$$\overline{\mathbf{Q}}_f = \begin{bmatrix} FA_b \\ FS_b \\ d_b FS_b + FM_b \\ FA_e \\ FS_e \\ -d_e FS_e + FM_e \end{bmatrix} \tag{9.36}$$

The procedure for analysis essentially remains the same as developed previously, except that the modified expressions for the stiffness matrices $\bar{\mathbf{k}}$ (Eq. (9.35)) and fixed-end force vectors $\overline{\mathbf{Q}}_f$ (Eq. (9.36)) are used (instead of $\mathbf{k}$ and $\mathbf{Q}_f$, respectively), for members with offset connections.

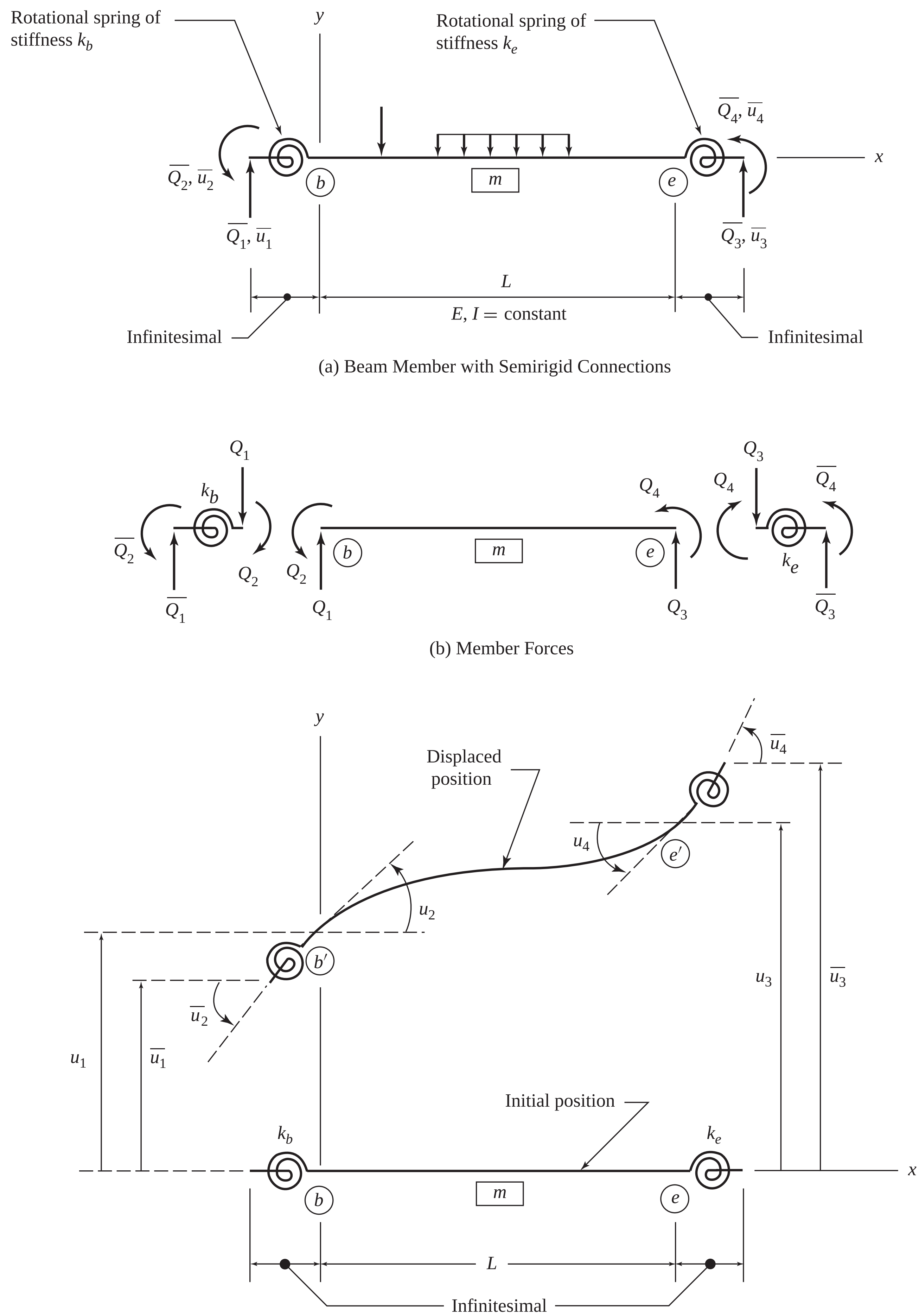
9.6 SEMIRIGID CONNECTIONS

While rigid and hinged types of connections, as considered thus far in this text, are the most commonly used in structural designs, a third type of connection, termed the *semirigid connection,* is also recognized by some design codes, and can be used for designing such structures as structural steel building frames. Recall that the rotation of a rigidly connected member end equals the rotation of the adjacent joint, whereas the rotation of a hinged end of a member must be such that the moment at the hinged end is 0. A connection is considered to be semirigid if its rotational restraint is less than that of a perfectly rigid connection, but more than that of a frictionless hinged connection. In other words, the moment transmitted by a semirigid connection is greater than 0, but less than that transmitted by a rigid connection. For the purpose of analysis, a semirigid connection can be conveniently modeled by a rotational (torsional) spring with stiffness equal to that of the actual connection. In this section, we derive the stiffness relations for members of beams with semirigid connections at their ends. Such relationships for other types of framed structures can be determined by using a similar procedure.

Consider an arbitrary member of a beam, as shown in Fig. 9.14(a) on the next page. The member is connected to the joints adjacent to its ends b and e, by means of rotational springs of infinitesimal size representing the semirigid connections of stiffnesses k_b and k_e, respectively. As shown in this figure, $\overline{\mathbf{Q}}$ and $\bar{\mathbf{u}}$ represent the local end forces and end displacements, respectively, at the exterior ends of the rotational springs. Our objective is to express $\overline{\mathbf{Q}}$ in terms of $\bar{\mathbf{u}}$ and any external loading applied to the member.

