CALCULUS

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Course contents

A review of limits, continuity, and differentiation. Geometrical interpretation of derivative, indeterminate forms, L'Hopital's rule, asymptotes. Applications of single variable calculus: Critical points, derivative tests, optimization, concavity, curvature, Taylor's series, Newton's method. Mean value theorem.

Further techniques of integration, integration by reduction formulae. Importance of the fundamental theorem of calculus, Definite integral and its properties. Area enclosed between curves, arc lengths. volume of a solid of revolution.

Infinite sequences and series.

Pre-requisites: The course is intended for those students who have studied Mathematics to at least Year-12 level. Students taking this course are expected to have a working knowledge of the basic elements of **single-variable calculus**.

Recommended books

The following are considered as good reference books:

- George B. Thomas (latest edition), "Calculus".
- Stewart (latest edition), "Single Variable Calculus".
- Howard Anton (latest edition), "Calculus".
- Swokowski (latest edition), "Calculus".
- N. A. Shah, "Calculus and analytic geometry".
- S. M. Yusuf, "Calculus and analytic geometry".

Acknowledgement. Much of the material below is from different resources, in-particular, from the above recommended books.

Learning outcomes

The primary objective of this course is to provide students a modern introduction to **single variable calculus** that supports conceptual understanding with skills students will need to apply in the real world problems. It is vital to realize that learning how to apply mathematics is different from learning mathematics. There is less emphasis on building-up theory, and more emphasis on building-up the skills needed to formulate and solve mathematical problems.

By the end of this course, the key things students will learn are:

- Understand the key concepts of single variable calculus;
- Apply derivatives to find tangents and normals, carry out optimization, graphical analysis, concavity of a function, curvature of a function, and asymptotes of functions;
- Compute integrals by substitutions, tabular form, reduction formulae, improper forms, and definite integrals and then compute area, volume, arc length.

Expectations and workload

Calculus is assumed to use at least 10 hours per week of students time. The expected normal pattern of student study each week is

1. three hours of lectures;
2. three hours of lecture preparation and revision (reading the text books);
3. four hours of assignments and/or home work and preparing for test and exam.

Students are expected to attend all lectures and to come prepared. This means that students will have previewed the material and done any preliminary examples that have been set. Stay on top of the course material as it is covered. If you get behind, it could be difficult to catch up. Make use of the assistance available; ask for help as soon as you need it. Students are encouraged to collaborate with one another and to work together.

P.S. If you are in any doubt about the permissible degree of collaboration, please do not hesitate to come and see me in person.

Limit and continuity

Some basic notations and terminology

You may have come across the terms; **theorem, lemma, proposition**. In Mathematics, they refer to results that are true. By convention, the theorem is often reserved for more significant results, proposition for less significant ones and lemma are intermediate results used along the way to proving theorem.

A **variable** is just a label we use as a place-holder for something, for example, $y = 2x$, where x is independent and y is dependent variable. Hence, y is a function of x, i.e., $y(x)$. That something could be a fixed constant value or something that varies (a variable).

We often write members or elements of a set inside curly braces { }. i.e., { element 1, element 2, ... }. Where, "**...**" (putting 3 dots is the standard notation) is short hand form of "**so on**".

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$\in$ is used to say something is an element of (belongs to) a set.

$\notin$ is used to say something is not an element of a set.

$\subset$ is used to indicate a subset relationship.

$:$ or $|$ is used for "such that".

Note:- All sets are subsets of themselves.

Example:- $A = \{1, 2, 3, 4, 5, 6\}$.

The set A contains elements 1, 2, 3, 4, 5, and 6.

Example:- $\mathbb{N}$ represents the set of all natural numbers.

$\mathbb{Z}$ represents the set of all integer numbers.

$\mathbb{Q}$ represents the set of all rational numbers.

$\mathbb{R}$ represents the set of all real numbers.

Limit and continuity

$\mathbb{C}$ represents the set of all complex numbers.

ϕ or $\{\}$ is empty set (containing nothing).

Note:- We have that ϕ is a subset of $\mathbb{N}$, which is a subset of $\mathbb{Z}$, which is a subset of $\mathbb{Q}$, which is a subset of $\mathbb{R}$, which is a subset of $\mathbb{C}$, i.e.,

$\phi \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

Example:- The set A of all natural numbers that are even and are less than 12 can be written as

$$A = \{x \in \mathbb{N} : x \text{ is even } \& \ x < 12\}, \quad \text{or}$$

$$A = \{x \in \mathbb{N} \mid x \text{ is even } \& \ x < 12\}, \quad \text{or}$$

$$A = \{2, 4, 6, 8, 10\}.$$

Limit and continuity

Example:- The set

$$A = \{\{1,2\},\{1,3\},\{4,11\},12,30\}$$

is an example of a set containing other sets as well as numbers.

Example:- $\{1,2,4,4,4,5\}$ is the same set as $\{5,4,2,1\}$.

