

Chapter 3 - L^p Spaces

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Exercise 3.

Let's suppose that ϕ is not convex in (a, b) then there exist $x, y \in (a, b)$ and $t_0 \in [0, 1]$ such that

$$\phi((1 - t_0)x + t_0y) > (1 - t_0)\phi(x) + t_0\phi(y)$$

and assume $x < y$. Let's define $g, h : [0, 1] \rightarrow \mathbb{R}$ such that,

$$g(t) = \phi((1 - t)x + ty)$$

$$h(t) = (1 - t)\phi(x) + t\phi(y)$$

and let $f = h - g$. We define the nonempty sets $A = \{t \in [0, t_0) \mid f(t) = 0\}$ and $B = \{t \in (t_0, 1] \mid f(t) = 0\}$ given that $f(0) = f(1) = 0$, and let $t_a = \sup A$ and $t_b = \inf B$, we want to see that $t_a \in A$ and $t_b \in B$. Let $\{a_n\}$ and $\{b_n\}$ be sequences in A and B which converge to t_a and t_b respectively, by continuity of f we can see that

$$0 = \lim_{n \rightarrow \infty} f(a_n) = f\left(\lim_{n \rightarrow \infty} a_n\right) = f(t_a)$$

With this and the fact that $t_a < t_0 < t_b$ as $f(t_0) < 0$ we have that $t_a \in A$ and the proof for $t_b \in B$ is analogous. Now by the definition of t_a and t_b we have that $f(t) < 0$ for all $t \in (t_a, t_b)$. Finally we take $x' = (1 - t_a)x + t_a y$ and $y' = (1 - t_b)x + t_b y$ and by working algebraically over $f\left(\frac{t_a + t_b}{2}\right) < 0$ we arrive at the following inequality

$$\phi\left(\frac{x' + y'}{2}\right) > \frac{\phi(x') + \phi(y')}{2}$$

which is a clear contradiction, as we wanted. This concludes the proof.

Observation This proof shows that to see that convexity holds it suffices to show that given each pair of points $x, y \in (a, b)$ there exists a $t \in [0, 1]$ (not necessarily $\frac{1}{2}$) such that $\phi((1 - t)x + ty) \leq (1 - t)\phi(x) + t\phi(y)$.

Exercise 4

Observation

Let $0 < r < p < s < +\infty$ with $p - r = s - p$ which is the same as $2p = r + s$ then using Holder's inequality we get,

$$\|f\|_p^p = \|f^p\|_1 = \|f^{\frac{r}{2}} f^{\frac{s}{2}}\|_1 \leq \|f^{\frac{r}{2}}\|_2 \|f^{\frac{s}{2}}\|_2 = (\|f\|_r^r \|f\|_s^s)^{\frac{1}{2}}$$

thus,

$$\|f\|_p^p \leq \sqrt{\|f\|_r^r \|f\|_s^s}$$

Exercise 10. Given that $fg \geq 1$ and both f and g are positive we have $f \geq \frac{1}{g}$ and using Holder's inequality and the fact that $\mu(\Omega) = 1$,

$$1 = \|1\|_1 = \|g^{\frac{1}{2}} g^{-\frac{1}{2}}\|_1 \leq \|g^{\frac{1}{2}}\|_2 \|g^{-\frac{1}{2}}\|_2 = (\|g\|_1 \|g^{-1}\|_1)^{\frac{1}{2}} \leq (\|g\|_1 \|f\|_1)^{\frac{1}{2}}$$

thus,

$$1 \leq \int_{\Omega} f \, d\mu \int_{\Omega} g \, d\mu$$