PHYS201 – Midterm 1 – Solutions

SalahDin Rezk

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1. Problem 1

   (a) Write down a possible function of the time- and space-dependent electric field $E(x, y, z, t)$ of a circularly polarized light wave of frequency f and amplitude E_0 traveling in the positive z-direction in space. Indicate whether the function represents RHCP or LHCP.

   $$k = 2\pi\frac{f}{c} \quad \omega = 2\pi f \quad E(x,t) = E_0 e^{(kx-\omega t)i}. \tag{1}$$

   Since the wave is travelling in the positive z-direction, then $E(z,t) = E_0 e^{(kz-\omega t)i}$

   $$\begin{aligned} E(z,t) &= E_0 e^{(kz-\omega t)i} && (2)\\ &= E_0 e^{(2\pi\frac{f}{c}z-2\pi f t)i} && (3)\\ &= \cos\left(2\pi\frac{f}{c}z - 2\pi f t\right) + \sin\left(2\pi\frac{f}{c}z - 2\pi f t\right) i && (4)\\ &\implies \phi = \frac{\pi}{2}. && (5)\end{aligned}$$

   A phase difference of $\frac{\pi}{2}$ indicates RHCP.

2. Utilizing Snell's law of refraction, demonstrate that an object observed through a glass slide of specific thickness seems closer, with a distance shift given by $\Delta L' = d(1 - 1/n)$, where d represents the glass slide thickness, and n is its refractive index. The small angle approximation can be applied for the analysis.

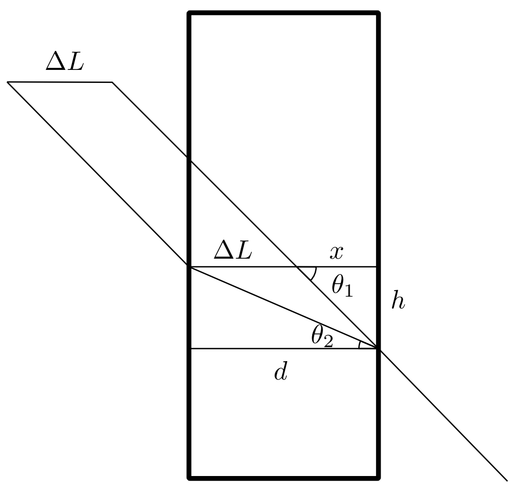

Figure 1

$$\Delta = d - x \tag{6}$$
$$x = h\cot\theta_1 \tag{7}$$
$$h = d\tan\theta_2. \tag{8}$$

Using small angle approximation:

$$\sin\theta \approx \theta \tag{9}$$
$$\cos\theta \approx 1 \tag{10}$$
$$\tan\theta \approx \theta. \tag{11}$$

For Snell's law:

$$\sin\theta_1 = n\sin\theta_2 \tag{12}$$
$$\theta_1 \approx n\theta_2. \tag{13}$$

Then:

$$h = d\theta_2 \tag{14}$$
$$x = \frac{d\theta_2}{\theta_1} \tag{15}$$
$$\Delta L = d - \frac{d\theta_2}{\theta_1} \tag{16}$$
$$= d - \frac{d\theta_2}{n\theta_2} \tag{17}$$
$$= d\left(1 - \frac{1}{n}\right). \tag{18}$$

3. Problem 2
   (a) Suppose a situation in which a uniform rope is suspended from a ceiling, with a mass of m and a length of L.
      i. Justify that the velocity of a transverse wave along the rope is dependent on y, the distance measured from the lower end, and is expressed as $v = \sqrt{gy}$.

$$v = \sqrt{\frac{T}{\mu}} \quad \mu = \frac{m}{L} \quad T = mg. \tag{19}$$

Since it is a rope of a mass:

$$\mu(y) = \frac{m}{y} \tag{20}$$
$$T(y) = \mu(y)yg \tag{21}$$
$$\tag{22}$$

$$v = \sqrt{\frac{\mu(y)mg}{\mu(y)}} \tag{23}$$
$$= \sqrt{mg}. \tag{24}$$

ii. What is the lowest refractive index required for the plastic rod to guarantee total reflection of any ray entering at the end?