We begin by writing, in explicit form, the previously derived relationship $\mathbf{Q} = \mathbf{k}\mathbf{u} + \mathbf{Q}_f$, between the end forces $\mathbf{Q}$ and the end displacements $\mathbf{u}$, which are defined at the actual ends b and e of the member. By using the expressions for $\mathbf{k}$ and $\mathbf{Q}_f$ from Eqs. (5.53) and (5.99), respectively, we write

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} FS_b \\ FM_b \\ FS_e \\ FM_e \end{bmatrix} \tag{9.37}$$



(a) Beam Member with Semirigid Connections

(b) Member Forces

(c) Member Displacements

Fig. 9.14

Figure 9.14(b) shows the forces $\overline{\mathbf{Q}}$ and $\mathbf{Q}$ acting at the exterior and interior ends, respectively, of the member's rotational springs. As the lengths of these springs are infinitesimal, equilibrium equations for the free bodies of the springs yield

$$\mathbf{Q} = \overline{\mathbf{Q}} \tag{9.38}$$

The displacements $\mathbf{u}$ and $\bar{\mathbf{u}}$ are depicted in Fig. 9.14(c) using an exaggerated scale. Because of the infinitesimal size of the springs, the translations of the spring ends are equal; that is,

$$u_1 = \bar{u}_1 \tag{9.39a}$$

$$u_3 = \bar{u}_3 \tag{9.39b}$$

The relationship between the rotations (u_2 and $\bar{u}_2$) of the two ends of the spring, at member end b, can be established by applying the spring stiffness relation:

$$\overline{Q}_2 = k_b\,(\bar{u}_2 - u_2)$$

from which,

$$u_2 = \bar{u}_2 - \frac{\overline{Q}_2}{k_b} \tag{9.39c}$$

Similarly, by using the stiffness relation for the spring attached to member end e, we obtain

$$u_4 = \bar{u}_4 - \frac{\overline{Q}_4}{k_e} \tag{9.39d}$$

To obtain the desired relationship between $\overline{\mathbf{Q}}$ and $\bar{\mathbf{u}}$, we now substitute Eqs. (9.38) and (9.39) into Eq. (9.37) to obtain the following equations.

