Chapter 3 - L^p Spaces

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Exercise Solutions

Exercise 3.3. Let's suppose that ϕ is not convex in (a,b) then there exist $x,y \in (a,b)$ and $t_0 \in [0,1]$ such that

$$\phi\left((1-t_0)\,x + t_0 y\right) > (1-t_0)\,\phi(x) + t_0\phi(y)$$

Define $g,h:[0,1]\to\mathbb{R}$ such that,

$$g(t) = \phi\left((1-t)\,x + ty\right)$$

$$h(t) = (1-t)\,\phi(x) + t\phi(y)$$

and let $f = h-g$. We define the nonempty sets $A = \{t \in [0,t_0\,) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f(0) = f(1) = 0$, and let $t_a = \sup A$ and $t_b = \inf B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in A and B which converge to t_a and t_b respectively, by continuity of f we can see that

$$0 = \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(t_a)$$

With this and the fact that $t_a < t_0 < t_b$ as $f(t_0) < 0$ we have that $t_a \in A$ and the proof for $t_b \in B$ is analogous. Now by the definition of t_a and t_b and continuity of f we have that $f(t) < 0$ for all $t \in (t_a, t_b)$. Finally we take $x' = (1-t_a)\,x + t_a y$ and $y' = (1-t_b)\,x + t_b y$ and by working algebraically over $f\left(\frac{t_a+t_b}{2}\right) < 0$ we arrive at the following inquality

$$\phi\left(\frac{x'+y'}{2}\right) > \frac{\phi(x') + \phi(y')}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

Observation This proof shows that to see that convexity holds it sufices to show that given each pair of points $x,y \in (a,b)$ there exists a $t \in [0,1]$ (not necessarily $\frac{1}{2}$) such that $\phi\left((1-t)\,x + ty\right) \le (1-t)\,\phi(x) + t\phi(y)$.

Exercise 3.4. Let f be a complex measurable function on X and μ a positive measure on X, define

$$\phi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty)$$

and now let $E = \{p : \phi(p) < \infty\}$ and assume $0 < \|f\|_\infty$. Let's begin by characterizing E. Let $0 < p < s < \infty$ and $x \in [0, \infty)$ then $x^p < x^s$ iff $1 < x$ and $x^s < x^p$ iff $x < 1$. With this in mind if $0 < r < p < s < \infty$ then $|f|^p \leq max\{|f|^r, |f|^s\} \leq |f|^r + |f|^s$ and we get,

$$\int_X |f|^p \, d\mu \leq \int_X |f|^r \, d\mu + \int_X |f|^s \, d\mu$$

Suppose that $s, r \in E$ then $p \in E$, which proves statement (a).

As a consequence of (a) we see that E is a connected set and with this in mind, supposing E is nonempty, we can see that E is an interval with endpoints $a = \inf E$ and $b = \sup E$, this interval will be closed or not if E contains them or not, but for sure $E^o = (a, b)$.

To prove that $\log \phi$ is convex in E^o we start by proving that the composition is well defined, in other words, ϕ is never zero. If $\phi(p) = 0$ for some $p \in E^o$ then $|f| = 0$ a.e. on X and this implies $\|f\|_\infty = 0$ contradicting our assumption, thus the composition is well defined and we proceed to prove convexity. Take $x, y \in (a, b)$ and $t \in (0, 1)$. By the properties of the logarithm we get,

$$(1-t)\log\phi(x) + t\log\phi(t) = \log\left(\phi(x)^{1-t}\,\phi(y)^t\right)$$

given that the logarithm is a nondecreasing function, convexity holds if and only if

$$\phi((1-t)x + ty) \leq \phi(x)^{1-t}\,\phi(y)^t$$

rewriting ϕ on both sides in terms of Lp norms we get the following,

$$\|f^{(1-t)x} f^{ty}\|_1 \leq \|f^{(1-t)x}\|_{(1-t)^{-1}} \|f^{ty}\|_{t^{-1}}$$

Given that $1 \leq (1-t)^{-1}$ and $1 \leq t^{-1}$ are conjugate exponents, the last inequallity holds by Holder's inequality and this concludes the proof.

To prove that ϕ is continous in E we start by noticing that $\log \phi$ is convex in (a, b) which implies convexity of ϕ in (a, b) which in turn implies continuity of ϕ in (a, b), so we only need to prove continuity on a and b, in case they are elements of E.

