Name: SalahDin Ahmed Salh Rezk
ID: 202201079
Date: December 11, 2023

Linear Algebra (MATH 201)

Assignment 3

---

1. Question 1

   (a) Let Q be a reflection transformation about the line $Y = mX$, such that $m = \tan\theta$. Find Q, and show that Q is a matrix transformation such that $Q\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$, where $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ is the homogeneous coordinates of the point $\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix}$ is the homogeneous coordinates of the reflected vector $\begin{pmatrix} x' \\ y' \end{pmatrix}$.

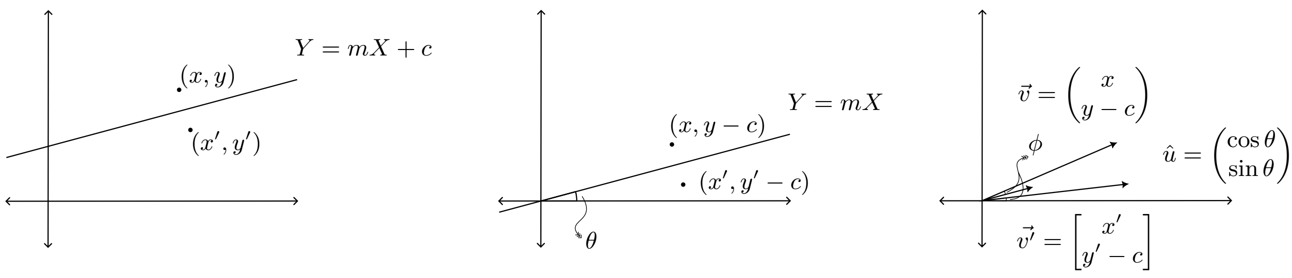

Figure 1

Solution.

1) Move the system to the origin.
2) Rotate the system by $-\theta$.
3) Reflect the system about the X-axis.
4) Reverse the rotation and the translation.

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$Q_2 = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta = \tan^{-1} m \tag{2}$$

$$Q_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

$$Q_4 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$Q_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

$$\begin{aligned}
Q &= Q_5Q_4Q_3Q_2Q_1 && (6)\\
&= Q_5Q_4Q_3 \begin{bmatrix} \cos(\theta) & \sin(\theta) & -c\sin(\theta) \\ -\sin(\theta) & \cos(\theta) & -c\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix} && (7)\\
&= Q_5Q_4 \begin{bmatrix} \cos(\theta) & \sin(\theta) & -c\sin(\theta) \\ \sin(\theta) & -\cos(\theta) & c\cos(\theta) \\ 0 & 0 & 1 \end{bmatrix} && (8)\\
&= Q_5 \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\cos(\theta)\sin(\theta) & -2c\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & -\cos^2(\theta) + \sin^2(\theta) & -c(-\cos^2(\theta) + \sin^2(\theta)) \\ 0 & 0 & 1 \end{bmatrix} && (9)\\
&= \begin{bmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\cos(\theta)\sin(\theta) & -2c\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & -\cos^2(\theta) + \sin^2(\theta) & c - c(-\cos^2(\theta) + \sin^2(\theta)) \\ 0 & 0 & 1 \end{bmatrix} && (10)\\
&= \begin{bmatrix} \cos 2\theta & \sin 2\theta & -c\sin 2\theta \\ \sin 2\theta & -\cos(2x) & 2c \\ 0 & 0 & 1 \end{bmatrix} && (11)\\
&\text{Substitute } \theta = \tan^{-1} m \\
&= \begin{bmatrix} \cos(2\arctan m) & \sin(2\arctan m) & -c\sin(2\arctan m) \\ \sin(2\arctan m) & -\cos(2\arctan m) & 2c \\ 0 & 0 & 1 \end{bmatrix} && (12)\\
&\text{Using } \cos(2\arctan m) = \frac{1-m^2}{1+m^2} \text{ and } \sin(2\arctan m) = \frac{2m}{1+m^2}
\end{aligned}$$

$$Q = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} & -c\frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & -\frac{1-m^2}{1+m^2} & 2c \\ 0 & 0 & 1 \end{bmatrix}. \tag{13}$$

■

(b) Reflect the given triangle with (2,6), (3,6), and (2.5,8) about the line $Y = \frac{1}{2}x + 3$.

