Phys CS 36: Homework #9

Dashiell Carrel

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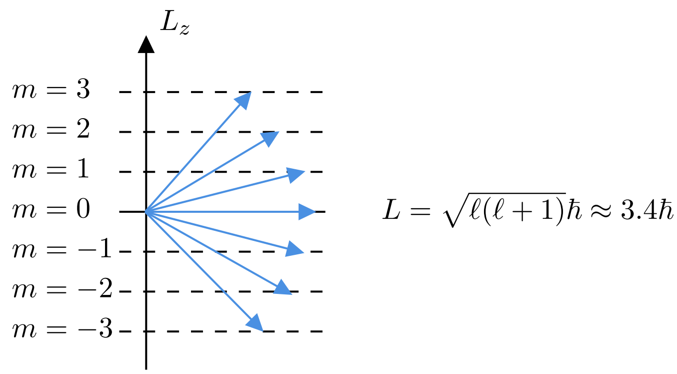
1 TZD Problem 8.25

Problem:
Solution:

(a) Vector Diagram

I do this as follows, for $\ell = 3$.



Evidently, angular momentum cannot be fully concentrated along the z axis.

(b) Orientations

There are seven distinct values of L_z that we can possibly measure, so it follows there are seven possible orientations for $\mathbf{L}$ as it concerns the diagram above.

(c) Angle

The smallest angle that occurs between $\mathbf{L}$ and the z axis is when $L_z = 3$, i.e,

$$\theta = \cos^{-1}(3/3.4) = 28.07°.$$

2 TZD Problem 8.29

Problem:
Solution:

(a) Equation and Solution

When $\ell = m = 0$, the θ equation may be written

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right) = 0.$$

Evidently, then, the choice of $\Theta = \mathit{const}$ will be a solution.

(b) A Second Solution

Since the given function is rather complicated, I will compute the first derivative all by itself:

$$\begin{aligned}\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} &= \left(\frac{1-\cos\theta}{1+\cos\theta}\right)\left(\frac{-\sin\theta}{(1-\cos\theta)^2} - \frac{\sin\theta(1+\cos\theta)}{(1-\cos\theta)^2}\right)\\ &= -\frac{\sin\theta}{1+\cos\theta} - \frac{\sin\theta}{1-\cos\theta} = -\sin\theta\left(\frac{2}{1-\cos^2\theta}\right) = -\frac{2}{\sin\theta}.\end{aligned}$$

Since $\sin\theta\ \mathrm{d}\Theta/\mathrm{d}\theta = \mathit{const}$, it is clear that this is a solution to the $\ell = m = 0$ DE. However, it diverges at $\theta = 0$ and $\theta = \pi$ and so it is not a healthy member of the Hilbert space.

(c) General Solution

Since only the constant solution is a member of the space, we consider, the only acceptable linearly combination is that with the coefficient on the solution in part (b) zero. Thus, the general solution to the $\ell = m = 0$ equation is

$$\Theta = \mathit{const}.$$

3 TZD Problem 8.40

Problem:
Solution:

(a) Radial Equation

When $\ell = n - 1$, the radial equation takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR) = \frac{2m}{\hbar^2}\left(-\frac{ke^2}{r} + \frac{n(n-1)\hbar^2}{2mr^2} - E\right)(rR).$$

(b) Verification

We set out now to verify the solution

$$R(r) = Ar^{n-1}e^{-r/a}.$$

This has second derivative (mixed with a factor of r)

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR) = \frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(Ar^n e^{-r/a}\right) = -\frac{2Anr^{n-1}e^{-r/a}}{a} + An(n-1)r^{n-2}e^{-r/a} + \frac{Ar^n e^{-r/a}}{a^2}.$$

On the right hand side, we have

$$-\frac{2mAke^2 r^{n-1}e^{-r/a}}{\hbar^2} + An(n-1)e^{n-2}e^{-r/a} - \frac{2mEA}{\hbar^2}r^n e^{-r/a}.$$

