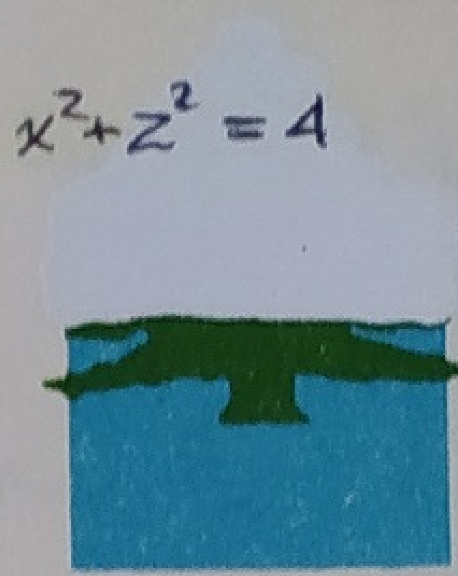
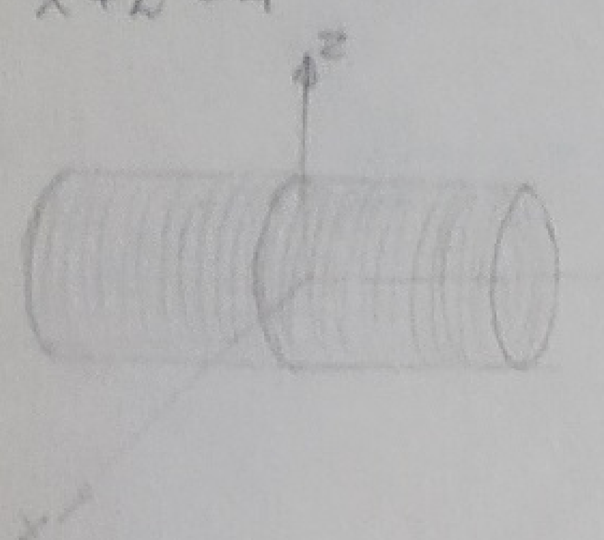
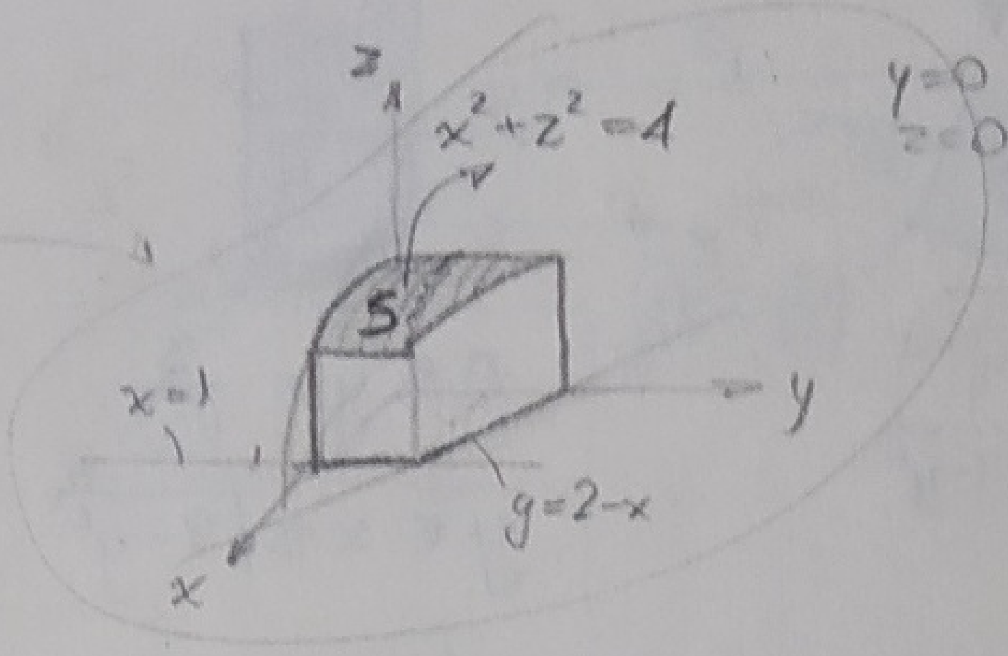
DE PARCIAL ⟶ SUPERFICIES ALABEADAS

3 ① Calcular el área de la porción de la superficie cilíndrica de ecuación $x^2+z^2=4$
limitada por: $x+y=2$, $\to y=2-x$
$x=1$,
los planos coordenados. $\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}$

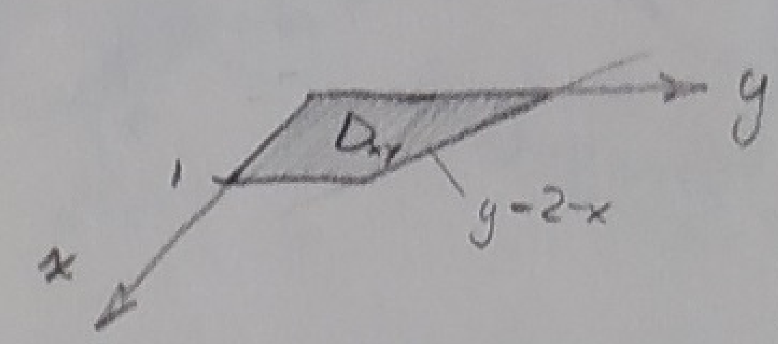


$x^2+z^2=4$





proyección en el plano xy



$0 \le x \le 1$
$0 \le y \le 2-x$

$x^2+z^2=4$
$z^2=4-x^2$
$z=\sqrt{4-x^2}$

$G(x,y,z) = x^2+z^2-4.$

$x^2+z^2=4$ define implícitamente a $z=f(x,y)=\sqrt{4-x^2}$

$G'_x = 2x$
$G'_y = 0$
$G'_z = 2z$

$\nabla G = (2x,\ 0,\ 2z) \longrightarrow \|\nabla G\| = \sqrt{(2x)^2+0^2+(2z)^2} = \sqrt{4x^2+4z^2} = \sqrt{4\cdot(x^2+z^2)} = 2\sqrt{x^2+z^2}$

$|G'_z| = 2z$

$$a(S) = \iint_{D_{xy}} \frac{\|\nabla G\|}{|G'_z|}\Bigg|_{z=\sqrt{4-x^2}} dx\,dy = \iint_{D_{xy}} \frac{2\sqrt{x^2+z^2}}{2z}\Bigg|_{z=\sqrt{4-x^2}} dx\,dy = \iint_{D_{xy}} \frac{\sqrt{x^2+(\sqrt{4-x^2})^2}}{\sqrt{4-x^2}}\,dx\,dy$$

$\sqrt{\cancel{x^2}+4-\cancel{x^2}}$
$\sqrt{4}$
2

$$= \iint_{D_{xy}} \frac{2}{\sqrt{4-x^2}}\,dx\,dy = 2\int_0^1 dx \int_0^{2-x} \frac{1}{\sqrt{4-x^2}}\,dy = 2\cdot\int_0^1 \left[\frac{y}{\sqrt{4-x^2}}\Bigg|_0^{2-x}\right] dx$$

$$= 2\cdot\int_0^1 \frac{2-x}{\sqrt{4-x^2}}\,dx$$

$$-\int_0^1 \frac{x-2}{\sqrt{4-x^2}}\,dx = -\left[\int_0^1 \frac{x}{\sqrt{4-x^2}}\,dx - \int_0^1 \frac{2}{\sqrt{4-x^2}}\,dx\right]$$

$$= -\left[-\sqrt{4-x^2}\Big|_0^1 - 2\cdot\arcsin\left(\frac{x}{2}\right)\Big|_0^1\right]$$

$$= -\left[-\sqrt{4-1^2}-\left(-\sqrt{4-0^2}\right) - 2\cdot\arcsin\left(\frac{1}{2}\right) - \left(2\cdot\arcsin\left(\frac{0}{2}\right)\right)\right]$$

