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Thermodynamics, Wave Motion and Optics

Assignment 3

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1. Determine the minimum thickness of a quartz plate needed to convert a left-hand circularly polarized light source at a wavelength of $\lambda_0 = 656$ nm to right-hand circularly polarized light. The refractive indices for quartz are $n_s = 1.551$ and $n_f = 1.542$.

   Solution. Since the light is circularly polarized, the phase difference is π.

$$\Delta\phi = \frac{2\pi}{\lambda_0}\Delta d \quad \Delta d = |n_s - n_f|d. \tag{1}$$

$$\pi = \frac{2\pi}{\lambda_0}|n_s - n_f|d \tag{2}$$

$$d = \frac{\lambda_0}{2|n_s - n_f|} \tag{3}$$

$$= \frac{656 \times 10^{-9}}{2|1.551 - 1.542|} \tag{4}$$

$$= 3.6 \times 10^{-5} \text{ m}. \tag{5}$$

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2. For each of the provided electromagnetic waves, please characterize their polarization state and provide a rationale for your classification

   (a) $E(z,t) = E_0 \cos(k_0 z - \omega_0 t)\hat{x} - E_0 \cos(k_0 z - \omega_0 t)\hat{y}$

   Solution. Linearly polarized, because the electric field vector is constant in magnitude and direction. ■

   (b) $E(z,t) = E_0 \sin[2\pi(z\lambda^{-1} - v_0 t)]\hat{x} - E_0 \sin[2\pi(z\lambda^{-1} - v_0 t)]\hat{y}$

   Solution. Linearly polarized, because the electric field vector is constant in magnitude and direction. ■

   (c) $E(z,t) = E_0 \sin(\omega_0 t - k_0 z)\hat{x} + E_0 \sin(\omega_0 t - k_0 z - \frac{\pi}{4})\hat{y}$

   Solution. Elliptically polarized, because the electric field vector is constant in magnitude and varying in direction with phase difference of $\dfrac{\pi}{4}$. ■

   (d) $E(z,t) = E_0 \sin(\omega_0 t - k_0 z)\hat{x} + E_0 \sin(\omega_0 t - k_0 z - \frac{\pi}{2})\hat{y}$

   Solution. Circularly polarized, because the electric field vector is constant in magnitude and varying in direction with phase difference of $\dfrac{\pi}{2}$. ■

3. The index of refraction of air at 300 K and 1 atmosphere pressure is 1.0003 in the middle of the visible spectrum. Assuming an isothermal atmosphere (of constant temperature) at 300 K, calculate by what factor the earth's atmosphere would have to be denser to cause light to bend around the earth with the earth's curvature at sea level. (In cloudless skies we could then watch sunset all night, in principle, but with an image of the sun drastically compressed vertically.) You may assume that the index of refraction n has the property that $n-1$ is proportional to the density. (Hint: Think of Fermat's Principle.) The $\frac{1}{e}$ height of this isothermal atmosphere is 8700 meters. (where $R = 6400 \times 10^3$ m is the earth's radius). i.e, $n(r) - 1 = \rho e^{-\frac{r-R}{8700}}$.

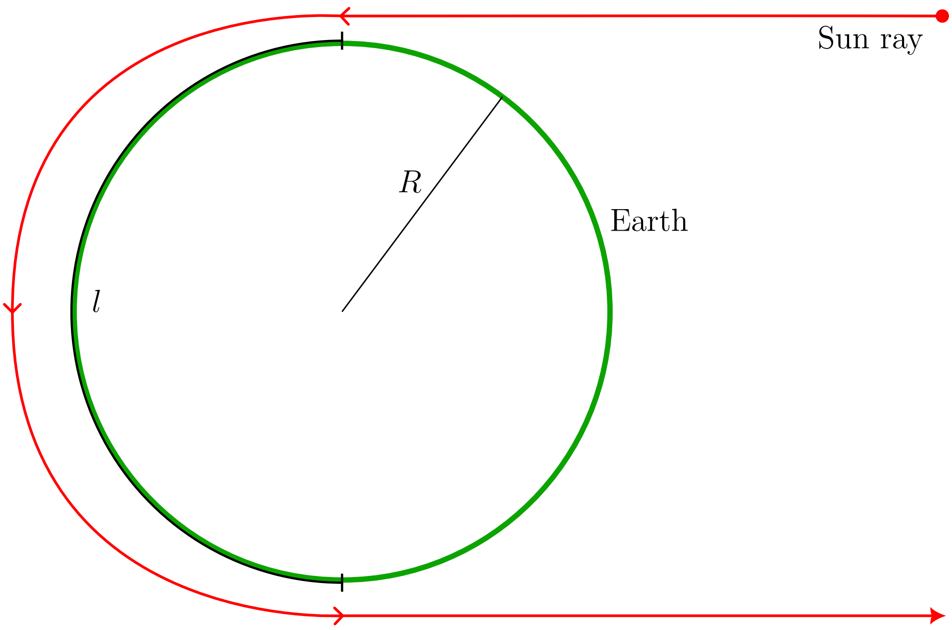

Figure 1

Solution.