By matching coefficients, we find that this is satisfied only if

$$E = -\frac{\hbar^2}{2ma^2} \quad \text{and} \quad a = \frac{n\hbar^2}{mke^2} = na_B$$

so that

$$E = -\frac{mk^2e^4}{2\hbar^2 n^2} = -\frac{E_R}{n^2}$$

4 TZD Problem 8.42

Problem:
Solution: The 1s orbital is the ground state orbital of the hydrogen atom., which takes the functional form

$$\psi_{110}(r,\theta,\phi) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}.$$

The average radius is therefore

$$\langle r \rangle = \frac{1}{\pi a_B^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} e^{-2r/a_B} r^3 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{4\pi}{\pi a_B^3} \int_0^{\infty} r^2 e^{-2r/a_B} \, \mathrm{d}r = \frac{3}{2} a_B.$$

Generally, the expected value refers to the mean, while the most probable value is the one with the greatest associated probability. This distinction is realized in Fig 8.18, where we can observer that the most probable value (of r) for an electron in the hydrogenic ground state is a_B, yet we just found that the expectation value of this variable is 3/2 this value.

5 TZD Problem 8.43

Problem:

Solution: The probability that the electron is found *inside* the Bohr radius is

$$P_{\text{in}} = \frac{1}{\pi a^3} \int_0^{2\pi} \int_0^{\pi} \int_0^{a_B} e^{-2r/a_B} r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi = \frac{4}{a_B^3} \int_0^{\infty} r^2 e^{-2r/a_B} \, \mathrm{d}r = \frac{e^2 - 5}{e^2}.$$

So the probability that the electron is found *outside* the Bohr radius is

$$P_{\text{out}} = 1 - P_{\text{in}} = \frac{5}{e^2} \approx 0.67.$$

6 TZD Problem 8.45

Problem:
Solution: For $n = 2$ and $l = 1$, the radial equation takes the form

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rR) = \frac{2m}{\hbar^2}\left(-\frac{ke^2}{r} + \frac{\hbar^2}{mr^2} - E\right)(rR).$$

From Problem 8.40, we know that this has (if you want verification, plug in $n = 2$ and $l = n - 1 = 1$) solution

$$R(r) = Are^{-r/2a_B}.$$

By imposing the normalization condition, we find

$$4\pi A^2 \int_0^\infty r^4 e^{-r/a_B}\,\mathrm{d}r = 96\pi a^5 A^2 = 1 \quad \longrightarrow \quad A = \frac{1}{4\sqrt{6\pi}a^{5/2}}.$$

7 Problem 1

Problem:
Solution:

(a) Spherical Coordinates

Recall that the angular momentum operator may be defined in terms of the gradient operator:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = -i\hbar \mathbf{r} \times \boldsymbol{\nabla}$$

where $\mathbf{r} = r\hat{\mathbf{r}}$ is the position operator. It thus follows that

$$\mathbf{L} = -i\hbar r \left[\hat{\mathbf{r}} \times \left(\hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \right) \right] = -i\hbar \left(\hat{\boldsymbol{\phi}} \frac{\partial}{\partial \theta} - \hat{\boldsymbol{\theta}} \frac{1}{\sin\theta} \frac{\partial}{\partial \phi} \right)$$

since $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \mathbf{0}, \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\theta}}$. Now,

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \quad \text{and} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}} \end{aligned}$$

so we may write

$$\mathbf{L} = -i\hbar \left[\left(-\sin\phi \frac{\partial}{\partial \theta} - \cos\phi\cot\theta \frac{\partial}{\partial \phi} \right) \hat{\mathbf{i}} + \left(\cos\phi \frac{\partial}{\partial \theta} - \sin\phi\cot\theta \frac{\partial}{\partial \phi} \right) \hat{\mathbf{j}} + \frac{\partial}{\partial \phi} \hat{\mathbf{k}} \right].$$

Thus, we can pick out:

$$\begin{aligned} L_x &= -i\hbar \left(-\sin\phi \frac{\partial}{\partial \theta} - \cos\phi\cot\theta \frac{\partial}{\partial \phi} \right), \\ L_y &= -i\hbar \left(\cos\phi \frac{\partial}{\partial \theta} - \sin\phi\cot\theta \frac{\partial}{\partial \phi} \right), \quad \text{and} \\ L_z &= -i\hbar \frac{\partial}{\partial \phi}. \end{aligned}$$

