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Ordinary Differential Equations (MATH 202)

Assignment 3

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1. Find Laplace Transform of the following functions:

   (a) $f(t) = (2t-1)^3$

   Solution.

$$\begin{aligned}
\mathscr{L}\left\{(2t-1)^3\right\} &= \mathscr{L}\left\{8t^3 - 12t^2 + 6t - 1\right\} && (1)\\
&= \frac{8 \cdot 3!}{s^4} - \frac{12 \cdot 2}{s^3} + \frac{6}{s^2} - \frac{1}{2} && (2)\\
&= \frac{48}{s^4} - \frac{24}{s^3} + \frac{6}{s^2} - \frac{1}{s}. && (3)
\end{aligned}$$

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   (b) $f(t) = e^{3t}\cos(6t)\cos(3t)$

   Solution.

$$\begin{aligned}
\mathscr{L}\left\{e^{3t}\cos(6t)\cos(3t)\right\} &= \mathscr{L}\left\{\frac{1}{2}e^{3t}\left(\cos(9t) + \cos(3t)\right)\right\} && (4)\\
&= \frac{1}{2}\left(\mathscr{L}\left\{e^{3t}\cos(9t) + \mathscr{L}\left\{e^{3t}\cos(3t)\right\}\right\}\right) && (5)\\
&= \frac{1}{2}\left[\frac{s-3}{(s-3)^2+9^2} + \frac{s-3}{(s-3)^2+3^2}\right]. && (6)
\end{aligned}$$

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   (c) $f(t) = te^{4t}\cos 3t$

   Solution.

$$\begin{aligned}
\mathscr{L}\left\{te^{4t}\cos 3t\right\} &= -\frac{d}{ds}\mathscr{L}\left\{e^{4t}\cos(3t)\right\} && (7)\\
&= -\frac{d}{ds}\left[\frac{s-4}{(s-4)^2+3^2}\right] && (8)\\
&= -\frac{d}{ds}\left[\frac{s-4}{(s-4)^2+9}\right] && (9)\\
&= \frac{2(s-4)}{\left[(s-4)^2+9\right]^2} - \frac{1}{(s-4)^2+9}. && (10)
\end{aligned}$$

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(d) $f(t) = t\cos(2t)\cosh(3t)$

Solution.

$$\begin{aligned}
\mathscr{L}\{t\cos(2t)\cosh(3t)\} &= -\frac{d}{ds}\mathscr{L}\{\cos(2t)\cosh(3t)\} && (11)\\
&= -\frac{d}{ds}\mathscr{L}\left\{\frac{e^{2ti}+e^{-2ti}}{2}\cdot\frac{e^{3t}+e^{-3t}}{2}\right\} && (12)\\
&= -\frac{d}{ds}\mathscr{L}\left\{\frac{1}{2}(e^{2ti}+e^{-2ti})\cdot(e^{3t}+e^{-3t})\right\} && (13)\\
&= -\frac{1}{4}\frac{d}{ds}\mathscr{L}\left\{(e^{2ti}+e^{-2ti})\cdot(e^{3t}+e^{-3t})\right\} && (14)\\
&= -\frac{1}{4}\frac{d}{ds}\mathscr{L}\left\{e^{t(3+2i)}+e^{t(3-2i)}+e^{t(-3+2i)}+e^{t(-3-2i)}\right\} && (15)\\
&= -\frac{1}{4}\frac{d}{ds}\left(\begin{array}{c}\frac{1}{s-3-2i}+\frac{1}{s-3+2i}\\ +\quad\frac{1}{s+3-2i}+\frac{1}{s+3+2i}\end{array}\right) && (16)\\
&= \frac{1}{4}\left(\begin{array}{c}\frac{1}{(s-3-2i)^2}+\frac{1}{(s-3+2i)^2}\\ +\quad\frac{1}{(s+3-2i)^2}+\frac{1}{(s+3+2i)^2}\end{array}\right) && (17)
\end{aligned}$$