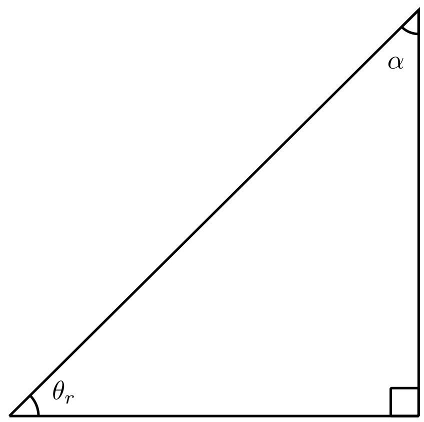

Figure 2

$$\sin\theta_i = n\sin\alpha \quad \sin\theta_c = \frac{1}{n}. \tag{25}$$

$$\begin{aligned}
\theta_r &> \theta_c & (26)\\
\cos\theta_r &< \cos\theta_c & (27)\\
\cos\theta_r &< \cos\sin^{-1}\frac{1}{n} & (28)\\
\cos\left(\frac{\pi}{2}-\alpha\right) &< \cos\sin^{-1}\frac{1}{n} & (29)\\
\sin\alpha &< \cos\sin^{-1}\frac{1}{n} & (30)\\
\sin\theta_i\cdot\frac{1}{n} &< \cos\sin^{-1}\frac{1}{n} & (31)\\
\sin\theta_i &< n\cos\sin^{-1}\frac{1}{n}. & (32)
\end{aligned}$$

$$\max(\sin\theta_i) = 1 \implies 1 < n\cos\sin^{-1}\frac{1}{n} \tag{33}$$

$$\begin{aligned}
1 &< n\cos\sin^{-1}\frac{1}{n} & (34)\\
\frac{1}{n} &< \cos\sin^{-1}\frac{1}{n} & (35)\\
\cos^{-1}\frac{1}{n} &< \sin^{-1}\frac{1}{n}. & (36)
\end{aligned}$$

$$\cos^{-1}x < \sin^{-1}x \implies \theta = \frac{\pi}{4} \implies x < \frac{1}{\sqrt{2}}. \tag{37}$$

$$\begin{aligned}
\frac{1}{n} &< \frac{1}{\sqrt{2}} & (38)\\
n &> \sqrt{2}. & (39)
\end{aligned}$$

4. Problem 3

(a) The rubber band variety employed within certain baseballs adheres to Hooke's law across a broad elongation range. A section of this material possesses an unstretched length L and a mass m. Upon applying a force F, the band extends an additional length ΔL.

i. Find the speed of transverse waves on this stretched rubber band in terms of m, ΔL, and the spring constant k.

$$v = \sqrt{\frac{T}{\mu}} \quad T = k\Delta L \quad \mu = \frac{m}{L} \tag{40}$$

Since the rope is stretched:

$$\mu = \frac{m}{L + \Delta L} \tag{41}$$

Then:

$$v = \sqrt{\frac{k\Delta L}{m/(L+\Delta L}} \tag{42}$$

$$= \sqrt{\frac{k}{m}\Delta L(L + \Delta L)}. \tag{43}$$

ii. Utilizing the result from part (a), illustrate that the time required for a transverse pulse to travel the length of the rubber band is proportionate to $L/\sqrt{\Delta L}$ when $\Delta L \ll L$ and remains constant when $\Delta L \gg L$.

$$v = \frac{d}{t} \tag{44}$$

$$= \frac{L + \Delta L}{t}. \tag{45}$$

$$t = \frac{L + \Delta L}{\sqrt{\frac{k}{m}\Delta L(L + \Delta L)}} \tag{46}$$

$$= \sqrt{\frac{k(L + \Delta L)}{m\Delta L}} \tag{47}$$

$$= \sqrt{\frac{k}{m}\left(\frac{L}{\Delta L} + 1\right)}. \tag{48}$$

$$\Delta L \gg L \iff v = \sqrt{\frac{k}{m}\left(\frac{0}{\Delta L} + 1\right)} \tag{49}$$

$$= \sqrt{\frac{k}{m}} \quad \text{(constant)} \tag{50}$$

$$\Delta L \ll L \iff v = \sqrt{\frac{k}{m}\left(\frac{L}{\Delta L}\right)} \tag{51}$$

$$\implies t \propto \frac{1}{\sqrt{\Delta L}}. \tag{52}$$

(b) An increase in the index of refraction of glass can be achieved through impurity diffusion, allowing for the creation of a lens with consistent thickness. Given a disk of radius a and thickness d, determine the radial variation of the index of refraction, $n(r)$, needed to produce a lens with a focal length F. Assume a thin lens ($d \ll a$).

