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Thermodynamics, Wave Motion and Optics
Study Assignment 1

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1. Suppose a situation in which a string is vibrating in its first harmonic mode of oscillation. The string is of linear density 0.0016 kg/m and stretched between two clamps 0.48 m apart. The tension in the string is made to be a linear function of time, such that the tension it is raised from 15 N to 25 N in a duration of 3.5 seconds. Calculate the number of oscillations it completes in this duration.

$$\mu = 0.0016 \text{ kg/m} \quad L = 0.48 \text{ m} \quad T_0 = 15 \text{ N} \quad T_1 = 25 \text{ N} \quad t = 3.5 \text{ s.} \tag{1}$$

$$f_0 = \frac{1}{2L}\sqrt{\frac{T_0}{\mu}} = \frac{1}{2(0.48)}\sqrt{\frac{15}{0.0016}} = 100 \text{ Hz} \tag{2}$$

$$f_1 = \frac{1}{2L}\sqrt{\frac{T_1}{\mu}} = \frac{1}{2(0.48)}\sqrt{\frac{25}{0.0016}} = 130 \text{ Hz.} \tag{3}$$

$$\overline{f} = \frac{f_0 + f_1}{2} = \frac{100 + 130}{2} = 115 \text{ Hz.} \tag{4}$$

$$N = \overline{f}t = 115 \times 3.5 = 402.5 \text{ oscillations.} \tag{5}$$

2. A boat is traveling at 4.0 m/s in the same direction as an ocean wave of wavelength 30 m and speed 6.8 m/s. If the boat is on the crest of a wave,

$$v_{\text{boat}} = 4.0 \text{ m/s} \quad \lambda = 30 \text{ m} \quad v_{\text{wave}} = 6.8 \text{ m/s.} \tag{6}$$

   (a) How much time will elapse until the boat is next on a crest?

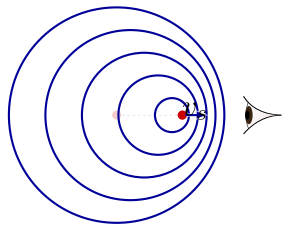
$$\Delta t = \frac{\lambda}{v_{\text{relative}}} = \frac{30}{6.8 - 4.0} = 11 \text{ s.} \tag{7}$$

   (b) Find the frequency of the waves relative to the shore and the boat

$$f_{\text{shore}} = \frac{v_{\text{wave}}}{\lambda} = \frac{6.8}{30} = 0.23 \text{ Hz} \tag{8}$$

$$f_{\text{boat}} = \frac{v_{\text{wave}} - v_{\text{boat}}}{\lambda} = \frac{6.8 - 4.0}{30} = 0.093 \text{ Hz.} \tag{9}$$

3. At a distance r away from a point source with power P_{avg}, the wave intensity is $I = \frac{P_{\text{avg}}}{4\pi r^2}$ show that at distance r straight in front of a point source with power P_{avg}, moving with constant speed v_s the wave intensity is
$$I = \frac{P_{\text{avg}}}{4\pi r^2}\left(\frac{v - v_s}{v}\right).$$



$$f' = f\left(\frac{v}{v - v_s}\right). \tag{10}$$

$$P_{\text{avg}} = \frac{1}{2}A\rho\omega^2 s_{\max}^2 v. \tag{11}$$

$$v = \lambda f. \tag{12}$$

$$\begin{aligned}
I = \frac{P}{A} = \frac{P_{\text{avg}}}{4\pi r^2} &= \frac{1}{2}\rho\omega^2 s_{\max}^2 v && (13)\\
I' &= \frac{1}{2}\rho\omega^2 s_{\max}^2 v' && (14)\\
&= \frac{1}{2}\rho\omega^2 s_{\max}^2 \lambda f' && (15)\\
&= \frac{1}{2}\rho\omega^2 s_{\max}^2 \lambda f\left(\frac{v}{v - v_s}\right) && (16)\\
&= \frac{1}{2}\rho\omega^2 s_{\max}^2 v\left(\frac{v}{v - v_s}\right) && (17)\\
&= \frac{P_{\text{avg}}}{4\pi r^2}\left(\frac{v}{v - v_s}\right). && (18)
\end{aligned}$$

4. Consider a string tied to a sinusoidal oscillator at P and running over a support at Q, and stretched by a block of mass m, as shown in the figure below. The separation L between P and Q is 1.2 m, and the frequency f of the oscillator is fixed at 120 Hz. The amplitude of the motion at P is small enough for that point to be considered a node. A node also exists at Q. A standing wave appears when the mass of the hanging block is 286.1 g or 447 g, but not for any intermediate mass. What is the linear density of the string?

