First Question

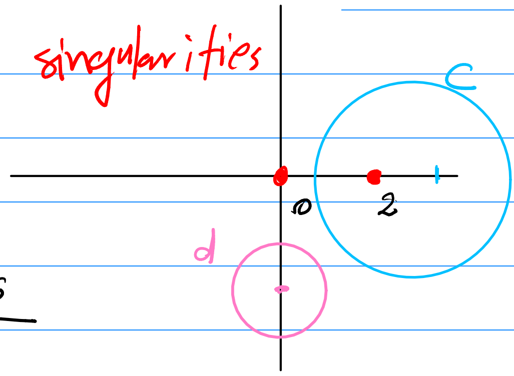
Evaluate the given integral along the indicated closed contour(s).

$$\oint_C \frac{2z+5}{z^2-2z}\,dz$$

$a)\ |z| = \frac{1}{2}$ $\qquad$ $b)\ |z+1| = 2$

$c)\ |z-3| = 2,$ $\qquad$ $d)\ |z+2i| = 1$

$$\int \frac{2z+5}{z^2-2z}\,dz = \int \frac{2z+5}{z(z-2)}\,dz$$



a) $\displaystyle\oint_{|z|=1/2} \frac{2z+5}{z(z-2)}\,dz = 2\pi i\,\frac{2(0)+5}{(0)-2} = \cancel{2}\pi i \cdot \frac{-5}{\cancel{2}}$

$= \boxed{-5\pi i}$

b) $\displaystyle\oint_{|z+1|=2} \frac{2z+5}{z(z-2)}\,dz = \boxed{-5\pi i}$ Similiar to a

c) $\displaystyle\oint_{|z-3|=2} \frac{2z+5}{z(z-2)}\,dz = \cancel{2}\pi i\,\frac{2(2)+5}{\cancel{2}} = \boxed{9\pi i}$

d) $\displaystyle\oint_{|z+2i|=1} \frac{2z+5}{z(z-2)}\,dz = 0$

Second Question

I) **Expand** $f(z) = \frac{z}{(z+1)(z-2)}$ **in a Laurent series valid for the indicated annular domain.**

a) $0 < |z+1| < 3$ $\Rightarrow 0 < \frac{|z+1|}{3} < 1$

b) $|z+1| > 3$ $\Rightarrow \frac{3}{|z+1|} < 1$

c) $1 < |z| < 2$ ✓

d) $0 < |z-2| < 3$

$$\frac{z}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2} = \frac{1}{3(z+1)} + \frac{2}{3(z-2)}$$

a)

$$\frac{2}{3}\cdot\frac{1}{z-2} = \frac{2}{3}\cdot\frac{1}{z+1-3} = \frac{2}{3}\cdot\frac{-1}{3-(z+1)} = \frac{-2}{9}\,\frac{1}{1-\frac{z+1}{3}} = \frac{-2}{9}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n$$

$$f(z) = \frac{1}{3z+3} - \frac{2}{9}\sum_{n=0}^{\infty}\left(\frac{z+1}{3}\right)^n \; ; \quad 0<|z+1|<3$$

b) $$\frac{2}{3}\cdot\frac{1}{z-2} = \frac{2}{3}\,\frac{1}{z+1-3} = \frac{2}{3}\,\frac{1}{z+1}\,\frac{1}{1-\frac{3}{z+1}}$$ $\frac{3}{|z+1|}<1$

$$= \frac{2}{3(z+1)}\sum_{n=0}^{\infty}\left(\frac{3}{z+1}\right)^n$$

$$= \frac{2}{3}\sum_{n=0}^{\infty}\frac{3^n}{(z+1)^{n+1}} = 2\sum_{n=0}\frac{3^{n-1}}{(z+1)^{n+1}}$$

$$f(z) = \frac{1}{3z+3} + 2\sum_{n=0}^{\infty}\frac{3^{n-1}}{(z+1)^{n+1}}$$

① $\left|\frac{z}{2}\right|<1$

c) $$\frac{2}{3}\,\frac{1}{z-2} = \frac{\cancel{2}}{3}\,\frac{-1}{\cancel{2}(1-\frac{z}{2})} = \frac{-1}{3}\cdot\frac{1}{1-\frac{z}{2}} = \frac{-1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

