TIF345/FYM345 Project 1: Cosmological Models

Markus Utterström Salar Ghanbari

January 27, 2025

1 Introduction

The accelerating expansion of the Universe is one of the most profound discoveries in modern cosmology. Type Ia supernovae, as standard candles, have played a crucial role in providing evidence for this phenomenon. By measuring the luminosity distances of these supernovae as a function of their redshift, it is possible to infer the parameters governing the Universe's expansion and the relative contributions of matter and dark energy.

This report focuses on two key tasks. The first task involves estimating the Hubble parameter (H_0) and the deceleration parameter (q_0) in the small-z regime ($z < 0.5$) using data from the SCP Union 2.1 dataset. The goal is to determine whether the data supports an accelerating universe and to validate the extraction of H_0 using Bayesian inference. The second task compares two cosmological models: ΛCDM (Lambda Cold Dark Matter) and wCDM (Cold Dark Matter with a dark energy equation of state parameter w), across the entire redshift range in the dataset. The aim is to identify which model better explains the data by evaluating model parameters, computing information criteria (AIC and BIC), and analyzing the posterior probability distribution for $\Omega_{M,0}$ in the ΛCDM model. All dimensionless values are reported without a unit.

2 Methodology

The analysis uses the SCP Union 2.1 dataset, which provides distance modulus measurements, redshifts, and associated uncertainties for a collection of Type Ia supernovae. The dataset was imported into Python, and the relevant columns, including redshift (z), distance modulus (μ), and distance modulus uncertainties (σ_μ), were extracted.

From the Friedmann equation and definitions of density parameters we arrive at the following expression for the Hubble parameter that depends of the redshift z: $H(z) = H_0\sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}} \equiv H_0 E(z)^{1/2}$

We assume a flat universe and thus set $\Omega_{k,0} = 0$. In this flat universe we get the following luminosity distance relation:

$$d_L = c(1+z)\int_0^z \frac{dz'}{H(z')}, \tag{1}$$

where c is the speed of light in vacuum. We can use this relation with the definition of distance modulus that is used to create the data of the SCP dataset:

$$\mu(z) = 5\log_{10}\left(\frac{c(1+z)}{H_0}\int_0^z \frac{1}{E(z')}dz'\right) + 25. \tag{2}$$

which in the small z regime can be Taylor expanded into

$$\mu_{\text{model}} = 5\log_{10}\left(\frac{c}{H_0}\left(z + \frac{1}{2}(1-q_0)z^2\right)\right) + 25, \tag{3}$$

where q_0 is a deceleration parameter and H_0 is the Hubble parameter.

2.1 Model Setup

For the first task, the Bayesian framework[1] was used to estimate the Hubble parameter H_0 and the deceleration parameter q_0 in the small-redshift regime ($z < 0.5$). This involved defining the priors, likelihood, and posterior distributions. The prior distributions were defined as follows: $H_0 \sim \mathcal{U}(50, 100)$, representing a uniform prior that reflects the plausible range for the Hubble parameter in units of km/s/Mpc; $q_0 \sim \mathcal{U}(-2, 2)$, a uniform prior covering a broader range of expected values for the deceleration parameter; and $\sigma^2 \sim \text{InvGamma}(\alpha = 2, \beta = 2)$, an inverse gamma prior chosen for the error scale. This updated prior configuration ensures flexibility in modeling uncertainties while maintaining minimal constraints on the parameters.

The likelihood function relates the observed distance modulus (μ_{obs}) to the theoretical model prediction (μ_{model}) and the measurement uncertainty (σ_μ). We incorporated weights based on the inverse squared uncertainties, $w = 1/\sigma_\mu$, normalized such that the sum of weights equals the number of data points, $w_{\text{norm}} = w \cdot (N_d / \sum w)$. This ensures that data points with higher uncertainty contribute less to the likelihood.

The likelihood is modelled as:

$$\mu_{\text{obs}} \sim \mathcal{N}(\mu_{\text{model}}, \frac{\sigma^2}{\sqrt{w_{\text{norm}}}}). \tag{4}$$

The posterior distribution is proportional to the product of the priors and the likelihood:

$$p(H_0, q_0, \sigma \mid \mu_{\text{obs}}) \propto p(H_0)p(q_0)p(\sigma)\prod_i \mathcal{N}(\mu_{\text{obs},i} \mid \mu_{\text{model},i}, \frac{\sigma^2}{\sqrt{w_{\text{norm},i}}}). \tag{5}$$

The model was implemented using the `pymc` package[2], with Markov Chain Monte Carlo (MCMC) sampling[2] conducted over 100,000 iterations and 10,000 tuning steps. This sampling approach ensured convergence and reliable estimation of the posterior distributions.

