FILTERS

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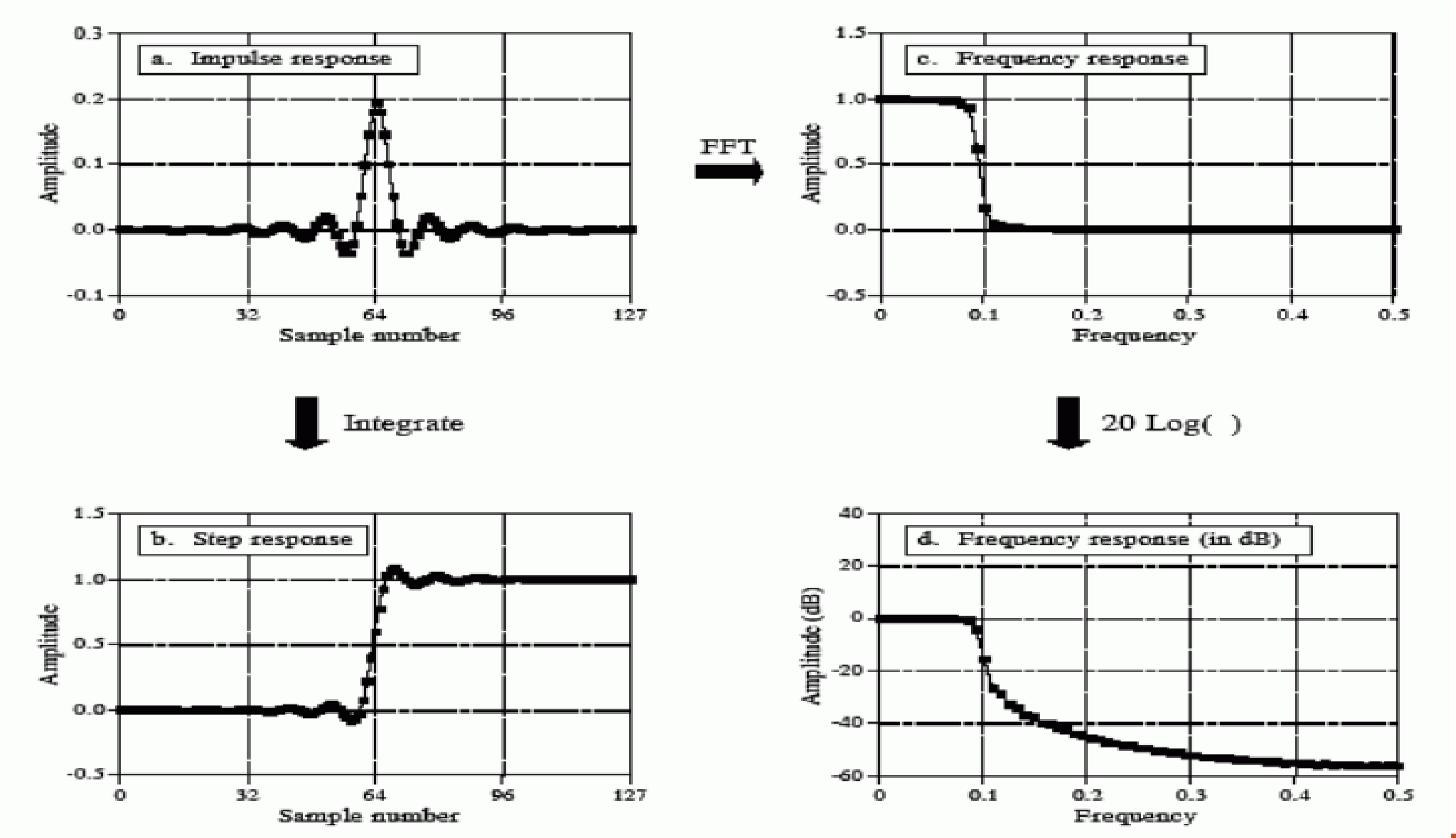
Filter Representations



ANALOG FILTER DESIGN (Butterworth filter)

