Question 1

An insurance company is randomly checking an institution's insurance claims for fraud by investigating 25 cases selected at random from the same institution. The company will not consider the institution fraudulent if at least 22 of those 25 cases were found true. Suppose that 95% of the cases from that institution are usually true.

1) Write the probability distribution of the claim cases that are true in the sample of 25 investigated claims.
2) What is the probability that the institution is considered fraudulent?
3) If the insurance company investigates 10 of its customer institutions, what is the probability that at least 3 institutions are found fraudulent?
4) What is the probability that the first institution found fraudulent is the seventh investigated institution?

1) Geometric Distribution: $f(x) = \binom{25}{x} 0.95^{x} \, 0.05^{25-x}$

2) Fradulant: $P(X < 22) = \sum_{x=0}^{21} f(x) = 0.0341$

3) $P_G(X \geq 3) = \sum_{x=3}^{10} g(x)$; $g(x) = \binom{10}{x} 0.0341^{x} \, 0.9659^{10-x}$

$= 0.004$

4) Negative Binomial Distribution: $P(N=7, K=1) = \binom{7-1}{1-1} \, 0.0341^{1} \, 0.9659^{7-1}$

$= 0.0278$

Question 2

In a warehouse for aeroplane maintenance, the number of aeroplanes requiring maintenance follows a Poisson distribution, and the average number of aeroplanes requiring maintenance is 10 aeroplanes per month.

1) What is the probability that exactly 10 aeroplanes require maintenance within a given month?
2) What is the probability that exactly 10 aeroplanes require maintenance within a given interval of 3 months?
3) What is the probability that no aeroplane requires maintenance within a given interval of 1 year?
4) Calculate and draw the CDF of the number of aeroplanes that require maintenance within an interval of 4 months?

$\lambda = 10/\text{month}$, Poisson : $f(x) = e^{-\lambda} \lambda^{x} / x!$

1) $P(X=10) = f(10) = e^{-10} \, 10^{10} / 10! = 0.1251$

2) $\lambda = 10 \, \frac{\text{plane}}{\text{month}} \times 3 \, \frac{\text{month}}{3\,\text{months}} = 30 \text{ planes}/3\text{ months}$

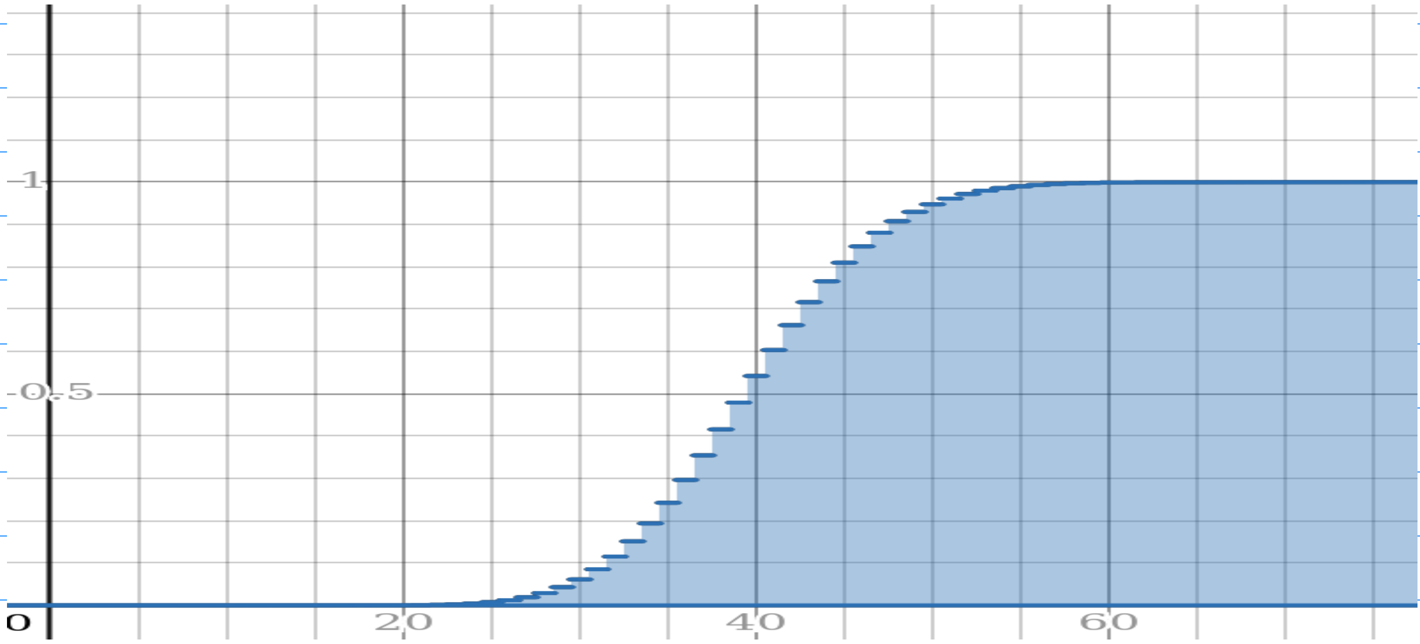
$P(X=10, \lambda=30) = f(10, \lambda=30) = e^{-30} \, 30^{10}/10! = 0.00001 5227$

3) $\lambda = 10 \, \frac{\text{plane}}{\text{month}} \times 12 \, \frac{\text{month}}{\text{year}} = 120 \, \frac{\text{plane}}{\text{year}}$; $P(X=0, \lambda=120) = 7.66 \times 10^{-53}$

$\approx 0\%$

4) $\lambda = 10 \times 4 = 40$, CDF: $F(x) = \sum_{n=0}^{x} e^{-\lambda} \lambda^n / n!$

$$= e^{-\lambda} \sum_{n=0}^{x} \lambda^n / n!$$



Question 3

In a comparison between a Samsung Galaxy S24 and an iPhone 15 Pro, it was found that the lifetime of a Galaxy phone is uniformly distributed with a mean of 1000 days and a standard deviation of 25 days, while the lifetime of the iPhone is normal distributed with a mean of 1100 days and a standard deviation of 100 days.