Recall that, elements of a set can only appear once and that the order does not matter.

Definition:- An **interval** is a set containing all real numbers between two bounds. Let a, b belong to the set of $\mathbb{R}$ and $a \leq b$. Here, a, b are those bounds. Then

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

is called a closed interval.

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$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

is called an open interval.

Example:- Let a, b belong to the set of $\mathbb{R}$ and $a < b$. Then

$$(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$$

is called as half-open half-closed interval.

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$$

is called as half-closed half-open interval.

Example:- Let $a \in \mathbb{R}$. Then we have

$$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$$

$$(a, \infty) = \{x \in \mathbb{R} : x > a\}$$

Limit and continuity

$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

Remarks:-

- Be-aware that in the previous example, we used the symbol ∞ to refer to the concept of infinity. Infinity is not a real number, but we are using it to symbolize the idea that things "go on forever" in a particular direction of the real number line.
- ∞ refers to the +ve direction.
- $-\infty$ refers to the -ve direction.
- Examining carefully the definitions of intervals involving the symbol ∞. Since ∞ is not a number, it does not make sense to use $\leq \infty$ or $\geq \infty$.

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Example:- Let $A = [2, 4)$, $B = (4, 6]$, and $C = [5, 7)$. Then we have

$$A \cup B = \{x \in \mathbb{R} : x \in [2, 6] \text{ and } x \neq 4\},$$

$$B \cap C = [5, 6],$$

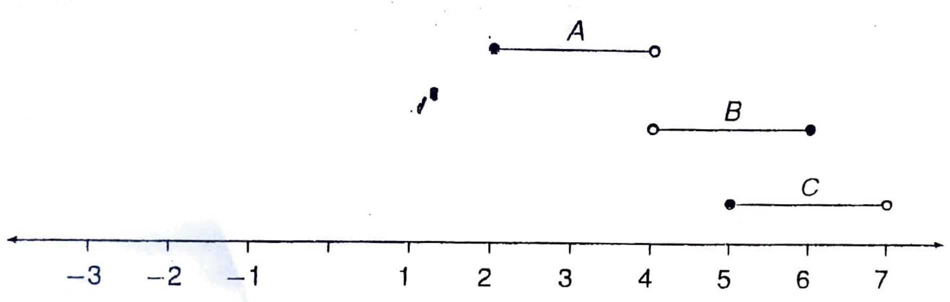
$$B \cup C = (4, 7),$$

$$A \cap C = \phi.$$

Notice that, the intersection of any two intervals is an interval. However, it is possible that the union of two intervals is not an interval. For example, here $A \cup B$ is not an interval as 4 is missing, i.e.,

$$A \cup B = \{x \in \mathbb{R} \mid x \in [2, 6] \ \& \ x \neq 4\}.$$

Limit and continuity



Note:- The empty set is also an interval as we can consider, for example, $4 < x < 4$. Since, there are no x values that satisfy these two bounds and we don't have to include the bounds themselves in the interval. Then the empty set satisfies the definition of being an interval. Similarly, $[4, 4] = \{4\}$ is an interval as it too satisfies the definition.

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Definition:- Let n be a natural number. An ordered n-tuple (n-tuple means n-dimensional components) is a sequence of real numbers $(v_1, v_2, v_3, \ldots, v_n)$, where $v_1, v_2, v_3, \ldots, v_n \in \mathbb{R}$. The set of all ordered n-tuples is called **n-space** and is denoted by $\mathbb{R}^n$.

Example:- The Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is a set of all ordered pairs (x, y) with $x, y \in \mathbb{R}$. It may also be represented as $f : \mathbb{R}^2 \to \mathbb{R}$ or $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Definition:- A **relation** is a set of ordered pairs.

Example:- $A = \{(2, 4), (1, 1), (0, 2), (1, 6)\}$ is a relation.

$$R = \{(x, y) \in \mathbb{R}^2 : y^2 = x^2\} \text{ is a relation.}$$

Definition:- A **function** is a relation

$$S = \{(a, b) \in A \times B : \text{each element of } A \text{ shows up only once.}\}$$

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if $f(x)$ is the rule that determines the method for picking how the pairs are made, then we write

$$f : A \to B;\ f(x) = \text{the rule used.}$$

The set A is called the domain of the function and B is called the co-domain (subtle difference to the range).

Remarks:-

- Functions are sometimes just given as $y = f(x)$ or called "f of x" to specify that y depends on the value of x using some rule f.
- Sometimes by convention, $f(x)$ is left out entirely and we are left to interpret things like $y = 2x$ to mean y is a function of x.

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Example:-

$$S = \{(1,2), (1,0), (4,1)\} \text{ is not a function.}$$

S can't define a function because 1 shows up twice in the first component of that relation.

$$R = \{(1,2), (3,2)\} \text{ is a function.}$$

R can define a function with the domain as $\{1, 3\}$.

