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Ordinary Differential Equations (MATH 202)

Assignment 1

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1. Obtain the general solution of the following differential equations:

   (a) $x\sin(y) + (x^2+1)\cos(y)y' = 0$

   Solution.

$$\left(x^2+1\right)\cos(y)y' = -x\sin(y) \tag{1}$$
$$\frac{\cos(y)}{\sin(y)}y' = -\frac{x}{x^2+1} \tag{2}$$
$$\frac{\cos(y)}{\sin(y)}\frac{dy}{dx} = -\frac{x}{x^2+1} \tag{3}$$
$$\frac{\cos(y)}{\sin(y)}dy = -\frac{x}{x^2+1}dx \tag{4}$$
$$\int\frac{\cos(y)}{\sin(y)}dy = -\int\frac{x}{x^2+1}dx \tag{5}$$

   Using substitution

$$u = \sin(y) \tag{6}$$
$$du = \cos(y)dy \tag{7}$$
$$\frac{du}{\cos(y)} = dy \tag{8}$$

   We have

$$\int\frac{\cos(y)}{u}\frac{du}{\cos(y)} = -\int\frac{x}{x^2+1}dx \tag{9}$$
$$\int\frac{1}{u}du = -\int\frac{x}{x^2+1}dx \tag{10}$$
$$\ln|u| = -\int\frac{x}{x^2+1}dx \tag{11}$$
$$\ln|\sin(y)| = -\int\frac{x}{x^2+1}dx \tag{12}$$

   Using substitution

$$u = x^2+1 \tag{13}$$
$$du = 2xdx \tag{14}$$
$$\frac{du}{2x} = dx \tag{15}$$

We have

$$
\begin{aligned}
\ln|\sin(y)| &= -\int \frac{x}{u}\frac{du}{2x} && (16)\\
&= -\frac{1}{2}\int \frac{1}{u}du && (17)\\
&= -\frac{1}{2}\ln|u|+C && (18)\\
&= -\frac{1}{2}\ln\left(x^2+1\right)+C && (19)\\
&= \ln\left(x^2+1\right)^{-\frac{1}{2}}+C && (20)\\
&= \ln\left(\frac{1}{\sqrt{x^2+1}}\right)+C && (21)\\
\sin(y) &= e^C\frac{1}{\sqrt{x^2+1}} && (22)\\
&= C\frac{1}{\sqrt{x^2+1}} && (23)\\
y &= \sin^{-1}\left(\frac{C}{\sqrt{x^2+1}}\right). && (24)
\end{aligned}
$$

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(b) $3x^2\tan(y)+1+(x^3\sec^2(y)-1)\frac{dy}{dx}=0$

Solution.

$$3x^2\tan(y)+1+\left(x^3\sec^2(y)-1\right)y'=0 \tag{25}$$

Using the technique of exact differential equations

$$
\begin{aligned}
M(x,y) &= 3x^2\tan(y)+1 && (26)\\
N(x,y) &= x^3\sec^2(y)-1 && (27)\\
\frac{\partial M}{\partial y} &= 3x^2\sec^2(y) && (28)\\
\frac{\partial N}{\partial x} &= 3x^2\sec^2(y) && (29)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, the given DE is exact.

$$
\begin{aligned}
f(x,y) &= \int M(x,y)dx+g(y) && (30)\\
&= \int (3x^2\tan(y)+1)dx+g(y) && (31)\\
&= x^3\tan(y)+x+g(y) && (32)\\
\frac{\partial f}{\partial y} &= x^3\sec^2(y)+g'(y) && (33)
\end{aligned}
$$

Comparing with $N(x,y)$, we have

$$
\begin{aligned}
g'(y) &= -1 && (34)\\
g(y) &= -y+C && (35)
\end{aligned}
$$

Then

$$f(x,y) = x^3\tan(y) + x - y + C \tag{36}$$

Since $f(x,y) = 0$, we have

$$C = x^3\tan(y) + x - y. \tag{37}$$

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(c) $\left(1 + e^{\frac{x}{y}}\right)dx + 2e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right)dy = 0$

Solution. No elementary method can be used, refer to WolframAlpha.

$$\int_1^{\frac{y}{x}} \frac{e^{\frac{1}{\xi}}(\xi - 1)}{\xi\left(e^{\frac{1}{\xi}}(2\xi - 1) + 1\right)}\,d\xi = c_1 - \frac{\log(x)}{2}.$$

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(d) $\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1$

Solution.

