Chapter 3 - L^p Spaces

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Exercise 3.

Let's assume that ϕ is not convex in (a, b) then there exist $x, y \in (a, b)$ and $t_0 \in [0, 1]$ such that

$$\phi\left(\left(1-t_0\right)x+t_0 y\right)>\left(1-t_0\right)\phi\left(x\right)+t_0\phi\left(y\right)$$

and assume $x < y$. Let's define $g, h : [0, 1] \to R$ such that,

$$g\left(t\right)=\phi\left(\left(1-t\right)x+ty\right)$$

$$h\left(t\right)=\left(1-t\right)\phi\left(x\right)+t\phi\left(y\right)$$

and let $f = h - g$. We define the nonempty sets $A = \{t \in [0, t_0\,) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f\left(0\right) = f\left(1\right) = 0$, and let $t_a = \sup A$ and $t_b = \inf B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in A and B which converge to t_a and t_b respectively, by continuity of f we can see that

$$0=\lim_{n\to\infty}f\left(a_n\right)=f\left(\lim_{n\to\infty}a_n\right)=f\left(t_a\right)$$

then $t_a \in A$ and the proof for $t_b \in B$ is analogous. As $f\left(t_0\right) < 0$ then $t_a < t_0 < t_b$ and by continuity of f and the definition of t_a and t_b we have that $f\left(t\right) < 0$ for all $t \in (t_a, t_b)$. Finally we take $x' = \left(1 - t_a\right)x + t_a y$ and $y' = \left(1 - t_b\right)x + t_b y$ and by working algebraically over $f\left(\frac{t_a+t_b}{2}\right) < 0$ we arrive at the following inquality

$$\phi\left(\frac{x'+y'}{2}\right)>\frac{\phi\left(x'\right)+\phi\left(y'\right)}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

Observation This proof shows that to see that convexity holds it sufices to show that given each pair of points $x, y \in (a, b)$ there exists a $t \in [0, 1]$ (not necessarily $\frac{1}{2}$) such that $\phi\left(\left(1-t\right)x+ty\right) \le \left(1-t\right)\phi\left(x\right)+t\phi\left(y\right)$.

Exercise 4

Observation

Let $0 < r < p < s < +\infty$ with $p - r = s - p$ which is the same as $2p = r + s$ then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

Exercise 10. Given that $fg \geq 1$ and both f and g are positive we have $f \geq \frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega) = 1$,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = \left(\|g\|_1 \|g^{-1}\|_1\right)^{\frac{1}{2}} \leq (\|g\|_1 \|f\|_1)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_\Omega f \, d\mu \int_\Omega g \, d\mu$$