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Date: October 29, 2023

MATH 201
Linear Algebra and Vector Geometry
Assignment 2

Question (1)

1.1 Find the determinant by row reduction to echelon form.

i. $\begin{vmatrix} 1 & 5 & -6 \\ 1 & 6 & 5 \\ -2 & -8 & 7 \end{vmatrix}$

$$\begin{vmatrix} 1 & 5 & -6 \\ 1 & 6 & 5 \\ -2 & -8 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & 11 \\ 0 & 2 & -5 \end{vmatrix} \tag{1}$$

$$= \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & 11 \\ 0 & 0 & -27 \end{vmatrix} \tag{2}$$

$$= 1 \times 1 \times (-27) = -27. \tag{3}$$

ii. $\begin{vmatrix} 1 & -2 & -1 & 0 & -2 \\ 0 & 3 & 8 & 5 & -7 \\ -2 & 4 & 2 & 5 & -1 \\ 1 & -5 & 5 & -9 & -7 \\ 0 & 3 & 8 & 10 & 4 \end{vmatrix}$

$$\begin{vmatrix} 1 & -2 & -1 & 0 & -2 \\ 0 & 3 & 8 & 5 & -7 \\ -2 & 4 & 2 & 5 & -1 \\ 1 & -5 & 5 & -9 & -7 \\ 0 & 3 & 8 & 10 & 4 \end{vmatrix}. \tag{4}$$

$$R_3 + 2R_1 \to R_3 \quad \begin{vmatrix} 1 & -2 & -1 & 0 & -2 \\ 0 & 3 & 8 & 5 & -7 \\ 0 & 0 & 0 & 5 & -5 \\ 1 & -5 & 5 & -9 & -7 \\ 0 & 3 & 8 & 10 & 4 \end{vmatrix} \tag{5}$$

$$R_4 - R_1 + R_2 \to R_4 \quad \begin{vmatrix} 1 & -2 & -1 & 0 & -2 \\ 0 & 3 & 8 & 5 & -7 \\ 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 14 & -4 & -12 \\ 0 & 3 & 8 & 10 & 4 \end{vmatrix} \tag{6}$$

$$R_5 - R_2 \to R_5 \quad \begin{vmatrix} 1 & -2 & -1 & 0 & -2 \\ 0 & 3 & 8 & 5 & -7 \\ 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 14 & -4 & -12 \\ 0 & 0 & 0 & 5 & 11 \end{vmatrix} \tag{7}$$

$$R_3 \leftrightarrow R_4 \quad \begin{vmatrix} 1 & -2 & -1 & 0 & -2 \\ 0 & 3 & 8 & 5 & -7 \\ 0 & 0 & 14 & -4 & -12 \\ 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 5 & 11 \end{vmatrix} \tag{8}$$

$$R_5 - R_4 \to R_5 \quad \begin{vmatrix} 1 & -2 & -1 & 0 & -2 \\ 0 & 3 & 8 & 5 & -7 \\ 0 & 0 & 14 & -4 & -12 \\ 0 & 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 0 & 16 \end{vmatrix}. \tag{9}$$

Because of the row exchange in (8), the sign of the determinant is changed.

$$\Delta = -(1 \times 3 \times 14 \times 5 \times 16) \tag{10}$$

$$= -3360. \tag{11}$$

1.2 Let U be a square matrix such that $U^\intercal U = I$. Show that $\det(U) = \pm 1$.

$$\det(U^\intercal U) = \det(I) \tag{12}$$

$$\det(U^\intercal)\det(U) = 1 \tag{13}$$

$$\det(U)\det(U) = 1 \tag{14}$$

$$\det(U)^2 = 1 \tag{15}$$

$$\det(U) = \pm 1. \tag{16}$$

Question (2)

