Chapter 1 - Abstract Integration

Salvador Castagnino
scastagnino@itba.edu.ar

Exercise Solutions

Useful Properties

Theorem 1.1 (Reverse Fatou's Lemma) Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequences of measurable functions with $f_n : X \to [0,\infty]$, if there exsits $g \in L^1(\mu)$ such that $f_n \le g$ for all $n \ge 1$ then

$$\limsup_{n\to\infty} \int_X f_n \, d\mu \le \int_X \limsup_{n\to\infty} f_n \, d\mu$$

Proof. As $f_n \le g$ for all $n \ge 1$ the functions $h_n = g - f_n$ are both measurable and nonnegative, thus applying Fatou's Lemma to h_n we get,

$$\int_X \liminf_{n\to\infty} (g - f_n) \, d\mu \le \liminf_{n\to\infty} \int_X g - f_n \, d\mu$$

rearranging both sides we get,

$$\int_X g \, d\mu - \int_X \limsup_{n\to\infty} f_n \, d\mu \le \int_X g \, d\mu - \limsup_{n\to\infty} \int_X f_n \, d\mu$$

Finally, as $\int_X g \, d\mu < \infty$ we can cancel on both sides and multiply by -1 getting the desired result.