② $\left|\frac{1}{z}\right|<1$

$$\frac{1}{3}\,\frac{1}{z+1} = \frac{1}{3z}\cdot\frac{1}{1+1/z} = \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$$

$$f(z) = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \frac{-1}{3}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}} - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$n \to n+1$

$$= \frac{-1}{3}\left(\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{z^{n+1}} + \sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right) = \frac{-1}{3}\left[1 + \sum_{n=1}^{\infty}(-1)^n z^{-n} + z^n 2^{-n}\right]$$

$\left|\frac{z-2}{3}\right|<1$

d) $$\frac{1}{3}\,\frac{1}{z+1} = \frac{1}{3}\,\frac{1}{z-2+3} = \frac{1}{9}\,\frac{1}{1+\frac{z-2}{3}} = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-2}{3}\right)^n$$

$$f(z) = \frac{1}{9}\sum_{n=0}^{\infty}\left(-\frac{z-2}{3}\right)^n + \frac{2}{3}\,\frac{1}{z-2}$$

II) Expand $f(z) = \frac{1}{(z-2)(z-1)^3}$ in a Laurent series valid for the indicated annular domain.

a) $0 < |z-2| < 1$

b) $0 < |z-1| < 1$

a)

$$\frac{1}{z-2}(z-2+1)^{-3} = \frac{1}{z-2}\sum_{n=0}^{\infty}\binom{-3}{n}(z-2)^n = \sum_{n=0}^{\infty}\binom{-3}{n}(z-2)^{n-1}$$

b)

$$\frac{1}{(z-1)^3} \quad \frac{1}{1+z-1} = \frac{-1}{(z-1)^3}\sum_{n=0}^{\infty}(z-1)^n = -\sum_{n=0}^{\infty}(z-1)^{n-3}$$

Third Question

Use Cauchy's residue theorem to evaluate the given integral along the indicated contour.

A)

$$\oint_C \frac{z+1}{z^2(z-2i)}dz$$

$a)\ |z| = 1$ $\qquad b)\ |z-2i| = 1$ $\qquad c)\ |z-2i| = 4$

$$\text{Res}\{f(z), 2i\} = \lim_{z\to 2i}\frac{z+1}{z^2} = \frac{1+2i}{-4} = \frac{-1}{4} - i\frac{1}{2}$$

$$\text{Res}\{f(z), 0\} = \lim_{z\to 0}\frac{d}{dz}\frac{z+1}{z-2i} = \frac{\cancel{z}-2i-\cancel{z}-1}{(z-2i)^2} = \frac{-1-2i}{-4} = \frac{1}{4}+i\frac{1}{2}$$

a) $= 2\pi i\left(\frac{1}{4}+\frac{1}{2}i\right) = \pi(-1+i/2)$

b) $= 2\pi i\left(\frac{-1}{4}-\frac{i}{2}\right) = \pi(1-i/2)$

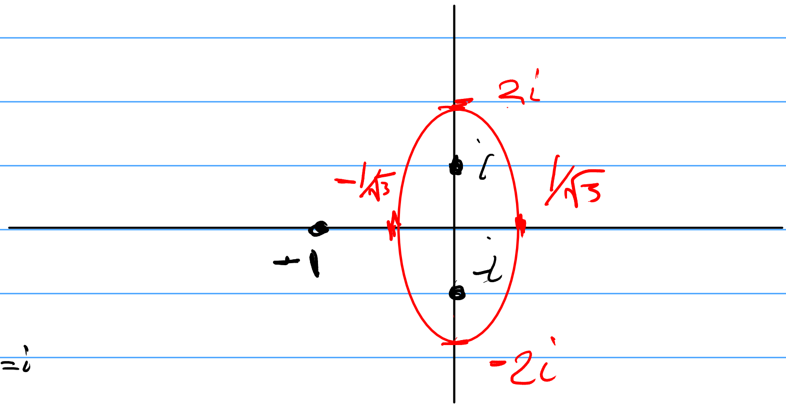
c) $= 0$ Since $\text{Res}_f\{2i\} = -\text{Res}_f\{0\}$