$$= -\left[-\sqrt{3}+2-\frac{1}{3}\pi - 0\right]$$

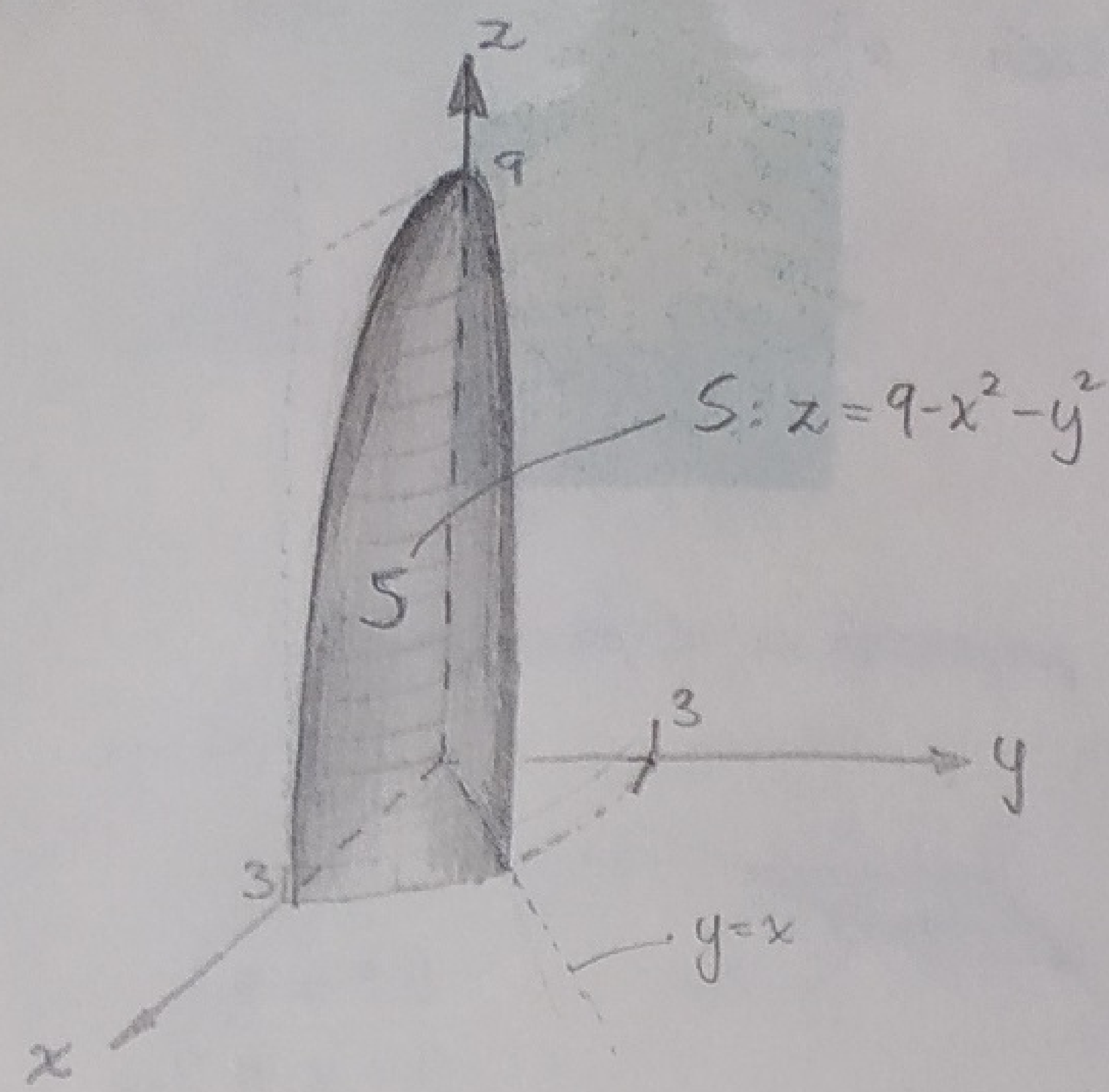
$$= \sqrt{3}-2+\frac{1}{3}\pi$$

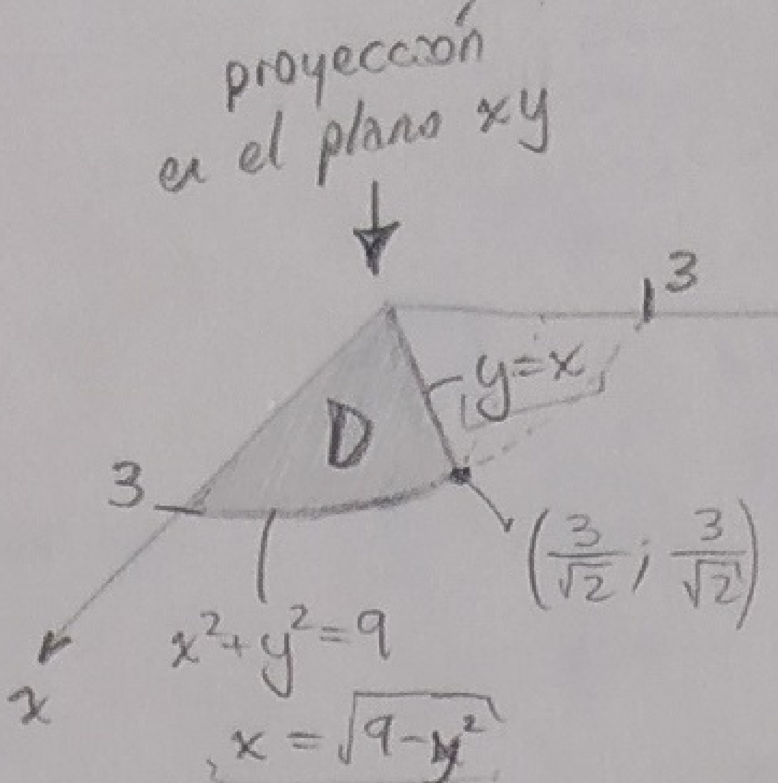
$$= 2\cdot\left(\sqrt{3}-2+\frac{1}{3}\pi\right)$$

$$= 2\sqrt{3}+\frac{2}{3}\pi-4$$

$$\boxed{a(S) \approx 1{,}5584}$$

5 ② Calcular el área de la porción del paraboloide $z = 9 - x^2 - y^2$ en el 1er octante con $y \le x$.





$z=0 \longrightarrow 0 = 9 - x^2 - y^2$

$x^2 + y^2 = 9 \quad \cap \quad y = x$

$x^2 + x^2 = 9$

$2x^2 = 9$

$x^2 = \frac{9}{2}$

$x = \frac{3}{\sqrt{2}} \quad y = \frac{3}{\sqrt{2}}$

$$\boxed{\begin{array}{c} 0 \le y \le \frac{3}{\sqrt{2}} \\ y \le x \le \sqrt{9-y^2} \end{array}}$$

$G(x,y,z) = z + x^2 + y^2 - 9$

$G'_x = 2x$

$G'_y = 2y$

$G'_z = 1 \longrightarrow |G'_z| = 1$

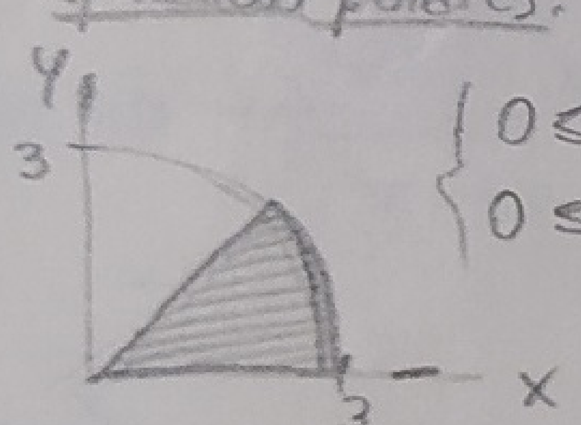
$\nabla G = (2x, 2y, 1) \longrightarrow \|\nabla G\| = \sqrt{(2x)^2 + (2y)^2 + 1^2} = \sqrt{4x^2 + 4y^2 + 1}$

