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Contents

1 Operations of Matrices

Definition 1.1: Matrix

It is basically a table of numbers arranged in rows and columns.

Example (Matrix).

$$A_{2\times 3} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

If we have a matrix in which the number of rows is equal to the number of columns, then it is called a square matrix.

Note:-

For symbol notation, we usually use the capital letters for matrices and the small ones for the entries:

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

1.1 Addition of Matrices

We can only add two matrices if they have the same dimensions.

Example.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

1.2 Scalar Multiplication of a Matrix

$$k \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}.$$

Example.

$$2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}.$$

1.3 Multiplication of Matrices

To be able to multiply two matrices, the number of columns of the first matrix must be equal to the number of rows of the second matrix.

$$A_{m\times n} \times B_{n\times p} = C_{m\times p}.$$

Example (Compute AB).

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}, B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2\times 4 + 3\times 1 & 2\times 3 + 3\times(-2) & 2\times 6 + 3\times 3 \\ 1\times 4 + (-5)\times 1 & 1\times 3 + (-5)\times(-2) & 1\times 6 + (-5)\times 3 \end{bmatrix} \\ &= \begin{bmatrix} 10 & -4 & 18 \\ 1 & 11 & 3 \end{bmatrix}. \end{aligned}$$

1.3.1 Row-vector rule

$$(AB)_{ij} = \text{row}_i(A) \cdot \vec{b}_j.$$

Example. If we want to find the 3rd element in the 1st row of AB, we dot product A's 1st row with B's 3rd column.

For $A_{m\times n}$ and $B_{n\times p}$,

- AB exist if $n = p$.
- AB exist if $m \times q$ matrix.
- $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Example (Find AB and BA).

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$$

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -3 \end{bmatrix} \\ BA &= \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 4 & -2 \end{bmatrix}. \end{aligned}$$

Note:-

$$AB \neq BA.$$

Example ($AB = 0$ while $A \neq 0$ and $B \neq 0$).

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Example (Find $(A+B)^2$, $A^2 + 2AB + B^2$).

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 0 & 7 \end{bmatrix} \\ B^2 &= \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 0 & 4 \end{bmatrix} \\ AB &= \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 4 & -4 \end{bmatrix} \\ 2AB &= \begin{bmatrix} 4 & 10 \\ 8 & -8 \end{bmatrix}. \end{aligned}$$

$$A^2 + B^2 + 2AB = \begin{bmatrix} 15 & 6 \\ 8 & -3 \end{bmatrix}.$$

Note:-

$$(A+B)^2 \neq A^2 + 2AB + B^2.$$
$$(A+B)^2 = A^2 + AB + BA + B^2.$$

Example ($AC = BC$ does not imply $A = B$).

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix}, C = \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}.$$

$$\begin{aligned} AC &= \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} \\ BC &= \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}. \end{aligned}$$

Example.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 4 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 & 4 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 2\times 2 + 3\times 5 + 4\times 1 \\ 5\times 2 + 4\times 5 - 4\times 1 \end{bmatrix} = \begin{bmatrix} 23 \\ 26 \end{bmatrix}.$$

$BA \longrightarrow$ does not exist because number of columns in B doesn't equal the number of rows in A

The Matrix multiplication identity is called I.

Example.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

$$AI_2 = A.$$

Example.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 1 & 2 \end{bmatrix}.$$

$$I_3 B = B.$$

Note:-

In R^n, I_n is a matrix with ones on the diagonal and it is the multiplication identity.

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $k(AB) = (kA)B = A(kB)$ where (k is a scalar)
- $A_{m\times n} I_n = A$
- $I_m A_{m\times n} = A$
- $A^n = AA \ldots A$

1.4 Transpose of Matrix

Definition 1.2: Matrix Transpose

Just interchange the matrix's rows and columns.

