Show that

A)

$$\int_{-1}^{1} x\,P_n(x)P_{n-1}(x)dx = \frac{2n}{4n^2-1}, n = 1, 2, 3, \dots$$

$$x P_n(x) = \frac{(n+1)P_{n+1}(x) + nP_{n-1}(x)}{2n+1}, \quad \int_{-1}^{1} P_n(x)P_m(x) = \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases}$$

$$\int_{-1}^{1} xP_n(x)P_{n-1}(x)dx = \int_{-1}^{1} \frac{(n+1)P_{n+1}(x)+nP_{n-1}(x)}{2n+1} P_{n-1}(x)dx$$

$$= \frac{1}{2n+1}\int_{-1}^{1} [\cancel{(n+1)P_{n+1}(x)} + nP_{n-1}(x)]\cancel{P_{n-1}(x)}dx$$

$$= \frac{n}{2n+1}\int_{-1}^{1} P_{n-1}^2(x)\,dx = \frac{n}{2n+1}\cdot\frac{2}{2(n-1)+1} = \frac{2n}{(2n+1)(2n-1)} = \frac{2n}{4n^2-1}$$

QED

B)

$$\int_{-1}^{1} x^2\,P_{n+1}(x)P_{n-1}(x)dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)},$$

$$n = 1, 2, 3, \dots$$

$$x P_n(x) = \frac{(n+1)P_{n+1}(x) + nP_{n-1}(x)}{2n+1}, \quad \int_{-1}^{1} P_n(x)P_m(x) = \begin{cases} 0, & n \neq m \\ \frac{2}{2n+1}, & n = m \end{cases}$$

$$\int_{1}^{1} x^2 P_{n+1}(x)P_{n-1}(x)dx = \int_{-1}^{1} \boxed{xP_{n+1}(x)}\boxed{xP_{n-1}(x)}dx$$

$$= \frac{1}{(2(n+1)+1)(2(n-1)+1)}\int_{-1}^{1}\left(\cancel{(n+2)P_{n+2}(x)} + (n+1)P_n(x)\right)\left(nP_n(x) + \cancel{(n-1)P_{n-2}(x)}\right)dx$$

$(2n+3)(2n-1)$

$$= \frac{n(n+1)}{(2n+3)(2n-1)}\int_{-1}^{1} P_n^2(x)dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$

QED

c)

$$\int_{-1}^{1}(1-x^2)\,P'_n(x)P'_m(x)dx = \begin{cases} 0 & \text{if } m \neq n \\ \dfrac{2n(n+1)}{2n+1} & \text{if } m = n \end{cases}$$

Legendre's Equation: $(1-x^2)y'' - 2xy' = -n(n+1)y$

$$D_x\left[(1-x^2)y'\right] = -n(n+1)y \quad ; \; y = P_n(x)$$

$$\int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx$$

integration by parts

$$= \cancel{(1-x^2)P_n'(x)P_m(x)\Big|_{-1}^{1}}^{\,0} - \int_{-1}^{1} D_x\left[(1-x^2)P_n'(x)P_m(x)\right]dx$$

$$= n(n+1)\int_{-1}^{1}P_n(x)P_m(x)\,dx = \begin{cases} 0, & n \neq m \\ \dfrac{2n(n+1)}{2n+1}, & n=m \end{cases}$$

QED

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n dx$$

Leibniz's theorem:

$$D_x^n(uv) = \sum_{k=0}^{n}\binom{n}{k}D_x^k u \cdot D_x^{n-k} v ,$$

Rodrigues' formula:

$$P_n(x) = \frac{D_x^n(x^2-1)^n}{2^n n!}$$

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{1}{2^n n!}\int_{-1}^{1} f(x)\,D_x^n(x^2-1)^n\,dx$$

Integration by parts

	D	I
$+$	$f(x)$	$D_x^n(x^2-1)^n$
$-$	$f'(x)$	$D_x^{n-1}(x^2-1)^n$
$+$	$f''(x)$	$D_x^{n-1}(x^2-1)^n$
	$\vdots$	$\vdots$

$$\boxed{(-1)^n f^{(n)} \qquad (x^2-1)^n}$$

$$= \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x^2-1)^n\,dx$$

All other terms will include (x^2-1) which evaluates to zero when substiting the limits

