The structure of currents in $\mathbb{R}^n$ and in metric spaces

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Plan

- Part 1: Review of Euclidean currents.
- Part 2: The metric currents of Ambrosio–Kirchheim.
- Part 3: Rectifiable decompositions of normal and metric currents: Solution of some open problems.
- Part 4: Digression on vector fields in metric spaces.
- Part 5: Sketch of the construction.

de Rham's currents 1

- Currents are the "right" way to generalize k-dimensional submanifolds of $\mathbb{R}^n$, providing compactness properties and a "good behavior" in limit processes (key concept: **normal current**).
- de Rham (1955): vector fields having distributional coefficients; generalize Schwartz's distributions (1951).
- The theory got momentum in the 1960s with the paper "Normal and integral currents" of Federer and Fleming: solution of **Plateau's Problem** (key concept: **integral currents**)
- Plateau's Problem: find the k-surface of minimal area/volume in $\mathbb{R}^n$ bounding a given (closed) $(k-1)$-surface.
- For $k = 2$ solved in the 1930s. General approach of Federer and Fleming: replace k-surface with integral currents.

de Rham's currents 2

Definition (deRham Current)

Let $\text{Smooth}_c(k, \mathbb{R}^n)$ denote the compactly supported smooth k-forms in $\mathbb{R}^n$ with the **smooth topology**. Then the space of k-dimensional currents $\text{Curr}(k, \mathbb{R}^n)$ is the dual of $\text{Smooth}_c(k, \mathbb{R}^n)$.

1. Example: integration wrt. to a locally (Lebesgue)-integrable k-field: $\xi : \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$:

$$T_\xi(\omega) = \int_{\mathbb{R}^n} \langle \omega, \xi \rangle \, d\mathcal{H}^n$$

2. Example: smooth k-surface. Let $\Phi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ smooth where U open with compact closure

$$[[\Phi]](\omega) = T_\Phi(\omega) = \int_U \langle \Phi^* \omega, e_1 \wedge \cdots \wedge e_k \rangle \, d\mathcal{H}^k$$

Boundary

1. What would be the boundary of Φ? Assume that ∂U is smooth and Φ extends smoothly to a neighbourhood of $\bar{U}$. We would use Stoke's Theorem (plug the differential of a $(k-1)$-form in $[[\Phi]]$:

$$\partial[[\Phi]](\omega) = \int_{\partial U} \langle \Phi^* \omega, \vec{\xi}_{\partial U} \rangle \, d\mathcal{H}^{k-1}$$

2. The boundary is defined to make Stoke's Theorem a tautology: $\partial T(\omega) = T(d\omega)$.

Mass

The "mass" of $[[\Phi]]$ should be the volume of the surface.

Definition (Mass in an open V)

The mass of T in an open set V is defined as:

$$M_V(T) = \sup\{T(\omega) \colon \omega \in \mathrm{Smooth_c}(k, \mathbb{R}^n), \|\omega\| \leq 1, \operatorname{spt}\omega \subset V\}$$

- If $M_{\mathbb{R}^n}(T) < \infty$ T has **finite mass** and is representable as a finite vector-valued Radon measure $T = \vec{\mu_T}$ or as a unit-norm Borel vector field times the **mass measure**: $T = \vec{T}\,\|T\|$. The **support** is the support of the mass measure.
- Even if T has finite mass, the boundary might just be a distributional object

Normal Currents

Definition (Normal Current)

A current N is normal ($N \in \mathrm{Norm}(k, \mathbb{R}^n)$) if both N and ∂N have finite mass (I will assume also compact support); equivalently we must have:

$$N(\omega) = \int_{\mathbb{R}^n} \langle \omega(x), \vec{N}(x) \rangle \, d\|N\|(x)$$

$$\partial N(\omega) = \int_{\mathbb{R}^n} \langle \omega(x), \partial\vec{N}(x) \rangle \, d\|\partial N\|(x)$$

- Morally, they are the geometric generalization of BV-functions
- **Compactness Theorem**: if $\{N_n\}_n$ is a sequence of normal currents with $\sup_n (M_{\mathbb{R}^n}(N_n) + M_{\mathbb{R}^n}(\partial N_n)) < \infty$ and $\mathrm{spt}\, N_n \subset K$, then a subsequence converges weak* to $N \in \mathrm{Norm}(k, \mathbb{R}^n)$.