1) If you are interested in a device that lasts for at least 1150 days, which device would you choose? Why?
2) What is the probability that both devices do not last for more than 1050 days?
3) For each device type, what is the lifetime that is 95% probable?
4) What is the probability that a Galaxy phone lasts for more than the average lifetime of an iPhone?

Uniform Dist: $\mu_1 = 1000$, $\sigma_1 = 25$

$$\mu = \frac{A+B}{2}, \quad \sigma^2 = \frac{(B-A)^2}{12} \Rightarrow \begin{cases} 1000 = \dfrac{A_1+B_1}{2} \\ 25^2 = \dfrac{(B_1-A_1)^2}{12} \end{cases} \Rightarrow \begin{matrix} A_1 = 956.69 \\ B_1 = 1043.3 \end{matrix}$$

$$f(x) = \begin{cases} 5\times10^{-4}, & 956.69 \leq x \leq 1043.3 \\ 0, & \text{o.w.} \end{cases}$$

$$A = \mu - \sigma\sqrt{3}$$

Normal Dist: $\mu_2 = 1100$, $\sigma_2 = 100$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = 4.0\times10^{-3}\, e^{-\frac{1}{2}\left(\frac{x-1100}{100}\right)^2}$$

1) Galaxy: $F(X \geq 1150) = 0$

iPhone: $F(X \geq 1150) = \int_{1150}^{\infty} f(t)dt$

$$= \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-1100}{100\sqrt{2}}\right)\Bigg|_{1150}^{\infty}$$

$$= \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{1}{2\sqrt{2}}\right) = \boxed{0.31}$$

$\Rightarrow$ iPhone because it has higher prob. of lasting at least 1150 days.

2) Galaxy: $F(X \leq 1050) = 1$

iPhone: $F(X \leq 1050) = \int_{-\infty}^{1050} f(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-1100}{100\sqrt{2}}\right)\Bigg|_{-\infty}^{1050}$

$$= \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{1}{2\sqrt{2}}\right) = \boxed{0.31}$$

3) Galaxy: $F(X \leq t) = 0.95 \Rightarrow t = A + 0.95(B-A) = \boxed{1038.9711}$

iPhone: $F(X \leq t) = 0.95 \Rightarrow t = \mu + Z\sigma = 1100 + 1.645 \times 100$

$$= \boxed{1264.5}$$

4) $P(X \geq 1100) = 0$

Question 4

Repeat Question 3 if the lifetime's are normally distributed instead of uniformly distributed.

$\mu_G = 1000$, $\sigma_G = 25$ | $\mu_I = 1100$, $\sigma_I = 100$

1) Still iPhone, 2) $F_G(X \leq 1050) = 0.9772$

$F_I(X \leq 1050) = 0.3085$

$\Rightarrow P = F_G \times F_I = 0.3015$

3) $t = \mu + Z\sigma$; for 0.95 $Z = 1.645$

Galaxy : $t = 1000 + 1.645 \times 25 = 1041.125$

4) $P(X \geq 1100) = 3.17 \times 10^{-5}$

Question 5

A random variable X is exponentially distributed with a variance of 25. Find:

1) $\mathcal{E}\{e^{tx}\}$
2) $\mathcal{E}\{3 - 2X\}$
3) $\mathcal{E}\{(3 - 2X)^2\}$
4) $P\{\mu_x - 2\sigma_x < X < \mu_x + 2\sigma_x\}$
5) $P\{\mu_x - 4\sigma_x < X < \mu_x + 4\sigma_x\}$

Exp. Dist : $f(x) = \lambda e^{-\lambda x} \Rightarrow \sigma^2 = \frac{1}{\lambda^2} \Rightarrow \lambda = \frac{1}{\sigma} = \frac{1}{\sqrt{25}}$

$= 0.2$

$$\mathcal{E}\{e^{tX}\} = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{x(t-\lambda)} dx = \frac{\lambda}{\lambda - t} = \frac{0.2}{0.2 - \lambda}$$

$$\mathcal{E}\{3 - 2X\} = \int_0^\infty (3 - 2x)\lambda e^{-\lambda x} dx = 3\lambda \int_0^\infty e^{-\lambda x} dx - 2\lambda \int_0^\infty x e^{-\lambda x} dx$$

$$= 3 \quad - 2/\lambda = -7$$

$$\mathcal{E}\{(3-2X)^2\} = \mathcal{E}\{9 - 12X + 4x^2\} = 9 - \frac{12}{\lambda} + \frac{8}{\lambda^2} = 149$$

$\mu_x = \frac{1}{\lambda} = 5$, $\sigma_x = \frac{1}{\lambda} = 5$

$$P\{\mu_x - 2\sigma_x < X < \mu_x + 2\sigma_x\} = P\{-5 < X < 15\} = P\{X < 15\}$$

$$= F(15) = 1 - e^{-15 \times 0.2} = 0.9502$$

$$P\{\mu_x - 4\sigma_x < X < \mu_x + 4\sigma_x\} = P\{X < 25\} = F(25) = 0.9934$$