Using a symbolic calculator to simplify the above expression, we get:

$$= \frac{s^6-5s^4-457s^2+845}{s^8-20s^6+438s^4-3380s^2+28561}. \qquad (18)$$

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2. Find the Inverse Laplace Transform of the following functions:

(a) $F(s) = \frac{s^2+1}{s^4-2s^3-s^2+2s}$

Solution.

$$\begin{aligned}
\mathscr{L}^{-1}\{F(s)\} &= \mathscr{L}^{-1}\left\{\frac{s^2+1}{s^4-2s^3-s^2+2s}\right\} && (19)\\
&= \mathscr{L}^{-1}\left\{\frac{s^2+1}{s(s-1)(s+1)(s-2)}\right\} && (20)\\
&= \mathscr{L}^{-1}\left\{\frac{A}{s}+\frac{B}{s-2}+\frac{C}{s+1}+\frac{D}{s-1}\right\} && (21)
\end{aligned}$$

$A = \frac{1}{2}, B = \frac{5}{6}, C = -\frac{1}{3}, D = -1$

$$\begin{aligned}
&= \mathscr{L}^{-1}\left\{\frac{1}{2s}+\frac{5}{6(s-2)}-\frac{1}{3(s+1)}-\frac{1}{s-1}\right\} && (22)\\
&= \frac{1}{2}+\frac{5}{6}e^{2t}-\frac{1}{3}e^{-t}-e^{t}. && (23)
\end{aligned}$$

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(b) $F(s) = \frac{s}{(s+2)(s^2+4)}$

Solution.

$$\mathscr{L}^{-1}\left\{F(s)\right\} = \mathscr{L}^{-1}\left\{\frac{s}{(s+2)\left(s^2+4\right)}\right\} \tag{24}$$

$$= \mathscr{L}^{-1}\left\{\frac{A}{s+2} + \frac{Bs+C}{s^2+4}\right\} \tag{25}$$

$A = \frac{1}{4}, B = -\frac{1}{4}, C = \frac{1}{2}$

$$= \mathscr{L}^{-1}\left\{\frac{1}{4(s+2)} + \frac{-\frac{1}{4}s+\frac{1}{2}}{(s^2+4)}\right\} \tag{26}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{4(s+2)} - \frac{s-2}{4(s^2+4)}\right\} \tag{27}$$

$$= \frac{1}{4}e^{-2t} - \frac{1}{4}(\cos 2t - \sin 2t). \tag{28}$$

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(c) $F(s) = \ln\left(\frac{s+1}{s-1}\right)$

Solution.

$$\mathscr{L}^{-1}\left\{F(s)\right\} = \mathscr{L}^{-1}\left\{\ln\left(\frac{s+1}{s-1}\right)\right\} \tag{29}$$

$$= \mathscr{L}^{-1}\left\{\ln(s+1) - \ln(s-1)\right\} \tag{30}$$

Using $\mathscr{L}\left\{t^n f(t)\right\}(s) = (-1)^n \dfrac{d^n F(s)}{ds^n}$

$$= -\frac{1}{t}\mathscr{L}^{-1}\left\{\frac{d}{ds}\left[\ln(s+1) - \ln(s-1)\right]\right\} \tag{31}$$

$$= -\frac{1}{t}\mathscr{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\} \tag{32}$$

$$= -\frac{1}{t}\left(e^{-t} - e^{t}\right) \tag{33}$$

$$= \frac{e^t - e^{-t}}{t} \tag{34}$$

$$= \frac{2\sinh t}{t}. \tag{35}$$

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(d) $\frac{s^2+a^2}{(s^2-a^2)^2}$ Use the convolution.

