# Chapter 4 - Elementary Hilbert Space Theory

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## Exercise Solutions

**Exercise 4.** We start by assuming that $\{u_n\}_{n\in\mathbb{N}}$ is a countable maximal orthonormal system in $H$, we want to see that $H$ is separable. Observe that it sufices to find a countable set $A$ such that $P \subset \overline{A}$, with $P$ the set of all finite linear combinations of elements in $\{u_n\}_{n\in\mathbb{N}}$. Define the set $A$ to be,

$$A = \left\{ \sum_{n=1}^{N} (q_n + ip_n)\, u_n : q_n, p_n \in \mathbb{Q},\ N \geq 1 \right\}$$

which is clearly countable. Take $x \in P$, we have $x = \sum_{n=1}^{N} c_n u_n$ for some $c_n$ complex numbers and some $N \geq 1$ . Observe that for every $1 \leq n \leq N$ there exist $\{q_{nk}\}_{k\in\mathbb{N}}$ and $\{p_{nk}\}_{k\in\mathbb{N}}$ sequences of rational numbers such that $(q_{nk} + ip_{nk}) \to c_n$ as $k$ goes to infinity. Then let $\epsilon > 0$ we can ask for a $k$ large enough such that $|\,(q_{nk} + ip_{nk}) - c_n| < \frac{\epsilon}{N}$ for all $1 \leq n \leq N$, we in turn have

$$\| \sum_{n=1}^{N} (q_{nk} + ip_{nk})\, u_n - \sum_{n=1}^{N} c_n u_n \| \leq \sum_{n=1}^{N} |\,(q_{nk} + ip_{nk}) - c_n| < \sum_{n=1}^{N} \frac{\epsilon}{N} < \epsilon$$

Given that every element in $P$ can be aproximated by elements in $A$ we have that, $P \subset \overline{A}$, which concludes the proof.

Now suppose that $\{x_n\}_{n\in\mathbb{N}}$ is countable and dense in $H$, we are going to build a countable maximal orthonormal system. Let $A$ be the set such that

$$\begin{cases} x_1 \in A \\ x_{n+1} \in A \text{ iff } x_{n+1} \notin [x_1, ..., x_n] \end{cases}$$

where $[x_1, ..., x_n]$ denotes the span of $\{x_1, ..., x_n\}$, let's see that $A$ is linearly independent. Suppose that for sume $v_1, ..., v_{m+1}$ in $A$ and $\alpha_1, ..., \alpha_m$ nonzero complex numbers we have that

$$\sum_{j=1}^{m} \alpha_j v_j = v_{m+1}$$

Every $v_j$ can be expressed as $x_{n_j}$, so let $v_k$ be such that $n_k$ is the greates index between all $n_j$, we then have

$$v_k = \frac{v_{m+1}}{\alpha_k} - \sum_{1 \leq n \neq k \leq m} \frac{\alpha_n v_n}{\alpha_k}$$

which clashes with the construction of A, thus A is linearly independent. By **Ex4.3** we can build from A a countable orthnormal set $\{u_n\}$ such that $[v_1, ..., v_n] = [u_1, ..., u_n]$ for all $n \geq 1$. Given that every $x_n$ can be expressed as a linear combination of elements in $A$, one can see that every $x_n$ will be in $[u_1, ..., u_m]$ for some $m$. Then we can see that every $x_n$ is an element of $\overline{P}$, with P the set of finite linear combinations of elements in $\{u_n\}$, which in turn imples that $P$ is dense in $H$ and concludes the proof.

## Useful Properties