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Date: November 30, 2023

Linear Algebra (MATH 201)

Bonus 2

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1. Prove that every square matrix can be written as a matrix multiplication of two symmetric matrices

Proof. Assume simpler case of 2×2 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix} \quad \text{such that} \quad A = BC. \tag{1}$$

$$BC = \begin{bmatrix} b_{11} \cdot c_{11} + b_{12} \cdot c_{12} & b_{11} \cdot c_{12} + b_{12} \cdot c_{22} \\ b_{21} \cdot c_{11} + b_{22} \cdot c_{12} & b_{21} \cdot c_{12} + b_{22} \cdot c_{22} \end{bmatrix}. \tag{2}$$

$$\implies \begin{cases} a_{11} = b_{11} \cdot c_{11} + b_{12} \cdot c_{12} \\ a_{12} = b_{11} \cdot c_{12} + b_{12} \cdot c_{22} \\ a_{21} = b_{21} \cdot c_{11} + b_{22} \cdot c_{12} \\ a_{22} = b_{21} \cdot c_{12} + b_{22} \cdot c_{22} \end{cases} \tag{3}$$

In this system of equations, we have 6 unknowns and 4 equations, thus we can choose any two variables to be free variables. Let's choose b_{11} and b_{22} to be free variables. Then we can solve the system of equations for b_{12} and b_{21}:

$$\begin{cases} b_{12} = \dfrac{a_{12} - b_{11} \cdot c_{12}}{c_{22}} \\ \\ b_{21} = \dfrac{a_{21} - b_{22} \cdot c_{12}}{c_{11}} \end{cases} \tag{4}$$

Thus, there always exists a solution for B and C such that $A = BC$, where B and C are symmetric matrices, such that $c_{22}, c_{11} \neq 0$.

This can be generalized to any $n \times n$ matrix as follows:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} \cdot c_{ij} \iff \begin{cases} a_{11} = b_{11} \cdot c_{11} + b_{12} \cdot c_{12} + \cdots + b_{1n} \cdot c_{1n} \\ a_{12} = b_{11} \cdot c_{12} + b_{12} \cdot c_{22} + \cdots + b_{1n} \cdot c_{2n} \\ \vdots \\ a_{1n} = b_{11} \cdot c_{1n} + b_{12} \cdot c_{2n} + \cdots + b_{1n} \cdot c_{nn} \\ a_{21} = b_{21} \cdot c_{11} + b_{22} \cdot c_{12} + \cdots + b_{2n} \cdot c_{1n} \\ \vdots \\ a_{nn} = b_{n1} \cdot c_{1n} + b_{n2} \cdot c_{2n} + \cdots + b_{nn} \cdot c_{nn} \end{cases}. \tag{5}$$

$$\implies \begin{cases} b_{12} = \dfrac{a_{12} - b_{11} \cdot c_{12} - \cdots - b_{1n} \cdot c_{2n}}{c_{22}} \\ b_{21} = \dfrac{a_{21} - b_{22} \cdot c_{12} - \cdots - b_{2n} \cdot c_{1n}}{c_{11}} \\ \vdots \\ b_{1n} = \dfrac{a_{1n} - b_{11} \cdot c_{1n} - \cdots - b_{1,n-1} \cdot c_{nn}}{c_{nn}} \\ b_{n1} = \dfrac{a_{n1} - b_{n2} \cdot c_{11} - \cdots - b_{n,n-1} \cdot c_{1n}}{c_{nn}} \end{cases} \tag{6}$$

Therefore, there always exists n^2 equations and $n(n+1)$ unknowns, thus we can choose n variables to be free variables. Then we can always solve the system of equations for the remaining $n^2 - n$ variables, such that $c_{ii} \neq 0$ for all $i \in \{1, 2, \cdots, n\}$. □

2. Given that A, B, and C are Three matrices...A commutes with B....B commutes with C, then prove that A commutes with some Polynomial matrix function of C.

Theorem 1. If the set of matrices considered is restricted to Hermitian matrices without multiple eigenvalues, then commutativity is transitive.

$$A \cdot B = B \cdot A \wedge B \cdot C = C \cdot B \implies A \cdot C = C \cdot A.$$

Under this restriction, the proof is trivial:

Proof. Let $P(C)$ be a polynomial matrix function of C, such that

$$P(C) = \sum_{i=0}^{n} a_i C^i. \tag{7}$$

As such, we are trying to prove that

$$A \cdot B = B \cdot A \wedge B \cdot C = C \cdot B \implies A \cdot P(C) = P(C) \cdot A \tag{8}$$

Then:

$$A \cdot P(C) \stackrel{?}{=} P(C) \cdot A \tag{9}$$

$$A \cdot \sum_{i=0}^{n} a_i C^i \stackrel{?}{=} \sum_{i=0}^{n} a_i C^i \cdot A \tag{10}$$

$$A(a_0 I + a_1 C + a_2 C^2 + \cdots + a_n C^n) \stackrel{?}{=} (a_0 I + a_1 C + a_2 C^2 + \cdots + a_n C^n) \cdot A \tag{11}$$

$$a_0 A + a_1 AC + a_2 AC^2 + \cdots + a_n AC^n \stackrel{?}{=} a_0 A + a_1 CA + a_2 C^2 A + \cdots + a_n C^n A \tag{12}$$

$$a_1 AC + a_2 AC^2 + \cdots + a_n AC^n \stackrel{?}{=} a_1 CA + a_2 C^2 A + \cdots + a_n C^n A \tag{13}$$

$$\implies \begin{cases} a_1 AC \stackrel{?}{=} a_1 CA \\ a_2 AC^2 \stackrel{?}{=} a_2 C^2 A \\ \vdots \\ a_n AC^n \stackrel{?}{=} a_n C^n A \end{cases} \implies \begin{cases} AC \stackrel{?}{=} CA \\ AC^2 \stackrel{?}{=} C^2 A \\ \vdots \\ AC^n \stackrel{?}{=} C^n A \end{cases} \tag{14}$$

Using Theorem 1, $AC^i = C^i A$ for all $i \in \mathbb{N}$, by induction.
Therefore, $A \cdot P(C) = P(C) \cdot A$. □

Note that this proof does not hold for non-Hermitian matrices or Hermitian matrices with multiple eigenvalues, as Theorem 1 does not hold in those cases.