Chapter 4 - Elementary Hilbert Space Theory

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Exercise Solutions

Exercise 4.4. We start by assuming that $\{u_n\}_{n\in\mathbb{N}}$ is a countable maximal orthonormal system in H, we want to see that H is separable. Observe that it sufices to find a countable set A such that $P \subset \overline{A}$, with P the set of all finite linear combinations of elements in $\{u_n\}_{n\in\mathbb{N}}$. Define the set A to be,

$$A = \left\{ \sum_{n=1}^{N} (q_n + ip_n)\, u_n : q_n, p_n \in \mathbb{Q},\ N \geq 1 \right\}$$

which is clearly countable. Take $x \in P$, we have $x = \sum_{n=1}^{N} c_n u_n$ for some c_n complex numbers and some $N \geq 1$. Observe that for every $1 \leq n \leq N$ there exist $\{q_{nk}\}_{k\in\mathbb{N}}$ and $\{p_{nk}\}_{k\in\mathbb{N}}$ sequences of rational numbers such that $(q_{nk} + ip_{nk}) \to c_n$ as k goes to infinity. Then let $\epsilon > 0$ we can ask for a k large enough such that $|\,(q_{nk} + ip_{nk}) - c_n| < \frac{\epsilon}{N}$ for all $1 \leq n \leq N$, we in turn have

$$\| \sum_{n=1}^{N} (q_{nk} + ip_{nk})\, u_n - \sum_{n=1}^{N} c_n u_n \| \leq \sum_{n=1}^{N} |\,(q_{nk} + ip_{nk}) - c_n| < \sum_{n=1}^{N} \frac{\epsilon}{N} < \epsilon$$

Given that every element in P can be aproximated by elements in A we have that, $P \subset \overline{A}$, which concludes the proof.

Now suppose that $\{x_n\}_{n\in\mathbb{N}}$ is countable and dense in H, we are going to build a countable maximal orthonormal system. Let A be the set such that

$$\begin{cases} x_1 \in A \\ x_{n+1} \in A \text{ iff } x_{n+1} \notin [x_1, ..., x_n] \end{cases}$$

where $[x_1, ..., x_n]$ denotes the span of $\{x_1, ..., x_n\}$, let's see that A is linearly independent. Suppose that for some $v_1, ..., v_{m+1}$ in A and $\alpha_1, ..., \alpha_m$ nonzero complex numbers we have that

$$\sum_{j=1}^{m} \alpha_j v_j = v_{m+1}$$

Every v_j can be expressed as x_{n_j}, so let v_k be such that n_k is the greates index between all n_j, we then have

$$v_k = \frac{v_{m+1}}{\alpha_k} - \sum_{1 \leq n \neq k \leq m} \frac{\alpha_n v_n}{\alpha_k}$$

which clashes with the construction of A, thus A is linearly independent. By **Ex4.3** we can build from A a countable orthnormal set $\{u_n\}$ such that $[v_1, ..., v_n] = [u_1, ..., u_n]$ for all $n \geq 1$. Given that every x_n can be expressed as a linear combination of elements in A, one can see that every x_n will be in $[u_1, ..., u_m]$ for some m. Then we can see that every x_n is an element of $\overline{P}$, with P the set of finite linear combinations of elements in $\{u_n\}$, which in turn imples that P is dense in H and concludes the proof.

Exercise 4.6. Let $U = \{u_n\}_{n \in \mathbb{N}}$ be an orthonormal set in H, let's prove the assertion **(a)**. Boundedness of U is more than clear and closedness and non-compacntess can be easily derived from that fact that $\|u_n - u_m\| = \sqrt{2}$ for all $1 \leq n < m$.

To prove assertion **(b)** (we go directly with the general case), we start by supposing that $\sum_{n=1}^{\infty} \delta_n^2 < \infty$ and proving that S is compact. In ordert to do this, let $\{x_n\}_{n \in \mathbb{N}}$ be a sequence in S, we will construct a convergent subsequence.

- Let $y_{n1} = \hat{x}_n(1)$, notation as in the book. Given that the closed disk $\overline{B}(0, \delta_1)$ is compact and $y_{n1} \in \overline{B}(0, \delta_1)$, we can find $y_{n_k 1}$ a convergent subsequence of y_{n1} which converges to a y_1 in that disk. We replace our original sequence x_n for the new subsequence x_{n_k}, define $z_1 = x_{n_1}$ and proceed be repeating what we just did but for the second terms (we also rename the new sequence x_{n_k} as x_n for the sake of simplicity).
- Notation as above we can find a convergent subsequence $y_{n_k 2}$ of the sequence of second terms which converges in $\overline{B}(0, \delta_2)$ to a y_2 in that disk. We again define $z_2 = x_{n_1}$ and replace x_n by it's subsequence x_{n_k}. Observe that the sequence of first terms still converges as it's a subsequence of a convergent sequence.

