

## Chapter 3 - $L^p$ Spaces

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### Exercise Solutions

**Exercise 3.3.** Let's suppose that  $\phi$  is not convex in  $(a, b)$  then there exist  $x, y \in (a, b)$  and  $t_0 \in [0, 1]$  such that

$$\phi((1 - t_0)x + t_0y) > (1 - t_0)\phi(x) + t_0\phi(y)$$

Define  $g, h : [0, 1] \rightarrow \mathbb{R}$  such that,

$$g(t) = \phi((1 - t)x + ty)$$

$$h(t) = (1 - t)\phi(x) + t\phi(y)$$

and let  $f = h - g$ . We define the nonempty sets  $A = \{t \in [0, t_0] \mid f(t) = 0\}$  and  $B = \{t \in (t_0, 1] \mid f(t) = 0\}$  given that  $f(0) = f(1) = 0$ , and let  $t_a = \sup A$  and  $t_b = \inf B$ , we want to see that  $t_a \in A$  and  $t_b \in B$ . Let  $\{a_n\}$  and  $\{b_n\}$  be sequences in  $A$  and  $B$  which converge to  $t_a$  and  $t_b$  respectively, by continuity of  $f$  we can see that

$$0 = \lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right) = f(t_a)$$

With this and the fact that  $t_a < t_0 < t_b$  as  $f(t_0) < 0$  we have that  $t_a \in A$  and the proof for  $t_b \in B$  is analogous. Now by the definition of  $t_a$  and  $t_b$  and continuity of  $f$  we have that  $f(t) < 0$  for all  $t \in (t_a, t_b)$ . Finally we take  $x' = (1 - t_a)x + t_a y$  and  $y' = (1 - t_b)x + t_b y$  and by working algebraically over  $f\left(\frac{t_a + t_b}{2}\right) < 0$  we arrive at the following inequality

$$\phi\left(\frac{x' + y'}{2}\right) > \frac{\phi(x') + \phi(y')}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

**Observation** This proof shows that to see that convexity holds it suffices to show that given each pair of points  $x, y \in (a, b)$  there exists a  $t \in [0, 1]$  (not necessarily  $\frac{1}{2}$ ) such that  $\phi((1 - t)x + ty) \leq (1 - t)\phi(x) + t\phi(y)$ .

**Exercise 3.4.** Let  $f$  be a complex measurable function on  $X$  and  $\mu$  a positive measure on  $X$ , define

$$\phi(p) = \int_X |f|^p d\mu = \|f\|_p^p \quad (0 < p < \infty)$$

and now let  $E = \{p : \phi(p) < \infty\}$  and assume  $0 < \|f\|_\infty$ . Let's begin by characterizing  $E$ . Let  $0 < p < s < \infty$  and  $x \in [0, \infty)$  then  $x^p < x^s$  iff  $1 < x$  and  $x^s < x^p$  iff  $x < 1$ . With this in mind if  $0 < r < p < s < \infty$  then  $|f|^p \leq \max\{|f|^r, |f|^s\} \leq |f|^r + |f|^s$  and we get,

$$\int_X |f|^p d\mu \leq \int_X |f|^r d\mu + \int_X |f|^s d\mu$$

Suppose that  $s, r \in E$  then  $p \in E$ , which proves statement (a).

As a consequence of (a) we see that  $E$  is a connected set and with this in mind, supposing  $E$  is nonempty, we can see that  $E$  is an interval with endpoints  $a = \inf E$  and  $b = \sup E$ , this interval will be closed or not if  $E$  contains them or not, but for sure  $E^o = (a, b)$ .

To prove that  $\log \phi$  is convex in  $E^o$  we start by proving that the composition is well defined, in other words,  $\phi$  is never zero. If  $\phi(p) = 0$  for some  $p \in E^o$  then  $|f| = 0$  a.e. on  $X$  and this implies  $\|f\|_\infty = 0$  contradicting our assumption, thus the composition is well defined and we proceed to prove convexity. Take  $x, y \in (a, b)$  and  $t \in (0, 1)$ . By the properties of the logarithm we get,

$$(1-t)\log \phi(x) + t\log \phi(y) = \log \left( \phi(x)^{1-t} \phi(y)^t \right)$$

given that the logarithm is a nondecreasing function, convexity holds if and only if

$$\phi((1-t)x + ty) \leq \phi(x)^{1-t} \phi(y)^t$$

rewriting  $\phi$  on both sides in terms of  $L_p$  norms we get the following,

$$\|f^{(1-t)x + ty}\|_1 \leq \|f^{(1-t)x}\|_{(1-t)^{-1}} \|f^{ty}\|_{t^{-1}}$$

Given that  $1 \leq (1-t)^{-1}$  and  $1 \leq t^{-1}$  are conjugate exponents, the last inequality holds by Holder's inequality and this concludes the proof.