Solution.

$$\mathscr{L}^{-1}\left\{\frac{s^2+a^2}{\left(s^2-a^2\right)^2}\right\} \tag{36}$$

$$=\mathscr{L}^{-1}\left\{\frac{s}{s^2-a^2}\cdot\frac{s}{s^2-a^2}+\frac{a}{s^2-a^2}\cdot\frac{a}{s^2-a^2}\right\} \tag{37}$$

$$=\mathscr{L}^{-1}\left\{\mathscr{L}^{-1}\left\{\cosh at\right\}\cdot\mathscr{L}^{-1}\left\{\cosh at\right\}+\mathscr{L}^{-1}\left\{\sinh at\right\}\cdot\mathscr{L}^{-1}\left\{\sinh at\right\}\right\} \tag{38}$$

$$=\cosh at*\cosh at+\sinh at*\sinh at \tag{39}$$

$$=\int_0^t\cosh a(t-\tau)\cosh a\tau d\tau+\int_0^t\sinh a(t-\tau)\sinh a\tau d\tau \tag{40}$$

$$=\int_0^t\cosh(at)+\cosh(at-2a\tau)d\tau+\int_0^t\cosh(at)-\cosh(at-2a\tau)d\tau \tag{41}$$

$$=2\int_0^t\cosh(at)d\tau \tag{42}$$

$$=2\sinh at. \tag{43}$$

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3. Solve the following IVPs using Laplace transform:

(a) $y''-4y'+4y=t^3e^{2t},\quad y(0)=0,\quad y'(0)=0$

Solution.

$$\mathscr{L}\left\{y''-4y'+4y\right\}=\mathscr{L}\left\{t^3e^{2t}\right\} \tag{44}$$

$$\left(s^2Y(s)-sy(0)-y'(0)\right)-4\left(sY(s)-y(0)\right)+4Y(s)=\frac{6}{(s-2)^4} \tag{45}$$

$$s^2Y(s)-4sY(s)+4Y(s)=\frac{6}{(s-2)^4} \tag{46}$$

$$Y(s)(s^2-4s+4)=\frac{6}{(s-2)^4} \tag{47}$$

$$Y(s)(s-2)^2=\frac{6}{(s-2)^4} \tag{48}$$

$$Y(s)=\frac{6}{(s-2)^6} \tag{49}$$

$$y(t)=\frac{1}{20}t^5e^{2t}. \tag{50}$$

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(b) $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, \quad y'(0) = 4$

Solution.

$$\mathscr{L}\{y'' - 2y' + 5y\} = \mathscr{L}\{1 + t\} \tag{51}$$

$$\left(s^2Y(s) - sy(0) - y'(0)\right) - 2\left(sY(s) - y(0)\right) + 5Y(s) = \frac{1}{s} + \frac{1}{s^2} \tag{52}$$

$$s^2Y(s) - 2sY(s) + 5Y(s) - 4 = \frac{1}{s} + \frac{1}{s^2} \tag{53}$$

$$Y(s)(s^2 - 2s + 5) = \frac{1}{s} + \frac{1}{s^2} + 4 \tag{54}$$

$$Y(s)[(s-1)^2 + 4] = \frac{1}{s} + \frac{1}{s^2} + 4 \tag{55}$$

$$Y(s) = \frac{1}{s[(s-1)^2 + 4]} + \frac{1}{s^2[(s-1)^2 + 4]} + \frac{4}{(s-1)^2 + 4} \tag{56}$$

$$y = 1 * \frac{1}{2}e^t \sin(2t) + t * \frac{1}{2}e^t \sin(2t) + e^t \sin(2t) \tag{57}$$

$$y(t) = \frac{1}{2}\int_0^t e^\tau \sin(2\tau)d\tau + \frac{1}{2}\int_0^t (t-\tau)e^\tau \sin(2\tau)d\tau + e^t \sin(2t) \tag{58}$$

$$y(t) = \frac{1}{10}\left(\mathrm{e}^t \sin(2t) - 2\mathrm{e}^t \cos(2t) + 2\right) - \frac{1}{50}\left(3e^t \sin(2t) + 4e^t \cos(2t) - 10t - 4\right) + e^t \sin(2t) \tag{59}$$

$$y(t) = \frac{1}{25}(5t + 26e^t \sin(2t) - 7e^t \cos(2t) + 7) \tag{60}$$

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(c) $y'' + y = \sqrt{2}\sin(\sqrt{2}t), \quad y(0) = 10, \quad y'(0) = 0$

Solution.

