مدينة زويل للعلوم والتكنولوجيا

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MATH 201

Linear Algebra and Vector Geometry

Open Courseware

Lecture #1

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Definition 1.1: Linear Equation

It is an equation where all the variables have a power of 1, with the general formula $a_1x_1 + a_2x_2 + a_3x_3..... = b$.

Example.

$$x = y; y = 2x + 3.$$

It is called a linear equation because, in 2-D, its solution is a point of intersection of two straight lines. and the two variables it can have are x and y. While in 3-D, the Solution is a plane produced by the intersections of straight lines.

Definition 1.2: A system of linear equations

It is a group of linear equations with the Same Shared Solution, which is solved simultaneously.

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3..... = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3..... = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3..... = b_3 \end{cases}$$

here is a system of three equations of three variables, more equations, and variables can be included in one system of equations.

Example.

$$x + 2y = 5.$$

$$x + 3y = 8.$$

$$x - x + 2y - 3y = 8 - 5 \Rightarrow y = 3$$

solving in one of the first two equations gives $x = -1$

we can say that

$$\begin{cases} x + 2y = 5 \\ y = 3 \end{cases} \quad and \quad \begin{cases} x + 2y = 5 \\ x + 3y = 8 \end{cases}.$$

are a row equivalent system, which means they can be changed into each other by elementary operations.

1.1 Representing a system of linear equations in a matrix

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ...a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ...a_{2n}x_n = b_2 \\ a_{n1}x_1 + a_{n2}x_2 + ...a_{nn}x_n = b_n \end{cases} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \cdots b_n \end{bmatrix}$$

in the last raw of the matrix a_{n1}, a_{n2}, a_{nn} represent the coefficient of the matrix.

Example.

$$\begin{cases} 2x + y + z = 1 \\ -y - 2x = -4 \\ -4x = -4 \end{cases} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ -2 & -1 & 0 & -4 \\ -4 & 0 & 0 & -4 \end{bmatrix}.$$

We can perform a number of operations on a matrix that will also not affect the solution.

1. Multiplication or division of any row by another one.
2. Adding any 2 rows to each other.
3. Interchanging any rows.

Note:-

this is called Gauss elimination with back substitution.

Example (system of two variables).

$$\begin{cases} 2x + y = 4 \\ x + 2y = 5 \end{cases} \Rightarrow \begin{cases} x + 2y = 5 \\ 2x + y = 4 \end{cases} \Rightarrow \begin{cases} -2x - 4y = -10 \\ 2x + y = 4 \end{cases} \Rightarrow \begin{cases} x + 2y = 5 \\ -3y = -6 \end{cases}$$

Now for a bigger system

$$\begin{cases} 2x + y + z = 1 \\ 6x + 2y + z = -1 \\ -2x + 2y + z = 7 \end{cases} \Rightarrow \begin{bmatrix} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{bmatrix}$$

We want to reach something like this:

$$2x + y + z = 1$$
$$()y + ()z = () \rightarrow \text{no x}$$
$$\text{or } ()z = () \rightarrow \text{no x or y}$$

So we keep x in the 1st equation and eliminate it from the rest, then we keep y in the 2nd equation and the 1st and eliminate it from the 3rd, and so on...

Example.

$$1 \begin{bmatrix} 2 & 1 & 1 & 1 \\ 6 & 2 & 1 & -1 \\ -2 & 2 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 x - 3 + R_2} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ -2 & 2 & 1 & 7 \end{bmatrix}$$

to eliminate 6 we multiply the 1st row by -3 and add it to the 2nd one

$$2 \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ -2 & 2 & 1 & 7 \end{bmatrix} \xrightarrow{R_1 + R_3} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{bmatrix}$$

$$3 \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 3 & 2 & 8 \end{bmatrix} \xrightarrow{R_2 x 3 + R_3} \begin{bmatrix} 2 & 1 & 1 & 1 \\ 0 & -1 & -2 & -4 \\ 0 & 0 & -4 & -4 \end{bmatrix} \rightarrow \text{This is called the echelon form}$$

so, we can now write it as

$$\begin{aligned} 2x + y + z &= 1 \\ -y - 2z = -4 \Rightarrow y &= 4 - 2z \\ -4z = -4 \Rightarrow z &= 1 \end{aligned}$$

by substitution, we can get the solution which is:

$$x = -1; y = 2; z = 1$$

Note:-

the echelon form should look like this:

$\begin{bmatrix} \blacksquare & \square & \square \\ \triangle & \blacksquare & \square \\ \triangle & \triangle & \blacksquare \end{bmatrix}$ $\rightarrow$where $\blacksquare$ is any non-zero(pivot) and $\square$ is any number.