To prove that  $\phi$  is continuous in  $E$  we start by noticing that  $\log \phi$  is convex in  $(a, b)$  which implies convexity of  $\phi$  in  $(a, b)$  which in turn implies continuity of  $\phi$  in  $(a, b)$ , so we only need to prove continuity on  $a$  and  $b$ , in case they are elements of  $E$ .

Let  $\{p_n\}$  be a sequence in  $(a, b)$  which converges to  $a$  and let's suppose  $a \in E$ , we will show that  $\phi(p_n)$  converges to  $\phi(a)$ , the proof for  $b$  is analogous. By linearity of the integral and properties of the absolute value we get

$$|\phi(p_n) - \phi(a)| = \left| \int_X |f^{p_n}| - |f^a| d\mu \right| \leq \int_X ||f^{p_n}| - |f^a|| d\mu \leq \int_X |f^{p_n} - f^a| d\mu$$

Also, by continuity of the exponential we know that  $f^{p_n}$  converges pointwise to  $f^a$  as  $n$  goes to infinity so it suffices to find a real function  $g \in L^1(\mu)$  such that  $|f^{p_n}| \leq g$ . Let  $g = |f^a| + |f^{p_M}|$  with  $p_M$  the greatest element of the sequence  $\{p_n\}$  which exists given the convergence of the sequence. Then we can see that  $a \leq p_n \leq p_M$  for all  $1 \leq n$  and as mentioned above we have

$$|f|^{p_n} \leq \max\{|f|^a, |f|^{p_M}\} \leq |f|^a + |f|^{p_M} = g$$

Finally by Lebesgue's Dominated Convergence Theorem we get that

$$\int_X |f^{p_n} - f^a| d\mu \rightarrow 0$$

which concludes the proof of (b).

**Exercise 3.5.** In order to prove that  $\|f\|_p \leq \|f\|_r$  if  $0 < p < r \leq \infty$  we start by looking at case in which both  $p$  and  $r$  are finite. For every  $x \in (0, \infty)$  the function  $x^c$  is twice differentiable and we have that

$$(x^c)'' = (x^{c-1}c)' = c(c-1)x^{c-2}$$

Observe that if  $1 \leq c$ , the expression above is nonnegative for all  $x \in (0, \infty)$  and thus  $x^c$  convex over that interval. With this in mind and the fact that  $\mu(\Omega) = 1$  and  $1 \leq \frac{r}{p}$  we will use Jensen's Inequality to prove the result. Let  $A = \{x \in \Omega | f(x) \neq 0\}$  (the integral over its complement is exactly 0 and in this way  $|f|(A) \subset (0, \infty)$ ), then

$$\|f\|_p = \left( \int_A |f|^p d\mu \right)^{\frac{r}{rp}} \leq \left( \int_A |f|^{\frac{rp}{p}} d\mu \right)^{\frac{1}{r}} = \|f\|_r$$

and this concludes the proof for the finite case.

Now let  $r = \infty$ , this case is much simpler. By definition of the essential supremum we have that  $|f| \leq \|f\|_\infty$  almost everywhere and using the fact that  $\mu(\Omega) = 1$  we get,

$$\|f\|_p = \left( \int_\Omega |f|^p d\mu \right)^{\frac{1}{p}} \leq \left( \int_\Omega \|f\|_\infty^p d\mu \right)^{\frac{1}{p}} = \|f\|_\infty$$

**Exercise 3.7.** Let  $0 < p < s < \infty$ , an example for the inclusion  $L^s(\mu) \subset L^p(\mu)$  can easily be constructed using **Ex.3.5**.

