# Real and Complex Analysis Solutions

Salvador Castagnino
scastagnino@itba.edu.ar

**Exercise 3.**

Let's assume that $\phi$ is not convex in $(a,b)$ then there exist $x, y \in (a,b)$ and $t_0 \in [0,1]$ such that

$$\phi\left((1-t_0)\,x + t_0 y\right) > (1-t_0)\,\phi(x) + t_0\phi(y)$$

and without loss of generality let's assume $x < y$. Let's define $g, h : [0,1] \to R$ such that,

$$g(t) = \phi\left((1-t)\,x + ty\right)$$

$$h(t) = (1-t)\,\phi(x) + t\phi(y)$$

and let $f = h - g$. We define the nonempty sets $A = \{t \in [0, t_0\,) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f(0) = f(1) = 0$, and let $t_a = \sup A$ and $t_b = \sup B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $A$ and $B$ which converge to $t_a$ and $t_b$ respectively, by continuity of $f$ we can see that

$$0 = \lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(t_a)$$

then $t_a \in A$ and the proof for $t_b \in B$ is analogous. As $f(t_0) < 0$ then $t_a < t_0 < t_b$ and by continuity of $f$ and the definition of $t_a$ and $t_b$ we have that $f(t) < 0$ for all $t \in (t_a, t_b)$. Finally we take $x' = (1-t_a)\,x + t_a y$ and $y' = (1-t_b)\,x + t_b y$ and by working algebraically over $f\left(\frac{t_a+t_b}{2}\right) < 0$ we arrive at the following inquality

$$\phi\left(\frac{x'+y'}{2}\right) > \frac{\phi(x') + \phi(y')}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

**Observation** This proof shows that to see that convexity holds it sufices to show that given each pair of points $x, y \in (a,b)$ there exists a $t \in [0,1]$ (not necessarily $\frac{1}{2}$ ) such that $\phi\left((1-t)\,x + ty\right) \leq (1-t)\,\phi(x) + t\phi(y)$.

**Exercise 4**

**Observation**

Let $0 < r < p < s < +\infty$ with $p - r = s - p$ which is the same as $2p = r + s$ then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

**Exercise 10.** Given that $fg \geq 1$ and both $f$ and $g$ are positive we have $f \geq \frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega) = 1$,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = \left(\|g\|_1 \|g^{-1}\|_1\right)^{\frac{1}{2}} \leq (\|g\|_1 \|f\|_1)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_\Omega f \, d\mu \int_\Omega g \, d\mu$$