Student Name: SalahDin Rezk
Student ID: 202201079
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MATH 201 - Linear Algebra and Vector Geometry
Assignment 1

Question (1)

1. Row reduce the matrix to echelon form. Identify the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$2R_1 - R_2 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 5 & 6 & 7 & 8 \end{array}\right] \tag{1}$$

$$-5R_1 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -4 & -8 & -12 \end{array}\right] \tag{2}$$

$$4R_2 + R_3 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{3}$$

$$\text{Pivot positions:} \quad \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{4}$$

$$\text{Pivot columns:} \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{5}$$

2. Suppose the system below is consistent for all possible values of f and g. What can you say about the coefficients c and d? Justify your answer.

$$x_1 + 25x_2 = f$$
$$cx_1 + dx_2 = g.$$

$$\left[\begin{array}{cc|c} 1 & 25 & f \\ c & d & g \end{array}\right] \tag{6}$$

$$R_2 - cR_1 \to R_2 \quad \left[\begin{array}{cc|c} 1 & 25 & f \\ 0 & d - c25 & g - cf \end{array}\right]. \tag{7}$$

$$d - c25 \neq 0. \tag{8}$$

If $d - c25 = 0$, then $g - cf = 0$, which means that the system is inconsistent, since infinitely many values of g and f does not satisfy the equality. □

3. Apply the elementary row operations to reduce the augmented matrix of the following systems to the reduced row echelon form. Hence, find its solution, general solution form, or state that it has no solution whenever appropriate,

(a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix}$

$$\frac{R_2 - 4R_1}{-3} \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 6 & 7 & 8 & 9 \end{array}\right] \tag{9}$$

$$\frac{R_3 - 6R_1}{-5} \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right] \tag{10}$$

$$R_3 - R_2 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{11}$$

$$x + 2y + 3z = 4 \tag{12}$$
$$y + 2z = 3. \tag{13}$$

$$x = 4 - 2y - 3z \tag{14}$$
$$y = 3 - 2z. \tag{15}$$

$$x = 4 - 2(3 - 2z) - 3z \tag{16}$$
$$= 4 - 6 + 4z - 3z \tag{17}$$
$$= z - 2. \tag{18}$$

$$s.s. = \left\{ \begin{bmatrix} z - 2 \\ 3 - 2z \\ z \end{bmatrix} \right\} \tag{19}$$
$$= \left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}. \tag{20}$$

(b) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{bmatrix}$

$$R_2 - 3R_1 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{array}\right] \tag{21}$$

$$R_3 - 5R_1 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{array}\right] \tag{22}$$

$$\frac{R_3 - 2R_2}{-4} \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & -5 \end{array}\right]. \tag{23}$$

$$x + 3y + 5z = 7 \tag{24}$$
$$-4y - 8z = -12 \tag{25}$$
$$0 = -5. \tag{26}$$

$$s.s. = \emptyset. \tag{27}$$

Question (2)

Determine if b is a linear combination of the vectors formed from the columns of the matrix A.

$$A = \begin{bmatrix} 1 & -6 & 4 \\ 0 & 6 & 7 \\ -3 & 18 & -12 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ -7 \\ -2 \end{bmatrix}$$

$$Ax = b \tag{28}$$
$$\begin{bmatrix} 1 & -6 & 4 \\ 0 & 6 & 7 \\ -3 & 18 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -2 \end{bmatrix} \tag{29}$$
$$\begin{bmatrix} x - 6y + 4z \\ 6y + 7z \\ -3x + 18y - 12z \end{bmatrix} = \begin{bmatrix} 2 \\ -7 \\ -2 \end{bmatrix}. \tag{30}$$

$$\left[\begin{array}{ccc|c} 1 & -6 & 4 & 2 \\ 0 & 6 & 7 & -7 \\ -3 & 18 & -12 & -2 \end{array}\right] \tag{31}$$

$$R_3 + 3R_1 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & -6 & 4 & 2 \\ 0 & 6 & 7 & -7 \\ 0 & 0 & 0 & -4 \end{array}\right]. \tag{32}$$

$$x - 6y + 4z = 2 \tag{33}$$
$$6y + 7z = -7 \tag{34}$$
$$0 = -4. \tag{35}$$

$$s.s. = \emptyset. \tag{36}$$

Vector b is not a linear combination of the vectors formed from the columns of the matrix A. $\square$