Rectifiable and Integral Currents 1

Definition (Rectifiable and Integral Currents)

A current T is rectifiable ($T \in \mathrm{Rect}(k, \mathbb{R}^n)$) is there are a k-rectifiable set Σ, $\theta \in L^1(\Sigma, \mathcal{H}^k)$ and an orientation τ_Σ of Σ such that:

$$T(\omega) = [[\Sigma, \tau_\Sigma, \theta]] = \int \langle \omega(x), \tau_\Sigma(\omega) \rangle \theta(x) \, d\mathcal{H}^k(x).$$

If θ is $\mathbb{Z}$-valued T is called **integer rectifiable** ($T \in \mathrm{IRect}(k, \mathbb{R}^n)$). If both T and ∂T are integer rectifiable and if T is normal, then T is called **integral** ($T \in \mathrm{Integ}(k, \mathbb{R}^n)$).

Rectifiable and Integral Currents 2

1. Example of Rectifiable: take $K \subset \mathbb{R}^k$ compact with $\mathcal{H}^k(K) > 0$ and $f : K \to \mathbb{R}^n$ Lipschitz with Df non-singular $\mathcal{H}^k$-a.e. Then take:

$$[[f]](\omega) = \int_K \langle \omega(f(x)), Df(x)(e_1 \wedge \cdots \wedge e_k) \rangle \, d\mathcal{H}^k(x).$$

2. Example of an integral current: take a $\mathbb{Z}$-sum of finitely many k-cubes. (For $k = 0$ that's the only possibility).

Ambrosio-Kirchheim metric currents

- de Giorgi: by density the action of normal currents can be extended to $(k+1)$-tuples of bounded Lipschitz functions:

$$(f_0, f_1, \cdots, f_k) \simeq f_0 df_1 \wedge \cdots \wedge df_k$$

- In a metric space X a k-current should act on $(k+1)$-tuples of bounded Lipschitz functions.
- $T(f_0, f_1, \cdots, f_k)$ should be multilinear alternating.
- T should have finite mass; $\|T\|$ is the minimal finite Radon measure μ such that:

$$|T(f_0, f_1, \cdots, f_k)| \leq \prod_{i=1}^{k} \mathbf{L}(f_i) \int_X |f_0| \, d\mu.$$

AK currents: weak* continuity

1. If we know the action of T on a dense set of "smooth functions" we should know it on all Lipschitz functions.
2. Look at the way you extend the action of normal currents in Euclidean space to Lipschitz k-forms.
3. **Weak* convergence for Lipschitz functions**: $f_n \to f$ weak* is $f_n \to f$ pointwise and $\sup_n \mathbf{L}(f_n) < \infty$
4. **AK weak*-continuity axiom:** if $f_{i,n} \to f_i$ weak* for each $i \in \{0, \cdots, k\}$:

$$\lim_{n\to\infty} T(f_{0,n}, f_{1,n}, \cdots, f_{k,n}) = T(f_0, f_1, \cdots, f_k)$$

5. AK develop mainly the theory of rectifiable and integral currents.

Geometric Applications

1. Wenger (2005): Isoperimetric inequalities for AK-integral currents in Banach spaces and simply connected metric spaces of non-positive curvature in the sense of Alexandrov and Busemann.
2. Wenger (2005): Used the isoperimetric inequality to solve Plateau's problem in this general setting.
3. Wenger (2006): Found the largest constant in an isoperimetric inequality which ensures that a geodesic metric space is Gromov hyperbolic.
4. Wenger (2011): Constructed the first examples of nilpotent groups without exactly polynomial Dehn functions.
5. Kleiner–Lang (in progress): study higher-rank hyperbolicity.

