On Choquet integral with respect to a rough measure

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Measure and Fuzzy measure

Definition

Let X be a set and Σ a σ-algebra over X. A set function $\mu : \Sigma \to \mathbb{R}_{\geq 0}$ is a measure on (X, Σ) if it satisfies the following axioms:

1. $\mu(\emptyset) = 0$.
2. $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ for all countable collections $\{E_i\}_{i=1}^{\infty}$ of pairwise disjoint sets in Σ.

Definition

Let X be a set. A set function $\mu : 2^X \to \mathbb{R}_{\geq 0}$ is a fuzzy measure on $(X, 2^X)$ if it satisfies the following axioms:

1. $\mu(\emptyset) = 0$.
2. $A \subseteq B$ implies $\mu(A) \leq \mu(B)$ for $A, B \in 2^X$.

Choquet integral

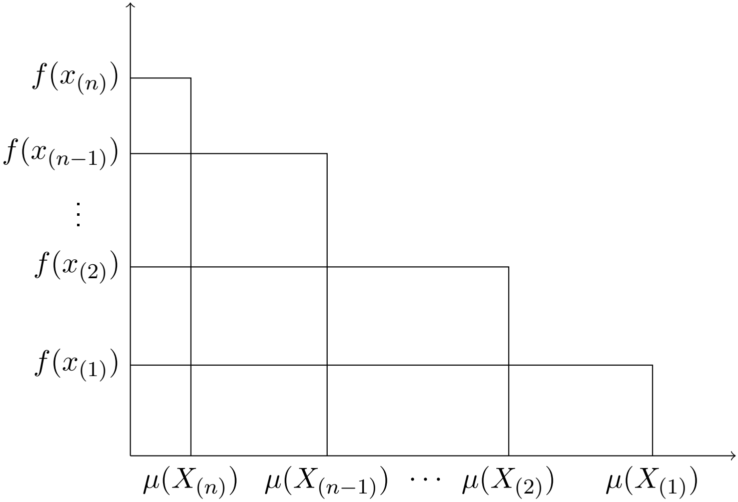
Definition

Let X be a finite, non-empty set whose elements are denoted by $x_1, \ldots, x_n$. Let μ be a fuzzy measure on $(X, 2^X)$. Then the Choquet integral $C_\mu(f)$ of $f : X \to [0,1]$ with respect to μ is defined by

$$C_\mu(f) := \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)}))\mu(X_{(i)}),$$

where $f(x_{(i)})$ indicates the the indices have been permuted so that $0 \leq f(x_{(1)}) \leq \cdots \leq f(x_{(n)})$, $X_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$ and $f(x_{(0)}) = 0$.

Interpretation of Choquet integral



Alternative definition

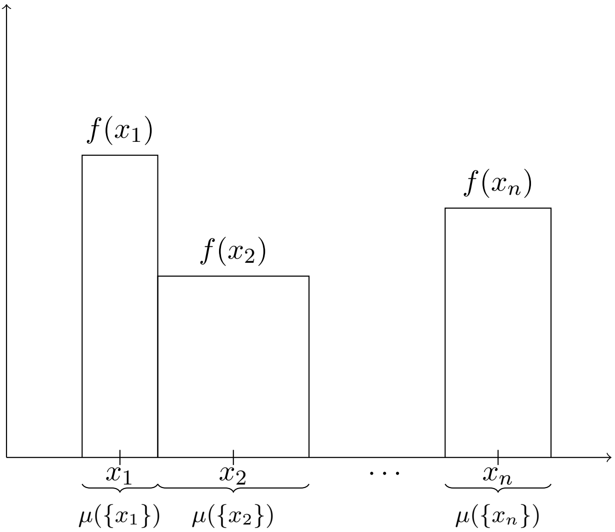
Proposition

Let X be a finite, non-empty set and μ be a fuzzy measure on $(X, 2^X)$. Then

$$C_\mu(f) = \sum_{i=1}^{n} f(x_{(i)})(\mu(X_{(i)}) - \mu(X_{(i+1)})),$$

where $f(x_{(i)})$ indicates the the indices have been permuted so that $0 \leq f(x_{(1)}) \leq \cdots \leq f(x_{(n)})$, $X_{(i)} = \{x_{(i)}, \ldots, x_{(n)}\}$ and $X_{(n+1)} = \emptyset$.

Integral with respect to a measure



Example (Grabisch (1996))

- Let us consider a problem of evaluating students in a high school with respect to three subjects: Mathematics (M), Physics (P) and Literature (L) on a scale from 0 to 10.
- Usually, this is done by a weighted sum. Suppose that this school is more scientifically oriented, so that the weights are $w_M = 3/8$, $w_P = 3/8$ and $w_L = 2/8$.