Question (3)

1. Let $v_1 = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -6 \\ 6 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 6 \\ -5 \\ 9 \end{bmatrix}$, does $\{v_1, v_2, v_3\}$ span $\mathbb{R}^3$? Why?

$$\left[\begin{array}{ccc|c} 0 & 0 & 6 & x \\ 0 & -6 & -5 & y \\ -3 & 6 & 9 & z \end{array}\right]. \tag{37}$$

$$R_3 \leftrightarrow R_1 \quad \left[\begin{array}{ccc|c} -3 & 6 & 9 & z \\ 0 & -6 & -5 & y \\ 0 & 0 & 6 & x \end{array}\right] \tag{38}$$
$$\frac{R_1}{-3} \to R_1 \quad \left[\begin{array}{ccc|c} 1 & -2 & -3 & -z \\ 0 & -6 & -5 & y \\ 0 & 0 & 6 & x \end{array}\right] \tag{39}$$
$$\frac{R_2}{-6} \to R_2 \quad \left[\begin{array}{ccc|c} 1 & -2 & -3 & -z \\ 0 & 1 & \frac{5}{6} & -\frac{y}{6} \\ 0 & 0 & 6 & x \end{array}\right] \tag{40}$$
$$R_1 + 2R_2 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & -\frac{z}{3} \\ 0 & 1 & \frac{5}{6} & -\frac{y}{6} \\ 0 & 0 & 6 & x \end{array}\right] \tag{41}$$
$$\frac{R_3}{6} \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & -\frac{z}{3} \\ 0 & 1 & \frac{5}{6} & -\frac{y}{6} \\ 0 & 0 & 1 & \frac{x}{6} \end{array}\right] \tag{42}$$
$$R_1 + \frac{1}{3}R_3 \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{z}{3} - \frac{x}{18} \\ 0 & 1 & \frac{5}{6} & -\frac{y}{6} \\ 0 & 0 & 1 & \frac{x}{6} \end{array}\right] \tag{43}$$
$$R_2 - \frac{5}{6}R_3 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{z}{3} - \frac{x}{18} \\ 0 & 1 & 0 & -\frac{y}{6} - \frac{5x}{36} \\ 0 & 0 & 1 & \frac{x}{6} \end{array}\right]. \tag{44}$$

$\{v_1, v_2, v_3\}$ spans $\mathbb{R}^3$ because the system of equations has a solution for all $x, y, z \in \mathbb{R}$. $\square$

2. For which value(s) of h, k does $\begin{bmatrix} h \\ k \end{bmatrix}$ lie in Span $\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$?

$$\left[\begin{array}{cc|c} 2 & 2 & h \\ -1 & 1 & k \end{array}\right]. \tag{45}$$

$$\begin{aligned}
R_1 + R_2 \to R_1 \quad & \left[\begin{array}{cc|c} 1 & 3 & h+k \\ -1 & 1 & k \end{array}\right] && (46) \\
R_2 + R_1 \to R_2 \quad & \left[\begin{array}{cc|c} 1 & 3 & h+k \\ 0 & 4 & 2h+2k \end{array}\right] && (47) \\
\frac{R_2}{4} \to R_2 \quad & \left[\begin{array}{cc|c} 1 & 3 & h+k \\ 0 & 1 & \frac{h+k}{2} \end{array}\right] && (48) \\
R_1 - 3R_2 \to R_1 \quad & \left[\begin{array}{cc|c} 1 & 0 & \frac{h-k}{2} \\ 0 & 1 & \frac{h+k}{2} \end{array}\right]. && (49)
\end{aligned}$$

$$s.s. = \mathbb{R}^2. \tag{50}$$

Since vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ spans $\mathbb{R}^2$, then any value for h and k lie in their span. □

3. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$. Denote the columns of A by a_1, a_2, a_3 and let $W = \text{Span}\{a_1, a_2, a_3\}$.

   (a) Is b in $\{a_1, a_2, a_3\}$?
   Vector b is not in the set $\{a_1, a_2, a_3\}$.

