I) Consider the boundary value problem

$$x^2y'' + xy' + \lambda y = 0, \quad 1 < x < e$$
$$y(1) = 0, \quad y(e) = 0.$$

a) Show that this BVP is equivalent to the following Sturm-Liouville problem

$$(xy')' + \frac{\lambda}{x}y = 0, \quad 1 < x < e$$
$$y(1) = 0, \quad y(e) = 0.$$

$$y'' + \frac{1}{x}y' + \frac{\lambda}{x^2}y = 0$$

$$\mu(x) = \exp\int \frac{1}{x}dx = e^{\ln x} = x$$

$$\frac{d}{dx}(xy') + \frac{\lambda}{x}y = 0 \qquad y(1)=0, \quad y(e)=0$$

b) Verify that the eigenvalues of the Sturm-Liouville problem are $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \ldots$ and the corresponding eigenfunctions are $\emptyset_n(x) = c_n \sin(n\pi \ln x)$.

$$u = \ln x \Rightarrow x = e^u$$

$$\frac{du}{dx} = \frac{1}{x} = \frac{1}{e^u}$$

$$\Downarrow$$

$$\frac{1}{e^u}\frac{d}{du}(y'(u)) + \frac{\lambda}{e^u}y = 0$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\frac{d}{dx} = \frac{1}{x}\frac{d}{du}$$

u-domain

$y(0) = 0$

$y(1) = 0$

$$\cancel{\frac{1}{e^u}}\left(\frac{d}{du}\frac{dy}{du} + \lambda y\right) = 0$$

let $y = e^{mu} \Rightarrow m^2 + \lambda = 0$, let $\lambda > 0 \Rightarrow m = \pm i\sqrt{\lambda}$

$$y = C_1 \cos\sqrt{\lambda}\, u + C_2 \sin\sqrt{\lambda}\, u \qquad y(0)=0, \quad y(1)=0$$

$C_1 = 0$

$C_2 \sin\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = n\pi \Rightarrow \phi_n(u) = C\sin(n\pi u)$

$\lambda_n = n^2\pi^2 \qquad \phi_n(x) = C\sin(n\pi \ln x)$

QED

c) **Show that the following orthogonality property**

$$\int_1^e \frac{1}{x}\emptyset_n(x)\emptyset_m(x)dx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \end{cases}$$

$$\int_1^e \frac{1}{x}\sin(n\pi \ln x)\sin(m\pi \ln x)\,dx$$

$v = \pi \ln x$

$dv = \frac{\pi}{x}dx$

$$\frac{1}{\pi}\int_0^\pi \sin(nv)\sin(mv)\,dv$$

$\sin x \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}$

$$= \frac{1}{2\pi}\int_0^\pi \cos(v(n-m)) - \cos(v(m+n))\,dv$$

if $n \neq m$

$$= \frac{1}{2\pi}\left[\frac{\sin(v(n-m))}{n-m} - \frac{\sin(v(n+m))}{n+m}\right]_0^\pi = 0$$

if $n = m$

$$= \frac{1}{2\pi}\int_0^\pi \cos(\cancel{v(0)})^{1} - \cos(2nv)\,dv$$

$$= \frac{1}{2\pi}\left[v + \frac{\sin(2vn)}{2n}\right]_0^\pi = \frac{1}{2}$$

$$\therefore \int_1^e \frac{1}{x}\phi_n(x)\phi_m(x)dx = \begin{cases} 0, & m \neq n \\ \frac{1}{2}, & m = n \end{cases}$$

II) **Verify that for the Sturm-Liouville problem**

$$(xy')' + \frac{\lambda}{x} y = 0, \qquad 1 < x < e^{2\pi}$$
$$y'(1) = 0, \qquad y'(e^{2\pi}) = 0$$

the eigenvalues are $\lambda_n = \frac{n^2}{4}$, $n = 0, 1, 2, 3, \ldots$ **and the corresponding eigenfunctions are** $\emptyset_n(x) = b_n \cos\left(\frac{n \ln x}{2}\right)$**. Show that**

$$\int_1^{e^{2\pi}} \frac{1}{x} \emptyset_n(x) \emptyset_m(x) dx = 0, \qquad m \neq n$$

