Home Work 1

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[1] Find the solution to the recurrence relation $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ for n >= 3, with the initial conditions $a_0 = 1$; $a_1 = 2$; and $a_2 = 3$.

Solution:

The characteristic equation of this recurrence relation is,

$r^3 - 2r^2 - 5r + 6 = 0$

Or, $(r-1)(r+2)(r-3) = 0$ // Using rational root test

Hence, the characteristic roots are,

$r_1 = 1$, $r_2 = -2$, and $r_3 = 3$

So, the solution is,

$a_n = A.(1)^n + B.(-2)^n + C.(3)^n$,where A, B, and C are constants.

Therefore,

$a_0 = A + B + C = 1$

$a_1 = A - 2B + 3C = 2$ and

$a_2 = A + 4B + 9C = 3$

Solving these equations, we get,

A = 5/6;

B = - (2/15);

C = 3/10;

So, the final solution is,

$a_n = 5/6 - (2/15).(-2)^n + (1/10)3^{(n+1)}$

[2] Let's modify the above recurrence relation to non-homogenous.

$$a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3} + 3^n$$

Solve the recurrence relation with boundary conditions of your own. Choice the initial values (and clearly state them at the beginning of the solution) so that you can find the constants for the solution easier.

Solution:

This is a non-linear homogeneous equation where its associated homogeneous equation,

$a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$,and

$f(n) = 3^n$

Let, the initial conditions are $a_0 = 1$, $a_1 = 47/10$, and $a_2 = 96/5$;

Now, the characteristic equation of its associated homogeneous relation is,

$r^3 - 2r^2 - 5r + 6 = 0$

So, $(r-1)(r+2)(r-3) = 0$

So, the characteristic roots are,

$r_1 = 1$, $r_2 = -2$, and $r_3 = 3$

Hence,

$a_h = A.(1)^n + B.(-2)^n + C.(3)^n$, where A, B, and C are constants

Since, $f(n) = 3^n$ and $r_3 = 3$

So, $a_t = Dn3^n$, where D is constant

After putting the solution in the recurrence relation,

We get,

$Dn3^n = 2D(n-1)3^{(n-1)} + 5D(n-2)3^{(n-2)} - 6D(n-3)3^{(n-3)} + 3^n$

Or, $Dn3^3 = 2D(n-1)3^2 + 5D(n-2).3 - 6D(n-3) + 3^3$

Or, $27Dn = 18Dn - 18D + 15Dn - 30D - 6Dn + 18D + 27$

Or, $D = 27/30$

Or, $D = 9/10$

So,

$$a_t = (9/10)n3^n = (n/10)3^{(n+2)}$$

Hence, the solution for the recurrence relation can be written as,

$$a_n = A + B.(-2)^n + C.(3)^n + (n/10)3^{(n+2)}$$

Using initial conditions,

$$a_0 = A + B + C = 1$$

$$a_1 = A -2B + 3C + 27/10 = 47/10$$

Or, $a_1 = A – 2B + 3C = 2$

$$a_2 = A + 4B + 9C + 81/5 = 96/5$$

Or, $a_2 = A + 4B + 9C = 3$

Solving these equations, we get,

$A = 5/6$;

$B = - (2/15)$;

$C = 3/10$;

So, the final solution is,

$$a_n = 5/6 – (2/15).(-2)^n + (1/10).3^{(n + 1)} + (n/10).3^{(n+2)}$$

[3] For the maximum sum problem, I gave you 4 algorithms with different complexity. This question is dealing with the one with a linear solution. (a) Rewrite the algorithm in a recursive way. (b) Write a recurrence relation to compute the time complexity of the recursive algorithm. (c) Solve the recurrence relation.

<u>Solution:</u>

(a)

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MaxSum(X, index=0, max_here=0, max_sum=0)
{
        If(index == arr_size)
                return max_sum;

        max_here = max(0, max_here + X[index]);
        max_sum = max(max_sum, max_here);

        return MaxSum(X, index+1, max_here, max_sum);
}
```

(b)

$T(n) = 1 + T(n-1)$,where $n>1$, $T(1) = 1$

(c)

We have,

$$
\begin{aligned}
T(n) &= 1 + T(n-1) \\
&= 1 + 1 + T(n-2) \\
&= 2 + 1 + T(n-3) \\
&\;\;\ldots\ldots\ldots\ldots \\
&= k + T(n-k) \\
&\;\;\ldots\ldots\ldots\ldots \\
&= n - 1 + T(1) \\
&= n - 1 + 1 \\
&= n
\end{aligned}
$$

So, $T(n) = \Theta(n)$

[4] Solve the following recurrence relation,

$$g(n) = 2g(\tfrac{n}{2}) - g(\tfrac{n}{4}) + 3$$

for all 2^i where i >= 2, g(1) = 1, and g(2) = 4.

<u>Solution</u>:

Let, $n = 2^k$,and $g(n) = a_k = g(2^k)$

Now, we can write the recurrence as,

$a_k = 2a_{k-1} - a_{k-2} + 3$, for $k >= 2$

$a_1 = 1$,and

$a_2 = 4$

Using operator E, we may write the equation,

$(E-1)^2 \langle a_k \rangle = \langle 3 \rangle$

The annihilator for the right-hand side is (E-1), so we obtain

$(E-1)^3 \langle a_k \rangle = \langle 0 \rangle$

Thus the characteristic root is 1 with a multiplicity of 3, so

$a_k = c_1 + c_2 k + c_3 k^2$

We have,

$a_2 - 2a_1 + a_0 = 3$

so, $a_0 = 1$

Solving for the constants, we obtain

$c_1 = 1$

$c_2 = -\frac{3}{2}$

$c_3 = \frac{3}{2}$

So,

$a_k = 1 - (3/2)k + (3/2)k^2$

The solution is,

$g(n) = 1 - (3/2)\lg n + (3/2)(\lg n)^2$

[5] Use the Principle of Induction to prove the following,

$$\sum_{i=0}^{n} x^i = \frac{1-x^{n+1}}{1-x}$$

for x ≠ 1 and integers n ≥ 0.

Solution:

Basis of Induction, n=0

$$\sum_{i=0}^{0} x^i = \frac{1-x^{0+1}}{1-x}$$

$$x^0 = \frac{1-x}{1-x} = 1$$

Inductive Hypothesis, n = k

$$\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$$

Inductive Step, n = k + 1

$$\sum_{i=0}^{k+1} x^i = \frac{1-x^{k+1+1}}{1-x}$$

Right side of the inductive step,

$$\frac{1-x^{k+2}}{1-x}$$

Left side of the inductive step,

$$\sum_{i=0}^{k+1} x^i$$

$$= \sum_{i=0}^{k} x^i + x^{k+1}$$

$$= \frac{1-x^{k+1}}{1-x} + x^{k+1}$$

$$= \frac{1-x^{k+1} + x^{k+1} - xx^{k+1}}{1-x}$$

$$= \frac{1-x^{k+2}}{1-x}$$

Which is identical to the right side.