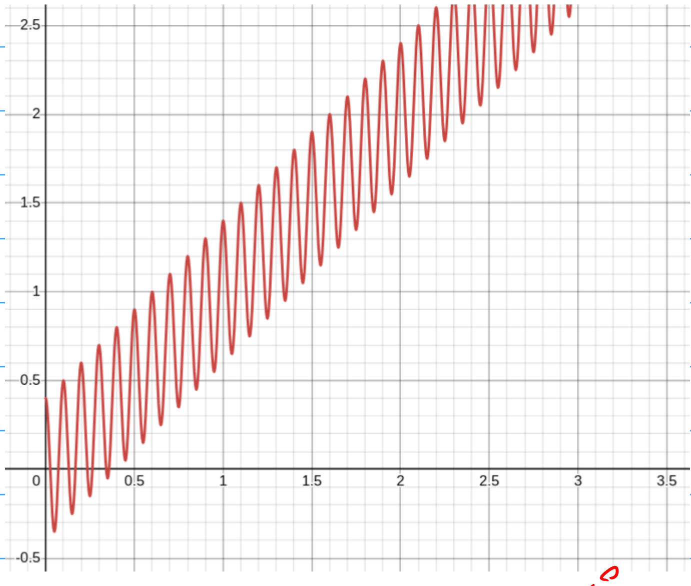
Question 1

Given a random process $x(t)$ defined as

$$x(t) = a|t| + b\cos(20\pi t), \qquad \text{where } a \sim \mathcal{U}(-3, 1), \quad b \sim \mathcal{N}(0, 5)$$

1) Sketch the ensemble of the random process.
   From your sketch, is the process stationary or non-stationary? **Why?**
2) Find the ensemble average, $\overline{x(t)}$.
3) Find the ACF, $R_x(t_1, t_2)$.
4) Justify your answer in the first part using the calculated average and ACF.

1)



2) $$\overline{x(t)} = E[x(t)] = E(a)|t| + \overset{0}{\cancel{E(b)}}\cos(20\pi t)$$

$$= \frac{1-3}{2}|t| = \boxed{-|t|}$$

3) $$R_x(t_1, t_2) = E[x(t_1)\, x(t_2)]$$

$$= E\{[a|t_1| + b\cos(20\pi t_1)][a|t_2| + b\cos(20\pi t_2)]\}$$

$$= E\{a^2|t_1||t_2| + \cancel{ab|t_1|\cos(20\pi t_2)} + \cancel{ab|t_2|\cos(20\pi t_1)} + \overset{5}{\cancel{b^2}}\cos(20\pi t_1)\cos(20\pi t_2)\}$$

$$= \overset{4/3}{\cancel{\frac{(1+3)^2}{12}}}|t_1||t_2| + 5\cos(20\pi t_1)\cos(20\pi t_2)$$

$$= \frac{4}{3}|t_1||t_2| + 5\cos(20\pi t_1)\cos(20\pi t_2)$$

$$= \boxed{\frac{4}{3}|t_1 t_2| + 5\cos(20\pi t_1)\cos(20\pi t_2)}$$

4) $\overline{x(t)} = -|t| \Rightarrow$ time-dependent $\Rightarrow$ non-stationary

$R_x(t_1, t_2) \Rightarrow$ ~~ $\Rightarrow$ ~~

$\tau := (t_2 - t_1)$

$A \sim \mathcal{U}\{2, 6, 10\}$

Question 2

A Random Process defined as

$$x(t) = A\cos(\omega_c t)$$

where ω_c is a random variable $\theta \in \{0, 4\pi, 8\pi, 16\pi\}$ with $f(\omega_c) = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}$, respectively, and A is a uniformly distributed discrete random variable, where $A \in \{2, 6, 10\}$. Both random variables are independent. Find:

1) The ensemble average of $x(t)$
2) The ACF of $x(t)$
3) The time average of $x(t)$
4) The time ACF of $x(t)$
5) Is the process WSS? Is it ergodic?
6) Find and draw the PSD of $X(t)$.

