Newtonian Mechanics in Geometric Algebra

SalahDin Rezk

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Disclaimer: This document does contain all different types of mistakes. Send whatever you find to s-salahdin.rezk@zewailcity.edu.eg. Thanks for your help.

Additional materials:-

- New Foundations for Classical Mechanics by David Hestenes.
- Clifford Algebra to Geometric Calculus by David Hestenes and Garret Sobczyk.
- A Swift Introduction to Geometric Algebra by sudgylacmoe.
- Essence of linear algebra by 3b1b.
- Visualizing quaternions by 3b1b.
- Multivariable Calculus by MIT OCW.
- Classical Mechanics by anaHr (Arabic).

Contents

Oct 4 2022

Lecture 1: Introduction

1.1 Physics as a tool

It is frequently the case that we forget that science as a whole is no more than a mere model that tries to reflect what reality is through our own perspective. It does not represent the truth nor claim to. It tries its best to be a useful tool for the human kind to progress and improve their own lives.

This section is not needed, it is more of a conceptual overlook of physics as a science.

This is more of my personal rant than an educated opinion.

Physics, therefore, is no exception. It is a collection of theories which failed to be proven wrong based on the empirical data we have. It is a tool that we use to understand the world around us and to predict the future; a *tool*, not a *truth*.

Thus no one should think too highly of physics as something holy and true in some absolute sense. Thus, we should think of our models as a distorted view of reality, and we should keep in mind that the hypothetical statements they make are not representative of the *true* nature of *existence* itself.

Debates on the true nature of existence is more of a philosophical thing. Physics, even if theoretical, is an experimental science, not humanities. It does not try to answer questions like dilemmas of god, free will, ideologies, etc. We only think of how we can improve our current theories to fit the reality we observe.

Furthermore, absurd theories (e.g string theory, quantum gravity, etc) are not profound physics theories. They are more of a set of arbitrary hypothesis bunched together to form a model that seem coherent yet does not new scarily conform to our reality. It is not to say that such theories are pseudoscience or they are not worth studying–even the theory of relativity was in such a state when it was first published. It is just that they are not *real* within our studying of the physical nature. They are more like philosophical speculations of the world; they may be true, but, for the most part, they are unreliable perspectives.

1.2 Formulation of a Mathematical Structure

Mathematical structures are made to help us, not to be enslaved to. Their formulation is like a game; it consists of a hierarchy of:

1. Goals
2. Rules
3. Definitions

Example. Cards Game

- Goals: winning (through gaining most cards).
- Rules:
    - Joker takes all, number takes its combination, etc.
    - Poker
    - etc
- Definitions: Ace of hearts, Knight of diamonds, etc.

Note:-

We can have the same game definitions with a different set of rules.

Definition 1.1: Logic

The process of using a set of definitions within a set of rules/constrains to reach a desired set of goals.

Note:-

Mathematical structures are not a truth within themselves; they are just tools to help us represent phenomenons in the real life efficiently.

Example. Useful Mathematical Structures

- $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ is a useful mathematical construct in some cases
  - You worked half half an hour yesterday and worked third of an hour today, **then** you worked a total of 50 minutes out of 60 minutes.
  - You won 1 out of 2 games yesterday and won 1 out of 3 games today, **then** you won a total of 2 out of 5 games
- $\frac{1}{2} + \frac{1}{3} = \frac{2}{5}$ is a plausible construct in most cases, but–clearly–not in all cases.

Definition 1.2: Truth

The combination of a set of *definitions* and a set of *rules*.

Definition 1.3: Validity

The inescapable conclustions formed by a given set of truth.

Example. Truths

- $\vec{v}_1 \times \vec{v}_2 = n|\vec{v}_1||\vec{v}_2| \sin\theta$ (We define the cross product as the perpendicular vector of the area formed by a parallelogram whose sides are formed from the vectors since it is a useful physical quantity).
- $\vec{v}_1 \times \vec{v}_2 = -\vec{v}_2 \times \vec{v}_1$ (We define the anti-commutative property of cross product since it represents rotational quantities).

Example. Validities

- $(\vec{v}_1 \times \vec{v}_2) \times \vec{v}_3 \neq \vec{v}_1 \times (\vec{v}_2 \times \vec{v}_3)$ (We can prove this using already defined proprieties and rules of the cross product).