$u = \ln x \Rightarrow x = e^u$

$\frac{du}{dx} = \frac{1}{x} = \frac{1}{e^u}$

$\Downarrow$

$\frac{1}{e^u} \frac{d}{du}(y'(u)) + \frac{\lambda}{e^u} y = 0$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$\frac{d}{dx} = \frac{1}{x} \frac{d}{du}$

u - domain

$y'(0) = 0$

$y'(2\pi) = 0$

$\cancel{\frac{1}{e^u}} \left(\frac{d}{du} \frac{dy}{du} + \lambda y \right) = 0$

let $y = e^{mu} \Rightarrow m^2 + \lambda = 0$, let $\lambda > 0 \Rightarrow m = \pm i\sqrt{\lambda}$

$y = C_1 \cos\sqrt{\lambda}\, u + C_2 \sin\sqrt{\lambda}\, u$

$y' = -C_1\sqrt{\lambda} \sin\sqrt{\lambda}\, u + C_2\sqrt{\lambda} \cos\sqrt{\lambda}\, u$

$y'(0) = 0$

$y'(2\pi) = 0$

$C_2 = 0$

$-C_1\sqrt{\lambda} \sin(2\pi\sqrt{\lambda}) = 0 \Rightarrow 2\pi\sqrt{\lambda} = n\pi$

$\lambda_n = n^2/4$

$y_n(u) = C \cdot \cos(u \cdot n/2)$

$y_n(x) = C \cdot \cos(n \ln x / 2)$

$$\int_0^{2\pi} \cos(un/2)\cos(um/2)\,du$$

$$\cos x \cos y = \frac{\cos(x-y)+\cos(x+y)}{2}$$

$$= \frac{1}{2}\int_0^{2\pi} \cos\left(\frac{u}{2}(n-m)\right) + \cos\left(\frac{u}{2}(n+m)\right)du$$

if $n \neq m$:

$$= \left[\frac{\sin\left(\frac{u}{2}(n-m)\right)}{n-m} + \frac{\sin\left(\frac{u}{2}(n+m)\right)}{n+m}\right]_0^{2\pi} = 0$$

if $n = m$:

$$= \frac{1}{2}\int_0^{2\pi} \cos\left(\overset{1}{\cancel{\frac{u}{2}(0)}}\right) + \cos\left(\overset{un}{\cancel{\frac{u}{2}(2n)}}\right)du$$

$$= \frac{1}{2}\int_0^{2\pi} 1 + \cos(un)\,du = \frac{1}{2}\left[u + \frac{1}{n}\sin(un)\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi) = \pi$$

$$\int_0^{e^{2\pi}} \frac{1}{x}\cos(n\ln x/2)\cos(m\ln x/2)\,dx = \begin{cases} 0, & n \neq m \\ \pi, & n = m \end{cases}$$

QED

Second Question

I) Given that $J_p'(\alpha_n b) = 0$ $(n = 1, 2, 3, \dots)$ for $p > -\frac{1}{2}$, show that

a) $\int_0^b xJ_p(\alpha_n x)J_p(\alpha_m x)dx = 0, \quad m \neq n$

b) $\int_0^b x\left[J_p(\alpha_n x)\right]^2 dx = \frac{\alpha_n^2 b^2 - p^2}{2\alpha_n^2}\left[J_p(\alpha_n b)\right]^2$

a)

$$x^2 y'' + xy' + (x^2 - \alpha^2)y = 0$$

$$x(xy'' + y') + (x^2 - \alpha^2)y = 0$$

$$x\frac{d}{dx}(xy') + (x^2 - \alpha^2)y = 0$$

$$\left[x\frac{d}{dx}\left(xJ_p'(\alpha_n x)\right) + (\alpha_n^2 x^2 - p^2)J_p(\alpha_n x) = 0\right] \times \frac{1}{x}J_p(\alpha_m x)$$

$$-$$

$$\left[x\frac{d}{dx}\left(xJ_p'(\alpha_m x)\right) + (\alpha_m^2 x^2 - p^2)J_p(\alpha_m x) = 0\right] \times \frac{1}{x}J_p(\alpha_n x)$$