$$\frac{dy}{dx} + x(x+y) = x^3(x+y)^3 - 1 \tag{38}$$

Using substitution

$$u = x + y \tag{39}$$
$$\frac{du}{dx} = 1 + \frac{dy}{dx} \tag{40}$$
$$\frac{dy}{dx} = \frac{du}{dx} - 1 \tag{41}$$

We have

$$\frac{du}{dx} - 1 + xu = x^3u^3 - 1 \tag{42}$$
$$\frac{du}{dx} + xu = x^3u^3 \tag{43}$$
$$u' + xu = x^3u^3 \tag{44}$$

Using the method of substitution in Bernoulli's equation

$$v = u^{-2} \tag{45}$$
$$v' = -2u^{-3}u' \tag{46}$$

Divide the DE by u^3

$$\frac{u'}{u^3} + \frac{xu}{u^3} = \frac{x^3u}{u^3} \tag{47}$$
$$u'u^{-3} + xu^{-2} = x^3 \tag{48}$$
$$-\frac{1}{2}v' + xv = x^3 \tag{49}$$
$$v' - 2xv = -2x^3 \tag{50}$$

Using the method of integrating factor

$$\mu(x) = e^{\int -2x dx} = e^{-x^2} \tag{51}$$

We have

$$\frac{d}{dx}\left(e^{-x^2}v\right) = -2x^3e^{-x^2} \tag{52}$$

$$e^{-x^2}v = \int -2x^3e^{-x^2}dx \tag{53}$$

Using substitution

$$w = -x^2 \tag{54}$$

$$dw = -2xdx \tag{55}$$

$$\frac{dw}{-2x} = dx \tag{56}$$

We have

$$e^{-x^2}v = \int -2x^3e^w\frac{dw}{-2x} \tag{57}$$

$$= \int x^2e^w dw \tag{58}$$

$$= -\int we^w dw \tag{59}$$

Using integration by parts

$$u = w, dv = e^w dw \tag{60}$$

$$du = dw, v = e^w \tag{61}$$

We have

$$e^{-x^2}v = -(we^w - e^w + C) \tag{62}$$

$$= e^w - we^w + C \tag{63}$$

$$= e^{-x^2} - (-x^2)e^{-x^2} + C \tag{64}$$

$$= e^{-x^2} + x^2e^{-x^2} + C \tag{65}$$

$$v = \frac{e^{-x^2} + x^2e^{-x^2} + C}{e^{-x^2}} \tag{66}$$

$$= 1 + x^2 + Ce^{x^2} \tag{67}$$

$$u^{-2} = 1 + x^2 + Ce^{x^2} \tag{68}$$

$$u = \pm\frac{1}{\sqrt{1 + x^2 + Ce^{x^2}}} \tag{69}$$

$$x + y = \pm\frac{1}{\sqrt{1 + x^2 + Ce^{x^2}}} \tag{70}$$

$$y = -x \pm \frac{1}{\sqrt{1 + x^2 + Ce^{x^2}}} \tag{71}$$

$$. \tag{72}$$

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(e) $y' + \frac{y}{x} = \frac{y^2}{x^2}$

Solution.

$$y' + \frac{y}{x} = \frac{y^2}{x^2} \tag{73}$$

Using the method of substitution in Bernoulli's equation

$$u = y^{-1} \tag{74}$$
$$u' = -y^{-2}y' \tag{75}$$

Divide the DE by y^2

$$\frac{y'}{y^2} + \frac{1}{xy} = \frac{1}{x^2} \tag{76}$$
$$-u' + \frac{1}{x}u = \frac{1}{x^2} \tag{77}$$
$$u' - \frac{1}{x}u = -\frac{1}{x^2} \tag{78}$$

Using the method of integrating factor

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x} \tag{79}$$

We have

$$\frac{1}{x}u' - \frac{1}{x^2}u = -\frac{1}{x^3} \tag{80}$$
$$\frac{d}{dx}\left(\frac{1}{x}u\right) = -\frac{1}{x^3} \tag{81}$$
$$\frac{1}{x}u = -\int \frac{1}{x^3}dx \tag{82}$$
$$= \frac{1}{2x^2} + C \tag{83}$$
$$u = \frac{1}{2x} + Cx \tag{84}$$
$$= \frac{1 + 2Cx^2}{2x} \tag{85}$$
$$= \frac{1 + Cx^2}{2x} \tag{86}$$
$$y^{-1} = \frac{1 + Cx^2}{2x} \tag{87}$$
$$y = \frac{2x}{1 + Cx^2}. \tag{88}$$

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(f) $\frac{dy}{dx} = \frac{2x+4y+1}{x+2y+3}$

Solution. Using substitution

$$u = x + 2y \tag{89}$$
$$\frac{du}{dx} = 1 + 2\frac{dy}{dx} \tag{90}$$
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 1\right) \tag{91}$$
$$\tag{92}$$