$$L = 1.2 \text{ m} \quad f = 120 \text{ Hz} \quad m_1 = 0.2861 \text{ kg} \quad m_2 = 0.447 \text{ kg}. \tag{19}$$

$$T = mg. \tag{20}$$

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}} \tag{21}$$

$$f_n = \frac{n}{2L}\sqrt{\frac{mg}{\mu}}. \tag{22}$$

$$\mu = \frac{n^2 mg}{4L^2 f_n^2}. \tag{23}$$

$$m = \frac{4L^2 f_n^2 \mu}{n^2 g} \implies m \propto \frac{1}{n^2}. \tag{24}$$

$$\frac{m_2}{m_1} = \frac{(n+1)^2}{n^2} \tag{25}$$

$$\frac{m_2}{m_1} = \left(\frac{n+1}{n}\right)^2 \tag{26}$$

$$\sqrt{\frac{m_2}{m_1}} = \frac{n+1}{n} \tag{27}$$

$$n\sqrt{\frac{m_2}{m_1}} = n+1 \tag{28}$$

$$4\sqrt{\frac{0.447}{0.2861}} \approx 5. \tag{29}$$

$$n_1 = 5 \tag{30}$$

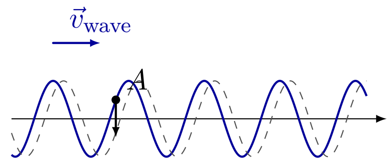
$$n_2 = 4. \tag{31}$$

$$\mu = \frac{n^2 mg}{4L^2 f_n^2} = \frac{5^2(0.2861)(9.8)}{4(1.2)^2(120)^2} = 8.45 \times 10^{-4} \text{ kg/m} \tag{32}$$

$$= \frac{4^2(0.447)(9.8)}{4(1.2)^2(120)^2} = 8.45 \times 10^{-4} \text{ kg/m}. \tag{33}$$

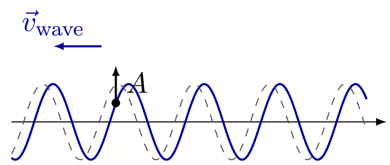
5. The Curve $y(x)$ at an instant of time for a wave travelling along x-axis on a string is shown below. The slope at the point A indicated on the curve happens to be 53°. Examine which of the following four statements hold in this situation. Show your detailed steps in assessing the statements.

   (a) The transverse velocity of a particle at A is positive if the wave is travelling along positive x-axis.



False.

   (b) The transverse velocity of a particle at A is positive if the wave is travelling along negative x-axis.



True.

   (c) The magnitude of transverse velocity of the particle at A is greater than the speed of the wave.

$$\frac{v_A}{v_{\text{wave}}} = \tan\theta \tag{34}$$
$$= \tan 53° \tag{35}$$
$$= 1.33 \tag{36}$$
$$v_A = 1.33 v_{\text{wave}}. \tag{37}$$

True.

   (d) The magnitude of transverse velocity of the particle at A is less than the speed of the wave.

As follows from (c) this statement is **False.**

6. Suppose you designed an oscillator with motion restricted in the x-axes, where the force on the system with mass m you manufactured to be proportional to cubic x.

$$F \propto x^3 \tag{38}$$
$$F = -kx^3. \tag{39}$$

(a) Find the potential energy function for the oscillator, you may assume no potential energy at $x = 0$.

$$U(x) = -\int F dx \tag{40}$$
$$U(x) = -\int (-kx^3) dx \tag{41}$$
$$U(x) = \frac{k}{4}x^4 + C \tag{42}$$
$$U(0) = 0 \implies C = 0 \implies U(x) = \frac{k}{4}x^4. \tag{43}$$

(b) The time required for the body to move from equilibrium at $x = 0$ to an amplitude A is 1/4 of a periodic time of the motion. Evaluate this duration and the periodic time.