Let  $X = (1, \infty)$  and  $m$  the Lebesgue Measure, we see that neither of the inclusions hold. On the one hand,  $\frac{1}{x} \notin L^1(1, \infty)$  and  $\frac{1}{x} \in L^2(1, \infty)$ . On the other hand,  $\frac{I_{(1,2)}}{x-1} \notin L^1(1, \infty)$  and  $\frac{I_{(1,2)}}{x-1} \in L^{\frac{1}{2}}(1, \infty)$  being  $I_A$  the indicator function of  $A$ . We will only prove the first of these last two statements (the second one can be proved in a similar manner choosing a convenient partition of  $(1, 2)$ ),

$$\int_1^\infty \frac{I_{(1,2)}}{x-1} dx = \int_1^2 \frac{1}{x-1} dx = \sum_{n=0}^\infty \int_{1+\frac{1}{e^{n+1}}}^{1+\frac{1}{e^n}} \frac{1}{x-1} dx$$

given that the antiderivative of  $\frac{1}{x-1}$  is  $\ln(x-1) + C$  the expression above equals

$$\sum_{n=0}^{\infty} \left[ \ln \left( \frac{1}{e^n} \right) - \ln \left( \frac{1}{e^{n+1}} \right) \right] = \sum_{n=0}^{\infty} \ln(e) = \infty$$

For the remaining inclusion we take  $X = \mathbb{N}$  with the Counting Measure over the power set of  $\mathbb{N}$ . Observe that all measurable functions in this case will be just sequences of complex numbers so if  $f = \{a_n\}$  we have,

$$\|f\|_p^p = \int_X |f|^p d\mu = \sum_{n=1}^{\infty} |a_n|^p$$

Suppose  $\{a_n\} \in \ell^p(\mathbb{N})$  then the sequence  $\{|a_n|^p\}$  converges to 0 and thus there exists a natural number  $M$  such that  $|a_n|^p < 1$  for all  $M < n$ . Given that  $1 < \frac{s}{p}$  we have that  $|a_n|^s = (|a_n|^p)^{\frac{s}{p}} < |a_n|^p$  for all  $n$  greater than  $M$  and with that in mind we get,

$$\sum_{n=1}^{\infty} |a_n|^s = \sum_{n=1}^M |a_n|^s + \sum_{n=M+1}^{\infty} |a_n|^s < \sum_{n=1}^M |a_n|^s + \sum_{n=M+1}^{\infty} |a_n|^p < \infty$$

As we wanted,  $\{a_n\} \in \ell^s(\mathbb{N})$ , which concludes the proof.

**Exercise 3.10.** Given that  $fg \geq 1$  and both  $f$  and  $g$  are positive we have  $f \geq \frac{1}{g}$  and using Holder's inequality and the fact that  $\mu(\Omega) = 1$ ,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = (\|g\|_1 \|g^{-1}\|_1)^{\frac{1}{2}} \leq (\|g\|_1 \|f\|_1)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_{\Omega} f d\mu \int_{\Omega} g d\mu$$

**Exercise 3.17.** We start by looking at the case in which  $1 \leq p < \infty$ , we want to see that

$$|\alpha - \beta|^p \leq 2^{p-1} (|\alpha|^p + |\beta|^p) \quad (1)$$

for every  $\alpha$  and  $\beta$  complex numbers not both zero (in that case the proof is trivial). Let  $X = [0, 1]$  and  $\mu$  be the Lebesgue Measure, we define the complex function  $f$  over  $[0, 1]$  such that

$$f(x) = \begin{cases} 2\alpha & \text{if } 0 \leq x < \frac{1}{2} \\ 2\beta & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Rearranging (1) we get,

$$|\alpha - \beta| \leq \left( \frac{1}{2} |2\alpha|^p + \frac{1}{2} |2\beta|^p \right)^{\frac{1}{p}} = \|f\|_p \quad (2)$$

Observing that  $\mu(X) = 1$  and  $0 < \|f\|_\infty$  by **Ex.3.5** we know that  $\|f\|_1 \leq \|f\|_p$  and we have

$$|\alpha - \beta| \leq |\alpha| + |\beta| = \|f\|_1 \leq \|f\|_p \quad (3)$$

which concludes the proof.