B)

$$I = \oint_C \frac{z}{(z+1)(z^2+1)}\,dz$$

C is the ellipse $6x^2 + y^2 = 4$.

$$= \oint_C \frac{z}{(z+1)(z-i)(z+i)}$$



$$\operatorname{Res}\{f(z), i\} = \left.\frac{z}{(z+1)(z+i)}\right|_{z=i}$$

$$= \frac{\cancel{i}\,(i-1)}{(i+1)(2\cancel{i})(i-1)} = \frac{1}{4} - \frac{1}{4}i$$

$$\operatorname{Res}\{f(z), -i\} = \frac{1}{4} + \frac{1}{4}i$$

$$\therefore\ I = 2\pi i\left(\frac{1}{4} + \cancel{\frac{1}{4}i} + \frac{1}{4} - \cancel{\frac{1}{4}i}\right) = \boxed{\pi i}$$

Fourth Question

Evaluate the given trigonometric integrals.

I)

$$I = \int^{2\pi} \frac{2i\cos^2\theta}{3 - 2i\sin\theta}\,d\theta$$

$d\theta = dz/iz$, $\cos\theta = \dfrac{z^2+1}{2z}$, $\sin\theta = \dfrac{z^2-1}{2zi}$

$$\oint_{|z|=1} \frac{(z^2+1)^2}{4z^2}\ \frac{2i\cancel{z}}{6zi - z^2 + 1}\ \frac{dz}{i\cancel{z}} = \frac{1}{2}\oint_{|z|=1} \frac{(z^2+1)^2}{z^2(-z^2+6zi+1)}\,dz$$

$$= \frac{-1}{2}\oint_{|z|=1} \frac{(z^2+1)^2}{z^2(z^2-6zi-1)}\,dz = \frac{-1}{2}\oint_{|z|=1} \frac{z^4+2z^2+1}{z^2(z^2-6zi-1)}\,dz =$$

$$\frac{-1}{2}\oint_{|z|=1} \frac{z^4+2z^2+1}{z^2(z^2-6zi-1)}dz = \frac{-1}{2}\oint \frac{z^4+2z^2+1}{z^2[z-i(3-2\sqrt{2})][z-i(3+2\sqrt{2})]}$$

$$Res\{f(z),0\} = \frac{d}{dz}\frac{z^4+2z^2+1}{[z-i(3-2\sqrt{2})][z-i(3+2\sqrt{2})]}\Bigg|_{z=0} = 6i$$

$$Res\{f(z), i(3-2\sqrt{2})\} = \frac{z^4+2z^2+1}{z^2[z-i(3+2\sqrt{2})]}\Bigg|_{z=i(3-2\sqrt{2})} = -4\sqrt{2}i$$

$$I = 2\pi i \times \frac{-1}{2} \times (6i-4\sqrt{2}i) \approx 0.3\pi i$$

Fifth Question

Evaluate the Cauchy principal value of the given improper integral.

