# CALCULUS ON METRIC SPACES: BEYOND THE POINCARÉ INEQUALITY

New Examples of Differentiability Spaces

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**Abstract:**
We discuss a framework introduced by J. Cheeger (1999) to differentiate Lipschitz maps defined on metric measure spaces which admit Poincaré inequalities, and discuss (the first) examples on which it is still possible to differentiate despite the infinitesimal geometry being incompatible with the Poincaré inequality.

# DIFFERENTIABILITY SPACES: WHY WE CARE?

# GEOMETRIC GROUP THEORY

- → Mostow (1968), "Quasi-conformal mappings in $n$-space and the rigidity of the hyperbolic space forms".
- → Pansu (1989), "Carnot-Caratheodory Metrics and quasi-Isometries of rank-one symmetric spaces"
- → Pansu's Rademacher Theorem: Any Lipschitz or quasi-Conformal map $f : \mathbb{G}_1 \to \mathbb{G}_2$ between Carnot groups is differentiable at $\mathcal{H}^{\dim \mathbb{G}_1}$-a.e. point. The derivative is a group homomorphism commuting with dilations.

# THE ABSTRACT POINCARÉ INEQUALITY (1)

- → Heinonen & Koskela (1998), "Quasiconformal maps in metric spaces with controlled geometry"
- → The key ingredients are **doubling measures** and **the Poincaré inequality** (PI-spaces)
- → Rigidity for Fuchsian buildings, Sobolev spaces, $p$-harmonic maps, ...

# THE ABSTRACT POINCARÉ INEQUALITY (2)

→ The classical $(1,p)$-Poincaré inequality:

$$\fint_B |u-u_B|\,d\mathcal{H}^N \leq C\operatorname{diam}(B)\left(\fint_B \|\nabla u\|^p\,d\mathcal{H}^N\right)^{1/p}. \quad (1)$$

→ In the metric setting: $\mathcal{H}^N \to \mu$, $\|\nabla u\| \to$ a "surrogate" $g$ (**upper gradient**).

→ If $u$ is Lipschitz can take:

$$\operatorname{Lip} u(x) = \limsup_{y\neq x\to x}\frac{|u(y)-u(x)|}{d(x,y)}. \quad (2)$$

# A FINITE-DIMENSIONALITY ARGUMENT

→ Cheeger (1999): If $\mu$ is doubling, at $\mu$-a.e. $f$ the blow-ups of $f$ are $p$-harmonic and the gradient has constant modulus **(generalized linearity)**.

→ Cheeger (1999): In a PI-space a linear subspace of generalized-linear functions is finite-dimensional.

→ Cheeger's Rademacher: Fix Lipschitz coordinate functions $\{\psi_i\}_{i=1}^N$; any $f$ is differentiable $\mu$ a.e.:

$$f(x') = f(x)+\sum_{i=1}^{N} a_i(x)\left(\psi_i(x') - \psi_i(x)\right)+o\left(d_X(x,x')\right). \quad (3)$$

# APPLICATIONS

- → Legitimates first-order calculus for PI-spaces (e.g. now $\mathrm{Lip} f$ is really $\|df\|$).
- → Provides non-embeddability results $f : X \to \mathbb{R}^n$.
- → The idea of differentiation has been generalized to other targets: $f : X \to L^\infty$ (**metric differentiation**)) and $f : X \to L^1$ (**the cut-metric representation** and a counterexample of Cheeger & Kleiner to the Goemans-Linial conjecture).

# CONSTRUCTING THE DERIVATIVE ALONG CURVES: ALBERTI REPRESENTATIONS

# DIFFERENTIABILITY SPACES

- → A metric measure space $(X, \mu)$ such that any Lipschitz $f : X \to \mathbb{R}$ is differentiable $\mu$-a.e. is a **differentiability space**.
- → Keith 2004, "A differentiabile structure for metric measure spaces": replace the Poincaré inequality with the "Lip-lip" inequality: $\mu$-a.e.:

$$\mathrm{Lip} f \leq K \mathrm{lip} f, \tag{4}$$

where

$$\mathrm{lip} f(x) = \liminf_{r \to 0} \sup_{y \in B(x,r)} \frac{|f(x) - f(y)|}{r}. \tag{5}$$

# DRAWBACKS

→ Both Cheeger & Keith's proof are non-constructive. No curves.

→ Cheeger proved better properties for PI-spaces (e.g. $\mathrm{Lip} f = \mathrm{lip} f$, stability of PI under blow-up).

→ No example of Keith's spaces not countable unions of positive-measure subsets of PI-spaces.

# THE PI-RECTIFIABILITY ``CONJECTURE''

→ **Strong form**, Cheeger-Kleiner-S. (2016): Any differentiability space is a countable union of positive-measure subsets of PI-spaces.

→ **Weak form**: A.e. the blow-ups/tangents of a differentiability space are PI-spaces.