Definition:- Sometimes people use the phrase **vertical line test** to see if the relation is a function. It says that if you graph the relation, i.e., plot all the points and if at any place you can run a vertical line through which intersects two or more points on the graph, then your relation can't be a function. On closer inspection, one should realize this test is really just seeing if the definition of a function is satisfied.

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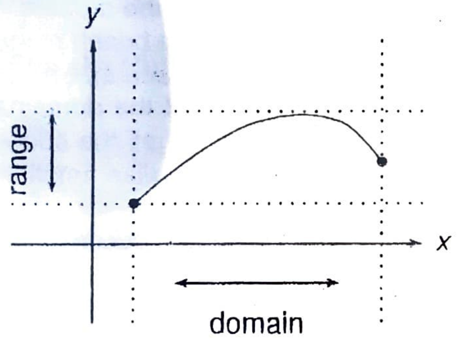
Example:- $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$
is a relation. It contains all of the points on a circle centered at (0, 0) of radius 1. It is not a function as $(0, 1) \in S$ and $(0, -1) \in S$.

We see that there are in-fact two pairs in this relation with 0 in the first component. Another way to see it is if we graphed it, then we could see that it violates the vertical line test.



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Definition:- The **natural domain** of a function is a set of all numbers $x \in \mathbb{R}$ which makes $f(x)$ a real number.

Example:-

- $f(x) = \frac{1}{x}$ has a natural domain of $x \neq 0$, i.e., set of real numbers excluding zero or $\mathbb{R} \setminus \{0\}$ or $(-\infty, 0)\ U\ (0, \infty)$.
- $f(x) = \ln x$ has a natural domain of $x > 0$, i.e., $\mathbb{R} \setminus (-\infty, 0]$ or $(0, \infty)$.
- $f(x) = \sqrt{x}$ has a natural domain of $x \geq 0$, i.e., $[0, \infty)$.
- $f(x) = \tan x$ has a natural domain of $\{x \in \mathbb{R} \quad x \neq n\pi - \frac{\pi}{2}$ for $n \in \mathbb{Z}\}$.

Question:- Where are the functions in the previous example undefined?

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Example:- The **range** of a function $f : A \to \mathbb{R}$ is the set $\{f(x) \in \mathbb{R} : x \in A\}$, i.e., the range of a function is the set of values that the function can output.

Example:-

- $f(x) = x$ has a range of $\mathbb{R}$.
- $f(x) = \sin x$ has a range of [-1, 1].
- $f(x) = \sqrt{x}$ has a range of $[0, \infty)$.

Note:- Special care must be taken when simplifying function. The simplified form may mislead you in to thinking the natural domain of the function is different from what it actually is. For example, let

$$f(x) = (1 + \sqrt{x})(1 - \sqrt{x}).$$

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The rule simplifies down to $f(x) = (1 - x)$. So, we might believe the natural domain is $\mathbb{R}$. However, this is not the case. The natural domain of $f(x)$ is $[0, \infty)$.

Definition:- A **piecewise function** is a function where different rules are used for different parts of the domain. You can think of a piecewise function as sticking together a bunch of other functions (where none of their domains overlap with each other). The domain of the piecewise function is the union of all of the domains of the other functions and the codomain is the union of all the codomains, and the range is the union of all the ranges. Piecewise functions are often written as

$$f(x) = \begin{cases} \text{rule 1}, & \text{some specific part of the domain,} \\ \text{rule 2}, & \text{another part of the domain,} \\ \vdots & \vdots \end{cases}$$

Limit and continuity

Example:-

Sketch a graph of the following function

$$f(x) = \begin{cases} 0, & x < 0, \\ x^2, & 0 \le x \le 1, \\ x - 1, & x > 1 \end{cases}$$

Definition:- The **absolute value** of a number is a function

$$|x| = \begin{cases} x, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

For any $c \in \mathbb{R}$, we can define a function $f(x) = |x - c|$ that allows us to measure the **distance** between the numbers c and x on the real number line.

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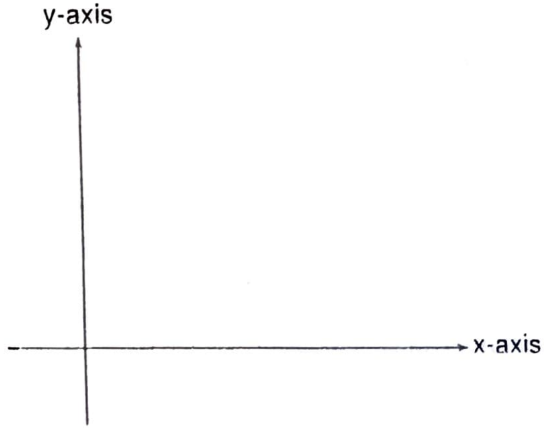

Figure: The absolute value function $y = |x|$.

