CIE 327 - Probability and Stochastic Processes

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Contents

Probability Theory

PART

I

Section 1

Probability

Definition 1 **Probability** quantifies uncertainty. It measures the likelihood of an event occurring.

Probability values lie between 0 and 1.

Section 2

Sample Space

Definition 2 The set of all possible outcomes of a statistical experiment is called the **Sample Space** S.

Each outcome is an element or sample point of S.

Example Flipping a coin and tossing a die

$$S = \{\text{H, T, 1, 2, 3, 4, 5, 6}\}$$

Section 3

Events

Definition 3 An **Event** is a subset of the sample space S.

- **Union:** $A \cup B$ contains all elements in A or B or both.
- **Intersection:** $A \cap B$ contains elements common to both A and B.
- **Complement:** A' contains elements in S but not in A.

Mutually exclusive events: $A \cap B = \emptyset$.

Section 4

Axioms of Probability

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. If A and B are disjoint: $P(A \cup B) = P(A) + P(B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Basic Rule: $P(A') = 1 - P(A)$.

PART

II

PART

III

Random Variables

Stochastic Processes

Section 5

Introduction to Random Processes

Definition 4 A **Random Process (RP)** is a collection of time functions (signals) corresponding to various outcomes of a random experiment. Each outcome is represented by a deterministic **sample function** (or realization).

An RP is essentially a time-dependent random signal.

Subsection 5.1

Random Process vs Random Variable

1. A collection of Random Variables $\rightarrow$ Random Process.
2. Sampling a Random Process $\rightarrow$ Random Variable.

Key Terms:

- **Ensemble:** Collection of all possible waveforms.
- **Sample Function:** A single waveform in the ensemble.

Random Variables (RVs) form the basis of RPs. Sampling an RP gives a single RV.

Sample functions are realizations of the RP.

Section 6

Statistics of Random Processes

Subsection 6.1

Mean of a RP

$$x(t) = \int_{-\infty}^{\infty} x(t) p(x;t) dx \tag{6.1}$$

This represents the average of all samples at each time.

The mean is derived from the first-order PDF of the RP.

Subsection 6.2

Auto-Correlation Function (ACF)

Definition 5 ACF measures the correlation between signal amplitudes at two distinct time instants:

$$R_x(t_i, t_j) = E[x(t_i)x(t_j)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i x_j p(x_i, x_j) dx_i dx_j \tag{6.2}$$

ACF reveals the dependence of the signal on itself over time.

Subsection 6.3

Time Statistics

- **Time Average:** $x(t)$ over an interval $[-T/2, T/2]$.
- **Time ACF:**

$$\tilde{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt \tag{6.3}$$

Time averages are particularly relevant for ergodic processes.

Section 7

Classification of Random Processes

Subsection 7.1

Stationary Random Processes

- A process is stationary if its statistical properties are time-invariant.
- Joint PDFs depend only on time differences, not on the absolute times.
- ACF: $R_x(t_i, t_j) = R_x(t_j - t_i) = R_x(\tau)$.

Stationary processes are time-invariant.

Subsection 7.2

Wide Sense Stationary (WSS)

- WSS requires:
  1. $\mu_x(t) =$ constant
  2. $R_x(t_i, t_j) = R_x(\tau)$
- Stationary processes are WSS, but not all WSS processes are stationary.

WSS is a less restrictive condition than full stationarity.

Subsection 7.3

Ergodicity

Definition 6 An **Ergodic Process** is one where time averages equal ensemble averages for any sample function:

$$\mu_x = \tilde{x}(t), \quad R_x(\tau) = \tilde{R}_x(\tau) \tag{7.1}$$

Ergodic processes allow representation using a single sample function.

Section 8

Power Spectral Density (PSD)

Definition 7 PSD represents the frequency content of a random process. For stationary processes:

$$S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau} d\tau \tag{8.1}$$

$$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) e^{j\omega\tau} d\omega \tag{8.2}$$

Properties:

- $R_x(\tau)$ is real and even $\Rightarrow S_x(\omega)$ is real and even.
- Mean square value: $R_x(0) = \int_{-\infty}^{\infty} S_x(f) df$.

PSD is derived using the Fourier Transform of the ACF.

The PSD quantifies how power is distributed over frequencies.

Section 9

Transmission of RP Through LTI Systems

Theorem 1 For an RP $x(t)$ applied to an LTI system with impulse response $h(t)$:

- Output RP: $y(t) = x(t) * h(t)$
- ACF: $R_y(\tau) = h(\tau) * h(-\tau) * R_x(\tau)$
- PSD: $S_y(\omega) = |H(\omega)|^2 S_x(\omega)$

LTI systems modify the RP's PSD based on the system's transfer function.

Section 10

Noise in Communication Systems

Definition 8 **White Gaussian Noise (WGN):**

- **Gaussian:** Zero-mean Gaussian distribution.
- **White:** Flat PSD.

WGN is the most common type of noise in communication systems.

Section 11

Binary Random Processes

Subsection 11.1

Polar NRZ Signaling

$$x(t) = \sum_n a_n p(t - nT_b - \alpha) \tag{11.1}$$

where $a_n \in \{A, -A\}$ and $p(t) = \mathrm{rect}(t - T_b/2)$.

Key Metrics:

- ACF: $R_x(\tau) = A^2(1 - |\tau|/T_b)$ for $|\tau| < T_b$.
- PSD: $S_x(\omega) = A^2 T_b \mathrm{sinc}^2\left(\frac{\omega T_b}{2\pi}\right)$.

The sinc function appears due to the rectangular pulse shaping.