1 Ecuaciones Diferenciales (1ra parte)

1 Determine el orden y, si existe, el grado de las siguientes EDOs. Reconozca las que son del tipo lineal:

a) $(y'')^2 - (y''') = y - (y')^2.$

ORDEN: 3.
GRADO: 1.

b) $y''' + x\cdot(y')^4 = 0.$

ORDEN: 3.
GRADO: 1.

c) $(1+x)\cdot(y'')^4 + 3y''' + 5x^2y = 0.$

ORDEN: 3.
GRADO: 1.

d) $y'' - 3\cdot \text{sen}(y') + y = x^3.$

ORDEN: 2.
~~GRADO: 1~~ No es polinómica → no tiene grado.

e) $3x\cdot dy - y\cdot dx = 0. \rightarrow 3x\cdot dy = y\cdot dx$

$\frac{dy}{dx} = \frac{y}{3x}$

$y'\cdot 3x - y = 0$
LINEAL

ORDEN: 1.
GRADO: 1.

f) $x\cdot y'' - 4\cdot y' + x - 1 = 0. \Rightarrow \frac{x\cdot y''}{x} - \frac{4y'}{x} = \frac{1}{x} - \frac{x}{x}$

$y'' - \frac{4}{x}y' = \frac{1}{x} - 1$
LINEAL

ORDEN: 2.
GRADO: 1.

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3 Halle la ED correspondiente a las siguientes familias de curvas:

a) $y^2 = 4ax. \Rightarrow 2y\cdot y' = 4a \rightarrow 2y\cdot y'\cdot x = 4ax \Rightarrow 2y\cdot y'\cdot x = y^2$

$\boxed{y = 2y'x}$

b) $x^2 + y^2 = r^2 \Rightarrow 2x + 2y\cdot y' = 0 \Rightarrow \cancel{2}y\cdot y' = -\cancel{2}x \Rightarrow \boxed{y' = -\frac{y}{x}}$

c) $y = \text{sen}(ax+b) \Rightarrow y' = \cos(ax+b)\cdot a \Rightarrow y'' = -\text{sen}(ax+b)\cdot a\cdot a \rightarrow y'' = -a^2\cdot\text{sen}(ax+b)$

$y'' = -a^2\cdot y$

Se despeja la constante para eliminarla...

$y^2 = \text{sen}^2(ax+b)$

$y'^2 = \cos^2(ax+b)\cdot a^2 \rightarrow \frac{y'^2}{a^2} = \cos^2(ax+b)$

$\rightarrow y^2 + \frac{y'^2}{a^2} = \text{sen}^2(ax+b) + \cos^2(ax+b)$

$y^2 + \frac{y'^2}{a^2} = 1$

$\frac{y'^2}{a^2} = 1 - y^2$

$a^2 = \frac{y'^2}{1-y^2}$

$y'' = -a^2\cdot y$

$y'' = \frac{y'^2}{1-y^2}\cdot y$

$\boxed{(1-y^2)\cdot y'' = y'^2\cdot y}$

(d) $y = a\cdot e^x + b\cdot x\cdot e^x \Rightarrow y = e^x(a+bx)$

$y' = \underline{e^x\cdot(a+bx)} + e^x\cdot b \rightarrow y' = y + e^x\cdot b$

$y'' = \underline{e^x\cdot(a+bx) + e^x\cdot b} + e^x\cdot b \rightarrow y'' = y' + e^x\cdot b$

$y' - y'' = y + \cancel{e^x\cdot b} - (y' + \cancel{e^x\cdot b})$

$y' - y'' = y - y'$

$$\boxed{y'' - 2y' + y = 0}$$

(e) $y = C_1\cdot x + C_2\cdot x^{-1} + C_3$

$y' = C_1 - C_2\cdot x^{-2}$

$y'' = 2\cdot C_2\cdot x^{-3}$

$y''' = -6\cdot C_2\cdot x^{-4}$

$\dfrac{y'''}{y''} = \dfrac{-6\cdot \cancel{C_2}\cdot x^{-4}}{2\cdot \cancel{C_2}\cdot x^{-3}} \Rightarrow \dfrac{y'''}{y''} = -\dfrac{3}{x} \rightarrow \boxed{x\cdot y''' + 3y'' = 0}$