Student	M	P	L	Weighted sum
A	9	8	5	7.625
B	5	6	9	6.375
C	7	8	7	7.375

Example (Grabisch (1996))

- If the school wants to favour well rounded students without weaknesses then student C should be considered better than student A.
- Usually, students that are good at mathematics are also good at physics (and vice versa), so that the evaluation is overestimated (underestimated) in the weighted sum for students good (bad) at mathematics and physics.
- We solve this by finding a suitable fuzzy measure μ and the Choquet integral.

Example (Grabisch (1996))

- Individually, we still want to emphasize science so we put the weights

$$\mu(\{M\}) = \mu(\{P\}) = 0.45,\ \ \mu(\{L\}) = 0.3.$$

- Since mathematics and physics overlap, the weight of $\{M, P\}$ should be lower than the sum:

$$\mu(\{M, P\}) = 0.5 < 0.45 + 0.45.$$

- Also we should reward those who are both good at science *and* literature so that the weight attributed to $\{M, L\}$ and $\{P, L\}$ be higher than the individual sum:

$$\mu(\{M, L\}) = \mu(\{P, L\}) = 0.9 > 0.45 + 0.3.$$

- $\mu(\emptyset) = 0$ and $\mu(\{M, P, L\}) = 1$ by definition.

Example (Grabisch (1996))

Now we are able to calculate the Choquet integral for each student.

Student	M	P	L
A	9	8	5
B	5	6	9
C	7	8	7

$$\begin{aligned} C_\mu(A) &= 5 \cdot \mu(\{L, P, M\}) + (8-5) \cdot \mu(\{P, M\}) + (9-8) \cdot \mu(\{M\}) \\ &= 5 \cdot 1 + 3 \cdot 0.5 + 1 \cdot 0.45 = 6.95 \end{aligned}$$

$$C_\mu(B) = 6.4$$

$$C_\mu(C) = 7.45$$

Example (Grabisch (1996))

Applying the Choquet integral to the three students we get the following result:

Student	M	P	L	Choquet integral
A	9	8	5	6.95
B	5	6	9	6.4
C	7	8	7	7.45

We have achieved our goal while ranking student B lower than A since science is rewarded higher.

Information systems

Let $S = (U, A)$ be an information system where U is a non-empty, finite set of objects and A is a non-empty finite set of attributes, where $a : U \to V_a$ for every $a \in A$. For each $B \subseteq A$, there is associated an equivelence relation $Ind_A(B)$ such that

$$Ind_A(B) = \{(x, y) \in U^2 \mid \forall a \in B \; a(x) = a(y)\}.$$

Example. Consider the following decision table.

X	a	e	X	a	e
x_1	0.20	0	x_6	0.45	1
x_2	0.45	1	x_7	0.46	1
x_3	0.45	1	x_8	0.40	1
x_4	0.11	0	x_9	0.41	1
x_5	0.10	0	x_{10}	0.20	0

Set approximations

The notation $[x]_B$ denotes equivelence classes of $Ind_A(B)$. For $X \subseteq U$ the set X can be approximated from the information contained in B by constructing B-lower and B-upper approximation denoted by $\underline{B}X$ and $\overline{B}X$ respectively where

$$\underline{B}X = \{x \mid [x]_B \subseteq X\},$$
$$\overline{B}X = \{x \mid [x]_B \cap X \neq \emptyset\}.$$

Example

X	a	e	X	a	e
x_1	0.20	0	x_6	0.45	1
x_2	0.45	1	x_7	0.46	1
x_3	0.45	1	x_8	0.40	1
x_4	0.11	0	x_9	0.41	1
x_5	0.10	0	x_{10}	0.20	0

Let $X = \{x_1\}, B = \{a\}$. Then $\underline{B}X = \emptyset$ and $\overline{B}X = \{x_1, x_{10}\}$.
Let $X = \{x_1, x_2, x_4, x_{10}\}, B = \{a\}$. Then $\underline{B}X = \{x_1, x_4, x_{10}\}$ and $\overline{B}X = \{x_1, x_2, x_3, x_4, x_6, x_{10}\}$.

Rough measure

Definition

Let $S = (U, A)$ be an information system, $X \subseteq U, B \subseteq A$ and $u \in U$. Let $\rho' : 2^X \to \mathbb{R}_{\geq 0}$ be an additive set function. Then we define $\rho_u : 2^X \to \mathbb{R}_{\geq 0}$ by $\rho_u(Y) = \rho'(Y \cap [u]_B])$ for $Y \in 2^X$. The set function ρ_u is called a *rough measure* relative to $U/Ind_A(B)$ and u.