Limit and continuity

Example:-

$$|-2| = 2,$$

$$|10.3| = 10.3.$$

Definition:- Given two functions f and g, we can **combine** them in several different ways:

1. Add them, i.e., $f(x) + g(x)$;
2. Subtract them, i.e., $f(x) - g(x)$;
3. Multiply them, i.e., $f(x)g(x)$;
4. Divide them, i.e., $f(x)/g(x)$.

Limit and continuity

Definition:- Given two functions f and g, the **composition of** f **and** g is a new function given by

$$f \circ g(x) = f(g(x))$$

Example:-

Let $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$. Then we have

$$(f \circ g)(x) = f(g(x)) = (\sqrt{x})^2 + 2 = x + 2$$

$$(g \circ f)(x) = g(f(x)) = \sqrt{(x^2 + 2)}$$

Remarks:-

1. Sometimes this composition is referred to as "f of g" or "f of g of x".
2. The domain of $f \circ g$ is the largest subset of Domain(g) so that Range(g) $\subset$ Domain(f).

Example:-

Let $f(x) = x^2 - 2$ and $g(x) = \sqrt{x}$. Then find $(f \circ g)(x)$ and its natural domain.

Solution:-

$$f \circ (g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 - 2 = x - 2.$$

Since $g(x) = \sqrt{x}$, for all $x \geq 0$, $\sqrt{x} \in \mathbb{R}$, and negative square roots are not real numbers. So, we have that the natural domain of g is $[0, \infty)$. The natural domain of f is $\mathbb{R}$, therefore the domain of $(f \circ g)(x)$ is $x \geq 0$.

Note:-

Note that $f \circ g$ is not the same as $g \circ f$.

Example:-

Let $f(x) = x^2 - 2$ and $g(x) = \sqrt{x}$. Then

$$(g \circ f)(x) = g(f(x)) = \sqrt{(x^2 - 2)}.$$

Since the natural domain of f is $\mathbb{R}$ and the natural domain of g is $[0, \infty)$, the natural domain of $g \circ f$ consists of all $x \in \mathbb{R}$ such that $f(x) = x^2 - 2$ lies inside $[0, \infty)$. Therefore,

$$x^2 - 2 \geq 0, \text{which gives } x^2 \geq 2.$$

From this we have

$$x \geq \sqrt{2} \text{ or } x \leq -\sqrt{2}.$$

Hence, the domain of $g \circ f$ is

$$(-\infty, -\sqrt{2}] \cup [\sqrt{2}, \infty).$$

Graph Transformations

There are many times when we know very well what the graph of a particular function looks like, and we want to know what the graph of a very similar function looks like. Here, we discuss some ways to draw graphs in these circumstances.

1. Right-left translation.

Let $c > 0$. Start with the graph of some function $f(x)$. Keep the $x-axis$ and $y-axis$ fixed, but move the graph c units to the right, or c units to the left (see the figures below.) You get the graphs of two new functions, i.e., $f(x-c)$ and $f(x+c)$, respectively. For example, moving the graph of the function $f(x)$ right by c units has the effect of replacing x by $x-c$ wherever it occurs in the formula, i.e., $f(x-c)$. For instance,

$$f(x) = x^2 + x \quad \Longrightarrow \quad f(x-1) = (x-1)^2 + (x-1) \;= x^2 - x.$$

The reasoning is similar if the graph is translated c units to the left.

2. Up-down translation.

The effect of up-down translation of the graph is much simpler to see. Start with the graph of some function $f(x)$ by considering $c > 0$. Keep the $x-axis$ and $y-axis$ fixed, but move the graph c units up, or c units down (see the figures below.) You get the graphs of two new functions, i.e., $f(x)+c$ and $f(x)-c$). For example, moving the graph of $f(x)$ up by c units has the effect of adding c units to each function value, and therefore gives us the graph of the function $f(x)+c$. The graph of the new function is easy to describe; just take every point in the graph of $f(x)$, and move it up a distance of c. That is, if (a,b) is a point in the graph of $f(x)$, then $(a, b+d)$ is a point in the graph of $f(x)+c$.

Example:- Sketch the graph of $f(x) = x^2$.

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The chart below describes how to use the graph of $f(x)$ to create the graph of some similar functions. Throughout the chart, $c > 1$, and (a, b) is a point in the graph of $f(x)$. Notice that, all of the "new functions" in the chart differ from $f(x)$ by some algebraic manipulation that happens after $f(x)$ plays its part as a function.

New function	How points in the graph of $f(x)$ become points of new graph	Visual effect
$f(x+c)$	$(a,b) \longmapsto (a+c,b)$	shift left by c units
$f(x-c)$	$(a,b) \longmapsto (a-c,b)$	shift right by c units
$f(cx)$	$(a,b) \longmapsto (\frac{1}{c}a,b)$	shrink horizontally by $\frac{1}{c}$ units
$f(\frac{1}{c}x)$	$(a,b) \longmapsto (ca,b)$	stretch horizontally by c units
$f(-x)$	$(a,b) \longmapsto (-a,b)$	flip over the $y-$ *axis*

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In a similar way to that of previous chart, we have another chart below.

