Chapter 3 - L^p Spaces

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Exercise 3. Let's suppose that ϕ is not convex in (a, b) then there exist $x, y \in (a, b)$ and $t_0 \in [0, 1]$ such that

$$\phi\left((1-t_0)\, x + t_0 y\right) > (1-t_0)\, \phi(x) + t_0 \phi(y)$$

Define $g, h : [0, 1] \to \mathbb{R}$ such that,

$$g(t) = \phi\left((1-t)\, x + ty\right)$$

$$h(t) = (1-t)\, \phi(x) + t\phi(y)$$

and let $f = h - g$. We define the nonempty sets $A = \{t \in [0, t_0\,) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f(0) = f(1) = 0$, and let $t_a = \sup A$ and $t_b = \inf B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in A and B which converge to t_a and t_b respectively, by continuity of f we can see that

$$0 = \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(t_a)$$

With this and the fact that $t_a < t_0 < t_b$ as $f(t_0) < 0$ we have that $t_a \in A$ and the proof for $t_b \in B$ is analogous. Now by the definition of t_a and t_b we have that $f(t) < 0$ for all $t \in (t_a, t_b)$. Finally we take $x' = (1-t_a)\, x + t_a y$ and $y' = (1-t_b)\, x + t_b y$ and by working algebraically over $f\left(\frac{t_a+t_b}{2}\right) < 0$ we arrive at the following inquality

$$\phi\left(\frac{x'+y'}{2}\right) > \frac{\phi(x') + \phi(y')}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

Observation This proof shows that to see that convexity holds it sufices to show that given each pair of points $x, y \in (a, b)$ there exists a $t \in [0, 1]$ (not necessarily $\frac{1}{2}$) such that $\phi\left((1-t)\, x + ty\right) \leq (1-t)\, \phi(x) + t\phi(y)$.

Exercise 4. Let f be a complex measurable function on X and μ a positive measure on X, define

$$\phi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty)$$

and now let $E = \{p : \phi(p) < \infty\}$ and assume $0 < \|f\|_\infty$. Let's begin by characterizing E. Let $0 < p < s < \infty$ and $x \in [0, \infty)$ then $x^p < x^s$ iff $1 < x$ and $x^s < x^p$ iff $x < 1$. With this in mind if $0 < r < p < s < \infty$ then $|f|^p \leq max\{|f|^r, |f|^s\} \leq |f|^r + |f|^s$ and we get,

$$\int_X |f|^p \, d\mu \leq \int_X |f|^r \, d\mu + \int_X |f|^s \, d\mu$$

Suppose that $s, r \in E$ then $p \in E$, which proves statement (a).

As a consequence of (a) we see that E is a connected set and with this in mind, supposing E is nonempty, we can see that E is an interval with endpoints $a = \inf E$ and $b = \sup E$, this interval will be closed or not if E contains them or not, but for sure $E^o = (a, b)$.

To prove that $\log \phi$ is convex in E^o we start by proving that the composition is well defined, in other words, ϕ is never zero. If $\phi(p) = 0$ for some $p \in E^o$ then $|f| = 0$ a.e. on X and this implies $\|f\|_\infty = 0$ contradicting our assumption, thus the composition is well defined and we proceed to prove convexity. Take $x, y \in (a, b)$ and $t \in (0, 1)$. By the properties of the logarithm we get,

$$(1-t)\log\phi(x) + t\log\phi(t) = \log\left(\phi(x)^{1-t}\phi(y)^t\right)$$

given that the logarithm is a nondecreasing function, convexity holds if and only if

$$\phi((1-t)x + ty) \leq \phi(x)^{1-t}\phi(y)^t$$

rewriting ϕ on both sides in terms of Lp norms we get the following,

$$\|f^{(1-t)x}f^{ty}\|_1 \leq \|f^{(1-t)x}\|_{(1-t)^{-1}}\|f^{ty}\|_{t^{-1}}$$

Given that $1 \leq (1-t)^{-1}$ and $1 \leq t^{-1}$ are conjugate exponents, the last inequallity holds by Holder's inequality and this concludes the proof.

To prove that ϕ is continous in E we start by noticing that $\log \phi$ is convex in (a, b) which implies convexity of ϕ in (a, b) which in turn implies continuity of ϕ in (a, b), so we only need to prove continuity on a and b, in case they are elements of E.

