Name: SalahDin Ahmed Salh Rezk
ID: 202201079
Date: December 24, 2023

Linear Algebra (MATH 201)

Assignment 4

---

1. Given that,

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 2 \end{pmatrix}$$

   (a) Is 3 an eigenvalue of A? If so, find its corresponding eigenspace.

   Solution.

$$\det(A - \lambda I) = 0 \tag{1}$$

$$\det \begin{pmatrix} 3-\lambda & 0 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & -1 & 2-\lambda \end{pmatrix} = 0 \tag{2}$$

$$(3-\lambda)((4-\lambda)(2-\lambda) - (-1)(-1)) = 0 \tag{3}$$

$$(3-\lambda)(\lambda^2 - 6\lambda + 9) = 0 \tag{4}$$

$$(3-\lambda)(\lambda - 3)^2 = 0 \tag{5}$$

$$\lambda = 3. \tag{6}$$

■

   Solution.

$$A - 3I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \tag{7}$$

$$R_2 \leftarrow R_2 + R_3 \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix} \tag{8}$$

$$R_3 \leftarrow -R_3 \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{9}$$

$$R_1 \leftrightarrow R_3 \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

$$\therefore \text{Eigenspace} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}. \tag{11}$$

■

(b) Express, if possible, e^A as a linear combination of I_3, A, A^2.

Solution.

$$e^A = \alpha_0 I_3 + \alpha_1 A + \alpha_2 A^2 \tag{12}$$
$$e^\lambda = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2 \tag{13}$$
$$\frac{d}{d\lambda} e^\lambda = \alpha_1 + 2\alpha_2 \lambda \tag{14}$$
$$\frac{d^2}{d\lambda^2} e^\lambda = 2\alpha_2. \tag{15}$$

$$e^3 = 2\alpha_2 \implies \alpha_2 = \frac{e^3}{2} \tag{16}$$
$$e^3 = \alpha_1 + 3e^3 \implies \alpha_1 = -2e^3 \tag{17}$$
$$e^3 = \alpha_0 - 6e^3 + \frac{9e^3}{2} \implies \alpha_0 = \frac{5e^3}{2}. \tag{18}$$

$$\therefore e^A = \frac{5e^3}{2} I_3 - 2e^3 A + \frac{e^3}{2} A^2. \tag{19}$$

■

(c) Find, if possible, an orthogonal matrix P, such that $A = PDP^T$.

Solution.

$$\because \dim(\text{Eigenspace}(A)) < \dim(A) \implies A \text{ is not diagonalizable} \tag{20}$$
$$\therefore \nexists P \text{ such that } A = PDP^T. \tag{21}$$

■

2. Use Cayley-Hamilton theorem to find the exponential matrix e^{At} such that:

$$A = \begin{pmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{pmatrix}.$$

Solution.

$$|A - \lambda I| = 0 \tag{22}$$

$$\begin{vmatrix} -6-\lambda & -11 & 16 \\ 2 & 5-\lambda & -4 \\ -4 & -5 & 10-\lambda \end{vmatrix} = 0. \tag{23}$$

$$\begin{aligned} \lambda^3 - \mathrm{Tr}(A)\lambda^2 + \mathrm{Tr}(\mathrm{Adj}(A))\lambda - |A| &= 0 \\ \lambda^3 - 9\lambda^2 + 26\lambda - 24 &= 0. \end{aligned}$$

$$\lambda = \{2, 3, 4\}. \tag{24}$$

$$e^{\lambda t} = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 \implies \begin{cases} e^{2t} &= \alpha_0 + 2\alpha_1 + 4\alpha_2 \\ e^{3t} &= \alpha_0 + 3\alpha_1 + 9\alpha_2 \\ e^{4t} &= \alpha_0 + 4\alpha_1 + 16\alpha_2 \end{cases} \tag{25}$$

$$\implies \left[\begin{array}{ccc|c} 1 & 2 & 4 & e^{2t} \\ 1 & 3 & 9 & e^{3t} \\ 1 & 4 & 16 & e^{4t} \end{array}\right] \tag{26}$$

$$\implies \begin{cases} \alpha_2 &= \frac{1}{2}(e^{4t} - 2e^{3t} + e^{2t}) \\ \alpha_1 &= -\frac{5}{2}e^{4t} + 6e^{3t} - \frac{7}{2}e^{2t} \\ \alpha_0 &= 3e^{4t} - 8e^{3t} + 6e^{2t} \end{cases}. \tag{27}$$

