vertices. In unweighted graphs, this equals the minimal number of hops; in weighted graphs, the minimal total weight.
- **Shortest Path:** A path whose length equals the distance between its endpoints.
- **Diameter:** The maximum distance between any two vertices in the graph; equivalently, the length of the longest shortest path.

Connected graph is an undirected graph where there exists a path between every pair of nodes

For directed graphs there is

- Strongly connected if for every pair there is a path
- Weakly connected if every pair there is a path **when you ignore edge directions**

If the largest component in a graph has a large fraction of the nodes, we call it the giant component

Two representations for graphs

1. Adjacency list: a mapping from each node to it's neighborhood
2. Adjacency matrix: $N \times N$ matrix A such that $A_{ij} = 1$ if link (i, j) exists, and 0 otherwise.

More about adjacency matrices

1. For undirected graphs they are symmetric
2. Number of walks between nodes i, j can be calculated as follows: $(A^l)_{ij}$ gives #walks of length l.

Some measures

Average path length:

$$h = \frac{1}{2E_{\max} \sum_{i,j \neq i} h_{ij}}$$

where h_ij is the distance from i to j and $E_{\max} = n(n-1)/2$

Clustering Coefficient

Let L_i be the number of links between neighbors of i. Then the clustering coefficient of node i is

$$c_i = \frac{2L_i}{k_i(k_i - 1)}$$

Global clustering coefficient: $C = \frac{1}{N} \sum_{i=1}^{N} c_i$

Properties of real networks

Real networks are:

- Scale-free (from p_k): they have hubs
- Small world (from h): average distance is very small
- Locally dense (from C): clustering is very high

We will try to have a process that can create a network with these properties

Model 1: Erdos-Renyi (ER)

I DO NOT CAREEEEE ABOUT THE DOTS ABOVE THEIR NAMES

Generative process: each of the $\binom{N}{2}$ possible edges, an edge is created with probability p **Degree Distribution:**

$$p_k = \binom{N-1}{k} p^k (1-p)^{N-1-k}$$

Mean degree is $\bar{k} = p(N-1)$, Variance is $(N-1)p(1-p)$

For large, sparse networks that simplifies to Poisson distribution:

$$p_k = e^{-\bar{k}} \frac{\bar{k}^k}{k!}$$

Clustering Coefficient: $E[C] = p = \frac{\bar{k}}{N-1}$, i.e. small

Average path length grows as: $O(\log N)$

Verdict:

1. Scale-free: NO, p_k is Poisson (should be power law)

2. Small world: YES, low h
3. Locally dense: NO, C is very low (should be high)

Model 2: Watts-Strogatz (WS) (Small World)

This model was made to fix the clustering problem.

Generative process:

1. Start with a regular ring lattice, where each node is connected to m nearest neighbors. Graph starts with high C and h
2. Go through each edge, and with prob p rewire one end of the edge to a randomly chosen node.

What happens for different p?

- $p = 0$: Just a regular lattice, high C and h
- $p = 1$: Just a random graph, low C and h
- $p = 0.01$: A few rewired links act as shortcuts, dropping h dramatically (to around $logN$), while keeping a high C

Verdict:

1. Scale-free: NO, p_k is peaked at around m
2. Small world: YES, low h
3. Locally dense: YES, C is high

We are almost there

Model 3: Barabasi-Albert (BA) Scale-Free Model

Generative Process

Two main new mechanisms:

1. **Growth:** The network is not a fixed size. It starts with a small "seed" network, and at each time step, a new node is added.
2. **Preferential Attachment:** New node connects to m existing nodes. The probability $\Pi(i)$ of it connecting to an existing node i is not uniform. It is proportional to that node's current degree k_i:

$$\Pi(i) = \frac{k_i}{\sum_j k_j}$$

Rich get richer, rich nodes have higher chance of getting new links

Verdict:

1. Scale-free: YES, naturally produces a power-law degree distribution (p_k around k^{-3})
2. Small world: YES, low h
3. Locally dense: KINDA, C is a typically lower than in real networks

Summary of Network Models

Model	Scale-Free?	Small World?	High Clustering?
Erdos-Renyi (ER)	No	Yes	No
Watts-Strogatz (WS)	No	Yes	Yes
Barabasi-Albert (BA)	Yes	Yes	Kinda

Importance of a node: Network Centrality Measures

Certain positions within the network give nodes more power or importance. We will explore different types

Classic (Non-Recursive) Centrality Measures

Degree Centrality

The number of other nodes n is connected to.

Meaning: A node with high degree centrality has high potential communication activity

Betweenness Centrality

Number of shortest paths connecting all pairs of other nodes that pass through n

Meaning: A node with high betweenness centrality acts as a "bridge" or "gatekeeper" and has significant control over the flow of information.

Closeness Centrality

Measures how "close" a node is to all other nodes in the network.

$$C(n) = \frac{N}{\sum_m^{d(m,n)}}$$

where $d(m, n)$ is distance between m and n

Meaning: Efficiency of information spread. It answers the question, "How fast can I reach everyone else from this node?"

Self-Consistent (Recursive) Centrality

A node's importance is determined by the importance of its neighbors. This creates a recursive, self-referential definition.

