Chapter 3 - L^p Spaces

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Exercise 3.

Let's suppose that ϕ is not convex in (a,b) then there exist $x, y \in (a,b)$ and $t_0 \in [0,1]$ such that

$$\phi\left(\left(1-t_0\right)x + t_0 y\right) > \left(1-t_0\right)\phi\left(x\right) + t_0\phi\left(y\right)$$

and assume $x < y$. Let's define $g, h : [0,1] \to R$ such that,

$$g\left(t\right) = \phi\left(\left(1-t\right)x + ty\right)$$

$$h\left(t\right) = \left(1-t\right)\phi\left(x\right) + t\phi\left(y\right)$$

and let $f = h - g$. We define the nonempty sets $A = \{t \in [0, t_0\,) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f\left(0\right) = f\left(1\right) = 0$, and let $t_a = \sup A$ and $t_b = \inf B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in A and B which converge to t_a and t_b respectively, by continuity of f we can see that

$$0 = \lim_{n\to\infty} f\left(a_n\right) = f\left(\lim_{n\to\infty} a_n\right) = f\left(t_a\right)$$

With this and the fact that $t_a < t_0 < t_b$ as $f\left(t_0\right) < 0$ we have that $t_a \in A$ and the proof for $t_b \in B$ is analogous. Now by the definition of t_a and t_b we have that $f\left(t\right) < 0$ for all $t \in (t_a, t_b)$. Finally we take $x' = \left(1-t_a\right)x + t_a y$ and $y' = \left(1-t_b\right)x + t_b y$ and by working algebraically over $f\left(\frac{t_a+t_b}{2}\right) < 0$ we arrive at the following inquality

$$\phi\left(\frac{x'+y'}{2}\right) > \frac{\phi\left(x'\right)+\phi\left(y'\right)}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

Observation This proof shows that to see that convexity holds it sufices to show that given each pair of points $x, y \in (a,b)$ there exists a $t \in [0,1]$ (not necessarily $\frac{1}{2}$) such that $\phi\left(\left(1-t\right)x + ty\right) \leq \left(1-t\right)\phi\left(x\right) + t\phi\left(y\right)$.

Exercise 4. Let f be a complex measurable function on X and μ a positive measura on X define

$$\phi\left(p\right)=\int_X |f|^p\, d\mu=\|f\|_p^p \quad (0<p<\infty)$$

and now let $E=\{p:\phi\left(p\right)<\infty\}$ and assume $0<\|f\|_\infty$. Let's begin by characterizing E. Let $0<p<s<\infty$ and $x\in[0,\infty)$ then $x^p<x^s$ iff $1<x$ and $x^s<x^p$ iff $x<1$. With this in mind if $0<r<p<s<\infty$ then $|f|^p\leq max\{|f|^r,|f|^s\}\leq|f|^r+|f|^s$ and we get,

$$\int_X |f|^p\, d\mu\leq\int_X |f|^r\, d\mu+\int_X |f|^s\, d\mu$$

Suppose that $p,r\in E$ then $p\in E$, which proves statement (a).

As a consequence of (a) we see that E is a connected set and with this in mind, supposing E is nonempty, we can see that E is an interval with endpoints $a=\inf E$ and $b=\sup E$, this interval will be closed or not if E contains them or not, but for sure $E^o=(a,b)$.

To prove that $\log\phi$ is convex in E^o we take $x,y\in(a,b)$ and $t\in[0,1]$. By the properties of the logarithm we get,

$$(1-t)\log\phi\left(x\right)+t\log\phi\left(t\right)=\log\left(\phi\left(x\right)^{1-t}\phi\left(y\right)^t\right)$$

given that the logarithm is a non decreasing function convexity holds if and only if

$$\phi\left((1-t)x+ty\right)\leq\phi\left(x\right)^{1-t}\phi\left(y\right)^t$$

rewriting ϕ on both sides in terms of Lp norms we get the following,

$$\|f^{(1-t)x}f^{ty}\|_1\leq\|f^{(1-t)x}\|_{(1-t)^{-1}}\|f^{ty}\|_{t^{-1}}$$

finally, given that $1\leq(1-t)^{-1}$ and $1\leq t^{-1}$ are conjugate exponents, the last inequallity hold by Holder's inequality and this concludes the proof.

Observation

Let $0<r<p<s<+\infty$ with $p-r=s-p$ which is the same as $2p=r+s$ then using Holder's inequality we get,

$$\|f\|_p^p=\|f^p\|_1=\|f^{\frac{r}{2}}f^{\frac{s}{2}}\|_1\leq\|f^{\frac{r}{2}}\|_2\|f^{\frac{s}{2}}\|_2=(\|f\|_r^r\|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p\leq\sqrt{\|f\|_r^r\|f\|_s^s}$$

Exercise 10. Given that $fg \geq 1$ and both f and g are positive we have $f \geq \frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega) = 1$,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = \left(\|g\|_1 \|g^{-1}\|_1\right)^{\frac{1}{2}} \leq (\|g\|_1 \|f\|_1)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_\Omega f \, d\mu \int_\Omega g \, d\mu$$