Butterworth Filter Response

- Amplitude response of an nth order Butterworth filter is given as,

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

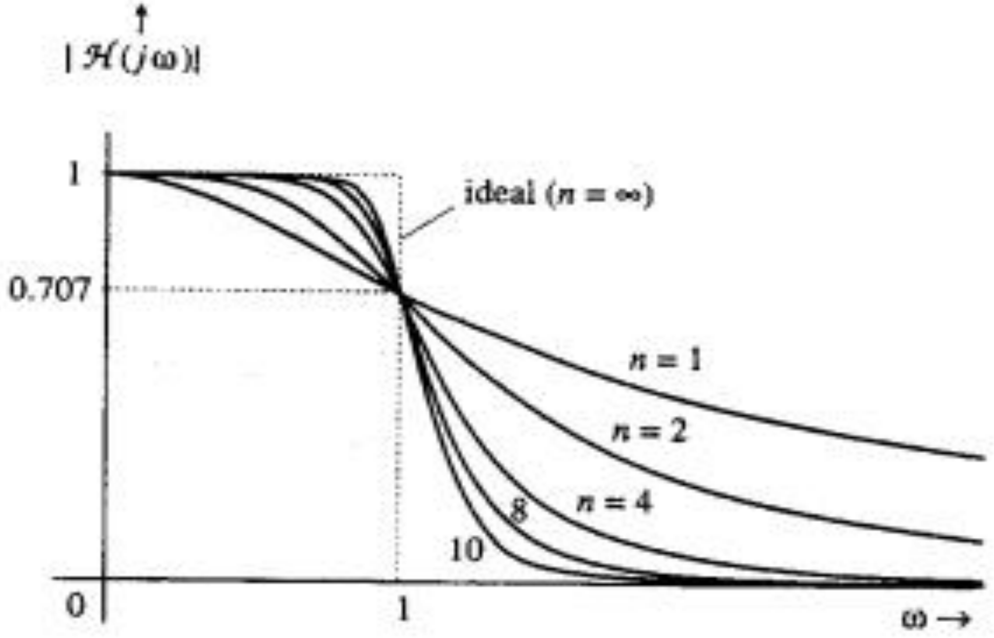


Fig. 7.20 Amplitude response of a normalized lowpass Butterworth filter.

Table for Normalized Butterworth filter, $\omega_c = 1$

Table 1: BUTTERWORTH POLYNOMIALS AND NORMALIZED LOWPASS BUTTERWORTH FILTERS

Order n	Butterworth polynomials $B_n(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.76536s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.6180s+1)(s^2+1.6180s+1)$

Normalized low-pass Butterworth filters $H_n(s)=\dfrac{1}{B_n(s)}$

Determination of Filter Order and Cutoff

- The order of the filter is determined using the equation below where G_s and G_p is the stopband and passband gain in dB. (Gains G_s at $\omega_s$and G_p at ω_p)

$$n = \frac{\log\left[\left(10^{-\hat{G}_s/10} - 1\right) / \left(10^{-\hat{G}_p/10} - 1\right)\right]}{2\log(\omega_s/\omega_p)}$$

- The cutoff frequency (in rad/s) is determined using

$$\omega_c = \frac{\omega_p}{\left[10^{-\hat{G}_p/10} - 1\right]^{1/2n}} \quad \text{or} \quad \omega_c = \frac{\omega_s}{\left[10^{-\hat{G}_s/10} - 1\right]^{1/2n}}$$

Frequency Scaling

- So far we have consider only normalized Butterworth filters with 3dB bandwidth and cut-off frequency $\omega_c = 1$.
- We can design filters for any other cut-off frequency by substituting s by s/ω_c.
- For example, the transfer function for a second-order Butterworth filter for $\omega_c = 100$ is given by:

$$H(s) = \frac{1}{\left(\frac{s}{100}\right)^2 + \sqrt{2}\left(\frac{s}{100}\right) + 1}$$

$$= \frac{1}{s^2 + 100\sqrt{2}s + 10^4}$$

Example #1

Butterworth Filter Design

Design a Butterworth lowpass filter to meet the specifications,

- *Passband gain to lie between 1 and Gp=0.794 (-2dB) for* $0 \leq \omega \leq 10.$
- *Stopband gain not exceed Gs=0.1 (-20dB) for* $\omega \geq 20.$