QED

Hence deduce that

$$\int_{-1}^{1} x^{n+2k} P_n(x)\,dx = \frac{(n+2k)!\,\Gamma(k+1/2)}{2^n(2k)!\,\Gamma(n+k+3/2)}, \quad k = 0, 1, 2, \ldots$$

$$\frac{(-1)^n}{2^n\, n!}\int_{-1}^{1} f^{(n)}(x^2-1)^n\,dx, \quad \beta(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$\int_{-1}^{1} x^{n+2k} P_n(x)\,dx = \frac{(-1)^n}{2^n\, n!}\int_{-1}^{1} D_x^n(x^{n+2k})(x^2-1)^n\,dx$$

$$= \frac{\cancel{(-1)^n}}{2^n n!}\cdot\frac{(n+2k)!}{(2k)!}\int_{-1}^{1} x^{2k}\underbrace{\cancel{(x^2-1)^n}}_{(1-x^2)^n}\,dx \qquad D_x^n(x^m) = \frac{m!}{(m-n)!}x^{m-n}\ ; \ m>n$$

$$= \frac{1}{2^n n!}\cdot\frac{(n+2k)!}{(2k)!}\int_{-1}^{1} x^{2k}(1-x^2)^n\,dx \longrightarrow$$

Even function as
$x^{2k} \times (1-x^2)^n = f(x)$
even $\times$ even or odd $=$ even

$$= \frac{(n+2k)!}{2^{n-1} n!\,(2k)!}\int_0^1 x^{2k}(1-x^2)^n\,dx$$

let $x = t^{1/2} \Rightarrow dx = \frac{1}{2}t^{-1/2}\,dt$, $\quad x=0 \to t=0$, $x=1 \to t=1$

$$= \frac{(n+2k)!}{2^n n!\,(2k)!}\int_0^1 t^{k-1/2}(1-t)^n\,dt$$

$x-1 = k-\frac{1}{2} \Rightarrow x = k+\frac{1}{2}$
$y-1 = n \Rightarrow y = n+1$

$$= \frac{(n+2k)!}{2^n n!\,(2k)!}\beta(k+\tfrac{1}{2}, n+1) = \frac{(n+2k)!}{2^n \cancel{n!}(2k)!}\cdot\frac{\Gamma(k+\frac{1}{2})\,\overset{n!}{\cancel{\Gamma(n+1)}}}{\Gamma(k+n+\underbrace{\cancel{1+\tfrac{1}{2}}}_{3/2})}$$

$$= \frac{(n+2k)!\,\Gamma(k+\frac{1}{2})}{2^n(2k)!\,\Gamma(n+k+\frac{3}{2})} \quad \text{QED}$$

Hermite's DE

$y'' - 2xy' + 2ny = 0$, $\quad H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{k!\,(n-2k)!}(2x)^{n-2k}$

$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$, $\quad H_n(x) = (-1)^n e^{x^2} D_x^n e^{-x^2}$

$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,dx = \begin{cases} 0, & n \neq m \\ 2^n n! \sqrt{\pi}, & n = m \end{cases}$, $\quad$ Leibniz's theorem: $D_x^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_x^k u \cdot D_x^{n-k} v$

A) Prove the orthogonality property.

$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,dx = (-1)^m \int_{-\infty}^{\infty} H_n(x) D_x^m e^{-x^2}\,dx$ $\quad$ Integration by Parts

$= \cancel{(-1)^m}(-1)^n \int_{-\infty}^{\infty} e^{-x^2} D_x^m H_n(x)\,dx$ $\quad$ All other terms will include e^{-x^2} which evaluates to zero given the limits

D	I
$H_n(x)$	$D_x^m e^{-x^2}$
⋮	⋮
$(-1)^m D_x^m H_n(x)$	e^{-x^2}

$\exists\, m > n$

$\because H_n(x) \in \mathbb{P}$ $\quad$ set of all polynomials

$\therefore D_x^m H_n(x) = 0$ $\quad$ 1

$\exists\, m = n : (-1)^n \int_{-1}^{1} e^{-x^2} D_x^n H_n(x)\,dx$

$H_n'(x) = (-1)^n D_x\left(e^{x^2} D_x^n e^{-x^2}\right)$

$= (-1)^n\left(D_x^n e^{-x^2} \cdot D_x e^{x^2} + e^{x^2} D_x^{n+1} e^{-x^2}\right)$

