Chapter 3 - L^p Spaces

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Exercise 3.

Let's suppose that ϕ is not convex in (a, b) then there exist $x, y \in (a, b)$ and $t_0 \in [0, 1]$ such that

$$\phi\left(\left(1 - t_0\right) x + t_0 y\right) > \left(1 - t_0\right) \phi\left(x\right) + t_0 \phi\left(y\right)$$

and assume $x < y$. Let's define $g, h : [0, 1] \to R$ such that,

$$g\left(t\right) = \phi\left(\left(1 - t\right) x + t y\right)$$

$$h\left(t\right) = \left(1 - t\right) \phi\left(x\right) + t \phi\left(y\right)$$

and let $f = h - g$. We define the nonempty sets $A = \{t \in [0, t_0\ ) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f\left(0\right) = f\left(1\right) = 0$, and let $t_a = \sup A$ and $t_b = \inf B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in A and B which converge to t_a and t_b respectively, by continuity of f we can see that

$$0 = \lim_{n \to \infty} f\left(a_n\right) = f\left(\lim_{n \to \infty} a_n\right) = f\left(t_a\right)$$

With this and the fact that $t_a < t_0 < t_b$ as $f\left(t_0\right) < 0$ we have that $t_a \in A$ and the proof for $t_b \in B$ is analogous. Now by the definition of t_a and t_b we have that $f\left(t\right) < 0$ for all $t \in \left(t_a, t_b\right)$. Finally we take $x' = \left(1 - t_a\right) x + t_a y$ and $y' = \left(1 - t_b\right) x + t_b y$ and by working algebraically over $f\left(\frac{t_a + t_b}{2}\right) < 0$ we arrive at the following inquality

$$\phi\left(\frac{x' + y'}{2}\right) > \frac{\phi\left(x'\right) + \phi\left(y'\right)}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

Observation This proof shows that to see that convexity holds it sufices to show that given each pair of points $x, y \in (a, b)$ there exists a $t \in [0, 1]$ (not necessarily $\frac{1}{2}$) such that $\phi\left(\left(1 - t\right) x + t y\right) \leq \left(1 - t\right) \phi\left(x\right) + t \phi\left(y\right)$.

Exercise 4. Let f be a complex measurable function on X and μ a positive measure on X, define

$$\phi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty)$$

and now let $E = \{p : \phi(p) < \infty\}$ and assume $0 < \|f\|_\infty$. Let's begin by characterizing E. Let $0 < p < s < \infty$ and $x \in [0, \infty)$ then $x^p < x^s$ iff $1 < x$ and $x^s < x^p$ iff $x < 1$. With this in mind if $0 < r < p < s < \infty$ then $|f|^p \leq max\{|f|^r, |f|^s\} \leq |f|^r + |f|^s$ and we get,

$$\int_X |f|^p \, d\mu \leq \int_X |f|^r \, d\mu + \int_X |f|^s \, d\mu$$

Suppose that $s, r \in E$ then $p \in E$, which proves statement (a).

As a consequence of (a) we see that E is a connected set and with this in mind, supposing E is nonempty, we can see that E is an interval with endpoints $a = \inf E$ and $b = \sup E$, this interval will be closed or not if E contains them or not, but for sure $E^o = (a, b)$.

To prove that $\log \phi$ is convex in E^o we start by proving that the composition is well defined, in other words, ϕ is never zero. If $\phi(p) = 0$ for some $p \in E^o$ then $|f| = 0$ a.e. on X and this implies $\|f\|_\infty = 0$ contradicting our assumption, thus the composition is well defined and we proceed to prove convexity. Take $x, y \in (a, b)$ and $t \in (0, 1)$. By the properties of the logarithm we get,

$$(1 - t) \log \phi(x) + t \log \phi(t) = \log \left(\phi(x)^{1-t} \phi(y)^t \right)$$

given that the logarithm is a nondecreasing function, convexity holds if and only if

$$\phi((1 - t)x + ty) \leq \phi(x)^{1-t} \phi(y)^t$$

rewriting ϕ on both sides in terms of Lp norms we get the following,

$$\|f^{(1-t)x} f^{ty}\|_1 \leq \|f^{(1-t)x}\|_{(1-t)^{-1}} \|f^{ty}\|_{t^{-1}}$$

Given that $1 \leq (1 - t)^{-1}$ and $1 \leq t^{-1}$ are conjugate exponents, the last inequallity holds by Holder's inequality and this concludes the proof.

To prove that ϕ is continous in E we start by noticing that $\log \phi$ is convex in (a, b) which implies convexity of ϕ in (a, b) which in turn implies continuity of ϕ in (a, b), so we only need to prove continuity on a and b, in case they are elements of E.