In Task 2, the ΛCDM and wCDM cosmological models are compared using the SCP Union 2.1 dataset for the full redshift range. The parameters for each model were estimated using maximum likelihood estimation, and model performance was evaluated using the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). In this task, weights inversely proportional to the measurement variance were used with an assumed variance of: $\sqrt{\text{avg}(\sigma_\mu)}$. This choice was made with the assumption that all errors in the dataset come from a Gaussian distribution.

The ΛCDM model assumes the Universe consists of matter $\Omega_{M,0}$ and dark energy $\Omega_{\Lambda,0}$, with $\Omega_{\Lambda,0} = 1 - \Omega_{M,0}$. The expansion rate is given by:

$$E(z) = \sqrt{\Omega_{M,0}(1+z)^3 + \Omega_{\Lambda,0}}, \tag{6}$$

where z is the redshift. The luminosity and the distance modulus are the same as in (2):

The parameter $\Omega_{M,0}$ was estimated by minimizing the negative log-likelihood:

$$\mathcal{L} = -\frac{1}{2}\sum_i \left(\frac{\mu_{\text{obs},i} - \mu_{\text{model},i}}{\sigma_{\mu,i}}\right)^2. \tag{7}$$

The wCDM model extends ΛCDM by introducing a dark energy equation of state parameter w, where $w = p/\rho$. The expansion rate is:

$$E(z) = \sqrt{\Omega_{M,0}(1+z)^3 + (1-\Omega_{M,0})(1+z)^{3(1+w)}}. \tag{8}$$

The luminosity distance and distance modulus are computed similarly to Eqs. (2). The parameters $\Omega_{M,0}$ and w were estimated by minimizing the negative log-likelihood in Eq. (7).

Model performance was evaluated using the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC):

$$\mathrm{AIC} = 2k - 2\ln(\mathcal{L}), \tag{9}$$
$$\mathrm{BIC} = k\ln(n) - 2\ln(\mathcal{L}), \tag{10}$$

where k is the number of free parameters, n is the number of data points, and $\ln(\mathcal{L})$ is the log-likelihood. Lower AIC and BIC values indicate better model performance.

For the ΛCDM model, the posterior distribution of $\Omega_{M,0}$ was extracted using Markov Chain Monte Carlo (MCMC) sampling. A uniform prior: $\Omega_{M,0} \sim \mathcal{U}(0,1)$, was applied, and the log-posterior is:

$$\ln P(\Omega_{M,0} \mid \mathcal{D}) \propto \ln P(\mathcal{D} \mid \Omega_{M,0}) + \ln P(\Omega_{M,0}), \tag{11}$$

where $P(\mathcal{D} \mid \Omega_{M,0})$ is the likelihood and $P(\Omega_{M,0})$ is the prior. For this task the known variance was used instead of weights, this means that all contributions to the likelihood was calculated with its own measurement variance.

3 Results

3.1 Task 1 Results

Figure 1 shows the trace plots and posterior distributions for H_0, q_0, and σ. The chains demonstrate good mixing and convergence, and the posterior distributions provide insights into the parameter estimates.

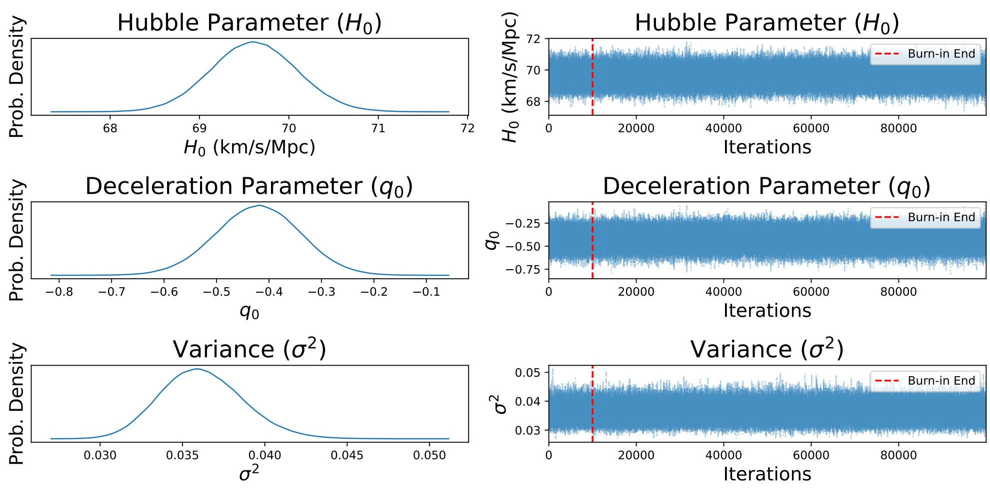

Figure 1: Trace plots (right) and posterior distributions (left) for the parameters H_0, q_0, and σ, respectively.