I)

$$I = \int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4}dx$$

$$\oint_C \frac{2z^2-1}{z^4+5z^2+4}dz = \oint \frac{2z^2-1}{(z^2+1)(z^2+4)}dz = \oint \frac{}{(z+i)\cancel{(z-i)}(z+2i)(z}$$

$$Res\{f,i\} = \frac{2z^2-1}{(z+i)(z^2+4)}\Bigg|_{z=i} = \frac{-i}{2}$$

$$Res\{f,2i\} = \frac{2z^2-1}{(z^2+1)(z+2i)}\Bigg|_{z=2i} = -\frac{3}{4}i$$

$$I = 2\pi i\left(\frac{-i}{2}-\frac{3}{4}i\right) = \frac{5}{2}\pi$$

Sixth Question

A) Let $a > 0$. Derive the following formula using **Fourier integral**

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{a^2+\omega^2}\,d\omega \qquad (x \geq 0)$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \underbrace{F_c(\omega)}_{I} \cos\omega x \, d\omega$$

$$I = \int_0^\infty e^{-ax}\cos\omega x\, d\omega =$$

	D	I
$+$	e^{-ax}	$\cos\omega x$
$-$	$-ae^{-ax}$	$\frac{1}{\omega}\sin\omega x$
$+$	$\boxed{+a^2e^{-ax}}$	$\boxed{-\frac{1}{\omega^2}\cos\omega x}$

$$\cancel{\frac{e^{-ax}\sin\omega x}{\omega}\Big|_0^\infty} - \underbrace{\frac{a}{\omega^2}e^{-ax}\cos\omega x\Big|_0^\infty}_{\frac{a}{\omega^2}}$$

$$-\frac{a^2}{\omega^2}\underbrace{\int_0^\infty e^{-ax}\cos\omega x}_{I}$$

$$I = \frac{a}{\omega^2} - \frac{a^2}{\omega^2} I \Rightarrow I\left(1+\frac{a^2}{\omega^2}\right) = \frac{a}{\omega^2}$$

$$I = \frac{a}{\omega^2\left(1+\frac{a^2}{\omega^2}\right)}$$

$$= \boxed{\frac{a}{\omega^2+a^2}}$$

$$e^{-ax} = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}\,d\omega \qquad \text{QED}$$

B) Find the **Fourier transform** of the function $f(x) = xe^{-x^2}$

$$\mathcal{F}\{xe^{-x^2}\} = \int_{-\infty}^{\infty} x e^{-x^2} e^{i\omega x}\, dx = F(\omega)$$

$$= \cancel{-\tfrac{1}{2} e^{i\omega x} e^{-x^2}\Big|_{-\infty}^{\infty}} + \frac{1}{2} i\omega \int_{-\infty}^{\infty} e^{i\omega x} e^{-x^2}$$

	D	I
$+$	$e^{i\omega x}$	xe^{-x^2}
$-$	$i\omega e^{i\omega x}$	$-\tfrac{1}{2} e^{-x^2}$

Gaussian function

$p = \frac{1}{2}$

$$F(\omega) = \boxed{\frac{1}{2} i\omega \sqrt{\pi}\, e^{-\omega^2/4}}$$

C) Use **Fourier transform** to solve the following **Heat equation**

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x,0) = xe^{-x^2}, \quad -\infty < x < \infty.$$

$u_t = u_{xx}$, $x \in]-\infty, \infty[$

$u(x,0) = xe^{-x^2}$

$\frac{\partial}{\partial t} U(\omega,t) = -\omega^2 U(\omega,t)$ $\qquad U(\omega,0) = \frac{1}{2} i\omega\sqrt{\pi}\, e^{-\omega^2/4}$

$\frac{\partial}{\partial t} U(\omega,t) + \omega^2 U(\omega,t) = 0$ $\qquad K = \frac{1}{2} i\omega \sqrt{\pi}\, e^{-\omega^2/4}$

$$U(\omega,t) = K e^{-\omega^2 t} = \frac{1}{2} i\omega\sqrt{\pi}\, e^{-\omega^2/4} e^{-\omega^2 t}$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2} i\omega\sqrt{\pi}\, e^{-\omega^2/4} e^{-\omega^2 t} e^{-i\omega x}\, d\omega$$