Oct 11 2022

Lecture 2: Vectors

2.1 Formulation

Definition 2.4: Vector

A mathematical object that has a:

- Magnitude
- Orientation
- Sense (sign)

We use arrows to represent vectors graphically, the choice of arrows is very similar to the choice of lengths to represent numbers.

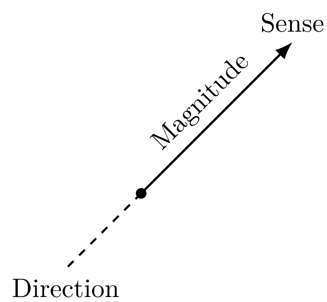

Figure 1: A vector represented graphically using an arrow.

Definition 2.5: Vector equality

For $\vec{a}$ to be equal to $\vec{b}$

- $|\vec{a}| = |\vec{b}|$
- $\vec{a} \parallel \vec{b}$
- Sense of $\vec{a}$ is the same as that of $\vec{b}$

The choice of such an equality, although looks natural, is a made construct. This specific equality is useful for our calculations. However, it is as real as $\frac{1}{2} = \frac{3}{6}$, which, even though represents the notion of ratios, is not *real* since, clearly, the process of cutting a pizza into 2 pieces is different from cutting it into 6 piece.

Note:-

Equality is defined in terms of the original definition.

2.2 Arithmetic

2.2.1 Addition

Definition 2.6: Vector addition

Vector $\vec{a}+\vec{b}$ starts at the end of $\vec{A}$ and ends at the end of $\vec{B}$

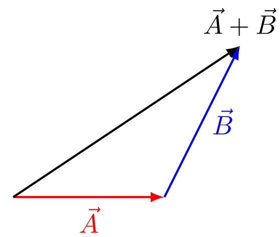

Figure 2: Sum of two vectors.

This definition is based on the practicality of such a geometrical construct according to real life observations. Moreover, this enables commutative and associative.

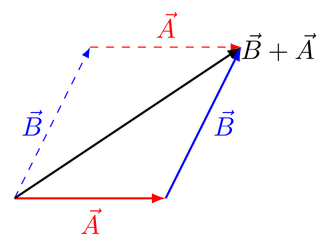

Figure 3: Commutative property of vector sum.

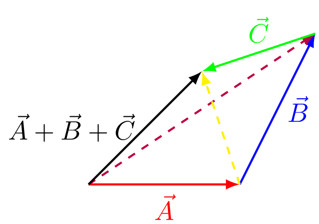

Figure 4: Associative property of vector sum.

We define the zero vector due to the usefulness of zero in normal arithmetic.

Definition 2.7: Zero Vector

$$\vec{v} + \vec{0} = \vec{v}.$$

Note:-

$$\vec{0} \neq 0.$$
$$|\vec{0}| = 0..$$

We also want $\vec{v}+(-\vec{v}) = \vec{0}$ to be true, which means that $-\vec{v}$ has to has the opposite sense of $\vec{v}$ but the same magnitude and direction, following the definition of addition and zero vector.

2.2.2 Scalar multiplication

Definition 2.8: Scalar multiplication

$c\vec{v}$ is a vector which has a $|c|$ times the magnitude of $\vec{v}$ in the same direction, and the same sense only if $c > 0$, otherwise, the opposite sense.

Note:-

$$d\vec{a} = \vec{a}d.$$
$$c(\vec{a} + \vec{b}) = c\vec{a} + c\vec{b}.$$
$$(c + d)\vec{a} = c\vec{a} + d\vec{a}.$$
$$c(d\vec{a}) = d(c\vec{a})..$$

2.2.3 Inner product

It is formulated to represent linear relations like $W = \vec{F} \cdot \vec{r}$, where only the components of the two vectors that are parallel to each other are relevant.

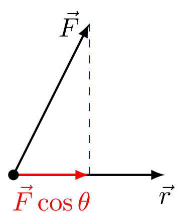

Figure 5: Inner product applied on work.

$$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}.$$

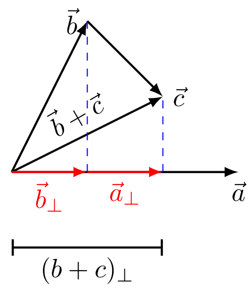

Figure 6: Distributive property of inner product.