**Exercise 3.18.** Let's start with statement **(a)**, for it we will assume that  $\mu(X) < \infty$ . Let  $A$  be a set in which  $\{f_n\}_{n \in \mathbb{N}}$  converges pointwise to  $f$  and  $\mu(X - A) = 0$  take  $\epsilon > 0$ . By Egorov's theorem there exists a set  $B \subset A$  in which the sequence converges uniformly and  $\mu(A - B) < \epsilon$ . By the uniform convergence of the sequence there exists  $N \in \mathbb{N}$  such that

$$|f_n(x) - f(x)| < \epsilon \quad \forall x \in B \quad \forall n > N$$

Given that the set of all points which don't satisfy this condition for  $n$  large enough is a subset of  $X - B$  its measure must be less than  $\epsilon$  which proves that  $\{f_n\}_{n \in \mathbb{N}}$  converges in measure to  $f$  and concludes the proof.

Now let's prove statement **(b)**, we won't need the assumption that  $\mu(X) < \infty$  for this one. Assume that  $\{f_n\}_{n \in \mathbb{N}}$  doesn't converge in measure to  $f$ , we start by looking at the case in which  $0 \leq p < \infty$ . By our assumption, there exists  $\epsilon > 0$  such that for every  $N \in \mathbb{N}$  there exists an  $n > N$  such that

$$\mu(\{x \in X : |f_n(x) - f(x)| > \epsilon\}) \geq \epsilon \quad (4)$$

Now let  $E_{\epsilon, n}$  denote the set in (4), keeping notation as above we have

$$\epsilon^{p+1} \leq \int_{E_{\epsilon, n}} \epsilon^p d\mu \leq \int_{E_{\epsilon, n}} |f_n - f|^p d\mu \leq \int_X |f_n - f|^p d\mu$$

and clearly  $\|f_n - f\|_p$  doesn't converge to zero which is a contradiction and concludes the proof for  $1 \leq p < \infty$  (the case in which  $p = \infty$  uses the same idea and won't be proven here). It calls my attention that we didn't need the

fact that  $f_n \in L^p(\mu)$  and makes me believe that there must be a mistake in the proof...

It only remains to prove (c), we won't need the assumption that  $\mu(X) < \infty$ . We start by taking a subsequence  $\{f_{n_j}\}_{j \in \mathbb{N}}$  such that

$$\mu(E_{2^{-j}, n_j}) < 2^{-j} \quad (5)$$

with  $E_{\epsilon, n}$  defined as in (4). Observe that for every  $\epsilon > 0$  and every  $\delta > 0$  we can ask for a  $m$  large enough such that,

$$\mu(A_m) := \mu\left(\bigcup_{j=m}^{\infty} E_{2^{-j}, n_j}\right) \leq \sum_{j=m}^{\infty} \mu(E_{2^{-j}, n_j}) < \epsilon \quad (6)$$

and

$$2^{-j} < \delta \quad \forall j \geq m \quad (7)$$

As defined in (6) we clearly have  $A_{m+1} \subset A_m$ . Now suppose that the sequence  $\{f_{n_j}\}_{j \in \mathbb{N}}$  doesn't converge pointwise to any point in a set  $B \subset X$  with  $\mu(B) > 0$  we can ask for an  $m$  large enough such that (6) holds for  $\epsilon = \min\{1, \mu(B)\}$  (we can't just pick  $\mu(B)$  as it could equal  $\infty$ ). In that case  $B$  won't be a subset of  $A_m$  and we can ask for a point  $x \in B - A_m$ .

Now take  $\gamma > 0$  and  $m' \geq m$  such that (7) holds for  $\delta = \gamma$  then  $x \in B - A_{m'}$  as  $A_{m'} \subset A_m$ . Given that if  $|f_{n_j}(y) - f(y)| > 2^{-m'}$  for some  $j \geq m'$  we have  $y \in A_{m'}$ , as  $x \notin A_{m'}$  it must be the case that for all  $j \geq m'$

$$|f_{n_j}(x) - f(x)| \leq 2^{-m'} < \delta = \gamma$$

and then we have that  $f_{n_j}(x) \rightarrow f(x)$  which contradicts our assumption and concludes the proof.

**Exercise 3.20.** Let  $\alpha, \beta \in \mathbb{R}$  and  $t \in [0, 1]$  we define

$$f(x) = \begin{cases} \alpha & \text{if } 0 \leq x < t \\ \beta & \text{if } t \leq x \leq 1 \end{cases}$$

which is clearly measurable and bounded. It only remains to replace  $f$  in the given inequality to prove the convexity of  $\phi$ .