$= \sqrt{4\left(x^2 + y^2 + \frac{1}{4}\right)}$

$= 2\sqrt{x^2 + y^2 + \frac{1}{4}}$

$$\boxed{a(S) = \iint_{D_{xy}} \frac{\|\nabla G\|}{|G'_z|}\Bigg|_{z=9-x^2-y^2}} dx\,dy = \iint_{D_{xy}} \frac{2\sqrt{x^2+y^2+\frac{1}{4}}}{1}\,dx\,dy = 2\int_0^{\frac{3}{\sqrt{2}}} dy \int_y^{\sqrt{9-y^2}} \sqrt{\underbrace{x^2+y^2}_{\rho^2}+\frac{1}{4}}\,dx$$

aplicando polares:



$\begin{cases} 0 \le \rho \le 3 \\ 0 \le \theta \le \frac{\pi}{4} \end{cases}$ $\begin{cases} x = \rho \cdot \cos\theta \\ y = \rho \cdot \operatorname{sen}\theta \end{cases}$

$x^2 + y^2 = \rho^2$

jacobiano

$$= 2\int_0^{\frac{\pi}{4}} d\theta \cdot \int_0^3 \sqrt{\rho^2 + \frac{1}{4}} \cdot \rho \cdot d\rho$$

$$= 2 \cdot \left[\theta \Big|_0^{\frac{\pi}{4}}\right] \cdot \left[\frac{\sqrt{\left(\rho^2+\frac{1}{4}\right)^3}}{3}\Bigg|_0^3\right]$$

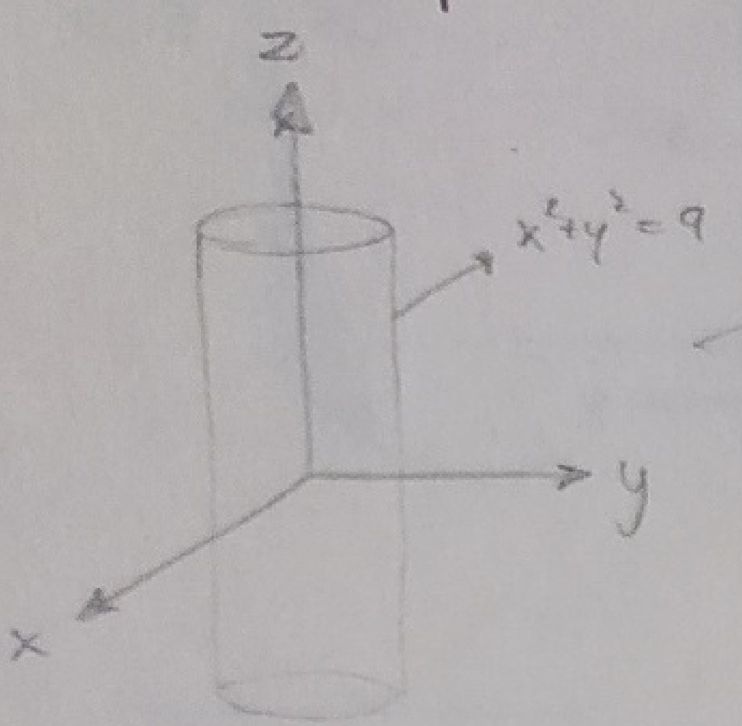
$$= 2 \cdot \frac{\pi}{4} \cdot \left[\frac{\sqrt{\left(9+\frac{1}{4}\right)^3}}{3} - \frac{\sqrt{\left(0+\frac{1}{4}\right)^3}}{3}\right]$$

$$= \frac{\pi}{2} \cdot \left(\frac{37\sqrt{37}}{24} - \frac{1}{24}\right)$$

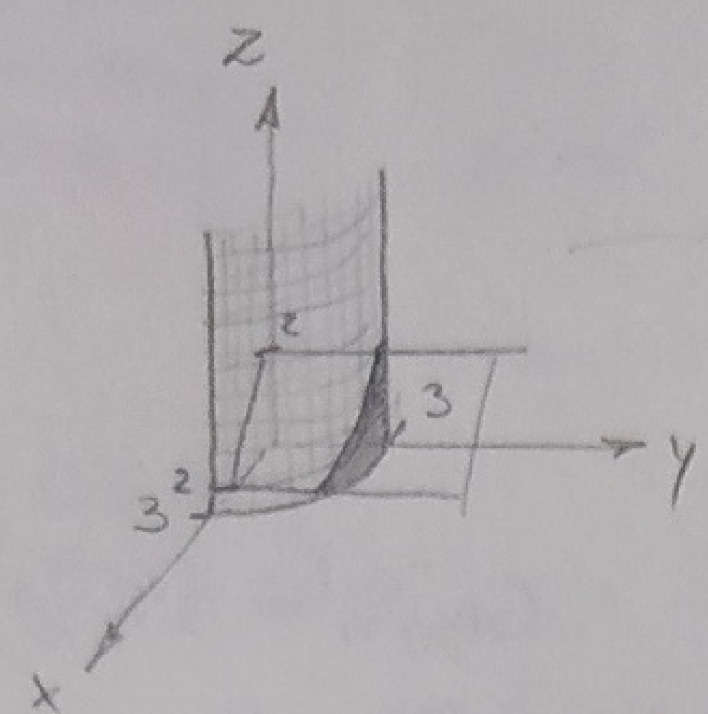
$$\boxed{\simeq 14{,}6649}$$

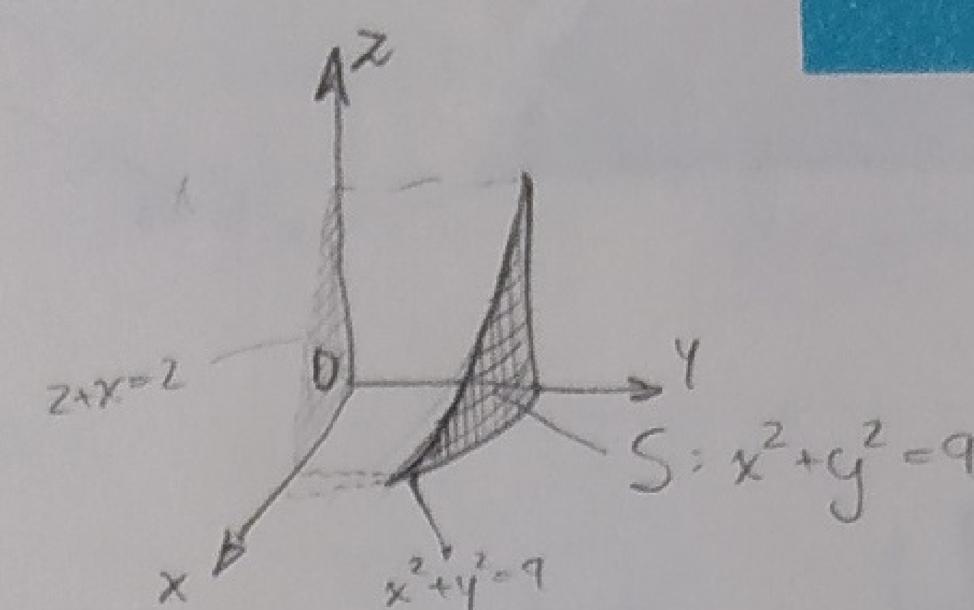
7 ③ Calcular el área de la porción de superficie cilíndrica $x^2+y^2=9$ limitado por $z+x\leq 2$ y el 1er octante.

