Navier-Stokes Equations: Numerical Solution for Steady-State Problems

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Math 302: Partial Differential Equations

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Introduction

- The Navier-Stokes equations describe fluid motion for liquids and gases.
- Key applications: weather modeling, aerodynamics, industrial fluid systems.
- Focus: Numerical solution for steady-state incompressible flows.

Governing Equations

Momentum:

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = -\nabla p + \mu \nabla^2 \mathbf{V}$$

Continuity:

$$\nabla \cdot \mathbf{V} = 0$$

- Models incompressible Newtonian fluids.
- Includes convection, diffusion, and pressure forces.

Non-dimensionalization

- Introduces Reynolds number: $\mathrm{Re} = \frac{\rho U L}{\mu}$.
- Simplified equations:

$$\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{V}, \quad \nabla \cdot \mathbf{V} = 0.$$

2D Navier-Stokes Equations

- **x-momentum:**

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\mathrm{Re}}\nabla^2 u.$$

- **y-momentum:**

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\mathrm{Re}}\nabla^2 v.$$

- **Continuity:**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Numerical Challenges

- Nonlinearity of the convective term.
- Coupling between velocity and pressure.
- Lack of a direct pressure equation.

SIMPLE Algorithm

1. Solve tentative velocity field from momentum equations.
2. Correct pressure using the Poisson equation:

$$\nabla \cdot \left(\mathbf{A}^{-1} \nabla p\right) = \nabla \cdot \left(\mathbf{A}^{-1} \mathbf{H}\right) .$$

3. Update velocity to satisfy continuity.
4. Iterate until convergence.

Python Implementation

- Discretized domain: Uniform grid (41x41 points).
- Finite-difference method for derivatives.
- Central difference for convection, five-point stencil for diffusion.
- Iterative solver for steady-state solution.

Boundary Conditions

- **Velocity:**
  - No-slip at walls: $u = 0, v = 0$.
  - Uniform inflow velocity at the top: $u = 1, v = 0$.
- **Pressure:** Neumann conditions (zero normal derivative).

Results Visualization

- Contour plots for pressure.
- Quiver plots for velocity vectors.
- Demonstrates steady-state flow field.

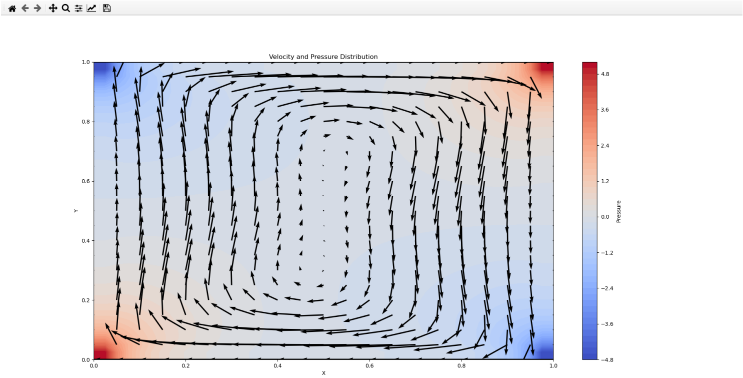

Figure: Example output of simulation.

Conclusion

- Successfully implemented the Navier-Stokes equations numerically.
- Used SIMPLE algorithm for velocity-pressure coupling.
- Achieved steady-state solution for a 2D incompressible flow.
- Future work: Extend to 3D and turbulent flows.