$$n(r) = 1 + \rho e^{-\frac{r-R}{8700}} \tag{6}$$

$$\frac{dn}{dr} = -\frac{\rho}{8700} e^{-\frac{r-R}{8700}}. \tag{7}$$

$$l = n(r) r\theta \tag{8}$$

$$\frac{dl}{dr} = \frac{dn}{dr} r\theta + n(r)\theta \tag{9}$$

$$\implies \frac{dn}{dr} = -\frac{n(r)}{r}. \tag{10}$$

$$\frac{n(r)}{r} = \frac{\rho}{8700} e^{-\frac{r-R}{8700}}. \tag{11}$$

$$r = R = 6.4 \times 10^6 \implies \frac{\rho \times 6.4 \times 10^6}{8700} = 1 + \rho \implies \rho = 1.36 \times 10^{-3} \tag{12}$$

$$n_0 - 1 = \rho_0 \implies \rho_0 = 3 \times 10^{-4}. \tag{13}$$

$$\frac{\rho}{\rho_0} = \frac{1.36 \times 10^{-3}}{3 \times 10^{-4}} = 4.53 \tag{14}$$

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4. A rod of diameter d is bent to take the shape shown in the figure below. The inner radius of the curved part of the rod is denoted by R. Find the limiting value for d/R for which all light rays entering normally at face A of the rod will totally come out ONLY from face B. The refractive index of the material of the rod is 1.5.

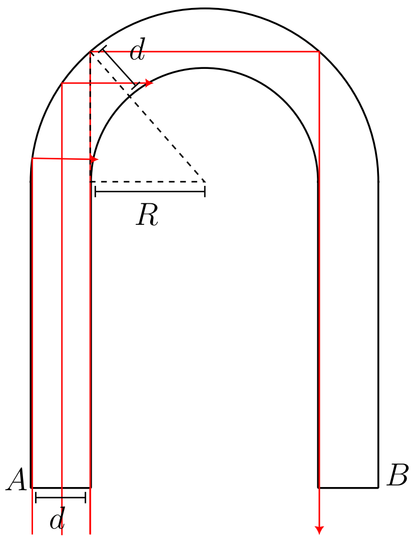

Figure 2

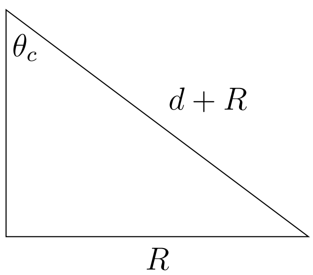

Figure 3

Solution.

$$\sin\theta_c = \frac{R}{d+R} \tag{15}$$
$$= \frac{1}{n}. \tag{16}$$

$$\frac{d}{R} = \frac{1}{\sin\theta_c} - 1 \tag{17}$$
$$= \frac{1}{\frac{1}{n}} - 1 \tag{18}$$
$$= n - 1. \tag{19}$$

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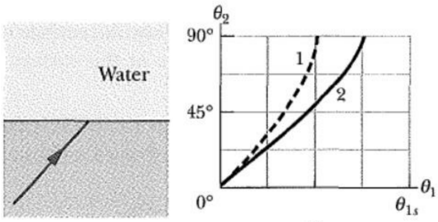
5. In the Figure 4a, a light ray in an underlying material is incident at angle θ_1 on a boundary with water, and some of the light refracts into the water. There are two choices of underlying material. For each, the angle of refraction θ_2 versus the incident angle θ_1 is given in the Figure 4b. The horizontal axis scale is set by $\theta_{1s} = 90°$.
Without calculation, determine whether the index of refraction of (a) material 1 and (b) material 2 is greater or less than the index of water ($n = 1.33$).
What is the index of refraction of (c) material 1 and (d) material 2?