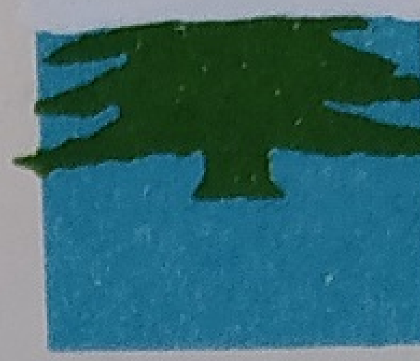
$y^2=\sqrt{9-x^2}$



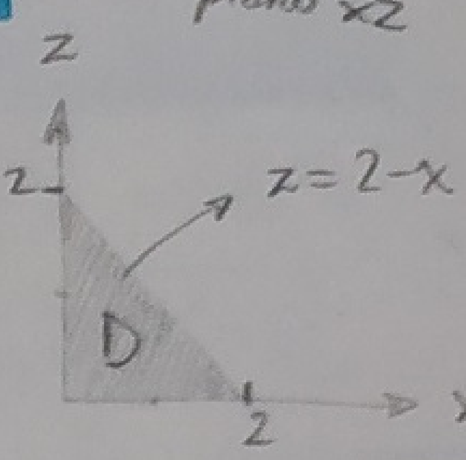
en el primer octante







proyección en el plano xz



$$\boxed{\begin{array}{l} 0\leq x\leq 2 \\ 0\leq z\leq 2-x \end{array}}$$

$G(x,y,z) = x^2+y^2-9$ $\quad \overline{\nabla} G = (2x, 2y, 0) \longrightarrow \|\nabla G\| = \sqrt{(2x)^2+(2y)^2} = \sqrt{4x^2+4y^2}$

$|G'_y| = 2y$ $\quad = \sqrt{4(x^2+y^2)} = 2\sqrt{x^2+y^2}$

$\cancel{x^2}-9-\cancel{x^2}$

$$a(S) = \iint_{D_{xz}} \frac{\|\nabla G\|}{|G'_y|}\Bigg|_{y=\sqrt{9-x^2}} dx\,dz = \iint_{D_{xz}} \frac{\cancel{2}\sqrt{x^2+y^2}}{\cancel{2}y}\Bigg|_{y=\sqrt{9-x^2}} dx\,dz = \iint_{D_{xz}} \frac{\sqrt{x^2+\sqrt{9-x^2}^{\,2}}}{\sqrt{9-x^2}}\,dx\,dz$$

$$= \iint_{D_{xz}} \frac{\sqrt{9}}{\sqrt{9-x^2}}\,dx\,dz = 3\int_0^2 dx \int_0^{2-x} \frac{1}{\sqrt{9-x^2}}\,dz = 3\int_0^2 \left[\frac{1}{\sqrt{9-x^2}}\,z\Big|_0^{2-x}\right] dx = 3\cdot\int_0^2 \left[\frac{2-x}{\sqrt{9-x^2}}\right] dx$$

$$\int_0^2 \frac{2}{\sqrt{9-x^2}}\,dx - \int_0^2 \frac{x}{\sqrt{9-x^2}}\,dx$$

$$\left[2\cdot \arcsen\left(\frac{x}{3}\right) - \left(-\sqrt{9-x^2}\right)\right]\Big|_0^2$$

$$\left[2\cdot \arcsen\left(\frac{x}{3}\right) + \sqrt{9-x^2}\right]\Big|_0^2$$

$$2\cdot \arcsen\left(\frac{2}{3}\right) + \sqrt{5} - (0+3)$$

$$2\cdot \arcsen\left(\frac{2}{3}\right) + \sqrt{5} - 3$$

$$= 3\int_0^2 \left(2\cdot \arcsen\left(\frac{2}{3}\right)+\sqrt{5}-3\right) dx = 3\cdot\left[\left(2\cdot \arcsen\left(\frac{2}{3}\right)+\sqrt{5}-3\right)x\right]_0^2$$

$$= 3\left[4\arcsen\left(\frac{2}{3}\right)+2\sqrt{5}-6\right]$$

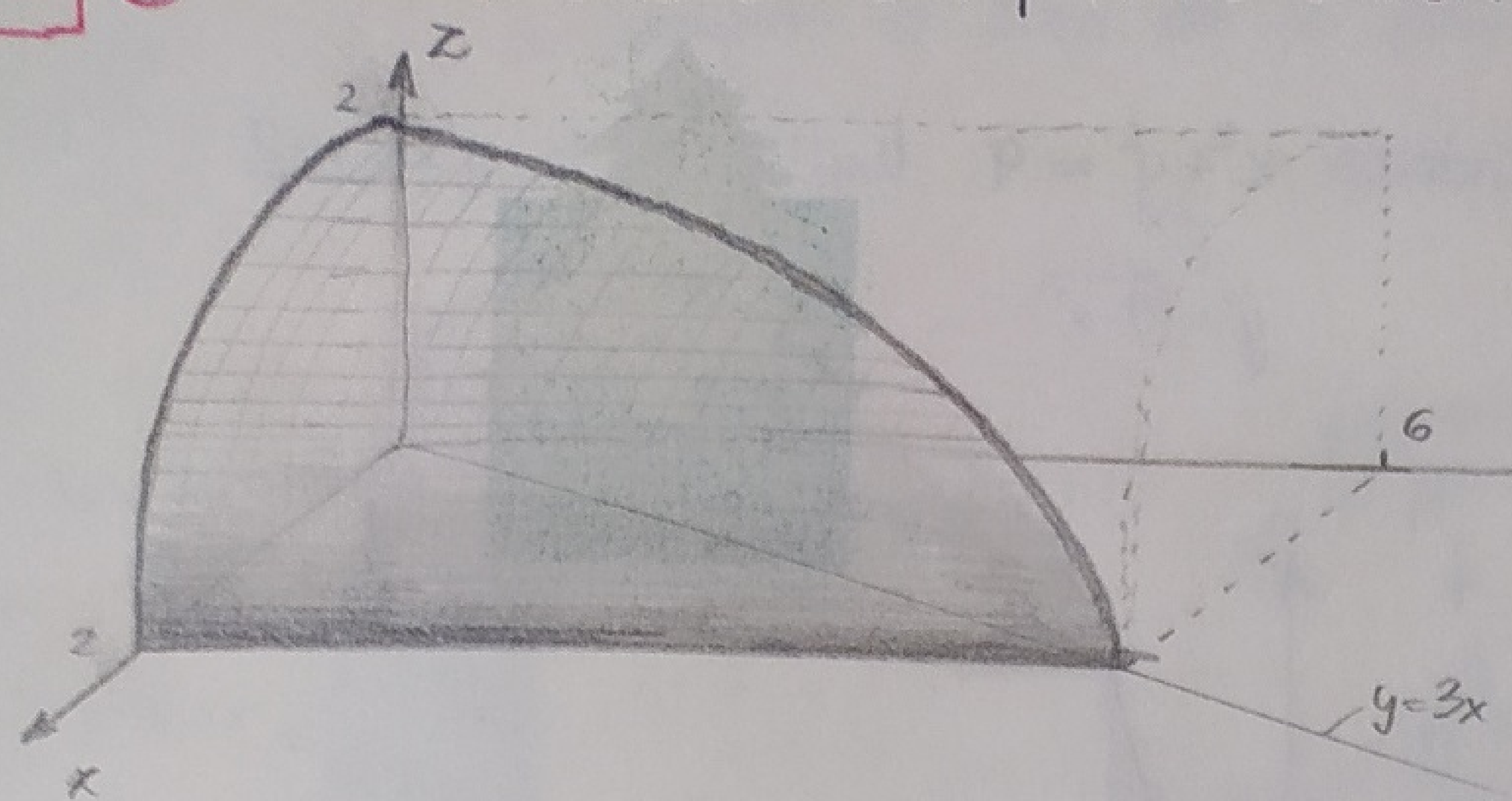
$$= 12\arcsen\left(\frac{2}{3}\right)+6\sqrt{5}-18$$

$$\boxed{a(S) \simeq 4{,}1731}$$

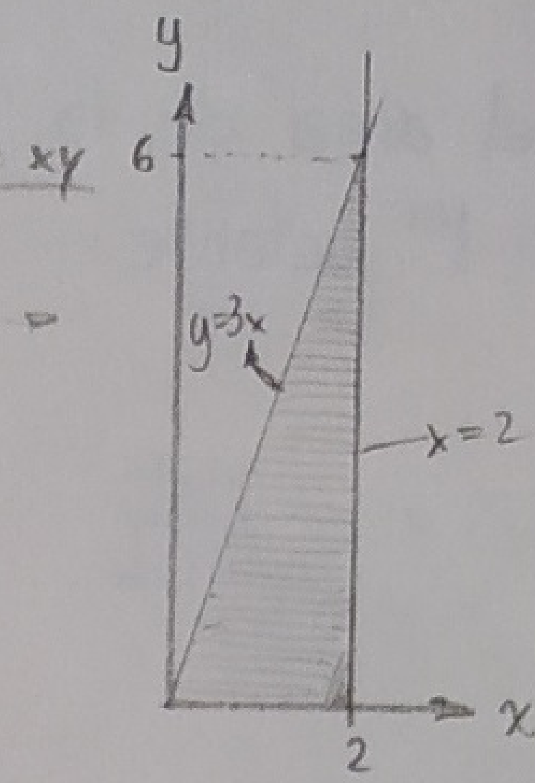
9 ① Hallar el área de la superficie cilíndrica $x^2+z^2=4$ limitada por $y\le 3x$, en el 1er octante.

