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Ordinary Differential Equations (MATH 202)

Assignment 1

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1. Obtain the general solution of the following DEs

(a) $y''' + y'' - 4y' + 2y = 0$

Solution.

$$y''' + y'' - 4y' + 2y = 0 \tag{1}$$
$$\lambda^3 + \lambda^2 - 4\lambda + 2 = 0 \tag{2}$$

By using the cubic formula, we get

$$\lambda_1 = 1 \tag{3}$$
$$\lambda_2 = -\sqrt{3} - 1 \tag{4}$$
$$\lambda_3 = \sqrt{3} - 1 \tag{5}$$

Then

$$y = c_1 e^x + c_2 e^{\left(\sqrt{3}-1\right)x} + c_3 e^{\left(-\sqrt{3}-1\right)x} \tag{6}$$

■

(b) $y^{(4)} + 4y^{(2)} = 0$

Solution.

$$\lambda^4 + 4\lambda^2 = 0 \tag{7}$$
$$\lambda^2\left(\lambda^2 + 4\right) = 0 \tag{8}$$
$$\lambda_1 = 0 \tag{9}$$
$$\lambda_2 = 0 \tag{10}$$
$$\lambda_3 = 2i \tag{11}$$
$$\lambda_4 = -2i \tag{12}$$

Then

$$y = c_1 + c_2 x + c_3 \cos(2x) + c_4 \sin(2x)\,. \tag{13}$$

■

(c) $x(x-2)y'' + 2(x-1)y' - 2y = 0$; use $y_1 = 1 - x$

Solution.

$$x(x-2)y'' + 2(x-1)y' - 2y = 0 \tag{14}$$
$$y'' + 2\frac{x-1}{x(x-2)}y' - \frac{2}{x(x-2)}y = 0. \tag{15}$$

$$y = c_1y_1 + c_2y_2 \tag{16}$$
$$y_2 = y_1 \int \frac{dx}{\mu(x) \cdot y_1^2} \tag{17}$$
$$\mu(x) = \exp \int p(x)dx \tag{18}$$
$$= \exp \int \frac{2(x-1)}{x(x-2)}dx \tag{19}$$
$$= \exp 2 \int \frac{(x-1)}{x(x-2)}dx \tag{20}$$
$$= \exp 2 \left(\int \frac{x}{x(x-2)}dx - \int \frac{1}{x(x-2)}dx \right) \tag{21}$$
$$= \exp 2 \left(\int \frac{dx}{(x-2)} - \int \frac{dx}{x(x-2)} \right) \tag{22}$$
$$= \exp 2 \left(\ln|x-2| - \int \frac{dx}{x(x-2)} \right) \tag{23}$$

Using partial fraction decomposition

$$= \exp 2 \left(\ln|x-2| - \int \frac{A}{x} + \frac{B}{x-2}dx \right) \tag{24}$$
$$= \exp 2 \left(\ln|x-2| - \int \frac{A}{x} + \frac{B}{x-2}dx \right) \tag{25}$$
$$= \exp 2 \left(\ln|x-2| - \int -\frac{1}{2x} + \frac{1}{2(x-2)}dx \right) \tag{26}$$
$$= \exp 2 \left(\ln|x-2| + \int \frac{1}{2x} - \frac{1}{2(x-2)}dx \right) \tag{27}$$
$$= \exp 2 \left(\ln|x-2| + \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x-2| \right) \tag{28}$$
$$= \exp 2 \left(\frac{1}{2}\ln|x-2| + \frac{1}{2}\ln|x| \right) \tag{29}$$
$$= \exp(\ln|x-2| + \ln|x|) \tag{30}$$
$$= \exp(\ln|x-2||x|) \tag{31}$$
$$= \exp(\ln[x(x-2)]) \tag{32}$$
$$= x(x-2) \tag{33}$$

