Example 2.7 Linear Column with Two Plastic Hinges

Consider that the SDOF column with two PHLs shown in Figure 2.9(a) is subjected to a lateral applied force F_o. Let the moment versus plastic rotation relationships of the two plastic hinges follow those plotted in Figure 2.9(b), where $m_{y1} = 2F_oL$, $m_{y2} = F_oL$, $K_{t1} = EI/L$, and $K_{t2} = 2EI/L$. These relationships can be summarized as:

$$\textit{if} \begin{cases} m_1 \le 2F_oL \\ m_1 > 2F_oL \end{cases}, \quad \textit{then} \begin{cases} \theta_1'' = 0 \\ m_1 = 2F_oL + (EI/L)\,\theta_1'' \end{cases} \tag{2.77a}$$

$$\textit{if} \begin{cases} m_2 \le F_oL \\ m_2 > F_oL \end{cases}, \quad \textit{then} \begin{cases} \theta_2'' = 0 \\ m_2 = F_oL + (2EI/L)\,\theta_2'' \end{cases} \tag{2.77b}$$

Based on the structure's configuration shown in Figure 2.9(a), the stiffness matrices are:

$$\mathbf{K} = \left[\frac{12EI}{L^3}\right], \quad \mathbf{K}' = \begin{bmatrix} \dfrac{6EI}{L^2} & \dfrac{6EI}{L^2} \end{bmatrix}, \quad \mathbf{K}'' = \begin{bmatrix} 4EI/L & 2EI/L \\ 2EI/L & 4EI/L \end{bmatrix} \tag{2.78}$$

and therefore,

$$\left[\begin{array}{c:cc} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ \hdashline 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{array}\right] \left[\begin{array}{c} x \\ \hdashline -\theta_1'' \\ -\theta_2'' \end{array}\right] = \left\{\begin{array}{c} F_o \\ \hdashline m_1 \\ m_2 \end{array}\right\} \tag{2.79}$$

First assume that the structure is linear, this gives $\theta_1'' = \theta_2'' = 0$. Then using the first equation of Eq. (2.79) gives

$$\frac{12EI}{L^3}x = F_o \tag{2.80}$$

Solving for the displacement x in Eq. (2.80) gives

$$x = F_oL^3/12EI \tag{2.81}$$

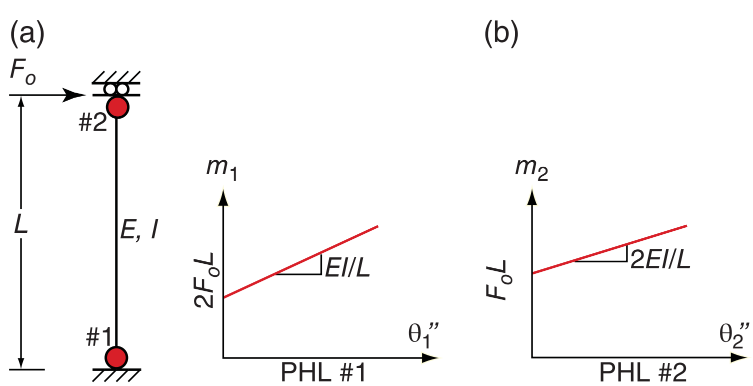

Figure 2.9 A SDOF system with two PHLs: (a) A single-degree-of-freedom system; (b) Moment versus plastic rotation relationships of the plastic hinges.

Then substituting Eq. (2.81) into the second and third equations of Eq. (2.79), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 \\ 6EI/L^2 \end{bmatrix} \left(\frac{F_o L^3}{12EI} \right) = \begin{Bmatrix} F_o L/2 \\ F_o L/2 \end{Bmatrix} \tag{2.82}$$

Comparing these moment demands in Eq. (2.82) with the yield moments in Eq. (2.77) shows that the plastic hinges have not yielded, which means the linear structure assumption is correct. Therefore, the final displacement and moments are presented in Eqs. (2.81) and (2.82), respectively.