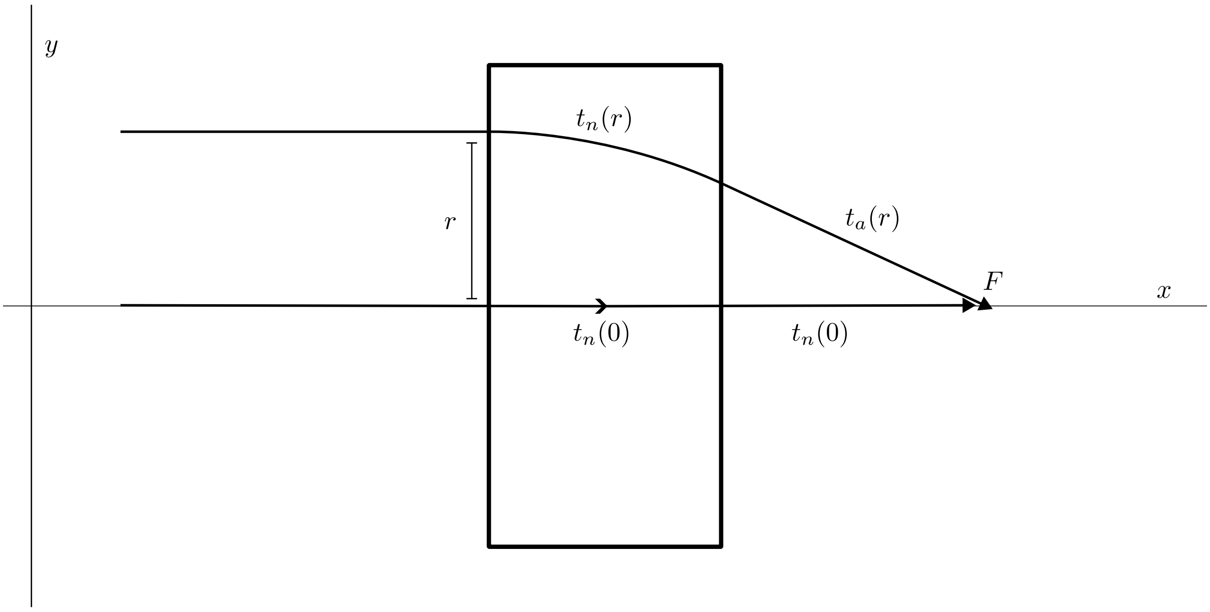

Figure 3

Since an image forms at F, then time taken for any light ray to reach F is the same. Let $t(r)$ be the time taken for a light ray to reach F from a distance r from the center of the lens, t_n is time taken in the lens, and t_a is time taken in air.

$$t = \frac{c}{d \cdot n} \tag{53}$$

$$\begin{aligned}
\Sigma t(r) &= \Sigma t(0) \quad &(54)\\
t_n(r) + t_a(r) &= t_n(0) + t_a(0) \quad &(55)\\
\frac{c}{d \cdot n(r)} + \frac{c}{\sqrt{F^2 + r^2}} &= \frac{c}{d \cdot n(0)} + \frac{c}{F} \quad &(56)\\
\frac{1}{d \cdot n(r)} + \frac{1}{\sqrt{F^2 + r^2}} &= \frac{1}{d \cdot n(0)} + \frac{1}{F} \quad &(57)
\end{aligned}$$

$$\begin{aligned}
\frac{1}{d \cdot n(r)} &= \frac{1}{d \cdot n(0)} + \frac{1}{F} - \frac{1}{\sqrt{F^2 + r^2}} \quad &(58)\\
\frac{1}{n(r)} &= \frac{1}{n(0)} + d\left(\frac{1}{F} - \frac{1}{\sqrt{F^2 + r^2}}\right). \quad &(59)
\end{aligned}$$

