Dense Right Ideals and Maximal Right Ring of Quotients

Dense Right Ideals

Let R be a ring and M a right R-module. A right ideal $I \subseteq R$ is said to be **dense** in R (with respect to M) if:

> For every $m \in M$ and every nonzero $r \in R$, there exists $i \in I$ such that $mi \neq 0$.

In particular, if $M = R_R$, the regular right module, then I is dense if for every nonzero $r \in R$, there exists $i \in I$ with $ri \neq 0$.

Properties

- If I is dense in R, then I is essential in R_R.
- If R is right nonsingular, the dense right ideals form a directed set.
- The intersection of dense right ideals is again dense.
- Every essential right ideal is dense; the converse may not hold.

Example

Let $R = \mathbb{Z}$. Then every nonzero ideal $(n) \subseteq \mathbb{Z}$ is dense. Indeed, for any nonzero $r \in \mathbb{Z}$ and any $n \neq 0$, we have $rn \neq 0$.

Maximal Right Ring of Quotients

Let $E(R_R)$ be the injective hull of R as a right R-module. Define:

$$Q^r_{\max}(R) := \mathrm{End}_R(E(R_R))$$

Properties

- $R \subseteq Q^r_{\max}(R)$
- $Q^r_{\max}(R)$ is right self-injective
- Every dense right ideal of R is essential in $Q^r_{\max}(R)$
- $Q^r_{\max}(R)$ is the largest right ring of quotients with respect to dense right ideals

References

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