$$v_x = \frac{dx}{dt}. \tag{44}$$

$$\frac{1}{2}mv_x^2 = \frac{k}{4}\left(A^4 - x^4\right) \tag{45}$$
$$\frac{dx}{\sqrt{A^4 - x^4}} = \sqrt{\frac{k}{2m}} dt \tag{46}$$
$$\int_0^A \frac{dx}{\sqrt{A^4 - x^4}} = \sqrt{\frac{k}{2m}} \int_0^{T/4} dt \tag{47}$$
$$\text{Let } u = \frac{x}{A} \implies dx = A du \tag{48}$$
$$\frac{1}{A}\int_0^1 \frac{du}{\sqrt{1 - u^4}} = \sqrt{\frac{k}{2m}} \frac{T}{4} \tag{49}$$
$$\frac{1.31}{A} = \sqrt{\frac{k}{2m}} \frac{T}{4} \tag{50}$$
$$T = \frac{7.41}{A}\sqrt{\frac{m}{k}}. \tag{51}$$

(c) Based on the calculations you found in part (b).

i. Does the periodic time depend on the amplitude A of oscillation?

For a nonlinear oscillator like this one, the period of oscillation **does** depend on the amplitude.

ii. Does your design produce a simple harmonic motion?

No, this design does not produce a simple harmonic motion. Simple harmonic motion occurs when the restoring force is proportional to displacement (not its cube). The motion will be more complex and will not follow the simple sinusoidal pattern of simple harmonic motion.

7. To examine wave phenomena on strings, your guru roommate friend promised you to bring you a wire that consists of two sections that have different linear densities that he saw in his hometown. To get everything well prepared and surprise him, you started analyzing such an interesting setup. You choose $x = 0$ to be the joining point of the two parts of the wire with linear mass densities μ_a and μ_b for before and after the knot point, respectively. Now suppose you let a wave generator to act at the left end of the wire generating a wave with displacement of the form $D(x,t) = A\sin[k_a(x - v_a t)]$.

   (a) Write down the wave functions of both the reflected and transmitted waves.

$$D_R(x,t) = A_R \sin[k_a(x + v_a t)] \tag{52}$$
$$D_T(x,t) = A_T \sin[k_b(x - v_b t)]. \tag{53}$$

   (b) Deduce a relation between A, A_T and A_R.

Use the continuity condition at the knot point.

$$D(0,t) + D_R(0,t) = D_T(0,t) \tag{54}$$
$$A + A_R = A_T. \tag{55}$$

   (c) Prove that $A_R = \left[\frac{v_a - v_b}{v_a + v_b}\right] A$.

Use the slope continuity condition at the knot point.

$$\frac{\partial D}{\partial x}(0,t) + \frac{\partial D_R}{\partial x}(0,t) = \frac{\partial D_T}{\partial x}(0,t) \tag{56}$$
$$k_a A - k_a A_R = k_b A_T \tag{57}$$
$$A - A_R = \frac{k_b}{k_a} A_T \tag{58}$$
$$A - A_R = \frac{v_a}{v_b} A_T \tag{59}$$
$$A - A_R = \frac{v_a}{v_b}(A + A_R) \tag{60}$$
$$. \tag{61}$$

   (d) Find A_T in terms of A and the wave numbers of the reflected and transmitted waves.

8. While you are studying for the exam, suppose you received a late call from your ZC nerd bestie asking you: What if a string is designed with a diameter that varies according to:

$$D = D_0 \sin(\pi \frac{x}{L}) \quad \text{for } 0 \leq x \leq L.$$

The string is of length L, constant mass density ρ_0, and put under constant tension T How can you find the time it takes for a transverse wave pulse to travel from one end to the other along the string.

$$A = \pi \frac{D^2}{4} \tag{62}$$
$$A = \pi \frac{D_0^2}{4} \sin^2(\pi \frac{x}{L}). \tag{63}$$

$$\mu = \frac{dm}{dx} \tag{64}$$
$$\mu = \frac{\rho_0 A x}{x} \tag{65}$$
$$\mu = \rho_0 A \tag{66}$$
$$\mu(x) = \rho_0 \pi \frac{D_0^2}{4} \sin^2(\pi \frac{x}{L}). \tag{67}$$