Solution.

$$m = \frac{1}{2} \implies \theta = \tan^{-1}\frac{1}{2} \quad c = 3. \tag{14}$$

$$P_1 = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} 2.5 \\ 8 \\ 1 \end{pmatrix}. \tag{15}$$

$$Q = \begin{bmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} & -c\frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & -\frac{2m}{1+m^2} & 2c \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & -\frac{12}{5} \\ \frac{4}{5} & -\frac{4}{5} & 6 \\ 0 & 0 & 1 \end{bmatrix}. \tag{17}$$

$$P_1' = QP_1 \tag{18}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & -\frac{12}{5} \\ \frac{4}{5} & -\frac{4}{5} & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} \tag{19}$$

$$= \begin{pmatrix} \frac{18}{5} \\ \frac{14}{5} \\ 1 \end{pmatrix} \tag{20}$$

$$P_2' = QP_2 \tag{21}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & -\frac{12}{5} \\ \frac{4}{5} & -\frac{4}{5} & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} \tag{22}$$

$$= \begin{pmatrix} \frac{21}{5} \\ \frac{18}{5} \\ 1 \end{pmatrix} \tag{23}$$

$$P_3' = QP_3 \tag{24}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{4}{5} & -\frac{12}{5} \\ \frac{4}{5} & -\frac{4}{5} & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 2.5 \\ 8 \\ 1 \end{pmatrix} \tag{25}$$

$$= \begin{pmatrix} 5.5 \\ 1.6 \\ 1 \end{pmatrix}. \tag{26}$$

The reflected triangle coordinates are $(3.6, 2.8)$, $(4.2, 3.6)$, and $(5.5, 1.6)$. ■

2. Question 2

   (a) Find a basis for the subspace spanned by the given vectors. What is the dimension of the subspace?

$$\begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ -3 \\ 7 \end{bmatrix}$$

Solution.

$$V = \operatorname{Col}\left(\begin{bmatrix} 1 & -3 & 2 & 4 \\ -3 & 9 & -1 & 5 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \right). \tag{27}$$

$$R_2 \leftarrow R_2 + 3R_1 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 5 & 17 \\ 2 & -6 & 4 & -3 \\ -4 & 12 & 2 & 7 \end{bmatrix} \tag{28}$$

$$R_3 \leftarrow R_3 - 2R_1 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 5 & 17 \\ 0 & 0 & 0 & -11 \\ -4 & 12 & 2 & 7 \end{bmatrix} \tag{29}$$

$$R_4 \leftarrow R_4 + 4R_1 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 5 & 17 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 10 & 23 \end{bmatrix} \tag{30}$$

$$R_4 \leftarrow R_4 - 2R_3 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 5 & 17 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & -11 \end{bmatrix} \tag{31}$$

$$R_4 \leftarrow R_4 - R_3 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 5 & 17 \\ 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

$$R_3 \leftarrow R_3 / -11 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 5 & 17 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

$$R_2 \leftarrow R_2 - 17R_3 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{34}$$

$$R_2 \leftarrow R_2 / 5 \quad \begin{bmatrix} 1 & -3 & 2 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

$$R_1 \leftarrow R_1 - 4R_3 \quad \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

$$R_1 \leftarrow R_1 - 2R_2 \quad \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

$$B_V = \left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ -6 \\ 12 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 4 \\ 2 \end{bmatrix} \right\}. \tag{38}$$

■

(b) Suppose a 4×7 matrix A has 3 pivot columns. Is $\text{Col}(A) = \mathbb{R}^3$? What is the dimension of $\text{Nul}(A)$? Explain your answers.

Solution. False, $\text{Col}(A)$ does not equal $\mathbb{R}^3$, because the number of rows is 4, and thus it will span a 3-dimensional subspace in $\mathbb{R}^4$. The dimension of $\text{Nul}(A)$ is 4, because the dimension of the nullspace of a matrix is equal to the number of free variables in the reduced echelon form of the matrix, which is equal to number of columns minus number of pivots $(7-3=4)$. Since there are 4 free variables in the reduced echelon form of A, the dimension of $\text{Nul}(A)$ is 4. ■

3. Respond as comprehensively as possible and justify your answer.

(a) Suppose F is a 5×5 matrix whose column space is not equal to $\mathbb{R}^5$. What can be said about $\text{Nul}(F)$?