Substituting back

$$y_2 = (1-x) \int \frac{dx}{x(x-2)(1-x)^2} \tag{34}$$

Using partial fraction decomposition

$$= (1-x)\int \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(1-x)^2}dx \tag{35}$$
$$= (1-x)\int -\frac{1}{2x} + \frac{1}{2(x-2)} - \frac{1}{(1-x)^2}dx \tag{36}$$
$$= (1-x)\left(-\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| - \frac{1}{1-x}\right) \tag{37}$$
$$= (1-x)\left(\frac{1}{2}\ln\frac{|x-2|}{|x|} - \frac{1}{1-x}\right) \tag{38}$$
$$= (1-x)\left(\frac{1}{2}\ln\left|\frac{x-2}{x}\right| - \frac{1}{1-x}\right) \tag{39}$$
$$= (1-x)\left(\ln\sqrt{\left|\frac{x-2}{x}\right|} - \frac{1}{1-x}\right) \tag{40}$$
$$= (1-x)\ln\sqrt{\left|\frac{x-2}{x}\right|} - 1 \tag{41}$$

Then

$$y = c_1(1-x) + c_2\left((1-x)\ln\sqrt{\left|\frac{x-2}{x}\right|} - 1\right). \tag{42}$$

■

(d) $y'' - 4y = \sin^2(x)$

Solution.

$$y'' - 4y = \sin^2(x) \tag{43}$$
$$\lambda^2 - 4 = 0 \tag{44}$$
$$\lambda^2 = 4 \tag{45}$$
$$\lambda_1 = 2 \tag{46}$$
$$\lambda_2 = -2 \tag{47}$$

The homogenous solution is

$$\boxed{y_h = c_1e^{2x} + c_2e^{-2x}} \tag{48}$$

For the particular solution

$$y'' - 4y = \sin^2(x) \tag{49}$$
$$= \frac{1}{2} - \frac{1}{2}\cos(2x) \tag{50}$$

Then

$$y_p = A + B\cos(2x) \tag{51}$$
$$y_p' = -2B\sin(2x) \tag{52}$$
$$y_p'' = -4B\cos(2x) \tag{53}$$
$$\tag{54}$$

Substituting back

$$-4B\cos(2x) - 4(A + B\cos(2x)) = \frac{1}{2} - \frac{1}{2}\cos(2x) \tag{55}$$

Then

$$-4A = \frac{1}{2} \tag{56}$$

$$\boxed{A = -\frac{1}{8}} \tag{57}$$

$$-4B - 4B = -\frac{1}{2} \tag{58}$$

$$-8B = -\frac{1}{2} \tag{59}$$

$$\boxed{B = \frac{1}{16}} \tag{60}$$

The particular solution is

$$y_p = -\frac{1}{8} + \frac{1}{16}\cos(2x) \tag{61}$$

The general solution is

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{8} + \frac{1}{16}\cos(2x). \tag{62}$$

■

(e) $y'' - 4y' + 3y = x$; use $y_1 = e^{3x}$

Solution.

$$y = c_1y_1 + c_2y_2 \tag{63}$$

$$y_2 = y_1\int \frac{dx}{\mu(x)\cdot y_1^2} \tag{64}$$

$$\mu(x) = \exp\int p(x)dx \tag{65}$$

$$= \exp\int -4dx \tag{66}$$

$$= \exp -4x \tag{67}$$

$$= e^{-4x} \tag{68}$$

Substituting back

$$y_2 = e^{3x}\int \frac{e^{4x}}{e^{6x}}dx \tag{69}$$

$$= e^{3x}\int e^{-2x}dx \tag{70}$$

$$= e^{3x}\left(-\frac{1}{2}e^{-2x}\right) \tag{71}$$

$$= -\frac{1}{2}e^{x} \tag{72}$$

Then

$$y = c_1 e^{3x} + c_2 e^{x}. \tag{73}$$

■

(f) $y'' + 5y' + 6y = e^{2x}\cos x$

Solution.

$$\begin{aligned}
y'' + 5y' + 6y &= e^{2x}\cos x &(74)\\
\lambda^2 + 5\lambda + 6 &= 0 &(75)\\
(\lambda+2)(\lambda+3) &= 0 &(76)\\
\lambda_1 &= -2 &(77)\\
\lambda_2 &= -3 &(78)
\end{aligned}$$

The homogenous solution is

$$\boxed{y_h = c_1 e^{-2x} + c_2 e^{-3x}} \tag{79}$$

For the particular solution

$$\begin{aligned}
y'' + 5y' + 6y &= e^{2x}\cos x &(80)\\
&= e^{2x}\left(\frac{e^{ix}+e^{-ix}}{2}\right) &(81)\\
&= \frac{1}{2}e^{2x+ix} + \frac{1}{2}e^{2x-ix} &(82)\\
&= \frac{1}{2}e^{x(2+i)} + \frac{1}{2}e^{x(2-i)} &(83)
\end{aligned}$$