Magnitude Response for Previous Example

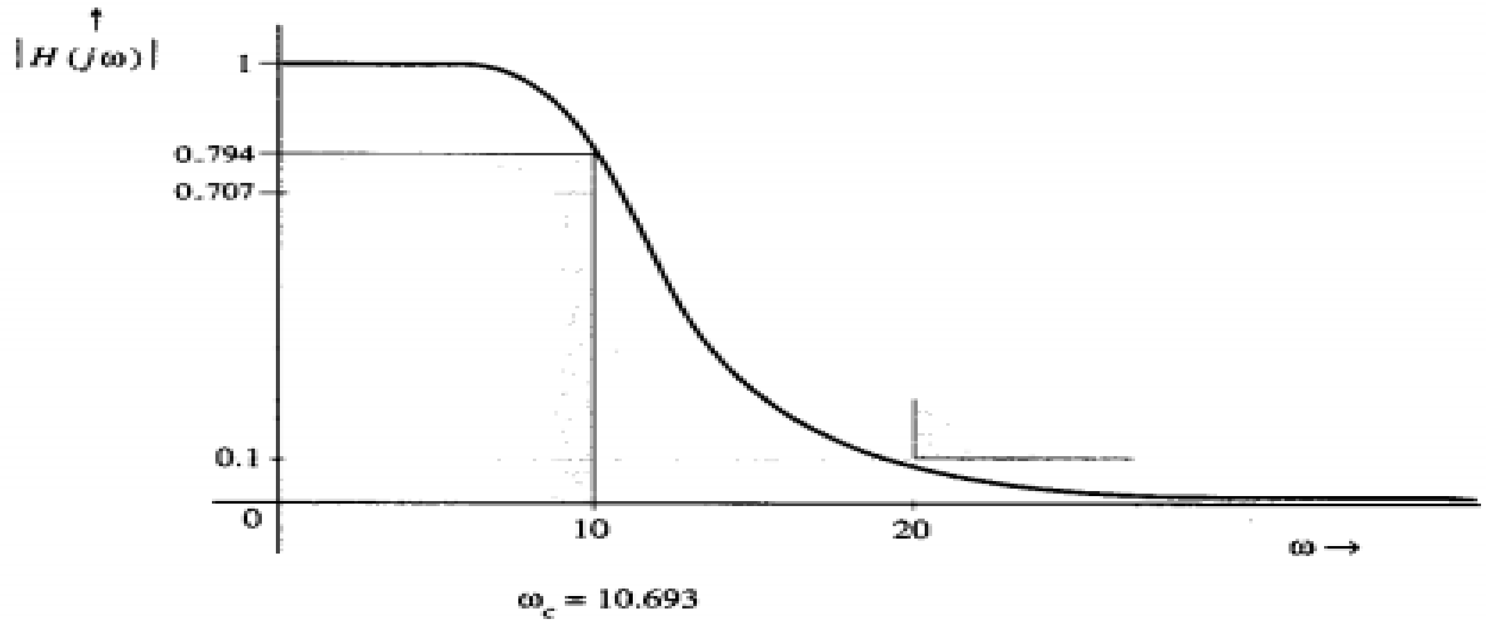


Fig. 7.23 Amplitude response of the lowpass Butterworth filter in Example 7.6.

The amplitude response of this filter is given by Eq. (7.31) with $n = 4$ and $\omega_c = 10.693$

$$|H(j\omega)| = \frac{1}{\sqrt{(\frac{\omega}{10.693})^8 + 1}}$$

Example #2

Design an analog Butterworth filter that has a -2dB or better cutoff frequency at 20 rad/sec and *at least* 10 dB attenuation at 30 rad/sec.

Solution:

The critical requirements are:

$$\Omega_1 = 20, \quad K_1 = -2, \quad \Omega_2 = 30, \quad K_2 = -10,$$

Substituting these requirements into (1):

$$n = \left\lceil \frac{\log_{10}[(10^{-(-2)/10} - 1)/(10^{-(-10)/10} - 1)]}{2\log_{10}(20/30)} \right\rceil = \lceil 3.3709 \rceil = 4$$

Example #2 (continue)

Using this value of n in (2) to exactly satisfy the -2dB requirement gives:

$$\Omega_c = 20/(10^{0.2} - 1)^{1/8} = 21.3868$$

The *normalised* Butterworth lowpass filter ($\Omega_c = 1$) for $n = 4$, can be found in Table 1 as:

$$H_4(s) = \frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)}$$

Applying a normalised lowpass to lowpass transformation, $s \to s/\Omega_c$ with $\Omega_c = 21.3868$ gives the desired transfer function as:

Example #2 (continue)

$$H(s) = H_4(s)\big|_{s \to s/21.3868}$$

$$= \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 0.76536\left(\frac{s}{21.3868}\right) + 1\right]} \times \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 1.84776\left(\frac{s}{21.3868}\right) + 1\right]}$$

$$= \frac{0.209210 \times 10^6}{(s^2 + 16.3686s + 457.394) \cdot (s^2 + 39.5176s + 457.394)}$$

DIGITAL FILTERS (FIR & IIR)

LTI Filtering

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ Y(\omega) &= H(\omega)X(\omega) \end{aligned}$$

where

$$\begin{aligned} x(n) &\overset{\mathcal{F}}{\longleftrightarrow} X(\omega) \\ h(n) &\overset{\mathcal{F}}{\longleftrightarrow} H(\omega) \\ y(n) &\overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) \end{aligned}$$

$$\begin{aligned} H(\omega) &= |H(\omega)|e^{j\Theta(\omega)} \\ |Y(\omega)| &= |H(\omega)||X(\omega)| \\ \angle Y(\omega) &= \Theta(\omega) + \angle X(\omega) \end{aligned}$$

Causal Finite Impulse Response (FIR) Filters

Definition: a discrete-time finite impulse response (FIR) filter is one in which the associated impulse response has finite duration.

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=0}^{M-1} h(k)x(n-k)
\end{aligned}
$$

- lower limit of $k = 0$ is from causality requirement
- upper limit of $0 \leq M - 1 < \infty$ is from the finite duration requirement; in this case the support is M consecutive points starting at time 0 and ending at $M - 1$

Causal Infinite Impulse Response (IIR) Filters

Definition: a discrete-time infinite impulse response (IIR) filter is one in which the associated impulse response has infinite duration.