New function	How points in the graph of $f(x)$ become points of new graph	Visual effect
$f(x)+c$	$(a,b) \longmapsto (a,b+c)$	shift up by c units
$f(x)-c$	$(a,b) \longmapsto (a,b-c)$	shift down by c units
$cf(x)$	$(a,b) \longmapsto (a,cb)$	stretch vertically by c units
$\frac{1}{c}f(x)$	$(a,b) \longmapsto (a,\frac{1}{c}b)$	shrink vertically by $\frac{1}{c}$ units
$-f(x)$	$(a,b) \longmapsto (a,-b)$	flip over the $x-$ *axis*

Limit and continuity

Geometry of inverse functions

Informally, two functions $f(x)$ and $g(x)$ are said to be inverses if each reverses, or undoes the other. More precisely, we can define as
Definition:- Two functions $f(x)$ and $g(x)$ are said to be inverses if for all x in the domain of $g(x)$

$$f \circ g(x) = f(g(x)) = x,$$

and for all x in the domain of $f(x)$

$$g \circ f(x) = g(f(x)) = x.$$

Example:- $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses, since $(x^3)^{1/3} = x$ and $(x^{1/3})^3 = x$.

Example:- $f(x) = x^2$ and $g(x) = x^{1/2}$ are not inverses, since $(x^2)^{1/2} = x$ is not true that $(x^{1/2})^2 = x$ for all real values of x. For instance, take $x = -2$. However, what happens when we consider the values of $x \geq 0$?

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Generally, we look for functions $y = f(x)$ and $g(y) = x$ for which $g(f(x)) = x$. If this is the case, then g is the inverse of f and we write $g = f^{-1}$ and f is the inverse of g written as $f = g^{-1}$.

How are the graphs of a function and its inverse related? We start by graphing $f(x) = \sqrt{x}$. Next we want to graph the inverse of f, which is $g(y) = x$. But this is exactly the graph we just drew. To compare the graphs of the functions f and f^{-1} we have to exchange x and y in the equation for f^{-1}. So to compare $f(x) = x$ to its inverse we replace y's by x's and graph $g(x) = x^2$.

Figure: The graph of $f^{-1}(x) = x^2$ is the reflection of the graph of $f(x) = \sqrt{x}$ across the line $y = x$.

In general, if you have the graph of a function $f(x)$ you can find the graph of f^{-1} by exchanging the $x-$ and $y-$coordinates of all the points on the graph. In other words, the graph of f^{-1} is the reflection of the graph of f across the line $y = x$. This suggests that if $\frac{dy}{dx}$ is the slope

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of a line tangent to the graph of f, then

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

is the slope of a line tangent to the graph of f^{-1}.

Example:- Find the inverse of

$$f(x) = \frac{(2x-6)}{(3x+7)}.$$

Solution:- First we write $x = (2y-6)/(3y+7)$ and then solve for y as

$$x = \frac{2y-6}{3y+7}$$
$$x(3y+7) = 2y-6$$
$$3xy + 7x = 2y - 6$$

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$$7x + 6 = 2y - 3xy$$
$$7x + 6 = y(2 - 3x)$$
$$\frac{7x+6}{2-3x} = y$$

Finally, we say that $f^{-1}(x) = \frac{7x+6}{2-3x}$.

Theorem (Inverse function theorem):- Let A be an open interval and let $f : A \to \mathbb{R}$ be injective and differentiable. If $f'(x) \neq 0$ for every $x \in A$ then f^{-1} is differentiable on $f(A)$ and

$$(f^{-1})'(x) = [f'(f^{-1}(x))]^{-1} = 1/f'(f^{-1}(x)).$$

Proof:- Fix $b \in f(A)$. Then there exists a unique $a \in A$, such that, $f(a) = b$. For $y \neq b$, let $x = f^{-1}(y)$. Since f is differentiable, it follows that f and hence f^{-1} are continuous.

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Then

$$\lim_{y \to b} \frac{f^{-1}(b) - f^{-1}(y)}{b - y} = \lim_{x \to a} \frac{a - x}{f(a) - f(x)} = \frac{1}{f'(a)}.$$

In Leibniz notation, this can be written as $\frac{dx}{dy} = \frac{1}{dy/dx}$, which is easy to remember since it looks like ordinary fractional algebra. We could also prove the above result using implicit differentiation.

