Chapter 4 - Elementary Hilbert Space Theory

Salvador Castagnino
scastagnino@itba.edu.ar

Exercise Solutions

Exercise 4. We start by assuming that $\{u_n\}_{n\in\mathbb{N}}$ is a countable maximal orthonormal system in H, we want to see that H is separable. Observe that it sufices to find a countable set A such that $P \subset \overline{A}$, with P the set of all finite linear combinations of elements in $\{u_n\}_{n\in\mathbb{N}}$. Define the set A to be,

$$A = \left\{ \sum_{n=1}^{N} (q_n + ip_n)\, u_n : q_n, p_n \in \mathbb{Q},\ N \geq 1 \right\}$$

which is clearly countable. Take $x \in P$, we have $x = \sum_{n=1}^{N} c_n u_n$ for some c_n complex numbers and some $N \geq 1$. Observe that for every $1 \leq n \leq N$ there exist $\{q_{nk}\}_{k\in\mathbb{N}}$ and $\{p_{nk}\}_{k\in\mathbb{N}}$ sequences of rational numbers such that $(q_{nk} + ip_{nk}) \to c_n$ as k goes to infinity. Then let $\epsilon > 0$ we can ask for a k large enough such that $|\,(q_{nk} + ip_{nk}) - c_n| < \frac{\epsilon}{N}$ for all $1 \leq n \leq N$, we in turn have

$$\|\sum_{n=1}^{N} (q_{nk} + ip_{nk})\, u_n - \sum_{n=1}^{N} c_n u_n\| \leq \sum_{n=1}^{N} |\,(q_{nk} + ip_{nk}) - c_n| < \sum_{n=1}^{N} \frac{\epsilon}{N} < \epsilon$$

Given that every element in P can be aproximated by elements in A we have that, $P \subset \overline{A}$, which concludes the proof.

Now suppose that $\{x_n\}_{n\in\mathbb{N}}$ is countable and dense in H, we are going to build a countable maximal orthonormal system. Let A be the set such that

$$\begin{cases} x_1 \in A \\ x_{n+1} \in A \text{ iff } x_{n+1} \notin [x_1, ..., x_n] \end{cases}$$

where $[x_1, ..., x_n]$ denotes the span of $\{x_1, ..., x_n\}$, let's see that A is linearly independent. Suppose that for sume $v_1, ..., v_{m+1}$ in A and $\alpha_1, ..., \alpha_m$ nonzero complex numbers we have that

$$\sum_{j=1}^{m} \alpha_j v_j = v_{m+1}$$

Every v_j can be expressed as x_{n_j}, so let v_k be such that n_k is the greates index between all n_j, we then have

$$v_k = \sum_{1 \leq n \neq k \leq m} \frac{\alpha_n v_n}{\alpha_k} - \frac{v_{m+1}}{\alpha_k}$$

which clashes with the construction of A, thus A is linearly independent. By **Ex4.3** we can build from A a countable orthnormal set $\{u_n\}$ such that $[v_1, ..., v_n] = [u_1, ..., u_n]$ for all $n \geq 1$. Given that every x_n can be expressed as a linear combination of elements in A, one can see that every x_n will be in $[u_1, ..., u_m]$ for some m. Then we can see that every x_n is an element of $\overline{P}$, with P the set of finite linear combinations of elements in $\{u_n\}$, which in turn imples that P is dense in H and concludes the proof.

Useful Properties