Discrete random variable

Ex: $S = \{HH, HT, TH, TT\}$

Random variable is a function that maps number to sample space.

$x = $ no. $H \Rightarrow x = \{0, 1, 2\}$

$P(x=2) = 1/4$, $P(x=1) = 1/2$

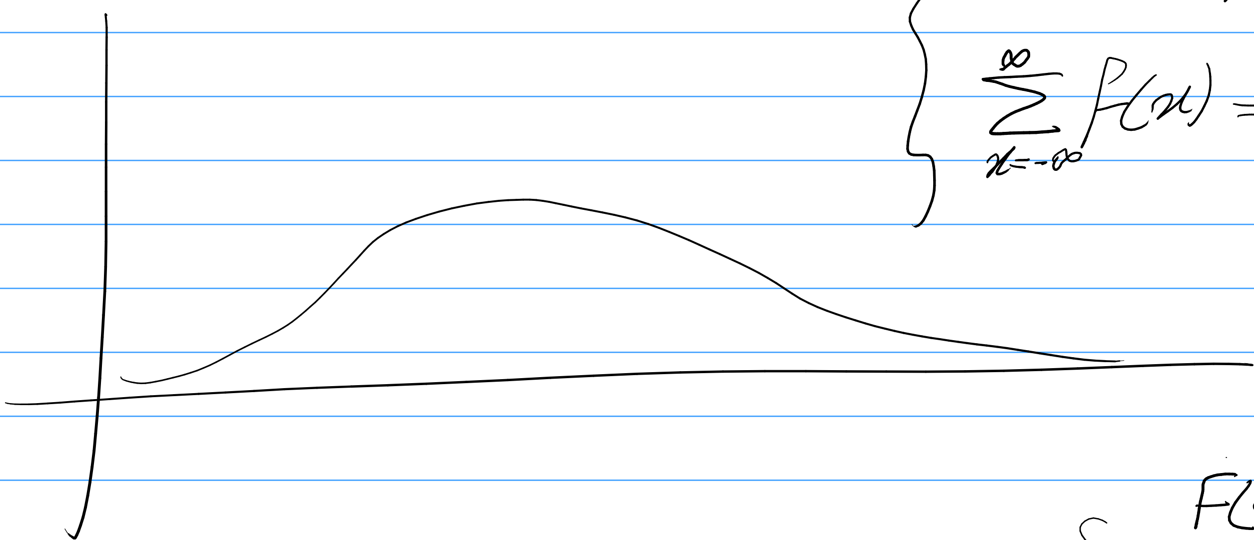
HH HT, TH

Discrete means countable.

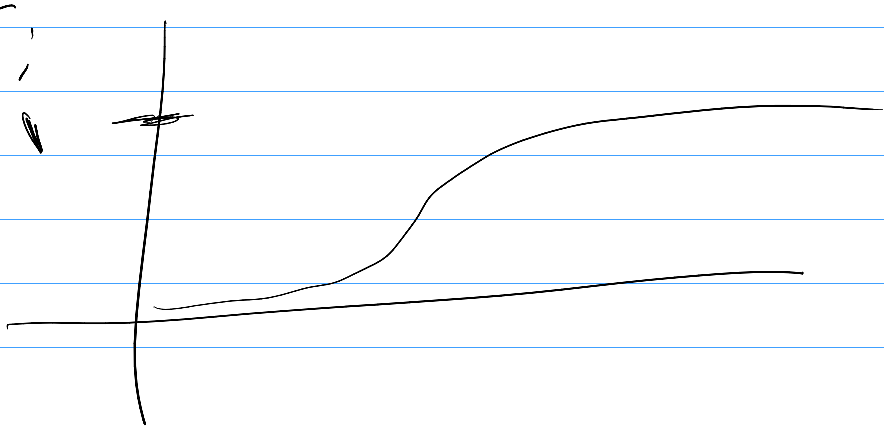
PMF: $(x, f(x))$

$f(x) = P\{X = x\}$

$$\begin{cases} f(x) \geq 0 \\ \sum_{x=-\infty}^{\infty} f(x) = 1 \end{cases}$$



CDF:



$F(x) = \sum_{x} f(x)$

$F(\infty) = 1$

$F(-\infty) = 0$

$F(x) \leq F(x + \varepsilon)$

$$\boxed{f(x) = \Delta F(x) = F(x+\varepsilon) - F(x)}$$

Mean: $\mu = \sum x f(x)$

$\mathcal{E}[X]$

Variance: $\sigma^2 = \sum (x-\mu)^2 f(x)$

Continuous Random Variable.

