תרגיל בית 2:

שאלה 1:

א- נתון $\hat{\theta}$ אומד בשיטת המומנטים של θ לכן לפי ההגדרה קיימת פונקציה g רציפה וגם מתקיים:

$$\theta = g(\mu_1,\dots,\mu_n)$$

והאומד $\hat{\theta}$ בשיטת המומנטים עבור θ הינו:

$$\hat{\theta} = g(M_1,\dots,M_n)$$

לכן עבור פונקציה $\tilde{g}=\tau\circ g$ (הרכבת הפונקציה τ על g) מתקיים:

$$(1)\quad \tau(\hat{\theta}) = \tau(g(M_1,\dots,M_n)) = \tau\circ g(M_1,\dots,M_n)$$

$$(2)\quad \tau(\theta) = \tau(g(\mu_1,\mu_2,\dots,\mu_n)) = \tau\circ g(\mu_1,\dots,\mu_n)$$

לכן $\tau(\hat{\theta})$ הינו אומד בשיטת המומנטים של $\tau(\theta)$ לפי ההגדרה.

ב- אם T_1,T_2 הם אומדים בשיטת המומנטים של θ אז קיימים שני פונקציות f ו- g כך ש:

רציפות

$$\theta = f(t_1^1, t_2^1, \dots, t_n^1)$$

$$\theta = g(t_1^2, t_2^2, \dots, t_n^2)$$

והאומדים T_1 ו- T_2 בשיטת המומנטים עבור

$$T_1 = f(T_1^1, T_2^1, \dots, T_n^1)$$

$$T_2 = g(T_2^1\ T_2^2, \dots, T_n^2)$$

נשים לב:

$$T_3 = \frac{T_1+T_2}{2} = \frac{1}{2}\left(f(T_1^1,T_2^1,\dots,T_n^2) + g(T_2^1,T_2^2,\dots,T_n^2)\right)$$

לכן

$$\frac{1}{2}\left(f(t_1^1,t_2^1,\dots,t_n^1) + g(t_1^2,t_2^2,\dots,t_n^2)\right) = \frac{1}{2}(\theta+\theta) = \frac{1}{2}\cdot 2\theta = \theta$$

לכן $T_3=\frac{T_1+T_2}{2}$ הוא גם אומד בשיטת המומנטים של θ.

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$$E(\bar{X})=\frac{2}{\lambda}=M_1=\bar{X}\Rightarrow\boxed{\frac{1}{\lambda}=\frac{\bar{X}}{2}}$$

$$var(X)=\frac{2}{\lambda^2}=M_2-M_1^2\Rightarrow$$

[illegible] c:

1c

$$T_1=\left(\frac{\sum_{i=1}^{n}X_i}{n}\right)^2=(\bar{X})^2=\left(\frac{2}{\lambda}\right)^2=\frac{4}{\lambda^2}$$

אזי $c_1=\frac{1}{4}$.

$$T_2=\frac{\sum_{i=1}^{n}X_i^2}{n}=\frac{\sum_{i=1}^{n}X_i^2+\bar{X}^2-\bar{X}^2}{n}=\frac{\sum_{i=1}^{n}X_i^2-\bar{X}^2}{n}+\frac{\sum_{i=1}^{n}\bar{X}^2}{n}=$$

$$\frac{\sum_{i=1}^{n}(X_i^2-\bar{X}^2)}{n}+\bar{X}^2=\frac{2}{\lambda^2}+\left(\frac{2}{\lambda}\right)^2=\frac{2}{\lambda^2}+\frac{4}{\lambda^2}=\frac{6}{\lambda^2}$$

אזי $c_2=\frac{1}{6}$

$$T_3=\frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{n}=\frac{2}{\lambda^2}\Rightarrow\boxed{c_3=\frac{1}{2}}$$