Then

$$\begin{aligned}
y_p &= Ae^{x(2+i)} + Be^{x(2-i)} &(84)\\
y_p' &= (2+i)Ae^{x(2+i)} + (2-i)Be^{x(2-i)} &(85)\\
y_p'' &= (2+i)^2Ae^{x(2+i)} + (2-i)^2Be^{x(2-i)} &(86)\\
&= (4+4i-1)Ae^{x(2+i)} + (4-4i-1)Be^{x(2-i)} &(87)\\
&= (3+4i)Ae^{x(2+i)} + (3-4i)Be^{x(2-i)} &(88)
\end{aligned}$$

Substituting back

$$\begin{array}{rl} (3+4i)Ae^{x(2+i)} & +(3-4i)Be^{x(2-i)} \\ +5\big[(2+i)Ae^{x(2+i)} & +(2-i)Be^{x(2-i)}\big] \\ +6\big[Ae^{x(2+i)} & +Be^{x(2-i)}\big] \end{array} = \frac{1}{2}e^{x(2+i)} + \frac{1}{2}e^{x(2-i)} \tag{89}$$

$$\begin{array}{rl} (3+4i)Ae^{x(2+i)} & +(3-4i)Be^{x(2-i)} \\ +(10+5i)Ae^{x(2+i)} & +(10-5i)Be^{x(2-i)} \\ +6Ae^{x(2+i)} & +6Be^{x(2-i)} \end{array} = \frac{1}{2}e^{x(2+i)} + \frac{1}{2}e^{x(2-i)} \tag{90}$$

$$(19+9i)Ae^{x(2+i)} + (19-9i)Be^{x(2+i)} = \frac{1}{2}e^{x(2+i)} + \frac{1}{2}e^{x(2-i)} \tag{91}$$

$$\implies \begin{cases} (19+9i)A &= \frac{1}{2} \\ (19-9i)B &= \frac{1}{2} \end{cases} \tag{92}$$

$$\begin{cases} A &= \frac{1}{2(19+9i)} = \frac{19-9i}{884} \\ B &= \frac{1}{2(19-9i)} = \frac{19+9i}{884} \end{cases} \tag{93}$$

The particular solution is

$$y_p = \frac{19-9i}{884}e^{x(2+i)} + \frac{19+9i}{884}e^{x(2-i)} \tag{94}$$

$$= \begin{array}{l} \frac{19-9i}{884}e^{2x}(\cos x + i\sin x) \\ + \frac{19+9i}{884}e^{2x}(\cos x - i\sin x) \end{array} \tag{95}$$

$$= \begin{array}{l} \frac{19}{884}e^{2x}\cos x + i\frac{19}{884}e^{2x}\sin x \\ -i\frac{9}{884}e^{2x}\cos x + \frac{9}{884}e^{2x}\sin x \\ +\frac{19}{884}e^{2x}\cos x - i\frac{19}{884}e^{2x}\sin x \\ +i\frac{9}{884}e^{2x}\cos x + \frac{9}{884}e^{2x}\sin x \end{array} \tag{96}$$

$$\boxed{= \frac{9}{442}e^{2x}\sin x + \frac{19}{442}e^{2x}\cos x} \tag{97}$$

The general solution is

$$\boxed{y = c_1e^{-2x} + c_2e^{-3x} + \frac{9}{442}e^{2x}\sin x + \frac{19}{442}e^{2x}\cos x.} \tag{98}$$

■

(g) $y'' + y = \sec x \tan x$

Solution.

$$y'' + y = \sec x \tan x \tag{99}$$
$$\lambda^2 + 1 = 0 \tag{100}$$
$$\lambda^2 = -1 \tag{101}$$
$$\lambda_1 = i \tag{102}$$
$$\lambda_2 = -i \tag{103}$$

The homogenous solution is

$$\boxed{y_h = c_1\cos x + c_2\sin x} \tag{104}$$

For the particular solution

$$y_p = u_1y_1 + u_2y_2 \tag{105}$$
$$u_i = \int \frac{W_i}{W}dx \tag{106}$$
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{107}$$
$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \tag{108}$$
$$= \cos^2 x + \sin^2 x \tag{109}$$
$$= 1 \tag{110}$$

$$
\begin{aligned}
W_1 &= \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} && (111)\\
&= \begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix} && (112)\\
&= -\sin x \sec x \tan x && (113)\\
&= -\tan^2 x && (114)\\
W_2 &= \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} && (115)\\
&= \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix} && (116)\\
&= \cos x \sec x \tan x && (117)\\
&= \tan x && (118)\\
& && (119)
\end{aligned}
$$

Then

$$
\begin{aligned}
y_p &= \cos x \int \frac{-\tan^2 x}{1} dx + \sin x \int \frac{\tan x}{1} dx && (120)\\
&= -\cos x\,(\tan x - x) - \sin x \ln|\cos x| && (121)\\
&= -\cos x \tan x + x \cos x - \sin x \ln|\cos x| && (122)\\
&= -\sin x + x\cos x - \sin x \ln|\cos x| && (123)\\
& && (124)
\end{aligned}
$$

The general solution is

$$\boxed{y = c_1 \cos x + c_2 \sin x - \sin x + x \cos x - \sin x \ln|\cos x|} \qquad (125)$$

■

(h) $y'' + 3y' + 2y = \frac{1}{1+e^x}$

Solution.

$$
\begin{aligned}
y'' + 3y' + 2y &= \frac{1}{1+e^x} && (126)\\
\lambda^2 + 3\lambda + 2 &= 0 && (127)\\
(\lambda+1)(\lambda+2) &= 0 && (128)\\
\lambda_1 &= -1 && (129)\\
\lambda_2 &= -2 && (130)
\end{aligned}
$$

The homogenous solution is

$$\boxed{y_h = c_1 e^{-x} + c_2 e^{-2x}} \qquad (131)$$

For the particular solution

$$y_p = u_1 y_1 + u_2 y_2 \tag{132}$$
$$u_i = \int \frac{W_i}{W} dx \tag{133}$$
$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{134}$$
$$= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} \tag{135}$$
$$= e^{-x} \cdot -2e^{-2x} - e^{-2x} \cdot -e^{-x} \tag{136}$$
$$= -2e^{-3x} + e^{-3x} \tag{137}$$
$$= -e^{-3x} \tag{138}$$
$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \tag{139}$$
$$= \begin{vmatrix} 0 & e^{-2x} \\ \frac{1}{1+e^x} & -2e^{-2x} \end{vmatrix} \tag{140}$$
$$= -\frac{1}{1+e^x} e^{-2x} \tag{141}$$
$$= -\frac{e^{-2x}}{1+e^x} \tag{142}$$
$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} \tag{143}$$
$$= \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{1}{1+e^x} \end{vmatrix} \tag{144}$$
$$= e^{-x} \cdot \frac{1}{1+e^x} \tag{145}$$
$$= \frac{e^{-x}}{1+e^x} \tag{146}$$

Then

$$y_p = e^{-x} \int \frac{-e^{-2x}}{1+e^x} \cdot \frac{1}{-e^{-3x}} dx + e^{-2x} \int \frac{e^{-x}}{1+e^x} \cdot \frac{1}{-e^{-3x}} dx \tag{147}$$
$$= e^{-x} \int \frac{e^{-2x}}{1+e^x} \cdot e^{3x} dx - e^{-2x} \int \frac{e^{-x}}{1+e^x} \cdot e^{3x} dx \tag{148}$$
$$= e^{-x} \int \frac{e^x}{1+e^x} dx + e^{-2x} \int \frac{e^{2x}}{1+e^x} dx \tag{149}$$
$$= e^{-x} \int \frac{e^x}{1+e^x} dx + e^{-2x} \int \frac{e^{2x}}{1+e^x} dx \tag{150}$$

Let $u = 1 + e^x$ then $du = e^x dx$

$$= e^{-x} \int \frac{1}{u} du + e^{-2x} \int \frac{u-1}{u} du \tag{151}$$
$$= e^{-x} \ln u + e^{-2x} (u - \ln u) \tag{152}$$

Substituting u

$$= e^{-x} \ln(1+e^x) + e^{-2x} (1 + e^x - \ln(1+e^x)) \tag{153}$$
$$= e^{-x} \ln(1+e^x) + e^{-2x} + e^{-x} - e^{-2x} \ln(1+e^x) \tag{154}$$