Let $\{p_n\}$ be a sequence in (a, b) which converges to a and let's suppose $a \in E$, we will show that $\phi(p_n)$ converges to $\phi(a)$, the proof for b is analogous. By linearity of the integral and properties of the absolute value we get

$$|\phi(p_n) - \phi(a)| = |\int_X |f^{p_n}| - |f^a| \, d\mu| \leq \int_X ||f^{p_n}| - |f^a|| \, d\mu \leq \int_X |f^{p_n} - f^a| \, d\mu$$

Also, by continuity of the exponential we know that f^{p_n} converges pointwise to f^a as n goes to infinity so it sufices to find a real function $g \in L^1(\mu)$ such that $|f^{p_n}| \leq g$. Let $g = |f^a| + |f^{p_M}|$ with p_M the greatest element of the sequence

$\{p_n\}$ which exists given the convergence of the sequence. Then we can see that $a \leq p_n \leq p_M$ for all $1 \leq n$ and as mentioned above we have

$$|f|^{p_n} \leq \max\{|f|^a, |f|^{p_M}\} \leq |f|^a + |f|^{p_M} = g$$

Finally by Lebesgue's Dominated Convergence Theorem we get that

$$\int_X |f^{p_n} - f^a| \, d\mu \to 0$$

which concludes the proof of (b).

Observation

Let $0 < r < p < s < +\infty$ with $p - r = s - p$ which is the same as $2p = r + s$ then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

Exercise 5. In order to prove that $\|f\|_p \leq \|f\|_r$ if $0 < p < r \leq \infty$ we start by looking at case in which both p and r are finite. For every $x \in (0, \infty)$ the function x^c is twice differentiable and we have that

$$(x^c)'' = \left(x^{c-1} c\right)' = c\,(c-1)\, x^{c-2}$$

Observe that if $1 \leq c$, the expression above is nonnegative for all $x \in (0, \infty)$ and thus x^c convex over that interval. With this in mind and the fact that $\mu(\Omega) = 1$ and $1 \leq \frac{r}{p}$ we will use Jensen's Inequality to prove the result. Let $A = \{x \in \Omega | f(x) \neq 0\}$ (the integral over it's complement is exactly 0 and in this way $|f|(A) \subset (0, \infty)$), then

$$\|f\|_p = \left(\int_A |f|^p \, d\mu\right)^{\frac{r}{rp}} \leq \left(\int_A |f|^{\frac{rp}{p}} \, d\mu\right)^{\frac{1}{r}} = \|f\|_r$$

and this concludes the proof for the finite case.

Now let $r = \infty$, this case is much simpler. By definition of the essential supremum we have that $|f| \leq \|f\|_\infty$ almost everywhere and using the fact that $\mu(\Omega) = 1$ we get,

$$\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{\frac{1}{p}} \leq \left(\int_\Omega \|f\|_\infty^p \, d\mu\right)^{\frac{1}{p}} = \|f\|_\infty$$

Exercise 7. Let $0 < p < s < \infty$, an example for the inclusion $L^s(\mu) \subset L^p(\mu)$ can easily be constructed using Ex.5.

Let $X = (1, \infty)$ and m the Lebesgue Measure, we see that neither of the inclusions hold. On the one hand, $\frac{1}{x} \notin L^1(1, \infty)$ and $\frac{1}{x} \in L^2(1, \infty)$. On the other hand, $\frac{I_{(1,2)}}{x-1} \notin L^1(1, \infty)$ and $\frac{I_{(1,2)}}{x-1} \in L^{\frac{1}{2}}(1, \infty)$ being I_A the indicator function of A. We will only prove the first of these last two statements (the second one can be proved in a simlar manner chosing a convenient partition of $(1, 2)$),

$$\int_1^\infty \frac{I_{(1,2)}}{x-1}\, dx = \int_1^2 \frac{1}{x-1}\, dx = \sum_{n=0}^{\infty} \int_{1+\frac{1}{e^{n+1}}}^{1+\frac{1}{e^n}} \frac{1}{x-1}\, dx$$

given that the antiderivative of $\frac{1}{x-1}$ is $\ln(x-1) + C$ the expression above equals

$$\sum_{n=0}^{\infty} \left[\ln\left(\frac{1}{e^n}\right) - \ln\left(\frac{1}{e^{n+1}}\right) \right] = \sum_{n=0}^{\infty} \ln(e) = \infty$$