Let's use implicit differentiation to find the derivative of the inverse function.

$$y = f(x)$$

$$f^{-1}(y) = x$$

$$\frac{d}{dx}(f^{-1}(y)) = \frac{d}{dx}(x) = 1$$

By the chain rule, we have

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$$\frac{d}{dx}(f^{-1}(y))\frac{dy}{dx} = 1,$$

so,

$$\frac{d}{dx}(f^{-1}(y)) = \frac{1}{\frac{dy}{dx}} = 1.$$

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Assumed knowledge:-

- Radian measure for angles.
- The domain of $f \circ g$ is the largest subset of Domain(g) so that Range(g) $\subset$ Domain(f).
- $360^\circ = 2\pi$ radians.
- The functions sin, cos, tan, csc, sec, and cot.
- Triangles for angels $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$.
- Pythagoras theorem.

Limit and continuity

Limit of a function.

Limit is one of the most powerful ideas ever created in mathematics to observe the behavior of functions. Heavy hitting mathematical theories, like, differentiation and integration owe their existence to the idea of the limit.,

Suppose we wish to observe the behavior of a function around certain input values to the function, i.e., values in the domain. Apart from evaluating the function at that point, we could also look at what the function is doing when we make the input values closer and closer to the input value we want to study. If a strong pattern emerges and the function looks like it is converging to a value, we would reasonably expect the function to take that value at the point itself. Figuring out these expectations is in essence what the theory of limits is all about.

Previously, you may have come across a definition of the limit being

Limit and continuity

described as what value a function tends to as x approaches a particular point. This description is good for understanding the basic idea we are trying to capture with a limit, but without actually specifying what "approaches" or "tends to" actually mean, any calculations often relied on a leap of faith that functions behave nicely. However, many functions behave in surprising ways, so we need a rigorous way to study the concept of limit. The obvious place to start is to make the definition rigorous.

Definition:-

Let $f : (a, b) \to \mathbb{R}$ be a function on the open interval (a, b), $c \in [a, b]$, and $L \in \mathbb{R}$. If for all real numbers ε with $\varepsilon > 0$, there exists a real number δ (δ can depend on ε) with $\delta > 0$, such that for all

$$x \in (c - \delta, c + \delta) \text{ with } x \neq c,$$

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we have that

$$f(x) \in (L - \varepsilon, L + \varepsilon),$$

then we say that the limit of f exists at c and the limit is L. **Sometimes,** it is written as

$$\lim_{x \to c} f(x) = L,$$

or

$$f(x) \to L, \quad \text{as } x \to c.$$

This definition might look scary at first, but a lot of it is just detail to make it mathematically precise.

Remarks:-

1. Notice that, although the limit is defined at individual points, it is telling us the behavior of what's happening around the neighbors of that point, and not at the point itself.

Limit and continuity

- Note that $|x-c|<d$ is equivalent to $x \in (c-d, c+d)$, so quite often you will the definition of limit formulated using $0<|x-c|<\delta$ and $|f(x)-L|<\varepsilon$ instead.

Example:-

Let $f(x)=3x+1$, $\varepsilon>0$ and $\delta=\frac{\varepsilon}{3}$. Then for all x such that $0<|x-5|<\delta$, we have

$$|f(x)-16|=|3x+1-16|=|3x-15|=3|x-5|<3\delta=3\frac{\varepsilon}{3}=\varepsilon.$$

Therefore,

$$\lim_{x\to 5} f(x)=16.$$

Limit and continuity

Definition:-

If we replace $x \in (c-\delta, c+\delta)$ in the limit definition by $(c, c+\delta)$, then we have the **right sided limit** and is written as

$$\lim_{x\to c^+} f(x)=L.$$

If we replace $x \in (c-\delta, c+\delta)$ in the limit definition by $(c-\delta, c)$, then we have the **left sided limit** and is written as

$$\lim_{x\to c^-} f(x)=L.$$

Notice the + and − superscripts on the c. If a right side limit exists at c, then it is the value the function is approaching by only looking at neighboring points which are larger than c. If a left side limit exists at c, then it is the value the function is approaching by only looking at neighboring points which are smaller than c.

Limit and continuity

Theorem:-

$\lim_{x\to c} f(x) = L$ exists if and only if $\lim_{x\to c^+} f(x) = L$ and $\lim_{x\to c^-} f(x) = L$ exist, and

$$\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = L.$$

That is, the limit of f at c exists if and only if the left sided limit exists at c, the right sided limit exists at c, and the limits are all the same values.

Definition:-

1. What is happening to the function $f(x) = \frac{|x|}{x}$ as $x \to 0$?
2. What is happening to the function $f(x) = \frac{x}{x+|x|}$ as $x \to 0$?

Solution-1:-

Follow the limit by using the definition of mode function. Here,

Differentiation
A review of limits, continuity and differentiation

Limit and continuity

$$f(x) = \frac{|x|}{x} = \frac{\pm x}{x},$$

$$f(x) = \frac{x}{x}, \qquad f(x) = \frac{-x}{x},$$

$$\lim_{x\to 0^+} f(x) = \frac{x}{x}, \qquad \lim_{x\to 0^-} f(x) = \frac{-x}{x},$$

$$\lim_{x\to 0^+} f(x) = 1, \qquad \lim_{x\to 0^-} f(x) = -1.$$

Thus the limit of the given function does not exist as $x \to 0$.

Approaching infinity.

Consider $f(x) = \frac{1}{x}$, defined on its natural domain.

As x approaches 0 from the right, the value of $f(x)$ is increasing without bound. We write

Limit and continuity

$$\lim_{x \to 0^+} f(x) = \infty$$

to denote that $f(x)$ increases towards infinity as x approaches from the right.

As x approaches 0 from the left, the value of $f(x)$ is decreasing without bound. We write

$$\lim_{x \to 0^-} f(x) = -\infty$$

to denote that $f(x)$ decreases towards infinity as x approaches from the left.

Recall that in the definition of the limit, the limit value L needed to be a real number. Since ∞ and $-\infty$ are not real numbers, so no limit exist. Even if they were real numbers, the right and left sided limits are not equal. Hence, $f(x)$ has no limit as $x \to 0$.

Limit and continuity

Limits of some "building block" functions.

The term building block here refers to some simple functions which can be glued together to construct more complicated functions. For example, $f(x) = 4x^2 - x$ is just the function $g(x) = x$ glued together with itself in a special way. In fact,

$$f(x) = 4x^2 - x = 4g(x)^2 - g(x).$$

Proposition:-

Let a and b be real numbers. Then we have

$$\lim_{x \to b} a = a, \qquad \lim_{x \to b} x = b,$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

Limit and continuity

Laws of limits.

Let $f(x), g(x)$ be two function. Assume $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$. Then we have

$$\lim_{x\to a} (f(x) \pm g(x)) = L_1 \pm L_2,$$

$$\lim_{x\to a} (f(x) \times g(x)) = L_1 \times L_2,$$

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{L_1}{L_2}, \quad \text{provided } L_2 \neq 0,$$

$$\lim_{x\to a} \left(\sqrt[n]{f(x)}\right) = \sqrt[n]{L_1}.$$

Limit and continuity

Theorem:-

For any polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n,$$

and any real number a, we have

$$\lim_{x\to a} p(x) = c_0 + c_1 a + c_2 a^2 + \dots + c_n a^n = p(a).$$

Definition:-

A **rational function** is a function of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where, $p(x)$ and $q(x)$ are polynomials.

Limit and continuity

Example:-

$$f(x) = \frac{x^3 - 10x + 5}{x^2 - 4}$$

is a rational function. The natural domain of $f(x)$ is all real numbers excluding -2 and 2, i.e., $\mathbb{R}\backslash\{-2, 2\}$.

Proposition:-

Let

$$f(x) = \frac{p(x)}{q(x)},$$

be a rational function and a be any real number. Then we have

- If $q(a) \neq 0$ then $\lim_{x\to a} f(x) = f(a)$.
- If $q(a) = 0$ and $p(a) \neq 0$ then $\lim_{x\to a} f(x)$ does not exist (as a real number).

Limit and continuity

- If $q(a) = 0$ and $p(a) = 0$ then $\lim_{x\to a} f(x)$ may sometimes exist (as a real number).

Example:-

$$\lim_{x\to 1} \frac{x^2+1}{x-1} \text{ does not exist}$$

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x - 3} = \lim_{x\to 3} \frac{(x-3)(x+2)}{x-3} = \lim_{x\to 3} \frac{x+2}{1} = 5.$$

Limits at infinity.

Consider what happens to the value of the function $f(x) = \frac{3x}{x^2+1}$ as the value of x gets larger. We observe that the numerator $3x$ starts to get insignificant as x gets larger and larger, and the whole fraction is converging towards zero.

Limit and continuity

Note:-

If the value of $f(x)$ eventually just keeps growing larger and larger without bounds, then we write

$$\lim_{x\to\infty} f(x) = \infty.$$

Notice that, the limit in this case does not actually exist, because ∞ is not a real number, but we write this just as shorthand to note that the function will eventually grow to be larger than any number we choose.

Proposition:-

Let $p > 0$. Then

$$\lim_{x\to\infty} \frac{1}{x^p} = \lim_{x\to-\infty} \frac{1}{x^p} = 0.$$

Limit and continuity

Proposition:-

Let n be a positive integer. Then

$$\lim_{x\to\infty} x^n = \infty.$$

If n is also an odd number. Then

$$\lim_{x\to-\infty} x^n = -\infty,$$

otherwise, if n is even then we have

$$\lim_{x\to-\infty} x^n = \infty.$$

Remarks:- With polynomial fractions, as $x \to \infty$ or $x \to -\infty$ it will be highest term that will dominate the behavior. We can exploit this idea to evaluate limits of rational functions by finding the highest power term in the denominator, and dividing every single component of top and bottom of fraction by this highest power to find the limit.