$= (-1)^n\left[D_x^n e^{-x^2}\, 2x e^{x^2} + e^{x^2} D_x^{n+1} e^{-x^2}\right]$

$= (-1)^n\left[2x e^{x^2} D_x^n e^{-x^2} - 2e^{x^2} D_x^n\left(x e^{-2x^2}\right)\right]$

$= (-1)^n\left[2x e^{x^2} D_x^n e^{-x^2} - 2e^{x^2} \sum_{k=0}^{n} \binom{n}{k} \underbrace{D_x^k x}_{=0 \text{ if } k>1} D_x^{n-k} e^{-x^2}\right]$

$= (-1)^n\left[\cancel{2x e^{x^2} D_x^n e^{-x^2}} - \cancel{2x e^{x^2} D_x^n e^{-x^2}} - 2n e^{x^2} D_x^{n-1} e^{-x^2}\right]$

$= -(-1)^n 2n e^{x^2} D_x^{n-1} e^{-x^2}$

$= (-1)^{n-1} 2n e^{x^2} D_x^{n-1} e^{-x^2} = \boxed{2n H_{n-1}(x)}$

$$H_n^{(n)}(x) = 2n\, D_x^n H_{n-1}(x) = 2^2 n(n-1) D_x^{n-1} H_{n-2}(x)$$
$$= \boxed{2^n n!}$$

$$\int_{-\infty}^{\infty} e^{-x^2} D_x^n H_n(x)\, dx = 2^n n! \int_{-\infty}^{\infty} e^{-x^2} dx = 2^n n! \sqrt{\pi} \quad (2)$$

From (1) and (2) $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0, & n \neq m \\ 2^n n! \sqrt{\pi}, & n = m \end{cases}$

B) **Derive the recurrence relation**

$$xH_n(x) = nH_{n-1}(x) + \frac{1}{2}H_{n+1}(x)$$

$y'' - 2x y' + 2ny = 0$

$H_n'(x) = 2n H_{n-1}(x)$

$H_n''(x) - 2x H_n'(x) + 2n H_n(x) = 0$

$H_n''(x) = 4n(n-1) H_{n-2}(x)$

$$\cancel{4}\cancel{n}(n-1) H_{n-2}(x) - \cancel{4} x \cancel{n} H_{n-1}(x) + \underset{1/2}{\cancel{2}}\cancel{n} H_n(x) = 0$$

$n \rightarrow n+1$

$$n H_{n-1}(x) - x H_n(x) + \frac{1}{2} H_{n+1}(x) = 0$$

$$\Rightarrow x H_n(x) = n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x) \quad \text{QED}$$

C) **Show that**

$$\int_{-\infty}^{\infty} x\, e^{-x^2} H_n(x) H_m(x) dx = \begin{cases} \sqrt{\pi}\, 2^{n-1}(n-1)!, & m = n-1 \\ \sqrt{\pi}\, 2^n\, (n+1)!, & m = n+1 \\ 0, & \textit{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} x\, e^{-x^2} H_n(x) H_m(x)\, dx$$

$$x H_n(x) = n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x)$$

$$= \int_{-\infty}^{\infty} \left[n H_{n-1}(x) + \frac{1}{2} H_{n+1}(x) \right] e^{-x^2} H_m(x)\, dx$$

$$= n \int_{-\infty}^{\infty} e^{-x^2} H_{n-1}(x) H_m(x)\, dx + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} H_{n+1}(x) H_m(x)\, dx$$

let $m=n-1$: $n\int_{-\infty}^{\infty} e^{-x^2}H_{n-1}(x)H_{n-1}(x)\,dx + \cancel{\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}H_{n+1}(x)H_{n-1}(x)\,dx}$

$$= \sqrt{\pi}\,2^{n-1}\,n(n-1)! = \sqrt{\pi}\,2^{n-1}\,n! \quad (1)$$

let $m=n+1$: $\cancel{n\int_{-\infty}^{\infty} e^{-x^2}H_{n-1}(x)H_{n+1}(x)\,dx} + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}H_{n+1}(x)H_{n+1}(x)\,dx$

$$= \sqrt{\pi}\,\frac{\cancel{2^{n+1}}}{\cancel{2}}\,(n+1)! = \sqrt{\pi}\,2^{n}(n+1)! \quad (2)$$