For the remaining inclusion we take $X = \mathbb{N}$ with the Counting Measure over the power set of $\mathbb{N}$. Observe that all measurable functions in this case will be just sequences of complex numbers so if $f = \{a_n\}$ we have,

$$\|f\|_p^p = \int_X |f|^p\, d\mu = \sum_{n=1}^{\infty} |a_n|^p$$

Suppose $\{a_n\} \in \ell^p(\mathbb{N})$ then the sequence $\{|a_n|^p\}$ converges to 0 and thus there exists a natural number M such that $|a_n|^p < 1$ for all $M < n$. Given that $1 < \frac{s}{p}$ we have that $|a_n|^s = (|a_n|^p)^{\frac{s}{p}} < |a_n|^p$ for all n greater than M and with that in mind we get,

$$\sum_{n=1}^{\infty} |a_n|^s = \sum_{n=1}^{M} |a_n|^s + \sum_{n=M+1}^{\infty} |a_n|^s < \sum_{n=1}^{M} |a_n|^s + \sum_{n=M+1}^{\infty} |a_n|^p < \infty$$

As we wanted, $\{a_n\} \in \ell^s(\mathbb{N})$, which concludes the proof.

Exercise 10. Given that $fg \geq 1$ and both f and g are positive we have $f \geq \frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega) = 1$,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = \left(\|g\|_1 \|g^{-1}\|_1\right)^{\frac{1}{2}} \leq (\|g\|_1 \|f\|_1)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_\Omega f\, d\mu \int_\Omega g\, d\mu$$

Exercise 17. We start by looking at the case in which $1 \leq p < \infty$ and α, we want to see that

$$|\alpha - \beta|^p \leq 2^{p-1} \left(|\alpha|^p + |\beta|^p\right) \tag{1}$$

for every α and β complex numbers not both zero (in that case the proof is tirivial). Let $X = [0,1]$ and μ be the Lebesgue Measure, we define the complex function f over $[0,1]$ such that

$$f(x) = \begin{cases} 2\alpha & \text{if } 0 \leq x < \frac{1}{2} \\ 2\beta & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Rearrangin (1) we get,

$$|\alpha - \beta| \leq \left(\frac{1}{2}|2\alpha|^p + \frac{1}{2}|2\beta|^p\right)^{\frac{1}{p}} = \|f\|_p \tag{2}$$

Observing that $\mu(X) = 1$ and $0 < \|f\|_\infty$ by Ex.5 we know that $\|f\|_1 \leq \|f\|_p$ and we have

$$|\alpha - \beta| \leq |\alpha| + |\beta| = \|f\|_1 \leq \|f\|_p \tag{3}$$

which concludes the proof.

Exercise 20. Let $\alpha, \beta \in \mathbb{R}$ and $t \in [0,1]$ we define

$$f(x) = \begin{cases} \alpha & \text{if } 0 \leq x < t \\ \beta & \text{if } t \leq x \leq 1 \end{cases}$$

which is clearly measurable and bounded. It only remains to replace f in the given inequality to prove the convexity of ϕ.

Exercise 23. Let μ be a positive measure with $\mu(X) < \infty$ and f a complex measurable function with $0 < \|f\|_\infty < \infty$ define the sequence

$$a_n = \int_X |f|^n \, d\mu = \|f\|_n^n$$

By the given hypothesis we have that first, the hypothesis of Ex.4.e hold and second, $0 < a_n < \infty$ for every $1 \leq n$ and thus the quotients $\frac{a_{n+1}}{a_n}$ are well defined and we have the inequality,

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} a_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} a_n^{\frac{1}{n}} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$$

Looking at Ex.4.e we see that is sufices to show that $\frac{a_{n+1}}{a_n}$ converges in $[0,\infty]$ because in that case both limits will exist and equal $\|f\|_\infty$ as $a_n^{\frac{1}{n}} = \|f\|_n$. In order to prove this, we will show that $\frac{a_{n+1}}{a_n}$ is a nondecreasing sequence.

Observing that $0 < n < n+1 < n+2 < \infty$ and that $2n = (n+1) + (n+2)$ we can use the observation presented at the end of Ex4 and we have,

$$\|f\|_n^n \leq \sqrt{\|f\|_{n+1}^{n+1} \|f\|_{n+2}^{n+2}}$$

rearranging the terms we get,

$$\frac{a_{n+1}}{a_n} \leq \frac{a_{n+2}}{a_{n+1}}$$

which proves that the sequence $\frac{a_{n+1}}{a_n}$ is nondecreasing, as we wanted. This concludes the proof.