$$\Downarrow$$

$$J_p(\alpha_m x)\frac{d}{dx}\left(xJ_p'(\alpha_n x)\right) - J_p(\alpha_n x)\frac{d}{dx}\left(xJ_p'(\alpha_m x)\right)$$

$$+ \frac{1}{\cancel{x}}(\alpha_n^2 x^{\cancel{2}} \cancel{- p^2})J_p(\alpha_n x)J_p(\alpha_m x) - \frac{1}{\cancel{x}}(\alpha_m^2 x^{\cancel{2}} \cancel{- p^2})J_p(\alpha_m x)J_p(\alpha_n x) = 0$$

$$\Downarrow$$

$$J_p(\alpha_m x)\frac{d}{dx}\left(xJ_p'(\alpha_n x)\right) - J_p(\alpha_n x)\frac{d}{dx}\left(xJ_p'(\alpha_m x)\right)$$

$$= (\alpha_m^2 - \alpha_n^2)\, x\, J_p(\alpha_m x)J_p(\alpha_n x)$$

$$\Big\downarrow \text{Integrate}$$

$$\int_0^b xJ_p(\alpha_m x)J_p(\alpha_n x)dx = \frac{1}{\alpha_m^2 - \alpha_n^2}\left[\int_0^b J_p(\alpha_m x)\frac{d}{dx}\left(xJ_p'(\alpha_n x)\right)dx - \int_0^b J_p(\alpha_n x)\frac{d}{dx}\left(xJ_p'(\alpha_m x)\right)dx\right]$$

$$\int_0^b x J_p(\alpha_m x) J_p(\alpha_n x)\,dx = \frac{1}{\alpha_m^2-\alpha_n^2}\left[\int_0^b J_p(\alpha_m x)\frac{d}{dx}\left(x J_p'(\alpha_n x)\right)dx - \int_0^b J_p(\alpha_n x)\frac{d}{dx}\left(x J_p'(\alpha_m x)\right)dx\right]$$

Integrating by parts

$$\int_0^b x J_p(\alpha_m x) J_p(\alpha_n x)\,dx = \frac{1}{\alpha_m^2-\alpha_n^2}\left[\cancel{x J_p'(\alpha_n x) J_p(\alpha_m x)\Big|_0^b} - \cancel{\int_0^b x J_p'(\alpha_n x) J_p'(\alpha_m x)\,dx} + \cancel{x J_p'(\alpha_m x) J_p(\alpha_n x)\Big|_0^b} + \cancel{\int_0^b x J_p'(\alpha_n x) J_p'(\alpha_m x)\,dx}\right]$$

$$= \frac{1}{\alpha_m^2\ \alpha_n^2} \times 0 = 0$$

QED

b) Similar to previous problem, but $m=n$:

$$\int_0^b x J_p^2(\alpha_n x)\,dx = \frac{\alpha_n^2 b^2 - p^2}{2\alpha_n^2}$$

QED

II) Given that $hJ_p(\alpha_n b) + \alpha_n J'_p(\alpha_n b) = 0$ $(n = 1, 2, 3, \dots)$ for constant h and $p > -\frac{1}{2}$, show that

a) $\int_0^b xJ_p(\alpha_n x)J_p(\alpha_m x)dx = 0, \quad m \neq n$

b) $\int_0^b x\left[J_p(\alpha_n x)\right]^2 dx = \frac{(\alpha_n^2+h^2)b^2-p^2}{2\alpha_n^2}\left[J_p(\alpha_n b)\right]^2$

$h J_p(\alpha_n b) + \alpha_n J_p'(\alpha_n b) = 0$

a) $\int_0^b x J_p(\alpha_n x) J_p(\alpha_m x)\, dx$ Using Bessel orthogonality formula from previous

b) Similar to previous problem, but $m=n$:

$$\int_0^b x J_p^2(\alpha_n x)\, dx = \frac{(\alpha_n^2+h^2)b^2-p^2}{2\alpha_n^2} J_p^2(\alpha_n b)$$ QED

Third Question

I)

a) Find the eigenvalues and eigenfunctions of the boundary-value problem

$$y'' + y' + \lambda y = 0, \quad y(0) = 0, \; y(2) = 0$$

b) Put the differential equation in self-adjoint form.

