On the Mason-Stothers theorem

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Goal

Theorem (Mason-Stothers)

Let f, g, h be non-constant relatively prime polynomials satisfying $f + g = h$. Then

$$\deg f, \deg g, \deg h \leq n_0(fgh) - 1.$$

Justification

Theorem 1.1

Let n be an integer ≥ 3. There is no solution to the equation

$$u^n + v^n = w^n$$

with non-constant relatively prime polynomials u, v, w.

Polynomials

We will work over an algebraically closed field F of characteristic 0, the complex numbers if you wish.
Let $f(x) \in K[x]$ be a non-zero polynomial, with its factorization

$$f(x) = c\prod_{i=1}^{r}(x-\alpha_i)^{m_i} = c(x-\alpha_1)^{m_1}\dots(x-\alpha_r)^{m_r}, \qquad (1)$$

with a non-zero constant c, and the distinct roots α_i $(i = 1, \dots, r)$.

Polynomials

It is convenient to write the factorization of $f(x)$ in the form

$$f(x) = (x - \alpha)^{m(\alpha)} g(x)$$

where $g(\alpha) \neq 0$ and $m(\alpha)$ is the multiplicity of α. If $\alpha = \alpha_k$ for some index k, then

$$f(x) = c(x - \alpha_k)^{m_k} \prod_{i \neq k} (x - \alpha_i)^{m_i},$$

and $g(x) = c \prod_{i \neq k} (x - \alpha_i)^{m_i}$.

Statement

Lemma 3.1

Let $f(x)$ be a polynomial over an algebraically closed field. Let α be a root of f with multiplicity $m(\alpha)$. Then the multiplicity of α in $f'(x)$ is $m(\alpha) - 1$.

Proof

Proof.

Write $f(x) = (x-\alpha)^m g(t)$ with $g(\alpha) \neq 0$. By taking the derivative of f, we get.

$$\begin{aligned} f'(x) &= (x-\alpha)^m g'(x) + m(x-\alpha)^{m-1} g(x) \\ &= (x-\alpha)^{m-1}((x-\alpha)g'(x) + mg(t)) = (x-\alpha)^{m-1} h(x) \end{aligned}$$

where $h(x) = ((x-\alpha)g'(x) + mg(x))$. Obviously $h(\alpha) \neq 0$. So that $m-1$ is the highest power such that $(x-\alpha)^{m-1}$ divides $f'(x)$.

□

Statement

Define $n_0(f)$ to be the number of distinct roots of f.

Corollary 1

Let f be a non-constant polynomial. Suppose that $f(t)$ has the factorization (1)*. Then the $g.c.d.$ (f, f') is*

$$(f, f') = c_1 \prod_{i=1}^{r} (x - \alpha)^{m_i - 1},$$

with some constant c_1. In particular,

$$\deg(f, f') = \deg f - n_0(f).$$

Significance

It is immediate that if f, g are non-zero polynomials, then

$$n_0(fg) \leq n_0(f) + n_0(g).$$

If in addition f, g are coprime, then equality holds.
It is obvious that $\deg f$ can be very large, but $n_0(f)$ may be small.

Example 4.1

$$f(x) = (x - \alpha)^{1000}$$

The polynomial f has degree 1000, but $n_0(f) = 1$.

The Mason-Stothers theorem gives a remarkable additive condition under which the degree cannot be large.

Formulation

Theorem (Mason-Stothers)

Let f, g, h be non-constant relatively prime polynomials satisfying $f + g = h$ Then

$$\deg f, \deg g, \deg h \leq n_0(fgh) - 1.$$

Proof

Proof.

We first note the identity

$$f'g - fg' = f'h - fh'.$$

This trivially follows from $f' + g' = h'$.
We have $f'g - fg' \neq 0$, otherwise g would divide g' since f, g are relatively prime and non-constant.
Then we notice that the g.c.d. (f, f') divides the left side and so does (g, g'), and (h, h') divides the right side, which is equal to the left side. Therefore, since f, g, h are relatively prime,

$$\text{the product } (f, f')(g, g')(h, h') \text{ divides } f'g - fg'.$$

Proof

Proof.

This yields an inequality between the degress, namely,

$$\begin{aligned} \deg(f, f') + \deg(g, g') + \deg(h, h') &\leq \deg(f'g - fg') \\ &\leq \deg f + \deg g - 1 \end{aligned} \tag{2}$$

We now use Corrolary 1 for each of f, g, h :

$$\begin{aligned} \deg(f, f') &= \deg f - n_0(f) \\ \deg(g, g') &= \deg g - n_0(g) \\ \deg(h, h') &= \deg h - n_0(h) \end{aligned}$$

Proof

Proof.

We substitue these equalities in (2) and, after simplifying, get the following:

$$\deg h \leq n_0(f) + n_0(g) + n_0(h) - 1 = n_0(fgh) - 1.$$

The result for f, g follows from the fact that $f'g - fg' = f'h - fh' = h'g - hg'$ and by changing the appropriate value in (2). □

Fermat's analogue

Theorem

Let n be an integer ≥ 3. There is no solution to the equation

$$u^n + v^n = w^n$$

with non-constant relatively prime polynomials u, v, w.

Proof

Proof.

Let $f = u^n, g = v^n$, and $h = w^n$. Then by the Mason-Stothers theorem

$$\deg u^n \leq n_0(u^n v^n w^n) - 1.$$

However, $\deg u^n = n \cdot \deg u$ and $n_0(u^n) = n_0(f) \leq \deg u$. Hence

$$n \cdot \deg u \leq \deg u + \deg v + \deg w - 1.$$

Similarly, we can obtain the analogous inequalities for v and w.
Adding the three inequalities yields

$$n(\deg uvw) \leq 3(\deg uvw) - 3 < 3(\deg uvw).$$

Cancelling $\deg uvw$ yields $n < 3$, thus proving the theorem. □

Reference

Lang, Serge. *Undergraduate Algebra.* 3rd ed. New York: Springer-Verlang, 2005. pp. 105-169. ISBN0-387-22025-9.