1)

$$E[A] = \frac{2+6+10}{3} = 6$$

$$E[\cos(\omega_c t)] = \sum_{\omega_c, \theta}^{N} f(\omega_c)\cos\theta = \frac{1}{2}\cancel{\cos} 0 + \frac{1}{4}\cos 4\pi t + \frac{1}{8}\cos 8\pi t + \frac{1}{8}\cos 16\pi t$$

$$= \frac{1}{2} + \frac{1}{4}\cos 4\pi t + \frac{1}{8}\cos 8\pi t + \frac{1}{8}\cos 16\pi t$$

$$\therefore E[A\cos\omega_c t] = 3\left(1 + \tfrac{1}{2}\cos 4\pi t + \tfrac{1}{8}\cos 8\pi t + \tfrac{1}{8}\cos 16\pi t\right)$$

2) $R_x(t_1, t_2) = E[x(t_1)x(t_2)] = E[A^2\cos\omega_c t_1 \cos\omega_c t_2]$

let $t_1 = t$ and $t_2 = t + \tau$

$$E\{A^2\} = \sum_i^N \frac{1}{N} A_i^2 = \frac{2^2 + 6^2 + 10^2}{3} = \frac{140}{3}$$

$$= E\{A^2\}\, E\{\cos\omega_c t \cos[\omega_c(t+\tau)]\}$$

$$= \frac{140}{3}\,\frac{1}{2}\, E\{\cos[\omega_c(2t+\tau)] + \cos\omega_c\tau\}$$

$$= \frac{140}{3} \frac{1}{2} E\left\{ \cos[\omega_c(2t+\tau)] \right\} + E\left\{ \cos \omega_c \tau \right\}$$

$$= 70 \left[2 + \tfrac{1}{2}\cos 4\pi\tau + \tfrac{1}{8}\cos 8\pi\tau + \tfrac{1}{8}\cos 16\pi\tau) + \tfrac{1}{2}\cos 4\pi(2t+\tau) + \tfrac{1}{8}\cos 8\pi(2t+\tau) + \tfrac{1}{8}\cos 16\pi(2t+\tau) \right]$$

3) $\widetilde{x(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \cancel{A\cos \omega_c t}\, dt = 0$

4) $\Rightarrow R_x(\tau) = A^2/2 \; \cos(\omega_c t)$

5) Not WSS because of the time dependance of R_x and variation in $\bar{x}$

6) $S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau$

$$= \int_{-\infty}^{\infty} 3(1 + \tfrac{1}{2}\cos 4\pi t + \tfrac{1}{8}\cos 8\pi t + \tfrac{1}{8}\cos 16\pi t) e^{-j\omega t} dt$$

$$= 3\left(F\{1\}(\omega) + \tfrac{1}{2} F\{\cos 4\pi t\}(\omega) + \tfrac{1}{8} F\{\cos 8\pi t\}(\omega) + \tfrac{1}{8} F\{\cos 16\pi t\}(\omega) \right)$$

$$= 3\delta(\omega) + \frac{3}{4}[\delta(\omega+2) + \delta(\omega-2)] + \frac{3}{16}[\delta(\omega+4) + \delta(\omega-4)] + \frac{3}{16}[\delta(\omega+8) + \delta(\omega-8)]$$

Question 3

Two random processes, $x(t)$ and $y(t)$, are defined as

$$x(t) = a(t) + m(t), \qquad \text{where } m(t) = \sin(\omega_c t + \theta)$$

$$y(t) = b(t) \times m(t), \qquad \text{where } m(t) = \sin(\omega_c t + \theta)$$

where ω_c is constant and θ is a random variable where $\theta \sim \mathcal{U}(0, 2\pi)$.
The process $a(t)$ is a WSS random process with zero mean and auto correlation function $R_a(\tau) = e^{-|\tau|}$.
Also, $b(t)$, is a WSS random process with zero mean and auto correlation function $R_b(\tau) = 3 + \frac{\sin(4\pi\tau)}{\pi\tau}$.
The two processes $a(t)$ and $b(t)$ are independent of each other and of θ. Find:

1) The ACF of $x(t)$.
2) The ACF of $y(t)$.
3) Are $x(t)$ and $y(t)$ WSS? Why?
4) The ensemble average and the ACF of $w(t) = x(t) + y(t)$.
5) The average total power of $w(t)$.