Example 2.8 Nonlinear Column with Yielding in One Plastic Hinge

Consider again the SDOF column as shown in Figure 2.9(a) with the moment versus plastic rotation relationships shown in Figure 2.9(b). Now let the lateral applied force be equal to $3F_o$. The matrix equation used for solving the problem is similar to Eq. (2.79) except for the applied load, i.e.

$$\left[\begin{array}{c:cc} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ \hdashline 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{array}\right] \left\{\begin{array}{c} x \\ \hdashline -\theta_1'' \\ -\theta_2'' \end{array}\right\} = \left\{\begin{array}{c} 3F_o \\ \hdashline m_1 \\ m_2 \end{array}\right\} \tag{2.83}$$

Again, the structure is first assumed to be linear, this gives $\theta_1'' = \theta_2'' = 0$. Then using the first equation of Eq. (2.83) gives

$$\frac{12EI}{L^3} x = 3F_o \tag{2.84}$$

Solving for the displacement x in Eq. (2.84) gives

$$x = F_o L^3 / 4EI \tag{2.85}$$

Then substituting Eq. (2.85) into the second and third equations of Eq. (2.83), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 \\ 6EI/L^2 \end{bmatrix} \left(\frac{F_o L^3}{4EI} \right) = \begin{Bmatrix} 3F_o L/2 \\ 3F_o L/2 \end{Bmatrix} \tag{2.86}$$

Comparing these moment demands in Eq. (2.86) with the yield moments in Eq. (2.77) shows that PHL #2 has yielded, but PHL #1 remains elastic. This means the original assumption that the structure is linear is incorrect.

Now assume that PHL #2 has yielded while PHL #1 remains elastic. This gives

$$\theta_1'' = 0, \quad m_2 = F_o L + (2EI/L)\, \theta_2'' \tag{2.87}$$

Substituting Eq. (2.87) into Eq. (2.83) and extracting the first and third equations of the matrix equation gives

$$\begin{bmatrix} 12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L+2EI/L \end{bmatrix} \begin{Bmatrix} x \\ -\theta_2'' \end{Bmatrix} = \begin{Bmatrix} 3F_o \\ F_oL \end{Bmatrix} \tag{2.88}$$

Solving for Eq. (2.88) gives

$$\begin{bmatrix} x \\ -\theta_2'' \end{bmatrix} = \begin{bmatrix} 12EI/L^3 & 6EI/L^2 \\ 6EI/L^2 & 6EI/L \end{bmatrix}^{-1} \begin{bmatrix} 3F_o \\ F_oL \end{bmatrix} = \begin{Bmatrix} F_oL^3/3EI \\ -F_oL^2/6EI \end{Bmatrix} \tag{2.89}$$

Then substituting Eq. (2.89) into the second and third equations of Eq. (2.83), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 & 2EI/L \\ 6EI/L^2 & 4EI/L \end{bmatrix} \begin{Bmatrix} F_oL^3/3EI \\ -F_oL^2/6EI \end{Bmatrix} = \begin{Bmatrix} 5F_oL/3 \\ 4F_oL/3 \end{Bmatrix} \tag{2.90}$$

Comparing the moment m_1 with the corresponding yield moment in Eq. (2.77) shows that PHL #1 remains elastic. This means the original assumption that PHL #2 has yielded and PHL #1 remains elastic is correct. Therefore, in summary,

$$x = \frac{F_oL^3}{3EI}, \quad \begin{bmatrix} \theta_1'' \\ \theta_2'' \end{bmatrix} = \begin{Bmatrix} 0 \\ F_oL^2/6EI \end{Bmatrix}, \quad \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 5F_oL/3 \\ 4F_oL/3 \end{Bmatrix} \tag{2.91}$$

Example 2.9 Nonlinear Column with Yielding in All Plastic Hinges

Consider again the SDOF column as shown in Figure 2.9(a) with the moment versus plastic rotation relationships shown in Figure 2.9(b). Now let the lateral applied force be equal to $5F_o$. The matrix equation used for solving the problem is similar to Eq. (2.79) except for the applied load, i.e.

$$\left[\begin{array}{c:cc} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ \hdashline 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{array}\right] \left\{\begin{array}{c} x \\ \hdashline -\theta_1'' \\ -\theta_2'' \end{array}\right\} = \left\{\begin{array}{c} 5F_o \\ \hdashline m_1 \\ m_2 \end{array}\right\} \tag{2.92}$$