$z^2=4-x^2$

$z=\sqrt{4-x^2}$



proyección sobre xy



$$\begin{aligned} 0 &\le x \le 2 \\ 0 &\le y \le 3x \end{aligned}$$

$$\boxed{a(S)} = \iint_{D_{xy}} \frac{\|\nabla G\|}{G'_z}\bigg|_{z=f(x,y)} dx\,dy$$

$G_{(x,y,z)} = x^2+z^2-4$

$\nabla G = (2x, 0, 2z) \longrightarrow \|\nabla G\| = \sqrt{(2x)^2+0^2+(2z)^2} = \sqrt{4x^2+4z^2} = 2\sqrt{x^2+z^2}$

$G'_z = 2z$

$|G'_z| = 2z.$

$$= \iint_{D_{xy}} \frac{2\sqrt{x^2+z^2}}{2z}\bigg|_{z=\sqrt{4-x^2}} dx\,dy$$

$$= \iint_{D_{xy}} \frac{\sqrt{x^2+(\cancel{\sqrt{4-x^2}})^{\cancel{2}}}}{\sqrt{4-x^2}}\,dx\,dy = \iint_{D_{xy}} \frac{\sqrt{\cancel{x^2}+4-\cancel{x^2}}}{\sqrt{4-x^2}}\,dx\,dy = 2\iint_{D_{xy}} \frac{1}{\sqrt{4-x^2}}\,dx\,dy$$

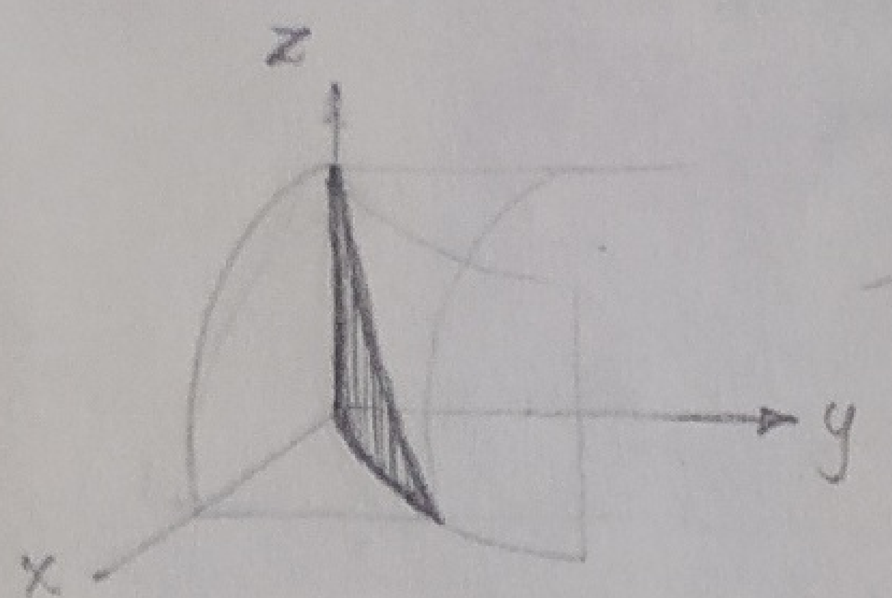
$$= 2\int_0^2 dx \int_0^{3x} \frac{1}{\sqrt{4-x^2}}\,dy = 2\int_0^2 dx \left[\frac{y}{\sqrt{4-x^2}}\Bigg|_0^{3x}\right] = 2\int_0^2 \frac{3x}{\sqrt{4-x^2}}\,dx = 6\int_0^2 \frac{x}{\sqrt{4-x^2}}\,dx$$

$$= 6\left[-\sqrt{4-x^2}\,\Big|_0^2\right] = 6\cdot\left[\left(-\sqrt{4-2^2}\right)-\left(-\sqrt{4-0^2}\right)\right]$$

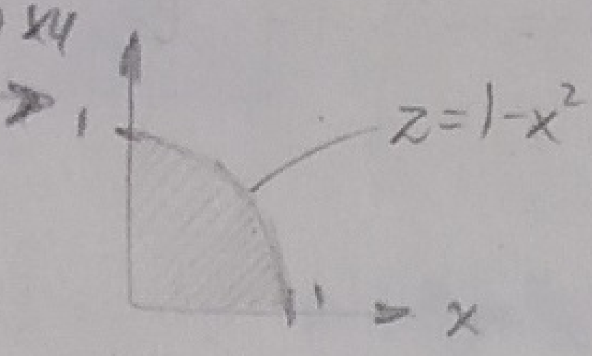
$$= 6\cdot\left[-(-2)\right]$$

$$= \boxed{12}$$

10 ① Calcular el área de la porción de cilindro $y=x^2$ limitado por $z \leq 1-x^2$ en el 1^{er} octante.

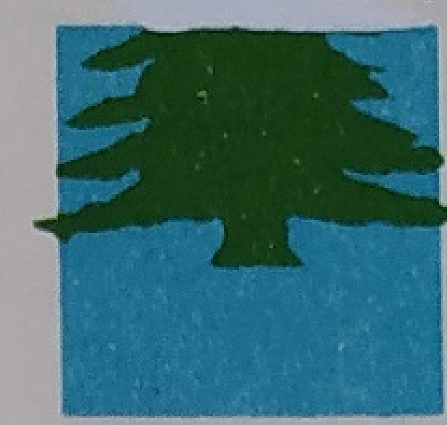


proyección en el plano xy



$$0 \leq x \leq 1$$
$$0 \leq z \leq 1-x^2$$

$$G_{(x,y,z)} = x^2 - y$$
$$\nabla G = (2x; -1; 0)$$
$$|G'_y| = |-1| = 1$$



$$\|\nabla G\| = \sqrt{(2x)^2+(-1)^2} = \sqrt{4x^2+1} = 2\sqrt{x^2+\tfrac{1}{4}}$$

$$\boxed{a(S)} = \iint_{D_{xz}} \frac{\|\nabla G\|}{|G'_y|}\Bigg|_{y=f(x,z)} dx\, dz$$

$$= \iint_{D_{xz}} \frac{2\sqrt{x^2+\frac{1}{4}}}{1}\, dx\, dz$$

$$= 2 \cdot \int_0^1 dx \cdot \int_0^{1-x^2} \sqrt{x^2+\tfrac{1}{4}}\, dz$$

$$\sqrt{x^2+\tfrac{1}{4}}\; z\Big|_0^{1-x^2} = \sqrt{x^2+\tfrac{1}{4}} \cdot (1-x^2)$$

$$= 2 \cdot \int_0^1 (1-x^2) \cdot \sqrt{x^2+\tfrac{1}{4}}\, dx$$

$$= 2\left[\int_0^1 \sqrt{x^2+\tfrac{1}{4}}\, dx - \int_0^1 x^2\sqrt{x^2+\tfrac{1}{4}}\, dx\right]$$

$$\int_0^1 \sqrt{x^2+\tfrac{1}{4}}\, dx = \frac{x\sqrt{x^2+\frac{1}{4}}}{2} + \frac{\frac{1}{4}}{2}\ln\left(x+\sqrt{x^2+\tfrac{1}{4}}\right)\Bigg|_0^1 = 0{,}7395$$

$$\int_0^1 x^2\sqrt{x^2+\tfrac{1}{4}}\, dx = \frac{x\sqrt{(x^2+\frac{1}{4})^3}}{4} - \frac{\frac{1}{4}x\sqrt{x^2+\frac{1}{4}}}{8} - \frac{\frac{1}{2}}{8}\ln\left(x+\sqrt{x^2+\tfrac{1}{4}}\right)\Bigg|_0^1 = 0{,}3032$$