We have

$$\frac{1}{2}\left(\frac{du}{dx}-1\right)=\frac{2u+1}{u+3} \tag{93}$$
$$\frac{1}{2}\left(u'-1\right)=\frac{2u+1}{u+3} \tag{94}$$
$$(u'-1)(u+3)=4u+2 \tag{95}$$
$$u'u+3u'-u-3=4u+2 \tag{96}$$
$$u'u+3u'=4u+2+u+3 \tag{97}$$
$$u'u+3u'=5u+5 \tag{98}$$
$$u'(u+3)=5u+5 \tag{99}$$
$$u'=\frac{5u+5}{u+3} \tag{100}$$
$$\frac{du}{dx}=\frac{5u+5}{u+3} \tag{101}$$
$$\int\frac{u+3}{5u+5}du=\int dx \tag{102}$$
$$\frac{1}{5}\int\frac{u+3}{u+1}du=\int dx \tag{103}$$
$$\tag{104}$$

$$\frac{1}{5}\left(\int\frac{u}{u+1}du+\int\frac{3}{u+1}du\right)=x+C \tag{105}$$
$$2\ln|u+1|+u=5x+C \tag{106}$$
$$2\ln|x+2y+1|+x+2y=5x+C \tag{107}$$
$$2\ln|x+2y+1|+2y=4x+C \tag{108}$$
$$\ln|x+2y+1|+y=2x+C. \tag{109}$$

A separable form can be obtained by using Lambert W function

$$y=W\left(-e^{\frac{5x}{2}+C-1}\right)+\frac{1}{2}(-x-1),$$

where $W(xe^x)=x$. There is also trivial solution

$$y=-\frac{x+1}{2}.$$

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(g) $y'+y+x+x^2+x^3=0$

Solution.

$$y'+y=-x-x^2-x^3 \tag{110}$$
$$y'+y=-(x+x^2+x^3) \tag{111}$$

Using the method of integrating factor

$$\mu(x)=e^{\int 1dx}=e^x \tag{112}$$

We have

$$e^x y' + e^x y = -e^x(x + x^2 + x^3) \tag{113}$$
$$\frac{d}{dx}(e^x y) = -e^x(x + x^2 + x^3) \tag{114}$$
$$e^x y = \int -e^x(x + x^2 + x^3)dx \tag{115}$$
$$e^x y = -\int e^x(x + x^2 + x^3)dx \tag{116}$$
$$e^x y = -\left(\int xe^x dx + \int x^2 e^x dx + \int x^3 e^x dx\right) \tag{117}$$

Using integration by parts

$$u = x^n, dv = e^x dx \tag{118}$$
$$du = nx^{n-1}dx, v = e^x \tag{119}$$

We have

$$\begin{aligned} e^x y = -\big(&xe^x - e^x + x^2e^x - 2x^x + 2e^x \\ &+x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C\big) \end{aligned} \tag{120}$$
$$\begin{aligned} y = -e^{-x}\big(&xe^x - e^x + x^2e^x - 2x^x + 2e^x \\ &+x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C\big) \end{aligned} \tag{121}$$
$$y = -x^3 + 2x^2 - 5x + 5 + Ce^{-x}. \tag{122}$$

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2. Solve the following IVPs:

(a) $(ye^{xy} + 4y^3) + (xe^{xy} + 12xy^2 - 2y)\, y' = 0 \quad y(0) = 2$

Solution. Using the method of exact differential equations

$$M(x,y) = ye^{xy} + 4y^3 \tag{123}$$
$$N(x,y) = xe^{xy} + 12xy^2 - 2y \tag{124}$$
$$\frac{\partial M}{\partial y} = e^{xy} + xye^{xy} + 12y^2 \tag{125}$$
$$\frac{\partial N}{\partial x} = e^{xy} + xye^{xy} + 12y^2 \tag{126}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given DE is exact.

$$f(x,y) = \int M(x,y)dx + g(y) \tag{127}$$
$$= \int (ye^{xy} + 4y^3)dx + g(y) \tag{128}$$
$$= e^{xy} + 4xy^3 + g(y) \tag{129}$$
$$\frac{\partial f}{\partial y} = xe^{xy} + 12xy^2 + g'(y) \tag{130}$$

Comparing with $N(x, y)$, we have

$$g'(y) = -2y \tag{131}$$
$$g(y) = -y^2 + C \tag{132}$$

Then

$$f(x, y) = e^{xy} + 4xy^3 - y^2 + C \tag{133}$$

Since $f(x, y) = 0$, we have

$$C = e^{xy} + 4xy^3 - y^2 \tag{134}$$

Using the initial condition

$$C = e^{0 \cdot 2} + 4 \cdot 0 \cdot 2^3 - 2^2 \tag{135}$$
$$= 1 - 4 = -3 \tag{136}$$