(f) $y = b\cdot a^x$

$y' = \underline{b\cdot a^x}\cdot \ln a \rightarrow y' = y\cdot \ln a \Rightarrow \ln a = \dfrac{y'}{y}$

$y'' = \underline{b\cdot a^x\cdot \ln a}\cdot \ln a \Rightarrow y'' = y'\cdot \ln a \Rightarrow y'' = y'\cdot \dfrac{y'}{y} \Rightarrow \boxed{y\cdot y'' = y'^2}$

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7 Halle, según corresponda, la SG o la SP de las siguientes EDs:

(a) $y' = \dfrac{x^2+1}{2-y}$, con $y(-3) = 4$.

$\dfrac{dy}{dx} = \dfrac{x^2+1}{2-y}$

$\int (2-y)\,dy = \int (x^2+1)\cdot dx$

$2y - \frac{1}{2}y^2 = \frac{1}{3}x^3 + x + c$

$\boxed{-\frac{1}{2}y^2 + 2y - \frac{1}{3}x^3 - x = c}$ (SG)

$-\frac{1}{2}(4)^2 + 2(4) - \frac{1}{3}(-3)^3 - (-3) = c$

$c = 3$

$\boxed{-\frac{1}{2}y^2 + 2y - \frac{1}{3}x^3 - x = 3}$ (SP)

(b) $x\dfrac{dy}{dx} - y = 2x^2 y$

$x\cdot\dfrac{dy}{dx} = 2x^2 y + y$

$x\cdot\dfrac{dy}{dx} = y(2x^2+1)$

$\dfrac{dy}{y} = \dfrac{2x^2+1}{x}\,dx$

$\int \frac{1}{y}\,dy = \int \frac{2x^{\cancel{2}}}{\cancel{x}}dx + \int \frac{1}{x}\,dx$

$\ln|y| = x^2 + \ln|x| + c$

$|y| = e^{\ln|x| + x^2 + c}$

$y = e^{\ln|x|}\cdot e^{x^2}\cdot e^c$

$\boxed{y = |x|\cdot e^{x^2}\cdot c}$ (SG)

c) $y' = 2x\sqrt{y-1}$

$\frac{dy}{dx} = 2x \cdot (y-1)^{\frac{1}{2}}$

$\frac{dy}{(y-1)^{1/2}} = 2x \cdot dx$

$\int (y-1)^{-\frac{1}{2}} dy = \int 2x \cdot dx$

$\frac{(y-1)^{\frac{1}{2}}}{\frac{1}{2}} = x^2 + c$

$\boxed{2\sqrt{y-1} = x^2 + c}$ (SG)

d) $x^2 dy = \frac{(x^2+1)\,dx}{3y^2+1}$ con $y(1) = 2.$

$(3y^2+1)\,dy = \frac{x^2+1}{x^2}\,dx$

$\int (3y^2+1)\,dy = \int \left(1 + \frac{1}{x^2}\right) dx$

$\boxed{y^3 + y = x - \frac{1}{x} + c}$ (SG)

$(2)^3 + (2) = (1) - \frac{1}{(1)} + c$

$8 + 2 = \cancel{1} - \cancel{1} + c$

$c = 10$

$\boxed{y^3 + y = x - \frac{1}{x} + 10}$

e) $y' = \frac{x}{\sqrt{x^2+9}}$ con $y(4) = 2.$

$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+9}}$

$\int dy = \frac{1}{2}\int \frac{2x}{\sqrt{x^2+9}}\,dx$

$u = x^2+9$

$du = 2x \cdot dx$

$\int \frac{2x}{\sqrt{x^2+9}}\,dx$

$\int \frac{du}{\sqrt{u}} = \int u^{-1/2} \cdot du$

$\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \cdot du$

$2\sqrt{u} + c$

$2\sqrt{x^2+9} + c$

$y = \frac{1}{2} 2\sqrt{x^2+9} + c$

$\boxed{y = \sqrt{x^2+9} + c}$ (SG)