(a) (b)

Figure 4

Solution.

$$\theta_1 > \theta_2 \implies n_1 < n_2. \tag{20}$$

$$\begin{aligned}
1.33 \sin(45°) &= n_1 \sin(90°) && (21)\\
n_1 &= 1.33 \sin(45°) && (22)\\
&= 1.33 \frac{\sqrt{2}}{2} && (23)\\
1.33 \sin(67.5°) &= n_2 \sin(90°) && (24)\\
n_2 &= 1.33 \sin(67.5°) && (25)\\
&= 1.33 \frac{\sqrt{2+\sqrt{2}}}{2}. && (26)
\end{aligned}$$

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6. A 4.00-m-long pole stands vertically in a lake having a depth of 2.00 m. The Sun is 40.0° above the horizontal. Determine the length of the pole's shadow on the bottom of the lake. Take the index of refraction for water to be 1.33.

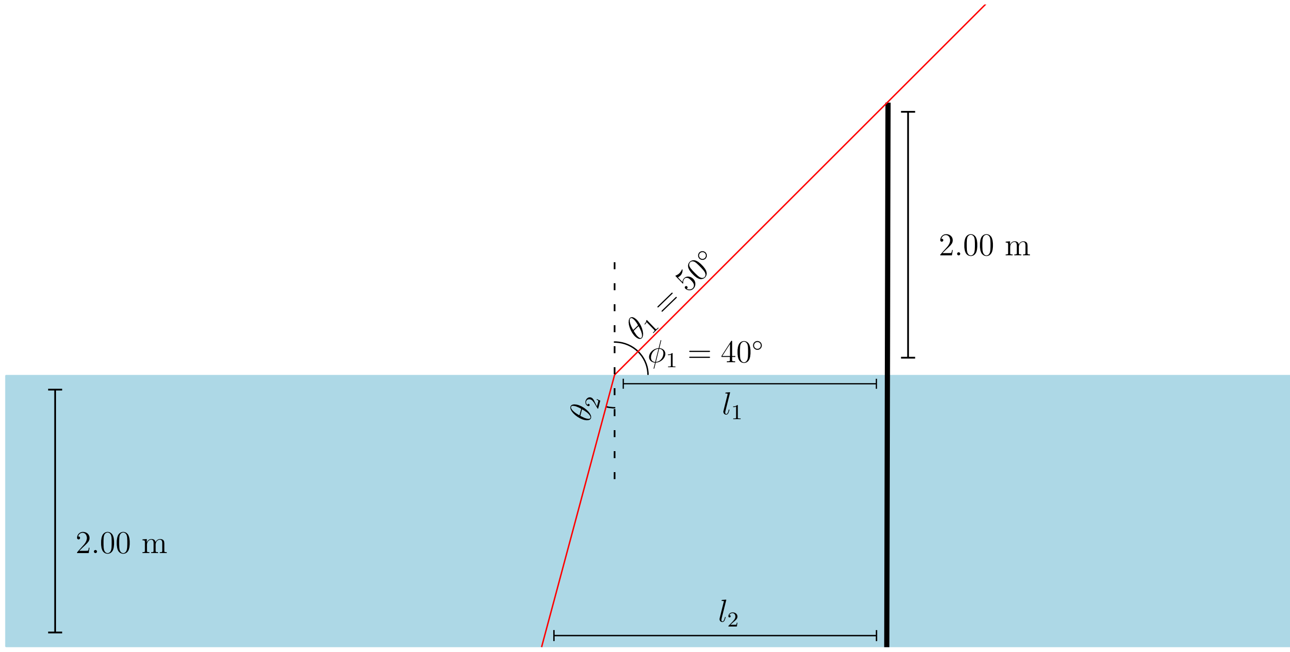

Figure 5

Solution.

$$\begin{aligned} l_1 &= 2\cot\phi_1 &(27)\\ &= 2\cot(40°) &(28)\\ &= 2.38 \text{ m.} &(29)\end{aligned}$$

$$\begin{aligned} 1.33\sin\theta_2 &= \sin\theta_1 &(30)\\ \theta_2 &= \sin^{-1}\left(\frac{\sin\theta_1}{1.33}\right) &(31)\\ &= \sin^{-1}\left(\frac{\sin(50°)}{1.33}\right) &(32)\\ &= 35.2°. &(33)\end{aligned}$$

$$\begin{aligned} l_2 - l_1 &= 2\cot(\theta_2) &(34)\\ &= 2\cot(35.2°) &(35)\\ &= 2.84 \text{ m.} &(36)\end{aligned}$$

$$\begin{aligned} l_2 &= l_1 + 2.84 &(37)\\ &= 5.22 \text{ m.} &(38)\end{aligned}$$

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