Then

$$0 = e^{xy} + 4xy^3 - y^2 + 3. \tag{137}$$

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(b) $(3xy + 3y - 4)dx + (1 + x)^2 dy = 0 \quad y(0) = 1$

Solution. Using the method of non-exact differential equations integrating factor

$$M(x, y) = 3xy + 3y - 4 \tag{138}$$
$$N(x, y) = (1 + x)^2 \tag{139}$$
$$\frac{\partial M}{\partial y} = 3x + 3 \tag{140}$$
$$\frac{\partial N}{\partial x} = 2(1 + x) \tag{141}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, the given DE is non-exact.

$$\mu(x) = e^{\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} dx} \tag{142}$$
$$= e^{\int \frac{3x+3-2(1+x)}{(1+x)^2} dx} \tag{143}$$
$$= e^{\int \frac{x+1}{(1+x)^2} dx} \tag{144}$$
$$= e^{\int \frac{1}{1+x} dx} \tag{145}$$
$$= e^{\ln|1+x|} \tag{146}$$
$$= 1 + x \tag{147}$$

We have

$$(3xy + 3y - 4)(1 + x)dx + (1 + x)^3 dy = 0 \tag{148}$$
$$3xy + 3y - 4 + 3x^2y + 3xy - 4x + (1 + x)^3 \frac{dy}{dx} = 0 \tag{149}$$
$$6xy + 3y - 4 + 3x^2y - 4x + (1 + x)^3 y' = 0 \tag{150}$$

Using the method of exact differential equations

$$
\begin{aligned}
M(x,y) &= 6xy + 3y - 4 + 3x^2y - 4x && (151)\\
N(x,y) &= (1+x)^3 && (152)\\
\frac{\partial M}{\partial y} &= 6x + 3 + 3x^2 && (153)\\
\frac{\partial N}{\partial x} &= 3(1+x)^2 && (154)\\
&= 3 + 6x + 3x^2 && (155)
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given DE is exact.

$$
\begin{aligned}
f(x,y) &= \int M(x,y)dx + g(y) && (156)\\
&= \int (6xy + 3y - 4 + 3x^2y - 4x)dx + g(y) && (157)\\
&= 3x^2y + 3xy - 4x + x^3y - 2x^2 + g(y) && (158)\\
\frac{\partial f}{\partial y} &= 3x^2 + 3x + x^3 + g'(y) && (159)
\end{aligned}
$$

Comparing with the first $N(x,y)$, we have

$$
\begin{aligned}
g'(y) &= -x^3 - 2x^2 - x + 1 && (160)\\
g(y) &= -x^3y - 2x^2y - xy + y + C && (161)\\
& && (162)
\end{aligned}
$$

Then

$$
\begin{aligned}
f(x,y) &= 3x^2y + 3xy - 4x + x^3y - 2x^2 - x^3y - 2x^2y - xy + y + C && (163)\\
&= x^2y - 2x^2 + 2xy - 4x + y + C && (164)
\end{aligned}
$$

Since $f(x,y) = 0$, we have

$$
\begin{aligned}
0 &= x^2y - 2x^2 + 2xy - 4x + y + C && (165)\\
C &= x^2y - 2x^2 + 2xy - 4x + y && (166)
\end{aligned}
$$

Using the initial condition

$$
\begin{aligned}
C &= 0 - 0 + 2\cdot 0 - 4\cdot 0 + 1 && (167)\\
&= 1 && (168)
\end{aligned}
$$

Then

$$
1 = x^2y - 2x^2 + 2xy - 4x + y. \quad (169)
$$

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(c) $\frac{dy}{dx} + y\ln(y) = ye^{-3x} \quad y(1) = 1$

Solution. Using substitution

$$u = \ln(y) \tag{170}$$
$$e^u = y \tag{171}$$
$$\frac{du}{dx} = \frac{1}{y}\frac{dy}{dx} \tag{172}$$
$$\frac{dy}{dx} = y\frac{du}{dx} \tag{173}$$
$$= e^u\frac{du}{dx} \tag{174}$$