Repeating this process an arbirary number of times we build a sequence z_n in S and a sequence of numbers y_n in $\overline{B}(0, \delta_n)$ for each n. Given that $0 \leq |y_n| \leq \delta_n$ for all $n \geq 1$ we have that $\sum_{n=1}^{\infty} |y_n|^2 < \infty$ and thus there exists a $y \in S$ such that $\hat{y}(n) = y_n$ for all $n \geq 1$. By the definition of the sequences z_n and y_n, given $\epsilon > 0$ we can find $N, M \in \mathbb{N}$ large enough such that,

$$\sum_{n=N}^{\infty} |\hat{z}_m(n) - y_n|^2 \leq \sum_{n=N}^{\infty} 4\delta_n^2 < \frac{\epsilon}{2} \tag{1}$$

$$|\hat{z}_m(n) - y_n|^2 < \frac{\epsilon}{2^{n+1}} \quad \forall\ 1 \leq n < N \tag{2}$$

for all $m \geq M$. With this in mind we can see that $\sum_{n=1}^{\infty} |\hat{z}_m(n) - y_n|^2 = \|z_m - y\|^2 < \epsilon$ for all $m \geq M$ which shows that z_n converges to y in S and concludes the proof.

Now suppose that S is compact and that $\sum_{n=1}^{\infty} \delta_n^2 = \infty$, we will get to a contradiction. Define $x_k = \sum_{n=1}^{\infty} c_{kn} u_n$ with

$$c_{kn} = \begin{cases} \delta_n & \text{if } n \leq k \\ 0 & \text{else} \end{cases}$$

Clearly $x_k \in S$ for all $k \geq 1$ and $\|x_k\|^2 \to \sum_{n=1}^{\infty} \delta_n^2 = \infty$ but given that $\|x_k\| \leq \|x_k - x\| + \|x\|$ for every $x \in S$ no subsequence of x_k can converge in S which implies that S is not compact, contradicting our assumption and concluding the proof.

Finally to prove assertion **(c)** we observe that $\frac{r}{2} u_n \in B(0, r)$ for all $r > 0$ and all $n \geq 1$. With this in mind, and the fact that $\|u_n\|$ has no covergent subsequence, the sequences $\|\frac{r}{2} u_n\|$ have no convergent subsequence in $\overline{B}(0, r)$ which in turn shows that 0 has no neighbourhood with compact closure in H which concludes the proof.

Exercise 11. We claim that the set $E = \{f_n\}_{n \in \mathbb{N}}$ with $f_n(x) = \sin(nx) I_{[0, \pi + \frac{\pi}{n}]}$ (where I_A stands for the indicator function of A) is closed in $L^2(T)$ and has no element of smallest norm. It can be easily verified that E is a subset of $L^2(T)$, we proceed to prove the other assertions.

We start by proving that E is closed, to do this we will prove that the sequence $\{f_n\}_{n \in \mathbb{N}}$ has no convergent subsequence in $L^2(T)$ and thus E equals it's closure. Before starting the proof, observe that most properties of $L^2(\mu)$ spaces hold for $L^2(T)$ as their norms are just a scalar multiplication appart, we won't give the proofs for them here. Suppose that there exists a subsequence $\{f_{n_j}\}_{j \in \mathbb{N}}$ convergent in $L^2(T)$, by Th3.12 this subsequence has a subsequence $\{f_{n_k}\}_{k \in \mathbb{N}}$ which converges pointwise a.e in $[0, 2\pi]$. Given that $f_n(x) = \sin(nx)$ for all $x \in [0, \pi]$, this conclusion contradicts the result obtained in Ex4.10 saying that the set in which $\sin(n_k x)$ converges has null measure and proves our assertion.

Now, to prove that E has no element of smallest norm let's prove that $\inf_{n \geq 1} \|f_n\|_2 = \frac{1}{2}$ and that this infimum is not attained in E. To do this we start by observing that

$$\left\{ \frac{1}{2\pi} \int_0^{\pi} \sin^2(nx)\ dx \right\}^{\frac{1}{2}} = \frac{1}{2} \tag{3}$$

for all $n \in \mathbb{N}$ and that $f_n(x) = \sin(nx) I_{[0, \pi]} + \sin(nx) I_{(\pi, \pi + \frac{\pi}{n}]}$. Now using (3) we can write

$$\|f_n\|_2 = \left\{ \frac{1}{2\pi} \int_0^{\pi} \sin^2(nx)\ dx + \frac{1}{2\pi} \int_{\pi}^{\pi + \frac{\pi}{n}} \sin^2(nx)\ dx \right\}^{\frac{1}{2}} > \frac{1}{2}$$

which shows that $\frac{1}{2}$ is smaller than the norm of every element in E. With this in mind and the fact that $\sin(x)$ is a bounded function using Minkowski's inequality we get,

$$0 < \|f_n\|_2 - \frac{1}{2} \leq \|\sin(nx)\, I_{(\pi,\pi+\frac{\pi}{n}]}\|_2 \to 0$$

This final assertion shows that E has no element of smallest norm which concludes the proof.

Useful Properties