Example.

$$A = \begin{bmatrix} 2 & 4 & -5 \\ 1 & -3 & 2 \end{bmatrix} \longrightarrow A^T = \begin{bmatrix} 2 & 1 \\ 4 & -3 \\ -5 & 2 \end{bmatrix}.$$

$$B = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \longrightarrow B^T = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix}.$$

Example (Find $(AB)^T$ and B^TA^T).

$$A = \begin{bmatrix} 2 & 4 & -5 \\ 1 & -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 2 & 4 & -5 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 5 & -4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \end{bmatrix}.$$

Note:-

$$(AB)^T = B^TA^T.$$

- $(ABC)^T = C^T(AB)^T = C^TB^TA^T$
- $(A^T)^T = A$
- $(AT)^T = B^TA^T$
- $(A+B)^T = A^T + B^T$
- $k(A)^T = (kA)^T$

Example (Find AA^T and A^TA).

$$A = \begin{bmatrix} 2 & 4 & -5 \\ 1 & -3 & 2 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 2 & 4 & -5 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 45 & -20 \\ -20 & 14 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 \\ 4 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & -5 \\ 1 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 & -8 \\ 5 & 25-26 & \\ -8 & -26 & 29 \end{bmatrix}.$$

Note:-

$$AA^T \neq A^TA.$$

Both are square matrices.

Both are symmetric matrices.

$(AA^T)^T = AA^T$

1.5 The Trace of Matrix

Only square matrices have a trace.

Definition 1.3

$$Tr(A) = \text{sum of the diagonal elements.}$$

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 1 & -3 & 7 \\ 4 & 3 & 6 \end{bmatrix} \longrightarrow Tr(A) = 1 + (-3) + 6 = 4.$$

Example (Find $Tr(AB)$, $Tr(BA)$, and $Tr(A+B)$).

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}.$$

$$Tr(A) = 3, Tr(B) = 4, Tr(A) + Tr(B) = 8.$$

$$A + B = \begin{bmatrix} 6 & 4 \\ 0 & 2 \end{bmatrix} \longrightarrow Tr(A+B) = 8.$$

Note:-

$$Tr(A+B) = Tr(A) + Tr(B).$$

Example (Find $2Tr(A)$, $Tr(2A)$, $Tr(AB)$, $Tr(BA)$).

$$A = \begin{bmatrix} 4 & 5 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}.$$

$$2Tr(A) = 2 \times 3 = 6.$$

$$2A = \begin{bmatrix} 8 & 6 \\ 4 & -2 \end{bmatrix} \longrightarrow Tr(2A) = 6.$$

$$Tr(AB) = Tr(\begin{bmatrix} 8 & 2 \\ 4 & -4 \end{bmatrix}) = 4.$$

$$Tr(AB) = Tr(\begin{bmatrix} 6 & 7 \\ 4 & -2 \end{bmatrix}) = 4.$$

Note:-

$$\left.\begin{matrix} Tr(kA) = kTr(A) \\ Tr(AB) = TR(BA) \end{matrix}\right\} \longrightarrow \text{Always True.}$$

$$Tr(ABC) = Tr(CAB) = Tr(BCA).$$

1.5.1 Trace Properties

- $Tr(A) = \sum_{i=1}^{n} a_{ii}\rho = 8$
- $Tr(A+B) = Tr(A) + Tr(B)$
- If k is scalar, $Tr(kA) = kTr(A)$
- $Tr(AB) = Tr(BA)$
- $Tr(A^T) = Tr(A)$

1.6 Special Matrix Shapes

Row vector	$V_{1\times 3} = \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$		
Column vector	$W_{2\times 1} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$		
Upper triangular	$U_{3\times 3} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & -6 \end{bmatrix}$		
Lower triangular	$\begin{bmatrix} 4 & 0 & 0 \\ 3 & 6 & 0 \\ 9 & 7 & -6 \end{bmatrix}$		
Diagonal Matrix	$D_{3\times 3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -6 \end{bmatrix}$		
Identity Matrix (A special case of the Diagonal matrix)	$U_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		
Symmetric Matrix	$A_{3\times 3} = \begin{bmatrix} 1 & 2 & -5 \\ 2 & 4 & 5 \\ -3 & 5 & 0 \end{bmatrix}$	$(A^T = A)$	$a_{12} = a_{21}$ $a_{13} = a_{31}$ $a_{23} = a_{32}$
Skew-Symmetric Matrix	$A_{3\times 3} = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$	$(A^T = -A)$	$a_{12} = -a_{21}$ $a_{13} = -a_{31}$ $a_{23} = -a_{32}$