Again, the structure is first assumed to be linear, this gives $\theta_1''=\theta_2''=0$. Then using the first equation of Eq. (2.92) gives

$$\frac{12EI}{L^3}x = 5F_o \tag{2.93}$$

Solving for the displacement x in Eq. (2.93) gives

$$x = 5F_oL^3/12EI \tag{2.94}$$

Then substituting Eq. (2.94) into the second and third equations of Eq. (2.92), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 \\ 6EI/L^2 \end{bmatrix} \left(\frac{5F_oL^3}{12EI} \right) = \begin{Bmatrix} 5F_oL/2 \\ 5F_oL/2 \end{Bmatrix} \tag{2.95}$$

Comparing these moment demands in Eq. (2.95) with the yield moments in Eq. (2.77) shows that both PHLs #1 and #2 have yielded. This means the original assumption that the structure is linear is incorrect.

Now assume that both PHLs #1 and #2 have yielded. This gives

$$m_1 = 2F_oL + (EI/L)\,\theta_1'', \quad m_2 = F_oL + (2EI/L)\,\theta_2'' \tag{2.96}$$

Substituting Eq. (2.96) into Eq. (2.92) gives

$$\begin{bmatrix} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L + EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L + 2EI/L \end{bmatrix} \begin{Bmatrix} x \\ -\theta_1'' \\ -\theta_2'' \end{Bmatrix} = \begin{Bmatrix} 5F_o \\ 2F_oL \\ F_oL \end{Bmatrix} \tag{2.97}$$

Solving for Eq. (2.97) gives

$$\begin{Bmatrix} x \\ -\theta_1'' \\ -\theta_2'' \end{Bmatrix} = \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ 6EI/L^2 & 5EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 6EI/L \end{bmatrix}^{-1} \begin{Bmatrix} 5F_o \\ 2F_oL \\ F_oL \end{Bmatrix} = \begin{Bmatrix} 16F_oL^3/15EI \\ -3F_oL^2/5EI \\ -7F_oL^2/10EI \end{Bmatrix} \tag{2.98}$$

Then substituting Eq. (2.98) into the second and third equations of Eq. (2.92), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{bmatrix} \begin{Bmatrix} 16F_oL^3/15EI \\ -3F_oL^2/5EI \\ -7F_oL^2/10EI \end{Bmatrix} = \begin{Bmatrix} 13F_oL/5 \\ 12F_oL/5 \end{Bmatrix} \tag{2.99}$$

These moments can similarly be obtained using Eq. (2.96), i.e.

$$m_1 = 2F_oL + \left(\frac{EI}{L}\right)\left(\frac{3F_oL^2}{5EI}\right) = \frac{13F_oL}{5} \tag{2.100a}$$

$$m_2 = F_oL + \left(\frac{2EI}{L}\right)\left(\frac{7F_oL^2}{10EI}\right) = \frac{12F_oL}{5} \tag{2.100b}$$

Comparing the moment demands m_1 and m_2 with the corresponding yield moments in Eq. (2.77) shows that both PHLs #1 and #2 have yielded. This means the original assumption is correct. Therefore, in summary,

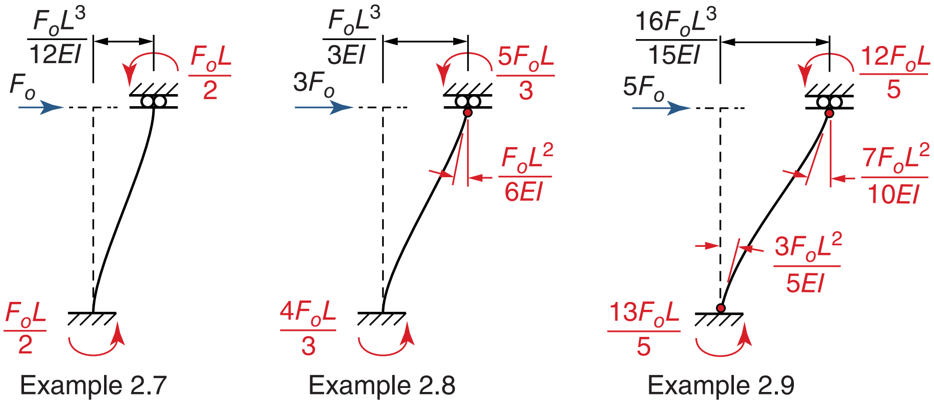

Figure 2.10 Graphical illustration of the responses of the SDOF system.