c) Give an orthogonality relation.

a)

let $y = e^{mx} \Rightarrow m^2 + m + \lambda = 0 \Rightarrow m = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$

$1-4\lambda = 0$: $m_1 = m_2 = \frac{-1}{2}$

$\therefore y = C_1 e^{-\frac{1}{2}x} + C_2 x e^{-\frac{1}{2}x} \Rightarrow C_1 = C_2 = 0$ Trivial

$1-4\lambda > 0$: $m = \frac{-1 \pm \sqrt{1-4\lambda}}{2}$

$\therefore y = C_1 \exp\left(\frac{-1+\sqrt{1-4\lambda}}{2} x\right) + C_2 \exp\left(\frac{-1-\sqrt{1-4\lambda}}{2} x\right)$

$\Rightarrow C_1 = C_2 = 0$ Trivial

$1-4\lambda < 0$: $m = \frac{-1}{2} \pm \frac{1}{2} i \sqrt{1-4\lambda}$

$\therefore y = e^{-\frac{1}{2}x}\left(c_1 \cos\left(\frac{x}{2}\sqrt{1-4\lambda}\right) + C_2 \sin\left(\frac{x}{2}\sqrt{1-4\lambda}\right)\right)$

$\Rightarrow C_1 = 0$, $e^{-1} C_2 \sin\left(\sqrt{1-4\lambda}\right) = 0 \Rightarrow \sqrt{1-4\lambda} = n\pi$

$\lambda_n = \frac{1-n^2\pi^2}{4}$

$\therefore y_n(x) = C_1 e^{-\frac{1}{2}x} \sin\left(\frac{x}{2}\sqrt{1-4\lambda}\right)$

b) $y'' + y' + \lambda y = 0$ $\quad \mu(x) = e^{\int dx} = e^x$

$\frac{d}{dx}\left(e^x y'\right) + e^x \lambda y = 0$

$r(x) = e^x$, $q(x) = 0$, $p(x) = e^x$

c) $\sin\left(\frac{x}{2}\sqrt{1-4\lambda_n}\right) = \sin\left(\frac{x}{2}\sqrt{1-\cancel{4}\left(\frac{1-n^2\pi^2}{\cancel{4}}\right)}\right) = \sin\left(\frac{x}{2}n\pi\right)$

$$\int_0^2 e^x \sin\left(\tfrac{x}{2}n\pi\right)\sin\left(\tfrac{x}{2}m\pi\right)dx$$

$\sin x \sin y = \frac{\cos(x-y)-\cos(x+y)}{2}$

$$=\frac{1}{2}\int_0^2 e^x\left[\sin\left(\frac{x}{2}\pi(n-m)\right)-\sin\left(\frac{x}{2}\pi(n+m)\right)\right]dx$$

$$=\frac{1}{2}\left(\int_0^2 e^x\sin\left(\frac{x}{2}\pi(n-m)\right)dx-\int_0^2 e^x\sin\left(\frac{x}{2}\pi(n+m)\right)dx\right)$$

if $n=m$

$$=\frac{1}{2}\left(\int_0^2 e^x \cancel{\sin\left(\frac{x}{2}\pi(n-n)\right)} dx-\int_0^2 e^x\sin\left(\frac{x}{2}\pi\cdot 2n\right)dx\right) = \frac{-1}{2}\int_0^2 e^x \sin(2n\pi x)dx$$

$$=\frac{-1}{2}\left[\frac{e^x}{1+4n^2\pi^2}\left(\cancel{\sin(2n\pi x)}-2n\pi\cos(2n\pi x)\right)\right]_0^2$$

$$=\frac{1}{2}$$

II) Consider the regular Sturm-Liouville problem

$$\frac{d}{dx}[(1+x^2)y'] + \frac{\lambda}{1+x^2}y = 0, \qquad y(0) = 0, \quad y(1) = 0$$

i) **Find the eigenvalues and eigenfunctions of the boundary-value problem. [Hint: Let $x = \tan\theta$ and then use the Chain Rule.]**

ii) **Give an orthogonality relation.**