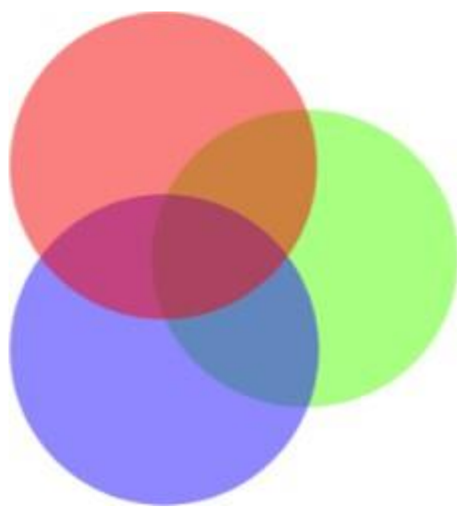
Fuzzy Logic

Basic concepts & Fuzzy rules

Lecture 3



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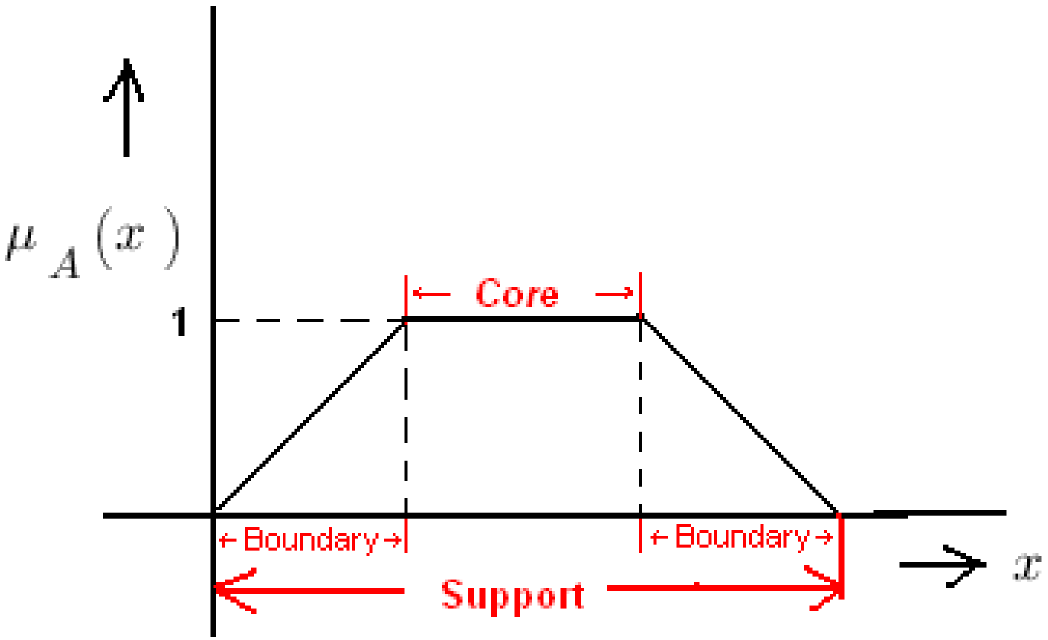
Features of the Membership Function

- **Core:** comprises those elements *x* of the universe such that $\mu_a(x) = 1$.
- **Support :** region of the universe that is characterized by nonzero membership.
- **Boundary :** boundaries comprise those elements *x* of the universe such that $0 < \mu_a(x) < 1$

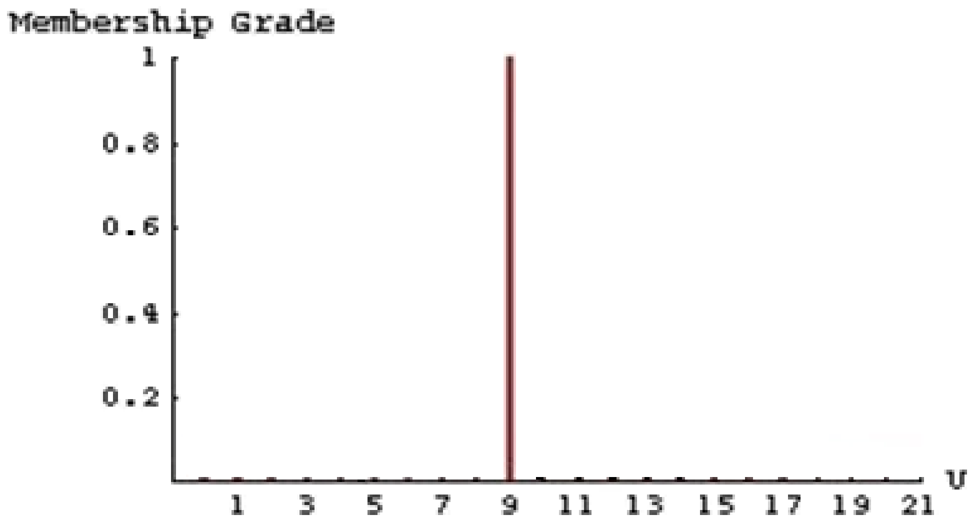


Basic concepts and terminology

The **support** of a fuzzy set A in the universal set X is a crisp set that contains all the elements of X that have nonzero membership values in A, that is,

$$\mathrm{supp}(A) = \{x \in X \mid A(x) > 0\}$$

A fuzzy **singleton** is a fuzzy set whose support is a single point in X.



Basic concepts and terminology

A **crossover point** of a fuzzy set is a point in X whose membership value to A is equal to 0.5.

The **height**, $h(A)$ of a fuzzy set A is the largest membership value attained by any point. If the height of a fuzzy set is **equal to one**, it is called a **normal** fuzzy set, otherwise it is **subnormal**.

Basic concepts and terminology

An **α-cut** of a fuzzy set A is a **crisp set** ${}^{\alpha}A$ that contains all the elements in X that have membership value in A greater than or equal to α.

$${}^{\alpha}A = \{x \mid A(x) \geq \alpha\}$$

A **strong α-cut** of a fuzzy set A is a crisp set ${}^{\alpha+}A$ that contains all the elements in X that have membership value in A **strictly** greater than α.

$${}^{\alpha+}A = \{x \mid A(x) > \alpha\}$$

Basic concepts and terminology

We observe that the strong α-cut ${}^{0+}A$ is equivalent to the support supp(A).

The 1-cut ${}^{1}A$ is often called the **core** of A.
Note! Sometimes the highest non-empty α-cut ${}^{h(A)}A$ is called the core of A. (in the case of subnormal fuzzy sets, this is different).

The word **kernel** is also used for both of the above definitions. (Total confusion!)

Basic concepts and terminology

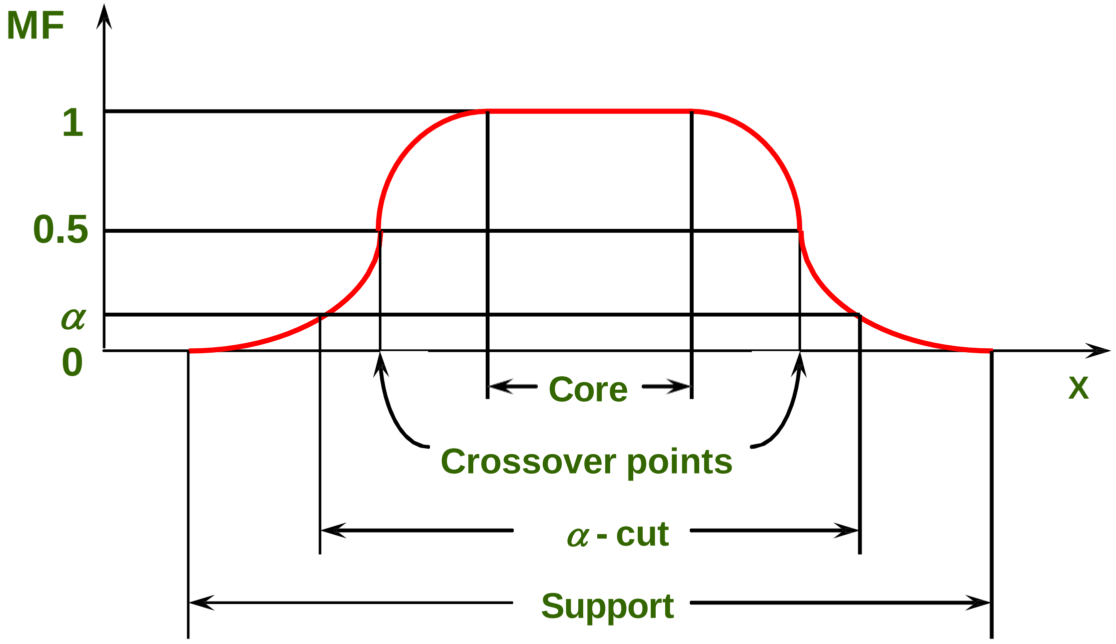
The ordering of the values of α in $[0, 1]$ is **inversely** preserved by set inclusion of the corresponding α-cuts as well as strong α-cuts. That is, for any fuzzy set A and $\alpha_1 < \alpha_2$ it holds that ${}^{\alpha_2}A \subseteq {}^{\alpha_1}A$.

All α-cuts and all strong α-cuts form two distinct families of **nested** crisp sets.

The set of all levels $\alpha \in [0, 1]$ that represent distinct α-cuts of a given fuzzy set A is called the **level set** of A.

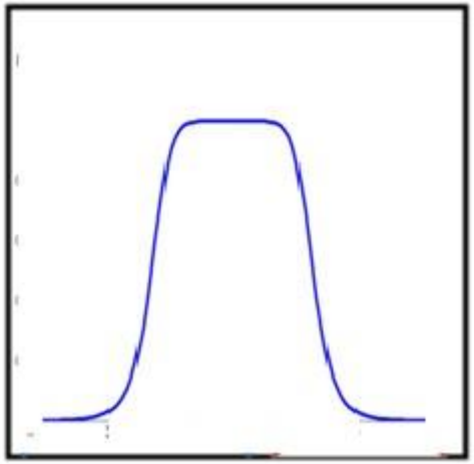
$$\Lambda(A) = \{\alpha \mid A(x) = \alpha \text{ for some } x \in X\}.$$

MF Terminology

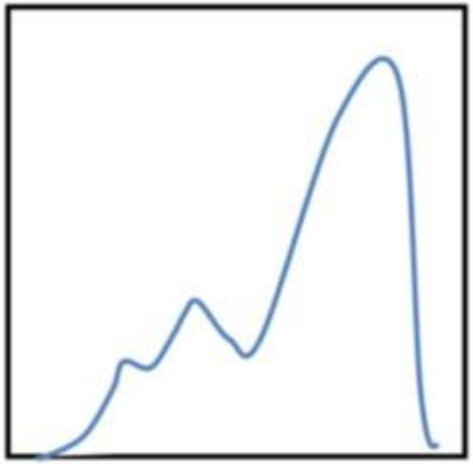


Convexity

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\,[\mu_A(x_1), \mu_A(x_2)] \qquad \lambda \in [0,1]$$



Convex



Non-convex

basic concepts

- The standard complement of fuzzy set A with respect to the universal set X is defined for all $x \in X$ by the equation

$$\overline{A}(x) = 1 - A(x)$$

- **For the standard complement, clearly, membership grades of equilibrium points are 0.5.**
  - Elements of X for which $\overline{A}(x) = A(x)$ are called equilibrium points of A.
  - For example, the equilibrium points of A_2 in Fig. 1.7 are 27.5 and 52.5.

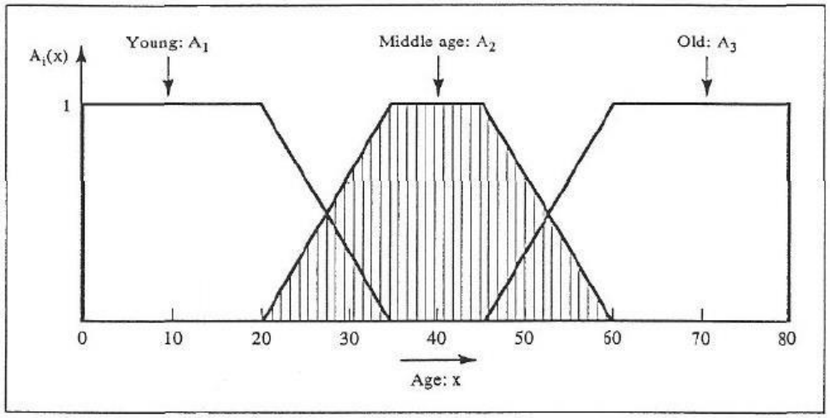


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

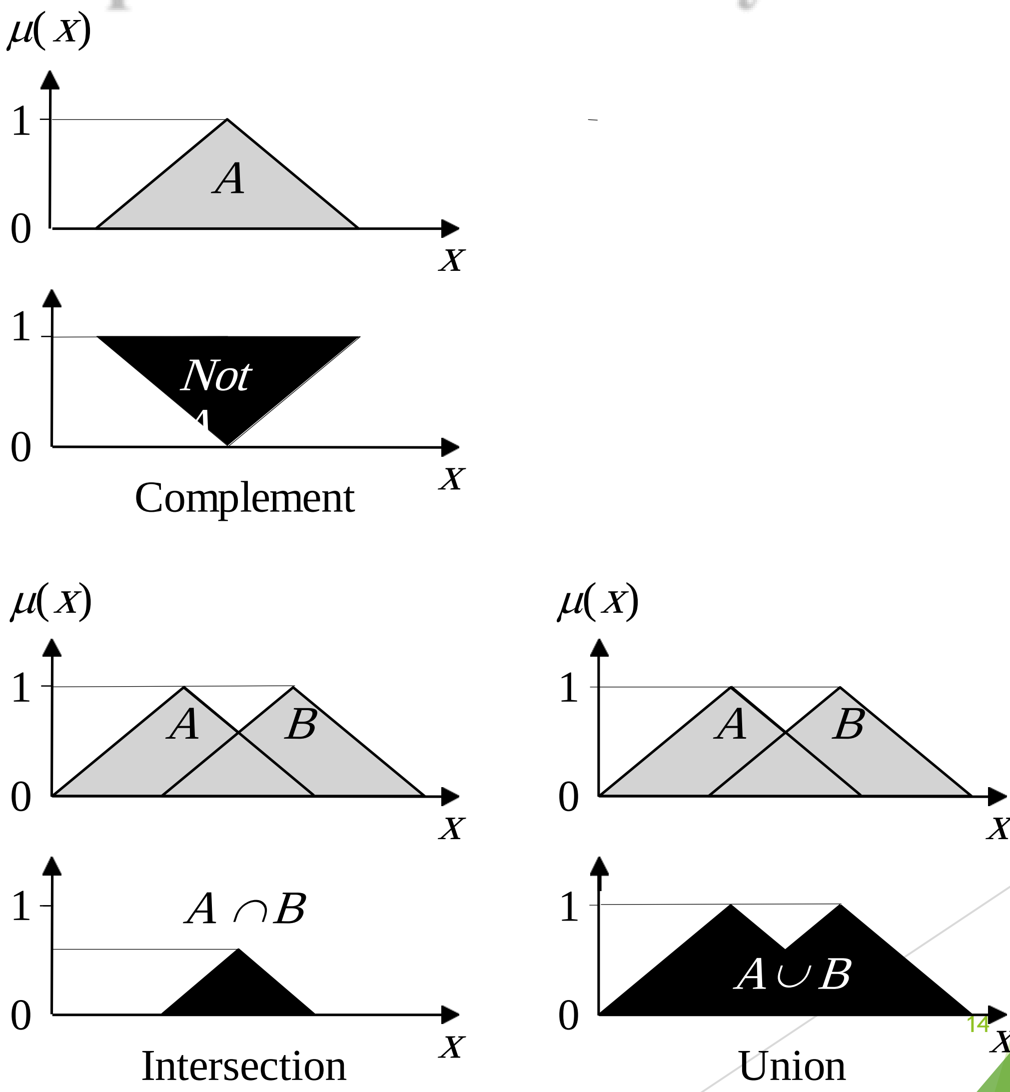
Basic concepts

- Discussions:
  - Normality may be lost when we operate on fuzzy sets by the standard operations of intersection and complement.
  - The fuzzy intersection and fuzzy union will satisfies all the properties of the Boolean lattice listed in Table 1.1 except the low of contradiction and the low of excluded middle.

TABLE 1.1 FUNDAMENTAL PROPERTIES OF CRISP SET OPERATIONS

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and $\varnothing$	$A \cup X = X$
	$A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$
	$A \cap X = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$
Law of excluded middle	$A \cup \overline{A} = X$
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

Operations of Fuzzy Sets



Properties of Fuzzy Sets

- Equality of two fuzzy sets
- Inclusion of one set into another fuzzy set
- Cardinality of a fuzzy set
- An empty fuzzy set
- α-cuts (alpha-cuts)

Equality

- Fuzzy set A is considered equal to a fuzzy set B, IF AND ONLY IF (iff):

$$\mu_A(x) = \mu_B(x), \;\; \forall x \in X$$

$A = 0.3/1 + 0.5/2 + 1/3$

$B = 0.3/1 + 0.5/2 + 1/3$

therefore $A = B$

Inclusion

- Inclusion of one fuzzy set into another fuzzy set. Fuzzy set $A \subseteq X$ is included in (is a subset of) another fuzzy set, $B \subseteq X$:

$$\mu_A(x) \leq \mu_B(x), \ \forall x \in X$$

Consider $X = \{1, 2, 3\}$ and sets A and B

$A = 0.3/1 + 0.5/2 + 1/3;$

$B = 0.5/1 + 0.55/2 + 1/3$

then A is a subset of B, or $A \subseteq B$

Cardinality

- Cardinality of a non-fuzzy set, Z, is the number of elements in Z. BUT the cardinality of a fuzzy set A, the so-called SIGMA COUNT, is expressed as a SUM of the values of the membership function of A, $\mu_A(x)$:

$$card_A = \mu_A(x_1) + \mu_A(x_2) + \ldots \mu_A(x_n) = \sum \mu_A(x_i), \quad \text{for } i=1..n$$

Consider $X = \{1, 2, 3\}$ and sets A and B

$A = 0.3/1 + 0.5/2 + 1/3$

$B = 0.5/1 + 0.55/2 + 1/3$

$card_A = 1.8$

$card_B = 2.05$

Empty Fuzzy Set

- A fuzzy set A is empty, IF AND ONLY IF:

$$\mu_A(x) = 0, \ \forall x \in X$$

Consider X = {1, 2, 3} and set

A = 0/1 + 0/2 + 0/3

then A is *empty*

Alpha-cut

- An α-cut or α-level set of a fuzzy set $A \subseteq X$ is an ORDINARY SET $A_\alpha \subseteq X$, suchthat:

  $$A_\alpha = \{\mu_A(x) \geq \alpha,\ \forall x \in X\}.$$

  Consider $X = \{1, 2, 3\}$ and set

  $A = 0.3/1 + 0.5/2 + 1/3$

  then $A_{0.5} = \{2, 3\}$,

  $A_{0.1} = \{1, 2, 3\}$,

  $A_1 = \{3\}$

Fuzzy Set Math Operations

- $aA = \{a\mu_A(x), \forall x \in X\}$

  Let a=0.5, and

  A= {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  aA= {0.25/a, 0.15/b, 0.1/c, 0.5/d}

- $A^a = \{\mu_A(x)^a, \forall x \in X\}$

  Let a=2, and

  A= {0.5/a, 0.3/b, 0.2/c, 1/d}

  then

  A^a= {0.25/a, 0.09/b, 0.04/c, 1/d}

- …

Fuzzy Sets Examples

- Consider two fuzzy subsets of the set X, $X = \{a, b, c, d, e\}$

  referred to as A and B

  $$A = \{1/a, 0.3/b, 0.2/c\ 0.8/d, 0/e\}$$

  and

  $$B = \{0.6/a, 0.9/b, 0.1/c, 0.3/d, 0.2/e\}$$

Fuzzy Sets Examples

- Support:

$$supp(A) = \{a, b, c, d\}$$
$$supp(B) = \{a, b, c, d, e\}$$

- Core:

$$core(A) = \{a\}$$
$$core(B) = \{\}$$

- Cardinality:

$$card(A) = 1+0.3+0.2+0.8+0 = 2.3$$
$$card(B) = 0.6+0.9+0.1+0.3+0.2 = 2.1$$

Fuzzy Sets Examples

- Complement:

  A= {1/a, 0.3/b, 0.2/c 0.8/d, 0/e}

  $\neg A$= {0/a, 0.7/b, 0.8/c 0.2/d, 1/e}

- Union:

  $A \cup B$ = {1/a, 0.9/b, 0.2/c, 0.8/d, 0.2/e}

- Intersection:

  $A \cap B$ = {0.6/a, 0.3/b, 0.1/c, 0.3/d, 0/e}

Fuzzy Sets Examples

- $\underline{aA}$:

  for a=0.5

  aA = {0.5/a, 0.15/b, 0.1/c, 0.4/d, 0/e}

- $\underline{A^a}$:

  for a=2

  A^a = {1/a, 0.09/b, 0.04/c, 0.64/d,0/e}

- $\underline{a\text{-cut}}$:

  $A_{0.2}$ = {a, b, c, d}

  $A_{0.3}$ = {a, b, d}

  $A_{0.8}$ = {a, d}

  A_1 = {a}

Homework

For

A ={0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}

B ={0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}

Draw the Fuzzy Graph of A and B

Then, calculate the following:

- Support, Core, Cardinality, and Complement for A and B independently
- Union and Intersection of A and B
- the new set C, if $C = A^2$
- the new set D, if $D = 0.5 \times B$
- the new set E, for an alpha cut at $A_{0.5}$

Solutions

A ={0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}
B ={0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}

Support

Supp(A) = {a, b, c, d}
Supp(B) = {b, c, d, e}

Core

Core(A) = {c}
Core(B) = {}

Cardinality

Card(A) = 0.2 + 0.4 + 1 + 0.8 + 0 = 2.4
Card(B) = 0 + 0.9 + 0.3 + 0.2 + 0.1 = 1.5

Complement

Comp(A) = {0.8/a, 0.6/b, 0/c, 0.2/d, 1/e}
Comp(B) = {1/a, 0.1/b, 0.7/c, 0.8/d, 0.9/e}

Solutions

A ={0.2/a, 0.4/b, 1/c, 0.8/d, 0/e}
B ={0/a, 0.9/b, 0.3/c, 0.2/d, 0.1/e}

Union

$A \cup B$ = {0.2/a, 0.9/b, 1/c, 0.8/d, 0.1/e}

Intersection

$A \cap B$ = {0/a, 0.4/b, 0.3/c, 0.2/d, 0/e}

$C=A^2$

C = {0.04/a, 0.16/b, 1/c, 0.64/d, 0/e}

$D = 0.5 \times B$

D = {0/a, 0.45/b, 0.15/c, 0.1/d, 0.05/e}

$E = A_{0.5}$

E = {c, d}

Alternative Notation

$$A = \{(x, \mu_A(x)) \mid x \in U\}$$

U: discrete universe → $A = \sum_{x_i \in U} \mu_A(x_i) / x_i$

U: continuous universe → $A = \int_U \mu_A(x) / x$

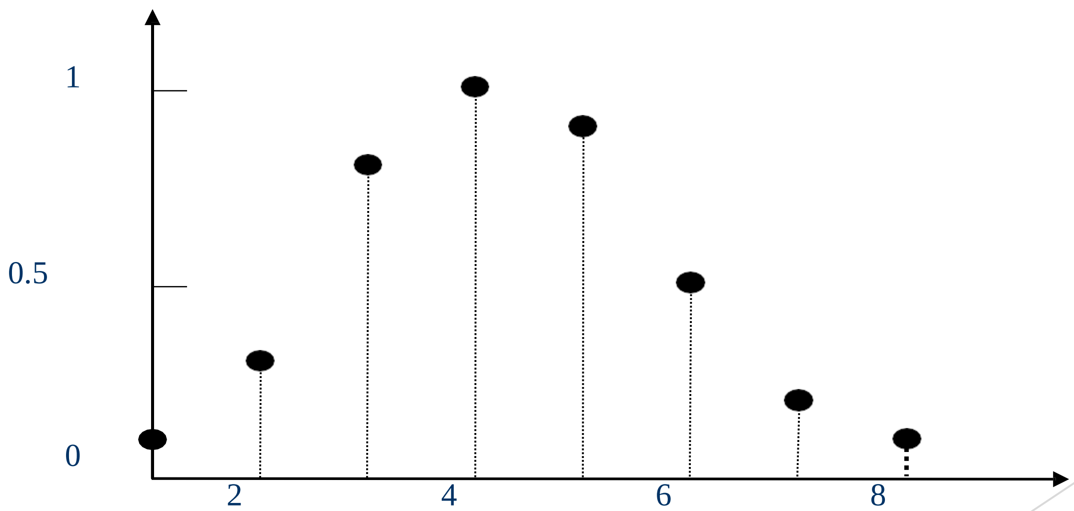
Note that $\sum$ and integral signs stand for the union of membershipgrades;
" / " stands for a marker and does not imply division.

Example (Discrete Universe)

$$X = \{1, 2, 3, 4, 5, 6, 7, 8\}$$ — # courses a student may take in a semester.

$$A = \left\{ \frac{0.1}{1} + \frac{0.3}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{0.9}{5} + \frac{0.5}{6} + \frac{0.2}{7} + \frac{0.1}{8} \right\}$$ — appropriate # courses taken

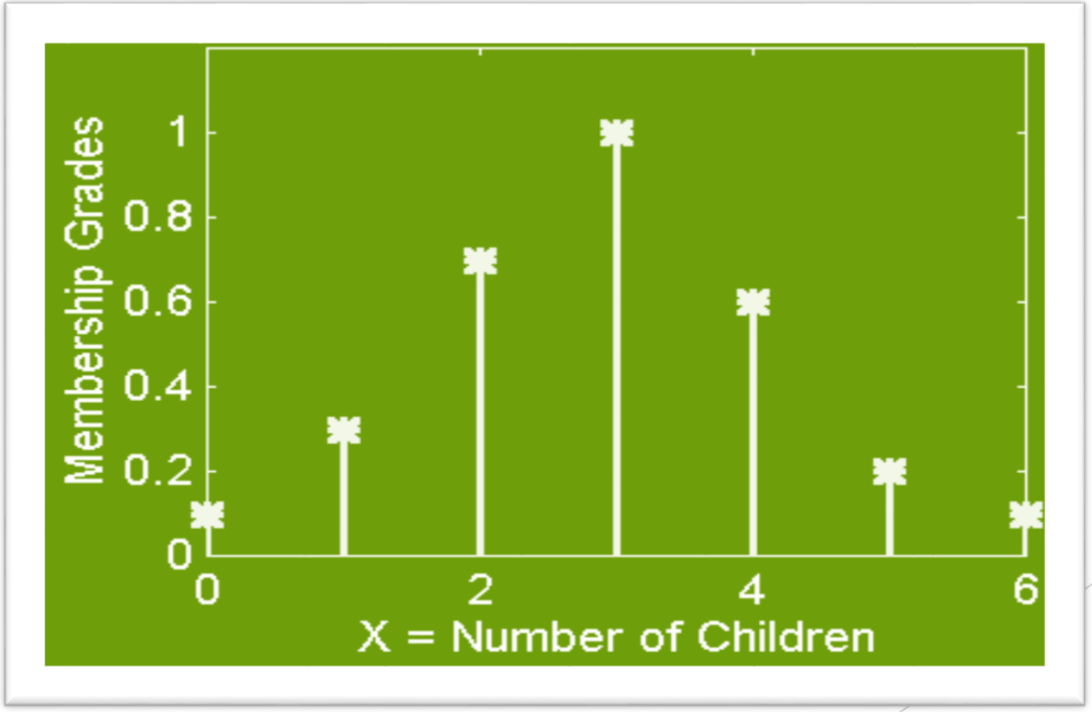
$\mu_A(x)$



x : # courses

Fuzzy Sets with Discrete Universes

- Fuzzy set C = "desirable city to live in"
  - X = {SF, Boston, LA} (discrete and nonordered)
  - C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}
- Fuzzy set A = "sensible number of children"
  - X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)
  - A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}



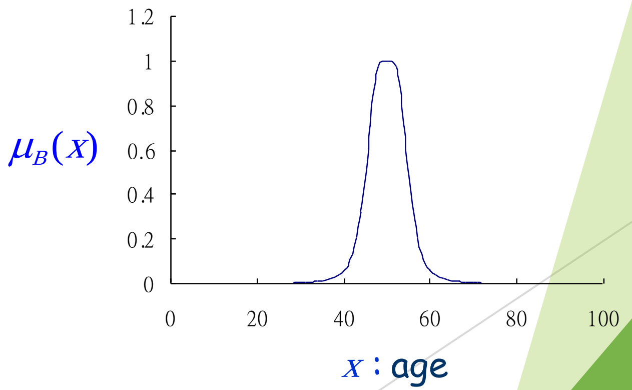
Example (Continuous Universe)

X: the set of positive real numbers —— possible ages

$$B = \{(x, \mu_B(x)) \mid x \in X\}$$

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{5}\right)^4}$$

about 50 years old



Alternative Representation:

$$B = \int_{R+} \frac{1}{1+\left(\frac{x-50}{5}\right)^4} \Big/ x$$

Fuzzy sets: basic concepts

- Consider three fuzzy sets that represent the concepts of a young, middle-aged, and old person. The membership functions are defined on the interval [0,80] as follows:

$$A_1(x) = \begin{cases} 1 & \text{when } x \leq 20 \\ (35 - x)/15 & \text{when } 20 < x < 35 \\ 0 & \text{when } x \geq 35 \end{cases}$$

young

$$A_2(x) = \begin{cases} 0 & \text{when either } x \leq 20 \text{ or } \geq 60 \\ (x - 20)/15 & \text{when } 20 < x < 35 \\ (60 - x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \leq x \leq 45 \end{cases}$$

middle-aged

$$A_3(x) = \begin{cases} 0 & \text{when } x \leq 45 \\ (x - 45)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } x \geq 60 \end{cases}$$

old

1.4 Fuzzy sets: basic concepts

$$A_2(x) = \begin{cases} 0 & \text{when either } x \leq 20 \text{ or } \geq 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60-x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \leq x \leq 45 \end{cases}$$

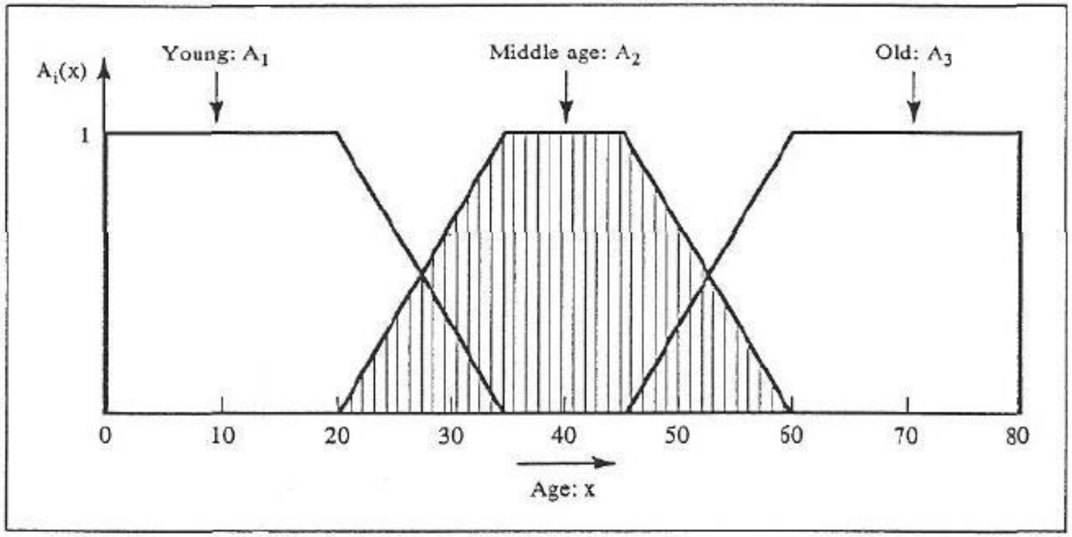


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION A_2 (FIG. 1.7) BY FUNCTION D_2 OF THE FORM: $D_2 : \{0, 2, 4, \ldots, 80\} \rightarrow [0, 1]$

x	$D_2(x)$
$x \notin \{22, 24, \ldots, 58\}$	0.00
$x \in \{22, 58\}$	0.13
$x \in \{24, 56\}$	0.27
$x \in \{26, 54\}$	0.40
$x \in \{28, 52\}$	0.53
$x \in \{30, 50\}$	0.67
$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \ldots, 44\}$	1.00

Fuzzy sets: basic concepts

- For example:

$$A_2(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or } \ge 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60-x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}$$

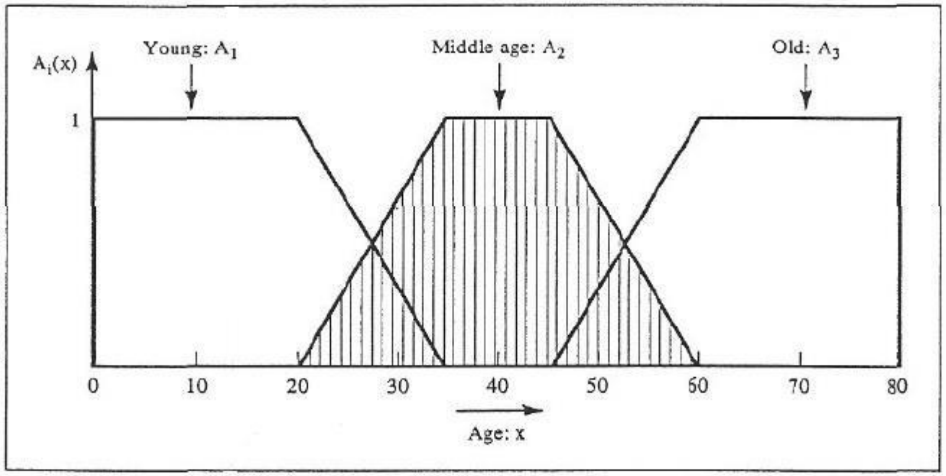


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

$$\begin{aligned}
{}^0A_1 &= {}^0A_2 = {}^0A_3 = [0, 80] = X; \\
{}^\alpha A_1 &= [0, 35 - 15\alpha], {}^\alpha A_2 = [15\alpha + 20, 60 - 15\alpha], {}^\alpha A_3 = [15\alpha + 45, 80] \text{ for all } \\
&\quad \alpha \in [0, 1]; \\
{}^{\alpha+}A_1 &= (0, 35 - 15\alpha), {}^{\alpha+}A_2 = (15\alpha + 20, 60 - 15\alpha), {}^{\alpha+}A_3 = (15\alpha + 45, 80) \text{ for all } \\
&\quad \alpha \in [0, 1]; \\
{}^{1+}A_1 &= {}^{1+}A_2 = {}^{1+}A_3 = \varnothing.
\end{aligned}$$

Fuzzy sets: basic concepts

- A level set of A:
  - The set of all levels α∈[0,1] that represent distinct α-cuts of a given fuzzy set A.

$$\Lambda(A) = \{\alpha \mid A(x) = \alpha \text{ for some } x \in X\},$$

  - For example:

$$\Lambda(A_1) = \Lambda(A_2) = \Lambda(A_3) = [0, 1], \text{ and}$$
$$\Lambda(D_2) = \{0, 0.13, 0.27, 0.4, 0.53, 0.67, 0.8, 0.93, 1\}.$$

$$A_2(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or } \ge 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60-x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}$$

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$x \in \{36, 38, \ldots, 44\}$	1.00

1.4 Fuzzy sets: basic concepts

- The properties of α-cut and strong α-cut
  - For any fuzzy set A and pair α_1 , $\alpha_2 \in [0,1]$ of distinct values such that $\alpha_1<\alpha_2$, we have

$$ {}^{\alpha_1}A \supseteq {}^{\alpha_2}A \text{ and } {}^{\alpha_1+}A \supseteq {}^{\alpha_2+}A $$

$$ {}^{\alpha_1}A \cap {}^{\alpha_2}A = {}^{\alpha_2}A, \quad {}^{\alpha_1}A \cup {}^{\alpha_2}A = {}^{\alpha_1}A $$

$$ {}^{\alpha_1+}A \cap {}^{\alpha_2+}A = {}^{\alpha_2+}A, \quad {}^{\alpha_1+}A \cup {}^{\alpha_2+}A = {}^{\alpha_1+}A $$

  - All α-cuts and all strong α-cuts of any fuzzy set form two distinct families of nested crisp sets.

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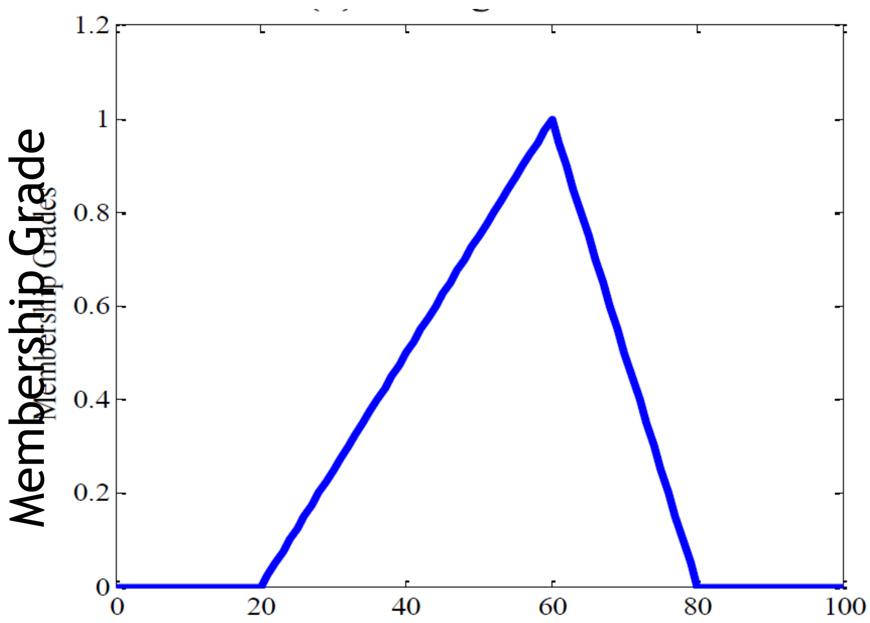
Membership Functions

- Triangular membership function

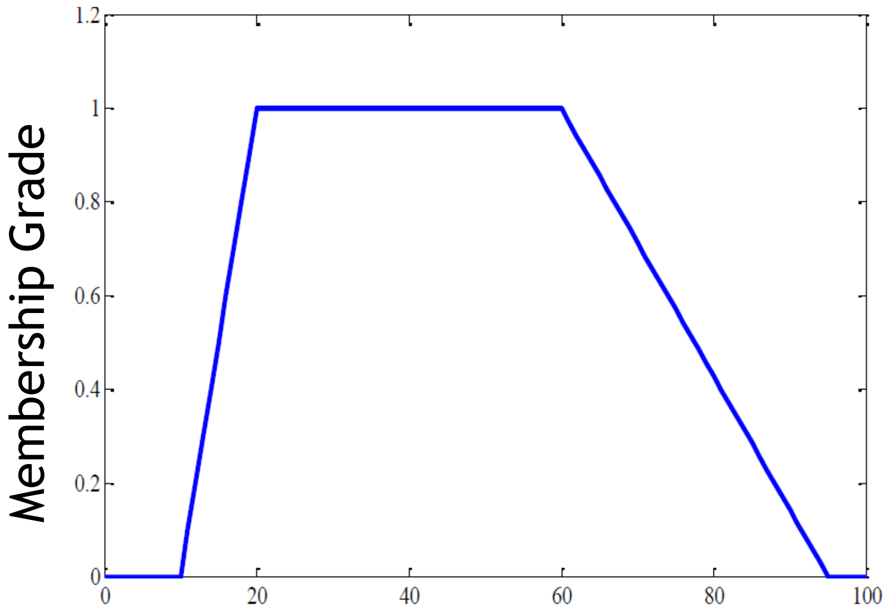
$$Triangular(x, a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$



Membership Functions

- Trapezoidal membership function

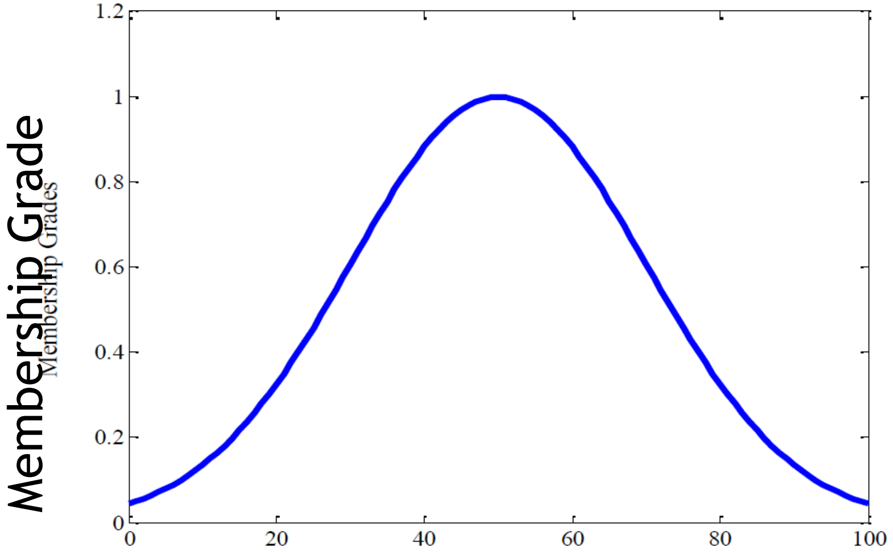
$$Trapezoidal(x, a, b, c, d) = \max(\min\left(\frac{x-a}{b-a}, \frac{d-x}{d-c}\right), 0)$$



Membership Functions

- Gaussian membership function

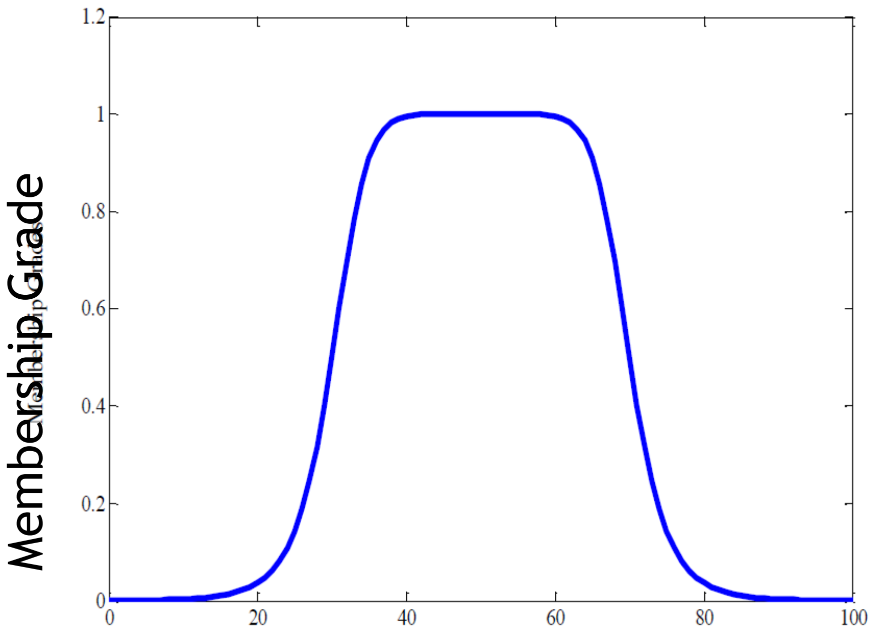
$$Gaussian(x, c, \sigma) = e^{\frac{-1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



Membership Functions

- Generalized bell membership function

$$gbell(x, a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



Fuzzy Rules

- In 1973, Lotfi Zadeh published his second most influential paper. This paper outlined a new approach to analysis of complex systems, in which Zadeh suggested capturing human knowledge in **fuzzy rules**.

- A fuzzy rule can be defined as a conditional statement in the form:

IF	x	is A
THEN	y	is B

- where x and y are linguistic variables; and A and B are linguistic values determined by fuzzy sets on the universe of discourses X and Y, respectively.

Classical Vs Fuzzy Rules

- A classical IF-THEN rule uses binary logic, for example,

<u>Rule: 1</u>
IF speed is > 100
THEN stopping_distance is long

<u>Rule: 2</u>
IF speed is < 40
THEN stopping_distance is short

- The variable speed can have any numerical value between 0 and 220 km/h, but the linguistic variable stopping_distance can take either value long or short. In other words, classical rules are expressed in the black-and-white language of Boolean logic.

Classical Vs Fuzzy Rules

- We can also represent the stopping distance rules in a fuzzy form:

<u>Rule: 1</u>
IF speed is fast
THEN stopping_distance is long

<u>Rule: 2</u>
IF speed is slow
THEN stopping_distance is short

- In fuzzy rules, the linguistic variable speed also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as slow, medium and fast. The universe of discourse of the linguistic variable stopping_distance can be between 0 and 300 m and may include such fuzzy sets as short, medium and long.

Classical Vs Fuzzy Rules

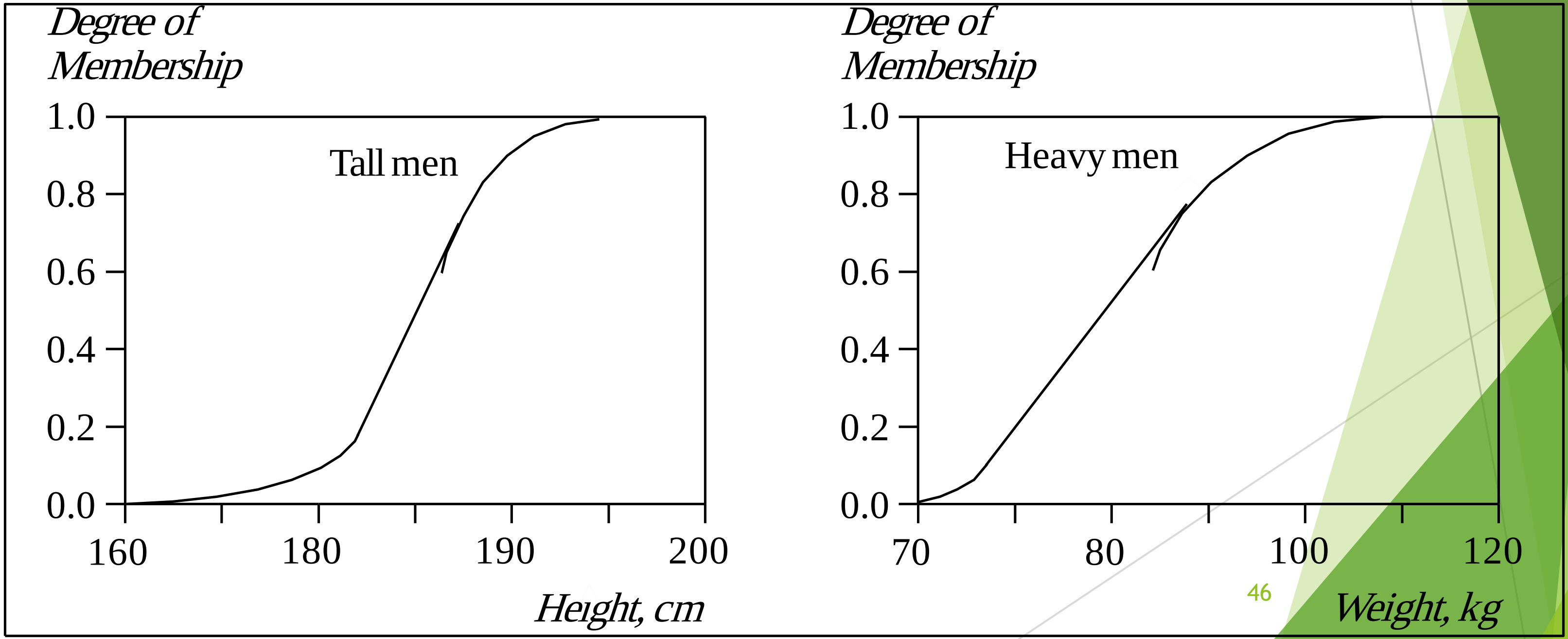
- Fuzzy rules relate fuzzy sets.
- In a fuzzy system, all rules fire to some extent, or in other words they fire partially. If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

Firing Fuzzy Rules

- These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:

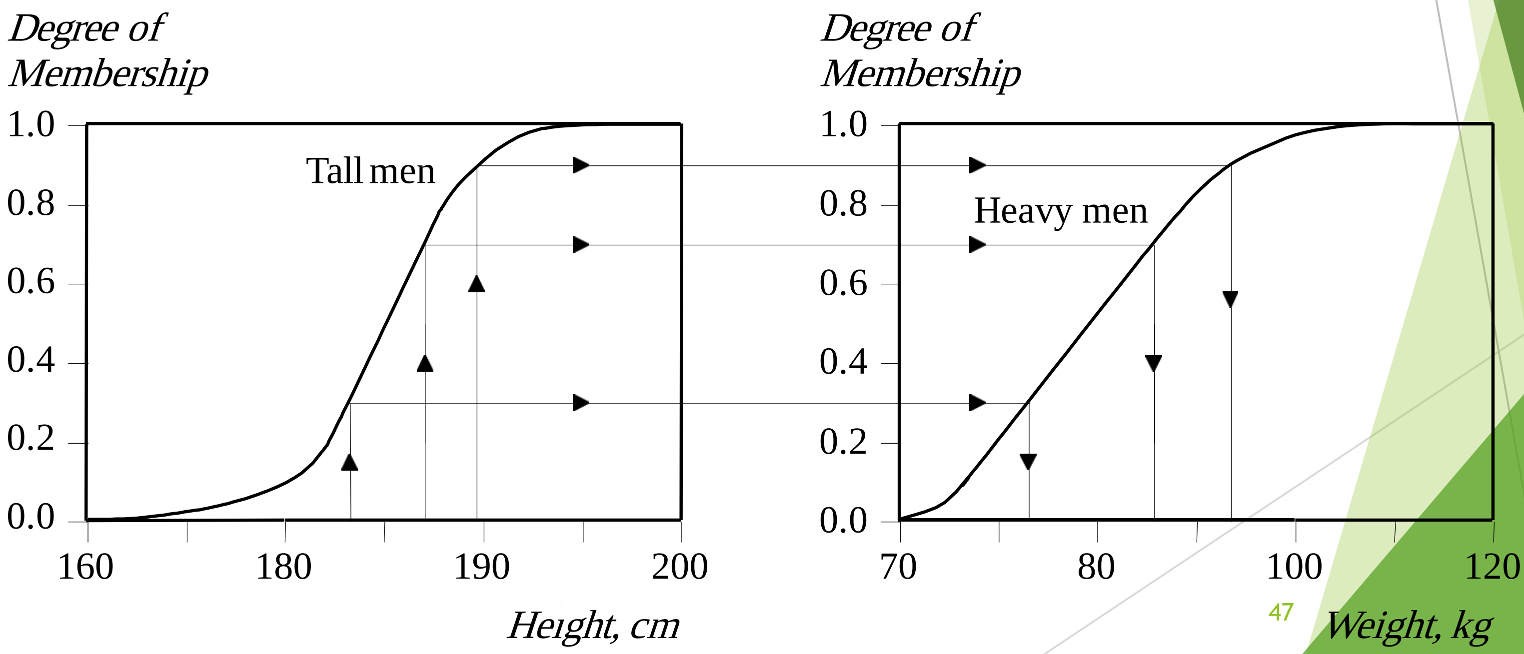
  IF height is tall

  THEN weight is heavy



Firing Fuzzy Rules

- The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called **monotonic selection**.



Firing Fuzzy Rules

- A fuzzy rule can have multiple antecedents, for example:

IF	project_duration is long
AND	project_staffing is large
AND	project_funding is inadequate
THEN	risk is high

IF	service is excellent
OR	food is delicious
THEN	tip is generous

- The consequent of a fuzzy rule can also include multiple parts, for instance:

IF	temperature is hot
THEN	hot_water is reduced;
	cold_water is increased

Fuzzy Sets Example

- Air-conditioning involves the delivery of air which can be warmed or cooled and have its humidity raised or lowered.

- An air-conditioner is an apparatus for controlling, especially lowering, the temperature and humidity of an enclosed space. An air-conditioner typically has a fan which blows/cools/circulates fresh air and has cooler and the cooler is under thermostatic control. Generally, the amount of air being compressed is proportional to the ambient temperature.

- Consider Johnny's air-conditioner which has five control switches: COLD, COOL, PLEASANT, WARM and HOT. The corresponding speeds of the motor controlling the fan on the air-conditioner has the graduations: MINIMAL, SLOW, MEDIUM, FAST and BLAST.

Fuzzy Sets Example

- The rules governing the air-conditioner are as follows:

RULE 1:

IF TEMP is COLD THEN SPEED is MINIMAL

RULE 2:

IF TEMP is COOL THEN SPEED is SLOW

RULE 3:

IF TEMP is PLEASANT THEN SPEED is MEDIUM

RULE 4:

IF TEMP is WARM THEN SPEED is FAST

RULE 5:

IF TEMP is HOT THEN SPEED is BLAST

Fuzzy Sets Example

The **temperature** graduations are related to Johnny's perception of ambient temperatures.

where:

Y : *temp* value belongs to the set $(0<\mu_A(x)<1)$

Y* : *temp* value is the ideal member to the set $(\mu_A(x)=1)$

N : temp value is not a member of the set $(\mu_A(x)=0)$

Temp (^{0}C).	COLD	COOL	PLEASANT	WARM	HOT
0	Y*	N	N	N	N
5	Y	Y	N	N	N
10	N	Y	N	N	N
12.5	N	Y*	N	N	N
15	N	Y	N	N	N
17.5	N	N	Y*	N	N
20	N	N	N	Y	N
22.5	N	N	N	Y*	N
25	N	N	N	Y	N
27.5	N	N	N	N	Y
30	N	N	N	N	Y*

Fuzzy Sets Example

Johnny's perception of the **speed** of the motor is as follows:

where:

Y : *temp* value belongs to the set ($0<\mu_A(x)<1$)

Y* : *temp* value is the ideal member to the set ($\mu_A(x)=1$)

N : temp value is not a member of the set ($\mu_A(x)=0$)

Rev/sec (RPM)	MINIMAL	SLOW	MEDIUM	FAST	BLAST
0	Y*	N	N	N	N
10	Y	N	N	N	N
20	Y	Y	N	N	N
30	N	Y*	N	N	N
40	N	Y	N	N	N
50	N	N	Y*	N	N
60	N	N	N	Y	N
70	N	N	N	Y*	N
80	N	N	N	Y	Y
90	N	N	N	N	Y
100	N	N	N	N	Y*

Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the **temperature** are:

- COLD:

  for $0 \le t \le 10$ $\mu_{COLD}(t) = -t / 10 + 1$

- SLOW:

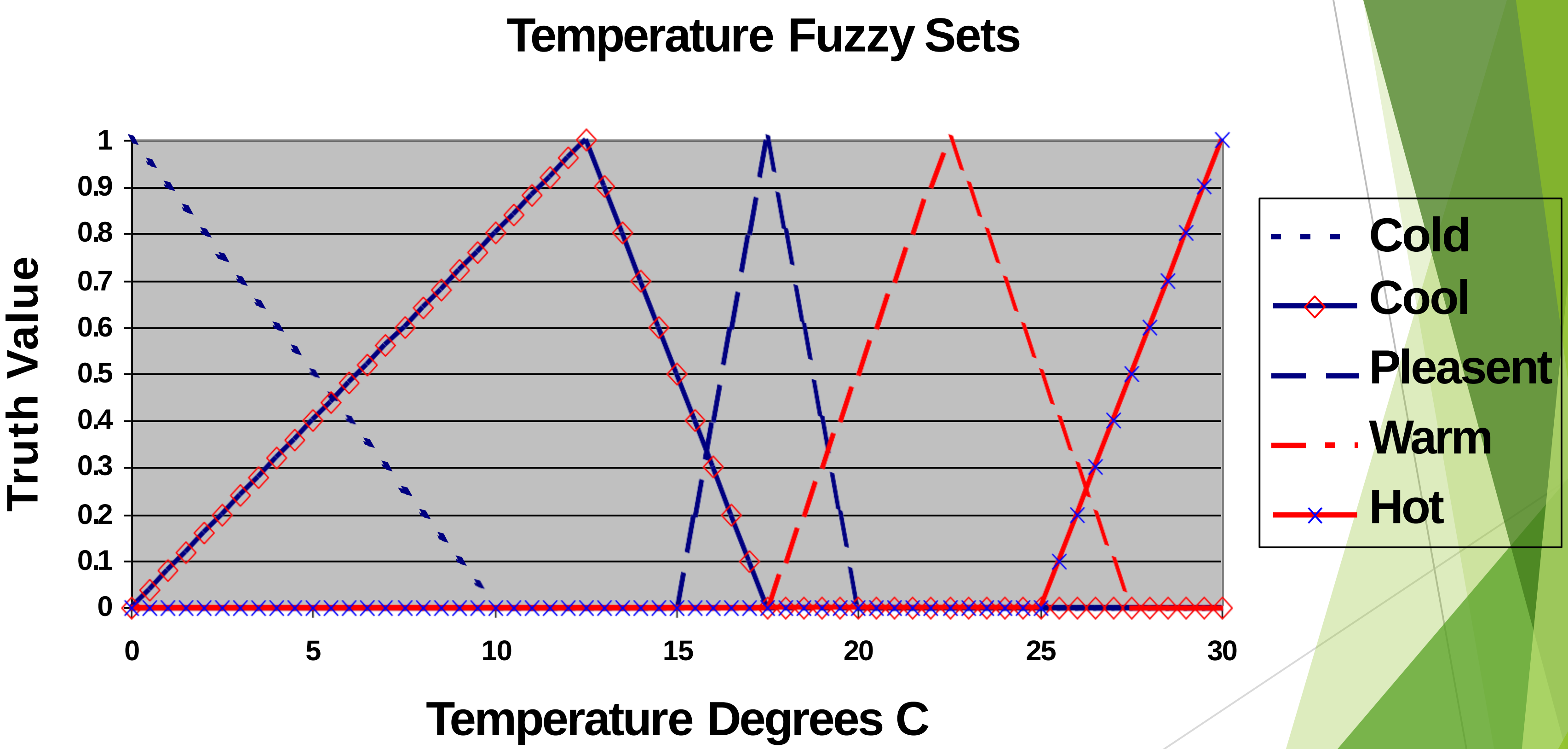
  for $0 \le t \le 12.5$ $\mu_{SLOW}(t) = t / 12.5$

  for $12.5 \le t \le 17.5$ $\mu_{SLOW}(t) = -t / 5 + 3.5$

- etc… all based on the linear equation:

  $y = ax + b$

Fuzzy Sets Example



Fuzzy Sets Example

- The analytically expressed membership for the reference fuzzy subsets for the **Speed** are:

- MINIMAL:

  for $0 \leq v \leq 30$ $\mu_{COLD}(t) = -v/30 + 1$

- SLOW:

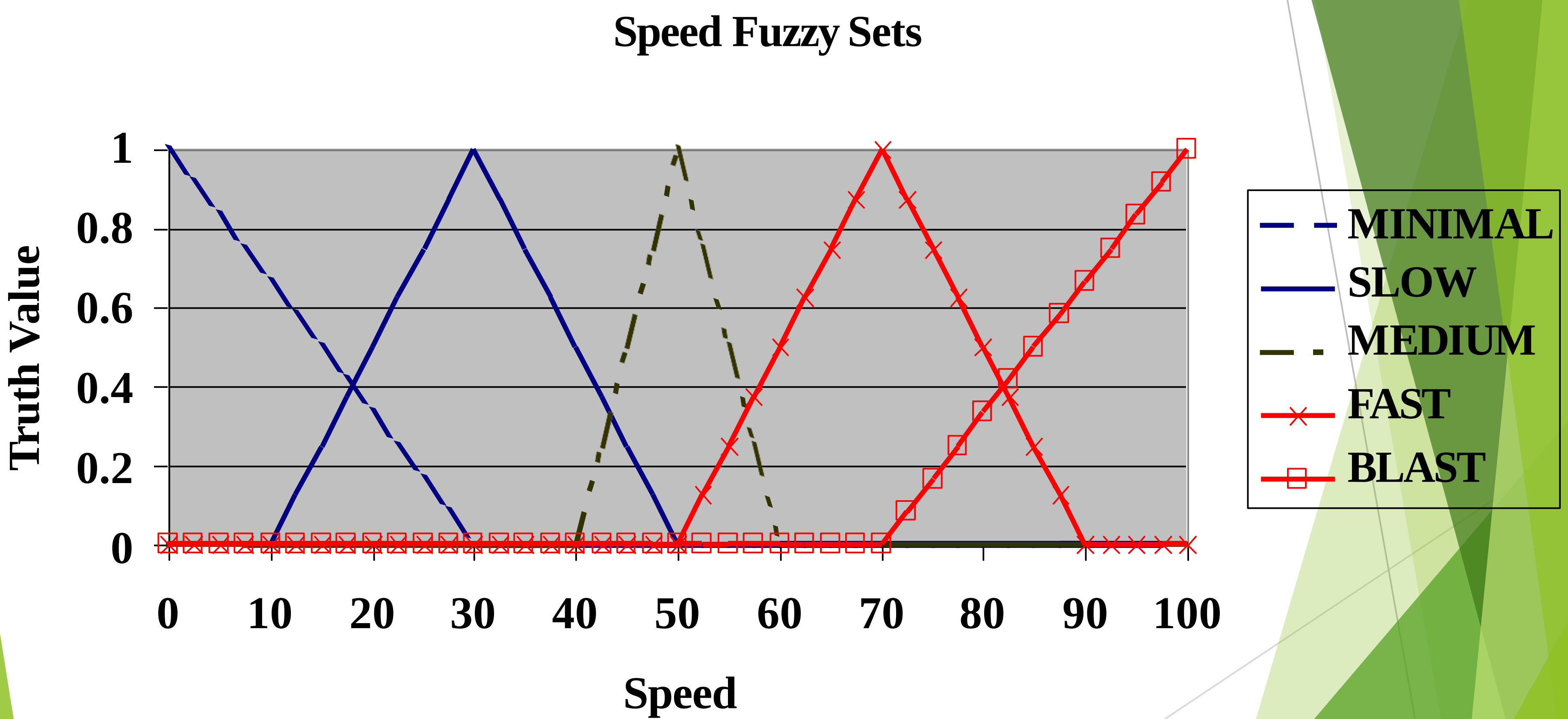
  for $10 \leq v \leq 30$ $\mu_{SLOW}(t) = v/20 - 0.5$

  for $30 \leq v \leq 50$ $\mu_{SLOW}(t) = -v/20 + 2.5$

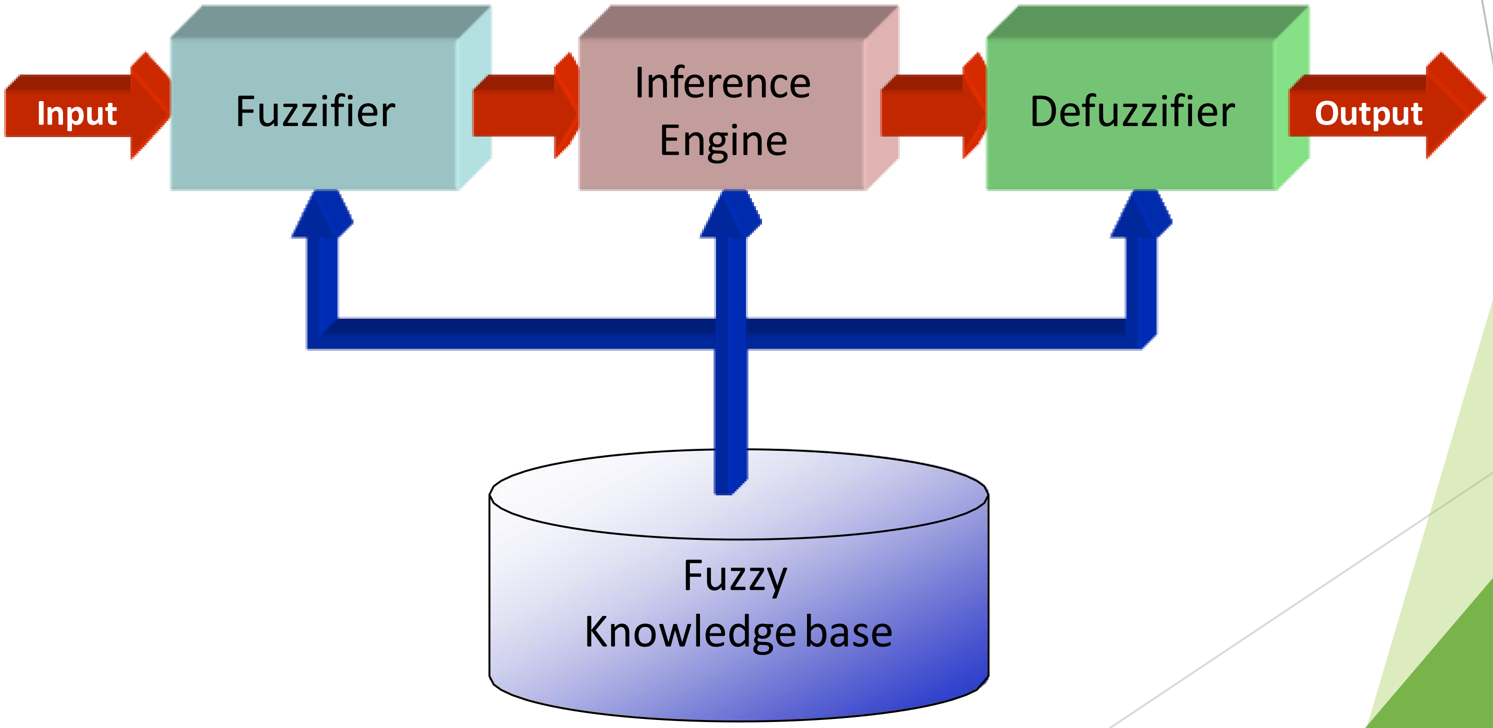
- etc… all based on the linear equation:

  $y = ax + b$

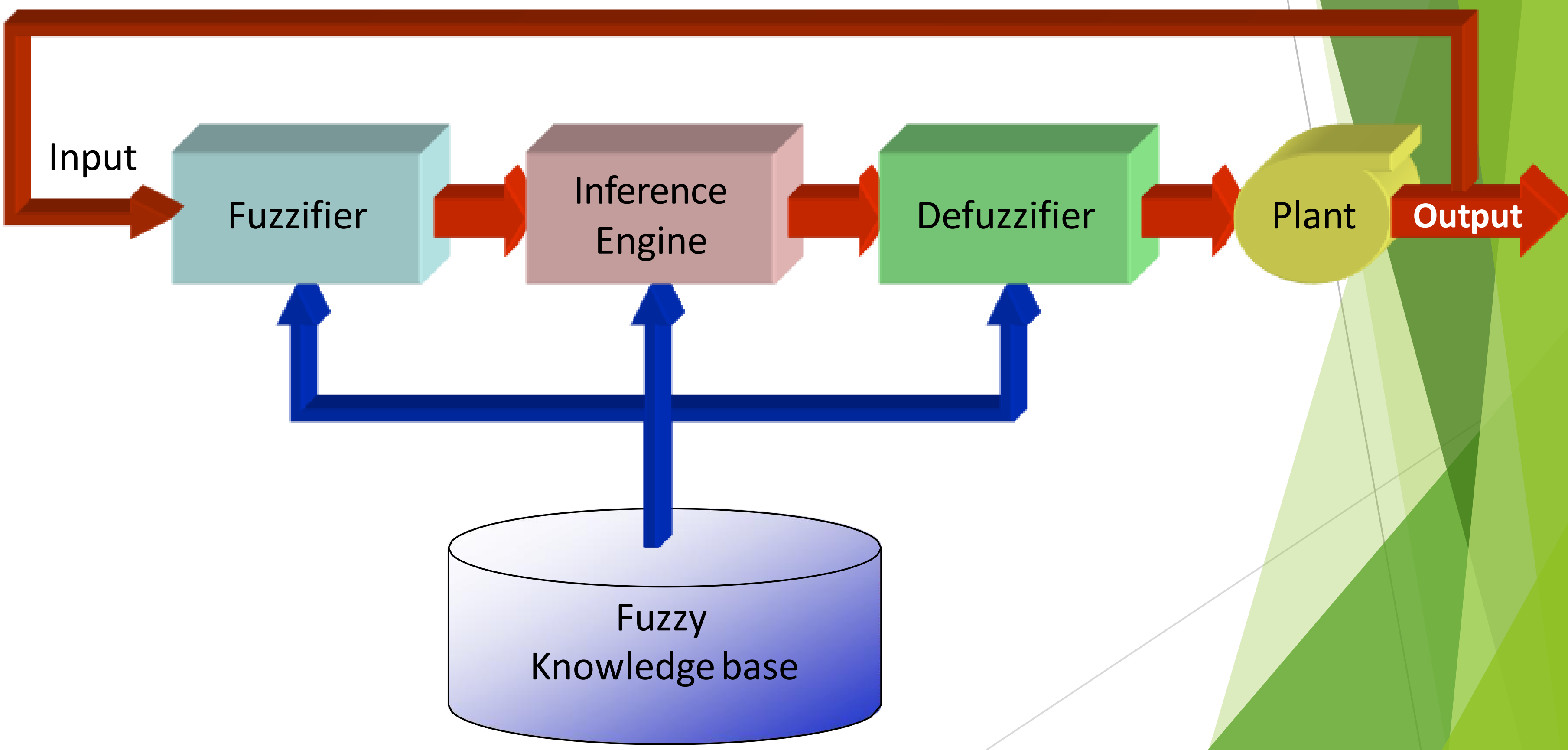
Fuzzy Sets Example



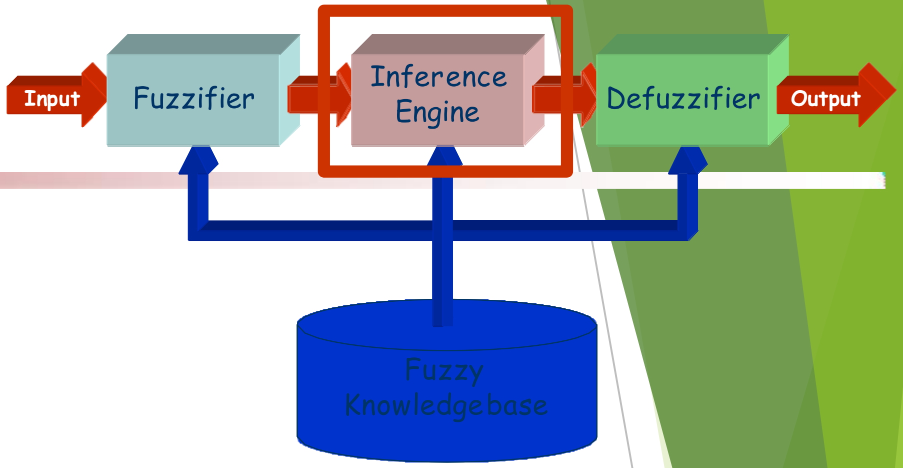
Fuzzy Systems



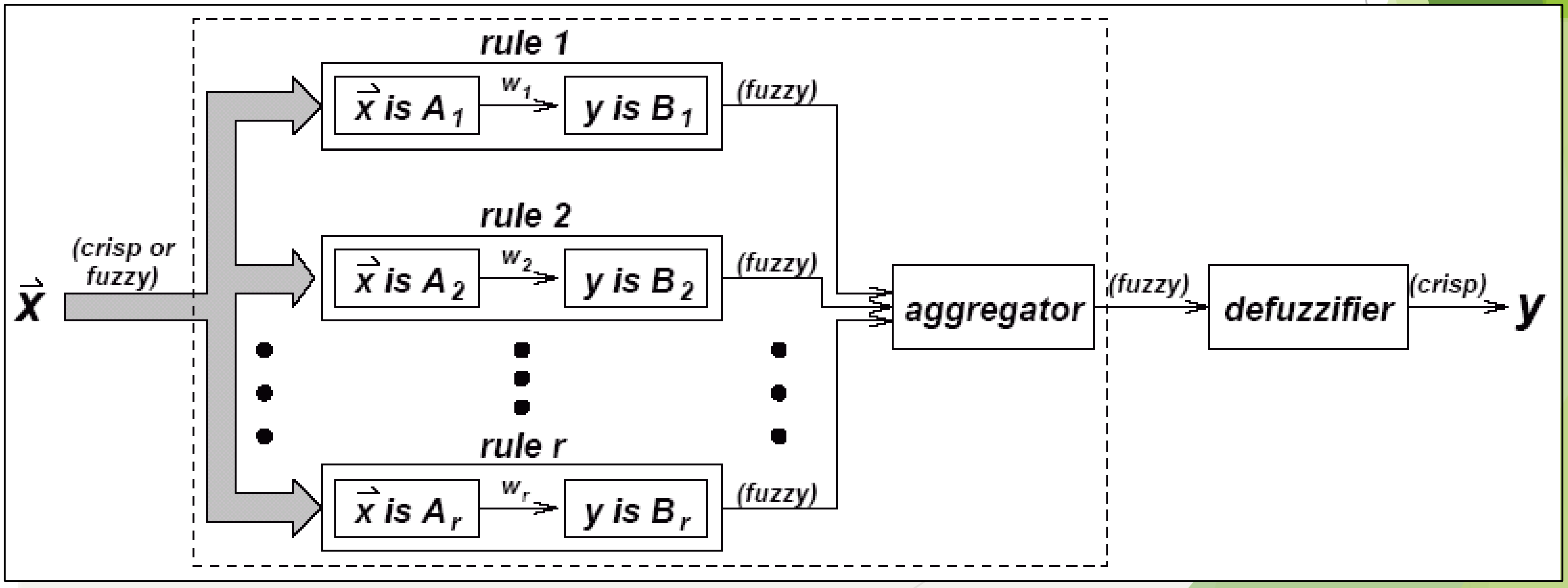
Fuzzy Control Systems



Inference Engine



Using If-Then type fuzzy rules converts the fuzzy input to the fuzzy output.



Outline

- Architecture of Fuzzy Inference
- Mamdani fuzzy inference
- Sugeno fuzzy inference
- Case study
- Fuzzy tools

Mamdani fuzzy inference

- The Mamdani-style fuzzy inference process is performed in four steps:

1. Fuzzification of the input variables,
2. Rule evaluation;
3. Aggregation of the rule outputs
4. Defuzzification.

Examine a simple two-input one-output problem that includes three rules:

Rule: 1
IF x is $A3$
OR y is $B1$
THEN z is $C1$

Rule: 1
IF *project_funding* is *adequate*
OR *project_staffing* is *small*
THEN *risk is low*

Rule: 2
IF x is $A2$
AND y is $B2$
THEN z is $C2$

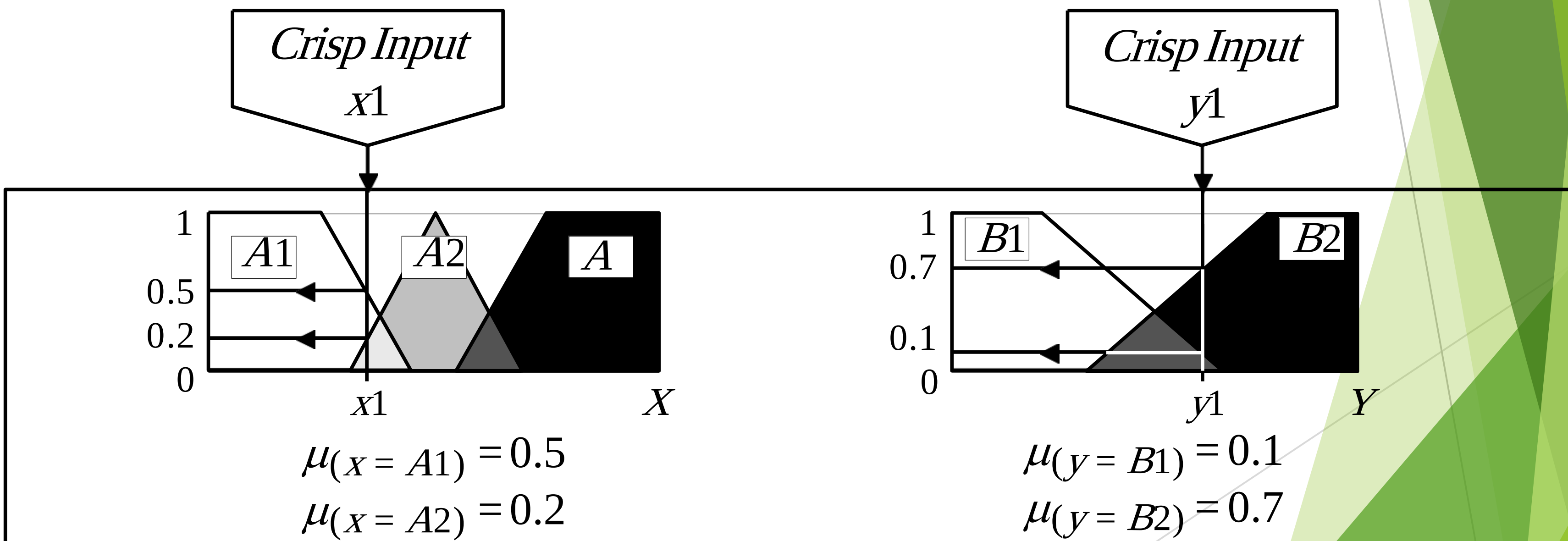
Rule: 2
IF *project_funding is marginal*
AND *project_staffing is large*
THEN *risk is normal*

Rule: 3
IF x is $A1$
THEN z is $C3$

Rule: 3
IF *project_funding is inadequate*
THEN *risk is high*

Step 1: Fuzzification

- The first step is to take the crisp inputs, x1 and y1 (project funding and project staffing), and determine the degree to which these inputs belong to each of the appropriate fuzzy sets.



Step 2: Rule Evaluation

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5$, $\mu_{(x=A2)} = 0.2$, $\mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules. If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation. This number (the truth value) is then applied to the consequent membership function.

Rule Evaluation

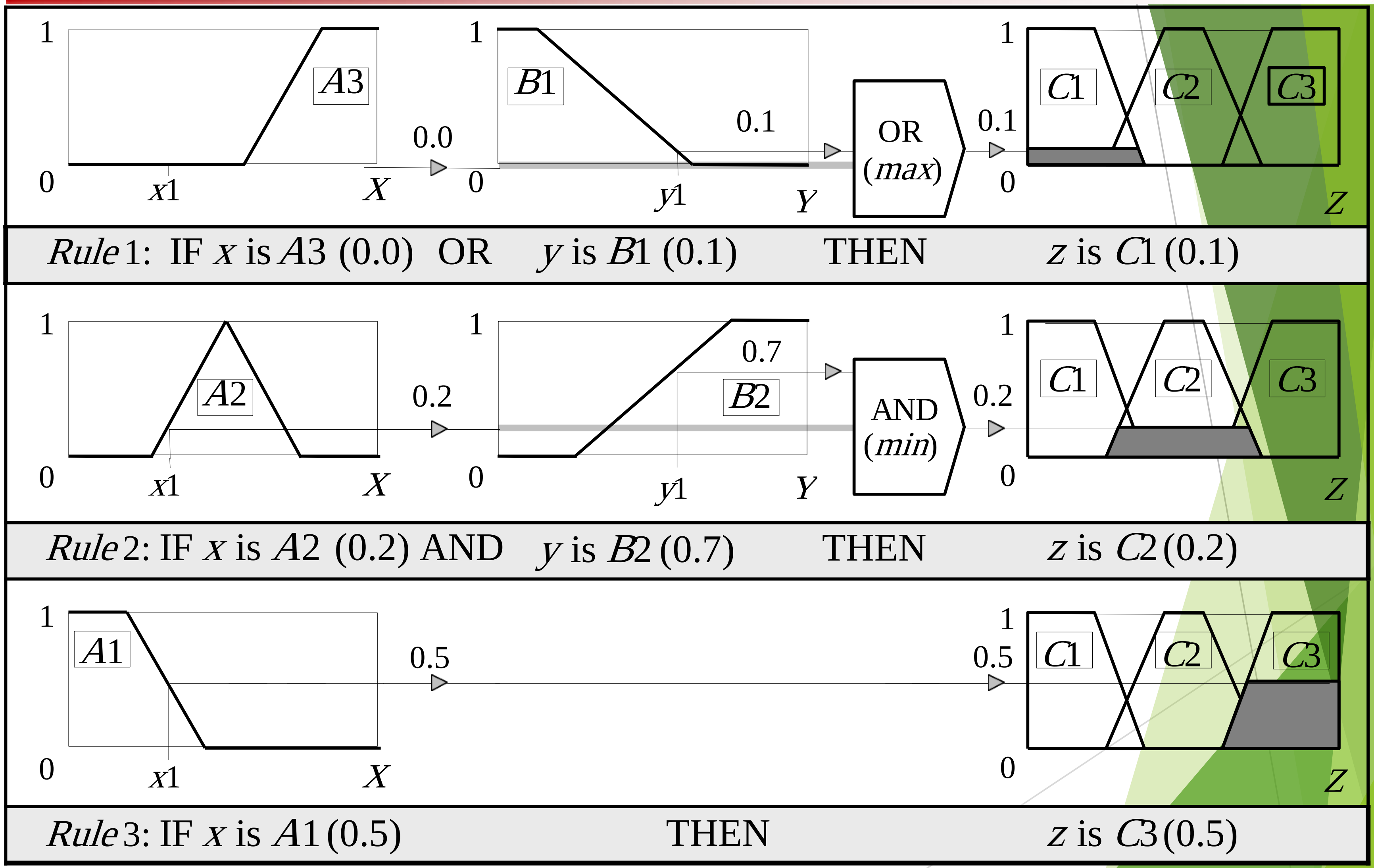
To evaluate the disjunction of the rule antecedents, we use the **OR fuzzy operation**. Typically, fuzzy expert systems make use of the classical fuzzy operation **union**:

$$\mu_{A \cup B}(x) = max\,[\mu_A(x),\, \mu_B(x)]$$

Similarly, in order to evaluate the conjunction of the rule antecedents, we apply the **AND fuzzy operation intersection**:

$$\mu_{A \cap B}(x) = min\,[\mu_A(x),\, \mu_B(x)]$$

Mamdani-style rule evaluation

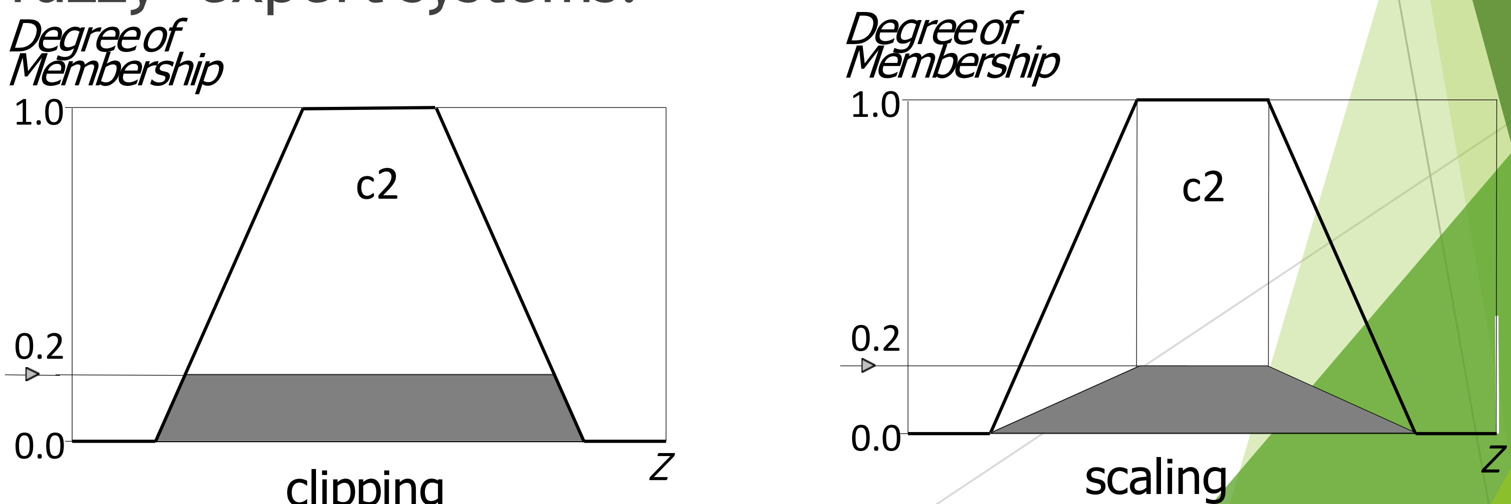


Result of the antecedent evaluation can be applied to the membership function of the consequent: 1.Clipping

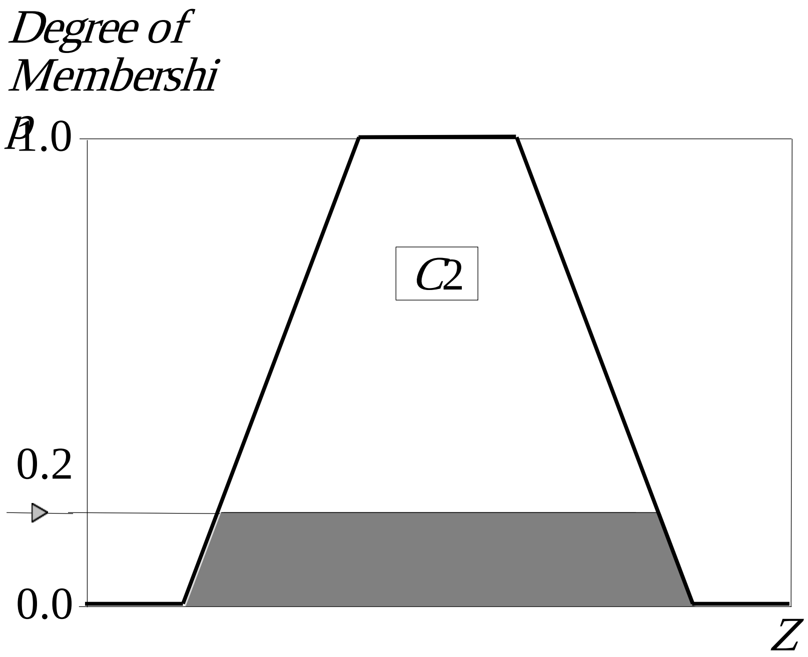
- The most common method of correlating the rule consequent with the truth value of the rule antecedent is to cut the consequent membership function at the level of the antecedent truth. This method is called clipping.
- Since the top of the membership function is sliced, the clipped fuzzy set loses some information. However, clipping is still often preferred because it involves less complex and faster mathematics, and generates an aggregated output surface that is easier to defuzzify.

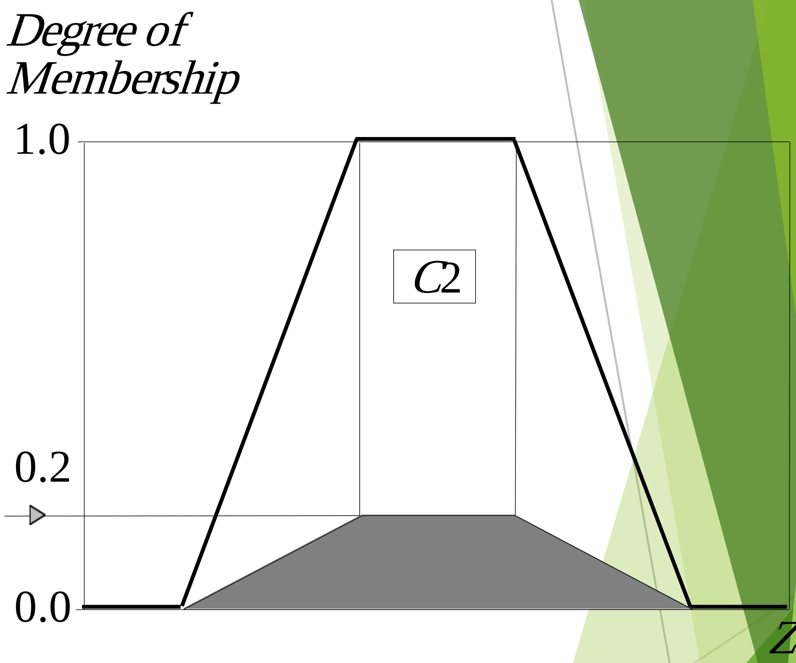
2.Scaling

- While clipping is a frequently used method, scaling offers a better approach for preserving the original shape of the fuzzy set.
- The original membership function of the rule consequent is adjusted by multiplying all its membership degrees by the truth value of the rule antecedent. This method, which generally loses less information, can be very useful in fuzzy expert systems.



Clipped and scaled membership functions



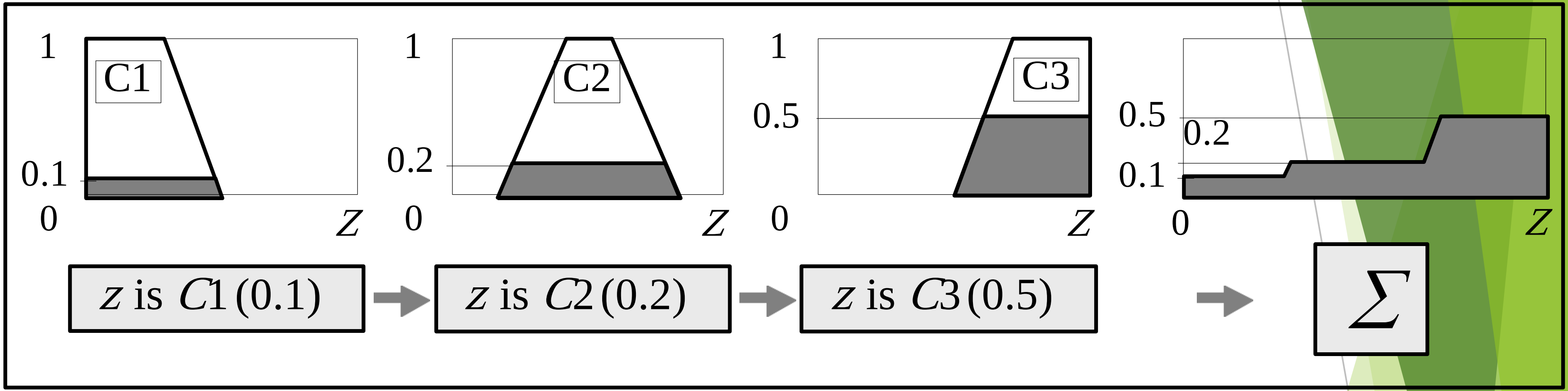


Step 3: Aggregation of the rule outputs

- Aggregation is the process of unification of the outputs of all rules. We take the membership functions of all rule consequents previously clipped or scaled and combine them into a single fuzzy set.

- The input of the aggregation process is the list of clipped or scaled consequent membership functions, and the output is one fuzzy set for each output variable.

Aggregation of the rule outputs



Step 4: Defuzzification

- The last step in the fuzzy inference process is defuzzification. Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.
- The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.

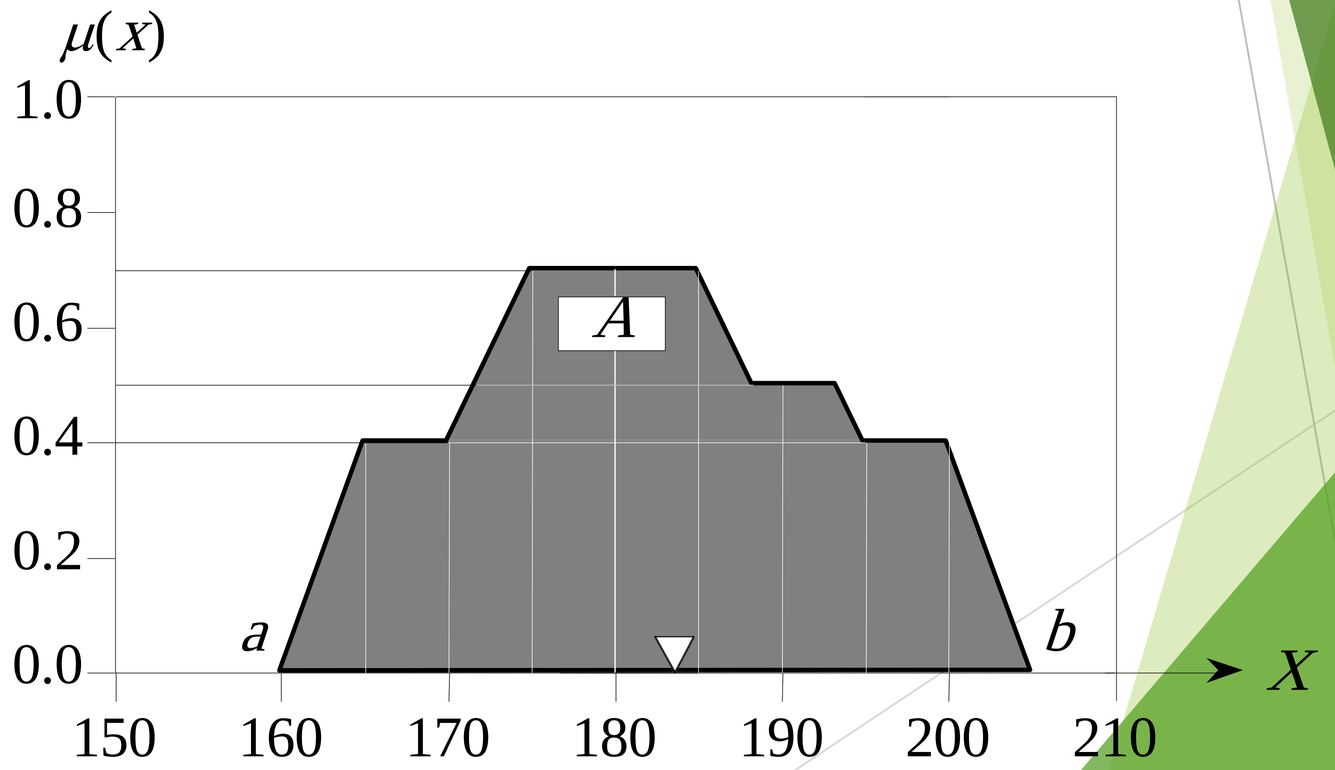
- There are several defuzzification methods, but probably the most popular one is the **centroid technique**. It finds the point where a vertical line would slice the aggregate set into two equal masses. Mathematically this **centre of gravity (COG)** can be expressed as:

$$COG = \frac{\int_a^b \mu_A(x)\, x\, dx}{\int_a^b \mu_A(x)\, dx}$$

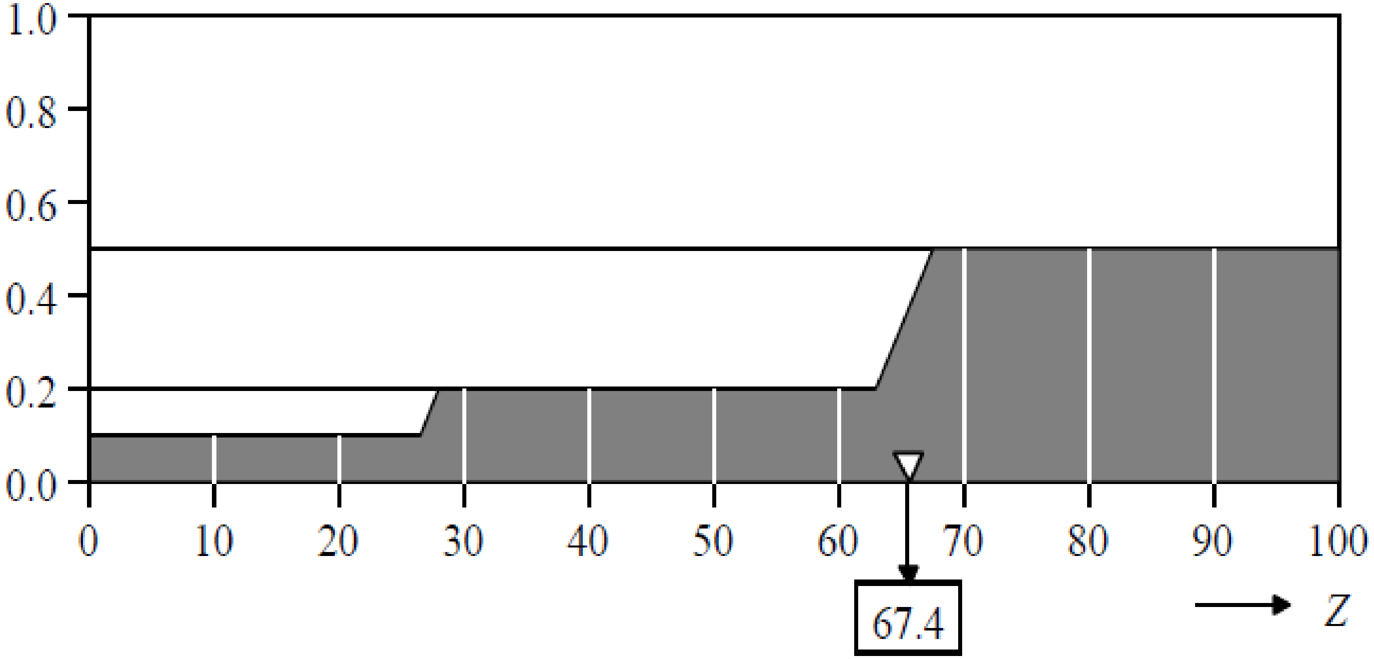
- Centroid defuzzification finds a point representing the centre of gravity of the aggregated fuzzy set *A*, on the interval [*a, b*].
- A reasonable estimate can be obtained by calculating it over a sample of points.

$$COG = \frac{\sum_{a}^{b} \mu_A(x)\, x}{\sum_{a}^{b} \mu_A(x)}$$

- Centroid defuzzification method finds a point representing the centre of gravity of the fuzzy set, A, on the interval, ab.
- A reasonable estimate can be obtained by calculating it over a sample of points.

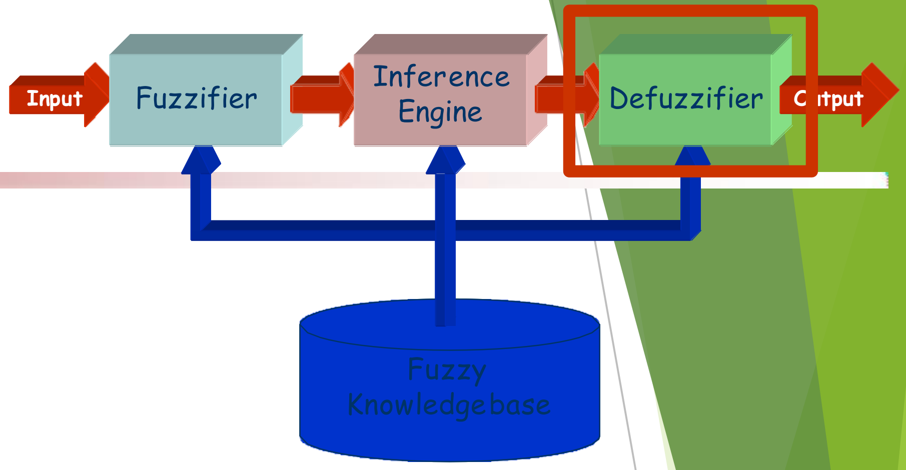


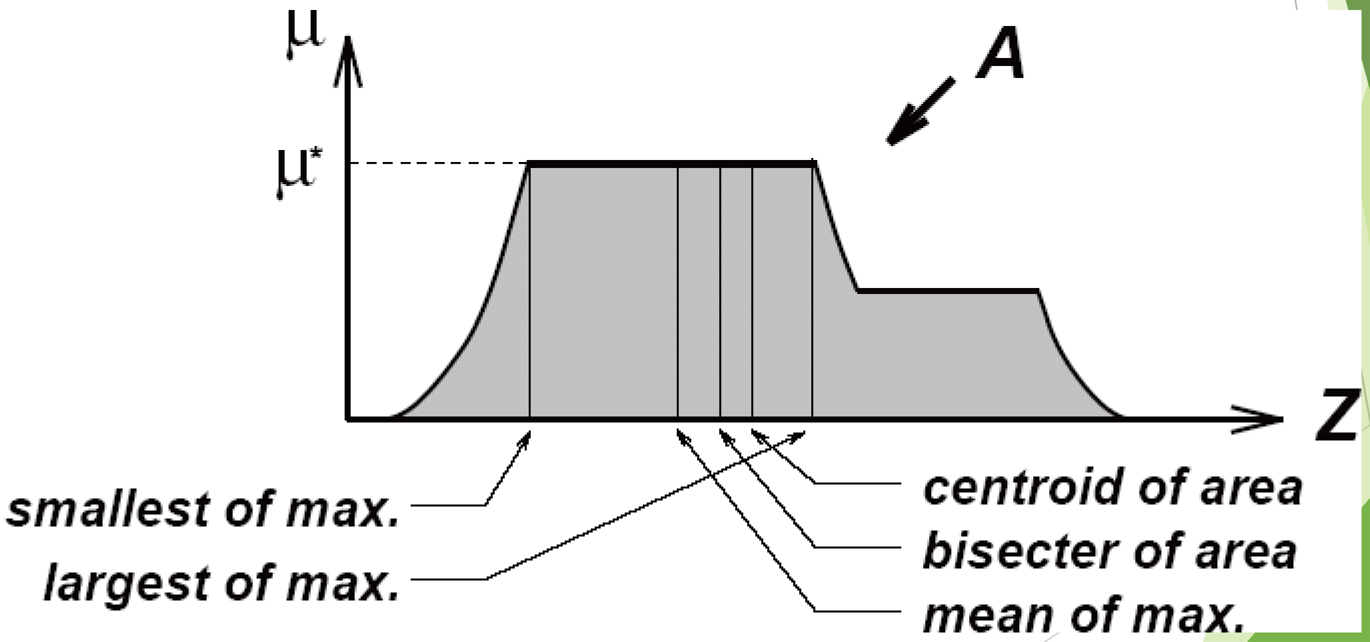
Centre of gravity (COG)



$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$

Defuzzifier





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