   (b) Is b in W ?

$$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right]. \tag{51}$$

$$\begin{aligned}
R_1 \leftrightarrow R_3 \quad & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{array}\right] && (52) \\
R_2 + R_1 \to R_2 \quad & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 6 & 6 & 6 \\ 2 & 0 & 6 & 10 \end{array}\right] && (53) \\
R_3 - 2R_1 \to R_3 \quad & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 6 & 6 & 6 \\ 0 & 4 & 4 & 4 \end{array}\right] && (54) \\
R_2 \leftrightarrow R_3 \quad & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 6 & 6 & 6 \end{array}\right] && (55) \\
R_3 - \frac{3}{2}R_2 \to R_3 \quad & \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]. && (56)
\end{aligned}$$

Since the system is consistent, then b is in W. □

Question (4)

Describe the solutions of the first system of equations below in parametric vector form. Provide a geometric comparison with the solution set of the second system of equations below.

$$\begin{aligned} 4x_1 + 4x_2 + 8x_3 &= 16 \\ -12x_1 - 12x_2 - 24x_3 &= -48 \\ -7x_2 - 21x_3 &= 14 \end{aligned} \qquad \begin{aligned} 4x_1 + 4x_2 + 8x_3 &= 0 \\ -12x_1 - 12x_2 - 24x_3 &= 0 \\ -7x_2 - 21x_3 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 8 & 16 \\ -12 & -12 & -24 & -48 \\ 0 & -7 & -21 & 14 \end{array}\right]. \tag{57}$$

$$\frac{R_1}{4} \to R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ -12 & -12 & -24 & -48 \\ 0 & -7 & -21 & 14 \end{array}\right] \tag{58}$$

$$R_2 + 12R_1 \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -7 & -21 & 14 \end{array}\right] \tag{59}$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -7 & -21 & 14 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{60}$$

$$\frac{R_2}{-7} \to R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]. \tag{61}$$

$$x + y + 2z = 4 \tag{62}$$

$$y + 3z = -2 \tag{63}$$

$$x = 4 - y - 2z \tag{64}$$

$$y = -2 - 3z. \tag{65}$$

$$x = 4 - (-2 - 3z) - 2z \tag{66}$$

$$x = 4 + 2 + 3z - 2z \tag{67}$$

$$x = 6 + z. \tag{68}$$

$$s.s._1 = \left\{ \begin{bmatrix} 6 \\ -2 \\ 0 \end{bmatrix} + z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\} \tag{69}$$

$$s.s_2 = \left\{ z \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \right\}. \tag{70}$$

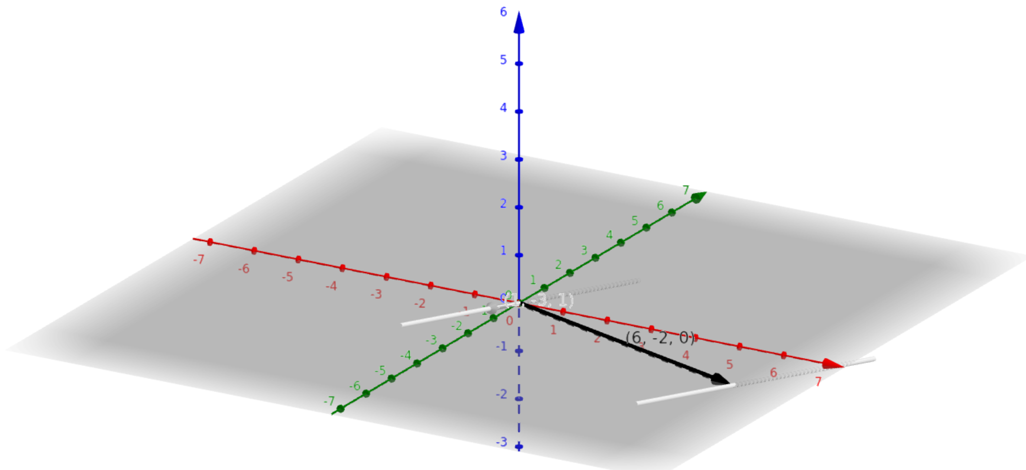

Figure 1: Vector equations results in two parallel lines as there is only one degree of freedom.