let $m\ne\begin{cases} n-1 \\ n+1\end{cases}$: $\cancel{n\int_{-\infty}^{\infty} e^{-x^2}H_{n-1}(x)H_m(x)\,dx} + \cancel{\frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}H_{n+1}(x)H_m(x)\,dx} = 0$

$$\therefore\ n\int_{-\infty}^{\infty} e^{-x^2}H_{n-1}(x)H_m(x)\,dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}H_{n+1}(x)H_m(x)\,dx = \begin{cases} \sqrt{\pi}\,2^{n-1}n!, & m=n-1 \\ \sqrt{\pi}\,2^{n}(n+1)!, & m=n+1 \\ 0, & \text{ow} \end{cases}$$

D) Show that

$$\int_x^{\infty} t^{n+1} e^{-t^2} P_n\left(\frac{x}{t}\right) dt = \frac{1}{2^{n+1}} e^{-x^2} H_n(x)$$

$D_x^n e^{-x^2} = (-1)^n e^{-x^2} H_n(x)$, $\quad D_x^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D_x^k u \cdot D_x^{n-k} v$

$P_n(x) = \frac{1}{2^n n!} D_x^n (x^2-1)^n$

$\int_x^{\infty} t^{n+1} e^{-t^2} P_n\left(\frac{x}{t}\right) dt$ $\quad$ let $t = x/u$, $dt = -x/u^2$, $\quad t=x,\ u=1$; $t=\infty,\ u=0$

$x\int_0^1 x^{n+1}/u^{n+1}\, e^{-(x/u)^2} P_n(u) \frac{1}{u^2}\, du$

$x^{n+2}\int_0^1 u^{-(n+3)} e^{-(x/u)^2} P_n(u)\, du$

$\frac{x^{n+2}}{2^n n!}\int_0^1 u^{-(n+3)} e^{-(x/u)^2} D_u^x (u^2-1)^n\, du\, (-1)^n$

D	I
$u^{-(n+3)} e^{-(x/u)^2}$	$D_u^n (u^2-1)^n$
$(-1)^n D_x^n\left(u^{-n-3} e^{-x^2/u^2}\right)$	$(u^2-1)^n$

$$= \frac{x^{n+2}}{2^n\, n!} \int_0^1 (u^2-1)^n D_x^n \left[e^{-\left(\frac{x}{u}\right)^2} u^{-(n+3)} \right] du\, (-1)^n$$

$$= \frac{x^{n+2}}{2^n\, n!} \int_0^1 (u^2-1)^n \sum_{k=0}^{n} \binom{n}{k} D_x^k e^{-\left(\frac{x}{u}\right)^2} D_x^{n-k} u^{-(n+3)}\, du\, (-1)^n$$

$$= \frac{x^{n+2}}{2^n\, n!} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n+k} \int_0^1 (u^2-1)^n H_k\left(\frac{x^2}{u^2}\right) e^{-x^2/u^2}$$

$$= \frac{\cancel{x^{n+2}}}{2^n\, \cancel{n!}} \cdot \frac{\cancel{n!}}{2} \frac{e^{-x^2} H_n(x)}{\cancel{x^{n+2}}} = \boxed{\frac{1}{2^{n+1}} e^{-x^2} H_n(x)}$$

QED

I) Evaluate

$$\left(J_{3/2}(x)\right)^2 + \left(J_{-3/2}(x)\right)^2$$

$2\nu J_\nu(x) = x\left[J_{\nu-1}(x) + J_{\nu+1}(x)\right]$, $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x$, $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$

$\nu = \frac{1}{2}$

$$\frac{J_{\frac{1}{2}}(x)}{x} = J_{-\frac{1}{2}}(x) + J_{\frac{3}{2}}(x) \Longrightarrow J_{\frac{3}{2}}(x) = \frac{J_{\frac{1}{2}}(x)}{x} - J_{-\frac{1}{2}}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$$

$$J_{\frac{3}{2}}^2(x) = \frac{2}{\pi x}\left(\frac{\sin x}{x} - \cos x\right)^2$$

$$= \frac{2}{\pi x}\left(\frac{\sin^2 x}{x^2} - \frac{2\sin x\cos x}{x} + \cos^2 x\right)$$