$$\overline{Q}_1 = \frac{EI}{L^3}\left[12\bar{u}_1 + 6L\left(\bar{u}_2 - \frac{\overline{Q}_2}{k_b}\right) - 12\bar{u}_3 + 6L\left(\bar{u}_4 - \frac{\overline{Q}_4}{k_e}\right)\right] + FS_b \tag{9.40a}$$

$$\overline{Q}_2 = \frac{EI}{L^3}\left[6L\bar{u}_1 + 4L^2\left(\bar{u}_2 - \frac{\overline{Q}_2}{k_b}\right) - 6L\bar{u}_3 + 2L^2\left(\bar{u}_4 - \frac{\overline{Q}_4}{k_e}\right)\right] + FM_b \tag{9.40b}$$

$$\overline{Q}_3 = \frac{EI}{L^3}\left[-12\bar{u}_1 - 6L\left(\bar{u}_2 - \frac{\overline{Q}_2}{k_b}\right) + 12\bar{u}_3 - 6L\left(\bar{u}_4 - \frac{\overline{Q}_4}{k_e}\right)\right] + FS_e \tag{9.40c}$$

$$\overline{Q}_4 = \frac{EI}{L^3}\left[6L\bar{u}_1 + 2L^2\left(\bar{u}_2 - \frac{\overline{Q}_2}{k_b}\right) - 6L\bar{u}_3 + 4L^2\left(\bar{u}_4 - \frac{\overline{Q}_4}{k_e}\right)\right] + FM_e \tag{9.40d}$$

Next, we solve Eqs. (9.40b) and (9.40d) simultaneously, to express $\overline{Q}_2$ and $\overline{Q}_4$ in terms of $\bar{u}_1$ through $\bar{u}_4$. This yields

$$\overline{Q}_2 = \frac{EIr_b}{L^3 R}[6L(2-r_e)\bar{u}_1 + 4L^2(3-2r_e)\bar{u}_2 - 6L(2-r_e)\bar{u}_3 + 2L^2 r_e \bar{u}_4] + \frac{r_b}{R}[(4-3r_e)FM_b - 2(1-r_e)FM_e] \tag{9.41a}$$

$$\overline{Q}_4 = \frac{EIr_e}{L^3 R}[6L(2-r_b)\bar{u}_1 + 2L^2 r_b \bar{u}_2 - 6L(2-r_b)\bar{u}_3 + 4L^2(3-2r_b)\bar{u}_4] + \frac{r_e}{R}[(4-3r_b)FM_e - 2(1-r_b)FM_b] \tag{9.41b}$$

in which r_b and r_e denote the dimensionless rigidity parameters defined as

$$\boxed{r_i = \frac{k_i L}{EI + k_i L} \qquad i = b,\ e} \tag{9.42}$$

and

$$\boxed{R = 12 - 8r_b - 8r_e + 5r_b r_e} \tag{9.43}$$

Finally, by substituting Eqs. (9.41) into Eqs. (9.40a) and (9.40c), we determine expressions for $\overline{Q}_1$ and $\overline{Q}_3$ in terms of $\bar{u}_1$ through $\bar{u}_4$. Thus,

$$\begin{aligned}\overline{Q}_1 = {} & \frac{EI}{L^3 R}[12(r_b + r_e - r_b r_e)\bar{u}_1 + 6Lr_b(2-r_e)\bar{u}_2 \\ & -12(r_b + r_e - r_b r_e)\bar{u}_3 + 6Lr_e(2-r_b)\bar{u}_4] \\ & + FS_b - \frac{6}{LR}[(1-r_b)(2-r_e)FM_b + (1-r_e)(2-r_b)FM_e]\end{aligned} \tag{9.44a}$$

$$\begin{aligned}\overline{Q}_3 = {} & \frac{EI}{L^3 R}[-12(r_b + r_e - r_b r_e)\bar{u}_1 - 6Lr_b(2-r_e)\bar{u}_2 \\ & +12(r_b + r_e - r_b r_e)\bar{u}_3 - 6Lr_e(2-r_b)\bar{u}_4] \\ & + FS_e + \frac{6}{LR}[(1-r_b)(2-r_e)FM_b + (1-r_e)(2-r_b)FM_e]\end{aligned} \tag{9.44b}$$