Solution. $\dim(\text{Nul}(F)) > 0$ (the nullspace $Fx = 0$ is nontrivial—the columns are linearly dependent). ■

(b) If B is a 7×7 matrix and $\text{Col}(B) = R^7$, what can be said about solutions of equations of the form $Bx = b$ for b in $\mathbb{R}^7$?

Solution. B has a solution for every b in $\mathbb{R}^7$ (system is consistent). ■

(c) What can be said about the shape of $m \times n$ matrix A when the columns of A form a basis for $\mathbb{R}^m$?

Solution. $m \geq n$ (number of rows is greater than or equal to the number of columns–the matrix is "tall" or "full-rank" with m linearly independent columns). ■

4. Question 4

(a) Find the vector x determined by the given coordinate vector $[x]_\beta$, and the given basis β, illustrate your answer graphically.

$$\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\}, [x]_\beta = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Solution.

$$\beta[x]_\beta = x \tag{39}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}. \tag{40}$$

■

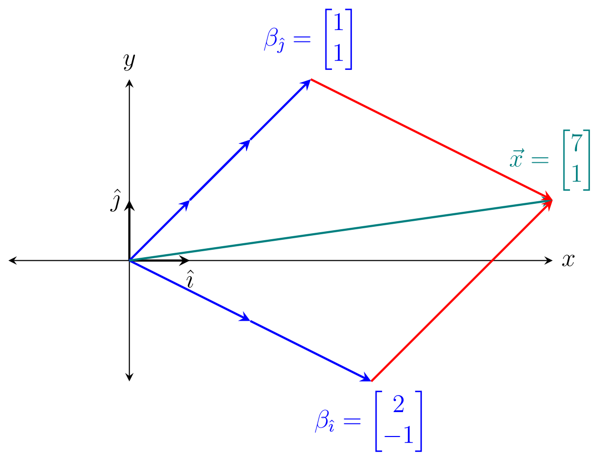

Figure 2

(b) Find the LU factorization of matrix $A = \begin{bmatrix} 4 & -8 & 8 & -4 \\ 16 & -29 & 27 & -14 \\ -1 & -10 & 18 & -4 \end{bmatrix}$.

Solution.

$$\begin{bmatrix} 4 & -8 & 8 & -4 \\ 16 & -29 & 27 & -14 \\ -1 & -10 & 18 & -4 \end{bmatrix} \tag{41}$$

$$\xrightarrow{R_2 = R_2 - 4R_1} \begin{bmatrix} 4 & -8 & 8 & -4 \\ 0 & 3 & -5 & 2 \\ -1 & -10 & 18 & -4 \end{bmatrix} \tag{42}$$

$$\xrightarrow{R_3 = R_3 + \frac{1}{4}R_1} \begin{bmatrix} 4 & -8 & 8 & -4 \\ 0 & 3 & -5 & 2 \\ 0 & -12 & 20 & -5 \end{bmatrix} \tag{43}$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \begin{bmatrix} 4 & -8 & 8 & -4 \\ 0 & 3 & -5 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}. \tag{44}$$

$$U = \begin{bmatrix} 4 & -8 & 8 & -4 \\ 0 & 3 & -5 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \tag{45}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{4} & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \tag{46}$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -\frac{1}{4} & -4 & 1 \end{bmatrix} \tag{47}$$

$$. \tag{48}$$

■

5. Given that,

$$A = \begin{pmatrix} 3 & -6 & -4 \\ -3 & 2 & 3 \\ 6 & 8 & -4 \end{pmatrix} \quad U = \begin{pmatrix} 3 & -6 & -4 \\ 0 & -4 & -1 \\ 0 & 0 & -1 \end{pmatrix} \tag{49}$$

If A is reduced to the row echelon form U using only row replacement operations,

(a) Find L such that $A = LU$.

Solution.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \tag{50}$$

$$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \tag{51}$$

■

(b) Find $|A|$.

Solution.

$$|A| = |LU| \tag{52}$$
$$= |L||U| \tag{53}$$
$$= 3 \cdot (-4) \cdot (-1) \tag{54}$$
$$= 12. \tag{55}$$

■

(c) Find a basis for Col A.