$$e^{At} = \alpha_0 I_3 + \alpha_1 A + \alpha_2 A^2 \tag{28}$$

$$e^{At} = (3e^{4t} - 8e^{3t} + 6e^{2t})I_3 + (-\frac{5}{2}e^{4t} + 6e^{3t} - \frac{7}{2}e^{2t})A + (\frac{1}{2}(e^{4t} - 2e^{3t} + e^{2t}))A^2. \tag{29}$$

$$A^2 = \begin{bmatrix} -50 & -69 & 108 \\ 14 & 23 & -28 \\ -26 & -31 & 56 \end{bmatrix} \tag{30}$$

$$\begin{aligned} e^{At} = (3e^{4t} - 8e^{3t} + 6e^{2t}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (-\frac{5}{2}e^{4t} + 6e^{3t} - \frac{7}{2}e^{2t}) \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix} \\ + (\frac{1}{2}(e^{4t} - 2e^{3t} + e^{2t})) \begin{bmatrix} -50 & -69 & 108 \\ 14 & 23 & -28 \\ -26 & -31 & 56 \end{bmatrix} \end{aligned} \tag{31}$$

$$= \begin{bmatrix} -7e^{4t} + 6e^{3t} + e^{2t} & -7e^{4t} + 3e^{3t} + 4e^{2t} & 14e^{4t} - 12e^{3t} - 2e^{2t} \\ 2e^{4t} - 2e^{3t} & 2e^{4t} - e^{3t} & -4e^{4t} + 4e^{3t} \\ -3e^{4t} + 2e^{3t} + e^{2t} & -3e^{4t} + e^{3t} + 2e^{2t} & 6e^{4t} - 4e^{3t} - e^{2t} \end{bmatrix}. \tag{32}$$

■

3. Given that:

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}.$$

(a) Show that A is an idempotent matrix (i.e. $AA = A$).

Solution.

$$AA = A \iff \lambda = 0 \vee \lambda = 1. \tag{33}$$

$$|A - \lambda I| = 0 \tag{34}$$

$$\begin{vmatrix} 2-\lambda & -2 & -4 \\ -1 & 3-\lambda & 4 \\ 1 & -2 & -3-\lambda \end{vmatrix} = 0 \tag{35}$$

$$\lambda^3 - \mathrm{Tr}(A)\lambda^2 + \mathrm{Tr}(\mathrm{Adj}(A))\lambda - |A| = 0 \tag{36}$$

$$\lambda^3 - 2\lambda^2 + \lambda = 0 \tag{37}$$

$$\lambda = \{0, 1, 1\}. \tag{38}$$

$$\therefore AA = A \tag{39}$$

■

(b) Use part (a) to solve the initial value problem.

$$x' = 2x - 2y - 4z$$
$$y' = -x + 3y + 4z$$
$$z' = x - 2y - 3z$$

Such that, $x(0) = y(0) = z(0) = 1$

Solution.

$$e^{At} = \alpha_0 I_3 + \alpha_1 A + \alpha_2 A^2 \tag{40}$$

Since A is idempotent i.e. $A^2 = A$,

$$= \alpha_0 I_3 + \alpha_1 A \tag{41}$$

Using Cayley-Hamilton:

$$e^{\lambda t} = \alpha_0 + \alpha_1 \lambda \tag{42}$$

$$e^0 = \alpha_0 + \alpha_1 \times 0 \implies \alpha_0 = 1 \tag{43}$$

$$e^t = 1 + \alpha_1 \times 1 \implies \alpha_1 = e^t - 1 \tag{44}$$

$$. \tag{45}$$

$$e^{At} = I_3 + (e^t - 1)A \tag{46}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (e^t - 1)\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} 1 + 2(e^t - 1) & -2(e^t - 1) & -4(e^t - 1) \\ -1(e^t - 1) & 1 + 3(e^t - 1) & 4(e^t - 1) \\ 1(e^t - 1) & -2(e^t - 1) & 1 + -3(e^t - 1) \end{bmatrix} \tag{48}$$

$$= \begin{bmatrix} 2e^t - 1 & -2e^t + 2 & -4e^t + 4 \\ -e^t + 1 & 3e^t - 2 & 4e^t - 4 \\ e^t - 1 & -2e^t + 2 & -3e^t + 4 \end{bmatrix}. \tag{49}$$

$$
\begin{aligned}
r &= e^{At} r_0 && (50)\\
&= \begin{bmatrix} 2e^t-1 & -2e^t+2 & -4e^t+4 \\ -e^t+1 & 3e^t-2 & 4e^t-4 \\ e^t-1 & -2e^t+2 & -3e^t+3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} && (51)\\
&= \begin{bmatrix} -4e^t+5 \\ 6e^t-5 \\ -4e^t+5 \end{bmatrix}. && (52)
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= -4e^t+5 && (53)\\
y(t) &= 6e^t-5 && (54)\\
z(t) &= -4e^t+5. && (55)
\end{aligned}
$$

■

4.

(a) Let A be a diagonalizable $n \times n$ matrix show that if the multiplicity of an eigenvalue λ is n, then $A = \lambda I$.

Solution.

$$
\begin{aligned}
A &= PD_\lambda P^{-1} && (56)\\
&= P(\lambda I)P^{-1} && (57)\\
&= \lambda PIP^{-1} && (58)\\
&= \lambda I. && (59)
\end{aligned}
$$

■

(b) Use part (a) to show that the matrix $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable.

Solution.

$$
\begin{aligned}
|A - \lambda I| &= 0 && (60)\\
(3-\lambda)^2 &= 0 && (61)\\
&\implies \lambda_1 = \lambda_2 = 3. && (62)
\end{aligned}
$$

Using (a), $3I$ is the only matrix of multiplicity 2 that is diagonalizable

$$
\begin{aligned}
\because \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} &\neq \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} && (63)\\
\therefore \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} &\text{ is not diagonalizable.} && (64)
\end{aligned}
$$

■

5.

(a) The eigenvalues (λ) of a diagonalizable square matrix A together with their multiplicities are given in the following table.

λ	**Multiplicity**
1	1
2	3
3	2

i. Find the dimension of A

Solution.

$$\begin{aligned} \dim(A) &= \sum_{k=1}^{n} \mathrm{Mult}(\lambda_k) && (65) \\ &= 1+3+2 && (66) \\ &= 6. && (67) \end{aligned}$$

■

ii. Find the Trace and determinant of A.

Solution.

$$\begin{aligned} \mathrm{Tr}(A) &= \sum_{k=1}^{n} \lambda_k && (68) \\ &= 1+3\times 3+3\times 2 && (69) \\ &= 13. && (70) \end{aligned}$$

$$\begin{aligned} |A| &= \prod_{k=1}^{n} \lambda_k && (71) \\ &= 1\times 2^3\times 3^2 && (72) \\ &= 72. && (73) \end{aligned}$$

■

iii. Find the characteristic polynomial of A.

Solution.

$$(\lambda-1)(\lambda-2)^3(\lambda-3)^2 = 0. \qquad (74)$$

■

iv. Find $|e^{At}|$.

Solution.

$$\begin{aligned} |e^{At}| &= |PD_{e^{\lambda t}}P^{-1}| && (75) \\ &= |P||D_{e^{\lambda t}}||P^{-1}| && (76) \\ &= |D_{e^{\lambda t}}| && (77) \\ &= e^{\sum_{k=1}^{n} \lambda_k t} && (78) \\ &= e^{\mathrm{Tr}(A)t} && (79) \\ &= e^{13t}. && (80) \end{aligned}$$

■

(b) Solve the differential equations

$$\frac{dy}{dt} = 2x - 3y; \frac{dx}{dt} = 3x - 4y.$$

for $x(0) = 1; y(0) = 0$

Solution.

$$|A - \lambda I| = 0 \tag{81}$$

$$\begin{vmatrix} 2-\lambda & -3 \\ -3 & 4-\lambda \end{vmatrix} = 0 \tag{82}$$

$$-(3-\lambda)(3+\lambda) + 8 = 0 \tag{83}$$

$$\lambda = \pm 1. \tag{84}$$

$$\text{At } \lambda = 1 \quad \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{85}$$

$$\text{At } \lambda = -1 \quad \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \implies v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{86}$$

$$e^{At} r_0 \tag{87}$$

$$e^{At} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{88}$$

$$P D_{e^{\lambda t}} P^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{89}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{90}$$

$$\begin{bmatrix} 2e^t - e^{-t} & -2e^t + 2e^{-t} \\ e^t - e^{-t} & -e^t + 2e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{91}$$

$$\begin{bmatrix} 2e^t - e^{-t} \\ e^t - e^{-t} \end{bmatrix}. \tag{92}$$

$$x(t) = 2e^t - e^{-t} \tag{93}$$

$$y(t) = e^t - e^{-t} \tag{94}$$

■