National University of Computer and Emerging Sciences

Lahore Campus

Discrete Structures (CS1005)

Date: 4th November 2025

Course Instructors

Dr. Saeeda Zia

Dr. Imran Nadeem

Dr. Aziz Ur Rehman

Ms. Laila Yawar

Ms. Namra Absar

Mr. Amjad Ali

Muhammad Adeel

Sessional-II Exam

Total Time: 1 Hour

Total Marks: 30

Total Questions: 03

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Roll No Section Student Signature

CLO # 2: Apply fundamental concepts of number theory, such as divisibility, greatest common divisors, modular arithmetic, prime numbers, congruences, and cardinality of sets.

Q. No 1:

i) Show that the set of real numbers is an uncountable set. [5]

Solution: Book Page No. 183, Example # 5.

ii) State the Division Algorithm. Apply Division algorithm to find the quotient and remainder when -257 is divided by 12. [2]

Solution: Statement of Division Algorithm:

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$.

By the Division Algorithm, given $a = -257$ and $d = 12$, we need to find integers q and r such that $-257 = 12q + r$ with $0 \leq r < 12$. By inspection, we get $q = -22$ and $r = 7$.

iii) Use the extended Euclidean algorithm to express $gcd(65, 40)$ as a linear combination of 65 and 40. [3]

Solution: We need to find the integers x and y such that

$$gcd(65, 40) = 65x + 40y$$

First, we find $gcd(65, 40)$. Successive uses of the division algorithm give:

$$65 = 40(1) + 25$$

$$40 = 25(1) + 15$$

$$25 = 15(1) + 10$$
$$15 = 10(1) + 5$$
$$10 = 5(2) + 0$$

Hence, $gcd(65, 40) = 5$, because 5 is the last nonzero remainder.

As the quotients are $q_1 = 1$, $q_2 = 1$, $q_3 = 1$, $q_4 = 1$, and $q_5 = 0$. The desired Bézout coefficients are the values of s_5 and t_5 generated by the extended Euclidean algorithm, where $s_0 = 1$, $s_1 = 0$, $t_0 = 0$ and $t_1 = 1$, and

$s_j = s_{j-2} - q_{j-1}s_{j-1}$ and $t_j = t_{j-2} - q_{j-1}t_{j-1}$

For $j = 2,3,4,5$, we find that

$s_2 = s_0 - q_1 s_1 = 1 - 1 . 0 = 1$, $s_3 = s_1 - q_2 s_2 = 0 - 1 . 1 = -1$, $s_4 = s_2 - q_3 s_3 = 1 - 1 . -1 = 2$, $s_5 = s_3 - q_4 s_4 = -1 - 1 . 2 = -3$ and

$t_2 = t_0 - q_1 t_1 = 0 - 1 . 1 = -1$, $t_3 = t_1 - q_2 t_2 = 1 - 1 . -1 = 2$, $t_4 = t_2 - q_3 t_3 = -1 - 1 . 2 = -3$, $t_5 = t_3 - q_4 t_4 = 2 - 1 . -3 = 5$

Because $s_5 = -3$ and $t_5 = 5$, we see that $gcd(65, 40) = 5 = -3 . 65 + 5 . 40$.

iv) Find all solutions of the congruence $4x \equiv 5 \pmod 9$. [3]

Solution: First, we find the inverse of 4 modulo 9. Since $gcd(4, 9) = 1$. Therefore, the inverse of 4 modulo 9 exists. By the Euclidean algorithm, we will write $gcd(4, 9) = 1$ as a linear combination of 4 and 9 as follows:

$$9 = 4 . 2 + 1 \quad \rightarrow (a)$$
$$4 = 1 . 4 + 0 \quad \rightarrow (b)$$

From equation (a), $1 = 1 . 9 + (-2) . 4$.

From this we have $(-2) . 4 \equiv 1 \pmod 9$. This implies that $-2 \equiv 7$ is the inverse of 4 modulo 9. Multiplying both sides of the congruence by 7 shows that

$$28\,x \equiv 35 \pmod 9$$

Because $28 \equiv 1 \pmod 9$ and $35 \equiv 8 \pmod 9$, it follows that $x \equiv 8 \pmod 9$. For any integer t, all solutions of the congruences are $x = 8 + 9t$.

v) Use Fermat's little theorem to find the remainder when 5^{2003} is divided by 11. [2]

Solution: By Fermat's little theorem, $5^{10} \equiv 1 \pmod{11}$.

So $(5^{10})^k \equiv 1 \pmod{11}$ for every positive integer k. We divide the exponent 2003 by 10, finding that $2003 = 10 . 200 + 3$. We now see that

$$5^{2003} = 5^{10 . 200 + 3} = (5^{10})^{200} . 5^3 \equiv (1)^{200} . 125 \equiv 4 \pmod{11}$$

It follows that 4 is the remainder when 5^{2003} is divided by 11.