Solution.

$$\text{Col } A = \text{Col } U \tag{56}$$
$$= \text{Col} \begin{bmatrix} 3 & -6 & -4 \\ 0 & -4 & -1 \\ 0 & 0 & -1 \end{bmatrix} \tag{57}$$
$$= \text{Span} \left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ -1 \end{bmatrix} \right\}. \tag{58}$$

■

(d) Find rank A.

Solution.

$$\text{rank } A = \text{rank } U \tag{59}$$
$$= 3. \tag{60}$$

■

(e) Find Nul A.

Solution.

$$\text{Nul } A = \text{Nul } U \tag{61}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{62}$$

■

6. Question 6

   (a) Let $M_{2\times 2}$ be the vector space of all 2×2 matrices, and define

$$T : M_{2\times 2} \to M_{2\times 2} \quad \text{by} \quad T(A) = A + A^T \quad \text{, where} \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

   i. Show that T is a linear transformation.

   Solution.

$$\begin{aligned}
T(A+B) &= (A+B) + (A+B)^T && (63)\\
&= (A+B) + (A^T + B^T) && (64)\\
&= A + A^T + B + B^T && (65)\\
&= T(A) + T(B) && (66)\\
T(cA) &= cA + (cA)^T && (67)\\
&= cA + cA^T && (68)\\
&= c(A + A^T) && (69)\\
&= cT(A). && (70)
\end{aligned}$$

■

   ii. Let B be any element of $M_{2\times 2}$ such that $B^\intercal = B$. Find a matrix A in $M_{2\times 2}$ such that $T(A) = B$.

   Solution.

$$\begin{aligned}
T(A) &= B && (71)\\
A + A^T &= B && (72)\\
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} &= \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} && (73)\\
\begin{bmatrix} 2a_{11} & a_{12} + a_{21} \\ a_{12} + a_{21} & 2a_{22} \end{bmatrix} &= \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} && (74)\\
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \begin{bmatrix} \frac{b_{11}}{2} & a_{12} \\ b_{12} - a_{12} & \frac{b_{22}}{2} \end{bmatrix}. && (75)
\end{aligned}$$

   Where a_{12} is a free variable. ■

   iii. Show that the range of T is the set of B in $M_{2\times 2}$ with the property that $B^\intercal = B$.

   Solution.

$$\begin{aligned}
T(A)^\intercal &= (A + A^\intercal)^\intercal && (76)\\
&= A^\intercal + (A^\intercal)^\intercal && (77)\\
&= A^\intercal + A && (78)\\
&= A + A^\intercal && (79)\\
&= T(A). && (80)
\end{aligned}$$

$$\therefore \text{Range } T = \{B \in M_{2\times 2} \mid B^\intercal = B\} \quad (81)$$

■

iv. Describe the kernel of T.

Solution.

$$T(A) = 0 \tag{82}$$
$$A + A^{\intercal} = 0 \tag{83}$$
$$A = -A^{\intercal} \tag{84}$$
$$. \tag{85}$$

The kernel of T is the set of all skew-symmetric matrices. ■

(b) Consider the polynomials $P_1(t) = 1 + t^2$ and $P_2(t) = 1 - t^2$. Is $\{p_1, p_2\}$ a linearly independent set in P_3? Why or why not ?

Solution.

$$c_1 P_1(t) + c_2 P_2(t) = 0 \tag{86}$$
$$c_1(1 + t^2) + c_2(1 - t^2) = 0 \tag{87}$$
$$c_1 + c_2 + c_1 t^2 - c_2 t^2 = 0 \tag{88}$$
$$c_1 + c_2 + t^2(c_1 - c_2) = 0 \tag{89}$$

$$\implies \begin{cases} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{cases} \implies \begin{cases} c_1 = 0 \\ c_2 = 0 \end{cases} . \tag{90}$$

Therefore, $\{p_1, p_2\}$ is a linearly independent set in P_3. ■

(c) Let H be the set of all vectors of the form $\begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find a vector V in $\mathbb{R}^3$ such that $H = \text{Span}\{V\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

Solution.

$$H = \text{Span}\left\{ \begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix} \right\} \tag{91}$$
$$= \left\{ \begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix} \mid t \in \mathbb{R} \right\} \tag{92}$$
$$= \left\{ t \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \mid t \in \mathbb{R} \right\} \tag{93}$$
$$= \text{Span}\left\{ \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \right\} . \tag{94}$$

This shows that H is a subspace of $\mathbb{R}^3$ because it is the span of a single vector. ■