The general solution is

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + e^{-x}\ln(1+e^x) + e^{-2x} + e^{-x} - e^{-2x}\ln(1+e^x)} \tag{155}$$

■

(i) $x^2y'' - 4xy' + 4y = 0$

Solution. Let $y = x^m$

$$y' = mx^{m-1} \tag{156}$$
$$y'' = m(m-1)x^{m-2} \tag{157}$$

Substituting back

$$x^2m(m-1)x^{m-2} - 4xmx^{m-1} + 4x^m = 0 \tag{158}$$
$$m(m-1)x^m - 4mx^m + 4x^m = 0 \tag{159}$$
$$m(m-1) - 4m + 4 = 0 \tag{160}$$
$$m^2 - m - 4m + 4 = 0 \tag{161}$$
$$m^2 - 5m + 4 = 0 \tag{162}$$
$$m_1 = 1 \tag{163}$$
$$m_2 = 4 \tag{164}$$
$$\boxed{y = c_1x + c_2x^4}. \tag{165}$$

■

(j) $(2x-5)^2y'' - 2(2x-5)y' + 4y = \frac{8x-20}{\ln^2(2x-5)+1}$

Solution. Using substitution $u = (2x-5)$

$$y' = \frac{dy}{du}\frac{du}{dx} \tag{166}$$
$$= \frac{dy}{du}\cdot 2 \tag{167}$$
$$y'' = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2} \tag{168}$$
$$= \frac{d^2y}{du^2}\cdot 4 + \frac{dy}{du}\cdot 0 \tag{169}$$
$$= 4\frac{d^2y}{du^2} \tag{170}$$

Substituting back

$$4u^2y'' - 4uy' + 4y = \frac{4u}{\ln^2 u + 1} \tag{171}$$

Using Cauchy-Euler method

$$4m^2 - 8m + 4 = 0 \tag{172}$$
$$4(m-1)^2 = 0 \tag{173}$$
$$m_1 = 1 \tag{174}$$
$$m_2 = 1 \tag{175}$$

$$\boxed{y_h = c_1 u + c_2 u \ln|u|} \tag{176}$$

For particular solution

$$4u^2 y'' - 4uy' + 4y = \frac{4u}{\ln^2 u + 1} \tag{177}$$

$$y'' - \frac{1}{u}y' + \frac{1}{u^2}y = \frac{1}{4u(\ln^2 u + 1)} \tag{178}$$

Using variation of parameters

$$y_p = u_1 y_1 + u_2 y_2 \tag{179}$$

$$u_i = \int \frac{W_i}{W} du \tag{180}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \tag{181}$$

$$= \begin{vmatrix} u & u\ln|u| \\ 1 & 1+\ln|u| \end{vmatrix} \tag{182}$$

$$= u(1+\ln|u|) - u\ln|u| \tag{183}$$

$$= u + u\ln|u| - u\ln|u| \tag{184}$$

$$= u \tag{185}$$

Where $u \neq 0$

$$W_1 = \begin{vmatrix} 0 & u\ln|u| \\ \frac{1}{4u(\ln^2 u+1)} & 1+\ln|u| \end{vmatrix} \tag{186}$$

$$= -\frac{1}{4u(\ln^2 u + 1)} u\ln|u| \tag{187}$$

$$= -\frac{\ln|u|}{4(\ln^2 u + 1)} \tag{188}$$

$$W_2 = \begin{vmatrix} u & 0 \\ 1 & \frac{1}{4u(\ln^2 u+1)} \end{vmatrix} \tag{189}$$

$$= \frac{1}{4u(\ln^2 u + 1)} u \tag{190}$$

$$= \frac{1}{4(\ln^2 u + 1)} \tag{191}$$

Then

$$y_p = u \int \frac{-\frac{\ln|u|}{4(\ln^2 u+1)}}{u} du + u\ln|u| \int \frac{\frac{1}{4(\ln^2 u+1)}}{u} du \tag{192}$$

$$= -u \int \frac{\ln|u|}{4u(\ln^2 u + 1)} du + u\ln|u| \int \frac{1}{4u(\ln^2 u + 1)} du \tag{193}$$

$$= -\frac{u}{4} \int \frac{\ln u}{u\ln^2 u + u} du + \frac{u\ln|u|}{4} \int \frac{1}{u(\ln^2 u + 1)} du \tag{194}$$

$$= -\frac{u}{4} \cdot \frac{\ln|\ln^2 u + 1|}{8} + \frac{u\ln|u|}{4} \tan^{-1}(\ln^2 u) \tag{195}$$

$$\tag{196}$$

Substituting back

$$y = c_1 u + c_2 u \ln|u| - \frac{u}{4} \cdot \frac{\ln|\ln^2 u + 1|}{8} + \frac{u \ln|u|}{4} \tan^{-1}(\ln^2 u) \tag{197}$$