2

$$E(c_1T_1)=\frac{1}{4}E(T_1)=\frac{1}{4}E(\bar{X}^2)=\frac{1}{4}\left[var(\bar{X})+E(\bar{X})^2\right]$$

$$=\frac{1}{4}\cdot\left[\frac{2}{\lambda^2 n}+\left(\frac{2}{\lambda}\right)^2\right]=\frac{1}{4}\left[\frac{2}{\lambda^2 n}+\frac{4}{\lambda^2}\right]=\frac{1}{2\lambda^2}\left[\frac{1}{n}+2\right]$$

$$=\frac{1}{2}\left[\frac{1+2n}{\lambda^2 n}\right]$$

$$E(c_2T_2)=c_2E(T_2)=\frac{1}{6}E(T_2)=\frac{1}{6}E\left(\frac{\sum_{i=1}^{n}X_i^2}{n}\right)=$$

$$\frac{1}{6}E\left(\frac{\sum_{i=1}^{n}(X_i^2-\bar{X})^2}{n}+\bar{X}^2\right)=\frac{1}{6}var(X)+\frac{1}{6}E(\bar{X}^2)=$$

$$\frac{1}{6}\cdot\left[\frac{2}{\lambda^2}+\left(\frac{2}{\lambda}\right)^2\right]=\frac{1}{6}\cdot\frac{6}{\lambda^2}=\frac{1}{\lambda^2}$$

ע"פ הרצאה 2 בסיכום.

$$E(c_3T_3)=c_3E(T_3)=c_3E\left(\frac{n-1}{n-1}\cdot\frac{1}{n}\cdot\sum_{i=1}^{n}(X_i-\bar{X})^2\right)=\frac{(n-1)}{n}\cdot var(X)\cdot c_3$$

$$=\frac{(n-1)}{n}\cdot\frac{2}{\lambda^2}\cdot c_3$$

$$\boxed{\frac{(n-1)}{n}\frac{1}{\lambda^2}}$$

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שאלה ה3:

א-

$$T_1 = \bar{X}(\bar{X}-1) = \mu_1(\mu_1-1) = \mu_1^2-\mu_1 = g(\mu_1)$$

$$\mu_1 = E(X) = \frac{1}{p}$$

$$g(\mu_1) = \left(\frac{1}{p}\right)^2 - \frac{1}{p} = \frac{1-p}{p^2} = var(X)$$

$$T_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2 \xrightarrow[\text{הוכחנו בהרצאה}]{} = \mu_2-\mu_1^2 = g(\mu_1,\mu_2)$$

$$\mu_1 = E(X) = \frac{1}{p}$$

$$\mu_2 = E(X^2) = var(X) + E(X)^2 \underset{X\sim Geo(p)}{=} \frac{1-p}{p^2} + \left(\frac{1}{p^2}\right) = \frac{2-p}{p^2}$$

$$g(\mu_1,\mu_2) = \mu_2-\mu_1^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2} = var(X)$$

$$T_3 = \frac{1}{2n}\sum_{i=1}^{n}X_i^2 - \frac{1}{2}\bar{X} = \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}X_i^2 - \bar{X}\right] = \frac{1}{2}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i^2-\bar{X}^2+\bar{X}^2) - \bar{X}\right]$$

$$= \frac{1}{2}\left[\underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i^2-\bar{X}^2)} + \frac{1}{n}\sum_{i=1}^{n}\bar{X}^2 - \bar{X}\right]$$

$$= \frac{1}{2}\left[\mu_2-\mu_1^2+\bar{X}^2-\bar{X}\right] = \frac{1}{2}\left[\mu_2-\mu_1^2+\mu_1^2-\mu_1\right]$$

$$= \frac{1}{2}\left[\mu_2-\mu_1\right] = f(\mu_1,\mu_2)$$

$$f(\mu_1,\mu_2) = \frac{1}{2}\left[\mu_2-\mu_1\right] = \frac{1}{2}\left[\frac{2-p}{p^2}-\frac{1}{p}\right] = \frac{1}{2}\left[\frac{2-p-p}{p^2}\right] =$$

$$\frac{1}{2}\cdot\frac{2-2p}{p^2} = \frac{1-p}{p^2} = var(X)$$

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ב

$$E(T_1) = E\left(\bar{X}(\bar{X}-1)\right) = E(\bar{X}^2 - \bar{X}) - E(\bar{X}^2) - E(\bar{X})$$

$$= var(\bar{X}) + E(\bar{X})^2 - E(\bar{X}) = \frac{1}{n} var(X) + E(X)^2 - E(X)$$

$$= \frac{1}{n}\left[\frac{(1-p)}{p^2}\right] + \left(\frac{1}{p}\right)^2 - \frac{1}{p} = \frac{1-p}{n \cdot p^2} + \frac{1}{p^2} - \frac{1}{p} =$$

$$\frac{1-p+n-np}{np^2} = \frac{(1-p)+n(1-p)}{np^2} = \frac{(1-p)}{p^2} \cdot \frac{n+1}{n} =$$