$(2) = \sqrt{(4)^2+9} + c$

$2 = 5 + c$

$c = 3$

$\boxed{y = \sqrt{x^2+9} + 3}$ (SP)

f) $y' = x \cdot y + x - 2y - 2$ con $y(0) = 2.$

$\frac{dy}{dx} = x(y+1) - 2(y+1)$

$\frac{dy}{dx} = (x-2) \cdot (y+1)$

$\int \frac{dy}{y+1} = \int (x-2)\,dx$

$\ln(y+1) = \frac{x^2}{2} - 2x + \ln c$

$e^{\ln(y+1)} = e^{\frac{x^2}{2} - 2x} \cdot e^{\ln c}$

$\boxed{y+1 = c \cdot e^{\frac{x^2}{2} - 2x}}$ (SG)

$(2) + 1 = c \cdot e^{\frac{(0)^2}{2} - 2(0)}$

$3 = c \cdot 1$

$c = 3$

$\boxed{y = 3 \cdot e^{\frac{x^2}{2} - 2x} - 1}$ (SP)

$\int (x-2)\,dx \rightarrow \frac{(x-2)^2}{2}$ $\quad \frac{x^2 - 4x + 4}{2}$ $\quad \frac{x^2}{2} - 2x + 2$

$\int x \cdot dx - \int 2\,dx = \frac{x^2}{2} - 2x$

13 Resuelva las siguientes EDLs de 1er orden:

ⓐ $xy' - y - x^3 = 0.$

$xy' - y = x^3$, EDL $\quad P(x) = -1$, $Q(x) = x^3$

Homogénea asociada: $xy' - y = 0$

$x \cdot \frac{dy}{dx} - y = 0$

$x \frac{dy}{dx} = y$

$\frac{dy}{y} = \frac{dx}{x}$

$\ln y = \ln x + \ln c$

$y_h = c \cdot x, \quad c \in \mathbb{R}$

$y_p = c(x) \cdot x$

$y_p' = c'(x) \cdot x + c(x)$

$xy' - y - x^3 = 0$

$x(c'(x) \cdot x + c) - cx - x^3 = 0$

$c'(x) \cdot x^2 \cancel{+ cx} \cancel{- cx} - x^3 = 0$

$c'(x) = \frac{x^3}{x^2}$

$\int c'(x) = \int x$

$c(x) = \frac{x^2}{2}$

$y_p = c(x) \cdot x$

$y_p = \frac{x^2}{2} \cdot x$

$y_p = \frac{1}{2} x^3$

$y_G = y_h + y_p \Rightarrow \boxed{y_G = c \cdot x + \frac{1}{2} x^3}$

ⓑ $y' + y \cdot \cos(x) = \operatorname{sen}(x) \cdot \cos(x)$ $\quad P(x) = \cos(x)$, $Q(x) = \operatorname{sen}(x) \cdot \cos(x)$

Homogénea asociada: $y' + y \cdot \cos(x) = 0$

$\frac{dy}{dx} + y \cdot \cos(x) = 0$

$\frac{dy}{dx} = -y \cdot \cos(x)$

$\int \frac{dy}{y} = \int \cos(x) \cdot dx$

$\ln y = -\operatorname{sen}(x) + c$

$y = e^{-\operatorname{sen}(x) + c}$

$y_h = c \cdot e^{-\operatorname{sen}(x)}$

$y_p = c(x) \cdot e^{-\operatorname{sen}(x)}$

$y_p' = c'(x) \cdot e^{-\operatorname{sen}(x)} - c(x) \cdot e^{-\operatorname{sen}(x)} \cdot \cos(x)$

$\hookrightarrow y_p' = e^{-\operatorname{sen}(x)} \cdot (c'(x) - c(x) \cdot \cos(x))$

$y' + y \cdot \cos(x) = \operatorname{sen}(x) \cdot \cos(x)$

$c'(x) \cdot e^{-\operatorname{sen}(x)} \cancel{- c(x) \cdot e^{-\operatorname{sen}(x)} \cdot \cos(x)} + \cancel{c(x) \cdot e^{-\operatorname{sen}(x)} \cos(x)} = \operatorname{sen}(x) \cdot \cos(x)$

$\int c'(x) dx = \int \frac{\operatorname{sen}(x) \cdot \cos(x)}{e^{-\operatorname{sen}(x)}} dx$