Remove absolute values as u is strictly > 0

$$= c_1 u + c_2 u \ln u - \frac{u}{4} \cdot \frac{\ln(\ln^2 u + 1)}{8} + \frac{u \ln u}{4} \tan^{-1}(\ln^2 u) \tag{198}$$

$$= c_1 u + c_2 u \ln u - \frac{1}{32} u \ln(\ln^2 u + 1) - \frac{1}{8} u \ln^2 u + \frac{1}{4} u \ln u \tan^{-1}(\ln^2 u) \tag{199}$$

Substitute back $u = 2x - 5$

$$\boxed{y = \begin{array}{l} c_1(2x-5) + c_2(2x-5)\ln(2x-5) \\ -\frac{1}{32}(2x-5)\ln(\ln^2(2x-5)+1) \\ -\frac{1}{8}(2x-5)\ln^2(2x-5) \\ +\frac{1}{4}(2x-5)\ln(2x-5)\tan^{-1}(\ln^2(2x-5)) \end{array} ; x > \frac{5}{2}}. \tag{200}$$

■

2. Given that the roots of the characteristic equation of a specific higher order DE with constant coefficients as
$$\lambda = (\pm 3, 2 \pm 3i, 3, 4, 3)$$
   (a) What is the order of this differential equation?

   Solution. The order of the differential equation is 7. ■

   (b) Write the form of the homogenous solution?

   Solution.
$$y = c_1 e^{3x} + c_2 e^{-3x} + c_3 e^{2x} \cos(3x) + c_4 e^{2x} \sin(3x) + c_5 x e^{3x} + c_6 e^{4x} + c_7 x^2 e^{3x}$$
   ■

3. Solve the following system of ODE's

$$(D+1)u-(D+1)v=e^t$$
$$(D-1)u+(2D+1)v=5.$$

Solution.

$$\left[\begin{array}{cc|c} D+1 & -D-1 & e^t \\ D-1 & 2D+1 & 5 \end{array}\right] \tag{201}$$

$$\xrightarrow{R_1=\frac{R_1}{D+1}} \left[\begin{array}{cc|c} 1 & -1 & \frac{e^t}{D+1} \\ D-1 & 2D+1 & 5 \end{array}\right] \tag{202}$$

$$\xrightarrow{R_1=\frac{R_2}{2D+1}} \left[\begin{array}{cc|c} 1+\frac{D-1}{2D+1} & 0 & \frac{e^t}{D+1}+\frac{5}{2D+1} \\ D-1 & 2D+1 & 5 \end{array}\right] \tag{203}$$

$$= \left[\begin{array}{cc|c} \frac{3D}{2D+1} & 0 & \frac{(2D+1)e^t+(D+1)5}{(D+1)(2D+1)} \\ D-1 & 2D+1 & 5 \end{array}\right] \tag{204}$$

$$\xrightarrow{R_1=R_1\cdot\frac{2D+1}{3D}} \left[\begin{array}{cc|c} 1 & 0 & \frac{(2D+1)e^t+(D+1)5}{(D+1)(3D)} \\ D-1 & 2D+1 & 5 \end{array}\right] \tag{205}$$

$$\tag{206}$$

Then

$$u=\frac{(2D+1)e^t+(D+1)5}{(D+1)(3D)} \tag{207}$$
$$(D+1)(3D)u=(2D+1)e^t+(D+1)5 \tag{208}$$
$$3D^2u+3Du=2De^t+e^t+5D+5 \tag{209}$$
$$3D^2u+3Du=3e^t+5 \tag{210}$$
$$\boxed{D^2u+Du=e^t+\frac{5}{3}} \tag{211}$$

$$u''+u'=e^t+\frac{5}{3} \tag{212}$$
$$\lambda^2+\lambda=0 \tag{213}$$
$$\lambda(\lambda+1)=0 \tag{214}$$
$$\lambda_1=0 \tag{215}$$
$$\lambda_2=-1 \tag{216}$$