Similarly $J_{\frac{-3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right)$

$$J_{\frac{-3}{2}}^2(x) = \frac{2}{\pi x}\left(\frac{\cos^2 x}{x^2} + \frac{2\sin x\cos x}{x} + \sin^2 x\right)$$

$$J_{\frac{3}{2}}^2(x) + J_{\frac{-3}{2}}^2(x) = \frac{2}{\pi x}\left(\cancel{\frac{\sin^2 x}{x^2}} - \cancel{\frac{2\sin x\cos x}{x}} + \cancel{\cos^2 x}\right) + \frac{2}{\pi x}\left(\cancel{\frac{\cos^2 x}{x^2}} + \cancel{\frac{2\sin x\cos x}{x}} + \cancel{\sin^2 x}\right)$$

$$= \boxed{\frac{2}{\pi x}\left(1 + \frac{1}{x^2}\right)}$$

II) Show that

$$\frac{d}{dx}\left(xJ_n(x)J_{n+1}(x)\right) = x\left[\left(J_n(x)\right)^2 - \left(J_{n+1}(x)\right)^2\right]$$

$$D_x\left(x^{n+1}x^{-n}J_n(x)J_{n+1}(x)\right)$$

$$D_x\left(x^{n+1}J_{n+1}(x)\cdot x^{-n}J_n(x)\right)$$

$$x^{n+1}J_{n+1}(x)\,D_x\left(x^{-n}J_n(x)\right) + x^{-n}J_n(x)\,D_x\left(x^{n+1}J_{n+1}(x)\right)$$

$$-\cancel{x^{n+1}}J_{n+1}^{2}(x)\,\cancel{x^{-n}}\,\cancel{J_{n+1}(x)} + \cancel{x^{-n}}J_n^{2}(x)\,\cancel{x^{n+1}}\,\cancel{J_n(x)}$$

$$= -xJ_{n+1}^2(x) + xJ_n^2(x) = x\left[J_n^2(x) - J_{n+1}^2(x)\right] \quad \text{QED}$$

III) **Show that**

$$\frac{d}{dx}\left(x^2 J_{n-1}(x) J_{n+1}(x)\right) = 2x^2 J_n(x) J_n'(x)$$

$$D_x\left[x^{n+1} x^{-n+1} J_{n-1}(x) J_{n+1}(x)\right]$$

$$= x^{n+1} J_{n+1}(x) D_x\left[x^{-n+1} J_{n-1}(x)\right] + x^{-n+1} J_{n-1}(x) D_x\left[x^{n+1} J_{n+1}(x)\right]$$

Given **11)** $\frac{d}{dx}\left(x^{v} J_v(x)\right) = x^{v} J_{v-1}(x)$ **and** $\frac{d}{dx}\left(x^{-v} J_v(x)\right) = -x^{-v} J_{v+1}$

$$= -x^{\cancel{n}+1} J_{n+1}(x)\, x^{-\cancel{n}+1} J_n(x) + x^{-\cancel{n}+1} J_{n-1}(x)\, x^{\cancel{n}+1} J_n(x)$$

$$= -x^2 J_{n+1}(x) J_n(x) + x^2 J_{n-1}(x) J_n(x)$$

$$= x^2 J_n(x)\left[J_{n-1}(x) - J_{n+1}(x)\right]$$

Given **12)** $xJ_v'(x) = vJ_v(x) - xJ_{v+1}(x)$ **and** $xJ_v'(x) = -vJ_v(x) + xJ_{v-1}(x)$.

$$= 2x^2 J_n(x) J_n'(x) \qquad QED$$

IV) **Orthogonality of Bessel functions. Show that if $m \neq n$**

$$I = \int_0^b x J_\vartheta(k_n x) J_\vartheta(k_m x)\, dx = 0 = -C$$

Using $x^2 y'' + xy' + (\alpha^2 x^2 - n^2) y = 0 \Rightarrow y(x) = c_1 J_n(\alpha x) + c_2 Y_n(\alpha x)$,

$xu'' + u' + (\alpha^2 x^2 - n^2) u / x = 0$	$xv'' + v' + (\alpha^2 x^2 - n^2) v / x = 0$
$D_x[xu'] + \frac{\alpha^2 x^2 - n^2}{x} u = 0$	$D_x[xv'] + \frac{\alpha^2 x^2 - n^2}{x} v = 0$
$v D_x(xu') + \frac{\alpha^2 x^2 - n^2}{x^2} uv = 0$	$u D_x(xv') + \frac{\alpha^2 x^2 - n^2}{x^2} uv = 0$
1	2