For $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $\quad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ Find

i. A^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{17}$$

$$R_2 - R_1 \to R_2 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{18}$$

$$R_3 - R_2 \to R_3 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{19}$$

$$R_4 - R_3 \to R_4 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 \end{array}\right]. \tag{20}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{21}$$

ii. $\left|2A^2\right|$

$$\begin{aligned} \left|2A^2\right| &= 2^4 \left|A^2\right| && (22) \\ &= 16 \left|A^2\right| && (23) \\ &= 16 \left|A\right|^2 && (24) \\ &= 16 \left(1\right)^2 && (25) \\ &= 16. && (26) \end{aligned}$$

iii. x such that $A^\intercal x = b$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]. \tag{27}$$

$$R_1 - R_2 \to R_1 \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \tag{28}$$

$$R_2 - R_3 \to R_2 \quad \left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \tag{29}$$

$$R_1 + R_3 \to R_1 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \tag{30}$$

$$R_1 - R_4 \to R_1 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \tag{31}$$

$$R_2 + R_4 \to R_2 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right] \tag{32}$$

$$R_3 - R_4 \to R_3 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]. \tag{33}$$

$$x = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 4 \end{bmatrix}. \tag{34}$$

Question (3)

If $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, a \in \mathbb{R}$

i. Find $Tr\left(C + 2C^{\intercal}\right)$.

$$C + 2C^{\intercal} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

$$= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 2a & 0 \\ 0 & a & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{36}$$

$$Tr\left(C + 2C^{\intercal}\right) = 3 + 3 + 3 + 3 \tag{37}$$

$$= 12. \tag{38}$$

ii. Find, if possible C^{-1}.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]. \tag{39}$$

$$R_3 - aR_2 \to R_3 \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -a & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \tag{40}$$

$$. \tag{41}$$

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{42}$$

iii. Find, if possible $\left|2C^{-1}\right|$, $\left|(2C)^{-1}\right|$.

$$\left|2C^{-1}\right| = 2^4 \left|C^{-1}\right| \tag{43}$$

$$= 16 \left|C^{-1}\right| \tag{44}$$

$$= 16\,(1) \tag{45}$$

$$= 16. \tag{46}$$

$$\left|(2C)^{-1}\right| = \frac{1}{2^4}\left|C^{-1}\right| \tag{47}$$

$$= \frac{1}{16}\left|C^{-1}\right| \tag{48}$$

$$= \frac{1}{16}(1) \tag{49}$$

$$= \frac{1}{16}. \tag{50}$$

iv. Derive a formula for C^n.

$$C^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & na & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

Question (4)

Mark each statement True or False. Justify each answer.

i. If A can be row reduced to the identity matrix, then A must be invertible.

True; since A can be row reduced to the identity matrix, A is row equivalent to the identity matrix and every matrix that is row equivalent to the identity is invertible. $\square$

ii. If A is invertible, then elementary row operations that reduce A to the identity In also reduce A^{-1} to I_n.

False; elementary row operations that reduce A to the identity I_n does not necessarily reduce A^{-1} to I_n. $\square$

iii. The columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible.

True; the columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible as the columns of A are linearly independent. $\square$

iv. If the $n \times n$ matrices E and F have the property that $EF = I$, then E and F commute.

True; if $EF = I$, then $E = F^{-1}$ and $F = E^{-1}$ so $EF = FE = I$. $\square$

v. The determinant of A_n is the product of the pivots in any echelon form U of A_n, multiplied by $(1)^r$, where r is the number of row interchanges made during row reduction from A_n to U.

False; $(-1)^r$ not $(1)^r$. $\square$

vi. $|-A| = -|A|$.

False; $|-A| = (-1)^n|A|$ and n could be even. $\square$

vii. If $A_{n\times n}$ is reduced to an upper triangular matrix U through row replacement operations only, then $|A| = |U|$.

False; $|A| = (-1)^r|U|$. $\square$

viii. If $A_{n\times n}$ is skew-symmetric and n is an odd positive integer, then A is non-invertible.

True; if $A^T = -A$ and n is an odd positive integer, then $|A| = |A^T| = |-A| = -|A| = 0$ so A is non-invertible. $\square$

ix. For any square matrices A, B, we have $Tr(BAB) = Tr(AB^2)$.

True; because of the cyclic property of trace. $\square$