$$\frac{n+1}{n} var(X) \geq var(X)$$

מתוקן: $T_1^* = \frac{n}{n+1} T_1$

$$E(T_2) = E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right) = E\left(\frac{1}{n}\sum_{i=1}^{n}\left[X_i - E(X) + E(X) - \bar{X}\right]^2\right)$$

$$= E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X))^2 - (\bar{X} - E(X))^2\right] = E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X))^2\right)$$

$$- E\left(\bar{X} - E(\bar{X})\right)^2 = var(X) - var(\bar{X}) = var(X) - \frac{1}{n} var(X)$$

$$= \frac{n-1}{n} var(X) \leq var(X)$$

מתוקן: $T_2^* = \frac{n}{n-1} T_2$

$$E(T_3) = E\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2 - \frac{1}{2}\bar{X}\right) = \frac{1}{2} E\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) - \frac{1}{2} E(\bar{X})$$

$$= \frac{1}{2}\left[E(X^2) - E(X)\right] = \frac{1}{2}\left[var(X) + E(X)^2 - E(X)\right]$$

$$= \frac{1}{2}\left[\frac{(1-p)}{p^2} + \frac{1}{p^2} - \frac{1}{p}\right] = \frac{1}{2}\left[\frac{1-p+1-p}{p^2}\right] = \frac{1}{2} \cdot \frac{2-2p}{p^2} =$$

$$\frac{1}{2} \cdot \frac{2(1-p)}{p^2} = \frac{1-p}{p^2} = var(X) = var(X)$$

$\boxed{T_3^* = T_3}$ אכן כבר מתוקן

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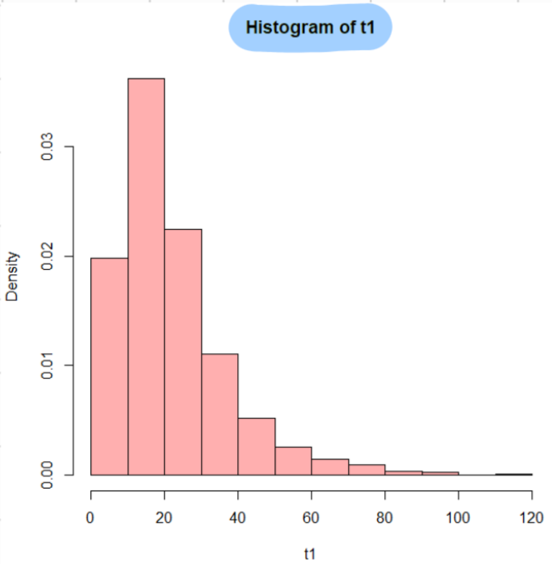
שאלה 3