We have

$$e^u\frac{du}{dx} + e^u u = e^u e^{-3x} \tag{175}$$

Divide the DE by e^u

$$\frac{du}{dx} + u = e^{-3x} \tag{176}$$

Using the method of integrating factor

$$\mu(x) = e^{\int 1dx} = e^x \tag{177}$$

We have

$$\frac{d}{dx}\left(e^x u\right) = e^x e^{-3x} \tag{178}$$
$$e^x u = \int e^{-2x}dx \tag{179}$$
$$u = e^{-x}\left(-\frac{e^{-2x}}{2} + C\right) \tag{180}$$
$$= Ce^{-x} - \frac{1}{2}e^{-3x} \tag{181}$$
$$\ln y = Ce^{-x} - \frac{1}{2}e^{-3x} \tag{182}$$
$$y = \exp\left(Ce^{-x} - \frac{1}{2}e^{-3x}\right) \tag{183}$$

Using the initial condition

$$0 = Ce^{-1} - \frac{1}{2}e^{-3} \tag{184}$$
$$\frac{1}{2}e^{-3} = Ce^{-1} \tag{185}$$
$$\frac{1}{2}e^{-2} = C \tag{186}$$

Then

$$y = \exp\left(\frac{1}{2}e^{-2-x} - \frac{1}{2}e^{-3x}\right). \tag{187}$$

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(d) $y'\left(x^2y^3+xy\right)=1 \quad y(2)=3$

Solution.

$$\frac{dy}{dx}=\frac{1}{x^2y^3+xy} \tag{188}$$

$$\frac{dx}{dy}=x^2y^3+xy \tag{189}$$

$$x'=x^2y^3+xy \tag{190}$$

Using the method of substitution in Bernoulli's equation

$$u=x^{-1} \tag{191}$$

$$u'=-x^{-2}x' \tag{192}$$

Divide the DE by x^2

$$x'x^{-2}=y^3+x^{-1}y \tag{193}$$

$$-u'=y^3+uy \tag{194}$$

$$u'=-y^3-uy \tag{195}$$

Using the method of integrating factor

$$\mu(y)=e^{\int -ydy} \tag{196}$$

$$=e^{-\frac{y^2}{2}} \tag{197}$$

We have

$$\frac{d}{dy}\left(e^{-\frac{y^2}{2}}u\right)=-e^{-\frac{y^2}{2}}y^3 \tag{198}$$

$$e^{-\frac{y^2}{2}}u=\int -e^{-\frac{y^2}{2}}y^3dy \tag{199}$$

$$u=-e^{\frac{y^2}{2}}\int e^{-\frac{y^2}{2}}y^3dy \tag{200}$$

Using substitution

$$v=\frac{y^2}{2} \tag{201}$$

$$dv=ydy \tag{202}$$

$$dy=\frac{dv}{y} \tag{203}$$

We have

$$u=-e^v\int e^{-v}y^3\frac{dv}{y} \tag{204}$$

$$=-e^v\int e^{-v}y^2dv \tag{205}$$

$$=-e^v\int e^{-v}2vdv \tag{206}$$

$$=-2e^v\int ve^{-v}dv \tag{207}$$

Using integration by parts

$$u = -2e^{v}\left(-ve^{-v} + e^{-v} + C\right) \tag{208}$$
$$= 2v - 2 + Ce^{v} \tag{209}$$
$$= y^2 - 2 + Ce^{\frac{y^2}{2}} \tag{210}$$
$$x^{-1} = y^2 - 2 + Ce^{\frac{y^2}{2}} \tag{211}$$
$$x = \frac{1}{y^2 - 2 + Ce^{\frac{y^2}{2}}} \tag{212}$$
$$2 = \frac{1}{3^2 - 2 + Ce^{\frac{3^2}{2}}} \tag{213}$$
$$\frac{1}{2} = 9 - 2 + Ce^{\frac{9}{2}} \tag{214}$$
$$Ce^{\frac{9}{2}} = \frac{1}{2} - 7 \tag{215}$$
$$C = -6.5e^{-\frac{9}{2}} \tag{216}$$
$$x = \frac{1}{y^2 - 2 - 6.5e^{\frac{y^2+9}{2}}}. \tag{217}$$

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(e) $x^2y' = 4x^2 + 7xy + 2y^2 \quad y(1) = 1$

Solution.

$$y' = 4 + 7\frac{y}{x} + 2\frac{y^2}{x^2} \tag{218}$$

Using substitution

$$u = \frac{y}{x} \tag{219}$$
$$y = ux \tag{220}$$
$$y' = u + xu' \tag{221}$$

We have

$$u + xu' = 4 + 7u + 2u^2 \tag{222}$$
$$xu' = 4 + 6u + 2u^2 \tag{223}$$
$$. \tag{224}$$

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(f) $(3x^2y + 2xy + y^3)\,dx + (x^2 + y^2)\,dy = 0 \quad y(0) = 0$