$c(x) = \int \operatorname{sen}(x) \cdot \cos(x) \cdot e^{\operatorname{sen}(x)} \cdot dx \rightarrow t = \operatorname{sen} x$, $dt = \cos x \, dt$

$\int t \cdot e^t \cdot dt$

$u = t \quad dv = e^t dt$
$du = dt \quad v = e^t$

$\downarrow \int u dv = uv - \int v du$

$t \cdot e^t - \int e^t \cdot dt$

$t e^t - e^t$

$e^t (t-1)$

$\hookrightarrow e^{\operatorname{sen} x} (\operatorname{sen} x - 1)$

$c(x) = e^{\operatorname{sen} x} \cdot (\operatorname{sen} x - 1)$

$y_p = c(x) \cdot e^{-\operatorname{sen} x} = \frac{c(x)}{e^{\operatorname{sen} x}}$

$y_p = \frac{\cancel{e^{\operatorname{sen} x}} \cdot (\operatorname{sen} x - 1)}{\cancel{e^{\operatorname{sen} x}}}$

$y_p = \operatorname{sen} x - 1$

$y_G = y_h + y_p \Rightarrow \boxed{y_G = c \cdot e^{-\operatorname{sen}(x)} + \operatorname{sen}(x) - 1}$

c) $(x^2+4)\cdot y' - 3xy = x$, halle la SP tal que $y(0)=1$.

I) $\frac{\cancel{x^2+4}}{\cancel{x^2+4}} y' - \frac{3x}{x^2+4} y = \frac{x}{x^2+4}$

$P(x) = -\frac{3x}{x^2+4}$

$Q(x) = \frac{x}{x^2+4}$

II) Homogénea asociada: $y' - \frac{3x}{x^2+4} y = 0$

(variables separables)

$\frac{dy}{dx} = \frac{3x}{x^2+4} y$

$\int \frac{dy}{y} = \int \frac{3x}{x^2+4} dx \rightarrow \frac{3}{2}\int \frac{2x}{x^2+4}dx$

$u = x^2+4$
$du = 2x\,dx$

$\frac{3}{2}\int \frac{u}{du}$

$\frac{3}{2}\ln u + c$

$\frac{3}{2}\ln(x^2+4)+c$

$\ln y = \frac{3}{2}\ln(x^2+4) + \ln c$

$\ln y = \ln(x^2+4)^{\frac{3}{2}} + \ln c$

$y_h = \boxed{c}\,(x^2+4)^{\frac{3}{2}}$

III) $y_p = \boxed{c(x)}\cdot(x^2+4)^{\frac{3}{2}}$

$y'_p = c'(x)\cdot(x^2+4)^{\frac{3}{2}} + c(x)\cdot\frac{3}{\cancel{2}}\cdot(x^2+4)^{\frac{1}{2}}\cdot \cancel{2}x$

$\hookrightarrow y'_p = c'(x)\cdot(x^2+4)^{\frac{3}{2}} + c(x)\cdot 3x\cdot(x^2+4)^{\frac{1}{2}}$

IV) $\underbrace{y'}_{y_p} - \frac{3x}{x^2+4}\underbrace{y}_{y_p} = \frac{x}{x^2+4}$

$c'(x)\cdot(x^2+4)^{\frac{3}{2}} \cancel{+ c(x)\cdot 3x\cdot(x^2+4)^{\frac{1}{2}}} - \cancel{\frac{3x\cdot c(x)\cdot(x^2+4)^{\frac{3}{2}}}{x^2+4}} = \frac{x}{x^2+4}$