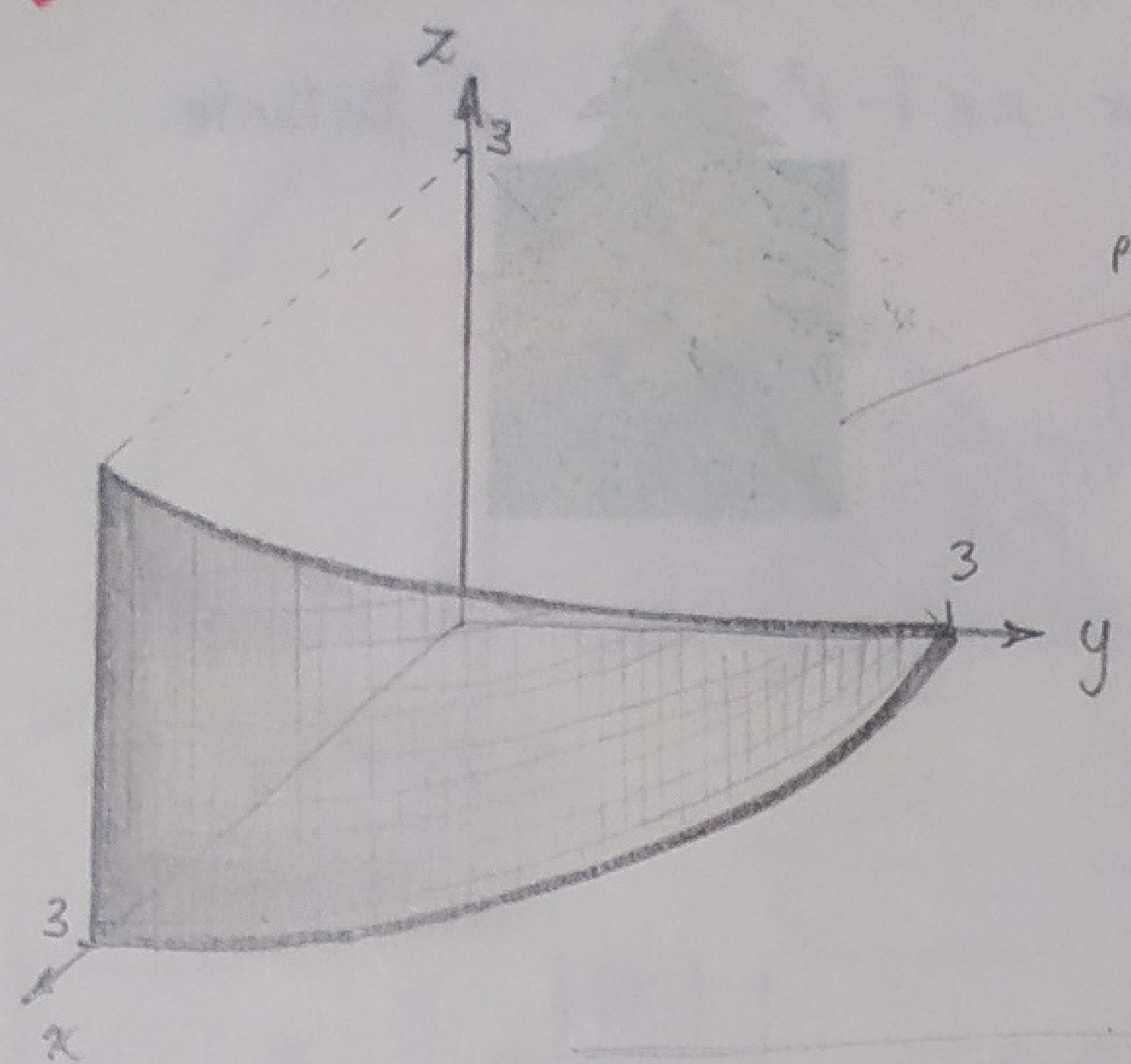
$$= 2\left(0{,}7395 - 0{,}3032\right)$$

$$\boxed{= 0{,}8726}$$

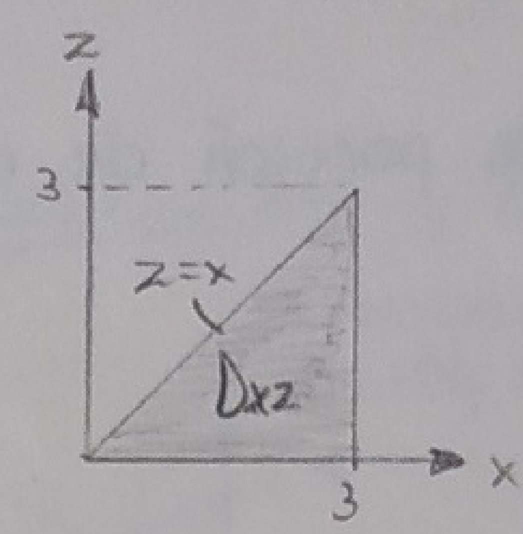
Calcular el área de la porción de cilindro $x^2+y^2=9$ en el 1er octante con $z \leq x$.

$y^2 = 9-x^2$

$y = \sqrt{9-x^2}$



proyección sobre el plano xz



$$\boxed{\begin{array}{l} 0 \leq x \leq 3 \\ 0 \leq z \leq x \end{array}}$$

$$G(x,y,z) = x^2+y^2-9 \longrightarrow \nabla G = (2x, 2y, 0)$$

$$\|\nabla G\| = \sqrt{(2x)^2+(2y)^2+0^2} = \sqrt{4(x^2+y^2)} = 2\sqrt{x^2+y^2}$$

$$G'_y = 2y \rightarrow |G'_y| = 2y$$

$$\sqrt{x^2+\sqrt{9-x^2}^{\,2}} = \sqrt{x^2+9-x^2} = \sqrt{9} = 3$$

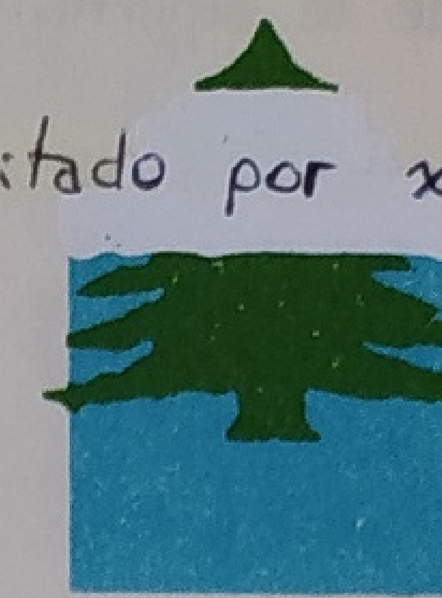
$$\boxed{a(S)} = \iint_{D_{xz}} \frac{\|\nabla G\|}{G'_y}\bigg|_{y=\sqrt{9-x^2}} dx\,dz = \iint_{D_{xz}} \frac{\cancel{2}\sqrt{x^2+y^2}}{\cancel{2}y}\bigg|_{y=\sqrt{9-x^2}} dx\,dz$$

$$= \iint_{D_{xz}} \frac{3}{\sqrt{9-x^2}}\,dx\,dz = 3\int_0^3 dx \int_0^x \frac{1}{\sqrt{9-x^2}}\,dz = 3\int_0^3 dx \left[\frac{z}{\sqrt{9-x^2}}\bigg|_0^x\right]$$

$$= 3\int_0^3 \frac{x}{\sqrt{9-x^2}}\,dx = 3\cdot\left[-\sqrt{9-x^2}\,\bigg|_0^3\right] = 3\left[(0)-(-3)\right]$$

$$\boxed{= 9}$$

4 ③ Calcular el área de la porción de plano de ecuación $x+y+z=6$ limitado por $x^2+2y^2 \leq 4$.