Limit and continuity

Continuity:-

Here, we aim to study a nice class of functions called as continuous functions. Previously, you may have been introduced to continuous functions by the idea that if you had to draw a graph of the function with a pen, then you would be able to do it without lifting the pen off the paper and causing a jump in the graph somewhere.

Unfortunately, some functions are hard to draw and others behave erratically, so we need a way to figure out whether a function is still continuous or not. Luckily, we can use the idea of limits to make this concept precise. The beauty of the definition is that it allows us to import almost all the results from the theory of limits and be able to say things about continuous functions essentially for free, i.e., without doing much extra work. This helps motivate the following definition of continuity.

Limit and continuity

Definition:-

Let $f : D \to \mathbb{R}$ be a function and $c, L \in \mathbb{R}$. Then we say that f is **continuous** at the point c if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$,

$$f(x) \in (L - \varepsilon, L + \varepsilon).$$

We say a function is **continuous everywhere**, or just **continuous** if this holds for all $c \in \mathbb{R}$, and continuous on an interval $[a, b]$ if it is continuous at every $x \in [a, b]$. Otherwise, we say that f has a **discontinuity** at $x = c$, or that f is **discontinuous at** c.

Note:-

You may have noticed that apart from not having $x \neq c$ requirement, everything else in the definition is identical to the definition of a limit. Mathematicians can thus reprove continuity results from limit results with hardly any more work.

Limit and continuity

Maths translator.

A function f is continuous at a point c if the following three conditions are met:

1. $f(c)$ is defined.
2. $\lim_{x\to c} f(x)$ exists, i.e., $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$ for some $L \in \mathbb{R}$.
3. $\lim_{x\to c} f(x) = f(c)$.

Remarks:-

1. All you need to remember is

$$\lim_{x\to c} f(x) = f(c)$$

as this statement captures all three conditions.

Limit and continuity

2. It is implied that if a function is continuous , then its natural domain is all of $\mathbb{R}$.
3. A function $y = f(x)$ is **continuous at the interior** c of its domain if

$$\lim_{x\to c} f(x) = f(c).$$

4. A function $y = f(x)$ is **continuous at a left end-point** a or is **continuous at a right end-point** b of its domain if

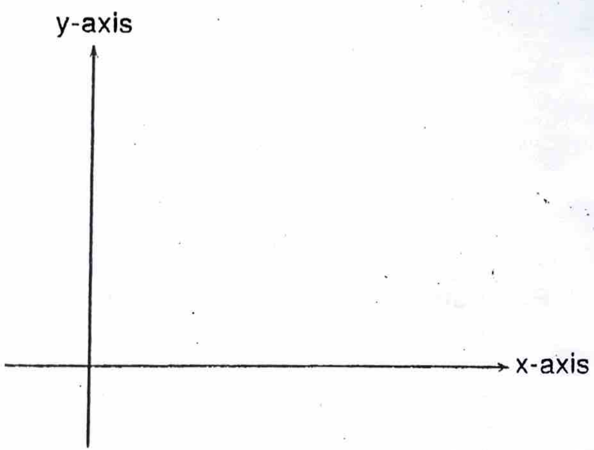
$$\lim_{x\to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x\to b^-} f(x) = f(b), \text{ respectively.}$$

5. Given how similar the definition of continuity and limits are, laws of limits can be reworded in the context of continuity to tell us that if two continuous functions have the same domain, then the sum and product of continuous functions is continuous.
6. If $f(x), g(x)$ are continuous functions, then $f(x)/g(x)$ is continuous everywhere where the denominator is non-zero, and has discontinuities everywhere where the denominator is zero.

Limit and continuity

Example:-

Classify the discontinuities at c. Why does each not satisfy continuity?



Limit and continuity

Example:-

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \neq 2 \\ 3, & \text{if } x = 2 \end{cases} \quad , f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x \neq 2 \\ 4, & \text{if } x = 2. \end{cases}$$

Theorem:-

Let f, g be two functions. If $\lim_{x\to c} g(x) = L$ and f is continuous at L. Then we have

$$\lim_{x\to c} f(g(x)) = f(L).$$

In other words,

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)).$$

The same applies if the limit above is replaced by a left or right side limit, or a limit at infinity.

Limit and continuity

Theorem:-

Let f, g be two functions, and $c \in \mathbb{R}$. Then

1. If g is continuous at c and f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.
2. If the function g is continuous everywhere and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.
3. The absolute value of a continuous function is again a continuous function.

Limit and continuity

Example:-

Let $f(x) = x^5$, $g(x) = \frac{x^2-2x-3}{x^2+4x+3}$. Then

$$\lim_{x\to -1} f(g(x)) = \lim_{x\to -1}\left(\frac{x^2-2x-3}{x^2+4x+3}\right)^5$$

$$\vdots$$

$$\lim_{x\to -1} f(g(x)) = -32$$

Proposition:-

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0.$$