PDF: $P(a<X<b) = \int_a^b f(x)\,dx$

CDF: $P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$= F(x)$

Moment Generating Function:

$M(t) = \mathcal{E}\{e^{tx}\}$

Common Discrete Probability Distributions

Bernoulli Trials :

$$f(x) = \begin{cases} p & x=1 \\ 1-p=q \; ; & x=0 \end{cases}$$

$\mu = p$

$\sigma^2 = pq$

$$M = E[e^{tx}] = \sum e^{tx} f(x) = pe^t + q$$

$$\frac{\partial}{\partial t} M \Big|_{t=0} = pe^t = p \;, \quad \frac{\partial^2}{\partial t^2} M \Big|_{t=0} = p$$

$$\sigma^2 = p - p^2 = p(1-p) = pq \quad \text{QED}$$

Binomial Experiment

- Independent / linear.
- Each trial should be Bernoulli trials
- " " " has the same p, q

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$\mu = np$

$\sigma^2 = npq$

$$M = E[e^{tx}] = E[e^{t(x_1+x_2+\cdots+x_n)}]$$
$$= E[e^{tx_1}] E[e^{tx_2}] \cdots$$
$$= (q+pe^t)^n$$
$$\Rightarrow \frac{\partial M}{\partial t}\Big|_{t=0} = npe^{\cancel{t}} (q + \cancel{pe^t})^{n-1} = np$$

Geometric Exp.

- Bin. Exp. but with success.

$$f(x) = p(1-p)^{x-1}, \quad \mu = 1/p, \quad \sigma^2 = q/p^2$$

$$M = \sum e^{xt} f(x) = e^{xt} p q^{x-1} = \frac{p}{1-p} \sum_{x=1}^{\infty} q^x e^{xt} = \frac{p}{1-p} \sum_{x=1}^{\infty} (qe^t)^x$$

$$= \frac{p}{1-p}\left(\frac{1}{1-qe^t} - 1\right)$$

$$= \frac{p}{\cancel{1-p}} \frac{\cancel{q}e^t}{1-qe^t} = \frac{pe^t}{1-qe^t}$$

Sum of Geometric Series Formula

$$S_n = a + \cancel{ar} + \cancel{\cdots} + \cancel{ar^{n-1}}$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{\cdots} + ar$$

$$S_n - rS_n = a + ar^{n-1} \Rightarrow S_n = \frac{a(1+r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r}; \ |r| < 1$$

$$\frac{\partial}{\partial t} M = \frac{p\cancel{e^t}(1-q\cancel{e^t}) - p\cancel{e^t}(-q\cancel{e^t})}{(1-q\cancel{e^t})^2}$$

$$\Big|_{t=0} = \frac{p(1-q) + pq}{(1-q)^2} = \frac{p - \cancel{qp} + \cancel{qp}}{(1-q)^2} = \frac{p}{(1-q)^{\cancel{2}}} = \frac{1}{1-q} = \frac{1}{p}$$

$$\frac{\partial^2 M}{\partial t^2}\Big|_{t=0} = \frac{\partial}{\partial t} \frac{pe^t - \cancel{pqe^{2t}} + \cancel{pqe^{2t}}}{(1-qe^t)^2} = \frac{\partial}{\partial t} \frac{pe^t}{(1-qe^t)^2}$$

$$= \frac{pe^t(1-qe^t)^2 + 2pe^t qe^t(\cancel{1}-qe^t)}{(1-qe^t)^4} = pe^t \cancel{-2pqe^{2t}} + pq^2e^{3t} \cancel{+2pqe^{2t}} - \cancel{2}pq^2e^{3t}$$

$$= \frac{p\cancel{e^t} - pq^2\cancel{e^{3t}}}{(1-q\cancel{e^t})^4} = \frac{p - pq^2}{(1-q)^4} = \frac{p(1-q^2)}{(1-q)^4} = \frac{p(1-(1-p)^2)}{}$$

$$\cancel{1} - \cancel{1} + 2p - p^2$$

$$= \frac{p^2(2-p)}{p^4} = \frac{2-p}{p^2} \Rightarrow \sigma^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = \frac{q}{p^2}$$

Poisson Exp.

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad \mu = \lambda, \quad \sigma^2 = \lambda$$

$$M = \sum \frac{e^{xt} e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \boxed{\Sigma (e^t \lambda)^x / x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