(b) Momentum Squared

To obtain L^2, I estimate that it will be easiest to compute L_x^2, L_y^2, and so forth individually. This goes as follows:

$$L_x^2 = -\hbar^2 \left(\sin^2\phi \frac{\partial^2}{\partial \theta^2} + \cos\phi\cot^2\theta \frac{\partial}{\partial \phi} \left(\cos\phi \frac{\partial}{\partial \phi} \right) + \sin\phi\cos\phi \frac{\partial}{\partial \theta} \left(\cot\theta \frac{\partial}{\partial \phi} \right) + \cos\phi\cot\theta \frac{\partial}{\partial \phi} \left(\sin\phi \frac{\partial}{\partial \theta} \right) \right).$$

For L_y^2, we have

$$L_y^2 = -\hbar^2 \left(\cos^2\phi \frac{\partial^2}{\partial \theta^2} + \sin\phi\cot^2\theta \frac{\partial}{\partial \phi} \left(\sin\phi \frac{\partial}{\partial \phi} \right) - \sin\phi\cos\phi \frac{\partial}{\partial \theta} \left(\cot\theta \frac{\partial}{\partial \phi} \right) - \sin\phi\cot\theta \frac{\partial}{\partial \phi} \left(\cos\phi \frac{\partial}{\partial \theta} \right) \right).$$

When we add these two together, it's clear that the third term drops out, and we get:

$$\begin{aligned}L_x^2 + L_y^2 = -\hbar^2 \bigg[\frac{\partial^2}{\partial\theta^2} &+ \cos\phi\cot^2\theta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) + \sin\phi\cot^2\theta\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) \\ &+ \cos\phi\cot\theta\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right) - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right)\bigg].\end{aligned}$$

It's useful to examine each of these terms individually:

$$\begin{aligned}\cos\phi\cot^2\theta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) &= \cos^2\phi\cot^2\theta\frac{\partial^2}{\partial\phi^2} - \cos\phi\sin\phi\cot^2\theta\frac{\partial}{\partial\theta} \\ \sin\phi\cot^2\theta\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) &= \sin^2\phi\cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cos\phi\sin\phi\cot^2\theta\frac{\partial}{\partial\theta}\end{aligned}$$

so,

$$\cos\phi\cot^2\theta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) + \sin\phi\cot^2\theta\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) = \cot^2\theta\frac{\partial^2}{\partial\phi^2}.$$

Likewise:

$$\begin{aligned}\cos\phi\cot\theta\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right) &= \cos\phi\sin\phi\cot\theta\frac{\partial^2}{\partial\phi\partial\theta} + \cos^2\phi\cot\theta\frac{\partial}{\partial\theta} \\ \sin\phi\cot\theta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right) &= \cos\phi\sin\phi\cot\theta\frac{\partial^2}{\partial\phi\partial\theta} - \sin^2\phi\cot\theta\frac{\partial}{\partial\theta}\end{aligned}$$

so,

$$\cos\phi\cot\theta\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\theta}\right) - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\theta}\right) = \cot\theta\frac{\partial}{\partial\theta}.$$

Putting everything back, we find

$$L_x^2 + L_y^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cot\theta\frac{\partial}{\partial\theta}\right).$$

Finally, adding a factor of $L_z^2 = -\hbar^2\,\partial^2/\partial\phi^2$ gives

$$\begin{aligned}L^2 = L_x^2 + L_y^2 + L_z^2 &= -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{\partial^2}{\partial\phi^2}\right) \\ &= -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \csc^2\theta\frac{\partial^2}{\partial\phi^2}\right).\end{aligned}$$