The joint posterior distributions of H_0 and q_0 are shown in Figure 2. The contour plots highlight the anti-correlation between H_0 and q_0. The posterior mean value for q_0 is -0.42, with a 95% credible interval of $[-0.58, -0.26]$.

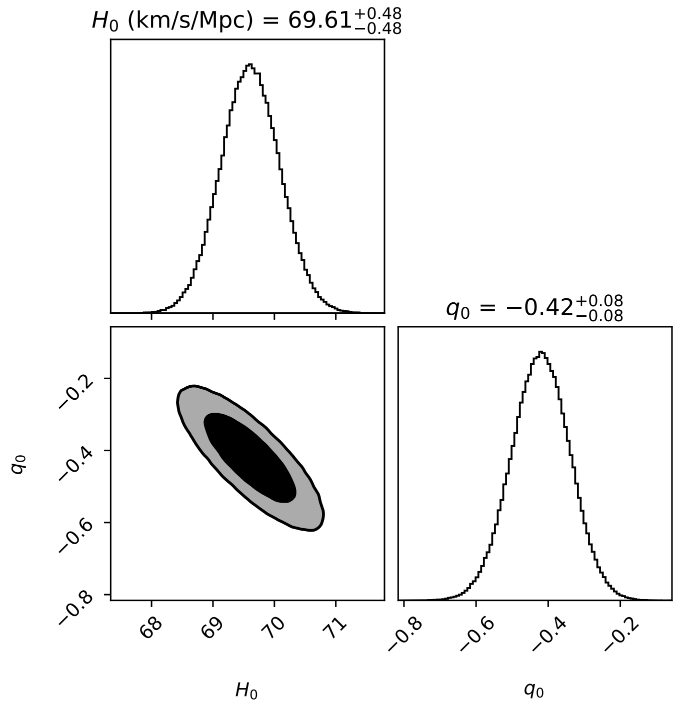

Figure 2: Corner plots showing joint posterior distributions of H_0 and q_0. Contours represent credible regions for confidence interval of 68% and 95%. On top of the histograms are the mean of the distributions with one standard deviation.

The posterior predictive plot of the distance modulus (μ) as a function of redshift (z) is presented in Figure 3. The red curve represents the posterior predictive mean. The lighter red colors represent the confidence intervals of 68% and 95%.

3.2 Task 2 Results

The best-fit parameters, AIC, and BIC values for ΛCDM and wCDM models are summarized in Table 1. For ΛCDM, the matter density parameter ($\Omega_{M,0}$) was estimated to be 0.279, with the remaining energy density attributed to Ω_Λ. For wCDM, the matter density parameter was 0.275, and the dark energy equation of state parameter (w) was -0.99.

Model	Best-fit Parameters	AIC	BIC
ΛCDM	$\Omega_{M,0} = 0.279$	77.3	81.6
wCDM	$\Omega_{M,0} = 0.275, w = -0.990$	79.3	88.0

Table 1: Best-fit parameters and model comparison metrics for ΛCDM and wCDM models.

Figure 4 shows the posterior distribution of $\Omega_{M,0}$ for the ΛCDM model. The posterior mean of $\Omega_{M,0}$ was found to be 0.28.