Taylor Series of e^x

$$\frac{\partial}{\partial t} M|_{t=0} = \lambda \cancel{e^t}\, \cancel{e^{\lambda(e^t-1)}} = \lambda$$

$$\frac{\partial^2}{\partial t^2} M|_{t=0} = \frac{\partial}{\partial t} \lambda e^t e^{\lambda(e^t - 1)} = \lambda \cancel{e^t}\, \cancel{e^{\lambda(e^t-1)}} + \lambda^2 \cancel{e^{2t}}\, \cancel{e^{\lambda(e^t-1)}}$$

$$= \lambda + \lambda^2$$

$$\sigma^2 = \lambda + \cancel{\lambda^2} - \cancel{\lambda^2} = \lambda$$

Continuous Uniform Distribution

$$f(x) = \begin{cases} 1/B-A, & [A, B] \\ 0, & ow \end{cases}$$

$$\mu = \frac{A+B}{2}, \quad \sigma^2 = \frac{(B-A)^2}{12}$$

$$M = \int_{-\infty}^{\infty} e^{xt} \frac{dx}{B-A} = \frac{1}{B-A} \int_A^B e^{xt} dx = \left[\frac{e^{xt}}{t}\right]_A^B = \frac{e^{Bt} - e^{At}}{t(B-A)}$$

Continuous Exponential Distribution

Time until first Poisson event. $T < t \Rightarrow 1-e^{-\lambda t}$

$f(x) = e^{-\lambda t}(\lambda t)^x / x!$ let $x=$ $F(t) := \text{CDF}$

$P(T \geq t) = P(X=0) = e^{-\lambda t}$, $P(T \leq t) = \boxed{1-e^{-\lambda t}}$

$$\text{PDF} = \frac{\partial}{\partial t}(1-e^{-\lambda t}) = \lambda e^{-\lambda t} \Rightarrow \boxed{f(t) = \lambda e^{-\lambda t}}$$

$$\int_0^\infty f(t)\,dt = \left[-e^{-\lambda t}\right]_0^\infty = 1$$

$$\mu = \mathcal{E}[t] = \int_0^\infty t\lambda e^{-\lambda t}\,dt = \lambda \int_0^\infty t e^{-\lambda t}\,dt$$

$$= -\frac{t}{\cancel{\lambda}} e^{-\lambda t} - \frac{e^{-\lambda t}}{\lambda^{\cancel{2}}} \Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

	D	I
		$e^{-\lambda t}$
+	t	$e^{-\lambda t}/\lambda$
−	1	$e^{-\lambda t}/\lambda^2$
+	0	

$$\mathcal{E}[t^2] = \int_0^\infty \lambda t^2 e^{-\lambda t} =$$

$$= \left[-t^2 e^{-\lambda t} - \frac{2t e^{-\lambda t}}{\lambda} - \frac{2e^{-\lambda t}}{\lambda^2}\right]_0^\infty$$

$$= \frac{2}{\lambda^2}$$

	D	I
		$e^{-\lambda t}$
+	t^2	$-e^{-\lambda t}/\cancel{\lambda}$
−	$2t$	$e^{-\lambda t}/\lambda^{\cancel{2}}$
+	2	$-e^{-\lambda t}/\lambda^{\cancel{3}\,2}$
−	0	

$$\sigma^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$M(k) = E[e^{tk}] = \int_0^\infty \lambda e^{tk} e^{-\lambda t} dt = \lambda \int_0^\infty e^{t(k-\lambda)} dt = \frac{\lambda}{k-\lambda} e^{t(k-\lambda)} \Big|_0^\infty$$

$$k < \lambda \Rightarrow k-\lambda < 0 \Rightarrow \lim_{t\to\infty} e^{t(k-\lambda)} = 0$$

$$\Rightarrow \frac{\lambda}{k-\lambda} e^{t(k-\lambda)} \Big|_0^\infty = -\frac{\lambda}{k-\lambda} = \boxed{\frac{\lambda}{\lambda-k}}$$

Example

In a large corporate computer network, user log-ons to the system can be modeled as a Poisson process with a mean of 30 log-ons per hour.

μ $t=60$

$$\lambda = \frac{\mu}{t} = \frac{1}{2}$$

1. What is the probability that there are no log-ons in an interval of 6 minutes? $P(X=0)\big|_{t=6} = e^{-\lambda t} = e^{-3}$
2. What is the probability that there are five log-ons in an interval of 6 minutes? $P(X=5)\big|_{t=6} = e^{-\lambda t}(\lambda t)^5/5! = \frac{81}{40} e^{-3} = 0.1008$
3. What is the probability that the time until the next log-on is between 2 and 3 minutes? $P(X=0 \mid 2 < T < 3) = e^{-2/2} - e^{-3/2} = 0.1447$
4. Determine the interval of time such that the probability that no log-on occurs in this interval is 0.9.
5. If no log-ons occurred during the last 10 minutes, what is the probability that next log-on occurs within the coming 3 minutes?

4) $P(X=0 \mid T \geq t') = 0.9$

$$1 - (1 - e^{-\lambda t'}) = 0.9 \Longrightarrow t' = -\ln 0.9/\lambda = -2\ln 0.9 = 0.2107$$

5) $P(X=0 \mid T_1 < 10,\ T_2 > 3) =$