$$v = \sqrt{\frac{T}{\mu}} \tag{68}$$
$$v = \sqrt{\frac{T}{\rho_0 \pi \frac{D_0^2}{4} \sin^2(\pi \frac{x}{L})}} \tag{69}$$
$$\frac{dx}{dt} = \sqrt{\frac{T}{\rho_0 \pi \frac{D_0^2}{4} \sin^2(\pi \frac{x}{L})}} \tag{70}$$
$$dt = \sqrt{\frac{\rho_0 \pi \frac{D_0^2}{4} \sin^2(\pi \frac{x}{L})}{T}}\, dx \tag{71}$$
$$\int_0^t dt = \int_0^L \sqrt{\frac{\rho_0 \pi \frac{D_0^2}{4} \sin^2(\pi \frac{x}{L})}{T}}\, dx \tag{72}$$
$$t = \sqrt{\frac{\rho_0 \pi D_0^2}{4T}} \int_0^L \sin(\pi \frac{x}{L})\, dx \tag{73}$$
$$t = \sqrt{\frac{\rho_0 \pi D_0^2}{4T}} \frac{2L}{\pi} \tag{74}$$
$$= D_0 \frac{L}{\pi} \sqrt{\frac{\rho_0 \pi}{T}} \tag{75}$$
$$= D_0 L \sqrt{\frac{\rho_0}{T\pi}}. \tag{76}$$

9. During the building of the Admin and Culture Complex in ZC, a deep well is formed in the construction area. A Phy201 student wanted to use a frequency generator of adjustable frequency to measure the depth of the well. If the student reported hearing two successive resonances at 51.87 Hz and 59.85 Hz:

$$v = 346 \text{ m/s} \quad f_n = \frac{(2n-1)\,v}{4L} \quad f_n = 51.87 \text{ Hz} \quad f_{n+1} = 59.85 \text{ Hz}. \tag{77}$$

$$L = \frac{nv}{4f_n} \tag{78}$$

$$\frac{(2n+1)v}{4f_{n+1}} = \frac{(2n-1)v}{4f_n} \tag{79}$$

$$\frac{2n+1}{f_{n+1}} = \frac{2n-1}{f_n} \tag{80}$$

$$\frac{2n+1}{59.85} = \frac{2n-1}{51.87} \tag{81}$$

$$n = 7. \tag{82}$$

(a) How deep is the well?

$$L = \frac{(2n-1)v}{4f_n} \tag{83}$$

$$L = \frac{(2\times 7-1)\times 346}{4\times 51.87} = 21.68 \text{ m} \tag{84}$$

$$= \frac{(2\times 8-1)\times 346}{4\times 59.85} = 21.68 \text{ m}. \tag{85}$$

(b) How many antinodes are in the standing wave at 51.87 Hz?

$$n = 7. \tag{86}$$

10. A non uniform string of mass 4.5 kg and length 1.5 m has a variable linear mass density given by $\mu = ke^x$, where x is the distance from one end of the string and k is a constant. Tension in the string is 15 N which is uniform. Find the time (in second) required for a pulse generated at one end of the string to travel to the other end.

$$\mu = ke^x \quad T = 15\text{ N} \quad L = 1.5\text{ m} \quad M = 4.5\text{ kg}. \tag{87}$$

$$\begin{aligned}
\mu &= ke^x && (88)\\
\frac{dm}{dx} &= ke^x && (89)\\
dm &= ke^x\,dx && (90)\\
\int_0^M dm &= \int_0^L ke^x\,dx && (91)\\
M &= ke^x\Big|_0^L && (92)\\
M &= ke^L - ke^0 && (93)\\
M &= ke^L - k && (94)\\
M &= k(e^L - 1) && (95)\\
k &= \frac{m}{e^L - 1}. && (96)
\end{aligned}$$

$$\begin{aligned}
v &= \sqrt{\frac{T}{\mu}} && (97)\\
v &= \sqrt{\frac{T}{\frac{me^x}{e^L-1}}} && (98)\\
v &= \sqrt{\frac{T(e^L-1)}{me^x}} && (99)\\
\frac{dx}{dt} &= \sqrt{\frac{T(e^L-1)}{me^x}} && (100)\\
\sqrt{e^x}\,dx &= \sqrt{\frac{T(e^L-1)}{m}}\,dt && (101)\\
\int_0^L \sqrt{e^x}\,dx &= \int_0^t \sqrt{\frac{T(e^L-1)}{m}}\,dt && (102)\\
2(e^{\frac{L}{2}} - 1) &= \sqrt{\frac{T(e^L-1)}{m}}t. && (103)
\end{aligned}$$

$$\begin{aligned}
t &= 2\sqrt{\frac{m}{T(e^L-1)}}(e^{\frac{L}{2}} - 1) && (104)\\
&= 2\sqrt{\frac{4.5}{15(e^{1.5}-1)}}(e^{\frac{1.5}{2}} - 1) && (105)\\
&= 0.66\text{ s}. && (106)
\end{aligned}$$