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Linear Algebra (MATH 201)

Assignment 5

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1. Let P_2 denotes the vector space of all polynomials of degree at most 2. Consider the following inner product on P_2,

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx..$$

(a) Find an orthogonal basis for P_2.

Solution. Consider the basis $\{1, x, x^2\}$:

$$u_1 = 1 \quad u_2 = x \quad u_3 = x^2. \tag{1}$$

$$\begin{aligned}
v_1 &= u_1 = 1 \qquad (2)\\
v_2 &= u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle}v_1 = x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} 1 dx} = x - 0 = x \qquad (3)\\
v_3 &= u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle}v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle}v_2 \qquad (4)\\
&= x^2 - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} 1 dx} - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1} x^2 dx}x = x^2 - \frac{1}{3} - 0 = x^2 - \frac{1}{3}. \qquad (5)
\end{aligned}$$

$$\{v_1, v_2, v_3\} = \{1, x, x^2 - \frac{1}{3}\} \tag{6}$$

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(b) Find the polynomial $\hat{p}(t) \in P_2$ which best approximates the function $f(t) = t^3$ on $[-1, 1]$.

Solution.

$$. \tag{7}$$

$$\hat{p}(t) = \frac{3}{5}t \tag{8}$$

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2. The total revenue (in millions of dollars) of a certain company from 2015 to 2018 are shown below,

x (Year 20-)	15	16	17	18
y (Revenue)	74	78	87	94

(a) Find the least squares regression line that best fits the data (Hint: use 2 digits only for the year variable x, i.e. 15, 16, ...).

Solution.

$$y = \begin{bmatrix} 74 \\ 78 \\ 87 \\ 94 \end{bmatrix} \quad X = \begin{bmatrix} 1 & 15 \\ 1 & 16 \\ 1 & 17 \\ 1 & 18 \end{bmatrix} \quad \beta = \begin{bmatrix} b \\ m \end{bmatrix}. \tag{9}$$

$$\begin{aligned} y &= X\beta && (10)\\ \beta &= X^{+}y && (11)\\ &= (X^T X)^{-1} X^T y. && (12) \end{aligned}$$

$$\begin{aligned} X^T X &= \begin{bmatrix} 4 & 66 \\ 66 & 1094 \end{bmatrix} && (13)\\ (X^T X)^{-1} &= \begin{bmatrix} 54.7 & -3.3 \\ -3.3 & 0.2 \end{bmatrix} && (14)\\ (X^T X)^{-1} X^T &= \begin{bmatrix} 5.2 & 1.9 & -1.4 & -4.7 \\ -0.3 & -0.1 & 0.1 & 0.3 \end{bmatrix} && (15)\\ (X^T X)^{-1} X^T y &= \begin{bmatrix} -30.6 \\ 6.9 \end{bmatrix}. && (16) \end{aligned}$$

$$b = -30.6 \quad m = 6.9 \implies y = 6.9x - 30.6. \tag{17}$$

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(b) Use the above model to predict the total revenue in 2019.

Solution.

$$y = 6.9(19) - 30.6 = 100.5. \tag{18}$$

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3.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \quad b = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Prove that the least squares solution for $Ax = b$ is given by the normal equations: $A^T Ax = A^T b$.

Solution.

$$A^T Ax = A^T b \tag{19}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{21}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2 \end{bmatrix} \tag{22}$$

$$x = \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}. \tag{23}$$

$$Ax = b \tag{24}$$

$$\tilde{x} = A^+ b \tag{25}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}^+ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{26}$$

$$= \begin{bmatrix} 0.5 & -1 & 0.5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \tag{27}$$

$$= \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix} \tag{28}$$

From (23) and (28) we can see that the least squares solution for $Ax = b$ is given by the normal equations: $A^T Ax = A^T b$ for the given cases of A and b. ■

(b) Find the least squares solution for $Ax = b$.

Solution. From the previous part:

$$x = \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}. \tag{29}$$

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(c) Find the closest vector to b in Col$A$.

Solution. By definition, the closest vector to b in Col$A$ is the least squares solution for $Ax = b$. From the previous part:

$$x = \begin{bmatrix} -\frac{2}{3} \\ \frac{4}{3} \end{bmatrix}. \tag{30}$$

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4. (a) Consider the inner product,

$$\langle p, q\rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + \ldots + p(t_n)q(t_n).$$

defined over P_n (the vector space of all polynomials of degree at most n). Let $\{1, t, t^2\}$ be the standard basis of P_2 and let $t_0 = -1, t_1 = 0, t_2 = 1$, find an orthonormal basis for P_2.

Solution.

$$u_1 = 1 \quad u_2 = t \quad u_3 = t^2. \tag{31}$$

$$v_1 = u_1 = 1 \tag{32}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = t - \frac{t_0^2 + t_1^2 + t_2^2}{t_0^2 + t_1^2 + t_2^2} = t - 0 = t \tag{33}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 \tag{34}$$

$$= t^2 - \frac{t_0^3 + t_1^3 + t_2^3}{t_0^2 + t_1^2 + t_2^2} - \frac{t_0^4 + t_1^4 + t_2^4}{t_0^4 + t_1^4 + t_2^4} t = t^2 - 0 - 0 = t^2. \tag{35}$$

$$\{v_1, v_2, v_3\} = \{1, t, t^2\} \tag{36}$$

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(b)