Rough membership function

Definition

Let $S = (U, A)$ be an information system, $B \subseteq A, u \in U$ and let $[u]_B$ be an equivelence class of an object $u \in U$ of $Ind_A(B)$. A set function given by

$$\mu_u^B : 2^U \to [0,1], \text{ where } \mu_u^B(X) = \frac{card(X \cap [u]_B)}{card([u]_B)}$$

for any $X \in 2^U$ is called a *rough membership set function*.

Proposition

Let $S = (U, A)$ be an information system. For every $u \in U$ and $B \subseteq A$ the function μ_u^B is a rough measure.

Rough integral

Definition

Let ρ be a rough measure on X where the elements of X are denoted by $x_1, \ldots, x_n$. The discrete rough integral of $f : X \to \mathbb{R}_{\geq 0}$ with respect to the rough measure ρ is defined by

$$\int_X f \; d\rho = \sum_{i=1}^{n} (f(x_{(i)}) - f(x_{(i-1)}))\rho(X_{(i)})$$

Example

Suppose that we have two identical sensors with five measurements each. We consider the measurement of the relevance of an sensor using a rough integral.

X	a	e	Y	a	e
x_1	0.20	0	y_1	0.45	1
x_2	0.45	1	y_2	0.46	1
x_3	0.45	1	y_3	0.40	1
x_4	0.11	0	y_4	0.41	1
x_5	0.10	0	y_5	0.20	0

We begin by constructing μ_u^e taking $u = x_2$ so that $[u]_e = \{x_2, x_3, y_1, y_2, y_3, y_4\}$.

Example

Since $a(x_1) = 0.20, a(x_2) = 0.45, a(x_3) = 0.45, a(x_4) = 0.11$ and $a(x_5) = 0.10$, we have $a(x_5) \leq a(x_4) \leq a(x_1) \leq a(x_2) \leq a(x_3)$.

After permutating the indices we get
$x_{(1)} = x_5,\ x_{(2)} = x_4,\ x_{(3)} = x_1,\ x_{(4)} = x_2,\ x_{(5)} = x_3$.

$X_{(1)} = \{x_{(1)}, x_{(2)}, x_{(3)}, x_{(4)}, x_{(5)}\}$.

$$\mu_u^e(X_{(1)}) = \frac{card(X_{(1)} \cap [u]_e)}{card([u]_e)} = \frac{card(\{x_2, x_3\})}{card(\{x_2, x_3, y_1, y_2, y_3, y_4\})} = \frac{1}{3}.$$

Example

We repeat the process for $X_{(2)} = \{x_{(2)}, x_{(3)}, x_{(4)}, x_{(5)}\}, X_{(3)}$, etc.

$\mu_u^e(X_{(2)}) = \mu_u^e(X_{(3)}) = \mu_u^e(X_{(4)}) = \frac{1}{3}$,

$\mu_u^e(X_{(5)}) = \frac{1}{6}$.

$\int_X a \, d\mu_u^e = 0.1 \cdot 1/3 + (0.11 - 0.1) \cdot 1/3 + (0.45 - 0.11) \cdot 1/3 + (0.45 - 0.45) \cdot 1/6 = 0.45/3 = 0.15$

After reordering the sensor values from Y, we similarly obtain $\int_Y a \, d\mu_u^e = 0.285$.

Example

From these two cases, it can be seen the relevance of attribute improves as the value of the rough integral increases in value. For a particular $[u]_e$, the rough integral measures the relevance of an attribute (sensor) for a particular table in a classification effort.

This raises the question: *Does the integral value reflect (to a sufficient degree) an expert opinion?* We must notice that the lowest sample values in a signal have considerable influence in the computation. In the case where a decision-maker (expert) focuses on values near or above a certain boundary, the integral does approximate an expert decision.

Potential uses

Suppose we have a decision system. Is it possible to determine which attribute is the decisive one in an expert's decision?

There has been an attempt to answer this question through defining a definite rough integral using upper and lower rough integrals.

Pattaraintakorn, Puntip & Peters, James & Ramanna, Sheela. (2009). Capacity-Based Definite Rough Integral and Its Application. 59. 299—308.

Unfortunately, this requires a more thorough look as the paper contains errors.

References

G. Choquet, Theory of capacities. Annales de l'Institut Fourier, 5, 1953, 131—295.

Grabisch, M., 1996. The application of fuzzy integrals in multicriteria decision making. European Journal of Operational Research 89,445—456.

Pawlak Z., Peters J.F., Skowron A., Suraj Z., Ramanna S., Borkowski M. (2003) Rough Measures, Rough Integrals and Sensor Fusion. In: Inuiguchi M., Hirano S., Tsumoto S. (eds) Rough Set Theory and Granular Computing. Studies in Fuzziness and Soft Computing, vol 125. Springer, Berlin, Heidelberg