**Exercise 3.23.** Let  $\mu$  be a positive measure with  $\mu(X) < \infty$  and  $f$  a complex measurable function with  $0 < \|f\|_{\infty} < \infty$  define the sequence

$$a_n = \int_X |f|^n d\mu = \|f\|_n^n$$

By the given hypothesis we have that first, the hypothesis of **Ex.3.4.e** hold and second,  $0 < a_n < \infty$  for every  $1 \leq n$  and thus the quotients  $\frac{a_{n+1}}{a_n}$  are well defined and we have the inequality,

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} a_n^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} a_n^{\frac{1}{n}} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$$

Looking at **Ex.3.4.e** we see that it suffices to show that  $\frac{a_{n+1}}{a_n}$  converges in  $[0, \infty]$  because in that case both limits will exist and equal  $\|f\|_\infty$  as  $a_n^{\frac{1}{n}} = \|f\|_n$ . In order to prove this, we will show that  $\frac{a_{n+1}}{a_n}$  is a nondecreasing sequence.

Observing that  $0 < n < n+1 < n+2 < \infty$  and that  $2(n+1) = n + (n+2)$  we can use **Obs.3.1** and we get,

$$\|f\|_{n+1}^{n+1} \leq \sqrt{\|f\|_n \|f\|_{n+2}^{n+2}}$$

rearranging the terms we get,

$$\frac{a_{n+1}}{a_n} \leq \frac{a_{n+2}}{a_{n+1}}$$

which proves that the sequence  $\frac{a_{n+1}}{a_n}$  is nondecreasing, as we wanted. This concludes the proof.

## Useful Properties

### Theorem 3.1 (Young's Inequality)

Let  $a, b \in \mathbb{R}_{\geq 0}$  and let  $1 < q, p < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$  we have

$$ab \leq \frac{a^q}{q} + \frac{b^p}{p}$$

*Proof.* If  $a$  or  $b$  equal 0 the proof is trivial, so assume they don't. Taking  $\ln$  on both sides we get

$$\frac{\ln(a^q)}{q} + \frac{\ln(b^p)}{p} \leq \ln\left(\frac{a^q}{q} + \frac{b^p}{p}\right)$$

which given that  $\frac{1}{p} + \frac{1}{q} = 1$  holds by the concavity of the logarithm and concludes the proof.

### Observation 3.1

Let  $0 < r < p < s < +\infty$  with  $p - r = s - p$  which is the same as  $2p = r + s$  then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

**Theorem 3.2** Let  $\phi : (a, b) \rightarrow \mathbb{R}$  be a convex function with  $a, b \in [-\infty, \infty]$  then  $\phi$  has lateral derivatives on every point in  $(a, b)$ .

*Proof.* By convexity of  $\phi$  for every  $a < x < y < b$  and  $a < z < b$  we have,

$$\frac{\phi(z) - \phi(x)}{z - x} \leq \frac{\phi(z) - \phi(y)}{z - y} \quad (8)$$

Let  $c \in (a, b)$  define  $f_c : (a, b) \rightarrow \mathbb{R}$  such that

$$f_c(x) = \begin{cases} \frac{\phi(c) - \phi(x)}{c - x} & \text{if } c \neq x \\ 0 & \text{if } c = x \end{cases}$$

by (8) we have that  $f_c$  is nondecreasing in  $(a, c)$  and in  $(c, b)$ . With this in mind we can see that the lateral limits of  $f_c$  exists in  $c$  and equal

$$\lim_{x \rightarrow c^-} f_c(x) = f'_-(c) = \sup_{a \leq x \leq c} \frac{\phi(c) - \phi(x)}{c - x}$$

$$\lim_{x \rightarrow c^+} f_c(x) = f'_+(c) = \inf_{c \leq x \leq b} \frac{\phi(c) - \phi(x)}{c - x}$$

This concludes the proof.

### **Corollary**

Let  $\phi : (a, b) \rightarrow \mathbb{R}$  with  $a, b \in [-\infty, \infty]$  a convex function then  $\phi$  is continuous over  $(a, b)$ .