$c'(x)\cdot(x^2+4)^{3/2}\cdot(x^2+4) = x$

$\int c'(x) = \int x\cdot(x^2+4)^{-\frac{5}{2}}$

$c(x) = \frac{1}{2}\int 2x\cdot(x^2+4)^{-\frac{5}{2}}dx$

$x^2+4 = w$
$2x\,dx = dw$

$\frac{1}{2}\int w^{-\frac{5}{2}}\cdot dw$

$\frac{1}{2}\cdot\frac{w^{-\frac{3}{2}}}{-\frac{3}{2}} + c$

$\frac{1}{\cancel{2}}\cdot -\frac{\cancel{2}}{3} w^{-\frac{3}{2}} + c$

$-\frac{1}{3}(x^2+4)^{-\frac{3}{2}} + c$

$c(x) = -\frac{1}{3}(x^2+4)^{-\frac{3}{2}} \cancel{+c}$

hago que $c=0$

No le pongo la constante porque estamos hallando una SP

V) $y_p = c(x)\cdot(x^2+4)^{\frac{3}{2}}$

$y_p = -\frac{1}{3}\cdot\cancel{(x^2+4)^{-\frac{3}{2}}}\cdot\cancel{(x^2+4)^{\frac{3}{2}}}$

$y_p = -\frac{1}{3}$

VI) $y_G = y_h + y_p$

$\boxed{y_G = c\cdot(x^2+4)^{3/2} - \frac{1}{3}}$ (SG)

Este ejercicio se podía haber resuelto por variables separables.

Como me piden una particular, averiguo el valor de la constante

VII) $(1) = c\,((0)^2+4)^{3/2} - \frac{1}{3}$

$\frac{4}{3} = c\cdot 8$

$c = \frac{1}{6}$

VIII) $\boxed{y_p = \frac{1}{6}(x^2+4)^{3/2} - \frac{1}{3}}$ (SP)

d) $\frac{dy}{dx} - 2\frac{y}{x} = x^2\cdot \text{sen}(3x)$

$y' - \frac{2}{x}y = x^2\cdot\text{sen}(3x)$

$P(x) = -\frac{2}{x}$

$Q(x) = x^2\cdot\text{sen}(3x)$

Homogénea asociada: $y' - \frac{2}{x}y = 0$

$\frac{dy}{dx} = \frac{2y}{x}$

$\int\frac{dy}{y} = 2\int\frac{dx}{x}$

$\ln|y| = 2\ln|x| + \ln c$

$y_h = \boxed{c}\,x^2$

$y_p = \boxed{c(x)}\cdot x^2$

$y'_p = c'(x)\cdot x^2 + c(x)\cdot 2x$

$\underbrace{y'}_{y_p} - \frac{2}{x}\underbrace{y}_{y_p} = x^2\,\text{sen}(3x)$

$c'(x)\cdot x^2 \cancel{+ c(x)\,2x} \cancel{- \frac{2}{x}\,c(x)\cdot x^2} = x^2\cdot\text{sen}(3x)$