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\
&= \sum_{k=0}^{\infty} h(k)x(n-k)
\end{aligned}
$$

- lower limit of $k = 0$ is from causality requirement
- necessary upper limit of ∞ is from the infinite duration requirement

Filter Representation using LCCDEs

Linear constant coefficient difference equations (LCCDEs) are an important class of filters that we consider in this course:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Time shifting:	$x(n-k)$	$z^{-k}X(z)$	ROC, except $z=0$ (if $k>0$)

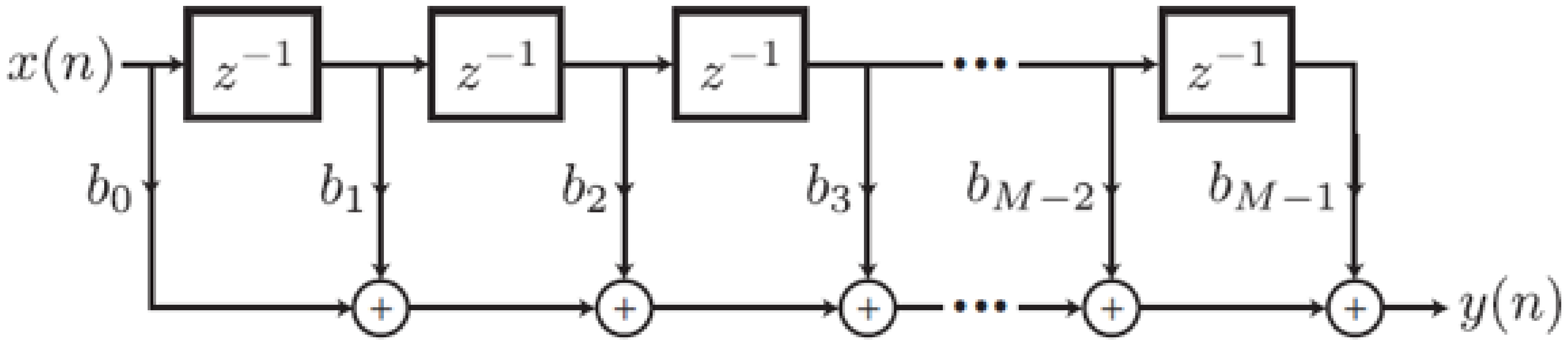
$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{\text{polynomial in } z}{\text{another polynomial in } z}$$

Depending on the values of N, M, a_k and b_k they can correspond to either FIR or IIR filters.

FIR LCCDEs

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

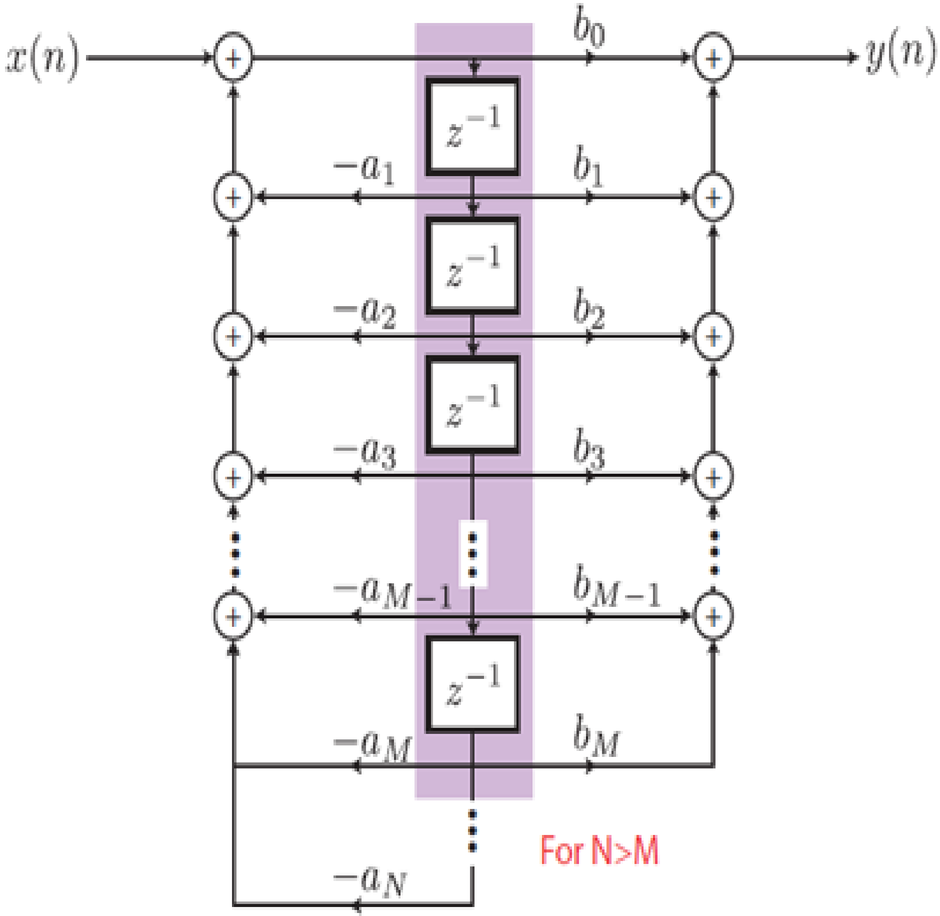
$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$