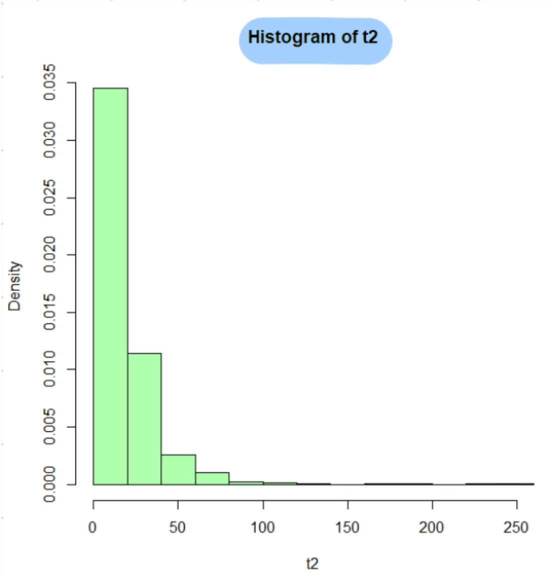
(2

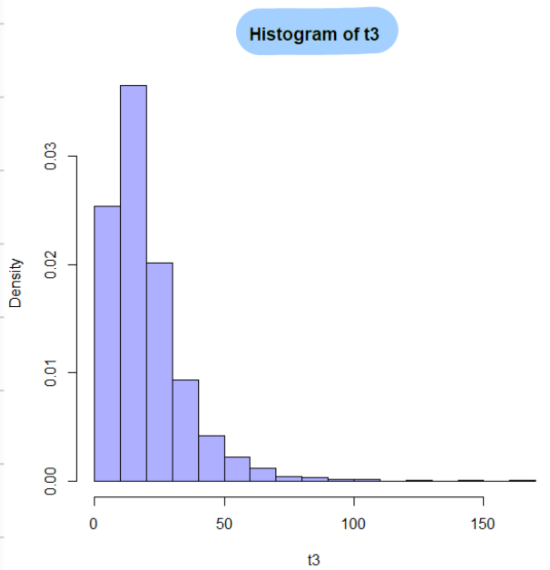
$$\sigma^2 = Var(Geo(0.2)) = \frac{1-0.2}{(0.2)^2} = 20$$

```r
N = 3000
n = 10
p = 0.2


mat = matrix(0, n, N)

for (i in 1:N) {
  sample = 1 + rgeom(n, p)
  mat[,i] = sample
}

t1 = rep(0, N)
t2 = rep(0, N)
t3 = rep(0, N)
m = rep(0, N)

T1 = function (arg1) {
  temp1 = mean(arg1)
  return(temp1*(temp1 - 1))
}

T2 = function(arg2) {
  mean2 = mean(arg2)
  temp2 = arg2 - mean2
  temp2 = temp2 ^ 2
  return(sum(temp2) / length(temp2))
}

T3 = function (arg3) {
  temp3 = arg3 ^ 2
  return((sum(temp3) / (2*length(temp3))) - 0.5 * mean(arg3))
}


for (i in 1:N) {
  t1[i] = T1(mat[, i])
  t2[i] = T2(mat[, i])
  t3[i] = T3(mat[, i])
  m[i] = mean(mat[, i])
}

mean(t1)
mean(t2)
mean(t3)
```

```
> mean(t1)
[1] 22.01214
> mean(t2)
[1] 18.01473
> mean(t3)
[1] 20.01343
```







עבור $d = 10$

מרחק בין t_i ל d $\Leftarrow$ $|t_i-\sigma^2|$

נרצה $P(|t_i-\sigma^2|\le d) = P(\sigma^2-d \le t_i \le \sigma^2+d)$

$$= F_{t_i}(\sigma^2+d) - F_{t_i}(\sigma^2-d)$$

```
> sum(t1 > 0 & t1 < 30) / length(t1)
[1] 0.7663333
> sum(t1 > 0 & t1 < 10) / length(t1)
[1] 0.198
> 0.7663333-0.198
```
$\longleftarrow F_{t_i}(\sigma^2+d)-F_{t_i}(\sigma^2-d)$

```
[1] 0.5683333
> sum(t1 > 10 & t1 < 30) / length(t1)
[1] 0.5683333
> sum(t1 > 10 & t1 < 30) / length(t1)
[1] 0.5683333
> sum(t2 > 10 & t2 < 30) / length(t2)
[1] 0.4743333
> sum(t3 > 10 & t3 < 30) / length(t3)
[1] 0.5663333
```

(3

.1

```
> snow <- c(1,1,3,3,3,4,5,5,5,5,5,5,5,5,6,10,10,10,15,15,15,15,20,20,20,20,20, 20,30,30,35,40,40,40,40,45,65)
> n.snow = length(snow)
>
> T1(snow)
[1] 278.279
> T2(snow)
[1] 233.0723
> T3(snow)
[1] 255.6757
>
> (n.snow/(n.snow+1)) * T1(snow)
[1] 270.9559
> (n.snow/(n.snow-1)) * T2(snow)
[1] 239.5465
> T3(snow)
[1] 255.6757
```
מתוקנים

.2

```
> snow <- c(1,1,3,3,3,4,5,5,5,5,5,5,5,5,6,10,10,10,15,15,15,15,20,20,20,20,20, 20,30,30,35,40,40,40,40,45,65)
> n.snow = length(snow)
>
> sqrt(T1(snow))
[1] 16.6817
> sqrt(T2(snow))
[1] 15.26671
> sqrt(T3(snow))
[1] 15.98986
>
> sqrt((n.snow/(n.snow+1)) * T1(snow))
[1] 16.46074
> sqrt((n.snow/(n.snow-1)) * T2(snow))
[1] 15.47729
> sqrt(T3(snow))
[1] 15.98986
```
מתוקנים

אני רואה כי ההבדל בין האומדנים לבין הגרסאות המתוקנות הוא בסדר גודל 10^{-1}