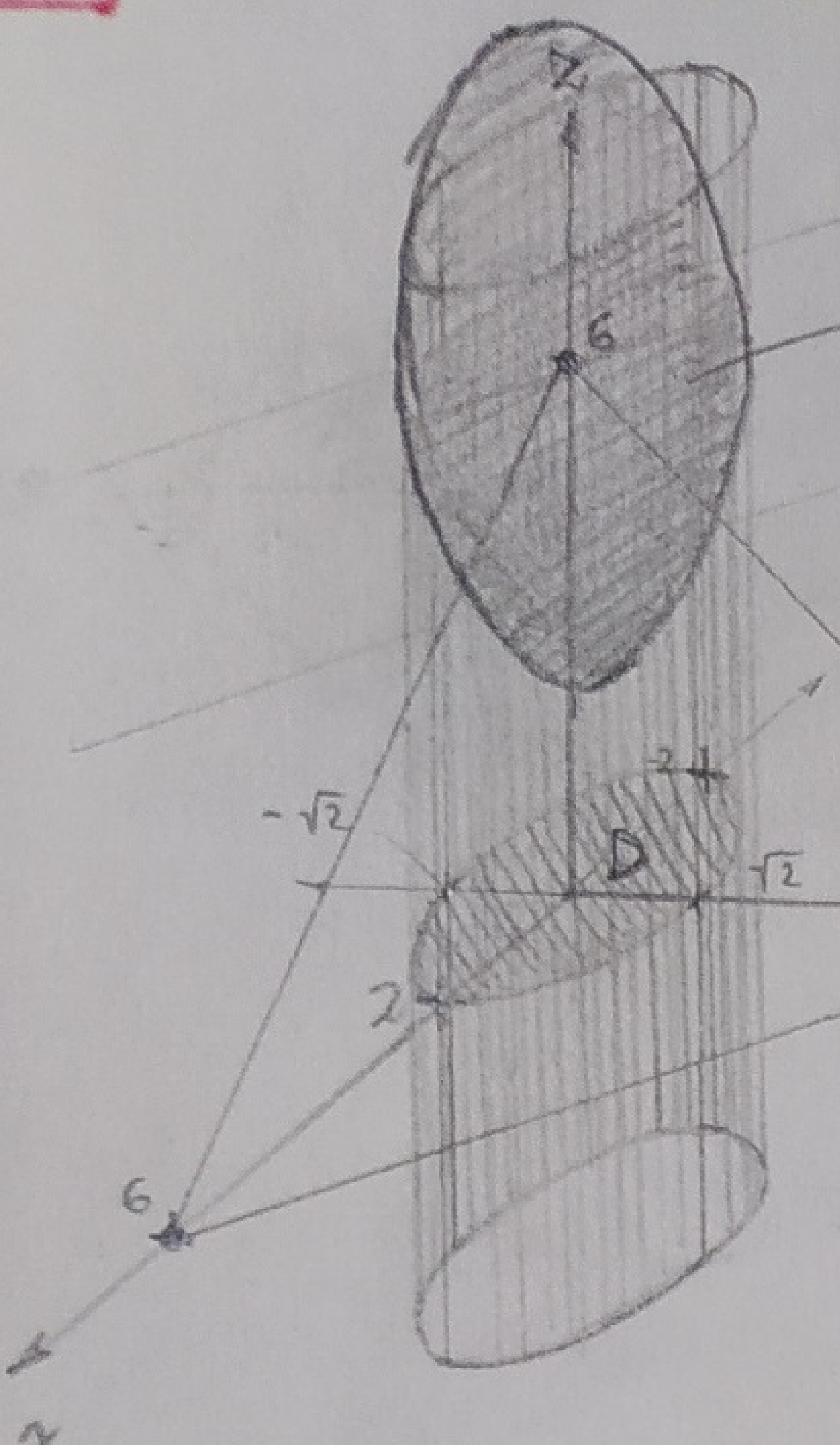


$z = 6-x-y$

$x+2y^2=4$

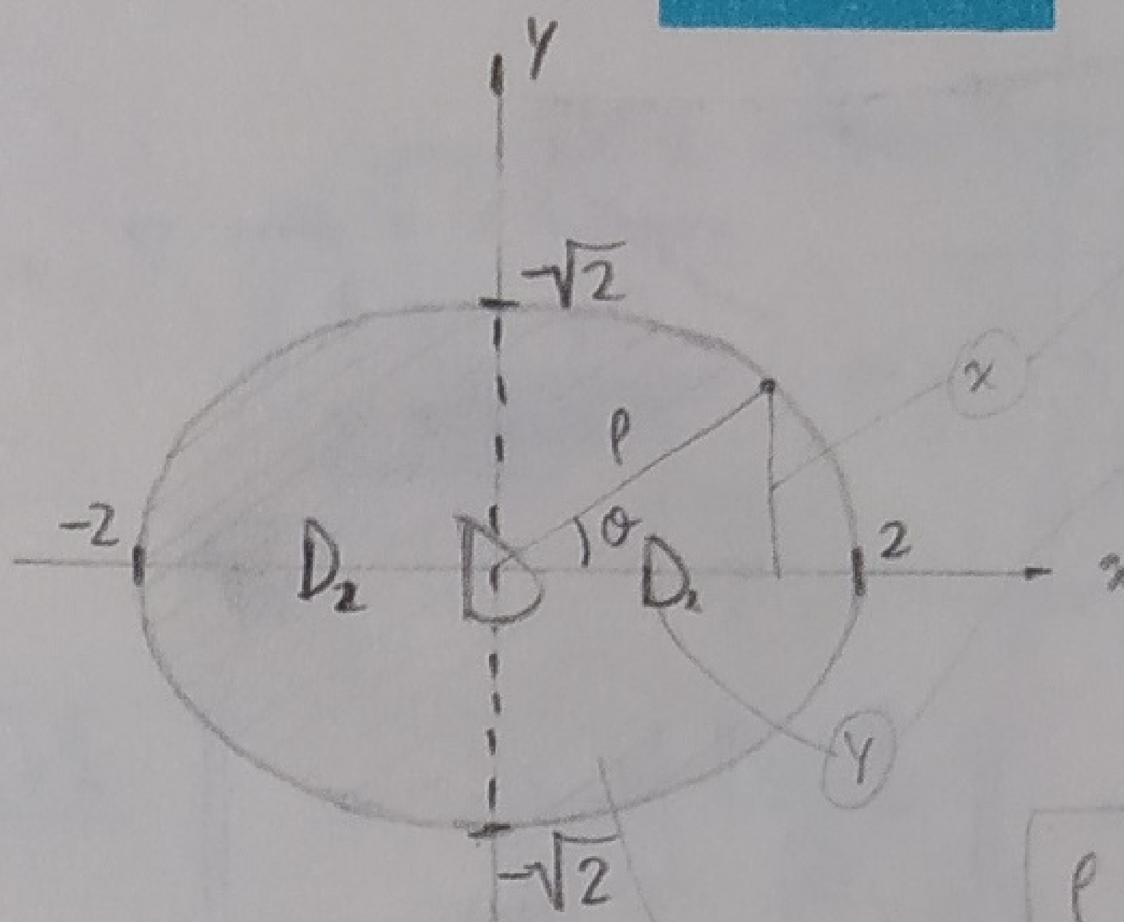
$\frac{x}{4}+\frac{y^2}{2}=1$

$\frac{x^2}{2^2}+\frac{y^2}{\sqrt{2}^2}=1$



$-2 \leq x \leq 2$

proyección en el plano xy



$\begin{cases} x = 2\cdot\cos\theta \\ y = \sqrt{2}\cdot \text{sen}\,\theta \end{cases}$