CLO #3: Apply mathematical induction to prove properties of sequences, Learn about recursive relations, relations and their properties.

Q. No 2:

i) Use principle of mathematical induction to prove that $7^n - 2^n$ is divisible by 5 for all $n \geq 1$.[2]

Solution: Let $P(n)$: 5 divides $7^n - 2^n$

Basic Step: For $n = 1$,

$$P(1): \; 7^1 - 2^1 = 5$$

Which is divisible by 5. So, $P(1)$ is true.

Inductive Step: Assume that $P(k)$ is true i.e. 5 divides $7^k - 2^k$. We show that $P(k+1)$ is true.

$$\begin{aligned} P(k+1): \qquad 7^{(k+1)} - 2^{(k+1)} &= 7\,.\,7^k - 2\,.\,2^k \\ &= 7\,.\,7^k - 7\,.\,2^k + 5\,.\,2^k \\ &= 7\,.\left(7^k - 2^k\right) + 5\,.\,2^k \end{aligned}$$

By inductive hypothesis, 5 divides $7^k - 2^k$ and obviously, 5 divides $5\,.\,2^k$. So, 5 divides $7^{(k+1)} - 2^{(k+1)} = 7\,.\left(7^k - 2^k\right) + 5\,.\,2^k$. This implies that $P(k+1)$ is true as desired.

ii) Use strong induction to prove that any integer $n \geq 2$ is either a prime or a product of primes.

Solution: Book Page No. 357, Example # 2.

CLO # 4: Apply fundamental counting principles to solve combinatorial problems.

Q. No 3: [10 = 2 + 2 +2 + 2 +2]

i) There are 6 buses from city A to city B. In how many ways can a man go from A to B and return by a different bus?

Solution: There are 6 different buses from city A to city B. A man can choose any one of these 6 buses to travel from A to B. However, since he must return by a different bus, he has only 5 choices for the return journey. Therefore, by the product rule, the total number of possible ways he can make the round trip is $6 \times 5 = 30$. Hence, the man can travel from A to B and return by a different bus in 30 ways.

ii) Show that among any group of five (not necessarily consecutive) integers, there are two with the same remainder when divided by 4.

Solution: When an integer is divided by 4, the possible remainders are 0,1,2,or 3. Think of these four remainders as four pigeonholes.

Now take any group of five integers (these need not be consecutive). Put each integer into the pigeonhole corresponding to its remainder upon division by 4. There are 5 integers (pigeons) and only 4 possible remainders (pigeonholes).

By the Pigeonhole Principle, at least one pigeonhole must contain at least two integers. That means at least two of the five integers have the same remainder when divided by 4.

Thus, among any group of five integers there are two with the same remainder when divided by 4.

iii) Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have more women than men?

Solution: For a 6-member committee, the number of women (W) must be greater than the number of men (M).

So, $W + M = 6$ and $W > M$

The number of 6-combinations are:

$$C(15,6) \,.\, C(10,0) + C(15,5) \,.\, C(10,1) + C(15,4) \,.\, C(10,2)$$

$= \binom{15}{6} . \binom{10}{0} + \binom{15}{5} . \binom{10}{1} + \binom{15}{4} . \binom{10}{2}$

= 96,460

iv) Use the binomial theorem to find the coefficient of $x^{16}y^{12}$ in the expansion of $(3x^4 - 2y^3)^8$.

Solution: By the binomial theorem,

$$\left(3x^4 + (-2y^3)\right)^8 = \sum_{j=0}^{8} \binom{8}{j} (3x^4)^{8-j}(-2y^3)^j = \sum_{j=0}^{8} \binom{8}{j} 3^{8-j}x^{32-4j}(-2)^j y^{3j} \rightarrow (a)$$

By comparing $x^{32-4j}y^{3j}$ with $x^{16}y^{12}$, we get $j = 4$. By taking $j = 4$ in (a), we get the coefficient of $x^{16}y^{12}$ as $\binom{8}{4}3^4(-2)^4 = 90{,}720$.

v) How many different strings can be made from the letters in *ABRACADABRA*, using all the letters?

Solution: The word "*ABRACADABRA*" has 11 letters in total, with some letters repeating. Here's the breakdown:

A appears 5 times $(n_1 = 5)$, B appears 2 times $(n_2 = 2)$, R appears 2 times $(n_3 = 2)$, C appears 1 time $(n_4 = 1)$, and D appears 1 time $(n_5 = 1)$.

The number of different permutations is

$$\frac{n!}{n_1!\ n_2!\ n_3!\ n_4!\ n_5!} = \frac{11!}{5!\,2!\,2!\,1!\,1!} = 83{,}160$$