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2 + 2} \tag{61}$$

$$s^2Y(s) - 10s + Y(s) = \frac{2}{s^2 + 2} \tag{62}$$

$$Y(s)(s^2 + 1) = \frac{2}{s^2 + 2} + 10s \tag{63}$$

$$. \tag{64}$$

$$\begin{aligned} Y(s) &= \frac{2}{(s^2+2)(s^2+1)} + \frac{10s}{s^2+1} && (65)\\ &= 2\left(\frac{(s^2+2) - (s^2+1)}{(s^2+2)(s^2+1)}\right) + \frac{10s}{s^2+1} && (66)\\ &= 2\left(\frac{1}{s^2+1} - \frac{1}{s^2+2}\right) + \frac{10s}{s^2+1} && (67)\\ y(t) &= 2\sin t - 2\sin(2t) + 10\cos t. && (68) \end{aligned}$$

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(d) $y'' + y = \delta(t) + H(t-4) + f(t), \quad y(0) = 0, \quad y'(0) = 0,$
where $f(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

Solution.

$$\begin{aligned} f(t) &= \cos t \cdot (H(t-0) - H(t-\pi)) + 0 \cdot H(t-\pi) && (69) \\ &= \cos t \cdot H(t) - \cos t \cdot H(t-\pi). && (70) \end{aligned}$$

$$\begin{aligned} y^{''} + y &= \delta(t) + H(t-4) + \cos t \cdot H(t) - \cos t \cdot H(t-\pi) && (71) \\ \mathscr{L}\left\{y^{''} + y\right\} &= \mathscr{L}\left\{\delta(t) + H(t-4) + \cos t \cdot H(t) - \cos t \cdot H(t-\pi)\right\} && (72) \\ &. && (73) \end{aligned}$$

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = 1 + \frac{e^{-4s}}{s} + \frac{s}{s^2+1} + \frac{s}{s^2+1} e^{-\pi s} \tag{74}$$

$$\begin{aligned} Y(s)(s^2+1) &= 1 + \frac{e^{-4s}}{s} + \frac{s}{s^2+1} + \frac{s}{s^2+1} e^{-\pi s} && (75) \\ Y(s) &= \frac{1}{s^2+1} + \frac{e^{-4s}}{s(s^2+1)} + \frac{s}{(s^2+1)^2} + \frac{e^{-\pi s}}{s^2+1} && (76) \\ &= \frac{1}{s^2+1} + e^{-4s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) + \frac{s}{(s^2+1)^2} + \frac{e^{-\pi s}}{s^2+1} && (77) \\ y &= \sin t + H(t-4)\left(1 - \cos(t-4)\right) + \frac{1}{2} t \sin t - H(t-\pi) \sin t. && (78) \end{aligned}$$

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4. Prove the following identities:

(a) $\mathscr{L}\left\{f''(t)\right\}(s) = s^2 F(s) - sf(0) - f'(0)$

Solution.

$$\begin{aligned} \mathscr{L}\left\{f''(t)\right\}(s) &= \int_0^\infty e^{-st} f''(t) dt && (79) \\ &= e^{-st} f'(t)\Big|_0^\infty - s \int_0^\infty e^{-st} f'(t) dt && (80) \\ &= -f'(0) + s\mathscr{L}\left\{f'(t)\right\}(s) && (81) \\ &= -f'(0) + s\left(\int_0^\infty e^{-st} f'(t) dt\right) && (82) \\ &= -f'(0) + s\left(e^{-st} f(t)\Big|_0^\infty - \int_0^\infty e^{-st} f(t) dt\right) && (83) \\ &= -f'(0) + s\left(f(0) - f(0) + \mathscr{L}\left\{f(t)\right\}(s)\right) && (84) \\ &= -f'(0) + s\left(f(0) + F(s)\right) && (85) \\ &= s^2 F(s) - sf(0) - f'(0). && (86) \end{aligned}$$