8 Problem 2

Problem:
Solution: Because $\mathbf{L} = \mathbf{r} \times \mathbf{p} = -i\hbar\mathbf{r} \times \boldsymbol{\nabla}$, we can rewrite its components in index notation as

$$L_k = -i\hbar\epsilon_{ijk}x_i\partial_j,$$

just as desired. We can thus commute the commutator:

$$\begin{aligned}
\left[L_i, L_j\right] &= -\hbar^2\left[\epsilon_{kli}x_k\partial_l(\epsilon_{mnj}x_m\partial_n) - \epsilon_{mnj}x_m\partial_n(\epsilon_{kli}x_k\partial_l)\right] \\
&= -\hbar^2\left[\epsilon_{kli}\epsilon_{mnj}x_k(\delta_{lm}\partial_n + \partial_{ln}) - \epsilon_{mnj}\epsilon_{kli}x_m(\delta_{nk}\partial_l + \partial_{nl})\right] \\
&= -\hbar^2(\epsilon_{kmi}\epsilon_{mnj}x_k\partial_n - \epsilon_{mkj}\epsilon_{kli}x_m\partial_l) \\
&= -\hbar^2(\epsilon_{kim}\epsilon_{jnm}x_k\partial_n - \epsilon_{mjk}\epsilon_{ilk}x_m\partial_l) \\
&= -\hbar^2(x_j\partial_i - \delta_{ij}x_k\partial_k - x_i\partial_j + \delta_{ij}x_mx_l) \\
&= -\hbar^2(x_j\partial_i - x_i\partial_j).
\end{aligned}$$

Note that

$$i\hbar\epsilon_{ijk}L_k = -\hbar^2(\epsilon_{ijk}\epsilon_{lmk}x_l\partial_m) = -\hbar^2(x_i\partial_j - x_j\partial_i)$$

and so we may conclude that

$$\left[L_i, L_j\right] = i\hbar\epsilon_{ijk}L_k.$$

Using this, we can show that $\left[L^2, L_i\right] = 0$ with out too much hassle. First, note that we can write $L^2 = L_jL_j$ so that

$$\begin{aligned}
L^2L_i = L_jL_jL_i &= L_j(L_iL_j + \left[L_j, L_i\right]) \\
&= (L_iL_j + \left[L_j, L_i\right])L_j + L_j\left[L_j, L_i\right] \\
&= L_iL_jL_j + i\hbar\epsilon_{jip/\hbar}L_kL_j + i\hbar\epsilon_{jip/\hbar}L_jL_k \\
&= L_iL_jL_j + i\hbar\epsilon_{jip/\hbar}L_kL_j - i\hbar\epsilon_{kij}L_jL_k = L_iL_jL_j = L_iL^2
\end{aligned}$$

so indeed $\left[L^2, L_i\right] = 0$.

9 Problem 3

Problem:
Solution: Suppose that the incident wave arrives from the right, then the wave function takes the general form

$$\psi(x) = \begin{cases} Ae^{ip/\hbar x} + Be^{-ip/\hbar x}, & x < 0 \\ Ce^{ip/\hbar x}, & x > 0. \end{cases}$$

As we found in Homework #8, the δ potential doesn't stop the wavefunction from being continuous anywhere, but it *does* impose the discontinuity condition

$$\Delta\left(\frac{\mathrm{d}\psi}{\mathrm{d}x}\right) = -\frac{2m\gamma\psi(0)}{\hbar^2}$$

on its derivative, at the origin (if the δ function is centered there). By first imposing the continuity condition, we see that

$$\psi(x) = \begin{cases} Ae^{ip/\hbar x} + Be^{-ip/\hbar x}, & x < 0 \\ (A+B)e^{ip/\hbar x}, & x > 0. \end{cases}$$

and then by using the discontinuity of the derivative at the origin, we obtain

$$(A-B)\frac{ip}{\hbar} - (A+B)\frac{ip}{\hbar} = -\frac{2mV_0(A+B)}{\hbar^2} \quad \longrightarrow \quad B = \frac{mV_0(A+B)}{\hbar ip} \quad \longrightarrow \quad \frac{B}{A} = \frac{mV_0/\hbar ip}{1 - mV_0/\hbar ip}.$$

The probability of reflection is therefore (see Homework #8, Problem 3)

$$R = \left(\frac{B}{A}\right)^2 = \frac{m^2V_0^2/p^2\hbar^2}{1 + m^2V_0^2/p^2\hbar^2}$$

and the transmission coefficient is

$$T = \left|1 + \frac{B}{A}\right|^2 = \left|\frac{1}{1 - mV_0/\hbar ip}\right|^2 = \frac{1}{1 + m^2V_0^2/\hbar^2p^2}.$$