Let $\{p_n\}$ be a sequence in (a, b) which converges to a and let's suppose $a \in E$, we will show that $\phi(p_n)$ converges to $\phi(a)$, the proof for b is analogous. By linearity of the integral and properties of the absolute value we get

$$|\phi(p_n) - \phi(a)| = |\int_X |f^{p_n}| - |f^a| \, d\mu| \leq \int_X ||f^{p_n}| - |f^a|| \, d\mu \leq \int_X |f^{p_n} - f^a| \, d\mu$$

Also, by continuity of the exponential we know that f^{p_n} converges pointwise to f^a as n goes to infinity so it sufices to find a real function $g \in L^1(\mu)$ such that $|f^{p_n}| \leq g$. Let $g = |f^a| + |f^{p_M}|$ with p_M the greatest element of the sequence $\{p_n\}$ which exists given the convergence of the sequence. Then we can see that $a \leq p_n \leq p_M$ for all $1 \leq n$ and as mentioned above we have

$$|f|^{p_n} \leq \max\{|f|^a, |f|^{p_M}\} \leq |f|^a + |f|^{p_M} = g$$

Finally by Lebesgue's Dominated Convergence Theorem we get that

$$\int_X |f^{p_n} - f^a| \, d\mu \to 0$$

which concludes the proof of (b).

Exercise 3.5. In order to prove that $\|f\|_p \leq \|f\|_r$ if $0 < p < r \leq \infty$ we start by looking at case in which both p and r are finite. For every $x \in (0, \infty)$ the function x^c is twice differentiable and we have that

$$(x^c)'' = \left(x^{c-1}c\right)' = c\,(c-1)\,x^{c-2}$$

Observe that if $1 \leq c$, the expression above is nonnegative for all $x \in (0, \infty)$ and thus x^c convex over that interval. With this in mind and the fact that $\mu(\Omega) = 1$ and $1 \leq \frac{r}{p}$ we will use Jensen's Inequality to prove the result. Let $A = \{x \in \Omega | f(x) \neq 0\}$ (the integral over it's complement is exactly 0 and in this way $|f|(A) \subset (0, \infty)$), then

$$\|f\|_p = \left(\int_A |f|^p \, d\mu\right)^{\frac{r}{rp}} \leq \left(\int_A |f|^{\frac{rp}{p}} \, d\mu\right)^{\frac{1}{r}} = \|f\|_r$$

and this concludes the proof for the finite case.

Now let $r = \infty$, this case is much simpler. By definition of the essential supremum we have that $|f| \leq \|f\|_\infty$ almost everywhere and using the fact that $\mu(\Omega) = 1$ we get,

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega \|f\|_\infty^p \, d\mu\right)^{\frac{1}{p}} = \|f\|_\infty$$

Exercise 3.7. Let $0 < p < s < \infty$, an example for the inclusion $L^s(\mu) \subset L^p(\mu)$ can easily be constructed using **Ex.3.5**.

Let $X = (1, \infty)$ and m the Lebesgue Measure, we see that neither of the inclusions hold. On the one hand, $\frac{1}{x} \notin L^1(1, \infty)$ and $\frac{1}{x} \in L^2(1, \infty)$. On the other hand, $\frac{I_{(1,2)}}{x-1} \notin L^1(1, \infty)$ and $\frac{I_{(1,2)}}{x-1} \in L^{\frac{1}{2}}(1, \infty)$ being I_A the indicator function of A. We will only prove the first of these last two statements (the second one can be proved in a simlar manner chosing a convenient partition of $(1, 2)$),

$$\int_1^\infty \frac{I_{(1,2)}}{x-1} \, dx = \int_1^2 \frac{1}{x-1} \, dx = \sum_{n=0}^\infty \int_{1+\frac{1}{e^{n+1}}}^{1+\frac{1}{e^n}} \frac{1}{x-1} \, dx$$

given that the antiderivative of $\frac{1}{x-1}$ is $\ln(x-1)+C$ the expression above equals

$$\sum_{n=0}^{\infty}\left[\ln\left(\frac{1}{e^n}\right)-\ln\left(\frac{1}{e^{n+1}}\right)\right]=\sum_{n=0}^{\infty}\ln(e)=\infty$$

For the remaining inclusion we take $X=\mathbb{N}$ with the Counting Measure over the power set of $\mathbb{N}$. Observe that all measurable functions in this case will be just sequences of complex numbers so if $f=\{a_n\}$ we have,

$$\|f\|_p^p=\int_X |f|^p\, d\mu=\sum_{n=1}^{\infty}|a_n|^p$$

Suppose $\{a_n\}\in\ell^p(\mathbb{N})$ then the sequence $\{|a_n|^p\}$ converges to 0 and thus there exists a natural number M such that $|a_n|^p<1$ for all $M<n$. Given that $1<\frac{s}{p}$ we have that $|a_n|^s=(|a_n|^p)^{\frac{s}{p}}<|a_n|^p$ for all n greater than M and with that in mind we get,

$$\sum_{n=1}^{\infty}|a_n|^s=\sum_{n=1}^{M}|a_n|^s+\sum_{n=M+1}^{\infty}|a_n|^s<\sum_{n=1}^{M}|a_n|^s+\sum_{n=M+1}^{\infty}|a_n|^p<\infty$$

As we wanted, $\{a_n\}\in\ell^s(\mathbb{N})$, which concludes the proof.

Exercise 3.10. Given that $fg\geq 1$ and both f and g are positive we have $f\geq\frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega)=1$,

$$1=\|1\|_1=\|g^{\frac{1}{2}}g^{-\frac{1}{2}}\|_1\leq\|g^{\frac{1}{2}}\|_2\|g^{-\frac{1}{2}}\|_2=\left(\|g\|_1\|g^{-1}\|_1\right)^{\frac{1}{2}}\leq\left(\|g\|_1\|f\|_1\right)^{\frac{1}{2}}$$

thus,

$$1\leq\int_\Omega f\, d\mu\int_\Omega g\, d\mu$$

Exercise 3.17. We start by looking at the case in which $1 \leq p < \infty$, we want to see that

$$|\alpha - \beta|^p \leq 2^{p-1}\left(|\alpha|^p + |\beta|^p\right) \tag{1}$$

for every α and β complex numbers not both zero (in that case the proof is tirivial). Let $X = [0,1]$ and μ be the Lebesgue Measure, we define the complex function f over $[0,1]$ such that

$$f(x) = \begin{cases} 2\alpha & \text{if } 0 \leq x < \frac{1}{2} \\ 2\beta & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Rearrangin (1) we get,

$$|\alpha - \beta| \leq \left(\frac{1}{2}|2\alpha|^p + \frac{1}{2}|2\beta|^p\right)^{\frac{1}{p}} = \|f\|_p \tag{2}$$

Observing that $\mu(X) = 1$ and $0 < \|f\|_\infty$ by **Ex.3.5** we know that $\|f\|_1 \leq \|f\|_p$ and we have

$$|\alpha - \beta| \leq |\alpha| + |\beta| = \|f\|_1 \leq \|f\|_p \tag{3}$$

which concludes the proof.

Exercise 3.18. Let's start with statement **(a)**, for it we will assume that $\mu(X) < \infty$. Let A be a set in which $\{f_n\}_{n\in\mathbb{N}}$ converges pointwise to f and $\mu(X - A) = 0$ take $\epsilon > 0$. By Egorov's theorem there exists a set $B \subset A$ in which the sequence converges uniformly and $\mu(A - B) < \epsilon$. By the uniform convergence of the sequence there exists $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \epsilon \qquad \forall\, x \in B \;\forall\, n > N$$

Given that the set of all points which don't satify this condition for n large enough is a subset of $X - B$ it's measure must be less than ϵ which proves that $\{f_n\}_{n\in\mathbb{N}}$ converges in measure to f and concludes the proof.

Now let's prove statement **(b)**, we won't need the assumption that $\mu(X) < \infty$ for this one. Assume that $\{f_n\}_{n\in\mathbb{N}}$ doesn't converge in measure to f, we start by looking at the case in which $0 \leq p < \infty$. By our assumption, there exists $\epsilon > 0$ such that for every $N \in \mathbb{N}$ there exists an $n > N$ such that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \geq \epsilon \tag{4}$$

Now let $E_{\epsilon,n}$ denote the set in (4), keeping notation as above we have

$$\epsilon^{p+1} \leq \int_{E_{\epsilon,n}} \epsilon^p \, d\mu \leq \int_{E_{\epsilon,n}} |f_n - f|^p \, d\mu \leq \int_X |f_n - f|^p \, d\mu$$

and clearly $\|f_n - f\|_p$ doesn't converge to zero which is a contradiction and concludes the proof for $1 \leq p < \infty$ (the case in which $p = \infty$ uses the same idea and won't we proven here). It calls my atention that we didn't need the fact that $f_n \in L^p(\mu)$ and makes me believe that there must be a mistake in the proof...

It only remains to prove **(c)**, we won't need the assumption that $\mu(X) < \infty$. We start by taking a subsequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ such that

$$\mu\left(E_{2^{-j},n_j}\right) < 2^{-j} \tag{5}$$

with $E_{\epsilon,n}$ defined as in (4). Observe that for every $\epsilon > 0$ and every $\delta > 0$ we can ask for a m large enough such that,

$$\mu(A_m) := \mu\left(\bigcup_{j=m}^{\infty} E_{2^{-j},n_j}\right) \leq \sum_{j=m}^{\infty} \mu\left(E_{2^{-j},n_j}\right) < \epsilon \tag{6}$$

and

$$2^{-j} < \delta \qquad \forall j \geq m \tag{7}$$

As defined in (6) we clearly have $A_{m+1} \subset A_m$. Now suposse that the sequence $\{f_{n_j}\}_{j\in\mathbb{N}}$ doesn't converge poitwise to any point in a set $B \subset X$ with $\mu(B) > 0$ we can ask for an m large enough such that (6) holds for $\epsilon = min\{1, \mu(B)\}$ (we can't just pick $\mu(B)$ as it could equal ∞). In that case B won't be a subset of A_m and we can ask for a point $x \in B - A_m$.

Now take $\gamma > 0$ and $m' \geq m$ such that (7) holds for $\delta = \gamma$ then $x \in B - A_{m'}$ as $A_{m'} \subset A_m$. Given that if $|f_{n_j}(y) - f(y)| > 2^{-m'}$ for some $j \geq m'$ we have $y \in A_{m'}$, as $x \notin A_{m'}$ it must be the case that for all $j \geq m'$

$$|f_{n_j}(x) - f(x)| \leq 2^{-m'} < \delta = \gamma$$

and then we have that $f_{n_j}(x) \to f(x)$ which contradicts our assumption and concludes the proof.

Exercise 3.20. Let $\alpha, \beta \in \mathbb{R}$ and $t \in [0,1]$ we define

$$f(x) = \begin{cases} \alpha & \text{if } 0 \leq x < t \\ \beta & \text{if } t \leq x \leq 1 \end{cases}$$

which is clearly measurable and bounded. It only remains to replace f in the given inequality to prove the convexity of ϕ.

Exercise 3.23. Let μ be a positive measure with $\mu(X) < \infty$ and f a complex measurable function with $0 < \|f\|_\infty < \infty$ define the sequence

$$a_n = \int_X |f|^n \, d\mu = \|f\|_n^n$$

By the given hypothesis we have that first, the hypothesis of **Ex.3.4.e** hold and second, $0 < a_n < \infty$ for every $1 \leq n$ and thus the quotients $\frac{a_{n+1}}{a_n}$ are well defined and we have the inequality,

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} a_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} a_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Looking at **Ex.3.4.e** we see that is sufices to show that $\frac{a_{n+1}}{a_n}$ converges in $[0, \infty]$ because in that case both limits will exist and equal $\|f\|_\infty$ as $a_n^{\frac{1}{n}} = \|f\|_n$. In order to prove this, we will show that $\frac{a_{n+1}}{a_n}$ is a nondecreasing sequence.

Observing that $0 < n < n+1 < n+2 < \infty$ and that $2(n+1) = n + (n+2)$ we can use **Obs.3.1** and we get,

$$\|f\|_{n+1}^{n+1} \leq \sqrt{\|f\|_n^n \|f\|_{n+2}^{n+2}}$$

rearranging the terms we get,

$$\frac{a_{n+1}}{a_n} \leq \frac{a_{n+2}}{a_{n+1}}$$

which proves that the sequence $\frac{a_{n+1}}{a_n}$ is nondecreasing, as we wanted. This concludes the proof.

Useful Properties

Theorem 3.1 (Young's Inequality)
Let $a, b \in \mathbb{R}_{\geq 0}$ and let $1 < q, p < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$ab \leq \frac{a^q}{q} + \frac{b^p}{p}$$

Proof. If a or b equal 0 the proof is trivial, so assum they don't. Taking ln on both sides we get

$$\frac{\ln(a^q)}{q} + \frac{\ln(b^p)}{p} \leq \ln\left(\frac{a^q}{q} + \frac{b^p}{p}\right)$$

which given that $\frac{1}{p} + \frac{1}{q} = 1$ holds by the concavity of the logarith and concludes the proof.

Observation 3.1
Let $0 < r < p < s < +\infty$ with $p - r = s - p$ which is the same as $2p = r + s$ then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

Teorem 3.2 Let $\phi : (a, b) \to \mathbb{R}$ be a convex function with $a, b \in [-\infty, \infty]$ then ϕ has lateral derivatives on every point in (a, b).

Proof. By convexity of ϕ for every $a < x < y < b$ and $a < z < b$ we have,

$$\frac{\phi(z) - \phi(x)}{z - x} \leq \frac{\phi(z) - \phi(y)}{z - y} \tag{8}$$

Let $c \in (a,b)$ define $f_c : (a,b) \to \mathbb{R}$ such that

$$f_c\left(x\right) = \begin{cases} \frac{\phi(c)-\phi(x)}{c-x} & \text{if } c \neq x \\ 0 & \text{if } c = x \end{cases}$$

by **(8)** we have that f_c is nondecreasing in (a,c) and in (c,b). With this in mind we can see that the lateral limis of f_c exists in c and equal

$$\lim_{x \to c^-} f_c\left(x\right) = f'_-\left(c\right) = \sup_{a \leq x \leq c} \frac{\phi(c) - \phi(x)}{c - x}$$

$$\lim_{x \to c^+} f_c\left(x\right) = f'_+\left(c\right) = \inf_{c \leq x \leq b} \frac{\phi(c) - \phi(x)}{c - x}$$

This concludes the proof.