Harmonic Ladder Optimization: RMSE & HSI vs Harmonia Position RMSE (AU) Optimum = 2.143687 AU0.6450 HSI (σ of % residuals) **Empirical optimum** 0.40 -Celtic 2AC/(1+A) - 0.6425 $\pi^{(2/3)}$ 0.35 - $\pi^{(2/3)} = 2.145029 \text{ AU}$ Celtic = 2.142136 AU- 0.6400 RMSE (AU) 0.30 0.25 - 0.6375 - 0.6350 Neptune resid = 0 at 2.14312- 0.6325 0.20 -- 0.6300 0.15 - 0.6275 2.12 2.13 2.14 2.15 2.16 2.17 2.18 Harmonia position $a_{\rm H}$ (AU) Per-planet Residuals across Harmonic Ladder (symmetric log scale) 100 -Jupiter Pluto **Uranus** Saturn Neptune 10^{-1} Residuals -10^{-1} -10^{0} -2.13 2.14 2.15 2.16 2.17 2.12 2.18 Harmonia position a_H (AU)