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Assignment no - 02

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1. Solve the following recurrence relation using the generating function method.

$h_n = 2h_{(n-1)} + n - 1$ for $n>0$, and $h_0 = 0$

Solution:

Let $A(z) = \sum_{n=0}^{\infty} h_n z^n$

$= \sum_{n=1}^{\infty} (2h_{n-1} + n - 1) z^n$

$= 2\sum_{n=1}^{\infty} h_{(n-1)} z^n + \sum_{n=1}^{\infty} n \,.\, z^n - \sum_{n=1}^{\infty} z^n$

$= 2z.A(z) + \frac{z}{(1-z)^2} - \frac{z}{1-z}$

$= \frac{z^2}{(1-2z)(1-z)^2}$

$= \frac{1}{1-2z} - \frac{1}{(1-z)^2}$

$= \sum_{n=0}^{\infty} 2^n \,.\, z^n - \sum_{n=0}^{\infty} (n+1) \,.\, z^n$

$= \sum_{n=0}^{\infty} (2^n - n - 1) z^n$

So, $h_n = 2^n - n - 1$

2. Solve the following recurrence relation by using range transformation.

$$n^2T(n) = 3(n-1)^2T(n-1) + 1, \qquad T(1) = 1$$

Solution:

Let,

$A_n = n^2T(n)$

So,

$A_n = 3A_{n-1} + 1$,for $n > 1$,

where $A_1 = 1$,and $A_0 = 0$

The characteristic equation,

$(r-1)(r-3) = 0$

And, characteristic roots are,

$r1 = 1$,and

$r2 = 3$

So, the equation can be written,

$A_n = c_1 + c_2 3^n$

Solving the constants, we obtain

$c_1 = -\frac{1}{2}$

$c_2 = \frac{1}{2}$

So,

$A_n = \left(\frac{1}{2}\right)3^n - \frac{1}{2}$

The solution is,

$T(n) = \frac{1}{2n^2}(3^n - 1)$

3. We have a 2xn chess board as shown in the figure below. We would like to cover such a board with 2x1 (vertical) dominoes or 1x2 (horizontal) dominoes completely. How many different ways can we do it? Derive a recurrence relation first and solve it using characteristic equation method.

Solution:

If the chess board is 2*1,
Then the board can be covered using 1 vertical domino.
So, Total ways = 1

If the chess board is 2*2,
Then the board can be covered using 2 vertical or 2 horizontal dominoes.
So, Total ways = 2

If the chess board is 2*3,
Then the board can be covered by number of ways of 2*2 + number of ways of 2*1.
So, Total ways = 2 + 1 = 3

So, we can express it as recurrence,

$a_n = a_{n-1} + a_{n-2}$, for $n>2$

Where, $a_1 = 1$, and $a_2 = 2$

The characteristic equation of this recurrence relation is,

$$r^2 - r - 1 = 0$$

So, the roots are,

$r_1 = \frac{1+\sqrt{5}}{2}$, and

$r_2 = \frac{1-\sqrt{5}}{2}$

So, the solution is,

$$a_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Therefore,

$$a_1 = 1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$a_2 = 2 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^2 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^2$$

Solving these equations, we get

$$C_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}$$

$$C_2 = \frac{3-\sqrt{5}}{5-\sqrt{5}}$$

So, the final equation is,

$$a_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{5-\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

4. We talked about the Huffman's algorithm in great detail including the proof. This is a slight modification of the Huffman's problem. The longest code produced by the algorithm can be pretty bad (depending on the input data of course). Propose an efficient way to reduce the height of the Huffman tree by one. The code produced from this tree should still be valid and optimal under the restriction. This, of course, assumes that the height can be reduced.

<u>Solution:</u>

Huffman coding algorithm takes two least frequency nodes and connects them to form a parent which has frequency equal sum of child nodes. In random frequency of symbols we need to compute the least two nodes to combine every time but in case of Fibonacci sequence of frequency, the sequence in Fibonacci series is same as sequence in Huffman coding.

Example: 1, 1, 2, 3, 5, 8
It will form a left skewed or right skewed tree.

Now if we slightly modify the last value from 8 to 7, then it will still generate skewed tree.

```
            19
           /   \
         12    7
        /  \
       7    5
      /  \
     4    3
    /  \
   2    2
  /  \
 1    1
```

In this tree, there are 5 levels.
We know, optimal cost of Huffman tree is, $\sum_i W_i \,.\, depth(i)$
So, total cost = 44

Now, we can reduce the depth by this way,

```
            19
           /   \
          7     12
         / \    / \
        4   3  5   7
       / \
      2   2
     / \
    1   1
```

In this tree, there are 4 levels, and total cost = 44

So, we can reduce the level of the tree with optimal cost in a certain condition that,

1. If the additive frequency of the Huffman tree equals the frequency of a single node.

2. It is not possible if the frequency of the node is 2.

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