Sub. ① from ②:

$$u D_x(x v') - v D_x(x u') + (\alpha_v - \alpha_u)\, x u v = 0$$

$$D_x[x(u v' - v u')] + (\alpha_v - \alpha_u)\, x u v = 0$$

; $u = J_v(k_m x)$, $\alpha_u = k_m$, $u = J_v(k_n x)$, $\alpha_u = k_n$

$n = v$

$k_n \neq k_m \Rightarrow \neq 0$

$$(k_n^2 - k_m^2)\, x J_v(k_n) J_v(k_m) = D_x\, x\left[J_v(k_m x) J_v'(k_n x) - J_v'(k_m x) J_v(k_n x)\right]$$

$$\Rightarrow I = \int_0^b x J_v(k_n) J_v(k_m) = \cancel{x\left[J_v(k_m x) J_v'(k_n x) - J_v'(k_m x) J_v(k_n x)\right]}\Big|_0^b$$

$= 0$ QED

V) **Show that**

$$J_n(x) = \frac{2(x/2)^{n-m}}{\Gamma(n-m)} \int_0^1 (1-t^2)^{n-m-1}\, t^{m+1} J_m(xt)\,dt,$$

$$n > m > -1, x > 0$$

Given 1) **The Bessel function of the first kind:**

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(1+v+n)} \left(\frac{x}{2}\right)^{2n+v}.$$

We will try to reach that form

$$\frac{2(x/2)^{n-m}}{\Gamma(n-m)} \int_0^1 (1-t^2)^{n-m-1}\, t^{m+1} J_m(xt)\,dt$$

$$= \frac{2(x/2)^{n-m}}{\Gamma(n-m)} \int_0^1 (1-t^2)^{n-m-1}\, t^{m+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+m+k)} \left(\frac{xt}{2}\right)^{2k+m} dt$$

$$= \frac{2(x/2)^{n-m}}{\Gamma(n-m)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+m+k)} \left(\frac{x}{2}\right)^{2k+m} \int_0^1 (1-t^2)^{n-m-1}\, t^{2k+2m+1}\,dt$$

$u = t^2,\ du = u^{-1}/2$

$$= \frac{\cancel{2}(x/2)^{n-m}}{\Gamma(n-m)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+m+k)} \left(\frac{x}{2}\right)^{2k+m} \frac{1}{\cancel{2}} \int_0^1 (1-u)^{n-m-1}\, u^{k+m}\,du$$

$$= \frac{(x/2)^{n-m}}{\cancel{\Gamma(n-m)}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\cancel{\Gamma(1+m+k)}} \left(\frac{x}{2}\right)^{2k+m} \frac{\cancel{\Gamma(n-m)}\,\cancel{\Gamma(1+k+m)}}{\Gamma(1+n+k)}$$

$$= \boxed{\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+n+k)} \left(\frac{x}{2}\right)^{2k+n}}$$

$$= J_n(x)$$

QED

A) Show that

$$\int_0^\infty e^{-bt}\, t^n J_n(at)\,dt = \frac{(2n)!}{2^n n!}(a^2+b^2)^{-n-1/2},$$

$$a, b > 0, n = 0, 1, 2, \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+k+n)} \left(\frac{a}{2}\right)^{2k+n} \int_0^\infty e^{-bt}\, t^{2k+2n}\,dt$$

let $u = bt$, $du = b\,dt$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+k+n)} \left(\frac{a}{2}\right)^{2k+n} b^{-2k-2n-1} \int_0^{\infty} e^{-u} u^{2k+2n}\, dt$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+k+n)} \left(\frac{a}{2}\right)^{2k+n} b^{-2k-2n-1} \Gamma(2k+2n+1)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(1+k+n)} \left(\frac{a}{2}\right)^{2k+n} b^{-2k-2n-1} (2k+2n)!$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+n)!} \left(\frac{a}{2}\right)^{2k+n} b^{-2k-2n-1} (2k+2n)!$$

I don't know how to continue.