IIR LCCDEs

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

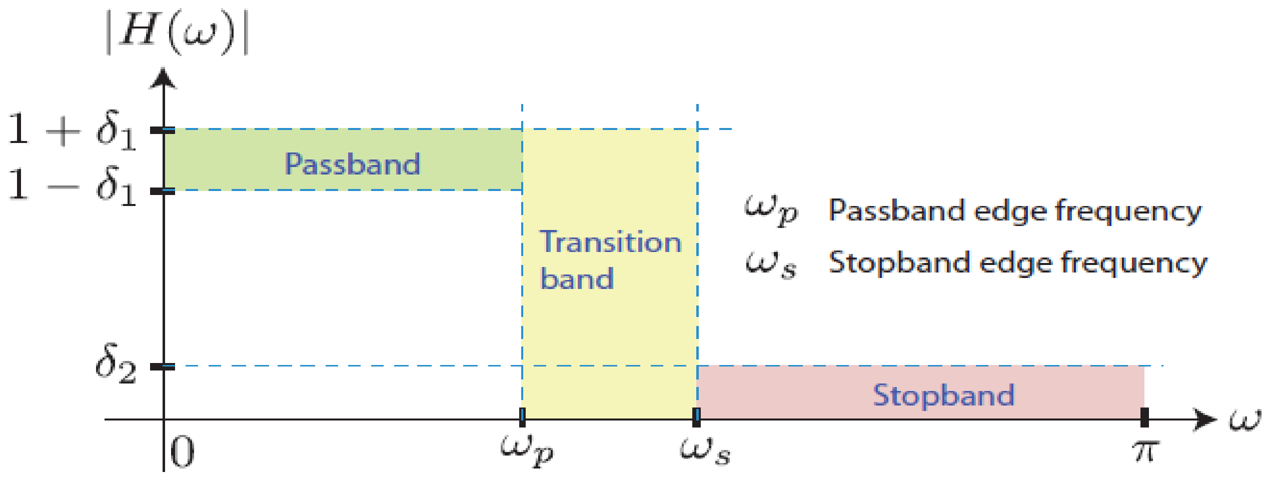


FIR vs IIR

	FIR	IIR
Impulse response	finite	infinite
System function	H(z) = N(z)	H(z) = N(z) / D(z)
Structure diagram	no feedback	have feedback
Phase response	Exact linear phase $h[n]= \pm h[n-N]$	_
Zero-poles	only have zeros	both zeros and poles

Digital Filter Design

- Desired filter characteristics are specified in the frequency domain in terms of desired magnitude and phase response of the filter; i.e., $H(\omega)$ is specified.



- Filter design involves determining the coefficients of a causal FIR or IIR filter that closely approximates the desired frequency response specifications.

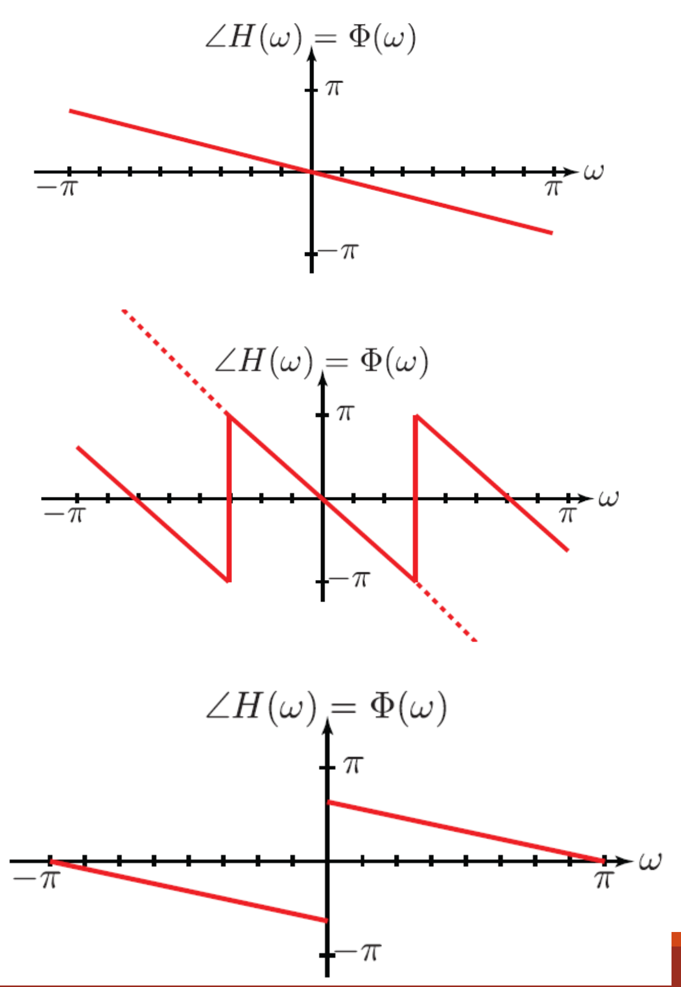
FIR Filter Design using Window Technique

Linear Phase FIR Filter

Time shifting: $x(n-k)$ $e^{-j\omega k}X(\omega)$

- Linear phase filters maintain the relative positioning of the sinusoids in the filter passband.
- This maintains the structure of the signal while removing unwanted frequency components.
- FIR filter is linear phase if it follows (M=sample number),

  1. $h(n) = \pm h(M-1-n)\ for\ n = 0,1,2,\ldots,M-1$
  2. $\alpha = \frac{M-1}{2}$



Design of FIR Filter using Window

- Use the table below to determine the number of sample, M.
- Choosing a proper window depends on the stopband gain which is approximately represented by the peak sidelobe.

Window type	Main lobe width	Peak sidelobe (dB)
Rectangular	$4\pi/M$	-13
Bartlett	$8\pi/M$	-25
Hanning	$8\pi/M$	-31
Hamming	$8\pi/M$	-41
Blackman	$12\pi/M$	-57

Design of FIR Filter using Window

1. Begin with a desired frequency response $H_d(\omega)$ that is linear phase with a delay of $(M-1)/2$ units in anticipation of forcing the filter to be length M.

Example:

$$H_d(\omega) = \begin{cases} 1 \cdot e^{-j\omega(M-1)/2} & 0 \leq |\omega| \leq \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Design of FIR Filter using Window

2. The corresponding impulse response is given by:

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(\omega)e^{j\omega n}d\omega$$

Example:

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-(M-1)/2)}d\omega$$

$$= \begin{cases} \frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} & n \neq \frac{M-1}{2} \\ \frac{\omega_c}{\pi} & n = \frac{M-1}{2} \end{cases} \quad \text{(if } M \text{ is odd)}$$

$$\frac{\sin\omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} \quad \text{(if } M \text{ is even)}$$

Design of FIR Filter using Window

3. Multiply $h_d(n)$ with a window of length M.

$$h(n) = h_d(n) \cdot w(n)$$

Example: rectangular window

$$\begin{aligned}
w(n) &= \begin{cases} 1 & n = 0, 1, \ldots, M-1 \\ 0 & \text{otherwise} \end{cases} \\
h(n) &= h_d(n) \cdot w(n) \\
&= \begin{cases} \frac{\sin \omega_c \left(n - \frac{M-1}{2}\right)}{\pi\left(n - \frac{M-1}{2}\right)} & 0 \leq n \leq M-1, n \neq \frac{M-1}{2} \\ \frac{\omega_c}{\pi} & n = \frac{M-1}{2} \text{ and } M \text{ is odd} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Window Function

Rectangular:

$$w(n) = \begin{cases} 1 & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

Hamming:

$$w(n) = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

Hanning:

$$w(n) = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

Blackman:

$$w(n) = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{M-1}\right) + 0.08\cos\left(\frac{4\pi n}{M-1}\right) & 0 \leq n \leq M-1 \\ 0 & \text{otherwise} \end{cases}$$

Example

FIR Filter Design

1) Design a linear phase FIR filter to approximate an ideal LPF with passband gain of unity, cutoff frequency of 850Hz and a sampling frequency of 5000Hz. The length of the impulse response should be 5. Use i) Rectangular window, ii) Hamming window.

2) Design a linear phase FIR filter using the window method to satisfy the specifications below:
   - $0.99 \leq |H(\omega)| \leq 1.01 \quad for \quad 0 \leq \omega \leq 0.19\pi$
   - $|H(\omega)| \leq 0.01 \quad for \quad 0.21\pi \leq \omega \leq \pi$
   - Cutoff frequency of 0.2π

IIR Filter Design using Impulse Invariance Method (IIM)

s-z Mapping (1)

To investigate the mapping between the s-plane and the z-plane implied by the sampling process, we rely on a generalization of the expression relating z-transform of $h(n)$ to the Laplace transform of $h_a(nT)$. This relationship is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$H(z)\Big|_{z=e^{sT}} = \sum_{n=-\infty}^{\infty} h(n) e^{-sTn}$$

$$z = e^{sT}$$

the s-z mapping

s-z Mapping (2)

Note that when, $s = j\Omega$

$$H(z)\Big|_{z=e^{sT}} = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a\left(s - j\frac{2\pi k}{T}\right)$$

reduces to

$$H(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H_a(j\Omega - jk\Omega_s)$$

$$\omega = \Omega T \qquad\qquad \Omega_s = \frac{2\pi}{T}$$

Frequency Mapping (IIM-Prewarp)

The general characteristic of the *s-z* mapping defined as $z = e^{sT}$ can be obtained by substituting

$$s = \sigma + j\Omega \qquad z = re^{j\omega}$$

With these substitutions we get

$$z = e^{(\sigma + j\Omega)T} = e^{\sigma T} e^{j\Omega T} = r\, e^{j\omega}$$

$$r = e^{\sigma T} \qquad \omega = \Omega T$$

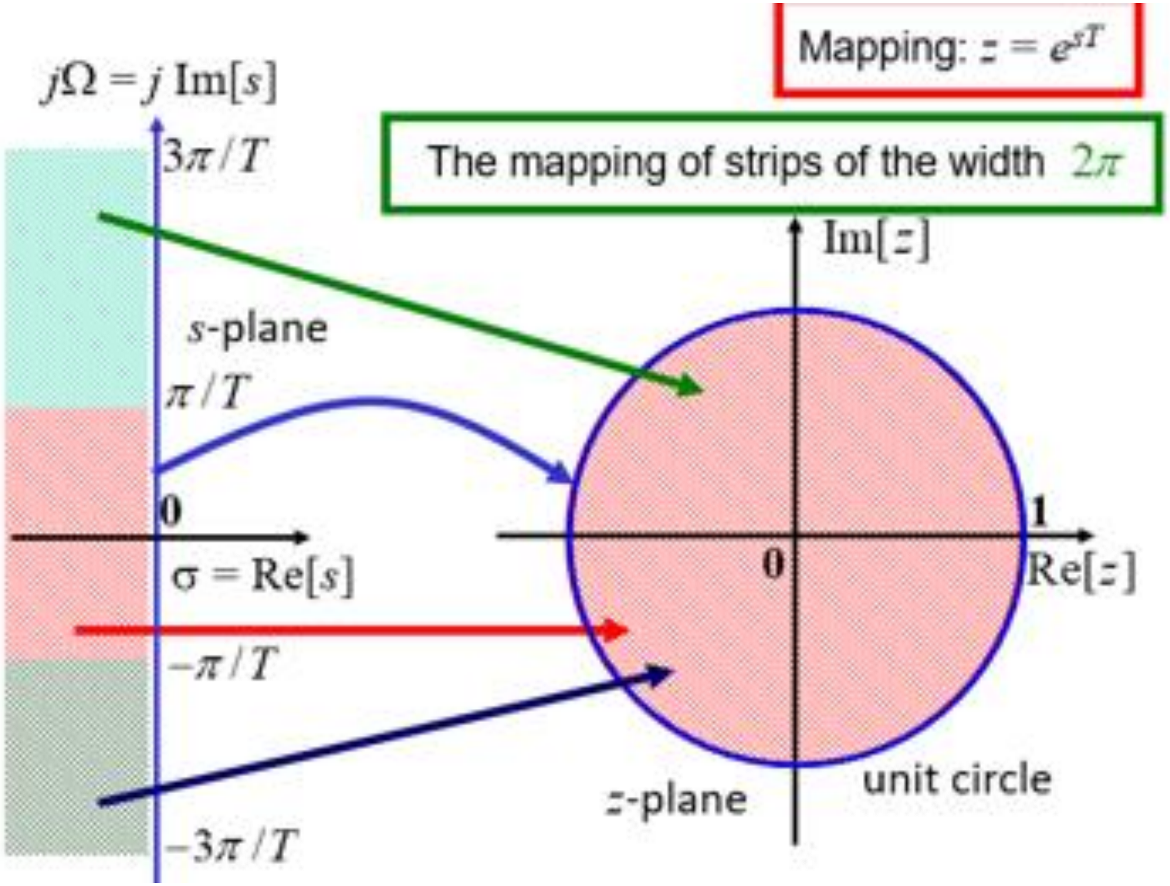
s-z Mapping (4)

a) $\sigma < 0 \rightarrow 0 < r < 1 \qquad \sigma > 0 \rightarrow r > 1 \qquad \sigma = 0 \rightarrow r = 1$

The left-half of *s*-plane is mapped inside the unite circle in *z*-plane and right-half of *s*-plane is mapped into points that fall outside the unit circle in *z*-plane. This is one of the desirable properties of a good *s-z* mapping.

b) $j\Omega$ -axis is mapped into the unit circle in *z*-plane as indicated above.

s-z Mapping (5)



Mapping: $z = e^{sT}$
The mapping of strips of the width 2π
$j\Omega = j\,\mathrm{Im}[s]$
$3\pi/T$
s-plane
π/T
0
$\sigma = \mathrm{Re}[s]$
$-\pi/T$
$-3\pi/T$
$\mathrm{Im}[z]$
1
0
$\mathrm{Re}[z]$
z-plane
unit circle

IIR Filter Design using IIM

The impulse response of the **analog** filter $h_a(t)$

1

$$h_a(t) = L^{-1}\{H_a(s)\}$$

$$= L^{-1}\left\{\sum_{k=1}^{N} \frac{c_k}{s+s_k}\right\} = \sum_{k=1}^{N} L^{-1}\left\{\frac{c_k}{s+s_k}\right\}$$

$$= \sum_{k=1}^{N} c_k e^{-s_k t} \rightarrow t = nT$$

The impulse response of the **digital** filter $h(nT)$:

2

$$h(n) = h_a(nT) = \sum_{k=1}^{N} c_k e^{-s_k nT}$$

$$n = 0,1,2,3,\ldots,\infty$$

The system function of the **digital** filter

3

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} \sum_{k=1}^{N} c_k e^{-s_k nT}$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{-s_k T} z^{-1}\right)^n$$

$$= \sum_{k=1}^{N} c_k \frac{1}{1 - e^{-s_k T} z^{-1}}$$

Transfer Function from s to z

Transfer function of the analog filter:

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s + s_k}$$

System function of the digital filter:

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{-s_k T} z^{-1}}$$

Comparing $H_a(s)$ and $H(z)$, we see that $H(z)$ can be obtained from $H_a(s)$ by using the mapping relation:

$$\frac{c_k}{s + s_k} \rightarrow \frac{c_k}{1 - e^{-s_k T} z^{-1}}$$

Example

IIR-IIM Filter Design

1) The transfer function of an analog system is given as $H_a(s) = \frac{s+1}{s^2+2s+5}$. Use the impulse invariance method to determine $H(z)$ for a discrete-time system such that $h(n) = h(nT), T = 1$.

IIR Filter Design using Bilinear Transformation

Bilinear Transform

bilinear transformation mapping is:

$$\boxed{s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

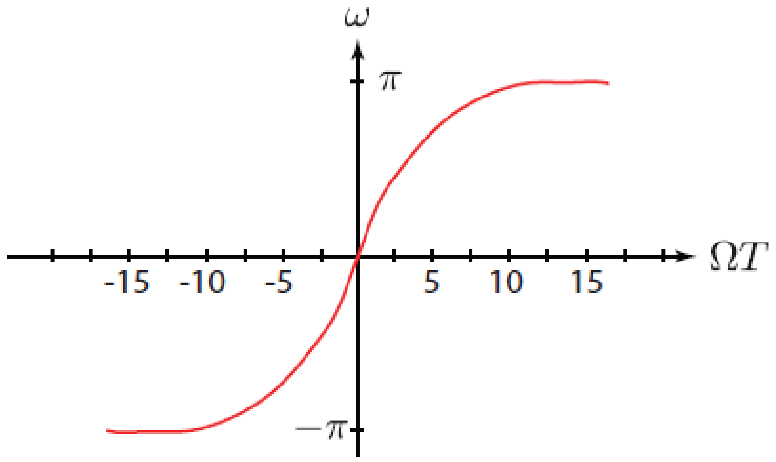
The mapping $s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ will work for any order of differential equation to convert $H_a(s)$ to $H(z)$.

General Methodology:

1. Start with $H_a(s)$ expression.
2. Determine T through the problem specifications.
3. $H(z) = H_a\left(\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right)$

$s \rightarrow z$ Mapping

For $s = j\Omega$ and $z = e^{j\omega}$:



The entire $-\infty < \Omega < \infty$ axis is mapped to $-\pi < \omega < \pi$. There is a huge compression of the frequency response at large Ω-values.

Frequency Mapping (BT-Prewarp)

- To derive the relation between ω and Ω

$$s = \frac{2}{T_d}\left(\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right) = \sigma + j\Omega = \frac{2}{T_d}\left[\frac{2e^{-j\omega/2} j\sin(\omega/2)}{2e^{-j\omega/2}\cos(\omega/2)}\right] = \frac{2j}{T_d}\tan\left(\frac{\omega}{2}\right)$$

which yields

$$\boxed{\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right)} \quad \text{or} \quad \boxed{\omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)}$$

Example

IIR-BT Filter Design

Design a digital low pass filter, H(z) using the bilinear transformation to satisfy the following equivalent analog specifications:

- Monotonic passband attenuation: -3.01dB or less at cutoff frequency 6 kHz.
- Monotonic stopband attenuation: -20 dB or greater at 9 kHz.
- Sampling rate of 24 kHz.

Extra Example

- Bilinear transform applied to Butterworth

$$0.89125 \le \left|H\left(e^{j\omega}\right)\right| \le 1 \qquad 0 \le |\omega| \le 0.2\pi$$

$$\left|H\left(e^{j\omega}\right)\right| \le 0.17783 \qquad 0.3\pi \le |\omega| \le \pi$$

- Apply bilinear transformation to specifications

$$0.89125 \le |H(j\Omega)| \le 1 \qquad 0 \le |\Omega| \le \frac{2}{T_d} \tan\left(\frac{0.2\pi}{2}\right)$$

$$|H(j\Omega)| \le 0.17783 \qquad \frac{2}{T_d} \tan\left(\frac{0.3\pi}{2}\right) \le |\Omega| < \infty$$

- We can assume $T_d=1$ and apply the specifications to

$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega / \Omega_c)^{2N}}$$

to get

$$1 + \left(\frac{2\tan 0.1\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.89125}\right)^2 \quad \text{and} \quad 1 + \left(\frac{2\tan 0.15\pi}{\Omega_c}\right)^{2N} = \left(\frac{1}{0.17783}\right)^2$$

Extra Example (cont..)

- Solve N and Ω_c

$$N = \frac{\log\left[\left(\left(\frac{1}{0.17783}\right)^2 - 1\right) \Big/ \left(\left(\frac{1}{0.89125}\right)^2 - 1\right)\right]}{2\log\left[\tan(0.15\pi)/\tan(0.1\pi)\right]} = 5.305 \cong 6$$

$$\Omega_c = 0.766$$

- The resulting transfer function has the following poles

$$s_k = \Omega_c e^{(j\pi/12)(2k+11)} \quad \text{for } k = 0,1,\ldots,11$$

- Resulting in

$$H_c(s) = \frac{0.20238}{(s^2 + 0.3996s + 0.5871)(s^2 + 1.0836s + 0.5871)(s^2 + 1.4802s + 0.5871)}$$

- Applying the bilinear transform yields

$$H(z) = \frac{0.0007378\left(1 + z^{-1}\right)^6}{\left(1 - 1.2686z^{-1} + 0.7051z^{-2}\right)\left(1 - 1.0106z^{-1} + 0.3583z^{-2}\right)} \times \frac{1}{\left(1 - 0.9044z^{-1} + 0.2155z^{-2}\right)}$$