Questions on the structure of currents

1. What is the structure of metric currents in $\mathbb{R}^n$? What is their relationship with classical currents?
2. What is the structure of normal currents in $\mathbb{R}^n$?
3. What is the structure of metric currents in general metric spaces? In particular, can you recover a "classical representation" $\vec{T}\|T\|$ for metric currents?
4. What is the structure of normal currents in metric spaces?

The flatness conjecture

1. AK (2000): For $\mathbb{R}^n$ normal and metric normal coincide.
2. **AK-flat(ness) conjecture** (2000): metric currents are the closure of normal currents in the flat topology:

$$F_{\mathbb{R}^n}(T) = \inf\{M_{\mathbb{R}^n}(R) + M_{\mathbb{R}^n}(S) : T = R + \partial S\}$$

3. Schioppa (2014): the flat conjecture holds for 1-metric currents (even in Banach spaces). Slightly stronger, can find $N_n \in \mathrm{Norm}(1, Bana)$ with $M_{Bana}(T - N_n) \to 0$.
4. Schioppa (2014) + de Philippis–Rindler (2016): the flat conjecture holds for n-dimensional metric currents in $\mathbb{R}^n$.
5. The flat conjecture for 1-dimensional metric currents is proven decomposing:

$$T = \int_{\mathrm{Rect}(1,Bana)} R_t \, dQ(R_t)$$

6. If you have a k-dimensional normal current ($k \geq 2$) in $\mathbb{R}^n$ (or even a metric space), does its support contain a piece of a k-surface?

The counterexamples

1. Schioppa (2016): for $k \geq 2$ constructed in $\mathbb{R}^{k+2}$ a k-dimensional simple normal current whose support is purely 2-unrectifiable.
2. They answer open questions asked by other people, Morgan, Alberti, Csörnyei and Preiss.
3. They are based on an earlier metric result.
4. Schioppa (2015): for $k \geq 2$ constructed a k-dimensional metric normal current N with simple k-vector field $\vec{N}$ and whose support $\operatorname{spt} N$ is purely 2-unrectifiable with topological and Assouad-Nagata dimension k.

Answer to Questions of Alberti and Morgan 1

1. Morgan (1984): If $T \in \mathrm{Norm}(k, \mathbb{R}^n)$ can you write:

$$T = \int_{\mathrm{Integ}(k,\mathbb{R}^n)} I_t \, dQ(I_t),$$

   possibly in a mass-efficient way

$$\|T\| = \int_{\mathrm{Integ}(k,\mathbb{R}^n)} \|I_t\| \, dQ(I_t)?$$

2. Motivation 1: no other examples of normal currents known at the time. ☹
3. Motivation 2: Study the Lavrentiev gap between normal and integral currents which are mass-minimizers with the same boundary. ☺

Answer to Questions of Alberti and Morgan 2

1. Zworski (1988): claimed that if $T = \xi\,\mathcal{H}^n$ where ξ is smooth, simple and non-involutive, than you cannot even get a rectifiable mass-efficient decomposition.
2. Alberti (1991): proved a remarkable Theorem on prescribing the gradients of functions on large sets; found rectifiable sets tangent to ξ and that Zworski argument is flawed (proof finished for non-existence of integral decompositions by Alberti and Massaccesi (2014)). ☹
3. Morgan revised (Alberti $\simeq$ 2005): If $T \in \mathrm{Norm}(k, \mathbb{R}^n)$ can you write:

$$T = \int_{\mathrm{Rect}(k,\mathbb{R}^n)} R_t \, dQ(R_t)?$$

Answer to Questions of Alberti and Morgan 3

1. YES for $k = 0, n$; ("soft"), Federer-Fleming ('60s) ☺
2. YES for $k = 1$, ("hard") Stanislav Smirnov (1993); Paolini and Eugene Stepanov (2013) ☺
3. YES for $k = n - 1$, ("hard") Federer-Fleming ('60s), Hardt-Pitts ('80s), Alberti–Massaccesi (2015) ☺
4. Schioppa (2016): NO for $2 \leq k \leq n - 2$. ☹

Link to differentiability

1. These counterexamples answer a question of Alberti-Csörnyei-Preiss concerning the structure of measures in $\mathbb{R}^n$ and the differentiability of Lipschitz functions.
2. In 2005–2010 ACP developped a differentiation theory for Lipschitz maps $f : \mathbb{R}^n \to \mathbb{R}$ wrt. singular measures (and even sets).
3. I need to introduce A-reps.
4. Morever, for the link to the flat conjecture and the work of de Philippis-Rindler, I need a digression on extending the notion of vector fields to metric spaces.