The homogenous solution is

$$\boxed{u_h=c_1+c_2e^{-t}} \tag{217}$$

For the particular solution

$$u_p=Ae^t+Bt \tag{218}$$
$$u_p'=Ae^t+B \tag{219}$$
$$u_p''=Ae^t \tag{220}$$

Substituting back

$$Ae^t + B + Ae^t = e^t + \frac{5}{3} \tag{221}$$

$$2Ae^t + B = e^t + \frac{5}{3} \tag{222}$$

$$\implies \begin{cases} 2A &= 1 \implies A = \frac{1}{2} \\ B &= \frac{5}{3} \end{cases} \tag{223}$$

The particular solution is

$$\boxed{u_p = \frac{1}{2}e^t + \frac{5}{3}t} \tag{224}$$

The general solution is

$$\boxed{u = c_1 + c_2 e^{-t} + \frac{1}{2}e^t + \frac{5}{3}t} \tag{225}$$

Substituting back

$$(D+1)\left(c_1 + c_2 e^{-t} + \frac{1}{2}e^t + \frac{5}{3}t\right) - (D+1)v = e^t \tag{226}$$

$$-c_2 e^{-t} + \frac{1}{2}e^t + \frac{5}{3} + c_1 + c_2 e^{-t} + \frac{1}{2}e^t + \frac{5}{3}t - (D+1)v = e^t \tag{227}$$

$$e^t + \frac{5}{3} + c_1 + \frac{5}{3}t - (D+1)v = e^t \tag{228}$$

$$-(D+1)v = -\frac{5}{3} - c_1 - \frac{5}{3}t \tag{229}$$

$$(D+1)v = \frac{5}{3} + c_1 + \frac{5}{3}t \tag{230}$$

$$Dv + v = \frac{5}{3} + c_1 + \frac{5}{3}t \tag{231}$$

$$v' + v = \frac{5}{3} + c_1 + \frac{5}{3}t \tag{232}$$

$$\lambda + 1 = 0 \tag{233}$$

$$\lambda = -1 \tag{234}$$

The homogenous solution is

$$\boxed{v_h = c_3 e^{-t}} \tag{235}$$

For the particular solution

$$v_p = At + B \tag{236}$$

$$v_p' = A \tag{237}$$

Substituting back

$$At + A + B = \frac{5}{3} + c_1 + \frac{5}{3}t \tag{238}$$

$$\implies \begin{cases} A &= \frac{5}{3} \\ B &= c_1 \end{cases} \tag{239}$$

The particular solution is

$$v_p = c_1 + \frac{5}{3}t \tag{240}$$

The general solution is

$$\boxed{v = c_1 + c_3 e^{-t} + \frac{5}{3}t} \tag{241}$$

Substituting back

$$(D-1)u + (2D+1)v = 5 \tag{242}$$

$$\begin{aligned} & (D-1)\left(c_1 + c_2 e^{-t} + \tfrac{1}{2}e^t + \tfrac{5}{3}t\right) \\ + \quad & (2D+1)\left(c_1 + c_3 e^{-t} + \tfrac{5}{3}t\right) \end{aligned} \quad = \tag{243}$$

$$\begin{aligned} -c_2 e^{-t} + \tfrac{1}{2}e^t + \tfrac{5}{3} - c_1 - c_2 e^{-t} - \tfrac{1}{2}e^t - \tfrac{5}{3}t \\ -2c_3 e^{-t} + \tfrac{10}{3} + c_1 + c_3 e^{-t} + \tfrac{5}{3}t \end{aligned} \quad = \tag{244}$$

$$-2c_2 e^{-t} + 5 - c_3 e^{-t} = \tag{245}$$

$$-2c_2 e^{-t} - c_3 e^{-t} = 0 \tag{246}$$

$$-c_3 e^{-t} = 2c_2 e^{-t} \tag{247}$$

$$\implies c_3 = -2c_2. \tag{248}$$

The solution to the system of ODE's is

$$\begin{cases} u &= c_1 + c_2 e^{-t} + \frac{1}{2}e^t + \frac{5}{3}t \\ v &= c_1 - 2c_2 e^{-t} + \frac{5}{3}t \end{cases} \tag{249}$$

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