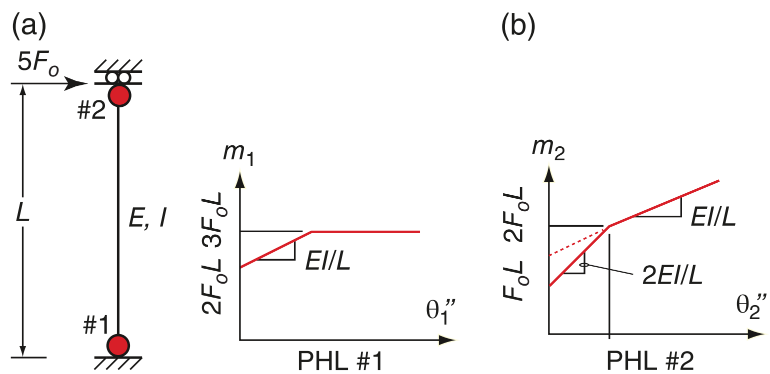

Figure 2.11 A SDOF system with two PHLs: (a) A single-degree-of-freedom system; (b) Moment versus plastic rotation relationships of the plastic hinges.

$$x = \frac{16F_oL^3}{15EI}, \quad \begin{bmatrix} \theta_1'' \\ \theta_2'' \end{bmatrix} = \begin{Bmatrix} 3F_oL^2/5EI \\ 7F_oL^2/10EI \end{Bmatrix}, \quad \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 13F_oL/5 \\ 12F_oL/5 \end{Bmatrix} \tag{2.101}$$

Figure 2.10 shows a summary of the results in Examples 2.7, 2.8, and 2.9.

Example 2.10 Nonlinear Column with Tri-linear Plastic Rotation Behavior

Consider that the SDOF column with two PHLs shown in Figure 2.11(a) is subjected to a lateral applied force $5F_o$. Let the moment versus plastic rotation relationships of the plastic hinges follow the tri-linear relationships plotted in Figure 2.11(b). These relationships can be expressed in equation forms as:

$$\text{if} \left\{ \begin{array}{c} m_1 \le 2F_oL \\ 2F_oL < m_1 \le 3F_oL, \\ m_1 > 3F_oL \end{array} \right. \quad \text{then} \left\{ \begin{array}{c} \theta_1'' = 0 \\ m_1 = 2F_oL + (EI/L)\,\theta_1'' \\ m_1 = 3F_oL \end{array} \right. \tag{2.102a}$$

$$\text{if} \left\{ \begin{array}{c} m_2 \le F_oL \\ F_oL < m_2 \le 2F_oL, \\ m_2 > 2F_oL \end{array} \right. \quad \text{then} \left\{ \begin{array}{c} \theta_2'' = 0 \\ m_2 = F_oL + (2EI/L)\,\theta_2'' \\ m_2 = 3F_oL/2 + (EI/L)\,\theta_2'' \end{array} \right. \tag{2.102b}$$

The matrix equation used for solving the problem can be obtained, just like the one in Eq. (2.92), as:

$$\left[\begin{array}{c:cc} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ \hdashline 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{array}\right]\left\{\begin{array}{c} x \\ \hdashline -\theta_1'' \\ -\theta_2'' \end{array}\right\}=\left\{\begin{array}{c} 5F_o \\ \hdashline m_1 \\ m_2 \end{array}\right\} \tag{2.103}$$