Under the assumption that $n(0) = 1$:

$$\begin{aligned}
\frac{1}{n(r)} &= 1 + d\left(\frac{1}{F} - \frac{1}{\sqrt{F^2 + r^2}}\right) \quad &(60)\\
n(r) &= \frac{1}{1 + d\left[F^{-1} - (F^2 + r^2)^{-1/2}\right]} \quad &(61)
\end{aligned}$$

5. Problem 4

(a) Suppose you designed a setup involving two point sources, S_1 and S_2, emitting sound waves with a wavelength λ of 2.0 m. The emissions are isotropic and synchronized. Suppose S_1 is placed at (0,0) and S_2 is placed at (0, −16.0 m). At any point P along the x-axis, the waves from S_2 and S_2 interfere following the superposition principle.

i. Plot the configuration in (x, y) the plan.

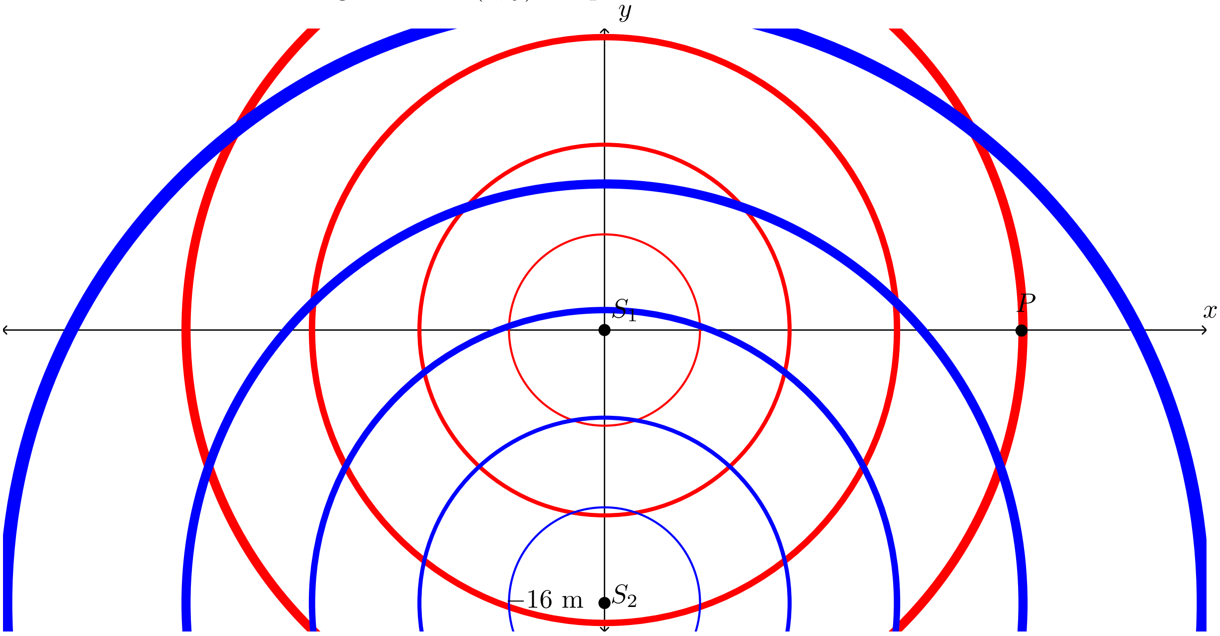

Figure 4

ii. When P is infinitely far away,

A. What is the phase difference at point P between the arriving waves from S_2 and S_2?

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x \tag{62}$$

$$= \frac{2\pi}{\lambda}\left(\sqrt{16^2 + x^2} - x\right) \tag{63}$$

$$\lim_{x\to\infty} = \frac{2\pi}{\lambda}(x - x) \tag{64}$$

$$= 0. \tag{65}$$

B. Consequently, what type of interference do they produce at point P?
Maximum Constructive Interference

iii. Suppose now we move point P along the x-axis toward the source S_1.