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(b) $\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \frac{F(s)}{s}$

Solution.

$$\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\}(s) = \int_0^\infty e^{-st}\left(\int_0^t f(\tau)d\tau\right)dt \tag{87}$$

$$= \frac{e^{-st}}{s}\int_0^t f(\tau)d\tau\Bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\left(\frac{d}{dt}\int_0^t f(\tau)d\tau\right)dt \tag{88}$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\left(\frac{d}{dt}\int_0^t f(\tau)d\tau\right)dt \tag{89}$$

$$= \frac{1}{s}\int_0^\infty e^{-st}\left(\frac{d}{dt}\int_0^t f(\tau)d\tau\right)dt \tag{90}$$

Using the Fundamental Theorem of Calculus

$$= \frac{1}{s}\int_0^\infty e^{-st}\left[f(t)\cdot\frac{d}{dt}t - f(0)\cdot\frac{d}{dt}0\right]dt \tag{91}$$

$$= \frac{1}{s}\int_0^\infty e^{-st}f(t)dt \tag{92}$$

$$= \frac{1}{s}\mathscr{L}\left\{f(t)\right\}(s) \tag{93}$$

$$= \frac{F(s)}{s}. \tag{94}$$

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(c) $\frac{d}{dt}(f * g) = f' * g + f(0)g(t)$

Solution.

$$\frac{d}{dt}(f * g) = \frac{d}{dt}\int_0^t f(\tau)g(t-\tau)d\tau \tag{95}$$

Using the Leibniz Rule

$$= \int_0^t \frac{d}{dt}f(\tau)g(t-\tau)d\tau + f(t)g(0) \tag{96}$$

$$= f * g' + f(t)g(0) \tag{97}$$

Using the definition of convolution

$$= f' * g + f(0)g(t) \tag{98}$$

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5. Use Laplace transform to solve the following system of linear DEs:

$$x'' + x - y = 0,$$
$$y'' + y - x = 0.$$

where

$$x(0) = 0, \quad y(0) = 0, \quad x'(0) = -2, \quad y'(0) = 1.$$

Solution.

Applying Laplace

$$s^2X(s) - sx(0) - x'(0) + X(s) - Y(s) = 0 \tag{99}$$
$$s^2Y(s) - sy(0) - y'(0) + Y(s) - X(s) = 0 \tag{100}$$
$$s^2X(s) + 2 + X(s) = Y(s) \tag{101}$$
$$X(s)(s^2 + 1) = Y(s) - 2 \tag{102}$$
$$s^2Y(s) - 1 + Y(s) = X(s) \tag{103}$$
$$Y(s)(s^2 + 1) = X(s) + 1 \tag{104}$$

Substitute the first equation into the second

$$X(s)(s^2 + 1)(s^2 + 1) = X(s) + 1 - 2 \tag{105}$$
$$X(s) = \frac{1 - 2(s^2 + 1)}{s^2(s^2 + 2)} \tag{106}$$
$$Y(s) = -\frac{s^2 + 2 + 3s^2 + s^4 + 2s^2}{2s^2(s^2 + 1)(s^2 + 2)}. \tag{107}$$

$$x(t) = -\frac{3}{2\sqrt{2}}\sin(\sqrt{2}t) - \frac{1}{2}t \tag{108}$$
$$y(t) = \frac{3}{2\sqrt{2}}\sin(\sqrt{2}t) - \frac{1}{2}t. \tag{109}$$

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