The structure is first assumed to be linear, this gives $\theta_1''=\theta_2''=0$. Then using the first equation of Eq. (2.103) gives

$$\frac{12EI}{L^3}x=5F_o \tag{2.104}$$

Solving for the displacement x in Eq. (2.104) gives

$$x=5F_oL^3/12EI \tag{2.105}$$

Then substituting Eq. (2.105) into the second and third equations of Eq. (2.103), the moments are calculated as

$$\left\{\begin{array}{c} m_1 \\ m_2 \end{array}\right\}=\left[\begin{array}{c} 6EI/L^2 \\ 6EI/L^2 \end{array}\right]\left(\frac{5F_oL^3}{12EI}\right)=\left\{\begin{array}{c} 5F_oL/2 \\ 5F_oL/2 \end{array}\right\} \tag{2.106}$$

Comparing these moment demands in Eq. (2.106) with the yield moments in Eq. (2.102) shows that both PHLs #1 and #2 have yielded. This means the original assumption that the structure is linear is incorrect.

Now assume that both PHLs #1 and #2 have yielded. Based on the moment demands in Eq. (2.106), it is also assumed that PHL #1 is at the first nonlinear slope and PHL #2 is at the second nonlinear slope. According to Eq. (2.102), this gives

$$m_1=2F_oL+(EI/L)\,\theta_1'', \qquad m_2=3F_oL/2+(EI/L)\,\theta_2'' \tag{2.107}$$

Substituting Eq. (2.107) into Eq. (2.103) gives

$$\left[\begin{array}{ccc} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L+EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L+EI/L \end{array}\right]\left\{\begin{array}{c} x \\ -\theta_1'' \\ -\theta_2'' \end{array}\right\}=\left\{\begin{array}{c} 5F_o \\ 2F_oL \\ 3F_oL/2 \end{array}\right\} \tag{2.108}$$

Solving for Eq. (2.108) gives

$$\left\{\begin{array}{c} x \\ -\theta_1'' \\ -\theta_2'' \end{array}\right\}=\left[\begin{array}{ccc} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ 6EI/L^2 & 5EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 5EI/L \end{array}\right]^{-1}\left\{\begin{array}{c} 5F_o \\ 2F_oL \\ 3F_oL/2 \end{array}\right\}=\left\{\begin{array}{c} 7F_oL^3/6EI \\ -2F_oL^2/3EI \\ -5F_oL^2/6EI \end{array}\right\} \tag{2.109}$$

Then substituting Eq. (2.109) into the second and third equations of Eq. (2.103), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{bmatrix} \begin{Bmatrix} 7F_oL^3/6EI \\ -2F_oL^2/3EI \\ -5F_oL^2/6EI \end{Bmatrix} = \begin{Bmatrix} 8F_oL/3 \\ 7F_oL/3 \end{Bmatrix} \tag{2.110}$$

These moments can similarly be obtained using Eq. (2.107), i.e.

$$m_1 = 2F_oL + \left(\frac{EI}{L}\right)\left(\frac{2F_oL^2}{3EI}\right) = \frac{8F_oL}{3} \tag{2.111a}$$

$$m_2 = \frac{3F_oL}{2} + \left(\frac{EI}{L}\right)\left(\frac{5F_oL^2}{6EI}\right) = \frac{7F_oL}{3} \tag{2.111b}$$

Comparing the moment demands m_1 and m_2 with the corresponding yield moments in Eq. (2.102) shows that PHL #1 is responding at the first nonlinear slope and PHL #2 is responding at the second nonlinear slope. This means the original assumption is correct. Therefore, in summary,

$$x = \frac{7F_oL^3}{6EI}, \qquad \begin{bmatrix} \theta_1'' \\ \theta_2'' \end{bmatrix} = \begin{Bmatrix} 2F_oL^2/3EI \\ 5F_oL^2/6EI \end{Bmatrix}, \qquad \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 8F_oL/3 \\ 7F_oL/3 \end{Bmatrix} \tag{2.112}$$

Example 2.11 Nonlinear Column with Softening Behavior

Consider again that the SDOF column with two PHLs shown in Figure 2.11(a) is subjected to a lateral applied force $5F_o$. But now, let the moment versus plastic rotation relationships of the plastic hinges follow:

$$\text{if } \begin{cases} m_1 \le 2F_oL \\ m_1 > 2F_oL \end{cases}, \quad \text{then } \begin{cases} \theta_1'' = 0 \\ m_1 = 2F_oL + (2EI/L)\,\theta_1'' \end{cases} \tag{2.113a}$$

$$\text{if } \begin{cases} m_2 \le F_oL \\ F_oL < m_2 \le 2F_oL \\ m_2 > 2F_oL \end{cases}, \quad \text{then } \begin{cases} \theta_2'' = 0 \\ m_2 = F_oL + (2EI/L)\,\theta_2'' \\ m_2 = 9F_oL/4 - (EI/2L)\,\theta_2'' \end{cases} \tag{2.113b}$$

Note that PHL #1 exhibits a bilinear hardening behavior, while PHL #2 exhibits a tri-linear softening behavior with a slope of $-EI/2L$ in the second nonlinear slope.

The matrix equation used for solving the problem can again be obtained as:

$$\left[\begin{array}{c|cc} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ \hline 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{array}\right] \left\{\begin{array}{c} x \\ \hline -\theta_1'' \\ -\theta_2'' \end{array}\right\} = \left\{\begin{array}{c} 5F_o \\ \hline m_1 \\ m_2 \end{array}\right\} \tag{2.114}$$

The structure is first assumed to be linear, this gives $\theta_1'' = \theta_2'' = 0$. Then using the first equation of Eq. (2.114) gives

$$\frac{12EI}{L^3}x = 5F_o \tag{2.115}$$

Solving for the displacement x gives

$$x = 5F_oL^3/12EI \tag{2.116}$$

Then substituting Eq. (2.116) into the second and third equations of Eq. (2.114), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 \\ 6EI/L^2 \end{bmatrix} \left(\frac{5F_oL^3}{12EI} \right) = \begin{Bmatrix} 5F_oL/2 \\ 5F_oL/2 \end{Bmatrix} \tag{2.117}$$

Comparing these moment demands in Eq. (2.117) with the yield moments in Eq. (2.113) shows that both PHLs #1 and #2 have yielded. This means the original assumption that the structure is linear is incorrect.

Now assume that both PHLs #1 and #2 have yielded. Based on the moment demands in Eq. (2.117), it is seen that the moment demand at PHL #2 has exceeded its maximum moment capacity, and therefore PHL #2 is assumed to reach its softening stage (i.e. the second nonlinear slope). According to Eq. (2.113), this gives

$$m_1 = 2F_oL + (2EI/L)\,\theta_1'', \quad m_2 = 9F_oL/4 - (EI/2L)\,\theta_2'' \tag{2.118}$$

Substituting Eq. (2.118) into Eq. (2.114) gives

$$\begin{bmatrix} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ 6EI/L^2 & 4EI/L + 2EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L - EI/2L \end{bmatrix} \begin{Bmatrix} x \\ -\theta_1'' \\ -\theta_2'' \end{Bmatrix} = \begin{Bmatrix} 5F_o \\ 2F_oL \\ 9F_oL/4 \end{Bmatrix} \tag{2.119}$$

Solving for Eq. (2.119) gives

$$\begin{Bmatrix} x \\ -\theta_1'' \\ -\theta_2'' \end{Bmatrix} = \begin{bmatrix} 12EI/L^3 & 6EI/L^2 & 6EI/L^2 \\ 6EI/L^2 & 6EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 7EI/2L \end{bmatrix}^{-1} \begin{Bmatrix} 5F_o \\ 2F_oL \\ 9F_oL/4 \end{Bmatrix} = \begin{Bmatrix} 13F_oL^3/6EI \\ -F_oL^2/EI \\ -5F_oL^2/2EI \end{Bmatrix} \tag{2.120}$$

Then substituting Eq. (2.120) into the second and third equations of Eq. (2.114), the moments are calculated as

$$\begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{bmatrix} 6EI/L^2 & 4EI/L & 2EI/L \\ 6EI/L^2 & 2EI/L & 4EI/L \end{bmatrix} \begin{Bmatrix} 13F_oL^3/6EI \\ -F_oL^2/EI \\ -5F_oL^2/2EI \end{Bmatrix} = \begin{Bmatrix} 4F_oL \\ F_oL \end{Bmatrix} \tag{2.121}$$