Vector fields in metric spaces

1. In general metric spaces there is no apriori differential structure.
2. Even in $\mathbb{R}^n$, if you fix a background measure, you don't want a classical Borel vector field!
3. One approach is functional analytic (from von Neuman Algebras) looking at vector fields as operators called (Weaver) derivations (Nik Weaver 2000).
4. Another is to look at tangents to 1-rectifiable sets (Ambrosio-Kirchheim $\simeq$ 2002). But how do you fit these tangents together?
5. One can use Alberti representations introduced to study BV functions.
6. Schioppa (2013): Turns out that Weaver derivations and Alberti representations are two ways of looking at the same thing. ☺

Alberti Representations

Definition

An Alberti representation of a Radon measure μ is a decomposition

$$\mu = \int_{\mathrm{Rect}(1,\mathbb{R}^n)} \|R_t\| \, dQ(R_t).$$

1. An A-rep (Q) is tangent to a vector field V if Q-a.e. R_t is tangent to V.
2. k-A-reps $\{(Q)_i\}_{i=1}^k$ are independent if they are tangent to k (1-dimensional) vector fields $\{V_i\}_{i=1}^k$ which are μ-a.e. independent.
3. You want to use normal currents: $\mu \ll \|N\|$, $N \in \mathrm{Norm}(1, \mathbb{R}^n)$ and

$$N = \int_{\mathrm{Rect}(1,\mathbb{R}^n)} R_t \, dQ(R_t)$$

Answer to a Question of Alberti-Csörnyei-Preiss 1

1. The ACP-theory uses the existence of k-independent A-reps.
2. ACP–Question ($\simeq$ 2010): If $k \geq 2$ and μ admits k-independent A-reps, is μ k-rectifiably representable?

$$\mu = \int_{\mathrm{Rect}(k,\mathbb{R}^n)} \|R_t\| \, dQ(R_t)$$

3. Link to currents: Schioppa (2014): the mass of a k-metric current admits k-independent A-reps. Moreover, one can recover a classical representation $\vec{T}\|T\|$.
4. Csörneyi–Jones (2011) announced: YES if $k = n$ ($\mu \ll \mathcal{H}^n$)
5. Mathé (2012) announced: NO if $n = 3$, $k = 2$.

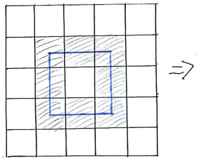
Answer to a Question of Alberti-Csörnyei-Preiss 2

1. de Philippis–Rindler (2016): YES if $k = n$; PDE + Fourier Analysis ☺
2. Schioppa (2016): NO if $2 \leq k \leq n-2$ and $n \geq 4$ ☹
3. Curious link between these works: for $k = n-2$ a normal current with singular mass has to be simple! (PDE + Linear Algebra), no Geometric Intuition ☹

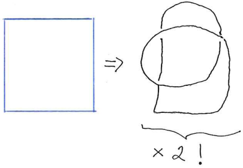
The argument – overview

1. Euclidean tools like Structure Theory for Rectifiable sets, Optimal Transport and Projection Theorems would help to prove YES. ☹
2. In metric spaces there is lots of flexibility to prove NO. ☺
3. One has then to see if the construction can be embedded in Euclidean spaces.

The metric ($K=2$) example

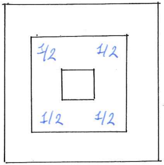


You take out the shaded area, replace it by a double cover & collapse the boundary to glue it back

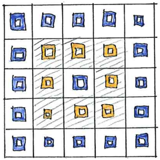


You have to split the measure on the doubled part!