→ Held belief to be true (had a paper rejected as the "expert" strongly believed in it), the PI-inequality should be "necessary" to have calculus.

# THE RADON-NIKODYM PROPERTY

- → An **RNP-Banach space** $B$: any Lipschitz $f : \mathbb{R} \to B$ is differentiable $\mathcal{H}^1$-a.e; $c_0$, $l^p$ $(1 \le p < \infty)$, $L^p([0,1])$ $(1 < p < \infty)$
- → Cheeger & Kleiner (2009): "Differentiability of Lipschitz maps from metric measure spaces to Banach spaces with the Radon–Nikodym property": if $(X, \mu)$ is PI, $f : X \to B$ is differentiabile
- → Study the derivative along curves; Preiss' observation: $\mu$ must admit an Alberti representation.

# ALBERTI REPRESENTATIONS

- → An Alberti representation is a Fubini-like decomposition of $\mu$:

$$\mu = \int_\Gamma g\mathcal{H}^1_\gamma \, dP(\gamma). \tag{6}$$

- → Bate (2014), "Structure of measures in Lipschitz differentiability spaces": characterizes differentiability spaces using Alberti representations
- → S. (2015–2016), shows that the "Lip-lip" inequality is an **equality, necessary and sufficient** to have differentiability, and that a.e. tangents of differentiability spaces are differentiability spaces.
- → Cheeger-Kleiner-S.: "Infinitesimal structure of differentiability spaces, and metric differentiation".

# FRAGMENTED CONNECTIVITY

- → To prove a PI-inequality it is necessary to construct families of curves joining points.
- → Bate & Li (2016), "The geometry of Radon-Nikodym Lipschitz differentiability spaces": if all Lipschitz $f : X \to \bigoplus_{l^1} l_n^\infty$ are differentiable, can connect points avoiding sets of low density.
- → Eriksson-Bique (2016), "Classifying Poincaré inequalities and the local geometry of RNP-differentiability spaces": RNP-differentiability spaces are PI-rectifiable.

# NEW EXAMPLES: AN ALTERNATIVE APPROACH TO QUANTITATIVE DIFFERENTIATION

# A PI-UNRECTIFIABLE EXAMPLE

- → S. (2016): An example of a differentiability space which is PI-unrectifiable.
- → $X$ has a disconnected tangent at a.e.
- → Any Lipschitz $f : X \to l^2$ is differentiable a.e.
- → For $\varepsilon > 0$ there is an a.e. **non**-differentiable $f : X \to l^{3+\varepsilon}$.
- → In the metric setting differentiability depends on the target!
- → Can “probe” Banach spaces with differentiability spaces.

# BUILDING BLOCKS

→ Non-quasiconvex diamonds.

→ Non-selfsimilar Inverse limit system (compare Laakso, Cheeger & Kleiner)

→ "Tritanopic" chromatic labels, Alberti representations and the horizonatl gradient.

# DISCONNECTED TANGENTS

- → At a generic point there is a disconnected tangent.
- → It is possible to join pairs of points by using "horizontal" paths and, if necessary, by using at most one "jump"
- → The jumps are unavoidable and break down the classical differentiation argument.

# CLASSICAL QUANTITATIVE DIFFERENTIATION

- → Jones (1988), “Lipschitz and bi-Lipschitz functions”
- → Quantify how $f : Q \subset \mathbb{R} \to \mathbb{R}$ is close to a “linear function” on dyadic subdivisions $Q_{n,j}$
- → The error:

$$\alpha(f, Q_{n,j}) = \frac{1}{\mathcal{H}^1(Q_{n,j})} \inf_{l} \sup_{x \in Q_{n,j}} |f(x) - l(x)| \tag{7}$$

- → A bound on “bad cubes”:

$$\sum_{Q_{n,j} : \alpha(f, Q_{n,j}) \geq \varepsilon} \mathcal{H}^1(Q_{n,j}) \lesssim \log(\varepsilon)\varepsilon^{-2}\mathcal{H}^1(Q). \tag{8}$$

# (TAIL)-RECURSIVE QUANTITATIVE DIFFERENTIATION

- Take $f : X \to l^2$, we would like to claim that if $Q$ is a quasiconvex diamond $1/n$-squeezed, $\|f(c_{green}) - f(c_{red})\|_2 = O(\varepsilon \operatorname{diam} Q/n)$.
- The error is $\varepsilon/n$, decreasing in the tail of the decomposition.
- The decomposition is no longer dyadic or self-similar.
- Using harmonic functions can prove:

$$\sum_{Q \text{ is } \varepsilon\text{-bad}} \frac{\varepsilon^2}{n_Q^3}\mu(Q) \lesssim \mathbf{L}(f)^2. \tag{9}$$

# FURTHER DIRECTIONS

- → Classify $p$: $f : X \to L^p$ is differentiable, $p = 3$ is critical.
- → The examples have analytic dimension $3$, lower to $1$.