Let $\{p_n\}$ be a sequence in (a, b) which converges to a and let's suppose $a \in E$, we will show that $\phi(p_n)$ converges to $\phi(a)$, the proof for b is analogous. By linearity of the integral and properties of the absolute value we get

$$|\phi(p_n) - \phi(a)| = \left| \int_X |f^{p_n}| - |f^a| \, d\mu \right| \leq \int_X ||f^{p_n}| - |f^a|| \, d\mu \leq \int_X |f^{p_n} - f^a| \, d\mu$$

Also, by continuity of the exponential we know that f^{p_n} converges pointwise to f^a as n goes to infinity so it sufices to find a real function $g \in L^1(\mu)$ such that $|f^{p_n}| \leq g$. Let $g = |f^a| + |f^{p_M}|$ with p_M the greatest element of the sequence $\{p_n\}$ which exists given the convergence of the sequence. Then we can see that $a \leq p_n \leq p_M$ for all $1 \leq n$ and as mentioned above we have

$$|f|^{p_n} \leq \max\{|f|^a, |f|^{p_M}\} \leq |f|^a + |f|^{p_M} = g$$

Finally by Lebesgue's Dominated Convergence Theorem we get that

$$\int_X |f^{p_n} - f^a| \, d\mu \to 0$$

which concludes the proof of (b).

Observation

Let $0 < r < p < s < +\infty$ with $p - r = s - p$ which is the same as $2p = r + s$ then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

Exercise 5. In order to prove that $\|f\|_p \leq \|f\|_r$ if $0 < p < r \leq \infty$ we start by looking at case in which both p and r are finite. For every $x \in (0, \infty)$ the function x^c is twice differentiable and we have that

$$(x^c)'' = \left(x^{c-1}c\right)' = c\,(c-1)\,x^{c-2}$$

Observe that if $1 \leq c$, the expression above is nonnegative for all $x \in (0, \infty)$ and thus x^c convex over that interval. With this in mind and the fact that $\mu(\Omega) = 1$ and $1 \leq \frac{r}{p}$ we will use Jensen's Inequality to prove the result. Let $A = \{x \in \Omega | f(x) \neq 0\}$ (the integral over it's complement is exactly 0 and in this way $|f|(A) \subset (0, \infty)$), then

$$\|f\|_p = \left(\int_A |f|^p \, d\mu\right)^{\frac{r}{rp}} \leq \left(\int_A |f|^{\frac{rp}{p}} \, d\mu\right)^{\frac{1}{r}} = \|f\|_r$$

and this concludes the proof for the finite case.

Now let $r = \infty$, this case is much simpler. By definition of the essential supremum we have that $|f| \leq \|f\|_\infty$ almost everywhere and using the fact that $\mu(\Omega) = 1$ we get,

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega \|f\|_\infty^p \, d\mu\right)^{\frac{1}{p}} = \|f\|_\infty$$

Exercise 7.

Let $0 < p < s < \infty$, an example for the inclusion $L^s(\mu) \subset L^p(\mu)$ can easily be constructed using Ex.5.

Let $X = (1, \infty)$ and m the Lebesgue Measure, we see thath neither of the inclusions hold observing that, on the one hand, $\frac{1}{x} \notin L^1(1, \infty)$ and $\frac{1}{x} \in L^2(1, \infty)$ and on the other, (the example i gave had an error, i'm working on it ...).

The remaining inclusion is more interesting, we take $X = N$ with the counting measure over the power set of N. Observe that all measurable functions in this case will be just sequences of complex numbers so if $f = \{a_n\}$ we have,

$$\|f\|_p^p = \int_X |f|^p \, d\mu = \sum_{n=1}^{\infty} |a_n|^p$$

Suppose $\{a_n\} \in \ell^p(N)$ then the sequence $\{|a_n|^p\}$ converges to 0 and thus there exists a natural number M such that $|a_n|^p < 1$ for all $M < n$. Given that $1 < \frac{s}{p}$ we have that $|a_n|^s = (|a_n|^p)^{\frac{s}{p}} < |a_n|^p$ for all n greater than M and with that in mind we get,

$$\sum_{n=1}^{\infty} |a_n|^s = \sum_{n=1}^{M} |a_n|^s + \sum_{n=M+1}^{\infty} |a_n|^s < \sum_{n=1}^{M} |a_n|^s + \sum_{n=M+1}^{\infty} |a_n|^p < \infty$$

As we wanted, $\{a_n\} \in \ell^p(N)$, which concludes the proof.

Exercise 10. Given that $fg \geq 1$ and both f and g are positive we have $f \geq \frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega) = 1$,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = \left(\|g\|_1 \|g^{-1}\|_1\right)^{\frac{1}{2}} \leq \left(\|g\|_1 \|f\|_1\right)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_\Omega f \, d\mu \int_\Omega g \, d\mu$$