1 General Theory

We first consider a very general type of theory: let $D \subset \mathbb{R}^n$ and $f(\mathbf{x})$, $g(\mathbf{x})$ be defined for $\mathbf{x} \in D$. Let $h(t, \mathbf{x})$ be defined for $\mathbf{x} \in D$ and $t \geq 0$. A wave in n dimensions on a geometry D with initial profiles $f(\mathbf{x})$ and $g(\mathbf{x})$, driven by a source $h(t, \mathbf{x})$ and free to move at its ends is thus characterized by

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \nabla^2 u = h(t, \mathbf{x}), & \mathbf{x} \in D,\ t \geq 0; \\ u(0, \mathbf{x}) = f(\mathbf{x}),\ u_t(0, \mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in D; \\ \dfrac{\partial u}{\partial n} = 0, & \mathbf{x} \in \partial D,\ t \geq 0. \end{cases} \tag{1.1}$$

where u is its amplitude at a particular time t and position $\mathbf{x}$. Typically, the goal is to obtain a solution of the form $u(t, \mathbf{x}) = v(t, \mathbf{x}) + w(t, \mathbf{x})$ where v solves the homogenous problem and boundary/initial conditions. Then, w need only satisfy

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2} - c^2 \nabla^2 u = h(t, \mathbf{x}), & \mathbf{x} \in D,\ t \geq 0; \\ w(0, \mathbf{x}) = f(\mathbf{x}),\ w_t(0, \mathbf{x}) = g(\mathbf{x}), & \mathbf{x} \in D; \\ \dfrac{\partial w}{\partial n} = 0, & \mathbf{x} \in \partial D,\ t \geq 0. \end{cases} \tag{1.2}$$

2 Time Periodic Source

In particular, we are interested in the case when the source varies sinuisoidally with time. For concreteness, we assume $h(t, \mathbf{x}) = \cos(\omega t) h(\mathbf{x})$ where $\omega > 0$ and $h(\mathbf{x}) \in L^2(D)$. For Neumann boundary conditions on the operator $-\nabla^2$, there are eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \quad \text{where} \quad \lim_{n \to \infty} = \infty$$

with corresponding eigenfunctions (that may be taken to be linearly independent) $u_1(\mathbf{x}), u_2(\mathbf{x}), \ldots, u_n(\mathbf{x}) \ldots$, all well-defined on D. This forms a complete basis for $L^2(D)$ and so we may write

$$h(\mathbf{x}) = \sum_{n=1}^{\infty} B_n u_n(\mathbf{x}) \quad \text{where} \quad B_n = \frac{\big\langle h(\mathbf{x}) \big| u_n(\mathbf{x}) \big\rangle}{\big\langle u_n(\mathbf{x}) \big| u_n(\mathbf{x}) \big\rangle}.$$

Similarly, we look for $w(t, \mathbf{x})$ of the form

$$w(t, \mathbf{x}) = \sum_{n=1}^{\infty} T_n(t) u_n(\mathbf{x})$$

and we may obtain the condition for $T_n(t)$ by substitution into the wave equation

$$\frac{\partial^2 w}{\partial t^2} - c^2 \nabla^2 u = h(t, \mathbf{x}) \quad \longleftrightarrow \quad \sum_{n=1}^{\infty} \left(\frac{\partial^2 T_n}{\partial t^2} + c^2 \lambda_n T_n(t) \right) u_n(\mathbf{x}) = \cos(\omega t) \sum_{n=1}^{\infty} B_n u_n(\mathbf{x}).$$

The linearly independent nature of the $u_n(\mathbf{x})$ allows us to conclude that

$$\frac{\partial^2 T_n}{\partial t^2} + c^2 \lambda_n T_n(t) = B_n \cos(\omega t)$$

and holds for $n \in \{1, 2, \ldots\}$. If we take the initial conditions $T(0) = T'(0) = 0$ to satisfy those imposed in (1), this second order DE can be solved with no ambiguity. Still, we segregate into two cases: the resonant ($\omega = c\sqrt{\lambda_n}$) case and the non-resonant ($\omega \neq c\sqrt{\lambda_n}$) case, which have respective solutions

$$T_n(t) = \frac{B_n t \sin\left(c\sqrt{\lambda_n} t\right)}{2c\sqrt{\lambda_n}} \quad \text{and} \quad T_n(t) = B_n \left(\frac{\cos(\omega t) - \cos\left(c\sqrt{\lambda_n} t\right)}{c^2 \lambda_n - \omega^2} \right)$$

so we have found the solution to (2):

$$w(t, \mathbf{x}) = \sum_{n=1}^{\infty} T_n(t) u_n(\mathbf{x}), \quad \text{where} \quad T_n(t) = B_n \begin{cases} \dfrac{\cos(\omega t) - \cos\left(c\sqrt{\lambda_n} t\right)}{c\lambda_n^2 - \omega^2}, & \omega \neq c\sqrt{\lambda_n}; \\ \dfrac{t \sin\left(c\sqrt{\lambda_n} t\right)}{2c\sqrt{\lambda_n}}, & \omega = c\sqrt{\lambda_m} \end{cases}.$$