These moments can similarly be obtained using Eq. (2.118), i.e.

$$m_1 = 2F_oL + \left(\frac{2EI}{L} \right) \left(\frac{F_oL^2}{EI} \right) = 4F_oL \tag{2.122a}$$

$$m_2 = \frac{9F_oL}{4} - \left(\frac{EI}{2L} \right) \left(\frac{5F_oL^2}{2EI} \right) = F_oL \tag{2.122b}$$

Comparing the moment demand for m_1 with the corresponding yield moment in Eq. (2.113) shows that PHL #1 has yielded. In addition, the combination of plastic rotation θ_2'' in Eq. (2.120) and moment demand m_2 in Eq. (2.121) indicates that the plastic hinge is responding in the softening stage and has not reached the limiting value of zero moment. Therefore, this means the original assumption is correct. In summary,

$$x = \frac{13F_oL^3}{6EI}, \quad \begin{bmatrix} \theta_1'' \\ \theta_2'' \end{bmatrix} = \begin{Bmatrix} F_oL^2/EI \\ 5F_oL^2/2EI \end{Bmatrix}, \quad \begin{Bmatrix} m_1 \\ m_2 \end{Bmatrix} = \begin{Bmatrix} 4F_oL \\ F_oL \end{Bmatrix} \tag{2.123}$$

Example 2.12 Nonlinear Two-Story Frame

Consider the two-story moment-resisting frame as shown in Figure 2.12. Assume that all the members are axially rigid, which therefore gives a total of 6 DOFs (i.e. $n = 6$, with 2 translations and 4 rotations) and 12 PHLs (i.e. $q = 12$). All the beams and columns have the same elastic modulus E. All columns have length L and moment of inertia I, while the beams have length $2L$ and moment of inertia $2I$. The frame is subjected to a lateral force F_o on the second floor and another lateral force F_o on the third floor.

For this two-story frame, the displacement, force, moment, and plastic rotation vectors are:

$$\mathbf{x} = \{x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6\}^T \tag{2.124a}$$

$$\mathbf{F} = \{F_o \ F_o \ 0 \ 0 \ 0 \ 0\}^T \tag{2.124b}$$

$$\mathbf{\Theta}'' = \{\theta_1'' \ \theta_2'' \ \theta_3'' \ \theta_4'' \ \theta_5'' \ \theta_6'' \ \theta_7'' \ \theta_8'' \ \theta_9'' \ \theta_{10}'' \ \theta_{11}'' \ \theta_{12}''\}^T \tag{2.124c}$$

$$\mathbf{m} = \{m_1 \ m_2 \ m_3 \ m_4 \ m_5 \ m_6 \ m_7 \ m_8 \ m_9 \ m_{10} \ m_{11} \ m_{12}\}^T \tag{2.124d}$$

The 6×6 stiffness matrix $\mathbf{K}$, the 6×12 stiffness matrix $\mathbf{K}'$, and the 12×12 stiffness matrix $\mathbf{K}''$ are:

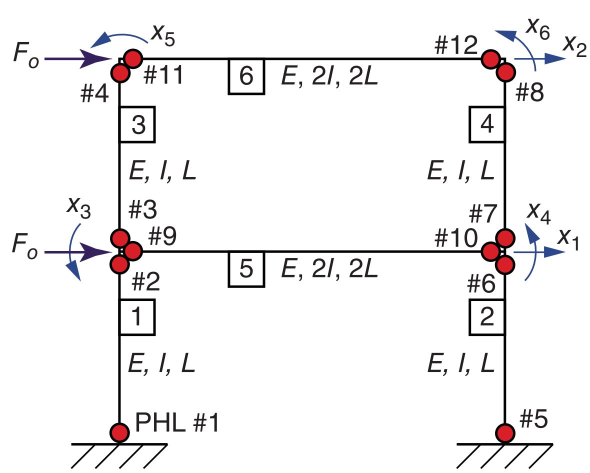

Figure 2.12 A two-story frame with 12 PHLs.