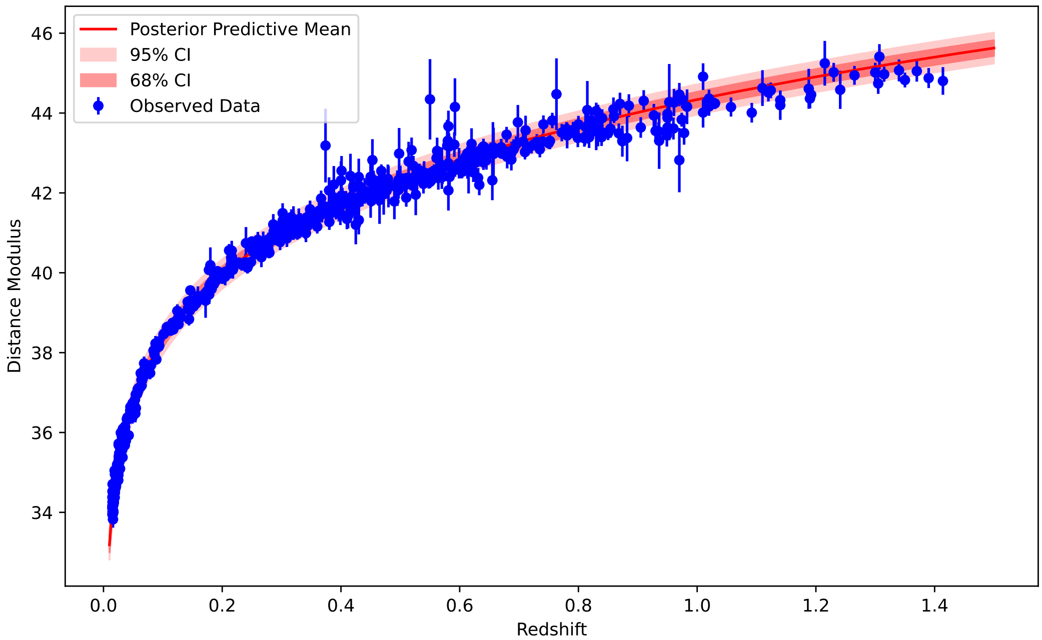

Figure 3: Posterior predictive plot of distance modulus (μ) versus redshift (z). The red curve represents the posterior predictive mean, and the shaded regions correspond to the 68% (darker) and 95% (lighter) credible intervals. Blue points indicate the observed data with error bars.

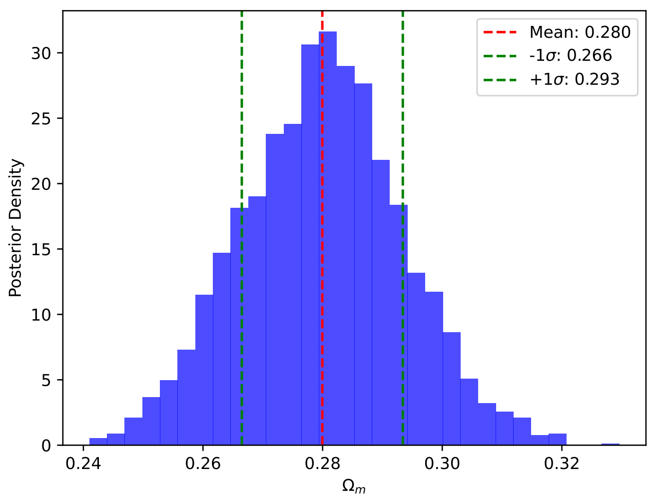

Figure 4: Posterior distribution of $\Omega_{M,0}$ for ΛCDM. The dashed red line indicates the posterior mean value of 0.28. The green lines show one standard deviation.

4 Discussion

The results from both tasks demonstrate the robustness of the SCP Union 2.1 dataset and the effectiveness of the Bayesian framework for cosmological parameter estimation. Task 1 strongly indicates an accelerating Universe $q_0 = -0.42$ with a confidence interval of 95%

below zero. This evidence is convincing for an accelerating universe. A posterior mean for $H_0 = 69.61\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$was provided, consistent with established values. The used dataset was generated with an assumed $H_0 = 70$. Task 2 extended the analysis to compare ΛCDM and wCDM models, with ΛCDM emerging as slightly preferred due to its lower AIC and BIC values, balancing model simplicity and fit. The posterior mean of $\Omega_{M,0}$ for ΛCDM suggests that approximately 28% of the Universe's density is matter, with 72% attributed to dark energy, reinforcing its role in driving the accelerating expansion.

In Task 1 the posterior predictive mean aligns closely with the observed data, demonstrating the model's ability to capture the relationship between distance modulus and redshift. The credible intervals encompass the majority of the data points, indicating that the model provides a robust fit to the observations.

The ΛCDM model exhibits slightly lower AIC and BIC values compared to the wCDM model, indicating a marginally better fit to the data while maintaining simplicity.

The posterior distribution for $\Omega_{M,0}$ is consistent with the best-fit estimate obtained through optimization. The presented value in this report does not match that of other more rigorous models, the Planck measurements got a value of 0.315 ± 0.007 [3] and from the dark energy survey [4], Ω_M for ΛCDM is 0.331 ± 0.038 and for wCDM, $\Omega_M = 0.321 \pm 0.018$ and $w = 0.987 \pm 0.059$. This might be due to the smaller amount of data used in this project and that radiation has been ignored in our models.

Better inference between q_0 and H_0 might be acquired from using more data for the Monte Carlo sampling. A more rigorous examination of the used priors might help improve the inference. There are many possible choices and here only an inverse gamma was used. This project uses only one model but more can be examined in the same manner as for ΛCDM and wCDM.

References

[1] Chalmers Physics Courses. Tif345 course notebooks repository, 2024. Accessed: Nov 19, 2024.

[2] Paul Erhart, Andreas Ekström, and Arkady Gonoskov. *Advanced Simulation and Machine Learning*. Lecture notes, 2024.

[3] N. Aghanim et al. Planck2018 results: Vi. cosmological parameters. *Astronomy amp; Astrophysics*, 641:A6, September 2020.

[4] T. M. C. Abbott et al. First cosmology results using type ia supernovae from the dark energy survey: Constraints on cosmological parameters. *The Astrophysical Journal Letters*, 872(2):L30, feb 2019.