Eigenvector centrality

The centrality of a node i is the scaled sum of centralities of its neighbors

$$c_i = \frac{1}{\lambda} \sum_{j \in N(i)}^{c_j}$$

Some math leads it to $A\mathbf{c} = \lambda\mathbf{c}$ where $\mathbf{c}$ is the principal eigenvector of the adjacency matrix

Computing eigvenector centrality

For a network with billions of nodes, we can't just "solve" this equation. We must compute it iteratively.

1. Power Method (Power Iteration)

This is the standard centralized algorithm. It uses iterative matrix multiplication to converge to the principal eigenvector.

1. **Initialize:** Choose an arbitrary nonzero vector $\mathbf{c}^{(0)}$ (e.g., all ones).
2. **Iterate:** Multiply by the adjacency matrix:
$$\mathbf{c}^{(t+1)} = A\mathbf{c}^{(t)}.$$
3. **Normalize:** To prevent numerical overflow, normalize at each step:
$$\mathbf{c}^{(t+1)} = \frac{A\mathbf{c}^{(t)}}{\|A\mathbf{c}^{(t)}\|}.$$

4. **Converge:** As $t \to \infty$, $\mathbf{c}^{(t)}$ converges to the principal eigenvector $\mathbf{c}_1$.

2. Gossip Algorithms (Decentralized Power Method)

Gossip algorithms implement the same idea in a distributed and asynchronous way, without a central coordinator.

- Each node i maintains its own estimate $c_i^{(t)}$.
- Nodes periodically *gossip* (push or pull) their current c_i values to their neighbors.
- Each node updates asynchronously:
$$c_i^{(t+1)} = \sum_{j \in N(i)} c_j^{(t)}.$$
Over time, these local updates collectively approximate the global power iteration, converging to the same principal eigenvector.

Alpha-Centrality

The idea is that the centrality is a combination of two things:

1. Recursive influence from its neighbors (scaled by α).
2. An "exogenous" or baseline importance e.

The defining equation is:
$$\mathbf{c} = \alpha A\mathbf{c} + \beta\mathbf{e}$$

Solving for $\mathbf{c}$:
$$\mathbf{c} = \beta(I - \alpha A)^{-1}\mathbf{e}$$

Since
$$(I - \alpha A)^{-1} = I + \alpha A + \alpha^2 A^2 + \ldots,$$
we can interpret each term as a contribution from paths of increasing length:

- $\beta I\mathbf{e}$ — baseline importance (paths of length 0),
- $\beta\alpha A\mathbf{e}$ — influence from direct neighbors (paths of length 1),
- $\beta\alpha^2 A^2\mathbf{e}$ — influence from neighbors-of-neighbors (paths of length 2),
- and so on for longer paths.

The parameter α controls the extent of influence propagation:

- For small α, only local structure matters (nearby nodes dominate).
- As α approaches $1/\lambda_1$ (where λ_1 is the largest eigenvalue of A), global structure dominates, and the measure converges to **eigenvector centrality**.

TODO (maybe): Add page rank stuff, and theres more math things there which looks like it wont come up... :)

Robots, or if you fancy: Cyber-Physical Multi-Agent Systems (CP MAS)

Systems of multiple interacting cyber-physical agents (robots, sensors) combining mechatronic structures, communication, and processing.

The near-future vision is of "Robots Everywhere"

Design Objectives of Robot Swarms

- **Scalability:** Performance should degrade gracefully as swarm size increases.
- **Robustness:** Tolerance to individual robot failures or environmental changes.
- **Flexibility / Adaptability:** Ability to handle different tasks or environments.
- **Distributed Control:** No single point of failure; decision-making is local.

Swarm Intelligence Design Approach

The primary design paradigm is bottom-up:

- Relatively simple individual controllers.
- Relatively complex interaction patterns.
- Relies mostly on locality of interactions and communications (decentralized and distributed).
- Leverages emergence and self-organization.

Downsides of Swarm Design

The complexity leads to challenges:

- Predictability and formal guarantees of swarm behavior can be challenging.
- Finite-time performance is often hard to characterize.
- Efficiency might be mediocre when heavily relying on full decentralization and self-organization.

CPMAS Taxonomy

Feature	**can be:**	**or can be:**
Members	Homogeneous (interchangeable units)	Heterogeneous (units with different skills/capabilities)
Coupling	Loosely coupled (agents operate independently; cooperation optional — e.g., speedup)	Tightly coupled (agents depend on each other; require coordination/cooperation)
Goals	Non-cooperative (agents maximize individual utility; equilibrium concepts apply)	Cooperative (agents maximize a global objective; social welfare / optimization concepts)
Control	Centralized control (single decision authority or planner)	Decentralized / distributed control (local decision-making; peer-to-peer coordination)
Coordination & Planning	Explicit (direct communication or sharing of plans among agents)	Implicit (coordination emerges through actions that influence others without direct communication)

Table 1: *Note:* The entries are *independent* options used to describe a system; real systems may combine any of these (e.g., a heterogeneous but loosely coupled group, or a homogeneous yet tightly coupled group).

Core issue:

Communications are fundamental but face issues:

- Global schemes won't scale for large swarms.
- Need for ad hoc networking in infrastructure-less environments (e.g., post-disaster).
- Challenges in deciding what to communicate, how frequently, and to whom.

AntHocNet:

An example of managing a Mobile Ad Hoc Network (MANET) using a hybrid Ant Colony Optimization (ACO) approach. It uses ant agents to set up full paths (reactive) and local information exchange to maintain and improve them (proactive).