Equations (9.41) and (9.44), which represent the modified stiffness relations for beam members with semirigid connections at both ends, can be expressed in matrix form:

$$\boxed{\overline{\mathbf{Q}} = \bar{\mathbf{k}}\bar{\mathbf{u}} + \overline{\mathbf{Q}}_f} \tag{9.45}$$

with

$$\bar{\mathbf{k}} = \frac{EI}{L^3R}\begin{bmatrix} 12(r_b + r_e - r_br_e) & 6Lr_b(2-r_e) & -12(r_b + r_e - r_br_e) & 6Lr_e(2-r_b) \\ 6Lr_b(2-r_e) & 4L^2r_b(3-2r_e) & -6Lr_b(2-r_e) & 2L^2r_br_e \\ -12(r_b + r_e - r_br_e) & -6Lr_b(2-r_e) & 12(r_b + r_e - r_br_e) & -6Lr_e(2-r_b) \\ 6Lr_e(2-r_b) & 2L^2r_br_e & -6Lr_e(2-r_b) & 4L^2r_e(3-2r_b) \end{bmatrix} \tag{9.46}$$

and

$$\overline{\mathbf{Q}}_f = \begin{bmatrix} FS_b - \dfrac{6}{LR}[(1-r_b)(2-r_e)FM_b + (1-r_e)(2-r_b)FM_e] \\ \dfrac{r_b}{R}[(4-3r_e)FM_b - 2(1-r_e)FM_e] \\ FS_e + \dfrac{6}{LR}[(1-r_b)(2-r_e)FM_b + (1-r_e)(2-r_b)FM_e] \\ \dfrac{r_e}{R}[-2(1-r_b)FM_b + (4-3r_b)FM_e] \end{bmatrix} \tag{9.47}$$

The $\bar{\mathbf{k}}$ matrix in Eq. (9.46) and the $\overline{\mathbf{Q}}_f$ vector in Eq. (9.47) represent the modified stiffness matrix and fixed-end force vector, respectively, for the members of beams with semirigid connections. It should be noted that these expressions for $\bar{\mathbf{k}}$ and $\overline{\mathbf{Q}}_f$ are valid for the values of the spring stiffness k_i ($i = b$ or e) ranging from 0, which represents a hinged connection, to infinity, which represents a rigid connection. From Eq. (9.42), we can see that as k_i varies from 0 to infinity, the value of the corresponding rigidity parameter r_i varies from 0 to 1. Thus, $r_i = 0$ represents a frictionless hinged connection, whereas $r_i = 1$ represents a perfectly rigid connection. The reader is encouraged to verify that when both r_b and r_e are set equal to 1, then $\bar{\mathbf{k}}$ (Eq. (9.46)) and $\overline{\mathbf{Q}}_f$ (Eq. (9.47)) reduce the $\mathbf{k}$ (Eq. (5.53)) and $\mathbf{Q}_f$ (Eq. (5.99)) for a beam member rigidly connected at both ends. Similarly, the expressions of $\mathbf{k}$ and $\mathbf{Q}_f$, derived in Chapter 7 for beam members with three combinations of rigid and hinged connections (i.e., $MT = 1$, 2, and 3), can be obtained from Eqs. (9.46) and (9.47), respectively, by setting r_b and r_e to 0 or 1, as appropriate.

The procedure for analysis of beams with rigid and hinged connections, developed previously, can be applied to beams with semirigid connections—provided that the modified member stiffness matrix $\bar{\mathbf{k}}$ (Eq. (9.46)) and fixed-end force vector $\overline{\mathbf{Q}}_f$ (Eq. (9.47)) are used in the analysis.

9.7 SHEAR DEFORMATIONS

The stiffness relations that have been developed thus far for beams, grids, and frames, do not include the effect of shear deformations of members. Such structures are generally composed of members with relatively large length-to-depth ratios, so that their shear deformations are usually negligibly small as compared to the bending deformations. However, in the case of beams, grids