Definition 2.9: Inner product

$$\vec{a} \cdot \vec{b} = |a||b| \cos\theta.$$

Note:-

Inner product results in a scalar. Thus, it can **not** be *commutative* ($\vec{a} \cdot \vec{b} \cdot \vec{c}$ is meaningless by definition). However, it **is** *associative* ($\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$).

2.2.4 Outer product

In contrast to inner product, outer product is formulated to represent rotational relations like $\vec{\tau} = \vec{F} \wedge \vec{r}$, where only the components of the two vectors that are perpendicular to each other are relevant.

However, it creates a new mathematical object, the bivector, which is oriented plane segment like how vectors are oriented line segments. They are related to pseudovectors in physics (e.g. angular velocity).

$$\vec{a} \wedge \vec{b} = \overset{\circ}{B}.$$

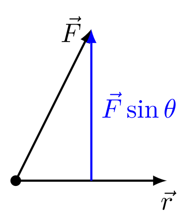

Figure 7: Outer product applied on torque.

Definition 2.10: Outer product

$$|\vec{a} \wedge \vec{b}| = |a||b| \sin\theta.$$

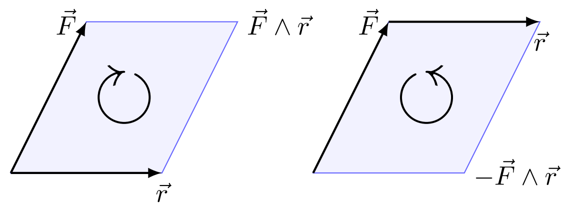

Figure 8: The bivector result of outer product.

The difference between clockwise and counterclockwise is represented through the anticommutative property ($\vec{a} \wedge \vec{b} = -\vec{b} \wedge \vec{a}$).

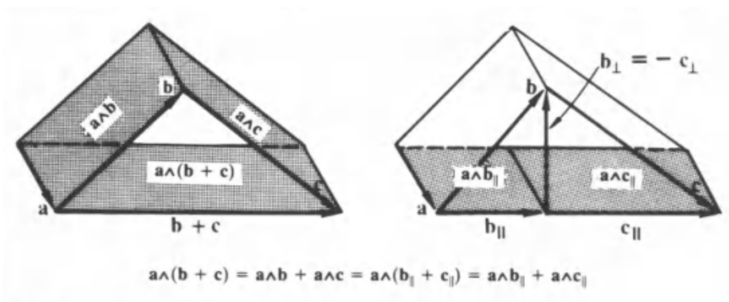

Figure 9: Distributive property of outer product

$$\begin{aligned}
\vec{b} + \vec{c} &= (\vec{b}_{\parallel} + \vec{b}_{\perp}) + (\vec{c}_{\parallel} + \vec{c}_{\perp}) \\
&= (\vec{b}_{\parallel} + \vec{b}_{\perp}) + (\vec{c}_{\parallel} - \vec{b}_{\perp}) \\
&= \vec{b}_{\parallel} + \vec{c}_{\parallel}.
\end{aligned}$$

$$\begin{aligned}
\vec{a} \wedge (\vec{b} + \vec{c}) &= \vec{a} \wedge (\vec{b}_{\parallel} + \vec{c}_{\parallel}) \\
&= \vec{a} \wedge \vec{b}_{\parallel} + \vec{a} \wedge \vec{c}_{\parallel} \\
&= \vec{a} \wedge \vec{b} + \vec{a} \wedge \vec{c}.
\end{aligned}$$

The next step after bivectors is trivectors. Just like vectors and bivectors but for three dimensions, they are oriented volume. In addition, they are closely related to pseudoscalers in physics (e.g. magnetic flux).

$$(\vec{a} \wedge \vec{b}) \wedge \vec{c} = \boldsymbol{T}.$$

However, we do not define tetravectors (4-vectors) because they are not helpful in the *current* dimensional space.