B) Express the general solution in terms of Bessel functions

$$i)\ 4x^2y'' + (1+4x)y = 0$$

Given **7)** $y'' + \frac{(1-2a)}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}\right)y = 0 \Rightarrow y(x) = x^a\left(c_1J_p(bx^c) + c_2Y_p(bx^c)\right)$

$$y'' + \frac{1+4x}{4x^2}y = 0 \Rightarrow \quad 1-2a=0 \Rightarrow \boxed{a=\tfrac{1}{2}}$$

$$\frac{1+4x}{4x^2} = b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}$$

$$\frac{1}{4x^2} + x^{-1} = b^2c^2x^{2c-2} + \frac{1/4-p^2c^2}{x^2}$$

$$\Rightarrow \frac{1}{4} - \cancel{p^2}^{\,0}c^2 = \frac{1}{4}$$ Can't be zero

$$\Rightarrow 2c-2=-1 \Rightarrow \boxed{c=\tfrac{1}{2}}$$

$$\Rightarrow b^2\tfrac{1}{4}=1 \Rightarrow \boxed{b=2}$$

$$y = \sqrt{x}\left[c_1 J_0(2\sqrt{x}) + C_2 J_0(2\sqrt{x})\right]$$

$$ii)\ x^2y'' + 5xy' + (9x^2 - 12)y = 0$$

Given $\boxed{7)\ y'' + \frac{(1-2a)}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}\right)y = 0 \Rightarrow y(x) = x^a\left(c_1J_p(bx^c) + c_2Y_p(bx^c)\right)}$

$$y'' + \frac{5}{x}y' + \frac{9x^2-12}{x^2}y = 0$$

$$1 - 2a = 5 \Rightarrow \boxed{a = -2}$$

$$b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2} = 9x^0 - \frac{12}{x^2}$$

$$\Rightarrow 2C - 2 = 0 \Rightarrow \boxed{C = 1}$$

$$\Rightarrow b^2 = 9 \Rightarrow \boxed{b = 3}$$

$$\Rightarrow 4 - p^2 = -12 \Rightarrow \boxed{p = 4}$$

$$y = \frac{1}{x^2}\left[C_1 J_4(3x) + C_2 Y_4(3x)\right]$$

C) Evaluate

$$\int x^2 I_1(x)dx$$

$$I_1(x) = i^{-1}J_1(ix) = -iJ_1(ix)$$

$$= -i\int x^2 J_1(ix)\,dx \qquad \text{let } u = ix,\ du = i\,dx$$

$$= \frac{\cancel{-i}}{i^{\cancel{3}}}\int u^2 J_1(u)du$$

$$= \cancel{\frac{-1}{i^2}}\int u^2 J_1(u)du = \int D_u\, u^2 J_2(u)du = u^2J_2(u) + C$$

$$= (xi)^2 J_2(xi) + C$$

$$= \boxed{x^2 I_2(x) + C}$$

D) **Show that**

$$K_n(x) = \frac{\pi}{2} i^{n+1}[J_n(ix) + iY_n(ix)]$$

[Hint: $i^{-2n} = e^{-n\pi i}$]

Given

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}.$$

$$K_n(x) = \frac{\pi}{2} \frac{I_{-n}(x) - I_n(x)}{\sin \pi n}$$

$$= \frac{\pi}{2} \frac{i^{n} J_{-n}(ix) - i^{-n} J_n(ix)}{\sin \pi n}$$

$$= \frac{\pi}{2} i^{n+1} \frac{-i J_{-n}(ix) - i^{-2n-1} J_n(ix)}{\sin \pi n}$$

Given

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}.$$

$$= \frac{\pi}{2} i^{n+1} \frac{i \sin \pi n Y_n(ix) - i \cos \pi n J_n(ix) - i^{-2n-1} J_n(ix)}{\sin \pi n}$$

$$= \frac{\pi}{2} i^{n+1} \frac{i \sin \pi n Y_n(ix) - i \cos \pi n J_n(ix) + i e^{-n\pi i} J_n(ix)}{\sin \pi n}$$

$$= \frac{\pi}{2} i^{n+1} \frac{i \cancel{\sin \pi n} Y_n(ix) + \cancel{\sin \pi n} J_n(ix)}{\cancel{\sin \pi n}}$$

$$= \frac{\pi}{2} i^{n+1} (J_n(ix) + i Y_n(ix))$$ QED