$c'(x)\cdot\cancel{x^2} = \cancel{x^2}\cdot\text{sen}(3x)$

$\int c'(x) = \frac{1}{3}\int\text{sen}(3x)3dx$

$w=3x$
$dw=3dx$

$\frac{1}{3}\int \text{sen}\,w\cdot dw = \frac{1}{3}(-\cos w)+c$

$= -\frac{1}{3}\cos(3x)$

$c(x) = -\frac{1}{3}\cos(3x)$

$y_p = c(x)\cdot x^2$

$y_p = -\frac{1}{3}\cos(3x)\cdot x^2$

$y_G = y_h + y_p$

$y_G = c\,x^2 - \frac{1}{3}\cos(3x)\cdot x^2$

$\boxed{y_G = x^2\left(c - \frac{1}{3}\cos(3x)\right)}$

20 Halle la familia de curvas ortogonal a la dada:

a) $y = 2x + c$

$y' = 2$

ED de la familia de tray. ortogonales: $-\dfrac{1}{y'} = 2$

$y' = -\dfrac{1}{2}$

$\dfrac{dy}{dx} = -\dfrac{1}{2}$

$\int dy = -\dfrac{1}{2}\int dx$

$\boxed{y = -\dfrac{1}{2}x + k}$

b) $y = c\cdot e^x$

$y' = c\cdot e^x$ $\Big\}\ y' = y$

$-\dfrac{1}{y'} = y$

$-\dfrac{dx}{dy} = y$

$\int y\cdot dy = -\int dx$

$\dfrac{1}{2}y^2 = -x + k$

$y^2 = -2x + 2k$

$\boxed{y^2 = -2x + k_1}$

c) $y(cx+1) = x \longrightarrow$

$cx + 1 = \dfrac{x}{y}$

$cx = \dfrac{x}{y} - 1$

$c = \dfrac{\frac{x}{y} - 1}{x}$

$y = \dfrac{x}{cx+1}$

$y' = \dfrac{cx+1 - x\cdot c}{(cx+1)^2}$

$y' = \dfrac{\left(\dfrac{\frac{x}{y}-1}{\cancel{x}}\right)\cancel{x} + 1 - \cancel{x}\left(\dfrac{\frac{x}{y}-1}{\cancel{x}}\right)}{\left(\left(\dfrac{\frac{x}{y}-1}{\cancel{x}}\right)x + 1\right)^2}$

$y' = \dfrac{\cancel{\frac{x}{y}} - 1 + 1 - \cancel{\frac{x}{y}} + 1}{\left(\frac{x}{y} - 1 + 1\right)^2}$

$y' = \dfrac{1}{\left(\frac{x}{y}\right)^2} = \dfrac{y^2}{x^2}$

$-\dfrac{1}{y'} = \dfrac{y^2}{x^2}$

$-\dfrac{dx}{dy} = \dfrac{y^2}{x^2}$

$\int -x^2 dx = \int y^2\cdot dy$

$-\dfrac{x^3}{3} + c_1 = \dfrac{y^3}{3} \longrightarrow \boxed{x^3 + y^3 = C_2}$

$\dfrac{y}{\text{sen}(2x)} = c$

d) $y = \ln(x + c) \longrightarrow e^y = x + c$

$y' = \dfrac{1}{x + C}$ Tengo que eliminar la constante

$y' = \dfrac{1}{e^y}$

$-\dfrac{1}{y'} = \dfrac{1}{e^y}$

ED de la familia de TO: $y' = -e^y$

$\dfrac{dy}{dx} = -e^y$

$\int \dfrac{dy}{e^y} = -\int dx$

$\int e^{-y}\cdot dy = -x + C_1$

$-e^{-y} = -x + C_1$

$e^{-y} = x - C_1$

$e^{-y} = x + C_2$

$-y = \ln(x + C_2)$

$\boxed{y = -\ln(x + C_2)}$

e) $y = c\cdot \text{sen}(2x)$

$y' = c\cdot\cos(2x)\cdot 2$

$y' = \dfrac{y}{\text{sen}(2x)}\cdot\cos(2x)$

$-\dfrac{1}{y'} = y\cdot\dfrac{\cos(2x)}{\text{sen}(2x)}$

$\dfrac{dx}{dy} = -y\dfrac{\cos(2x)}{\text{sen}(2x)}$

$\int \dfrac{\text{sen}(2x)}{\cos(2x)}dx = \int -y\cdot dy$

$\int \tan(2x)\cdot dx\cdot\dfrac{2}{2}$

$\dfrac{1}{2}\int \tan(2x)\cdot 2dx$

$\dfrac{1}{2}\int \tan u\, du$

$-\dfrac{1}{2}\ln(\cos u)$

$\int \tan(2x)\,dx = -\dfrac{1}{2}y^2 + c$

$\cancel{-\dfrac{1}{2}}\ln(\cos(2x)) = \cancel{-\dfrac{1}{2}}y^2 + c$

$\boxed{\ln(\cos(2x)) = y^2 + c}$

f) $(x+y)^2 = kx^2,\ k > 0$

$\dfrac{(x+y)^2}{x^2} = k$

$x + y = \sqrt{k\cdot x^2}$

$y = \sqrt{k}\cdot x - x$

$y' = \sqrt{k} - 1$

$y' = \sqrt{\dfrac{(x+y)^2}{x^2}} - 1$

$y' = \dfrac{x+y}{x} - 1$

$y' = \cancel{\dfrac{x}{x}} + \dfrac{y}{x} \cancel{- 1}$

$y' = \dfrac{y}{x}$

$-\dfrac{1}{y'} = \dfrac{y}{x}$

$-\dfrac{dx}{dy} = \dfrac{y}{x}$

$\int -x\,dx = \int y\cdot dy$

$\dfrac{x^2}{2} = \dfrac{y^2}{2} + c_1$

$\boxed{x^2 = y^2 + C_2}$