$$
\begin{aligned}
((\vec{a} \wedge \vec{b}) \wedge \vec{c}) \wedge \vec{d} &= 0 \\
\vec{a} \wedge \vec{b} \wedge \vec{c} \wedge \vec{d} &= 0 \\
\boldsymbol{T} \wedge \vec{d} &= 0.
\end{aligned}
$$

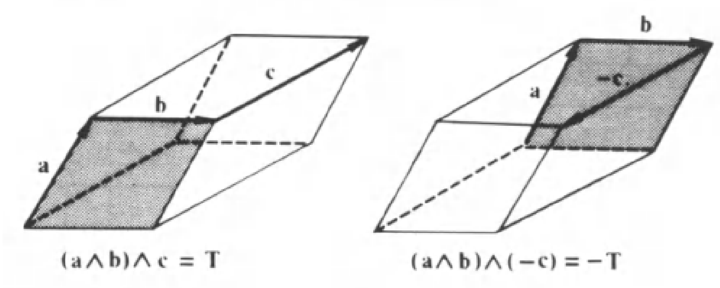

Figure 10: The trivector result of outer product.

Note:-

Outer product is associative: $(\vec{a} \wedge \vec{b}) \wedge \vec{c} = \vec{a} \wedge (\vec{b} \wedge \vec{c})$.

2.2.5 Geometric product

While the inner product captures the parallelism between vectors and the outer product captures the perpendicularity between vectors, none captures both. Therefore, the formulation of geometric product is needed.

Definition 2.11

$$\vec{a}\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}.$$

It is simply defined as their sum, although they are different mathematical objects, similar to how we define complex numbers $(a + bi)$. Following this definition we conclude the geometric product properties.

$$
\begin{aligned}
\vec{a}\vec{b} &\neq \vec{b}\vec{a} \\
&= \vec{a} \cdot \vec{b} - \vec{b} \wedge \vec{a}.
\end{aligned}
$$

$$
\begin{aligned}
\vec{a}(\vec{b} + \vec{c}) &= a \cdot (b + c) + a \wedge (b + c) \\
&= (a \cdot b + a \wedge b) + (a \cdot c + a \wedge c) \\
&= \vec{a}\vec{b} + \vec{a}\vec{c}.
\end{aligned}
$$

$$c\vec{a}\vec{b} = (c\vec{a})\vec{b} = a(c\vec{b}).$$

$$\begin{aligned}\vec{a}\vec{b} + \vec{b}\vec{a} &= (a \cdot b + \vec{a} \wedge \vec{b}) + (b \cdot a - \vec{a} \wedge \vec{b}) \\ &= 2a \cdot b \\ \therefore \vec{a} \cdot \vec{b} &= \frac{1}{2}(\vec{a}\vec{b} + \vec{b}\vec{a}).\end{aligned}$$

$$\begin{aligned}\vec{a}\vec{b} - \vec{b}\vec{a} &= (a \cdot b + \vec{a} \wedge \vec{b}) + (-b \cdot a + \vec{a} \wedge \vec{b}) \\ &= 2\vec{a} \wedge \vec{b} \\ \therefore \vec{a} \wedge \vec{b} &= \frac{1}{2}(\vec{a}\vec{b} - \vec{b}\vec{a}).\end{aligned}$$

2.2.6 Geometric division

Now we can define geometric division as $\boldsymbol{A_k^{-1} A_k} = 1$, where A_k is a multivector, and follow to conclude a formula.

$$\begin{aligned}\boldsymbol{A_k^{-1} A_k} &= 1 \\ \boldsymbol{A_k^{-1} A_k A_k} &= \boldsymbol{A_k} \\ \boldsymbol{A_k^{-1} A_k^2} &= \boldsymbol{A_k} \\ \boldsymbol{A_k^{-1}} |\boldsymbol{A_k}|^2 &= \boldsymbol{A_k} \\ \boldsymbol{A_k^{-1}} &= \frac{\boldsymbol{A_k}}{|\boldsymbol{A_k}|^2}\end{aligned}$$

.

Note:-

$$\begin{aligned}A_k A_k &= A_k^2 \\ &= A_k \cdot A_k + A_k \wedge A_k \\ &= |A_k||A_k| \cos 0 + |A_k||A_k| \sin 0 \\ &= |A_k||A_k| + 0 \\ &= |A_k|^2.\end{aligned}$$

Notes