Note:-

to check your answer substitute in all equations of the system.

Now let's see if every system of linear equations has a solution and if it does is it a unique one?

1.
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases} \Rightarrow \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix} \xrightarrow{R-1+R_2} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
so,
$$x_1 = 3; X_2 = 2$$
this system has a unique solution that represents a point of intersection of the two lines that are represented by the system's equations.

2.
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 2x_2 = 1 \end{cases} \Rightarrow \text{these are two coincident lines}$$
so the solution will be the line this system represents$\rightarrow$infinite points $\rightarrow$infinite solutions. For the solution, we write the general form $x_1 = 2x_2 + 1$

3. If the two equations of the system represent two parallel lines then there is no solution
$$\begin{cases} x_1 - 2x_1 = -1 \\ -x_1 + 2x_1 = 3 \end{cases} \Rightarrow \text{parallel lines}$$
$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 2 & 3 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow x_1 + 2x_2 = -1 \tag{1}$$
solving the equation gives 0=2 $\rightarrow$contradiction

Now for 3 equations system:

$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\rightarrow$ this is the echelon form, but there are pivots for x,y and not for z

$\begin{cases} x + y + 2x = 2 \\ 2y + 2z = 4 \end{cases}$ $\rightarrow$ any value of z would work $\rightarrow$ there are infinite solutions.

Note:-

when we don't find a pivot for one of the variables we call it"the pivot-free variable" and the others are "the pivot variables".

1. $\begin{bmatrix} \blacksquare & \square & \square & \square \\ \triangle & \blacksquare & \square & \square \\ \triangle & \triangle & \blacksquare & \square \end{bmatrix}$ $\rightarrow$unique solution

2. $\begin{bmatrix} \blacksquare & \square & \square & \square \\ \triangle & \blacksquare & \square & \square \\ \triangle & \triangle & \triangle & \triangle \end{bmatrix}$ $\rightarrow$infinite solution

3. $\begin{bmatrix} \blacksquare & \square & \square & \square \\ \triangle & \blacksquare & \square & \square \\ \triangle & \triangle & \triangle & \blacksquare \end{bmatrix}$ $\rightarrow$contradiction(no solution)

where $\blacksquare$ is any non-zero(pivot) and $\square$ is any number including zero.

1.2 Row echelon form:

1. All non-Zero rows are above zero rows.
2. First number of a row is on the right of the first number (pivot) in the row above.
3. All entries below each pivot are zero.

Note:

- If there are any free variables, that means that Now the system has infinite solutions.
- If there are any Contradictions, that means that the system has no solution.
- If all pivots are present (no. of pivots = of variables), then the system has one, unique solution.
- A system of equations is Consistent if it has any no of solutions.
- A system of equations is inconsistent if it has no solutions.

Example. $\begin{cases} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{cases} \rightarrow \begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{-2R_2+R_3} \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}$
$\xrightarrow{2R_2+R_3} \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix} \rightarrow 0 = 15$ Contradiction -no solution-.

Example. $\begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 1 & 2 & 4 \end{bmatrix}$ $\rightarrow$3 parallel planes which mean that there is no solution.

Note:-

the ratio between entries above each other $\neq 1$ always/ is not constant, so which means they are parallel, not coincident.

Example. $A \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} \to$ No solution $B \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 5 & 0 \end{bmatrix} \to$ Unique solution

$C \begin{bmatrix} 1 & 1 & 2 & 2 & 4 \\ 0 & 2 & 1 & 4 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \to$ No solution $D \begin{bmatrix} 1 & 1 & 2 & 2 & 4 \\ 0 & 2 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$ No solution

$E \begin{bmatrix} 1 & 1 & 2 & 2 & 4 \\ 0 & 2 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix} \to$ No solution

$F \begin{bmatrix} 1 & 1 & 2 & 2 & 4 \\ 0 & 2 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to$ infinte solutions $\to \begin{cases} x_1 + x_2 + x_2 + 2x_3 + 2x_4 = 4 \\ 2x_2 + x_3 + 4x_4 = 4 \end{cases}$

$G \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to$ Unique solution $\to 2x_2 = 1 \to x_2 = \frac{1}{2}, x_1 = \frac{3}{2}$

$H \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix} \to$ No solution $\to$ 2 variables-contradiction-

the general solution for example F is

- $x_2 = \frac{4 - x_3 - 4x_4}{2}$
- $x_1 = 4 - 2x_3 - 2x_4 - x_2 = 4 - 2x_3 - 2x_4 - \frac{4 - x_3 - 4x_4}{2}$