

On the Mason-Stothers theorem

Emils Kalugins

University of Latvia

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Goal

Theorem (Mason-Stothers)

Let f, g, h be non-constant relatively prime polynomials satisfying $f + g = h$. Then

$$\deg f, \deg g, \deg h \leq n_0(fgh) - 1.$$

Justification

Theorem 1.1

Let n be an integer ≥ 3 . There is no solution to the equation

$$u^n + v^n = w^n$$

with non-constant relatively prime polynomials u, v, w .

Polynomials

We will work over an algebraically closed field F of characteristic 0, the complex numbers if you wish.

Let $f(x) \in F[x]$ be a non-zero polynomial, with its factorization

$$f(x) = c \prod_{i=1}^r (x - \alpha_i)^{m_i} = c(x - \alpha_1)^{m_1} \dots (x - \alpha_r)^{m_r}, \quad (1)$$

with a non-zero constant c , and the distinct roots α_i ($i = 1, \dots, r$).

Polynomials

It is convenient to write the factorization of $f(x)$ in the form

$$f(x) = (x - \alpha)^{m(\alpha)} g(x)$$

where $g(\alpha) \neq 0$ and $m(\alpha)$ is the multiplicity of α . If $\alpha = \alpha_k$ for some index k , then

$$f(x) = c(x - \alpha_k)^{m_k} \prod_{i \neq k} (x - \alpha_i)^{m_i},$$

and $g(x) = c \prod_{i \neq k} (x - \alpha_i)^{m_i}$.

Statement

Lemma 3.1

Let $f(x)$ be a polynomial over an algebraically closed field. Let α be a root of f with multiplicity $m(\alpha)$. Then the multiplicity of α in $f'(x)$ is $m(\alpha) - 1$.

Proof

Proof.

Write $f(x) = (x - \alpha)^m g(x)$ with $g(\alpha) \neq 0$. By taking the derivative of f , we get.

$$\begin{aligned} f'(x) &= (x - \alpha)^m g'(x) + m(x - \alpha)^{m-1} g(x) \\ &= (x - \alpha)^{m-1} ((x - \alpha)g'(x) + mg(x)) = (x - \alpha)^{m-1} h(x) \end{aligned}$$

where $h(x) = ((x - \alpha)g'(x) + mg(x))$. Obviously $h(\alpha) \neq 0$. So that $m - 1$ is the highest power such that $(x - \alpha)^{m-1}$ divides $f'(x)$.



Statement

Define $n_0(f)$ to be the number of distinct roots of f .

Corollary 1

Let f be a non-constant polynomial. Suppose that $f(t)$ has the factorization (1). Then the g.c.d. (f, f') is

$$(f, f') = c_1 \prod_{i=1}^r (x - \alpha)^{m_i - 1},$$

with some constant c_1 . In particular,

$$\deg(f, f') = \deg f - n_0(f).$$

Significance

It is immediate that if f, g are non-zero polynomials, then

$$n_0(fg) \leq n_0(f) + n_0(g).$$

If in addition f, g are coprime, then equality holds.

It is obvious that $\deg f$ can be very large, but $n_0(f)$ may be small.

Example 4.1

$$f(x) = (x - \alpha)^{1000}$$

The polynomial f has degree 1000, but $n_0(f) = 1$.

The Mason-Stothers theorem gives a remarkable additive condition under which the degree cannot be large.

Formulation

Theorem (Mason-Stothers)

Let f, g, h be non-constant relatively prime polynomials satisfying $f + g = h$. Then

$$\deg f, \deg g, \deg h \leq n_0(fgh) - 1.$$

Proof

Proof.

We first note the identity

$$f'g - fg' = f'h - fh'.$$

This trivially follows from $f' + g' = h'$.

We have $f'g - fg' \neq 0$, otherwise g would divide g' since f, g are relatively prime and non-constant.

Then we notice that the g.c.d. (f, f') divides the left side and so does (g, g') , and (h, h') divides the right side, which is equal to the left side. Therefore, since f, g, h are relatively prime,

the product $(f, f')(g, g')(h, h')$ divides $f'g - fg'$.

Proof

Proof.

This yields an inequality between the degrees, namely,

$$\begin{aligned}\deg(f, f') + \deg(g, g') + \deg(h, h') &\leq \deg(f'g - fg') & (2) \\ &\leq \deg f + \deg g - 1\end{aligned}$$

We now use Corollary 1 for each of f, g, h :

$$\deg(f, f') = \deg f - n_0(f)$$

$$\deg(g, g') = \deg g - n_0(g)$$

$$\deg(h, h') = \deg h - n_0(h)$$

Proof

Proof.

We substitute these equalities in (2) and, after simplifying, get the following:

$$\deg h \leq n_0(f) + n_0(g) + n_0(h) - 1 = n_0(fgh) - 1.$$

The result for f, g follows from the fact that $f'g - fg' = f'h - fh' = h'g - hg'$ and by changing the appropriate value in (2).



Fermat's analogue

Theorem

Let n be an integer ≥ 3 . There is no solution to the equation

$$u^n + v^n = w^n$$

with non-constant relatively prime polynomials u, v, w .

Proof

Proof.

Let $f = u^n$, $g = v^n$, and $h = w^n$. Then by the Mason-Stothers theorem

$$\deg u^n \leq n_0(u^n v^n w^n) - 1.$$

However, $\deg u^n = n \cdot \deg u$ and $n_0(u^n) = n_0(f) \leq \deg u$. Hence

$$n \cdot \deg u \leq \deg u + \deg v + \deg w - 1.$$

Similarly, we can obtain the analogous inequalities for v and w .
Adding the three inequalities yields

$$n(\deg uvw) \leq 3(\deg uvw) - 3 < 3(\deg uvw).$$

Cancelling $\deg uvw$ yields $n < 3$, thus proving the theorem. □

Reference

Lang, Serge. *Undergraduate Algebra*. 3rd ed. New York: Springer-Verlang, 2005. pp. 105-169. ISBN0-387-22025-9.