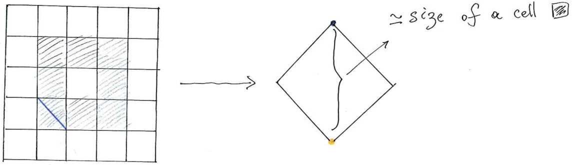
$\Downarrow$



Then you just iterate on subsquares



The fiber



1. 2-unrectifiability proven via a blow-up argument: X_∞ is self-similar and you cannot fit a plane inside ("soft" + topology)

2. Let's try to do the same in ℓ^2; simplest idea: find a biLipschitz map
$$F : X_\infty \hookrightarrow \ell^2$$

3. Cheeger & Kleiner (2013): $(X_\infty, \|N_\infty\|)$ is a PI-space $\Rightarrow$ Cheeger & Kleiner (2010) F would be differentiable $\Rightarrow$ F has to collapse fibers!

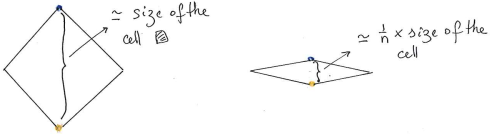
2-current in ℓ^2 (1)

1. Try F Lipschitz & a topological embedding. Have to find a rate of collapse of the fibers that allows 2-unrectifiability.

2. Ansatz : $\simeq 1/n$ at stage n; $\sum \frac{1}{n^2} < \infty$ but $\sum \frac{1}{n} = \infty$

3. F gets replaced by an inverse system F_n.

4.



5. If you always put new spirals in orthogonal directions,

$$|\nabla F_{n+1}| \sim \left(1 + \frac{1}{(n+1)^2}\right) |\nabla F_n|$$

so $F = F_\infty$ will be Lipschitz.

$\Pi(e_1, e_2) \leftarrow$ square

$\Pi(e_3, e_4) \leftarrow$ level 1 -"spirals"

$\vdots$

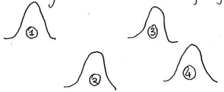
$\Pi(e_{2n+1}, e_{2n+2}) \leftarrow$ level n - "spirals"

2-current in $\ell^2(2)$

1. 2-unrectifiability ("haarol") : collapsing of fibers prevents a blow-up argument
2. Weak $*$ – continuity allows to estimate the intersection of a C-lipschitz surface projecting onto Step n- approximations
3. At step n we get holes of area $\gtrsim \dfrac{\gamma(G')}{n}$;
4. But we have to "skip" to reiterate $n \Rightarrow n_k$
5. Can show $\sum_{k=0}^{s} \frac{1}{n_k} \gtrsim \log \log s$

2-current in $\mathbb{R}^4$

1. Need always orthogonal directions
2. SVM, Kernel trick: you can kind of fit ℓ^2 into $\mathbb{R}^4$!



3. Destroys the self-similarity also for X_∞
4. Now projections are locally defined!

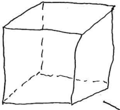
In ℓ^2:

Get a $(1+\varepsilon_n)$-Lipschitz projection

5. You can compose these projections if $\sum \varepsilon_n < \infty$!
6. weak* - continuity becomes even more crucial!

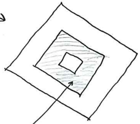
K - current in $\mathbb{R}^{K+2}$

1. Take $K = 3$ in $\mathbb{R}^5$
2. Split coordinates in two groups: $\underbrace{x_1, x_2, x_3}_{\text{3-cubes}}$; $\underbrace{x_4, x_5}_{\text{Spirals}}$

3. 

Project to : $\Pi(x_1, x_2)$ / $\Pi(x_2, x_3)$ / $\Pi(x_1, x_3)$ if iteration mod $3 \equiv$ 0 / 1 / 2

$$\begin{array}{lll} \Pi(x_1,x_2) & & 0 \\ \Pi(x_2,x_3) & \text{if iteration mod } 3 \equiv & 1 \\ \Pi(x_1,x_3) & & 2 \end{array}$$



take double cover for the inverse image