Chapter 3 - L^p Spaces

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Exercise 3.

Let's suppose that ϕ is not convex in (a, b) then there exist $x, y \in (a, b)$ and $t_0 \in [0, 1]$ such that

$$\phi ((1 - t_0) x + t_0 y) > (1 - t_0) \phi (x) + t_0 \phi (y)$$

and assume $x < y$. Let's define $g, h : [0, 1] \to R$ such that,

$$g (t) = \phi ((1 - t) x + ty)$$

$$h (t) = (1 - t) \phi (x) + t\phi (y)$$

and let $f = h - g$. We define the nonempty sets $A = \{t \in [0, t_0) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f(0) = f(1) = 0$, and let $t_a = \sup A$ and $t_b = \inf B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in A and B which converge to t_a and t_b respectively, by continuity of f we can see that

$$0 = \lim_{n \to \infty} f (a_n) = f \left(\lim_{n \to \infty} a_n \right) = f (t_a)$$

With this and the fact that $t_a < t_0 < t_b$ as $f(t_0) < 0$ we have that $t_a \in A$ and the proof for $t_b \in B$ is analogous. Now by the definition of t_a and t_b we have that $f(t) < 0$ for all $t \in (t_a, t_b)$. Finally we take $x' = (1 - t_a) x + t_a y$ and $y' = (1 - t_b) x + t_b y$ and by working algebraically over $f \left(\frac{t_a + t_b}{2} \right) < 0$ we arrive at the following inquality

$$\phi \left(\frac{x' + y'}{2} \right) > \frac{\phi (x') + \phi (y')}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

Observation This proof shows that to see that convexity holds it sufices to show that given each pair of points $x, y \in (a, b)$ there exists a $t \in [0, 1]$ (not necessarily $\frac{1}{2}$) such that $\phi ((1 - t) x + ty) \leq (1 - t) \phi (x) + t\phi (y)$.

Exercise 4. Let f be a complex measurable function on X and μ a positive measura on X define

$$\phi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \quad (0 < p < \infty)$$

and now let $E = \{p : \phi(p) < \infty\}$ and assume $0 < \|f\|_\infty$. Let's begin by characterizing E. Let $0 < p < s < \infty$ and $x \in [0, \infty)$ then $x^p < x^s$ iff $1 < x$ and $x^s < x^p$ iff $x < 1$. With this in mind if $0 < r < p < s < \infty$ then $|f|^p \leq max\{|f|^r, |f|^s\} \leq |f|^r + |f|^s$ and we get,

$$\int_X |f|^p \, d\mu \leq \int_X |f|^r \, d\mu + \int_X |f|^s \, d\mu$$

Suppose that $s, r \in E$ then $p \in E$, which proves statement (a).

As a consequence of (a) we see that E is a connected set and with this in mind, supposing E is nonempty, we can see that E is an interval with endpoints $a = \inf E$ and $b = \sup E$, this interval will be closed or not if E contains them or not, but for sure $E^o = (a, b)$.

To prove that $\log \phi$ is convex in E^o we take $x, y \in (a, b)$ and $t \in (0, 1)$. By the properties of the logarithm we get,

$$(1 - t) \log \phi(x) + t \log \phi(t) = \log\left(\phi(x)^{1-t} \phi(y)^t\right)$$

given that the logarithm is a nondecreasing function, convexity holds if and only if

$$\phi((1-t)x + ty) \leq \phi(x)^{1-t} \phi(y)^t$$

rewriting ϕ on both sides in terms of Lp norms we get the following,

$$\|f^{(1-t)x} f^{ty}\|_1 \leq \|f^{(1-t)x}\|_{(1-t)^{-1}} \|f^{ty}\|_{t^{-1}}$$

finally, given that $1 \leq (1-t)^{-1}$ and $1 \leq t^{-1}$ are conjugate exponents, the last inequallity hold by Holder's inequality and this concludes the proof.

To prove that ϕ is continous in E we start by noticing that $\log \phi$ is convex in (a, b) which implies convexity of ϕ in (a, b) which in turn implies continuity of ϕ in (a, b), so we only need to prove continuity on a and b, in case they are elements of E.

Let $\{p_n\}$ be a sequence in (a, b) which converges to a and let's suppose $a \in E$, we will show that $\phi(p_n)$ converges to $\phi(a)$, the proof for b is analogous. By linearity of the integral and properties of the absolute value we get

$$|\phi(p_n) - \phi(a)| = |\int_X |f^{p_n}| - |f^a| \, d\mu| \leq \int_X ||f^{p_n}| - |f^a|| \, d\mu \leq \int_X |f^{p_n} - f^a| \, d\mu$$

Also, by continuity of the exponential we know that f^{p_n} converges pointwise to f^a as n goes to infinity so it sufices to find a real function $g \in L^1(\mu)$ such that $|f^{p_n}| \leq g$. Let $g = |f^a| + |f^{p_M}|$ with p_M the greatest element of the sequence $\{p_n\}$ which exists given the convergence of the sequence. Then we can see that $a \leq p_n \leq p_M$ for all $1 \leq n$ and as mentioned above we have

$$|f|^{p_n} \leq \max\{|f|^a, |f|^{p_M}\} \leq |f|^a + |f|^{p_M} = g$$

Finally by Lebesgue's Dominated Convergence Theorem we get that

$$\int_X |f^{p_n} - f^a| \, d\mu \to 0$$

which concludes the proof of (b).

Observation

Let $0 < r < p < s < +\infty$ with $p - r = s - p$ which is the same as $2p = r + s$ then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

Exercise 10. Given that $fg \geq 1$ and both f and g are positive we have $f \geq \frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega) = 1$,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = (\|g\|_1 \|g^{-1}\|_1)^{\frac{1}{2}} \leq (\|g\|_1 \|f\|_1)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_\Omega f \, d\mu \int_\Omega g \, d\mu$$