A. Will the phase difference between the waves increase or decrease?

B. At what distance x will the waves exhibit a phase difference of (I) 0.5λ, (II) 1.0λ, and (III) 1.5λ?

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x \tag{66}$$

$$n\lambda = \frac{2\pi}{\lambda}\left(\sqrt{16^2 + x^2} - x\right) \tag{67}$$

$$\frac{n\lambda^2}{2\pi} = \sqrt{16^2 + x^2} - x. \tag{68}$$

$$\Delta\phi = 0.5\lambda \implies x = 401 \tag{69}$$

$$\Delta\phi = 1.0\lambda \implies x = 200 \tag{70}$$

$$\Delta\phi = 1.5\lambda \implies x = 134 \tag{71}$$

$$\tag{72}$$

6. A natural light beam of irradiance I_0 is incident upon a polaroid. The transmitted beam is incident upon a second polaroid whose transmission axis is aligned with the first at time $t = 0$, and rotated about the optical axis with an angular speed ω (radians per second).

   (a) Derive an expression for the transmitted irradiance $I(t)$ out of the second polaroid as a function of time, and as a fraction of I_0.

$$I_1 = \frac{1}{2} I_0 \quad I_2 = I_1 \cos^2\theta \quad \theta = \omega t \tag{73}$$

$$I(t) = \frac{1}{2} I_0 \cos^2(\omega t) \tag{74}$$

   (b) Plot this transmitted irradiance as a function of time. Be sure to indicate the scale of each axis accurately.

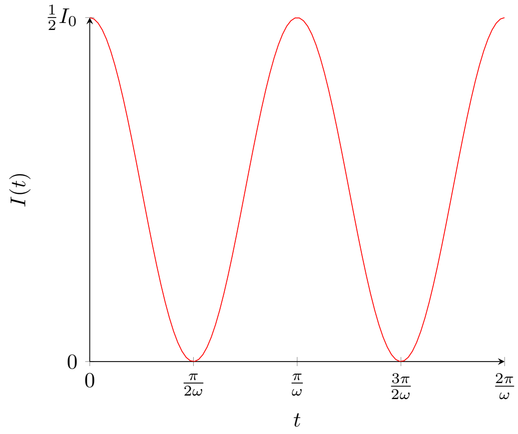

Figure 5

   (c) If a third polaroid were now placed to the right of the rotating polaroid, with its transmission axis oriented at 90° to that of the first (fixed) polaroid, how many maxima of the transmitted irradiance would occur per each complete (360°) revolution of the rotating polaroid?

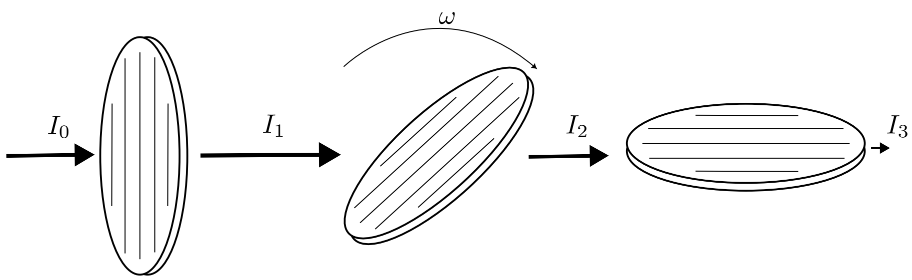

Figure 6

$$I(t) = \frac{1}{2} I_0 \cos^2(\omega t) \cos^2(\frac{\pi}{2} - \omega t) \tag{75}$$

$$= \frac{1}{2} I_0 \cos^2(\omega t) \sin^2(\omega t) \tag{76}$$

$$= \frac{1}{2} I_0 \left[\cos(\omega t) \sin(\omega t)\right]^2 \tag{77}$$

$$= \frac{1}{8} I_0 \left[\frac{\sin(2\omega t)}{2}\right]^2 \tag{78}$$

$$= \frac{1}{8} I_0 \sin^2(2\omega t) \tag{79}$$

$$\Longrightarrow \text{ 4 maxima per revolution.} \tag{80}$$