$\rho^2 = x^2+y^2$

$\rho^2 = (2\cos\theta)^2 + (\sqrt{2}\,\text{sen}\,\theta)^2$

$\rho^2 = 4\cos^2\theta + 2\,\text{sen}^2\theta$

$\boxed{\rho = \sqrt{4\cos^2\theta + 2\cdot\text{sen}^2\theta}}$

$0 \leq x \leq 2$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$G_{(x,y,z)} = x+y+z-6 \begin{cases} G'_x = 1 \\ G'_y = 1 \\ G'_z = 1 \end{cases}$

define implícitamente a $z = 6-x-y$

$$a(S) = \iint_{D_{xy}} \left.\frac{\|G'_x; G'_y; G'_z\|}{|G'_z|}\right|_{z=6-x-y} dx\,dy$$

$\|G'_x; G'_y; G'_z\| = \sqrt{G'^2_x + G'^2_y + G'^2_z}$

$= \sqrt{1^2+1^2+1^2}$

$= \sqrt{3}$

$|G'_z| = |1| = 1$

$$= \iint_{D_{xy}} \frac{\sqrt{3}}{1}\, dx\,dy$$

$$= \iint_{D_{xy}} \sqrt{3}\, dx\,dy$$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ | $\frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi$

$0 \leq \rho \leq \sqrt{4\cos^2\theta + 2\cdot\text{sen}^2\theta}$

$$= \sqrt{3}\int_0^{2\pi} d\theta \int_0^{\sqrt{4\cos^2\theta + 2\cdot\text{sen}\,\theta}} \rho\, d\rho$$

hay que "partir" el dominio y llevarlo al recinto de tipo 1.

Por un lado, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

y por el otro, $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

$\left.\frac{\rho^2}{2}\right|_0^{\sqrt{\sim}} = \frac{\left(\sqrt{4\cos^2\theta + 2\cdot\text{sen}^2\theta}\right)^2}{2} = 2\cdot\cos^2\theta + \text{sen}^2\theta$

$$= \sqrt{3}\int_0^{2\pi} \left(2\cdot\cos^2\theta + \text{sen}^2\theta\right) d\theta$$

$$= \sqrt{3}\left[\frac{3}{2}\theta + \frac{\text{sen}(2\theta)}{4}\right]\Bigg|_0^{2\pi}$$

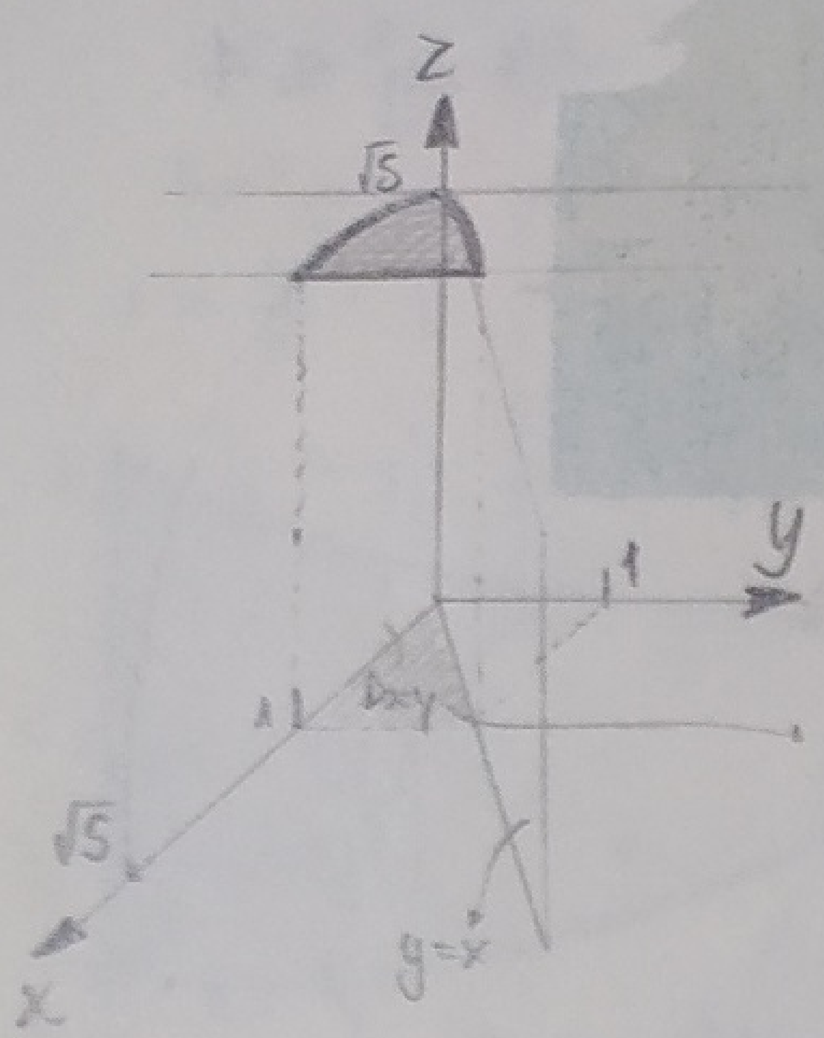
$$= \sqrt{3}\left[\left[\frac{3}{2}\theta + \frac{\text{sen}(2\theta)}{4}\right]\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \left[\frac{3}{2}\theta + \frac{\text{sen}(2\theta)}{4}\right]\Bigg|_{\frac{\pi}{2}}^{\frac{3}{2}\pi}\right]$$

$$= \sqrt{3}\left[\frac{3}{2}\pi + \frac{3}{2}\pi\right]$$

$$\boxed{= 3\sqrt{3}\,\pi}$$

Hallar el área de la superficie $x^2+z^2=5$ limitada por $0 \le y \le x$ $\left\{ \begin{array}{l} y \geq 0 \\ x \geq y \end{array} \right.$, $x \le 1$, $z \geq 0$.

$z = \sqrt{5-x^2}$



proyección en el plano xy

$0 \le x \le 1$

$0 \le y \le x$

$G(x,y,z) = x^2+z^2-5 \quad \nabla G = (2x, 0, 2z)$

$\|\nabla G\| = \sqrt{(2x)^2+(2z)^2} = 2\sqrt{x^2+z^2}$

$|G'_z| = 2z$

$\sqrt{x^2+\sqrt{5-x^2}^2} = \sqrt{\cancel{x^2}+5-\cancel{x^2}} = \sqrt{5}$

$$\boxed{a(S)} = \iint_{D_{xy}} \frac{\|\nabla G\|}{G'_z}\Big|_{z=f(x,y)} dx\,dy = \iint_{D_{xy}} \frac{2\sqrt{x^2+z^2}}{2z}\Big|_{z=\sqrt{5-x^2}} dx\,dy = \iint_{D_{xy}} \frac{\sqrt{5}}{\sqrt{5-x^2}}\,dx\,dy$$

$$= \sqrt{5}\int_0^1 dx \int_0^x \frac{1}{\sqrt{5-x^2}}\,dy = \sqrt{5}\int_0^1 dx \left[\frac{y}{\sqrt{5-x^2}}\Big|_0^x\right] =$$

$$= \sqrt{5}\int_0^1 \frac{x}{\sqrt{5-x^2}}\,dx = \sqrt{5}\left[-\sqrt{5-x^2}\Big|_0^1\right] = \sqrt{5}\Big[\underbrace{\left(-\sqrt{5-1}\right)}_{-2} - \underbrace{\left(-\sqrt{5-0}\right)}_{-(-\sqrt{5})}\Big]$$

$$= \sqrt{5}\left(-2+\sqrt{5}\right)$$

$$= 5-2\sqrt{5}$$

$$\boxed{\approx 0{,}5278}$$