If $\omega = c\sqrt{\lambda_m}$ for some $m \in \{1, 2, 3, \ldots\}$, we may then write

$$w(t, \mathbf{x}) = \sum_{n=1, n\neq m}^{\infty} B_n \left(\frac{\cos(\omega t) - \cos\left(c\sqrt{\lambda_n} t\right)}{c\lambda_n^2 - \omega^2} \right) u_n(\mathbf{x}) + B_m \left(\frac{t \sin\left(c\sqrt{\lambda_m} t\right)}{2c\sqrt{\lambda_m}} \right) u_m(\mathbf{x})$$

When $B_m \neq 0$, then the term with $n = m$ with dominate as $t = \infty$, and so the terminal behavior of $w(t, x)$ is described simply by

$$w(t, \mathbf{x}) = B_m \left(\frac{t \sin\left(c\sqrt{\lambda_m} t\right)}{2c\sqrt{\lambda_m}} \right) u_m(\mathbf{x}).$$

In particular, note that the nodal lines $\mathbf{x} \in D\,\text{s.t.}\,\forall t,\ w(t, \mathbf{x}) = 0$ are given equivalently by $\mathbf{x} \in D\text{s.t}\, u(\mathbf{x}) = 0$, so they can be determined *without* considering the time-dependence of the system. If the geometry is not driven at a resonant frequency, then there is nothing unique about any particular eigenfunction, and so we write

$$w(t, \mathbf{x}) = \sum_{n=1}^{\infty} B_n \left(\frac{\cos(\omega t) - \cos\left(c\sqrt{\lambda_n} t\right)}{c\lambda_n^2 - \omega^2} \right) u_n(\mathbf{x}).$$

In this case, nodal lines are much more difficult to determine analytically, but can be found using a computer.

3 Application to Square and Circular Plates

Because we tested square and circulate plates, and their geometries are relatively easy to solve, we see what kind of patterns and resonant frequencies theory predicts. For the square plate, we consider a domain of the kind $D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq a,\ 0 \leq a \leq b\}$ and the eigenvalue problem

$$\begin{cases} -\nabla^2 u(x, y) = \lambda u(x, y), & (x, y) \in D; \\ u_n(x, y) = 0, & (x, y) \in \partial D. \end{cases}$$

This can be solved with separation of variables, which yields the following eigenvalue/eigenfunction pairing

$$\lambda_{mn} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2, \qquad u_{mn}(x,y) = \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)$$

The resonant frequencies are therefore

$$\omega = c\sqrt{\lambda} = \frac{c\pi}{a}\sqrt{m^2+n^2} = \frac{c\pi}{a}, \frac{c\pi\sqrt{2}}{a}, \frac{c\pi\sqrt{5}}{a}, \ldots.$$

which, when using $c \sim 10^3$ m/s and $a \sim 10^{-1}$ m, all values appropriate to the plate we used, yields values far exceeding most frequencies we tested. Despite this, our experimentation did produce ample evidence that there *are* nodal lines below these theoretical frequencies, suggesting that this is either not a good model for our setup or that there is a strange linear combination of eigenfunctions that allows this to be true. Since the plates are far from ideal 2D membranes, it is likely that the first is true. Nonetheless, the theoretically expected nodal patterns are still bear some resemblance to those that we obtained experimentally. For the circular plate, we consider the domain $D = \{(r,\theta), | \, 0 \le r \le a, \, 0 \le \theta \le 2\pi\}$ along with the eigenvalue problem

$$\begin{cases} -\Delta u = -u_{rr} - \frac{1}{r}u_r - \frac{1}{r^2}u_{\theta\theta} = \lambda u, & (r,\theta) \in D \\ u_r(a,\theta) = 0, & 0 \le \theta \le 2\pi \end{cases}$$

where $u(r,\theta)$is 2π periodic in θ and continuous on D. By some sort of black magic, we can find the eigenvalue/eigenfunction pairing

$$\lambda_{mn} = (z_{mn}/a)^2, \qquad J_n\left(\frac{z_{mn}r}{a}\right)(A_{mn}\cos(n\theta) + B_{mn}\sin(n\theta))$$

where z_{mn} is the mth zero of the the derivative of J_n. I have no plans to compute the theoretically expected resonant frequencies, since they weren't especially useful for the analysis of our experimental results in the square geometry and the presence of z_{mn} make it all the more complicated to do so. However, it may still be interesting to qualitatively compare the expected nodal patterns with experiment, which I do below: