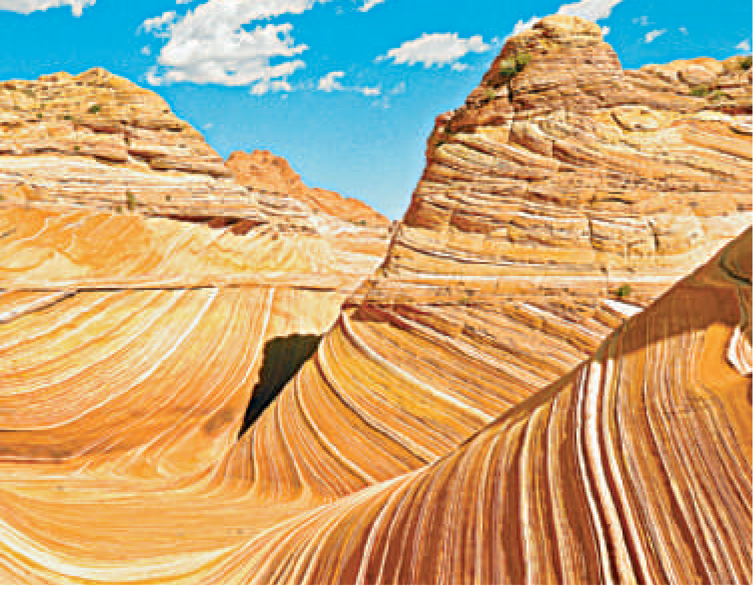


14

Partial Derivatives

OVERVIEW Many functions depend on more than one independent variable. For instance, the volume of a right circular cylinder is a function $V = \pi r^2 h$ of its radius and its height, so it is a function $V(r, h)$ of two variables r and h. The speed of sound through seawater is primarily a function of salinity S and temperature T. The surface area of the human body is a function of its height h and weight w. The monthly payment on a home mortgage is a function of the principal borrowed P, the interest rate i, and the term t of the loan.

In this chapter we extend the basic ideas of single-variable differential calculus to functions of several variables. Their derivatives are more varied and interesting because of the different ways the variables can interact. The applications of these derivatives are also more varied than for single-variable calculus, and in the next chapter we will see that the same is true for integrals involving several variables.

14.1 Functions of Several Variables

Real-valued functions of several independent real variables are defined analogously to functions in the single-variable case. Points in the domain are ordered pairs (triples, quadruples, n-tuples) of real numbers, and values in the range are real numbers as we have worked with all along.

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

If f is a function of two independent variables, we usually call the independent variables x and y and the dependent variable z, and we picture the domain of f as a region in the xy-plane (Figure 14.1). If f is a function of three independent variables, we call the independent variables x, y, and z and the dependent variable w, and we picture the domain as a region in space.

In applications, we tend to use letters that remind us of what the variables stand for. To say that the volume of a right circular cylinder is a function of its radius and height, we might write $V = f(r, h)$. To be more specific, we might replace the notation $f(r, h)$ by the formula

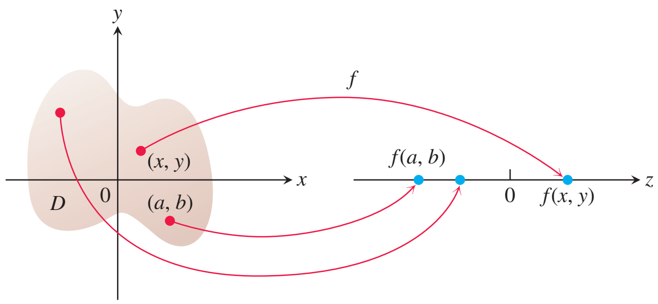


FIGURE 14.1 An arrow diagram for the function $z = f(x, y)$.

that calculates the value of V from the values of r and h, and write $V = \pi r^2 h$. In either case, r and h would be the independent variables and V the dependent variable of the function.

As usual, we evaluate functions defined by formulas by substituting the values of the independent variables in the formula and calculating the corresponding value of the dependent variable. For example, the value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point (3, 0, 4) is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

Domains and Ranges

In defining a function of more than one variable, we follow the usual practice of excluding inputs that lead to complex numbers or division by zero. If $f(x, y) = \sqrt{y - x^2}$, y cannot be less than x^2. If $f(x, y) = 1/(xy)$, xy cannot be zero. The domain of a function is assumed to be the largest set for which the defining rule generates real numbers, unless the domain is otherwise specified explicitly. The range consists of the set of output values for the dependent variable.

EXAMPLE 1

(a) These are functions of two variables. Note the restrictions that may apply to their domains in order to obtain a real value for the dependent variable z.

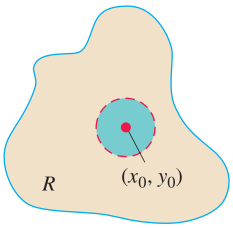
Function	Domain	Range
$z = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$z = \dfrac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$z = \sin xy$	Entire plane	$[-1, 1]$

(b) These are functions of three variables with restrictions on some of their domains.

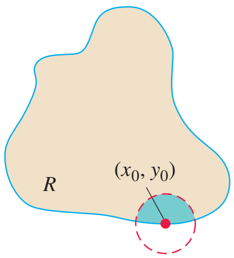
Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \dfrac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

Functions of Two Variables

Regions in the plane can have interior points and boundary points just like intervals on the real line. Closed intervals $[a, b]$ include their boundary points, open intervals (a, b) don't include their boundary points, and intervals such as $[a, b)$ are neither open nor closed.



(a) Interior point

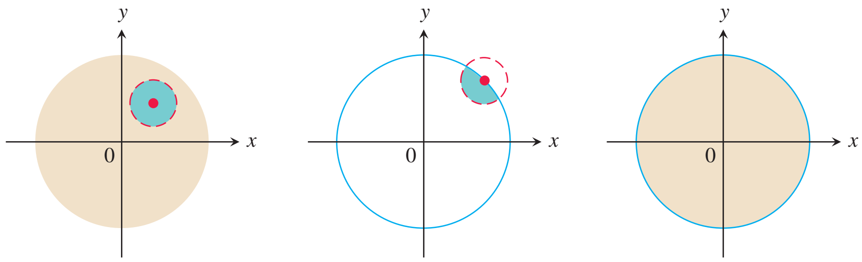


(b) Boundary point

FIGURE 14.2 Interior points and boundary points of a plane region R. An interior point is necessarily a point of R. A boundary point of R need not belong to R.

DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



$\{(x, y) \mid x^2 + y^2 < 1\}$ Open unit disk. Every point an interior point.

$\{(x, y) \mid x^2 + y^2 = 1\}$ Boundary of unit disk. (The unit circle.)

$\{(x, y) \mid x^2 + y^2 \leq 1\}$ Closed unit disk. Contains all boundary points.

FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

As with a half-open interval of real numbers $[a, b)$, some regions in the plane are neither open nor closed. If you start with the open disk in Figure 14.3 and add to it some, but not all, of its boundary points, the resulting set is neither open nor closed. The boundary points that *are* there keep the set from being open. The absence of the remaining boundary points keeps the set from being closed.

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of finite radius. A region is **unbounded** if it is not bounded.

Examples of *bounded* sets in the plane include line segments, triangles, interiors of triangles, rectangles, circles, and disks. Examples of *unbounded* sets in the plane include lines, coordinate axes, the graphs of functions defined on infinite intervals, quadrants, half-planes, and the plane itself.

EXAMPLE 2 Describe the domain of the function $f(x, y) = \sqrt{y - x^2}$.

Solution Since f is defined only where $y - x^2 \geq 0$, the domain is the closed, unbounded region shown in Figure 14.4. The parabola $y = x^2$ is the boundary of the domain. The points above the parabola make up the domain's interior. ■

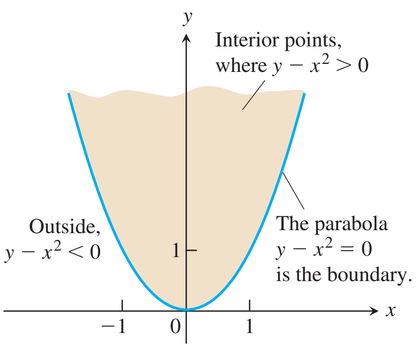


FIGURE 14.4 The domain of $f(x, y)$ in Example 2 consists of the shaded region and its bounding parabola.

Graphs, Level Curves, and Contours of Functions of Two Variables

There are two standard ways to picture the values of a function $f(x, y)$. One is to draw and label curves in the domain on which f has a constant value. The other is to sketch the surface $z = f(x, y)$ in space.

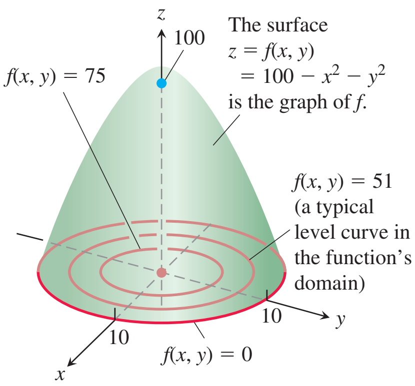


FIGURE 14.5 The graph and selected level curves of the function $f(x, y)$ in Example 3.

> **DEFINITIONS** The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $\boldsymbol{z = f(x, y)}$.

EXAMPLE 3 Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.

Solution The domain of f is the entire xy-plane, and the range of f is the set of real numbers less than or equal to 100. The graph is the paraboloid $z = 100 - x^2 - y^2$, the positive portion of which is shown in Figure 14.5.

The level curve $f(x, y) = 0$ is the set of points in the xy-plane at which

$$f(x, y) = 100 - x^2 - y^2 = 0, \qquad \text{or} \qquad x^2 + y^2 = 100,$$

which is the circle of radius 10 centered at the origin. Similarly, the level curves $f(x, y) = 51$ and $f(x, y) = 75$ (Figure 14.5) are the circles

$$f(x, y) = 100 - x^2 - y^2 = 51, \qquad \text{or} \qquad x^2 + y^2 = 49$$
$$f(x, y) = 100 - x^2 - y^2 = 75, \qquad \text{or} \qquad x^2 + y^2 = 25.$$

The level curve $f(x, y) = 100$ consists of the origin alone. (It is still a level curve.)

If $x^2 + y^2 > 100$, then the values of $f(x, y)$ are negative. For example, the circle $x^2 + y^2 = 144$, which is the circle centered at the origin with radius 12, gives the constant value $f(x, y) = -44$ and is a level curve of f. ■

The curve in space in which the plane $z = c$ cuts a surface $z = f(x, y)$ is made up of the points that represent the function value $f(x, y) = c$. It is called the **contour curve** $f(x, y) = c$ to distinguish it from the level curve $f(x, y) = c$ in the domain of f. Figure 14.6 shows the contour curve $f(x, y) = 75$ on the surface $z = 100 - x^2 - y^2$ defined by the function $f(x, y) = 100 - x^2 - y^2$. The contour curve lies directly above the circle $x^2 + y^2 = 25$, which is the level curve $f(x, y) = 75$ in the function's domain.

Not everyone makes this distinction, however, and you may wish to call both kinds of curves by a single name and rely on context to convey which one you have in mind. On most maps, for example, the curves that represent constant elevation (height above sea level) are called contours, not level curves (Figure 14.7).

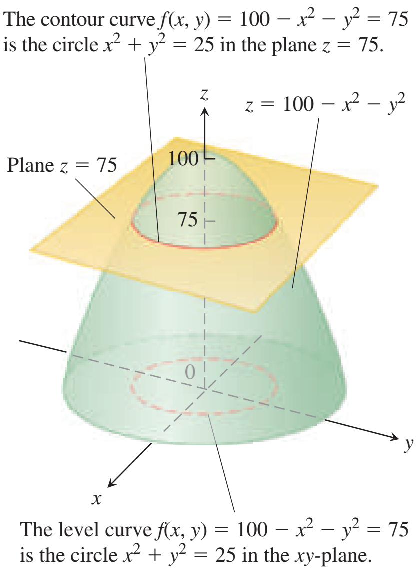


FIGURE 14.6 A plane $z = c$ parallel to the xy-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

Functions of Three Variables

In the plane, the points where a function of two independent variables has a constant value $f(x, y) = c$ make a curve in the function's domain. In space, the points where a function of three independent variables has a constant value $f(x, y, z) = c$ make a surface in the function's domain.

> **DEFINITION** The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

Since the graphs of functions of three variables consist of points $(x, y, z, f(x, y, z))$ lying in a four-dimensional space, we cannot sketch them effectively in our three-dimensional frame of reference. We can see how the function behaves, however, by looking at its three-dimensional level surfaces.

EXAMPLE 4 Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

FIGURE 14.7 Contours on Mt. Washington in New Hampshire. (Reprinted by permission of the Appalachian Mountain Club.)

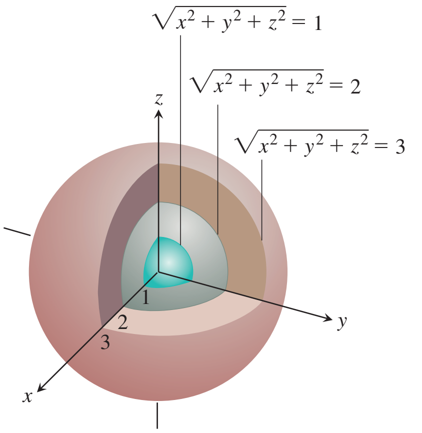
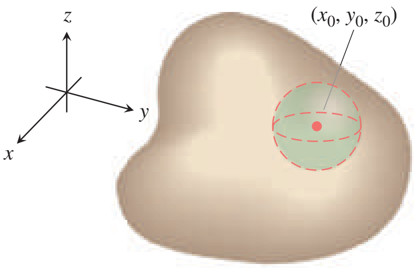


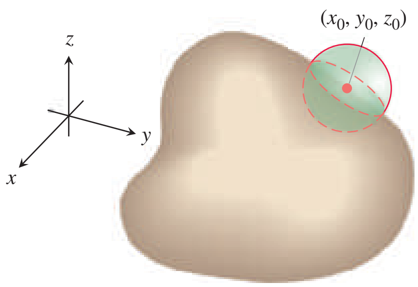
FIGURE 14.8 The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres (Example 4).

Solution The value of f is the distance from the origin to the point (x, y, z). Each level surface $\sqrt{x^2 + y^2 + z^2} = c$, $c > 0$, is a sphere of radius c centered at the origin. Figure 14.8 shows a cutaway view of three of these spheres. The level surface $\sqrt{x^2 + y^2 + z^2} = 0$ consists of the origin alone.

We are not graphing the function here; we are looking at level surfaces in the function's domain. The level surfaces show how the function's values change as we move through its domain. If we remain on a sphere of radius c centered at the origin, the function maintains a constant value, namely c. If we move from a point on one sphere to a point on another, the function's value changes. It increases if we move away from the origin and decreases if we move toward the origin. The way the values change depends on the direction we take. The dependence of change on direction is important. We return to it in Section 14.5. ■



(a) Interior point



(b) Boundary point

FIGURE 14.9 Interior points and boundary points of a region in space. As with regions in the plane, a boundary point need not belong to the space region R.

The definitions of interior, boundary, open, closed, bounded, and unbounded for regions in space are similar to those for regions in the plane. To accommodate the extra dimension, we use solid balls of positive radius instead of disks.

> **DEFINITIONS** A point (x_0, y_0, z_0) in a region R in space is an **interior point** of R if it is the center of a solid ball that lies entirely in R (Figure 14.9a). A point (x_0, y_0, z_0) is a **boundary point** of R if every solid ball centered at (x_0, y_0, z_0) contains points that lie outside of R as well as points that lie inside R (Figure 14.9b). The **interior** of R is the set of interior points of R. The **boundary** of R is the set of boundary points of R.
>
> A region is **open** if it consists entirely of interior points. A region is **closed** if it contains its entire boundary.

Examples of *open* sets in space include the interior of a sphere, the open half-space $z > 0$, the first octant (where x, y, and z are all positive), and space itself. Examples of *closed* sets in space include lines, planes, and the closed half-space $z \geq 0$. A solid sphere with part of its boundary removed or a solid cube with a missing face, edge, or corner point is *neither open nor closed*.

Functions of more than three independent variables are also important. For example, the temperature on a surface in space may depend not only on the location of the point $P(x, y, z)$ on the surface but also on the time t when it is visited, so we would write $T = f(x, y, z, t)$.

Computer Graphing

Three-dimensional graphing software makes it possible to graph functions of two variables. We can often get information more quickly from a graph than from a formula, since the surfaces reveal increasing and decreasing behavior, and high points or low points.

EXAMPLE 5 The temperature w beneath the Earth's surface is a function of the depth x beneath the surface and the time t of the year. If we measure x in feet and t as the number of days elapsed from the expected date of the yearly highest surface temperature, we can model the variation in temperature with the function

$$w = \cos(1.7 \times 10^{-2}t - 0.2x)e^{-0.2x}.$$

(The temperature at 0 ft is scaled to vary from $+1$ to -1, so that the variation at x feet can be interpreted as a fraction of the variation at the surface.)

Figure 14.10 shows a graph of the function. At a depth of 15 ft, the variation (change in vertical amplitude in the figure) is about 5% of the surface variation. At 25 ft, there is almost no variation during the year.

The graph also shows that the temperature 15 ft below the surface is about half a year out of phase with the surface temperature. When the temperature is lowest on the surface (late January, say), it is at its highest 15 ft below. Fifteen feet below the ground, the seasons are reversed. ■

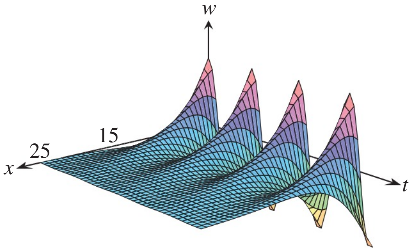
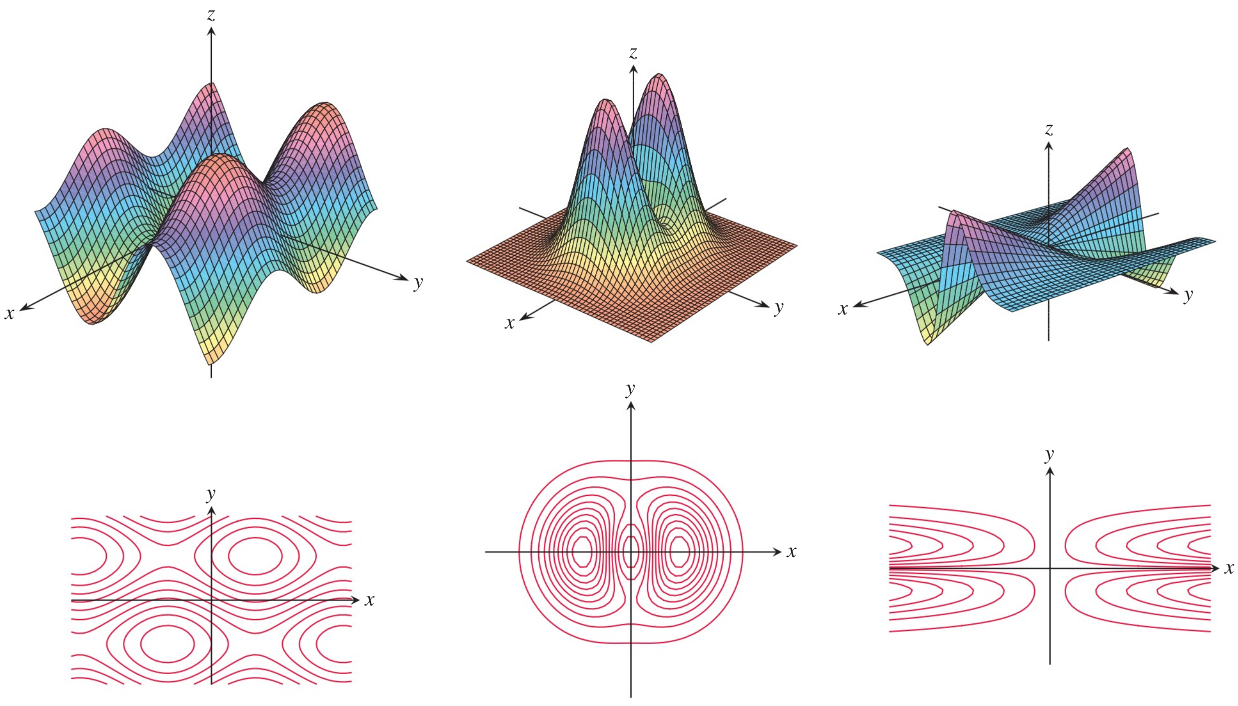


FIGURE 14.10 This graph shows the seasonal variation of the temperature below ground as a fraction of surface temperature (Example 5).

Figure 14.11 shows computer-generated graphs of a number of functions of two variables together with their level curves.



(a) $z = \sin x + 2\sin y$ (b) $z = (4x^2 + y^2)e^{-x^2-y^2}$ (c) $z = xye^{-y^2}$

FIGURE 14.11 Computer-generated graphs and level curves of typical functions of two variables.

Exercises 14.1

Domain, Range, and Level Curves

In Exercises 1–4, find the specific function values.

1. $f(x, y) = x^2 + xy^3$

a. $f(0, 0)$ **b.** $f(-1, 1)$

c. $f(2, 3)$ **d.** $f(-3, -2)$

2. $f(x, y) = \sin(xy)$

a. $f\left(2, \frac{\pi}{6}\right)$ **b.** $f\left(-3, \frac{\pi}{12}\right)$

c. $f\left(\pi, \frac{1}{4}\right)$ **d.** $f\left(-\frac{\pi}{2}, -7\right)$

3. $f(x, y, z) = \dfrac{x - y}{y^2 + z^2}$

a. $f(3, -1, 2)$ **b.** $f\left(1, \frac{1}{2}, -\frac{1}{4}\right)$

c. $f\left(0, -\frac{1}{3}, 0\right)$ **d.** $f(2, 2, 100)$

4. $f(x, y, z) = \sqrt{49 - x^2 - y^2 - z^2}$

a. $f(0, 0, 0)$ **b.** $f(2, -3, 6)$

c. $f(-1, 2, 3)$ **d.** $f\left(\frac{4}{\sqrt{2}}, \frac{5}{\sqrt{2}}, \frac{6}{\sqrt{2}}\right)$

In Exercises 5–12, find and sketch the domain for each function.

5. $f(x, y) = \sqrt{y - x - 2}$

6. $f(x, y) = \ln(x^2 + y^2 - 4)$

7. $f(x, y) = \dfrac{(x - 1)(y + 2)}{(y - x)(y - x^3)}$

8. $f(x, y) = \dfrac{\sin(xy)}{x^2 + y^2 - 25}$

9. $f(x, y) = \cos^{-1}(y - x^2)$

10. $f(x, y) = \ln(xy + x - y - 1)$

11. $f(x, y) = \sqrt{(x^2 - 4)(y^2 - 9)}$

12. $f(x, y) = \dfrac{1}{\ln(4 - x^2 - y^2)}$

In Exercises 13–16, find and sketch the level curves $f(x, y) = c$ on the same set of coordinate axes for the given values of c. We refer to these level curves as a contour map.

13. $f(x, y) = x + y - 1, \quad c = -3, -2, -1, 0, 1, 2, 3$

14. $f(x, y) = x^2 + y^2, \quad c = 0, 1, 4, 9, 16, 25$

15. $f(x, y) = xy, \quad c = -9, -4, -1, 0, 1, 4, 9$

16. $f(x, y) = \sqrt{25 - x^2 - y^2}, \quad c = 0, 1, 2, 3, 4$

In Exercises 17–30, **(a)** find the function's domain, **(b)** find the function's range, **(c)** describe the function's level curves, **(d)** find the boundary of the function's domain, **(e)** determine if the domain is an open region, a closed region, or neither, and **(f)** decide if the domain is bounded or unbounded.

17. $f(x, y) = y - x$ **18.** $f(x, y) = \sqrt{y - x}$

19. $f(x, y) = 4x^2 + 9y^2$ **20.** $f(x, y) = x^2 - y^2$

21. $f(x, y) = xy$ **22.** $f(x, y) = y/x^2$

23. $f(x, y) = \dfrac{1}{\sqrt{16 - x^2 - y^2}}$ **24.** $f(x, y) = \sqrt{9 - x^2 - y^2}$

25. $f(x, y) = \ln(x^2 + y^2)$ **26.** $f(x, y) = e^{-(x^2+y^2)}$

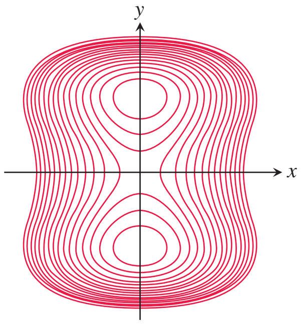
27. $f(x, y) = \sin^{-1}(y - x)$ **28.** $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$

29. $f(x, y) = \ln(x^2 + y^2 - 1)$ **30.** $f(x, y) = \ln(9 - x^2 - y^2)$

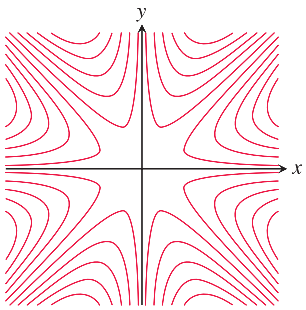
Matching Surfaces with Level Curves

Exercises 31–36 show level curves for the functions graphed in (a)–(f) on the following page. Match each set of curves with the appropriate function.

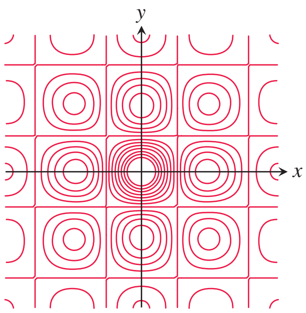
31.



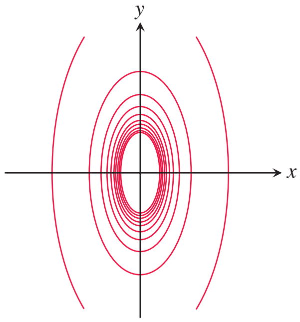
32.



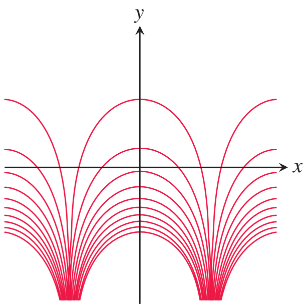
33.



34.



35.



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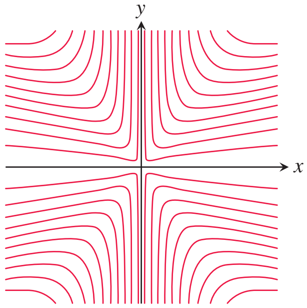
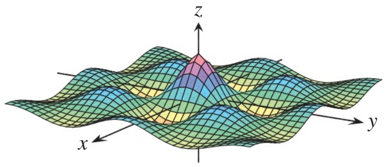
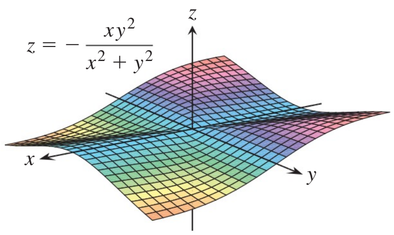


a.



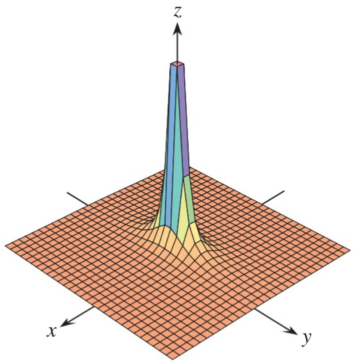
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2 + y^2}/4}$

b.



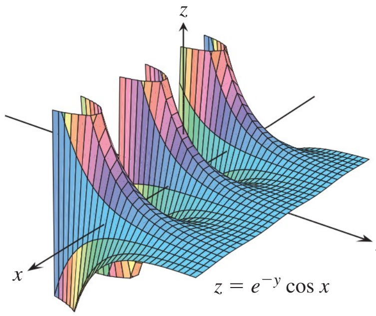
$z = -\dfrac{xy^2}{x^2 + y^2}$

c.



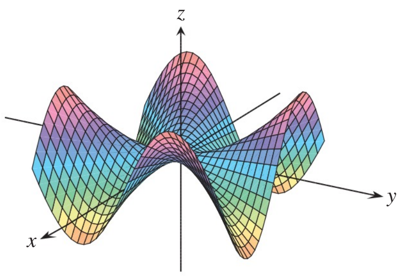
$z = \dfrac{1}{4x^2 + y^2}$

d.



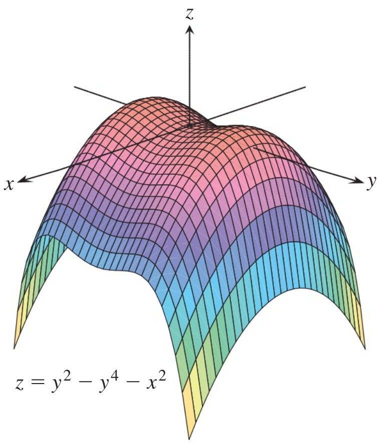
$z = e^{-y} \cos x$

e.



$z = \dfrac{xy(x^2 - y^2)}{x^2 + y^2}$

f.



$z = y^2 - y^4 - x^2$

Functions of Two Variables

Display the values of the functions in Exercises 37–48 in two ways: **(a)** by sketching the surface $z = f(x, y)$ and **(b)** by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.

37. $f(x, y) = y^2$

38. $f(x, y) = \sqrt{x}$

39. $f(x, y) = x^2 + y^2$

40. $f(x, y) = \sqrt{x^2 + y^2}$

41. $f(x, y) = x^2 - y$

42. $f(x, y) = 4 - x^2 - y^2$

43. $f(x, y) = 4x^2 + y^2$

44. $f(x, y) = 6 - 2x - 3y$

45. $f(x, y) = 1 - |y|$

46. $f(x, y) = 1 - |x| - |y|$

47. $f(x, y) = \sqrt{x^2 + y^2 + 4}$

48. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

Finding Level Curves

In Exercises 49–52, find an equation for and sketch the graph of the level curve of the function $f(x, y)$ that passes through the given point.

49. $f(x, y) = 16 - x^2 - y^2, \quad \left(2\sqrt{2}, \sqrt{2}\right)$

50. $f(x, y) = \sqrt{x^2 - 1}, \quad (1, 0)$

51. $f(x, y) = \sqrt{x + y^2 - 3}, \quad (3, -1)$

52. $f(x, y) = \dfrac{2y - x}{x + y + 1}, \quad (-1, 1)$

Sketching Level Surfaces

In Exercises 53–60, sketch a typical level surface for the function.

53. $f(x, y, z) = x^2 + y^2 + z^2$

54. $f(x, y, z) = \ln\left(x^2 + y^2 + z^2\right)$

55. $f(x, y, z) = x + z$

56. $f(x, y, z) = z$

57. $f(x, y, z) = x^2 + y^2$

58. $f(x, y, z) = y^2 + z^2$

59. $f(x, y, z) = z - x^2 - y^2$

60. $f(x, y, z) = \left(x^2/25\right) + \left(y^2/16\right) + \left(z^2/9\right)$

Finding Level Surfaces

In Exercises 61–64, find an equation for the level surface of the function through the given point.

61. $f(x, y, z) = \sqrt{x - y} - \ln z, \quad (3, -1, 1)$

62. $f(x, y, z) = \ln\left(x^2 + y + z^2\right), \quad (-1, 2, 1)$

63. $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \quad \left(1, -1, \sqrt{2}\right)$

64. $g(x, y, z) = \dfrac{x - y + z}{2x + y - z}, \quad (1, 0, -2)$

In Exercises 65–68, find and sketch the domain of f. Then find an equation for the level curve or surface of the function passing through the given point.

65. $f(x, y) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{x}{y}\right)^n, \quad (1, 2)$

66. $g(x, y, z) = \displaystyle\sum_{n=0}^{\infty} \frac{(x + y)^n}{n! z^n}, \quad (\ln 4, \ln 9, 2)$

67. $f(x, y) = \displaystyle\int_x^y \frac{d\theta}{\sqrt{1 - \theta^2}}, \quad (0, 1)$

68. $g(x, y, z) = \displaystyle\int_x^y \frac{dt}{1 + t^2} + \int_0^z \frac{d\theta}{\sqrt{4 - \theta^2}}, \quad \left(0, 1, \sqrt{3}\right)$

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for each of the functions in Exercises 69–72.

a. Plot the surface over the given rectangle.

b. Plot several level curves in the rectangle.

c. Plot the level curve of f through the given point.

69. $f(x, y) = x \sin \dfrac{y}{2} + y \sin 2x, \quad 0 \le x \le 5\pi, \quad 0 \le y \le 5\pi,$ $P(3\pi, 3\pi)$

70. $f(x, y) = (\sin x)(\cos y)e^{\sqrt{x^2+y^2}/8}, \quad 0 \le x \le 5\pi,$ $0 \le y \le 5\pi, \quad P(4\pi, 4\pi)$

71. $f(x, y) = \sin(x + 2\cos y), \quad -2\pi \le x \le 2\pi,$ $-2\pi \le y \le 2\pi, \quad P(\pi, \pi)$

72. $f(x, y) = e^{(x^{0.1} - y)} \sin(x^2 + y^2), \quad 0 \le x \le 2\pi,$ $-2\pi \le y \le \pi, \quad P(\pi, -\pi)$

Use a CAS to plot the implicitly defined level surfaces in Exercises 73–76.

73. $4 \ln(x^2 + y^2 + z^2) = 1$ **74.** $x^2 + z^2 = 1$

75. $x + y^2 - 3z^2 = 1$

76. $\sin\left(\dfrac{x}{2}\right) - (\cos y)\sqrt{x^2 + z^2} = 2$

Parametrized Surfaces Just as you describe curves in the plane parametrically with a pair of equations $x = f(t)$, $y = g(t)$ defined on some parameter interval I, you can sometimes describe surfaces in space with a triple of equations $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ defined on some parameter rectangle $a \le u \le b$, $c \le v \le d$. Many computer algebra systems permit you to plot such surfaces in *parametric mode*. (Parametrized surfaces are discussed in detail in Section 16.5.) Use a CAS to plot the surfaces in Exercises 77–80. Also plot several level curves in the xy-plane.

77. $x = u\cos v, \quad y = u \sin v, \quad z = u, \quad 0 \le u \le 2,$ $0 \le v \le 2\pi$

78. $x = u\cos v, \quad y = u\sin v, \quad z = v, \quad 0 \le u \le 2,$ $0 \le v \le 2\pi$

79. $x = (2 + \cos u)\cos v, \quad y = (2 + \cos u)\sin v, \quad z = \sin u,$ $0 \le u \le 2\pi, \quad 0 \le v \le 2\pi$

80. $x = 2\cos u \cos v, \quad y = 2\cos u \sin v, \quad z = 2\sin u,$ $0 \le u \le 2\pi, \quad 0 \le v \le \pi$

14.2 Limits and Continuity in Higher Dimensions

This section treats limits and continuity for multivariable functions. These ideas are analogous to limits and continuity for single-variable functions, but including more independent variables leads to additional complexity and important differences requiring some new ideas.

Limits for Functions of Two Variables

If the values of $f(x, y)$ lie arbitrarily close to a fixed real number L for all points (x, y) sufficiently close to a point (x_0, y_0), we say that f approaches the limit L as (x, y) approaches (x_0, y_0). This is similar to the informal definition for the limit of a function of a single variable. Notice, however, that if (x_0, y_0) lies in the interior of f's domain, (x, y) can approach (x_0, y_0) from any direction. For the limit to exist, the same limiting value must be obtained whatever direction of approach is taken. We illustrate this issue in several examples following the definition.

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

The definition of limit says that the distance between $f(x, y)$ and L becomes arbitrarily small whenever the distance from (x, y) to (x_0, y_0) is made sufficiently small (but not 0). The definition applies to interior points (x_0, y_0) as well as boundary points of the domain of f, although a boundary point need not lie within the domain. The points (x, y) that approach (x_0, y_0) are always taken to be in the domain of f. See Figure 14.12.

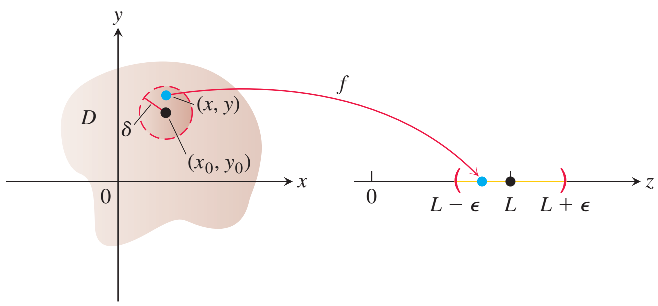


FIGURE 14.12 In the limit definition, δ is the radius of a disk centered at (x_0, y_0). For all points (x, y) within this disk, the function values $f(x, y)$ lie inside the corresponding interval $(L - \epsilon, L + \epsilon)$.

As for functions of a single variable, it can be shown that

$$\lim_{(x, y)\to(x_0, y_0)} x = x_0$$

$$\lim_{(x, y)\to(x_0, y_0)} y = y_0$$

$$\lim_{(x, y)\to(x_0, y_0)} k = k \qquad \text{(any number } k\text{)}.$$

For example, in the first limit statement above, $f(x, y) = x$ and $L = x_0$. Using the definition of limit, suppose that $\epsilon > 0$ is chosen. If we let δ equal this ϵ, we see that

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta = \epsilon$$

implies

$$\begin{aligned} \sqrt{(x - x_0)^2} &< \epsilon && (x - x_0)^2 \le (x - x_0)^2 + (y - y_0)^2 \\ |x - x_0| &< \epsilon && \sqrt{a^2} = |a| \\ |f(x, y) - x_0| &< \epsilon. && x = f(x, y) \end{aligned}$$

That is,

$$|f(x, y) - x_0| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$

So a δ has been found satisfying the requirement of the definition, and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = \lim_{(x,y)\to(x_0,y_0)} x = x_0.$$

As with single-variable functions, the limit of the sum of two functions is the sum of their limits (when they both exist), with similar results for the limits of the differences, constant multiples, products, quotients, powers, and roots.

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) \cdot g(x,y)) = L \cdot M$
5. *Quotient Rule:* $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

While we won't prove Theorem 1 here, we give an informal discussion of why it's true. If (x, y) is sufficiently close to (x_0, y_0), then $f(x, y)$ is close to L and $g(x, y)$ is close to M (from the informal interpretation of limits). It is then reasonable that $f(x, y) + g(x, y)$ is close to $L + M$; $f(x, y) - g(x, y)$ is close to $L - M$; $kf(x, y)$ is close to kL; $f(x, y)g(x, y)$ is close to LM; and $f(x, y)/g(x, y)$ is close to L/M if $M \neq 0$.

When we apply Theorem 1 to polynomials and rational functions, we obtain the useful result that the limits of these functions as $(x, y) \to (x_0, y_0)$ can be calculated by evaluating the functions at (x_0, y_0). The only requirement is that the rational functions be defined at (x_0, y_0).

EXAMPLE 1 In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the x- and y-values of the point being approached into the functional expression to find the limiting value.

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$ ■

EXAMPLE 2 Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} && \text{Multiply by a form equal to 1.} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} && \text{Algebra} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) && \text{Cancel the nonzero factor } (x - y). \\
&= 0(\sqrt{0} + \sqrt{0}) = 0 && \text{Known limit values}
\end{aligned}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (along which $x - y = 0$) is *not* in the domain of the function

$$f(x, y) = \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

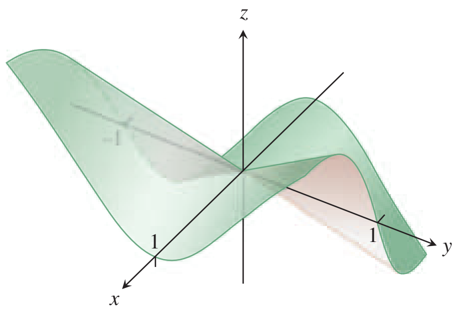


FIGURE 14.13 The surface graph shows the limit of the function in Example 3 must be 0, if it exists.

EXAMPLE 3 Find $\lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2 + y^2}$ if it exists.

Solution We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$, the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches $(0, 0)$, the value of the limit must be 0 (see Figure 14.13). To see if this is true, we apply the definition of limit.

Let $\epsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2 + y^2} < \delta$$

or

$$\frac{4|x|y^2}{x^2 + y^2} < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{x^2 + y^2} < \delta.$$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2 + y^2} \leq 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2 + y^2}. \qquad \frac{y^2}{x^2 + y^2} \leq 1$$

So if we choose $\delta = \epsilon/4$ and let $0 < \sqrt{x^2 + y^2} < \delta$, we get

$$\left| \frac{4xy^2}{x^2 + y^2} - 0 \right| \leq 4\sqrt{x^2 + y^2} < 4\delta = 4\left(\frac{\epsilon}{4}\right) = \epsilon.$$

It follows from the definition that

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2} = 0.$$

EXAMPLE 4 If $f(x, y) = \dfrac{y}{x}$, does $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ exist?

Solution The domain of f does not include the y-axis, so we do not consider any points (x, y) where $x = 0$ in the approach toward the origin $(0, 0)$. Along the x-axis, the value of the function is $f(x, 0) = 0$ for all $x \neq 0$. So if the limit does exist as $(x, y) \to (0, 0)$, the value of the limit must be $L = 0$. On the other hand, along the line $y = x$, the value of the function is $f(x, x) = x/x = 1$ for all $x \neq 0$. That is, the function f approaches the value 1 along the line $y = x$. This means that for every disk of radius δ centered at $(0, 0)$, the disk will contain points $(x, 0)$ on the x-axis where the value of the function is 0, and also points (x, x) along the line $y = x$ where the value of the function is 1. So no matter how small we choose δ as the radius of the disk in Figure 14.12, there will be points within the disk for which the function values differ by 1. Therefore, the limit cannot exist because we can take ϵ to be any number less than 1 in the limit definition and deny that $L = 0$ or 1, or any other real number. The limit does not exist because we have different limiting values along different paths approaching the point $(0, 0)$. ■

Continuity

As with functions of a single variable, continuity is defined in terms of limits.

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

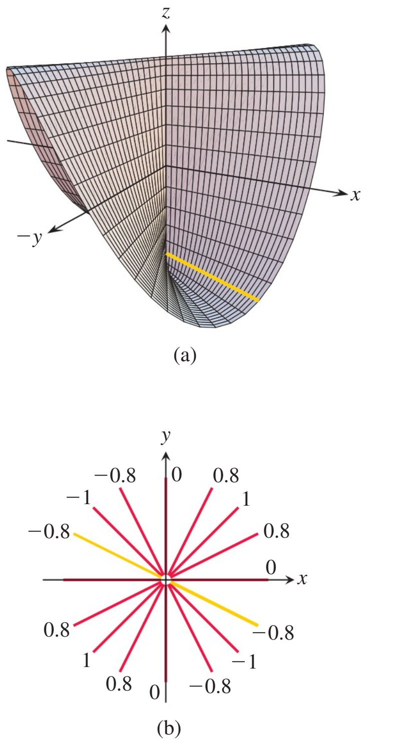


FIGURE 14.14 (a) The graph of

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$$

The function is continuous at every point except the origin. (b) The values of f are different constants along each line $y = mx$, $x \neq 0$ (Example 5).

As with the definition of limit, the definition of continuity applies at boundary points as well as interior points of the domain of f. The only requirement is that each point (x, y) near (x_0, y_0) be in the domain of f.

A consequence of Theorem 1 is that algebraic combinations of continuous functions are continuous at every point at which all the functions involved are defined. This means that sums, differences, constant multiples, products, quotients, and powers of continuous functions are continuous where defined. In particular, polynomials and rational functions of two variables are continuous at every point at which they are defined.

EXAMPLE 5 Show that

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

is continuous at every point except the origin (Figure 14.14).

Solution The function f is continuous at any point $(x, y) \neq (0, 0)$ because its values are then given by a rational function of x and y and the limiting value is obtained by substituting the values of x and y into the functional expression.

At (0, 0), the value of f is defined, but f, we claim, has no limit as $(x, y) \to (0, 0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

For every value of m, the function f has a constant value on the "punctured" line $y = mx$, $x \neq 0$, because

$$f(x, y)\Big|_{y=mx} = \frac{2xy}{x^2 + y^2}\Big|_{y=mx} = \frac{2x(mx)}{x^2 + (mx)^2} = \frac{2mx^2}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}.$$

Therefore, f has this number as its limit as (x, y) approaches (0, 0) along the line:

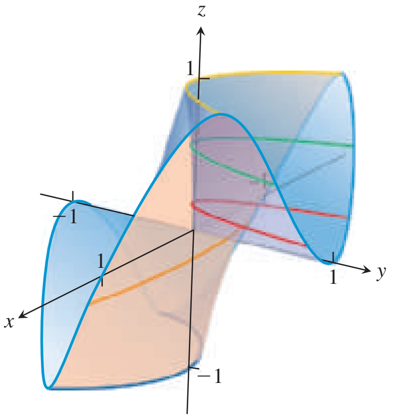
$$\lim_{\substack{(x, y)\to(0,0)\\ \text{along } y=mx}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[f(x, y)\Big|_{y=mx} \right] = \frac{2m}{1 + m^2}.$$

This limit changes with each value of the slope m. There is therefore no single number we may call the limit of f as (x, y) approaches the origin. The limit fails to exist, and the function is not continuous. ■

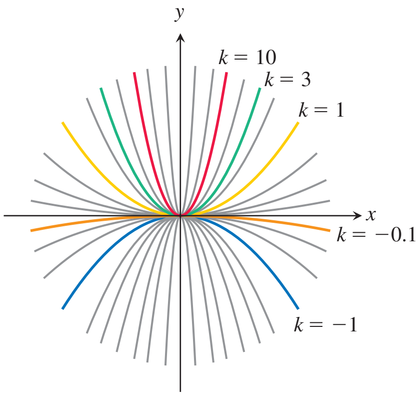
Examples 4 and 5 illustrate an important point about limits of functions of two or more variables. For a limit to exist at a point, the limit must be the same along every approach path. This result is analogous to the single-variable case where both the left- and right-sided limits had to have the same value. For functions of two or more variables, if we ever find paths with different limits, we know the function has no limit at the point they approach.

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ does not exist.



(a)



(b)

FIGURE 14.15 (a) The graph of $f(x, y) = 2x^2y/(x^4 + y^2)$. (b) Along each path $y = kx^2$ the value of f is constant, but varies with k (Example 6).

EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.15) has no limit as (x, y) approaches (0, 0).

Solution The limit cannot be found by direct substitution, which gives the indeterminate form $0/0$. We examine the values of f along parabolic curves that end at (0, 0). Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\Big|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\Big|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y)\to(0,0)\\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[f(x, y)\Big|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches (0, 0) along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches (0, 0) along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches (0, 0). ■

It can be shown that the function in Example 6 has limit 0 along every path $y = mx$ (Exercise 53). We conclude that

Having the same limit along all straight lines approaching (x_0, y_0) does not imply a limit exists at (x_0, y_0).

Whenever it is correctly defined, the composite of continuous functions is also continuous. The only requirement is that each function be continuous where it is applied. The proof, omitted here, is similar to that for functions of a single variable (Theorem 9 in Section 2.5).

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

For example, the composite functions

$$e^{x-y}, \qquad \cos \frac{xy}{x^2 + 1}, \qquad \ln \left(1 + x^2y^2\right)$$

are continuous at every point (x, y).

Functions of More Than Two Variables

The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables. Functions like

$$\ln (x + y + z) \qquad \text{and} \qquad \frac{y \sin z}{x - 1}$$

are continuous throughout their domains, and limits like

$$\lim_{P \to (1,0,-1)} \frac{e^{x+z}}{z^2 + \cos \sqrt{xy}} = \frac{e^{1-1}}{(-1)^2 + \cos 0} = \frac{1}{2},$$

where P denotes the point (x, y, z), may be found by direct substitution.

Extreme Values of Continuous Functions on Closed, Bounded Sets

The Extreme Value Theorem (Theorem 1, Section 4.1) states that a function of a single variable that is continuous throughout a closed, bounded interval $[a, b]$ takes on an absolute maximum value and an absolute minimum value at least once in $[a, b]$. The same holds true of a function $z = f(x, y)$ that is continuous on a closed, bounded set R in the plane (like a line segment, a disk, or a filled-in triangle). The function takes on an absolute maximum value at some point in R and an absolute minimum value at some point in R. The function may take on a maximum or minimum value more than once over R.

Similar results hold for functions of three or more variables. A continuous function $w = f(x, y, z)$, for example, must take on absolute maximum and minimum values on any closed, bounded set (solid ball or cube, spherical shell, rectangular solid) on which it is defined. We will learn how to find these extreme values in Section 14.7.

Exercises 14.2

Limits with Two Variables

Find the limits in Exercises 1–12.

1. $\displaystyle\lim_{(x, y) \to (0,0)} \frac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

2. $\displaystyle\lim_{(x, y) \to (0,4)} \frac{x}{\sqrt{y}}$

3. $\displaystyle\lim_{(x, y) \to (3,4)} \sqrt{x^2 + y^2 - 1}$

4. $\displaystyle\lim_{(x, y) \to (2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^2$

5. $\displaystyle\lim_{(x, y) \to (0,\pi/4)} \sec x \tan y$

6. $\displaystyle\lim_{(x, y) \to (0,0)} \cos \frac{x^2 + y^3}{x + y + 1}$

7. $\lim\limits_{(x,y)\to(0,\ln 2)} e^{x-y}$

8. $\lim\limits_{(x,y)\to(1,1)} \ln\left|1 + x^2y^2\right|$

9. $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$

10. $\lim\limits_{(x,y)\to(1/27,\,\pi^3)} \cos\sqrt[3]{xy}$

11. $\lim\limits_{(x,y)\to(1,\,\pi/6)} \dfrac{x \sin y}{x^2 + 1}$

12. $\lim\limits_{(x,y)\to(\pi/2,0)} \dfrac{\cos y + 1}{y - \sin x}$

Limits of Quotients

Find the limits in Exercises 13–24 by rewriting the fractions first.

13. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - 2xy + y^2}{x - y}$

14. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq y}} \dfrac{x^2 - y^2}{x - y}$

15. $\lim\limits_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \dfrac{xy - y - 2x + 2}{x - 1}$

16. $\lim\limits_{\substack{(x,y)\to(2,-4)\\ x\neq -4,\, x\neq x^2}} \dfrac{y + 4}{x^2y - xy + 4x^2 - 4x}$

17. $\lim\limits_{\substack{(x,y)\to(0,0)\\ x\neq y}} \dfrac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$

18. $\lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{x + y - 4}{\sqrt{x + y} - 2}$

19. $\lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \dfrac{\sqrt{2x - y} - 2}{2x - y - 4}$

20. $\lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x} - \sqrt{y + 1}}{x - y - 1}$

21. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2 + y^2\right)}{x^2 + y^2}$

22. $\lim\limits_{(x,y)\to(0,0)} \dfrac{1 - \cos(xy)}{xy}$

23. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{x^3 + y^3}{x + y}$

24. $\lim\limits_{(x,y)\to(2,2)} \dfrac{x - y}{x^4 - y^4}$

Limits with Three Variables

Find the limits in Exercises 25–30.

25. $\lim\limits_{P\to(1,3,4)} \left(\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}\right)$

26. $\lim\limits_{P\to(1,-1,-1)} \dfrac{2xy + yz}{x^2 + z^2}$

27. $\lim\limits_{P\to(\pi,\pi,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

28. $\lim\limits_{P\to(-1/4,\pi/2,2)} \tan^{-1} xyz$

29. $\lim\limits_{P\to(\pi,0,3)} ze^{-2y}\cos 2x$

30. $\lim\limits_{P\to(2,-3,6)} \ln\sqrt{x^2 + y^2 + z^2}$

Continuity for Two Variables

At what points (x, y) in the plane are the functions in Exercises 31–34 continuous?

31. a. $f(x, y) = \sin(x + y)$ **b.** $f(x, y) = \ln\left(x^2 + y^2\right)$

32. a. $f(x, y) = \dfrac{x + y}{x - y}$ **b.** $f(x, y) = \dfrac{y}{x^2 + 1}$

33. a. $g(x, y) = \sin\dfrac{1}{xy}$ **b.** $g(x, y) = \dfrac{x + y}{2 + \cos x}$

34. a. $g(x, y) = \dfrac{x^2 + y^2}{x^2 - 3x + 2}$ **b.** $g(x, y) = \dfrac{1}{x^2 - y}$

Continuity for Three Variables

At what points (x, y, z) in space are the functions in Exercises 35–40 continuous?

35. a. $f(x, y, z) = x^2 + y^2 - 2z^2$

b. $f(x, y, z) = \sqrt{x^2 + y^2 - 1}$

36. a. $f(x, y, z) = \ln xyz$ **b.** $f(x, y, z) = e^{x+y}\cos z$

37. a. $h(x, y, z) = xy\sin\dfrac{1}{z}$ **b.** $h(x, y, z) = \dfrac{1}{x^2 + z^2 - 1}$

38. a. $h(x, y, z) = \dfrac{1}{|y| + |z|}$ **b.** $h(x, y, z) = \dfrac{1}{|xy| + |z|}$

39. a. $h(x, y, z) = \ln\left(z - x^2 - y^2 - 1\right)$

b. $h(x, y, z) = \dfrac{1}{z - \sqrt{x^2 + y^2}}$

40. a. $h(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$

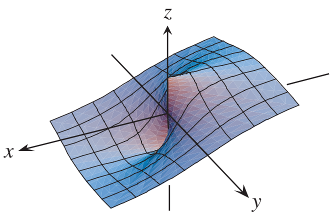
b. $h(x, y, z) = \dfrac{1}{4 - \sqrt{x^2 + y^2 + z^2 - 9}}$

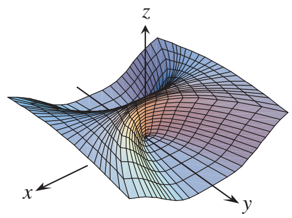
No Limit Exists at the Origin

By considering different paths of approach, show that the functions in Exercises 41–48 have no limit as $(x, y) \to (0, 0)$.

41. $f(x, y) = -\dfrac{x}{\sqrt{x^2 + y^2}}$

42. $f(x, y) = \dfrac{x^4}{x^4 + y^2}$





43. $f(x, y) = \dfrac{x^4 - y^2}{x^4 + y^2}$

44. $f(x, y) = \dfrac{xy}{|xy|}$

45. $g(x, y) = \dfrac{x - y}{x + y}$

46. $g(x, y) = \dfrac{x^2 - y}{x - y}$

47. $h(x, y) = \dfrac{x^2 + y}{y}$

48. $h(x, y) = \dfrac{x^2y}{x^4 + y^2}$

Theory and Examples

In Exercises 49 and 50, show that the limits do not exist.

49. $\lim\limits_{(x,y)\to(1,1)} \dfrac{xy^2 - 1}{y - 1}$

50. $\lim\limits_{(x,y)\to(1,-1)} \dfrac{xy + 1}{x^2 - y^2}$

51. Let $f(x, y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

a. $\lim\limits_{(x,y)\to(0,1)} f(x, y)$

b. $\lim\limits_{(x,y)\to(2,3)} f(x, y)$

c. $\lim\limits_{(x,y)\to(0,0)} f(x, y)$

52. Let $f(x, y) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$.

Find the following limits.

a. $\displaystyle\lim_{(x, y)\to(3, -2)} f(x, y)$

b. $\displaystyle\lim_{(x, y)\to(-2, 1)} f(x, y)$

c. $\displaystyle\lim_{(x, y)\to(0,0)} f(x, y)$

53. Show that the function in Example 6 has limit 0 along every straight line approaching (0, 0).

54. If $f(x_0, y_0) = 3$, what can you say about

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y)$$

if f is continuous at (x_0, y_0)? If f is not continuous at (x_0, y_0)? Give reasons for your answers.

The Sandwich Theorem for functions of two variables states that if $g(x, y) \leq f(x, y) \leq h(x, y)$ for all $(x, y) \neq (x_0, y_0)$ in a disk centered at (x_0, y_0) and if g and h have the same finite limit L as $(x, y) \to (x_0, y_0)$, then

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L.$$

Use this result to support your answers to the questions in Exercises 55–58.

55. Does knowing that

$$1 - \frac{x^2y^2}{3} < \frac{\tan^{-1} xy}{xy} < 1$$

tell you anything about

$$\lim_{(x, y)\to(0,0)} \frac{\tan^{-1} xy}{xy}?$$

Give reasons for your answer.

56. Does knowing that

$$2|xy| - \frac{x^2y^2}{6} < 4 - 4\cos\sqrt{|xy|} < 2|xy|$$

tell you anything about

$$\lim_{(x, y)\to(0,0)} \frac{4 - 4\cos\sqrt{|xy|}}{|xy|}?$$

Give reasons for your answer.

57. Does knowing that $|\sin(1/x)| \leq 1$ tell you anything about

$$\lim_{(x, y)\to(0,0)} y \sin\frac{1}{x}?$$

Give reasons for your answer.

58. Does knowing that $|\cos(1/y)| \leq 1$ tell you anything about

$$\lim_{(x, y)\to(0,0)} x \cos\frac{1}{y}?$$

Give reasons for your answer.

59. (*Continuation of Example 5.*)

a. Reread Example 5. Then substitute $m = \tan\theta$ into the formula

$$f(x, y)\Big|_{y=mx} = \frac{2m}{1 + m^2}$$

and simplify the result to show how the value of f varies with the line's angle of inclination.

b. Use the formula you obtained in part (a) to show that the limit of f as $(x, y) \to (0, 0)$ along the line $y = mx$ varies from -1 to 1 depending on the angle of approach.

60. Continuous extension Define $f(0, 0)$ in a way that extends

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$$

to be continuous at the origin.

Changing Variables to Polar Coordinates

If you cannot make any headway with $\lim_{(x, y)\to(0,0)} f(x, y)$ in rectangular coordinates, try changing to polar coordinates. Substitute $x = r\cos\theta$, $y = r\sin\theta$, and investigate the limit of the resulting expression as $r \to 0$. In other words, try to decide whether there exists a number L satisfying the following criterion:

Given $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad\Rightarrow\quad |f(r, \theta) - L| < \epsilon. \tag{1}$$

If such an L exists, then

$$\lim_{(x, y)\to(0,0)} f(x, y) = \lim_{r\to 0} f(r\cos\theta, r\sin\theta) = L.$$

For instance,

$$\lim_{(x, y)\to(0,0)} \frac{x^3}{x^2 + y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta}{r^2} = \lim_{r\to 0} r\cos^3\theta = 0.$$

To verify the last of these equalities, we need to show that Equation (1) is satisfied with $f(r, \theta) = r\cos^3\theta$ and $L = 0$. That is, we need to show that given any $\epsilon > 0$, there exists a $\delta > 0$ such that for all r and θ,

$$|r| < \delta \quad\Rightarrow\quad |r\cos^3\theta - 0| < \epsilon.$$

Since

$$|r\cos^3\theta| = |r||\cos^3\theta| \leq |r| \cdot 1 = |r|,$$

the implication holds for all r and θ if we take $\delta = \epsilon$.

In contrast,

$$\frac{x^2}{x^2 + y^2} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

takes on all values from 0 to 1 regardless of how small $|r|$ is, so that $\lim_{(x, y)\to(0,0)} x^2/(x^2 + y^2)$ does not exist.

In each of these instances, the existence or nonexistence of the limit as $r \to 0$ is fairly clear. Shifting to polar coordinates does not always help, however, and may even tempt us to false conclusions. For example, the limit may exist along every straight line (or ray) $\theta =$ constant and yet fail to exist in the broader sense. Example 5 illustrates this point. In polar coordinates, $f(x, y) = (2x^2y)/(x^4 + y^2)$ becomes

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + \sin^2\theta}$$

for $r \neq 0$. If we hold θ constant and let $r \to 0$, the limit is 0. On the path $y = x^2$, however, we have $r \sin\theta = r^2 \cos^2\theta$ and

$$f(r\cos\theta, r\sin\theta) = \frac{r\cos\theta\sin 2\theta}{r^2\cos^4\theta + (r\cos^2\theta)^2}$$

$$= \frac{2r\cos^2\theta\sin\theta}{2r^2\cos^4\theta} = \frac{r\sin\theta}{r^2\cos^2\theta} = 1.$$

In Exercises 61–66, find the limit of f as $(x, y) \to (0, 0)$ or show that the limit does not exist.

61. $f(x, y) = \dfrac{x^3 - xy^2}{x^2 + y^2}$

62. $f(x, y) = \cos\left(\dfrac{x^3 - y^3}{x^2 + y^2}\right)$

63. $f(x, y) = \dfrac{y^2}{x^2 + y^2}$

64. $f(x, y) = \dfrac{2x}{x^2 + x + y^2}$

65. $f(x, y) = \tan^{-1}\left(\dfrac{|x| + |y|}{x^2 + y^2}\right)$

66. $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

In Exercises 67 and 68, define $f(0, 0)$ in a way that extends f to be continuous at the origin.

67. $f(x, y) = \ln\left(\dfrac{3x^2 - x^2y^2 + 3y^2}{x^2 + y^2}\right)$

68. $f(x, y) = \dfrac{3x^2y}{x^2 + y^2}$

Using the Limit Definition

Each of Exercises 69–74 gives a function $f(x, y)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y),

$$\sqrt{x^2 + y^2} < \delta \quad \Rightarrow \quad |f(x, y) - f(0, 0)| < \epsilon.$$

69. $f(x, y) = x^2 + y^2, \quad \epsilon = 0.01$

70. $f(x, y) = y/(x^2 + 1), \quad \epsilon = 0.05$

71. $f(x, y) = (x + y)/(x^2 + 1), \quad \epsilon = 0.01$

72. $f(x, y) = (x + y)/(2 + \cos x), \quad \epsilon = 0.02$

73. $f(x, y) = \dfrac{xy^2}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \epsilon = 0.04$

74. $f(x, y) = \dfrac{x^3 + y^4}{x^2 + y^2}$ and $f(0, 0) = 0, \quad \epsilon = 0.02$

Each of Exercises 75–78 gives a function $f(x, y, z)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x, y, z),

$$\sqrt{x^2 + y^2 + z^2} < \delta \quad \Rightarrow \quad |f(x, y, z) - f(0, 0, 0)| < \epsilon.$$

75. $f(x, y, z) = x^2 + y^2 + z^2, \quad \epsilon = 0.015$

76. $f(x, y, z) = xyz, \quad \epsilon = 0.008$

77. $f(x, y, z) = \dfrac{x + y + z}{x^2 + y^2 + z^2 + 1}, \quad \epsilon = 0.015$

78. $f(x, y, z) = \tan^2 x + \tan^2 y + \tan^2 z, \quad \epsilon = 0.03$

79. Show that $f(x, y, z) = x + y - z$ is continuous at every point (x_0, y_0, z_0).

80. Show that $f(x, y, z) = x^2 + y^2 + z^2$ is continuous at the origin.

14.3 Partial Derivatives

The calculus of several variables is similar to single-variable calculus applied to several variables one at a time. When we hold all but one of the independent variables of a function constant and differentiate with respect to that one variable, we get a "partial" derivative. This section shows how partial derivatives are defined and interpreted geometrically, and how to calculate them by applying the rules for differentiating functions of a single variable. The idea of *differentiability* for functions of several variables requires more than the existence of the partial derivatives because a point can be approached from so many different directions. However, we will see that differentiable functions of several variables behave in the same way as differentiable single-variable functions, so they are continuous and can be well approximated by linear functions.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$ (Figure 14.16). This curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x; the vertical coordinate is z. The y-value is held constant at y_0, so y is not a variable.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of $f(x, y_0)$ with respect to x at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the d previously used. In the definition, h represents a real number, positive or negative.

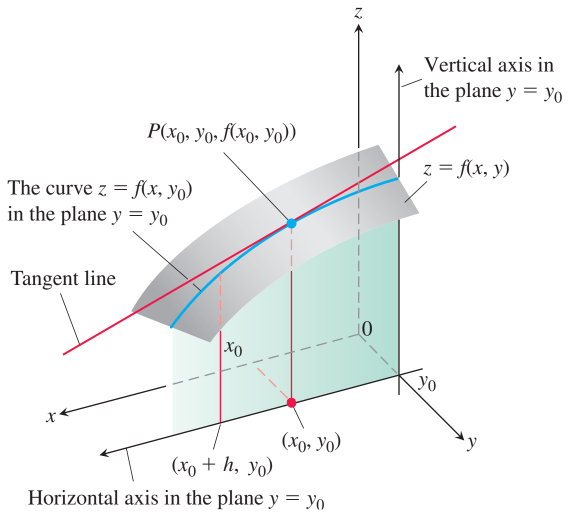


FIGURE 14.16 The intersection of the plane $y = y_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

DEFINITION The **partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

An equivalent expression for the partial derivative is

$$\left.\frac{d}{dx} f(x, y_0)\right|_{x=x_0}.$$

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at (x_0, y_0). (In Figure 14.16 this slope is negative.) The tangent line to the curve at P is the line in the plane $y = y_0$ that passes through P with this slope. The partial derivative $\partial f/\partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is held fixed at the value y_0.

We use several notations for the partial derivative:

$$\frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0), \qquad \left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \qquad \text{and} \qquad f_x,\ \frac{\partial f}{\partial x},\ z_x, \text{ or } \frac{\partial z}{\partial x}.$$

The definition of the partial derivative of $f(x, y)$ with respect to y at a point (x_0, y_0) is similar to the definition of the partial derivative of f with respect to x. We hold x fixed at the value x_0 and take the ordinary derivative of $f(x_0, y)$ with respect to y at y_0.

DEFINITION **The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

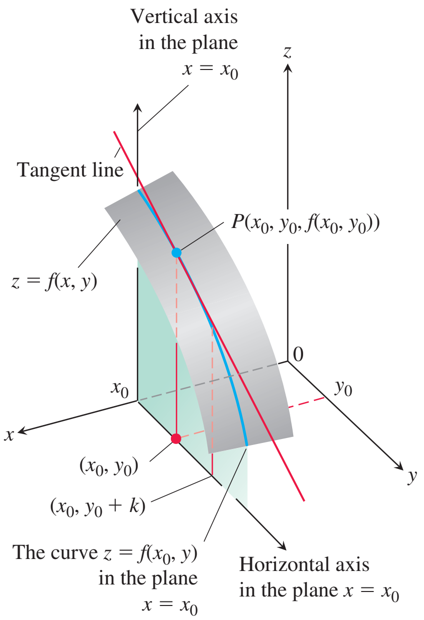


FIGURE 14.17 The intersection of the plane $x = x_0$ with the surface $z = f(x, y)$, viewed from above the first quadrant of the xy-plane.

The slope of the curve $z = f(x_0, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the vertical plane $x = x_0$ (Figure 14.17) is the partial derivative of f with respect to y at (x_0, y_0). The tangent line to the curve at P is the line in the plane $x = x_0$ that passes through P with this slope. The partial derivative gives the rate of change of f with respect to y at (x_0, y_0) when x is held fixed at the value x_0.

The partial derivative with respect to y is denoted the same way as the partial derivative with respect to x:

$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0), \qquad \frac{\partial f}{\partial y}, \qquad f_y.$$

Notice that we now have two tangent lines associated with the surface $z = f(x, y)$ at the point $P(x_0, y_0, f(x_0, y_0))$ (Figure 14.18). Is the plane they determine tangent to the surface at P? We will see that it is for the *differentiable* functions defined at the end of this section, and we will learn how to find the tangent plane in Section 14.6. First we have to learn more about partial derivatives themselves.

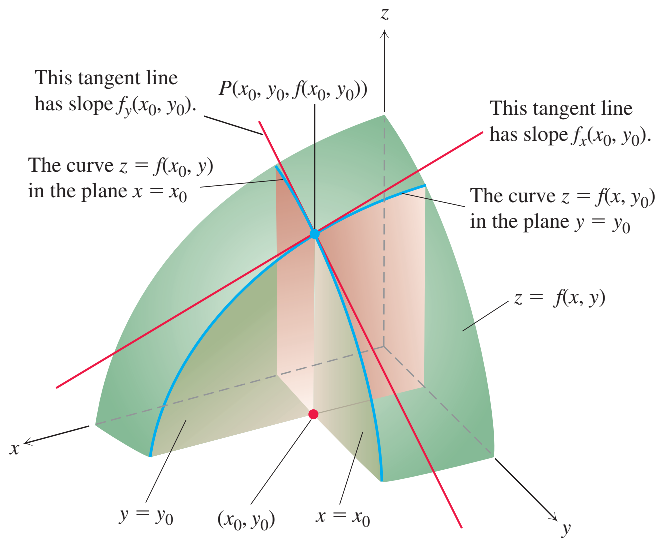


FIGURE 14.18 Figures 14.16 and 14.17 combined. The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

Calculations

The definitions of $\partial f/\partial x$ and $\partial f/\partial y$ give us two different ways of differentiating f at a point: with respect to x in the usual way while treating y as a constant and with respect to y in the usual way while treating x as a constant. As the following examples show, the values of these partial derivatives are usually different at a given point (x_0, y_0).

EXAMPLE 1 Find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution To find $\partial f/\partial x$, we treat y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$

The value of $\partial f/\partial x$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\partial f/\partial y$, we treat x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y - 1\right) = 0 + 3\cdot x\cdot 1 + 1 - 0 = 3x + 1.$$

The value of $\partial f/\partial y$ at $(4, -5)$ is $3(4) + 1 = 13$. ■

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y \sin xy) = y\frac{\partial}{\partial y}\sin xy + (\sin xy)\frac{\partial}{\partial y}(y)\\ &= (y\cos xy)\frac{\partial}{\partial y}(xy) + \sin xy = xy\cos xy + \sin xy.\end{aligned}$$ ■

EXAMPLE 3 Find f_x and f_y as functions if

$$f(x, y) = \frac{2y}{y + \cos x}.$$

Solution We treat f as a quotient. With y held constant, we get

$$\begin{aligned}f_x &= \frac{\partial}{\partial x}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y + \cos x)}{(y + \cos x)^2}\\ &= \frac{(y + \cos x)(0) - 2y(-\sin x)}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}.\end{aligned}$$

With x held constant, we get

$$\begin{aligned}f_y &= \frac{\partial}{\partial y}\left(\frac{2y}{y + \cos x}\right) = \frac{(y + \cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{dy}(y + \cos x)}{(y + \cos x)^2}\\ &= \frac{(y + \cos x)(2) - 2y(1)}{(y + \cos x)^2} = \frac{2\cos x}{(y + \cos x)^2}.\end{aligned}$$ ■

Implicit differentiation works for partial derivatives the way it works for ordinary derivatives, as the next example illustrates.

EXAMPLE 4 Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

Solution We differentiate both sides of the equation with respect to x, holding y constant and treating z as a differentiable function of x:

$$\begin{aligned}\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z &= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}\\ y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} &= 1 + 0\\ \left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} &= 1\\ \frac{\partial z}{\partial x} &= \frac{z}{yz - 1}.\end{aligned}$$ ■

With y constant, $\frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}$.

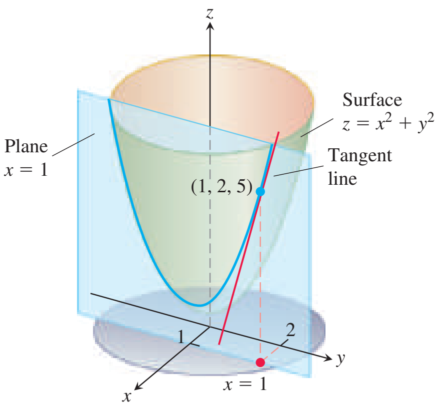


FIGURE 14.19 The tangent to the curve of intersection of the plane $x = 1$ and surface $z = x^2 + y^2$ at the point (1, 2, 5) (Example 5).

EXAMPLE 5 The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5) (Figure 14.19).

Solution The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = \left.2y\right|_{(1,2)} = 2(2) = 4.$$

As a check, we can treat the parabola as the graph of the single-variable function $z = (1)^2 + y^2 = 1 + y^2$ in the plane $x = 1$ and ask for the slope at $y = 2$. The slope, calculated now as an ordinary derivative, is

$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = \left.2y\right|_{y=2} = 4.$$

Functions of More Than Two Variables

The definitions of the partial derivatives of functions of more than two independent variables are like the definitions for functions of two variables. They are ordinary derivatives with respect to one variable, taken while the other independent variables are held constant.

EXAMPLE 6 If x, y, and z are independent variables and

$$f(x, y, z) = x \sin(y + 3z),$$

then

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x \sin(y + 3z)\right] = x \frac{\partial}{\partial z}\sin(y + 3z)$$
$$= x \cos(y + 3z)\frac{\partial}{\partial z}(y + 3z) = 3x \cos(y + 3z).$$

EXAMPLE 7 If resistors of R_1, R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

(Figure 14.20). Find the value of $\partial R/\partial R_2$ when $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$ ohms.

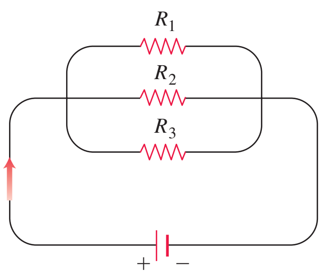


FIGURE 14.20 Resistors arranged this way are said to be connected in parallel (Example 7). Each resistor lets a portion of the current through. Their equivalent resistance R is calculated with the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Solution To find $\partial R/\partial R_2$, we treat R_1 and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$
$$-\frac{1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{R_2{}^2} + 0$$
$$\frac{\partial R}{\partial R_2} = \frac{R^2}{R_2{}^2} = \left(\frac{R}{R_2}\right)^2.$$

When $R_1 = 30$, $R_2 = 45$, and $R_3 = 90$,

$$\frac{1}{R} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{3 + 2 + 1}{90} = \frac{6}{90} = \frac{1}{15},$$

so $R = 15$ and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}.$$

Thus at the given values, a small change in the resistance R_2 leads to a change in R about $1/9$th as large. ■

Partial Derivatives and Continuity

A function $f(x, y)$ can have partial derivatives with respect to both x and y at a point without the function being continuous there. This is different from functions of a single variable, where the existence of a derivative implies continuity. If the partial derivatives of $f(x, y)$ exist and are continuous throughout a disk centered at (x_0, y_0), however, then f *is* continuous at (x_0, y_0), as we see at the end of this section.

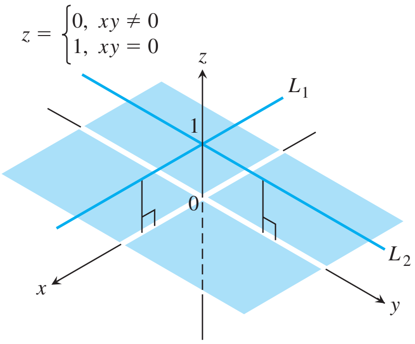


FIGURE 14.21 The graph of

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

consists of the lines L_1 and L_2 and the four open quadrants of the xy-plane. The function has partial derivatives at the origin but is not continuous there (Example 8).

EXAMPLE 8 Let

$$f(x, y) = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$$

(Figure 14.21).

(a) Find the limit of f as (x, y) approaches $(0, 0)$ along the line $y = x$.

(b) Prove that f is not continuous at the origin.

(c) Show that both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ exist at the origin.

Solution

(a) Since $f(x, y)$ is constantly zero along the line $y = x$ (except at the origin), we have

$$\lim_{(x,y)\to(0,0)} f(x, y)\Big|_{y=x} = \lim_{(x,y)\to(0,0)} 0 = 0.$$

(b) Since $f(0, 0) = 1$, the limit in part (a) proves that f is not continuous at $(0, 0)$.

(c) To find $\partial f/\partial x$ at $(0, 0)$, we hold y fixed at $y = 0$. Then $f(x, y) = 1$ for all x, and the graph of f is the line L_1 in Figure 14.21. The slope of this line at any x is $\partial f/\partial x = 0$. In particular, $\partial f/\partial x = 0$ at $(0, 0)$. Similarly, $\partial f/\partial y$ is the slope of line L_2 at any y, so $\partial f/\partial y = 0$ at $(0, 0)$. ■

Example 8 notwithstanding, it is still true in higher dimensions that *differentiability* at a point implies continuity. What Example 8 suggests is that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives. We define differentiability for functions of two variables (which is slightly more complicated than for single-variable functions) at the end of this section and then revisit the connection to continuity.

Second-Order Partial Derivatives

When we differentiate a function $f(x, y)$ twice, we produce its second-order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \qquad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy},$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \quad \text{and} \quad \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}.$$

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$

and so on. Notice the order in which the mixed partial derivatives are taken:

$\dfrac{\partial^2 f}{\partial x \partial y}$ Differentiate first with respect to y, then with respect to x.

$f_{yx} = (f_y)_x$ Means the same thing.

HISTORICAL BIOGRAPHY
Pierre-Simon Laplace
(1749–1827)

EXAMPLE 9 If $f(x, y) = x \cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x) = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x) = -x \sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

The Mixed Derivative Theorem

The order in which partial derivatives are taken can sometimes lead to different results.

You may have noticed that the "mixed" second-order partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

in Example 9 are equal. This is not a coincidence. They must be equal whenever f, f_x, f_y, f_{xy}, and f_{yx} are continuous, as stated in the following theorem. However, the mixed derivatives can be different when the continuity conditions are not satisfied (see Exercise 72).

THEOREM 2—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

HISTORICAL BIOGRAPHY
Alexis Clairaut
(1713–1765)

Theorem 2 is also known as Clairaut's Theorem, named after the French mathematician Alexis Clairaut, who discovered it. A proof is given in Appendix 9. Theorem 2 says that to calculate a mixed second-order derivative, we may differentiate in either order, provided the continuity conditions are satisfied. This ability to proceed in different order sometimes simplifies our calculations.

EXAMPLE 10 Find $\partial^2 w / \partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w/\partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x. However, if we interchange the order of differentiation and differentiate first with respect to x we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \qquad \text{and} \qquad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x \partial y = 1$ as well. We can differentiate in either order because the conditions of Theorem 2 hold for w at all points (x_0, y_0).

Partial Derivatives of Still Higher Order

Although we will deal mostly with first- and second-order partial derivatives, because these appear the most frequently in applications, there is no theoretical limit to how many times we can differentiate a function as long as the derivatives involved exist. Thus, we get third- and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx},$$

$$\frac{\partial^4 f}{\partial x^2 \partial y^2} = f_{yyxx},$$

and so on. As with second-order derivatives, the order of differentiation is immaterial as long as all the derivatives through the order in question are continuous.

EXAMPLE 11 Find f_{yxyz} if $f(x, y, z) = 1 - 2xy^2z + x^2y$.

Solution We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$\begin{aligned} f_y &= -4xyz + x^2 \\ f_{yx} &= -4yz + 2x \\ f_{yxy} &= -4z \\ f_{yxyz} &= -4. \end{aligned}$$

Differentiability

The concept of *differentiability* for functions of several variables is more complicated than for single-variable functions because a point in the domain can be approached along more than one path. In defining the partial derivatives for a function of two variables, we intersected the surface of the graph with vertical planes parallel to the xz- and yz-planes, creating a curve on each plane, called a *trace*. The partial derivatives were seen as the slopes of the two tangent lines to these trace curves at the point on the surface corresponding to the point (x_0, y_0) being approached in the domain. (See Figure 14.18.) For a differentiable function, it would seem reasonable to assume that if we were to rotate slightly one of these vertical planes, keeping it vertical but no longer parallel to its coordinate plane, then a smooth trace curve would appear on that plane that would have a tangent line at the point on the surface having a slope differing just slightly from what it was before (when the plane was parallel to its coordinate plane). However, the mere existence of the original partial derivative does not guarantee that result. For example, the surface might have a "fissure" in the direction of the new plane, so the trace curve is not even continuous at (x_0, y_0), let alone having a tangent line at the corresponding point on the curve. Just as having a limit in the x- and y-coordinate directions does not imply the function itself has a limit at (x_0, y_0), as we see in Figure 14.21, so is it the case that the existence of both partial derivatives is not enough by itself to ensure derivatives exist for trace curves in other vertical planes. For the existence of differentiability, a property is needed to ensure that no abrupt change occurs in the function resulting from small changes in the independent variables along any path approaching (x_0, y_0), paths along which *both* variables x and y are allowed to change, rather than just one of them at a time. We saw a way of thinking about the change in a function in Section 3.9.

In our study of functions of a single variable, we found that if a function $y = f(x)$ is differentiable at $x = x_0$, then the change Δy resulting in the change of x from x_0 to $x_0 + \Delta x$ is close to the change ΔL along the tangent line (or linear approximation L of the function f at x_0). That is, from Equation (1) in Section 3.9,

$$\Delta y = f'(x_0)\Delta x + \epsilon \Delta x$$

in which $\epsilon \to 0$ as $\Delta x \to 0$. The extension of this result is what we use to *define* differentiability for functions of two variables.

DEFINITION A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

The following theorem (proved in Appendix 9) and its accompanying corollary tell us that functions with *continuous* first partial derivatives at (x_0, y_0) are differentiable there, and they are closely approximated locally by a linear function. We study this approximation in Section 14.6.

THEOREM 3—The Increment Theorem for Functions of Two Variables Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

COROLLARY OF THEOREM 3 If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

If $z = f(x, y)$ is differentiable, then the definition of differentiability ensures that $\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$ approaches 0 as Δx and Δy approach 0. This tells us that a function of two variables is continuous at every point where it is differentiable.

THEOREM 4—Differentiability Implies Continuity If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

As we can see from Corollary 3 and Theorem 4, a function $f(x, y)$ must be continuous at a point (x_0, y_0) if f_x and f_y are continuous throughout an open region containing (x_0, y_0). Remember, however, that it is still possible for a function of two variables to be discontinuous at a point where its first partial derivatives exist, as we saw in Example 8. Existence alone of the partial derivatives at that point is not enough, but continuity of the partial derivatives guarantees differentiability.

Exercises 14.3

Calculating First-Order Partial Derivatives

In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1. $f(x, y) = 2x^2 - 3y - 4$

2. $f(x, y) = x^2 - xy + y^2$

3. $f(x, y) = (x^2 - 1)(y + 2)$

4. $f(x, y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

5. $f(x, y) = (xy - 1)^2$

6. $f(x, y) = (2x - 3y)^3$

7. $f(x, y) = \sqrt{x^2 + y^2}$

8. $f(x, y) = (x^3 + (y/2))^{2/3}$

9. $f(x, y) = 1/(x + y)$

10. $f(x, y) = x/(x^2 + y^2)$

11. $f(x, y) = (x + y)/(xy - 1)$

12. $f(x, y) = \tan^{-1}(y/x)$

13. $f(x, y) = e^{(x+y+1)}$

14. $f(x, y) = e^{-x}\sin(x + y)$

15. $f(x, y) = \ln(x + y)$

16. $f(x, y) = e^{xy}\ln y$

17. $f(x, y) = \sin^2(x - 3y)$

18. $f(x, y) = \cos^2(3x - y^2)$

19. $f(x, y) = x^y$

20. $f(x, y) = \log_y x$

21. $f(x, y) = \displaystyle\int_x^y g(t)\,dt \quad$ (g continuous for all t)

22. $f(x, y) = \displaystyle\sum_{n=0}^{\infty}(xy)^n \quad (|xy| < 1)$

In Exercises 23–34, find f_x, f_y, and f_z.

23. $f(x, y, z) = 1 + xy^2 - 2z^2$

24. $f(x, y, z) = xy + yz + xz$

25. $f(x, y, z) = x - \sqrt{y^2 + z^2}$

26. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

27. $f(x, y, z) = \sin^{-1}(xyz)$

28. $f(x, y, z) = \sec^{-1}(x + yz)$

29. $f(x, y, z) = \ln(x + 2y + 3z)$

30. $f(x, y, z) = yz\ln(xy)$

31. $f(x, y, z) = e^{-(x^2+y^2+z^2)}$

32. $f(x, y, z) = e^{-xyz}$

33. $f(x, y, z) = \tanh(x + 2y + 3z)$

34. $f(x, y, z) = \sinh(xy - z^2)$

In Exercises 35–40, find the partial derivative of the function with respect to each variable.

35. $f(t, \alpha) = \cos(2\pi t - \alpha)$

36. $g(u, v) = v^2 e^{(2u/v)}$

37. $h(\rho, \phi, \theta) = \rho\sin\phi\cos\theta$

38. $g(r, \theta, z) = r(1 - \cos\theta) - z$

39. Work done by the heart (Section 3.9, Exercise 61)

$$W(P, V, \delta, v, g) = PV + \frac{V\delta v^2}{2g}$$

40. Wilson lot size formula (Section 4.5, Exercise 53)

$$A(c, h, k, m, q) = \frac{km}{q} + cm + \frac{hq}{2}$$

Calculating Second-Order Partial Derivatives

Find all the second-order partial derivatives of the functions in Exercises 41–50.

41. $f(x, y) = x + y + xy$

42. $f(x, y) = \sin xy$

43. $g(x, y) = x^2y + \cos y + y\sin x$

44. $h(x, y) = xe^y + y + 1$

45. $r(x, y) = \ln(x + y)$

46. $s(x, y) = \tan^{-1}(y/x)$

47. $w = x^2\tan(xy)$

48. $w = ye^{x^2 - y}$

49. $w = x\sin(x^2y)$

50. $w = \dfrac{x - y}{x^2 + y}$

Mixed Partial Derivatives

In Exercises 51–54, verify that $w_{xy} = w_{yx}$.

51. $w = \ln(2x + 3y)$

52. $w = e^x + x\ln y + y\ln x$

53. $w = xy^2 + x^2y^3 + x^3y^4$

54. $w = x\sin y + y\sin x + xy$

55. Which order of differentiation will calculate f_{xy} faster: x first or y first? Try to answer without writing anything down.

a. $f(x, y) = x\sin y + e^y$

b. $f(x, y) = 1/x$

c. $f(x, y) = y + (x/y)$

d. $f(x, y) = y + x^2y + 4y^3 - \ln(y^2 + 1)$

e. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$

f. $f(x, y) = x\ln xy$

56. The fifth-order partial derivative $\partial^5 f/\partial x^2\partial y^3$ is zero for each of the following functions. To show this as quickly as possible, which variable would you differentiate with respect to first: x or y? Try to answer without writing anything down.

a. $f(x, y) = y^2x^4e^x + 2$

b. $f(x, y) = y^2 + y(\sin x - x^4)$

c. $f(x, y) = x^2 + 5xy + \sin x + 7e^x$

d. $f(x, y) = xe^{y^2/2}$

Using the Partial Derivative Definition

In Exercises 57–60, use the limit definition of partial derivative to compute the partial derivatives of the functions at the specified points.

57. $f(x, y) = 1 - x + y - 3x^2y,\quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(1, 2)$

58. $f(x, y) = 4 + 2x - 3y - xy^2,\quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2, 1)$

59. $f(x, y) = \sqrt{2x + 3y - 1},\quad \dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(-2, 3)$

60. $f(x, y) = \begin{cases} \dfrac{\sin(x^3 + y^4)}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0), \end{cases}$

$\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(0, 0)$

61. Let $f(x, y) = 2x + 3y - 4$. Find the slope of the line tangent to this surface at the point $(2, -1)$ and lying in the **a.** plane $x = 2$ **b.** plane $y = -1$.

62. Let $f(x, y) = x^2 + y^3$. Find the slope of the line tangent to this surface at the point $(-1, 1)$ and lying in the **a.** plane $x = -1$ **b.** plane $y = 1$.

63. Three variables Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f/\partial z$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial z$ at $(1, 2, 3)$ for $f(x, y, z) = x^2yz^2$.

64. Three variables Let $w = f(x, y, z)$ be a function of three independent variables and write the formal definition of the partial derivative $\partial f/\partial y$ at (x_0, y_0, z_0). Use this definition to find $\partial f/\partial y$ at $(-1, 0, 3)$ for $f(x, y, z) = -2xy^2 + yz^2$.

Differentiating Implicitly

65. Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation

$$xy + z^3x - 2yz = 0$$

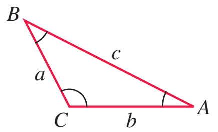
defines z as a function of the two independent variables x and y and the partial derivative exists.

66. Find the value of $\partial x/\partial z$ at the point $(1, -1, -3)$ if the equation

$$xz + y \ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivative exists.

Exercises 67 and 68 are about the triangle shown here.



67. Express A implicitly as a function of a, b, and c and calculate $\partial A/\partial a$ and $\partial A/\partial b$.

68. Express a implicitly as a function of A, b, and B and calculate $\partial a/\partial A$ and $\partial a/\partial B$.

69. Two dependent variables Express v_x in terms of u and y if the equations $x = v \ln u$ and $y = u \ln v$ define u and v as functions of the independent variables x and y, and if v_x exists. (*Hint:* Differentiate both equations with respect to x and solve for v_x by eliminating u_x.)

70. Two dependent variables Find $\partial x/\partial u$ and $\partial y/\partial u$ if the equations $u = x^2 - y^2$ and $v = x^2 - y$ define x and y as functions of the independent variables u and v, and the partial derivatives exist. (See the hint in Exercise 69.) Then let $s = x^2 + y^2$ and find $\partial s/\partial u$.

Theory and Examples

71. Let $f(x, y) = \begin{cases} y^3, & y \geq 0 \\ -y^2, & y < 0. \end{cases}$

Find f_x, f_y, f_{xy}, and f_{yx}, and state the domain for each partial derivative.

72. Let $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if } (x, y) \neq 0, \\ 0, & \text{if } (x, y) = 0. \end{cases}$

a. Show that $\dfrac{\partial f}{\partial y}(x, 0) = x$ for all x, and $\dfrac{\partial f}{\partial x}(0, y) = -y$ for all y.

b. Show that $\dfrac{\partial^2 f}{\partial y \partial x}(0, 0) \neq \dfrac{\partial^2 f}{\partial x \partial y}(0, 0)$.

The graph of f is shown on page 788.

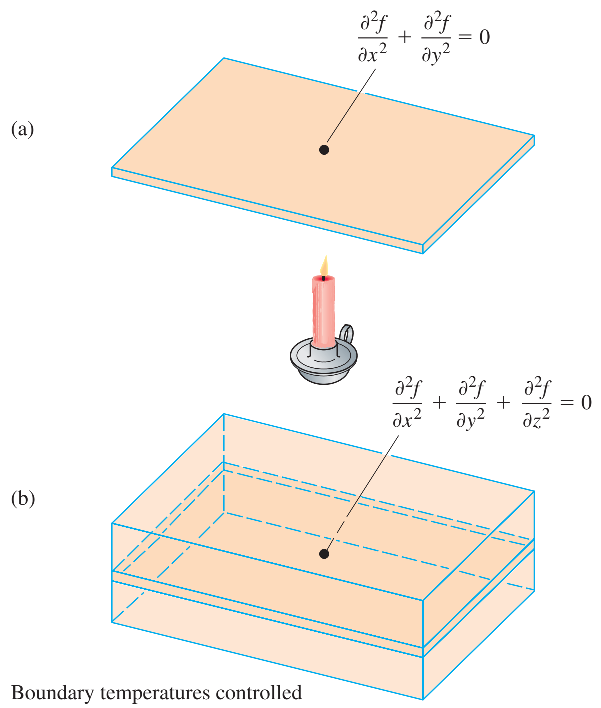
The **three-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

is satisfied by steady-state temperature distributions $T = f(x, y, z)$ in space, by gravitational potentials, and by electrostatic potentials. The **two-dimensional Laplace equation**

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

obtained by dropping the $\partial^2 f/\partial z^2$ term from the previous equation, describes potentials and steady-state temperature distributions in a plane (see the accompanying figure). The plane (a) may be treated as a thin slice of the solid (b) perpendicular to the z-axis.



Boundary temperatures controlled

Show that each function in Exercises 73–80 satisfies a Laplace equation.

73. $f(x, y, z) = x^2 + y^2 - 2z^2$

74. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z$

75. $f(x, y) = e^{-2y} \cos 2x$

76. $f(x, y) = \ln\sqrt{x^2 + y^2}$

77. $f(x, y) = 3x + 2y - 4$

78. $f(x, y) = \tan^{-1} \dfrac{x}{y}$

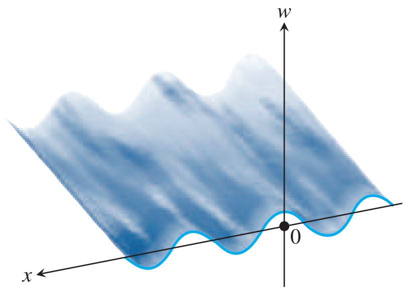
79. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

80. $f(x, y, z) = e^{3x+4y} \cos 5z$

The Wave Equation If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the **one-dimensional wave equation**

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2},$$

where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propagated.



In our example, x is the distance across the ocean's surface, but in other applications, x might be the distance along a vibrating string, distance through air (sound waves), or distance through space (light waves). The number c varies with the medium and type of wave.

Show that the functions in Exercises 81–87 are all solutions of the wave equation.

81. $w = \sin(x + ct)$

82. $w = \cos(2x + 2ct)$

83. $w = \sin(x + ct) + \cos(2x + 2ct)$

84. $w = \ln(2x + 2ct)$

85. $w = \tan(2x - 2ct)$

86. $w = 5\cos(3x + 3ct) + e^{x+ct}$

87. $w = f(u)$, where f is a differentiable function of u, and $u = a(x + ct)$, where a is a constant

88. Does a function $f(x, y)$ with continuous first partial derivatives throughout an open region R have to be continuous on R? Give reasons for your answer.

89. If a function $f(x, y)$ has continuous second partial derivatives throughout an open region R, must the first-order partial derivatives of f be continuous on R? Give reasons for your answer.

90. The heat equation An important partial differential equation that describes the distribution of heat in a region at time t can be represented by the *one-dimensional heat equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2}.$$

Show that $u(x, t) = \sin(\alpha x) \cdot e^{-\beta t}$ satisfies the heat equation for constants α and β. What is the relationship between α and β for this function to be a solution?

91. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$. (*Hint:* Use Theorem 4 and show that f is not continuous at $(0, 0)$.)

92. Let $f(x, y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise.} \end{cases}$

Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist, but f is not differentiable at $(0, 0)$.

14.4 The Chain Rule

The Chain Rule for functions of a single variable studied in Section 3.6 says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and dw/dt can be calculated by the formula

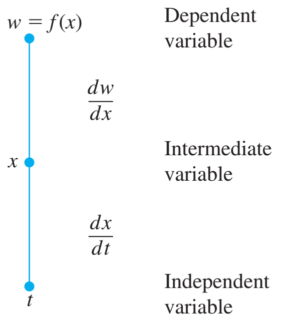
$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$

For this composite function $w(t) = f(g(t))$, we can think of t as the independent variable and $x = g(t)$ as the "intermediate variable," because t determines the value of x which in turn gives the value of w from the function f. We display the Chain Rule in a "branch diagram" in the margin on the next page.

For functions of several variables the Chain Rule has more than one form, which depends on how many independent and intermediate variables are involved. However, once the variables are taken into account, the Chain Rule works in the same way we just discussed.

To find dw/dt, we read down the route from w to t, multiplying derivatives along the way.

Chain Rule



$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

Each of $\frac{\partial f}{\partial x}, \frac{\partial w}{\partial x}, f_x$ indicates the partial derivative of f with respect to x.

Functions of Two Variables

The Chain Rule formula for a differentiable function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof The proof consists of showing that if x and y are differentiable at $t = t_0$, then w is differentiable at t_0 and

$$\left(\frac{dw}{dt}\right)_{t_0} = \left(\frac{\partial w}{\partial x}\right)_{P_0}\left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial w}{\partial y}\right)_{P_0}\left(\frac{dy}{dt}\right)_{t_0},$$

where $P_0 = (x(t_0), y(t_0))$. The subscripts indicate where each of the derivatives is to be evaluated.

Let Δx, Δy, and Δw be the increments that result from changing t from t_0 to $t_0 + \Delta t$. Since f is differentiable (see the definition in Section 14.3),

$$\Delta w = \left(\frac{\partial w}{\partial x}\right)_{P_0}\Delta x + \left(\frac{\partial w}{\partial y}\right)_{P_0}\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. To find dw/dt, we divide this equation through by Δt and let Δt approach zero. The division gives

$$\frac{\Delta w}{\Delta t} = \left(\frac{\partial w}{\partial x}\right)_{P_0}\frac{\Delta x}{\Delta t} + \left(\frac{\partial w}{\partial y}\right)_{P_0}\frac{\Delta y}{\Delta t} + \epsilon_1\frac{\Delta x}{\Delta t} + \epsilon_2\frac{\Delta y}{\Delta t}.$$

Letting Δt approach zero gives

$$\begin{aligned}\left(\frac{dw}{dt}\right)_{t_0} &= \lim_{\Delta t \to 0}\frac{\Delta w}{\Delta t} \\ &= \left(\frac{\partial w}{\partial x}\right)_{P_0}\left(\frac{dx}{dt}\right)_{t_0} + \left(\frac{\partial w}{\partial y}\right)_{P_0}\left(\frac{dy}{dt}\right)_{t_0} + 0\cdot\left(\frac{dx}{dt}\right)_{t_0} + 0\cdot\left(\frac{dy}{dt}\right)_{t_0}.\end{aligned}$$

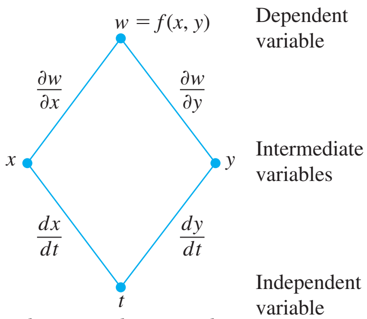
∎

Often we write $\partial w/\partial x$ for the partial derivative $\partial f/\partial x$, so we can rewrite the Chain Rule in Theorem 5 in the form

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$

To remember the Chain Rule, picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

However, the meaning of the dependent variable w is different on each side of the preceding equation. On the left-hand side, it refers to the composite function $w = f(x(t), y(t))$ as a function of the single variable t. On the right-hand side, it refers to the function $w = f(x, y)$ as a function of the two variables x and y. Moreover, the single derivatives dw/dt, dx/dt, and dy/dt are being evaluated at a point t_0, whereas the partial derivatives $\partial w/\partial x$ and $\partial w/\partial y$ are being evaluated at the point (x_0, y_0), with $x_0 = x(t_0)$ and $y_0 = y(t_0)$. With that understanding, we will use both of these forms interchangeably throughout the text whenever no confusion will arise.

The **branch diagram** in the margin provides a convenient way to remember the Chain Rule. The "true" independent variable in the composite function is t, whereas x and y are *intermediate variables* (controlled by t) and w is the dependent variable.

A more precise notation for the Chain Rule shows where the various derivatives in Theorem 5 are evaluated:

$$\frac{dw}{dt}(t_0) = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \frac{dx}{dt}(t_0) + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \frac{dy}{dt}(t_0).$$

EXAMPLE 1 Use the Chain Rule to find the derivative of

$$w = xy$$

with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \pi/2$?

Solution We apply the Chain Rule to find dw/dt as follows:

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} \\
&= \frac{\partial(xy)}{\partial x} \cdot \frac{d}{dt}(\cos t) + \frac{\partial(xy)}{\partial y} \cdot \frac{d}{dt}(\sin t) \\
&= (y)(-\sin t) + (x)(\cos t) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) \\
&= -\sin^2 t + \cos^2 t \\
&= \cos 2t.
\end{aligned}$$

In this example, we can check the result with a more direct calculation. As a function of t,

$$w = xy = \cos t \sin t = \frac{1}{2}\sin 2t,$$

so

$$\frac{dw}{dt} = \frac{d}{dt}\left(\frac{1}{2}\sin 2t\right) = \frac{1}{2} \cdot 2\cos 2t = \cos 2t.$$

In either case, at the given value of t,

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos\left(2 \cdot \frac{\pi}{2}\right) = \cos \pi = -1.$$

Functions of Three Variables

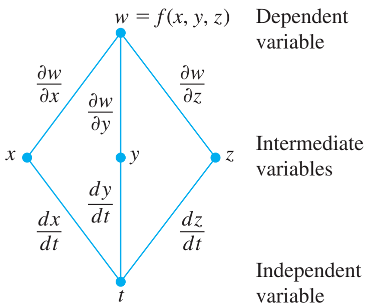
You can probably predict the Chain Rule for functions of three intermediate variables, as it only involves adding the expected third term to the two-variable formula.

THEOREM 6—Chain Rule for Functions of One Independent Variable and Three Intermediate Variables If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

The proof is identical with the proof of Theorem 5 except that there are now three intermediate variables instead of two. The branch diagram we use for remembering the new equation is similar as well, with three routes from w to t.

EXAMPLE 2 Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as t changes. What is the derivative's value at $t = 0$?

Solution Using the Chain Rule for three intermediate variables, we have

$$\begin{aligned}
\frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\
&= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\
&= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\
&= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t,
\end{aligned}$$

Substitute for the intermediate variables.

so

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

For a physical interpretation of change along a curve, think of an object whose position is changing with time t. If $w = T(x, y, z)$ is the temperature at each point (x, y, z) along a curve C with parametric equations $x = x(t)$, $y = y(t)$, and $z = z(t)$, then the composite function $w = T(x(t), y(t), z(t))$ represents the temperature relative to t along the curve. The derivative dw/dt is then the instantaneous rate of change of temperature due to the motion along the curve, as calculated in Theorem 6.

Functions Defined on Surfaces

If we are interested in the temperature $w = f(x, y, z)$ at points (x, y, z) on the earth's surface, we might prefer to think of x, y, and z as functions of the variables r and s that give the points' longitudes and latitudes. If $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$, we could then express the temperature as a function of r and s with the composite function

$$w = f(g(r, s), h(r, s), k(r, s)).$$

Under the conditions stated below, w has partial derivatives with respect to both r and s that can be calculated in the following way.

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

The first of these equations can be derived from the Chain Rule in Theorem 6 by holding s fixed and treating r as t. The second can be derived in the same way, holding r fixed and treating s as t. The branch diagrams for both equations are shown in Figure 14.22.

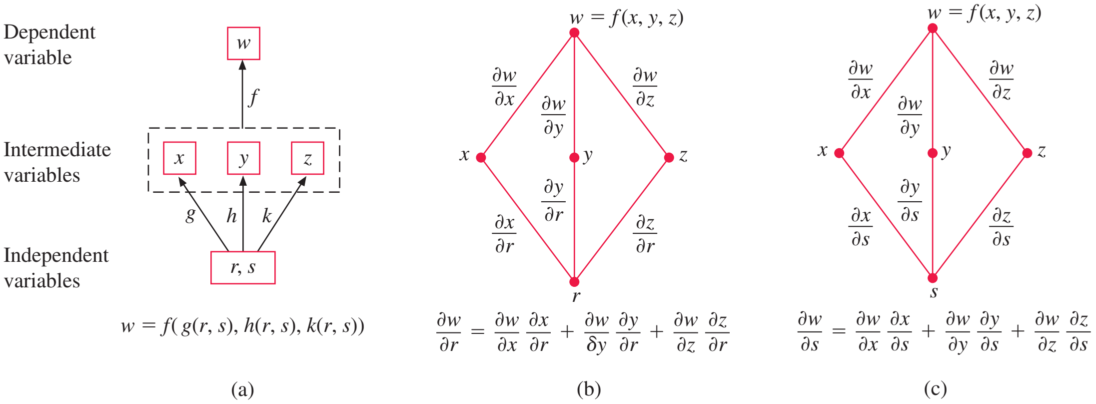


FIGURE 14.22 Composite function and branch diagrams for Theorem 7.

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution Using the formulas in Theorem 7, we find

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r$$

Substitute for intermediate variable z.

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

$$= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}. \quad \blacksquare$$

If f is a function of two intermediate variables instead of three, each equation in Theorem 7 becomes correspondingly one term shorter.

Chain Rule

If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}.$$

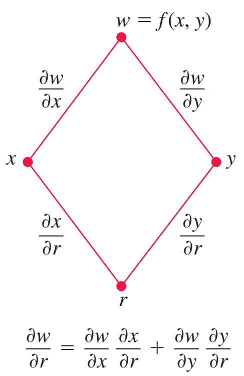


FIGURE 14.23 Branch diagram for the equation

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}.$$

Figure 14.23 shows the branch diagram for the first of these equations. The diagram for the second equation is similar; just replace r with s.

EXAMPLE 4 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s.$$

Solution The preceding discussion gives the following.

$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) \\ &= 4r\end{aligned} \qquad \begin{aligned}\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r - s) + 2(r + s) \\ &= 4s\end{aligned}$$

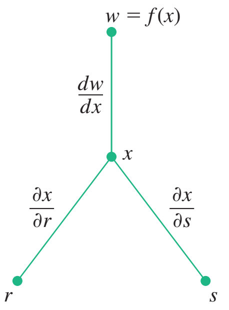
Substitute for the intermediate variables.

If f is a function of a single intermediate variable x, our equations are even simpler.

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}.$$

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

FIGURE 14.24 Branch diagram for differentiating f as a composite function of r and s with one intermediate variable.

In this case, we use the ordinary (single-variable) derivative, dw/dx. The branch diagram is shown in Figure 14.24.

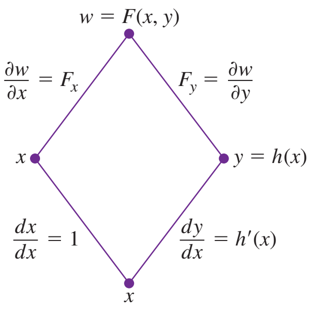
Implicit Differentiation Revisited

The two-variable Chain Rule in Theorem 5 leads to a formula that takes some of the algebra out of implicit differentiation. Suppose that

1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Since $w = F(x, y) = 0$, the derivative dw/dx must be zero. Computing the derivative from the Chain Rule (branch diagram in Figure 14.25), we find

$$\begin{aligned}0 = \frac{dw}{dx} &= F_x\frac{dx}{dx} + F_y\frac{dy}{dx} \qquad \text{Theorem 5 with } t = x \text{ and } f = F\\ &= F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.\end{aligned}$$



$$\frac{dw}{dx} = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}$$

FIGURE 14.25 Branch diagram for differentiating $w = F(x, y)$ with respect to x. Setting $dw/dx = 0$ leads to a simple computational formula for implicit differentiation (Theorem 8).

If $F_y = \partial w/\partial y \neq 0$, we can solve this equation for dy/dx to get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

We state this result formally.

THEOREM 8—A Formula for Implicit Differentiation Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}. \tag{1}$$

EXAMPLE 5 Use Theorem 8 to find dy/dx if $y^2 - x^2 - \sin xy = 0$.

Solution Take $F(x, y) = y^2 - x^2 - \sin xy$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy}$$

$$= \frac{2x + y\cos xy}{2y - x\cos xy}.$$

This calculation is significantly shorter than a single-variable calculation using implicit differentiation. ■

The result in Theorem 8 is easily extended to three variables. Suppose that the equation $F(x, y, z) = 0$ defines the variable z implicitly as a function $z = f(x, y)$. Then for all (x, y) in the domain of f, we have $F(x, y, f(x, y)) = 0$. Assuming that F and f are differentiable functions, we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ with respect to the independent variable x:

$$0 = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}$$

$$= F_x \cdot 1 + F_y \cdot 0 + F_z \cdot \frac{\partial z}{\partial x},$$

y is constant when differentiating with respect to x.

so

$$F_x + F_z \frac{\partial z}{\partial x} = 0.$$

A similar calculation for differentiating with respect to the independent variable y gives

$$F_y + F_z \frac{\partial z}{\partial y} = 0.$$

Whenever $F_z \neq 0$, we can solve these last two equations for the partial derivatives of $z = f(x, y)$ to obtain

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \tag{2}$$

An important result from advanced calculus, called the **Implicit Function Theorem**, states the conditions for which our results in Equations (2) are valid. If the partial derivatives F_x, F_y, and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0), and if for some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then the equation $F(x, y, z) = c$ defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by Equations (2).

EXAMPLE 6 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at $(0, 0, 0)$ if $x^3 + z^2 + ye^{xz} + z\cos y = 0$.

Solution Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Then

$$F_x = 3x^2 + zye^{xz}, \qquad F_y = e^{xz} - z\sin y, \qquad \text{and} \qquad F_z = 2z + xye^{xz} + \cos y.$$

Since $F(0, 0, 0) = 0$, $F_z(0, 0, 0) = 1 \neq 0$, and all first partial derivatives are continuous, the Implicit Function Theorem says that $F(x, y, z) = 0$ defines z as a differentiable function of x and y near the point $(0, 0, 0)$. From Equations (2),

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + zye^{xz}}{2z + xye^{xz} + \cos y} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{e^{xz} - z\sin y}{2z + xye^{xz} + \cos y}.$$

At $(0, 0, 0)$ we find

$$\frac{\partial z}{\partial x} = -\frac{0}{1} = 0 \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1.$$

■

Functions of Many Variables

We have seen several different forms of the Chain Rule in this section, but each one is just a special case of one general formula. When solving particular problems, it may help to draw the appropriate branch diagram by placing the dependent variable on top, the intermediate variables in the middle, and the selected independent variable at the bottom. To find the derivative of the dependent variable with respect to the selected independent variable, start at the dependent variable and read down each route of the branch diagram to the independent variable, calculating and multiplying the derivatives along each route. Then add the products found for the different routes.

In general, suppose that $w = f(x, y, \ldots, v)$ is a differentiable function of the intermediate variables $x, y, \ldots, v$ (a finite set) and the $x, y, \ldots, v$ are differentiable functions of the independent variables $p, q, \ldots, t$ (another finite set). Then w is a differentiable function of the variables p through t, and the partial derivatives of w with respect to these variables are given by equations of the form

$$\frac{\partial w}{\partial p} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial p} + \cdots + \frac{\partial w}{\partial v}\frac{\partial v}{\partial p}.$$

The other equations are obtained by replacing p by $q, \ldots, t$, one at a time.

One way to remember this equation is to think of the right-hand side as the dot product of two vectors with components

$$\underbrace{\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \ldots, \frac{\partial w}{\partial v}\right)}_{\substack{\text{Derivatives of } w \text{ with} \\ \text{respect to the} \\ \text{intermediate variables}}} \quad \text{and} \quad \underbrace{\left(\frac{\partial x}{\partial p}, \frac{\partial y}{\partial p}, \ldots, \frac{\partial v}{\partial p}\right)}_{\substack{\text{Derivatives of the intermediate} \\ \text{variables with respect to the} \\ \text{selected independent variable}}}.$$

Exercises 14.4

Chain Rule: One Independent Variable

In Exercises 1–6, **(a)** express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then **(b)** evaluate dw/dt at the given value of t.

1. $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$

2. $w = x^2 + y^2, \quad x = \cos t + \sin t, \quad y = \cos t - \sin t; \quad t = 0$

3. $w = \frac{x}{z} + \frac{y}{z}, \quad x = \cos^2 t, \quad y = \sin^2 t, \quad z = 1/t; \quad t = 3$

4. $w = \ln(x^2 + y^2 + z^2), \quad x = \cos t, \quad y = \sin t, \quad z = 4\sqrt{t}; \quad t = 3$

5. $w = 2ye^x - \ln z, \quad x = \ln(t^2 + 1), \quad y = \tan^{-1} t, \quad z = e^t; \quad t = 1$

6. $w = z - \sin xy, \quad x = t, \quad y = \ln t, \quad z = e^{t-1}; \quad t = 1$

Chain Rule: Two and Three Independent Variables

In Exercises 7 and 8, **(a)** express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

7. $z = 4e^x \ln y, \quad x = \ln(u\cos v), \quad y = u\sin v; \quad (u, v) = (2, \pi/4)$

8. $z = \tan^{-1}(x/y), \quad x = u\cos v, \quad y = u\sin v;$
$(u, v) = (1.3, \pi/6)$

In Exercises 9 and 10, **(a)** express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating. Then **(b)** evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v).

9. $w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$
$(u, v) = (1/2, 1)$

10. $w = \ln(x^2 + y^2 + z^2), \quad x = ue^v \sin u, \quad y = ue^v \cos u,$
$z = ue^v; \quad (u, v) = (-2, 0)$

In Exercises 11 and 12, **(a)** express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x, y, and z both by using the Chain Rule and by expressing u directly in terms of x, y, and z before differentiating. Then **(b)** evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z).

11. $u = \dfrac{p - q}{q - r}, \quad p = x + y + z, \quad q = x - y + z,$
$r = x + y - z; \quad (x, y, z) = \left(\sqrt{3}, 2, 1\right)$

12. $u = e^{qr}\sin^{-1} p, \quad p = \sin x, \quad q = z^2 \ln y, \quad r = 1/z;$
$(x, y, z) = (\pi/4, 1/2, -1/2)$

Using a Branch Diagram

In Exercises 13–24, draw a branch diagram and write a Chain Rule formula for each derivative.

13. $\dfrac{dz}{dt}$ for $z = f(x, y), \quad x = g(t), \quad y = h(t)$

14. $\dfrac{dz}{dt}$ for $z = f(u, v, w), \quad u = g(t), \quad v = h(t), \quad w = k(t)$

15. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = h(x, y, z), \quad x = f(u, v), \quad y = g(u, v),$
$z = k(u, v)$

16. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = f(r, s, t), \quad r = g(x, y), \quad s = h(x, y),$
$t = k(x, y)$

17. $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$ for $w = g(x, y), \quad x = h(u, v), \quad y = k(u, v)$

18. $\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial w}{\partial y}$ for $w = g(u, v), \quad u = h(x, y), \quad v = k(x, y)$

19. $\dfrac{\partial z}{\partial t}$ and $\dfrac{\partial z}{\partial s}$ for $z = f(x, y), \quad x = g(t, s), \quad y = h(t, s)$

20. $\dfrac{\partial y}{\partial r}$ for $y = f(u), \quad u = g(r, s)$

21. $\dfrac{\partial w}{\partial s}$ and $\dfrac{\partial w}{\partial t}$ for $w = g(u), \quad u = h(s, t)$

22. $\dfrac{\partial w}{\partial p}$ for $w = f(x, y, z, v), \quad x = g(p, q), \quad y = h(p, q),$
$z = j(p, q), \quad v = k(p, q)$

23. $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial s}$ for $w = f(x, y), \quad x = g(r), \quad y = h(s)$

24. $\dfrac{\partial w}{\partial s}$ for $w = g(x, y), \quad x = h(r, s, t), \quad y = k(r, s, t)$

Implicit Differentiation

Assuming that the equations in Exercises 25–28 define y as a differentiable function of x, use Theorem 8 to find the value of dy/dx at the given point.

25. $x^3 - 2y^2 + xy = 0, \quad (1, 1)$

26. $xy + y^2 - 3x - 3 = 0, \quad (-1, 1)$

27. $x^2 + xy + y^2 - 7 = 0, \quad (1, 2)$

28. $xe^y + \sin xy + y - \ln 2 = 0, \quad (0, \ln 2)$

Find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in Exercises 29–32.

29. $z^3 - xy + yz + y^3 - 2 = 0, \quad (1, 1, 1)$

30. $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} - 1 = 0, \quad (2, 3, 6)$

31. $\sin(x + y) + \sin(y + z) + \sin(x + z) = 0, \quad (\pi, \pi, \pi)$

32. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0, \quad (1, \ln 2, \ln 3)$

Finding Partial Derivatives at Specified Points

33. Find $\partial w/\partial r$ when $r = 1, s = -1$ if $w = (x + y + z)^2$, $x = r - s$, $y = \cos(r + s)$, $z = \sin(r + s)$.

34. Find $\partial w/\partial v$ when $u = -1, v = 2$ if $w = xy + \ln z$, $x = v^2/u$, $y = u + v$, $z = \cos u$.

35. Find $\partial w/\partial v$ when $u = 0, v = 0$ if $w = x^2 + (y/x)$, $x = u - 2v + 1$, $y = 2u + v - 2$.

36. Find $\partial z/\partial u$ when $u = 0, v = 1$ if $z = \sin xy + x\sin y$, $x = u^2 + v^2$, $y = uv$.

37. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = \ln 2$, $v = 1$ if $z = 5\tan^{-1} x$ and $x = e^u + \ln v$.

38. Find $\partial z/\partial u$ and $\partial z/\partial v$ when $u = 1$, $v = -2$ if $z = \ln q$ and $q = \sqrt{v + 3}\tan^{-1} u$.

Theory and Examples

39. Assume that $w = f(s^3 + t^2)$ and $f'(x) = e^x$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

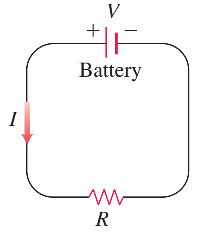
40. Assume that $w = f\left(ts^2, \dfrac{s}{t}\right)$, $\dfrac{\partial f}{\partial x}(x, y) = xy$, and $\dfrac{\partial f}{\partial y}(x, y) = \dfrac{x^2}{2}$.

Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

41. Changing voltage in a circuit The voltage V in a circuit that satisfies the law $V = IR$ is slowly dropping as the battery wears out. At the same time, the resistance R is increasing as the resistor heats up. Use the equation

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$$

to find how the current is changing at the instant when $R = 600$ ohms, $I = 0.04$ amp, $dR/dt = 0.5$ ohm/sec, and $dV/dt = -0.01$ volt/sec.



42. Changing dimensions in a box The lengths a, b, and c of the edges of a rectangular box are changing with time. At the instant in question, $a = 1$ m, $b = 2$ m, $c = 3$ m, $da/dt = db/dt = 1$ m/sec, and $dc/dt = -3$ m/sec. At what rates are the box's volume V and surface area S changing at that instant? Are the box's interior diagonals increasing in length or decreasing?

43. If $f(u, v, w)$ is differentiable and $u = x - y$, $v = y - z$, and $w = z - x$, show that

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0.$$

44. Polar coordinates Suppose that we substitute polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ in a differentiable function $w = f(x, y)$.

a. Show that

$$\frac{\partial w}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

and

$$\frac{1}{r}\frac{\partial w}{\partial\theta} = -f_x\sin\theta + f_y\cos\theta.$$

b. Solve the equations in part (a) to express f_x and f_y in terms of $\partial w/\partial r$ and $\partial w/\partial\theta$.

c. Show that

$$(f_x)^2 + (f_y)^2 = \left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial\theta}\right)^2.$$

45. Laplace equations Show that if $w = f(u, v)$ satisfies the Laplace equation $f_{uu} + f_{vv} = 0$ and if $u = (x^2 - y^2)/2$ and $v = xy$, then w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$.

46. Laplace equations Let $w = f(u) + g(v)$, where $u = x + iy$, $v = x - iy$, and $i = \sqrt{-1}$. Show that w satisfies the Laplace equation $w_{xx} + w_{yy} = 0$ if all the necessary functions are differentiable.

47. Extreme values on a helix Suppose that the partial derivatives of a function $f(x, y, z)$ at points on the helix $x = \cos t$, $y = \sin t$, $z = t$ are

$$f_x = \cos t, \qquad f_y = \sin t, \qquad f_z = t^2 + t - 2.$$

At what points on the curve, if any, can f take on extreme values?

48. A space curve Let $w = x^2e^{2y}\cos 3z$. Find the value of dw/dt at the point $(1, \ln 2, 0)$ on the curve $x = \cos t$, $y = \ln(t + 2)$, $z = t$.

49. Temperature on a circle Let $T = f(x, y)$ be the temperature at the point (x, y) on the circle $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$ and suppose that

$$\frac{\partial T}{\partial x} = 8x - 4y, \qquad \frac{\partial T}{\partial y} = 8y - 4x.$$

a. Find where the maximum and minimum temperatures on the circle occur by examining the derivatives dT/dt and d^2T/dt^2.

b. Suppose that $T = 4x^2 - 4xy + 4y^2$. Find the maximum and minimum values of T on the circle.

50. Temperature on an ellipse Let $T = g(x, y)$ be the temperature at the point (x, y) on the ellipse

$$x = 2\sqrt{2}\cos t, \qquad y = \sqrt{2}\sin t, \qquad 0 \le t \le 2\pi,$$

and suppose that

$$\frac{\partial T}{\partial x} = y, \qquad \frac{\partial T}{\partial y} = x.$$

a. Locate the maximum and minimum temperatures on the ellipse by examining dT/dt and d^2T/dt^2.

b. Suppose that $T = xy - 2$. Find the maximum and minimum values of T on the ellipse.

Differentiating Integrals Under mild continuity restrictions, it is true that if

$$F(x) = \int_a^b g(t, x)\,dt,$$

then $F'(x) = \displaystyle\int_a^b g_x(t, x)\,dt$. Using this fact and the Chain Rule, we can find the derivative of

$$F(x) = \int_a^{f(x)} g(t, x)\,dt$$

by letting

$$G(u, x) = \int_a^u g(t, x)\,dt,$$

where $u = f(x)$. Find the derivatives of the functions in Exercises 51 and 52.

51. $F(x) = \displaystyle\int_0^{x^2} \sqrt{t^4 + x^3}\,dt$ **52.** $F(x) = \displaystyle\int_{x^2}^1 \sqrt{t^3 + x^2}\,dt$

14.5 Directional Derivatives and Gradient Vectors

If you look at the map (Figure 14.26) showing contours within Yosemite National Park in California, you will notice that the streams flow perpendicular to the contours. The streams are following paths of steepest descent so the waters reach lower elevations as quickly as possible. Therefore, the fastest instantaneous rate of change in a stream's elevation above sea level has a particular direction. In this section, you will see why this direction, called the "downhill" direction, is perpendicular to the contours.

Directional Derivatives in the Plane

We know from Section 14.4 that if $f(x, y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve $x = g(t)$, $y = h(t)$ is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

At any point $P_0(x_0, y_0) = P_0(g(t_0), h(t_0))$, this equation gives the rate of change of f with respect to increasing t and therefore depends, among other things, on the direction of

FIGURE 14.26 Contours within Yosemite National Park in California show streams, which follow paths of steepest descent, running perpendicular to the contours. (*Source:* Yosemite National Park Map from U.S. Geological Survey, http://www.usgs.gov)

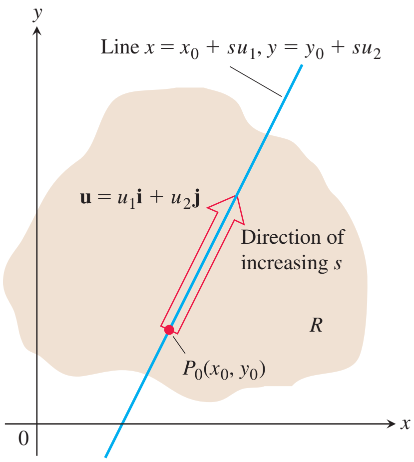


FIGURE 14.27 The rate of change of f in the direction of $\mathbf{u}$ at a point P_0 is the rate at which f changes along this line at P_0.

motion along the curve. If the curve is a straight line and t is the arc length parameter along the line measured from P_0 in the direction of a given unit vector $\mathbf{u}$, then df/dt is the rate of change of f with respect to distance in its domain in the direction of $\mathbf{u}$. By varying $\mathbf{u}$, we find the rates at which f changes with respect to distance as we move through P_0 in different directions. We now define this idea more precisely.

Suppose that the function $f(x, y)$ is defined throughout a region R in the xy-plane, that $P_0(x_0, y_0)$ is a point in R, and that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector. Then the equations

$$x = x_0 + su_1, \qquad y = y_0 + su_2$$

parametrize the line through P_0 parallel to $\mathbf{u}$. If the parameter s measures arc length from P_0 in the direction of $\mathbf{u}$, we find the rate of change of f at P_0 in the direction of $\mathbf{u}$ by calculating df/ds at P_0 (Figure 14.27).

DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \tag{1}$$

provided the limit exists.

The **directional derivative** defined by Equation (1) is also denoted by

$$(D_{\mathbf{u}}f)_{P_0}. \qquad \text{“The derivative of } f \text{ at } P_0 \text{ in the direction of } \mathbf{u}\text{”}$$

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the directional derivatives of f at P_0 in the $\mathbf{i}$ and $\mathbf{j}$ directions. This observation can be seen by comparing Equation (1) to the definitions of the two partial derivatives given in Section 14.3.

EXAMPLE 1 Using the definition, find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1, 2)$ in the direction of the unit vector $\mathbf{u} = (1/\sqrt{2})\mathbf{i} + (1/\sqrt{2})\mathbf{j}$.

Solution Applying the definition in Equation (1), we obtain

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s\to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s} \qquad \text{Eq. (1)}$$

$$= \lim_{s\to 0} \frac{f\left(1 + s \cdot \frac{1}{\sqrt{2}}, 2 + s \cdot \frac{1}{\sqrt{2}}\right) - f(1, 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{s}{\sqrt{2}}\right)^2 + \left(1 + \frac{s}{\sqrt{2}}\right)\left(2 + \frac{s}{\sqrt{2}}\right) - (1^2 + 1 \cdot 2)}{s}$$

$$= \lim_{s\to 0} \frac{\left(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}\right) + \left(2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\right) - 3}{s}$$

$$= \lim_{s\to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s} = \lim_{s\to 0} \left(\frac{5}{\sqrt{2}} + s\right) = \frac{5}{\sqrt{2}}.$$

The rate of change of $f(x, y) = x^2 + xy$ at $P_0(1, 2)$ in the direction $\mathbf{u}$ is $5/\sqrt{2}$. ■

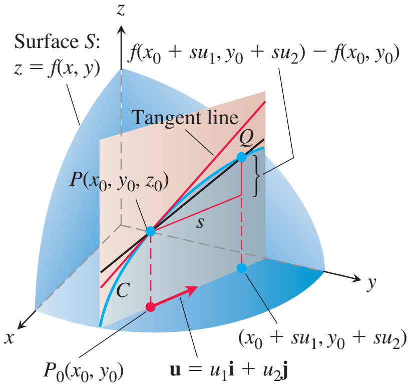


FIGURE 14.28 The slope of the trace curve C at P_0 is $\lim_{Q\to P}$ slope (PQ); this is the directional derivative

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (D_{\mathbf{u}} f)_{P_0}.$$

Interpretation of the Directional Derivative

The equation $z = f(x, y)$ represents a surface S in space. If $z_0 = f(x_0, y_0)$, then the point $P(x_0, y_0, z_0)$ lies on S. The vertical plane that passes through P and $P_0(x_0, y_0)$ parallel to $\mathbf{u}$ intersects S in a curve C (Figure 14.28). The rate of change of f in the direction of $\mathbf{u}$ is the slope of the tangent to C at P in the right-handed system formed by the vectors $\mathbf{u}$ and $\mathbf{k}$.

When $\mathbf{u} = \mathbf{i}$, the directional derivative at P_0 is $\partial f/\partial x$ evaluated at (x_0, y_0). When $\mathbf{u} = \mathbf{j}$, the directional derivative at P_0 is $\partial f/\partial y$ evaluated at (x_0, y_0). The directional derivative generalizes the two partial derivatives. We can now ask for the rate of change of f in any direction $\mathbf{u}$, not just the directions $\mathbf{i}$ and $\mathbf{j}$.

For a physical interpretation of the directional derivative, suppose that $T = f(x, y)$ is the temperature at each point (x, y) over a region in the plane. Then $f(x_0, y_0)$ is the temperature at the point $P_0(x_0, y_0)$ and $(D_{\mathbf{u}} f)_{P_0}$ is the instantaneous rate of change of the temperature at P_0 stepping off in the direction $\mathbf{u}$.

Calculation and Gradients

We now develop an efficient formula to calculate the directional derivative for a differentiable function f. We begin with the line

$$x = x_0 + su_1, \qquad y = y_0 + su_2, \tag{2}$$

through $P_0(x_0, y_0)$, parametrized with the arc length parameter s increasing in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$. Then by the Chain Rule we find

$$\begin{aligned}
\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} &= \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds} && \text{Chain Rule for differentiable } f \\
&= \left(\frac{\partial f}{\partial x}\right)_{P_0} u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0} u_2 && \text{From Eqs. (2), } dx/ds = u_1 \text{ and } dy/ds = u_2 \\
&= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}. && (3)
\end{aligned}$$

Equation (3) says that the derivative of a differentiable function f in the direction of $\mathbf{u}$ at P_0 is the dot product of $\mathbf{u}$ with the special vector, which we now define.

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f. Using the gradient notation, we restate Equation (3) as a theorem.

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 and $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution Recall that the direction of a vector $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$\begin{aligned}
f_x(2, 0) &= (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1 \\
f_y(2, 0) &= (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.
\end{aligned}$$

The gradient of f at $(2, 0)$ is

$$\nabla f\big|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

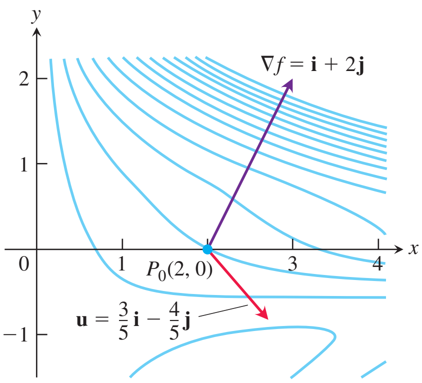


FIGURE 14.29 Picture ∇f as a vector in the domain of f. The figure shows a number of level curves of f. The rate at which f changes at $(2, 0)$ in the direction $\mathbf{u}$ is $\nabla f \cdot \mathbf{u} = -1$, which is the component of ∇f in the direction of unit vector $\mathbf{u}$ (Example 2).

(Figure 14.29). The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} \qquad \text{Eq. (4) with the } (D_{\mathbf{u}}f)_{P_0} \text{ notation}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

Evaluating the dot product in the brief version of Equation (4) gives

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f, and reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

As we discuss later, these properties hold in three dimensions as well as two.

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point $(1, 1)$, and

(b) decreases most rapidly at $(1, 1)$.

(c) What are the directions of zero change in f at $(1, 1)$?

Solution

(a) The function increases most rapidly in the direction of ∇f at $(1, 1)$. The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{(1)^2 + (1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

(b) The function decreases most rapidly in the direction of $-\nabla f$ at $(1, 1)$, which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at $(1, 1)$ are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.30.

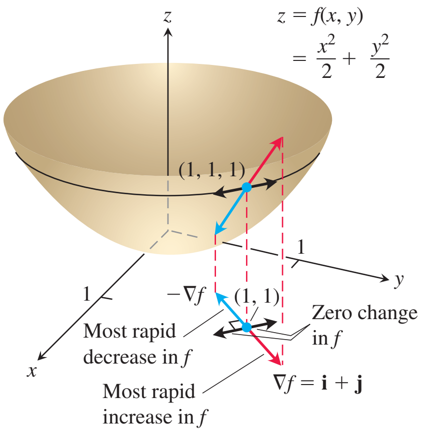


FIGURE 14.30 The direction in which $f(x, y)$ increases most rapidly at $(1, 1)$ is the direction of $\nabla f|_{(1,1)} = \mathbf{i} + \mathbf{j}$. It corresponds to the direction of steepest ascent on the surface at $(1, 1, 1)$ (Example 3).

Gradients and Tangents to Level Curves

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve part of a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

$$\frac{d}{dt}f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \qquad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0. \tag{5}$$

Equation (5) says that ∇f is normal to the tangent vector $d\mathbf{r}/dt$, so it is normal to the curve.

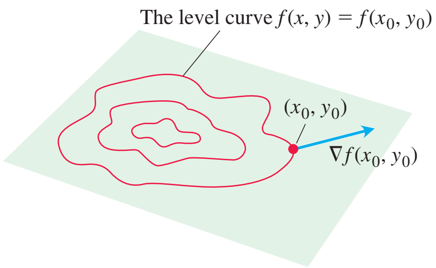


FIGURE 14.31 The gradient of a differentiable function of two variables at a point is always normal to the function's level curve through that point.

> At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.31).

Equation (5) validates our observation that streams flow perpendicular to the contours in topographical maps (see Figure 14.26). Since the downflowing stream will reach its destination in the fastest way, it must flow in the direction of the negative gradient vectors from Property 2 for the directional derivative. Equation (5) tells us these directions are perpendicular to the level curves.

This observation also enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients. The line through a point $P_0(x_0, y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0$$

(Exercise 39). If $\mathbf{N}$ is the gradient $(\nabla f)_{(x_0, y_0)} = f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}$, the equation gives the following formula.

> **Tangent Line to a Level Curve**
>
> $$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \tag{6}$$

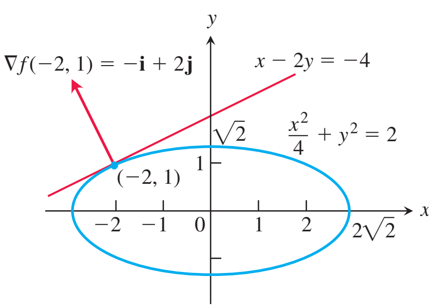


FIGURE 14.32 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

EXAMPLE 4 Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

(Figure 14.32) at the point $(-2, 1)$.

Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f\big|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

The tangent to the ellipse at $(-2, 1)$ is the line

$$(-1)(x + 2) + (2)(y - 1) = 0 \qquad \text{Eq. (6)}$$
$$x - 2y = -4.$$

If we know the gradients of two functions f and g, we automatically know the gradients of their sum, difference, constant multiples, product, and quotient. You are asked to establish the following rules in Exercise 40. Notice that these rules have the same form as the corresponding rules for derivatives of single-variable functions.

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

EXAMPLE 5 We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ — Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

$= 3y(\mathbf{i} - \mathbf{j}) + 3y\mathbf{j} + (3x - 6y)\mathbf{j}$ — $g\nabla f$ plus terms . . .

$= 3y(\mathbf{i} - \mathbf{j}) + (3x - 3y)\mathbf{j}$ — simplified.

$= 3y(\mathbf{i} - \mathbf{j}) + (x - y)3\mathbf{j} = g\nabla f + f\nabla g$ — Rule 4

Functions of Three Variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||u|\cos\theta = |\nabla f|\cos\theta,$$

so the properties listed earlier for functions of two variables extend to three variables. At any given point, f increases most rapidly in the direction of ∇f and decreases most rapidly in the direction of $-\nabla f$. In any direction orthogonal to ∇f, the derivative is zero.

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy\big|_{(1,1,0)} = -2, \qquad f_z = -1\big|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f\big|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f\big|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$ ■

The Chain Rule for Paths

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth path C, and $w = f(\mathbf{r}(t))$ is a scalar function evaluated along C, then according to the Chain Rule, Theorem 6 in Section 14.4,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

The partial derivatives on the right-hand side of the above equation are evaluated along the curve $\mathbf{r}(t)$, and the derivatives of the intermediate variables are evaluated at t. If we express this equation using vector notation, we have

The Derivative Along a Path

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t). \qquad (7)$$

What Equation (7) says is that the derivative of the composite function $f(\mathbf{r}(t))$ is the "derivative" (gradient) of the outside function f "times" (dot product) the derivative of the inside function $\mathbf{r}$. This is analogous to the "Outside-Inside" Rule for derivatives of composite functions studied in Section 3.6. That is, the multivariable Chain Rule for paths has exactly *the same form* as the rule for single-variable differential calculus when appropriate interpretations are given to the meanings of the terms and operations involved.

Exercises 14.5

Calculating Gradients

In Exercises 1–6, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$
2. $f(x, y) = \ln(x^2 + y^2), \quad (1, 1)$
3. $g(x, y) = xy^2, \quad (2, -1)$
4. $g(x, y) = \dfrac{x^2}{2} - \dfrac{y^2}{2}, \quad (\sqrt{2}, 1)$
5. $f(x, y) = \sqrt{2x + 3y}, \quad (-1, 2)$
6. $f(x, y) = \tan^{-1}\dfrac{\sqrt{x}}{y}, \quad (4, -2)$

In Exercises 7–10, find ∇f at the given point.

7. $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x, \quad (1, 1, 1)$
8. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1} xz, \quad (1, 1, 1)$
9. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), \quad (-1, 2, -2)$
10. $f(x, y, z) = e^{x+y}\cos z + (y + 1)\sin^{-1} x, \quad (0, 0, \pi/6)$

Finding Directional Derivatives

In Exercises 11–18, find the derivative of the function at P_0 in the direction of $\mathbf{u}$.

11. $f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$
12. $f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1), \quad \mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$
13. $g(x, y) = \dfrac{x - y}{xy + 2}, \quad P_0(1, -1), \quad \mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$
14. $h(x, y) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2), \quad P_0(1, 1),$ $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j}$
15. $f(x, y, z) = xy + yz + zx, \quad P_0(1, -1, 2), \quad \mathbf{u} = 3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$
16. $f(x, y, z) = x^2 + 2y^2 - 3z^2, \quad P_0(1, 1, 1), \quad \mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$
17. $g(x, y, z) = 3e^x \cos yz, \quad P_0(0, 0, 0), \quad \mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$
18. $h(x, y, z) = \cos xy + e^{yz} + \ln zx, \quad P_0(1, 0, 1/2),$ $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

In Exercises 19–24, find the directions in which the functions increase and decrease most rapidly at P_0. Then find the derivatives of the functions in these directions.

19. $f(x, y) = x^2 + xy + y^2, \quad P_0(-1, 1)$
20. $f(x, y) = x^2y + e^{xy}\sin y, \quad P_0(1, 0)$
21. $f(x, y, z) = (x/y) - yz, \quad P_0(4, 1, 1)$
22. $g(x, y, z) = xe^y + z^2, \quad P_0(1, \ln 2, 1/2)$
23. $f(x, y, z) = \ln xy + \ln yz + \ln xz, \quad P_0(1, 1, 1)$
24. $h(x, y, z) = \ln(x^2 + y^2 - 1) + y + 6z, \quad P_0(1, 1, 0)$

Tangent Lines to Level Curves

In Exercises 25–28, sketch the curve $f(x, y) = c$ together with ∇f and the tangent line at the given point. Then write an equation for the tangent line.

25. $x^2 + y^2 = 4, \quad (\sqrt{2}, \sqrt{2})$
26. $x^2 - y = 1, \quad (\sqrt{2}, 1)$
27. $xy = -4, \quad (2, -2)$
28. $x^2 - xy + y^2 = 7, \quad (-1, 2)$

Theory and Examples

29. Let $f(x, y) = x^2 - xy + y^2 - y$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f(1, -1)$ for which
    - **a.** $D_{\mathbf{u}} f(1, -1)$ is largest
    - **b.** $D_{\mathbf{u}} f(1, -1)$ is smallest
    - **c.** $D_{\mathbf{u}} f(1, -1) = 0$
    - **d.** $D_{\mathbf{u}} f(1, -1) = 4$
    - **e.** $D_{\mathbf{u}} f(1, -1) = -3$

30. Let $f(x, y) = \dfrac{(x - y)}{(x + y)}$. Find the directions $\mathbf{u}$ and the values of $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ for which
    - **a.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is largest
    - **b.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right)$ is smallest
    - **c.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 0$
    - **d.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = -2$
    - **e.** $D_{\mathbf{u}} f\left(-\frac{1}{2}, \frac{3}{2}\right) = 1$

31. **Zero directional derivative** In what direction is the derivative of $f(x, y) = xy + y^2$ at $P(3, 2)$ equal to zero?

32. **Zero directional derivative** In what directions is the derivative of $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ at $P(1, 1)$ equal to zero?

33. Is there a direction $\mathbf{u}$ in which the rate of change of $f(x, y) = x^2 - 3xy + 4y^2$ at $P(1, 2)$ equals 14? Give reasons for your answer.

34. **Changing temperature along a circle** Is there a direction $\mathbf{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is -3°C/ft? Give reasons for your answer.

35. The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$ and in the direction of $-2\mathbf{j}$ is -3. What is the derivative of f in the direction of $-\mathbf{i} - 2\mathbf{j}$? Give reasons for your answer.

36. The derivative of $f(x, y, z)$ at a point P is greatest in the direction of $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$. In this direction, the value of the derivative is $2\sqrt{3}$.
    - **a.** What is ∇f at P? Give reasons for your answer.
    - **b.** What is the derivative of f at P in the direction of $\mathbf{i} + \mathbf{j}$?

37. **Directional derivatives and scalar components** How is the derivative of a differentiable function $f(x, y, z)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$ related to the scalar component of $(\nabla f)_{P_0}$ in the direction of $\mathbf{u}$? Give reasons for your answer.

38. **Directional derivatives and partial derivatives** Assuming that the necessary derivatives of $f(x, y, z)$ are defined, how are $D_{\mathbf{i}}f$, $D_{\mathbf{j}}f$, and $D_{\mathbf{k}}f$ related to f_x, f_y, and f_z? Give reasons for your answer.

39. **Lines in the xy-plane** Show that $A(x - x_0) + B(y - y_0) = 0$ is an equation for the line in the xy-plane through the point (x_0, y_0) normal to the vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$.

40. **The algebra rules for gradients** Given a constant k and the gradients

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}, \qquad \nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k},$$

establish the algebra rules for gradients.

14.6 Tangent Planes and Differentials

In single-variable differential calculus we saw how the derivative defined the tangent line to the graph of a differentiable function at a point on the graph. The tangent line then provided for a linearization of the function at the point. In this section, we will see analogously how the gradient defines the *tangent plane* to the level surface of a function $w = f(x, y, z)$ at a point on the surface. In the same way as before, the tangent plane then provides for a linearization of f at the point and defines the total differential of the function.

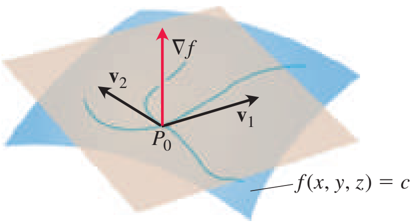


FIGURE 14.33 The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_0. The velocity vectors at P_0 therefore lie in a common plane, which we call the tangent plane at P_0.

Tangent Planes and Normal Lines

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, we found in Equation (7) of the last section that

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Since f is constant along the curve $\mathbf{r}$, the derivative on the left-hand side of the equation is 0, so the gradient ∇f is orthogonal to the curve's velocity vector $\mathbf{r}'$.

Now let us restrict our attention to the curves that pass through P_0 (Figure 14.33). All the velocity vectors at P_0 are orthogonal to ∇f at P_0, so the curves' tangent lines all lie in the plane through P_0 normal to ∇f. We now define this plane.

DEFINITIONS The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

From Section 12.5, the tangent plane and normal line have the following equations:

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \tag{1}$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \tag{2}$$

EXAMPLE 1 Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \qquad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.

Solution The surface is shown in Figure 14.34.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

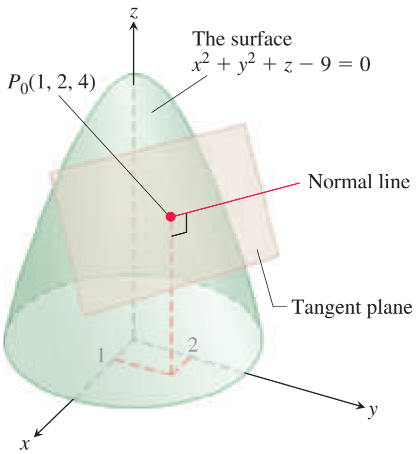


FIGURE 14.34 The tangent plane and normal line to this level surface at P_0 (Example 1).

To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$ is therefore the zero level surface of the function $F(x, y, z) = f(x, y) - z$. The partial derivatives of F are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1.$$

The formula

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

for the plane tangent to the level surface at P_0 therefore reduces to

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \tag{3}$$

EXAMPLE 2 Find the plane tangent to the surface $z = x \cos y - ye^x$ at $(0, 0, 0)$.

Solution We calculate the partial derivatives of $f(x, y) = x \cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$
$$f_y(0, 0) = (-x \sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (3)}$$

or

$$x - y - z = 0.$$

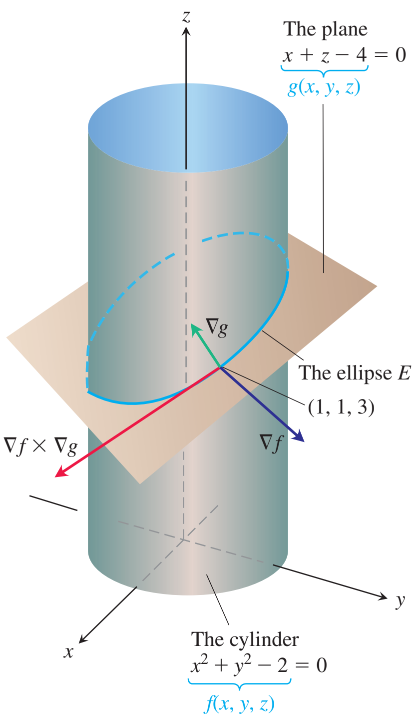


FIGURE 14.35 This cylinder and plane intersect in an ellipse E (Example 3).

EXAMPLE 3 The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (Figure 14.35). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.

Solution The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f\big|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g\big|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t.$$

Estimating Change in a Specific Direction

The directional derivative plays the role of an ordinary derivative when we want to estimate how much the value of a function f changes if we move a small distance ds from a point P_0 to another point nearby. If f were a function of a single variable, we would have

$$df = f'(P_0)\, ds. \qquad \text{Ordinary derivative} \times \text{increment}$$

For a function of two or more variables, we use the formula

$$df = (\nabla f\big|_{P_0} \cdot \mathbf{u})\, ds, \qquad \text{Directional derivative} \times \text{increment}$$

where $\mathbf{u}$ is the direction of the motion away from P_0.

Estimating the Change in f in a Direction $\mathbf{u}$

To estimate the change in the value of a differentiable function f when we move a small distance ds from a point P_0 in a particular direction $\mathbf{u}$, use the formula

$$df = \underbrace{(\nabla f\big|_{P_0} \cdot \mathbf{u})}_{\substack{\text{Directional}\\ \text{derivative}}} \underbrace{ds}_{\substack{\text{Distance}\\ \text{increment}}}$$

EXAMPLE 4 Estimate how much the value of

$$f(x, y, z) = y \sin x + 2yz$$

will change if the point $P(x, y, z)$ moves 0.1 unit from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

Solution We first find the derivative of f at P_0 in the direction of the vector $\overrightarrow{P_0P_1} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. The direction of this vector is

$$\mathbf{u} = \frac{\overrightarrow{P_0P_1}}{|\overrightarrow{P_0P_1}|} = \frac{\overrightarrow{P_0P_1}}{3} = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}.$$

The gradient of f at P_0 is

$$\nabla f\big|_{(0,1,0)} = ((y \cos x)\mathbf{i} + (\sin x + 2z)\mathbf{j} + 2y\mathbf{k})_{(0,1,0)} = \mathbf{i} + 2\mathbf{k}.$$

Therefore,

$$\nabla f\big|_{P_0} \cdot \mathbf{u} = (\mathbf{i} + 2\mathbf{k}) \cdot \left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) = \frac{2}{3} - \frac{4}{3} = -\frac{2}{3}.$$

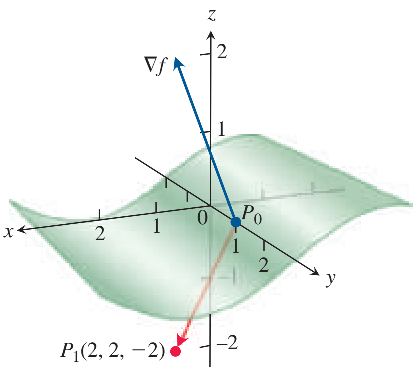


FIGURE 14.36 As $P(x, y, z)$ moves off the level surface at P_0 by 0.1 unit directly toward P_1, the function f changes value by approximately -0.067 unit (Example 4).

The change df in f that results from moving $ds = 0.1$ unit away from P_0 in the direction of **u** is approximately

$$df = (\nabla f|_{P_0} \cdot \mathbf{u})(ds) = \left(-\frac{2}{3}\right)(0.1) \approx -0.067 \text{ unit.}$$

See Figure 14.36.

How to Linearize a Function of Two Variables

Functions of two variables can be complicated, and we sometimes need to approximate them with simpler ones that give the accuracy required for specific applications without being so difficult to work with. We do this in a way that is similar to the way we find linear replacements for functions of a single variable (Section 3.9).

Suppose the function we wish to approximate is $z = f(x, y)$ near a point (x_0, y_0) at which we know the values of f, f_x, and f_y and at which f is differentiable. If we move from (x_0, y_0) to any nearby point (x, y) by increments $\Delta x = x - x_0$ and $\Delta y = y - y_0$ (see Figure 14.37), then the definition of differentiability from Section 14.3 gives the change

$$f(x, y) - f(x_0, y_0) = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where $\epsilon_1, \epsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$. If the increments Δx and Δy are small, the products $\epsilon_1 \Delta x$ and $\epsilon_2 \Delta y$ will eventually be smaller still and we have the approximation

$$f(x, y) \approx \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{L(x, y)}.$$

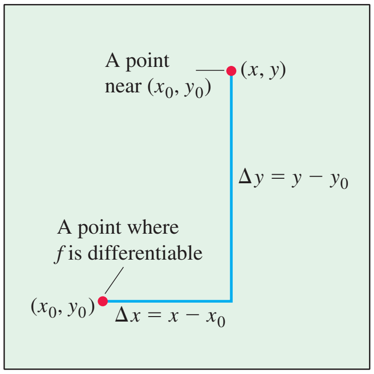


FIGURE 14.37 If f is differentiable at (x_0, y_0), then the value of f at any point (x, y) nearby is approximately $f(x_0, y_0) + f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$.

In other words, as long as Δx and Δy are small, f will have approximately the same value as the linear function L.

DEFINITIONS The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).

From Equation (3), we find that the plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point (x_0, y_0). Thus, the linearization of a function of two variables is a tangent-*plane* approximation in the same way that the linearization of a function of a single variable is a tangent-*line* approximation. (See Exercise 55.)

EXAMPLE 5 Find the linearization of

$$f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$$

at the point (3, 2).

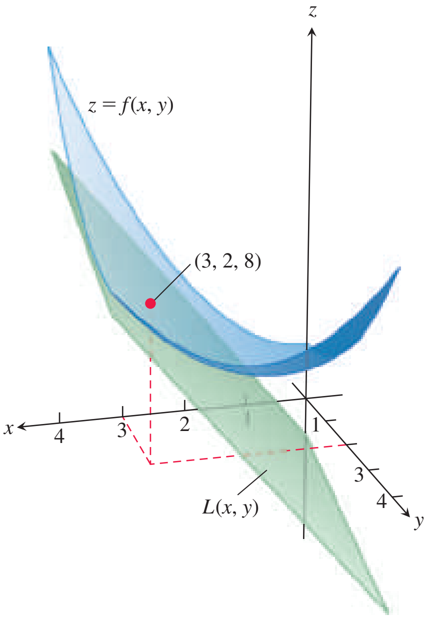


FIGURE 14.38 The tangent plane $L(x, y)$ represents the linearization of $f(x, y)$ in Example 5.

Solution We first evaluate f, f_x, and f_y at the point $(x_0, y_0) = (3, 2)$:

$$f(3, 2) = \left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = 8$$

$$f_x(3, 2) = \frac{\partial}{\partial x}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (2x - y)_{(3,2)} = 4$$

$$f_y(3, 2) = \frac{\partial}{\partial y}\left(x^2 - xy + \frac{1}{2}y^2 + 3\right)_{(3,2)} = (-x + y)_{(3,2)} = -1,$$

giving

$$\begin{aligned} L(x, y) &= f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= 8 + (4)(x - 3) + (-1)(y - 2) = 4x - y - 2. \end{aligned}$$

The linearization of f at $(3, 2)$ is $L(x, y) = 4x - y - 2$ (see Figure 14.38). ■

When approximating a differentiable function $f(x, y)$ by its linearization $L(x, y)$ at (x_0, y_0), an important question is how accurate the approximation might be.

If we can find a common upper bound M for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on a rectangle R centered at (x_0, y_0) (Figure 14.39), then we can bound the error E throughout R by using a simple formula (derived in Section 14.9). The **error** is defined by $E(x, y) = f(x, y) - L(x, y)$.

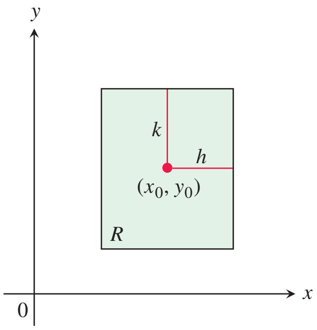


FIGURE 14.39 The rectangular region R: $|x - x_0| \le h$, $|y - y_0| \le k$ in the xy-plane.

The Error in the Standard Linear Approximation

If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \le \frac{1}{2}M(|x - x_0| + |y - y_0|)^2.$$

To make $|E(x, y)|$ small for a given M, we just make $|x - x_0|$ and $|y - y_0|$ small.

Differentials

Recall from Section 3.9 that for a function of a single variable, $y = f(x)$, we defined the change in f as x changes from a to $a + \Delta x$ by

$$\Delta f = f(a + \Delta x) - f(a)$$

and the differential of f as

$$df = f'(a)\Delta x.$$

We now consider the differential of a function of two variables.

Suppose a differentiable function $f(x, y)$ and its partial derivatives exist at a point (x_0, y_0). If we move to a nearby point $(x_0 + \Delta x, y_0 + \Delta y)$, the change in f is

$$\Delta f = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0).$$

A straightforward calculation from the definition of $L(x, y)$, using the notation $x - x_0 = \Delta x$ and $y - y_0 = \Delta y$, shows that the corresponding change in L is

$$\begin{aligned}\Delta L &= L(x_0 + \Delta x, y_0 + \Delta y) - L(x_0, y_0)\\ &= f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y.\end{aligned}$$

The **differentials** dx and dy are independent variables, so they can be assigned any values. Often we take $dx = \Delta x = x - x_0$, and $dy = \Delta y = y - y_0$. We then have the following definition of the differential or *total* differential of f.

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\, dx + f_y(x_0, y_0)\, dy$$

in the linearization of f is called the **total differential of f**.

EXAMPLE 6 Suppose that a cylindrical can is designed to have a radius of 1 in. and a height of 5 in., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution To estimate the absolute change in $V = \pi r^2 h$, we use

$$\Delta V \approx dV = V_r(r_0, h_0)\, dr + V_h(r_0, h_0)\, dh.$$

With $V_r = 2\pi rh$ and $V_h = \pi r^2$, we get

$$\begin{aligned}dV &= 2\pi r_0 h_0\, dr + \pi r_0{}^2\, dh = 2\pi(1)(5)(0.03) + \pi(1)^2(-0.1)\\ &= 0.3\pi - 0.1\pi = 0.2\pi \approx 0.63 \text{ in}^3\end{aligned}$$

■

EXAMPLE 7 Your company manufactures stainless steel right circular cylindrical molasses storage tanks that are 25 ft high with a radius of 5 ft. How sensitive are the tanks' volumes to small variations in height and radius?

Solution With $V = \pi r^2 h$, the total differential gives the approximation for the change in volume as

$$\begin{aligned}dV &= V_r(5, 25)\, dr + V_h(5, 25)\, dh\\ &= (2\pi rh)_{(5,25)}\, dr + (\pi r^2)_{(5,25)}\, dh\\ &= 250\pi\, dr + 25\pi\, dh.\end{aligned}$$

Thus, a 1-unit change in r will change V by about 250π units. A 1-unit change in h will change V by about 25π units. The tank's volume is 10 times more sensitive to a small change in r than it is to a small change of equal size in h. As a quality control engineer concerned with being sure the tanks have the correct volume, you would want to pay special attention to their radii.

In contrast, if the values of r and h are reversed to make $r = 25$ and $h = 5$, then the total differential in V becomes

$$dV = (2\pi rh)_{(25,5)}\, dr + (\pi r^2)_{(25,5)}\, dh = 250\pi\, dr + 625\pi\, dh.$$

Now the volume is more sensitive to changes in h than to changes in r (Figure 14.40).

The general rule is that functions are most sensitive to small changes in the variables that generate the largest partial derivatives. ■

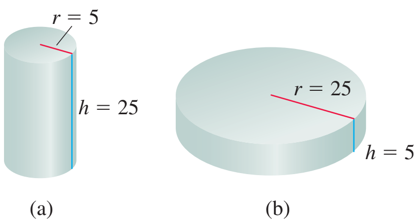


FIGURE 14.40 The volume of cylinder (a) is more sensitive to a small change in r than it is to an equally small change in h. The volume of cylinder (b) is more sensitive to small changes in h than it is to small changes in r (Example 7).

Functions of More Than Two Variables

Analogous results hold for differentiable functions of more than two variables.

1. The **linearization** of $f(x, y, z)$ at a point $P_0(x_0, y_0, z_0)$ is

$$L(x, y, z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0).$$

2. Suppose that R is a closed rectangular solid centered at P_0 and lying in an open region on which the second partial derivatives of f are continuous. Suppose also that $|f_{xx}|, |f_{yy}|, |f_{zz}|, |f_{xy}|, |f_{xz}|$, and $|f_{yz}|$ are all less than or equal to M throughout R. Then the **error** $E(x, y, z) = f(x, y, z) - L(x, y, z)$ in the approximation of f by L is bounded throughout R by the inequality

$$|E| \le \frac{1}{2}M(|x - x_0| + |y - y_0| + |z - z_0|)^2.$$

3. If the second partial derivatives of f are continuous and if x, y, and z change from x_0, y_0, and z_0 by small amounts dx, dy, and dz, the **total differential**

$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

gives a good approximation of the resulting change in f.

EXAMPLE 8 Find the linearization $L(x, y, z)$ of

$$f(x, y, z) = x^2 - xy + 3\sin z$$

at the point $(x_0, y_0, z_0) = (2, 1, 0)$. Find an upper bound for the error incurred in replacing f by L on the rectangular region

$$R: \quad |x - 2| \le 0.01, \qquad |y - 1| \le 0.02, \qquad |z| \le 0.01.$$

Solution Routine calculations give

$$f(2, 1, 0) = 2, \qquad f_x(2, 1, 0) = 3, \qquad f_y(2, 1, 0) = -2, \qquad f_z(2, 1, 0) = 3.$$

Thus,

$$L(x, y, z) = 2 + 3(x - 2) + (-2)(y - 1) + 3(z - 0) = 3x - 2y + 3z - 2.$$

Since

$$f_{xx} = 2, \qquad f_{yy} = 0, \qquad f_{zz} = -3\sin z, \qquad f_{xy} = -1, \qquad f_{xz} = 0, \qquad f_{yz} = 0,$$

and $|-3\sin z| \le 3\sin 0.01 \approx 0.03$, we may take $M = 2$ as a bound on the second partials. Hence, the error incurred by replacing f by L on R satisfies

$$|E| \le \frac{1}{2}(2)(0.01 + 0.02 + 0.01)^2 = 0.0016.$$

Exercises 14.6

Tangent Planes and Normal Lines to Surfaces

In Exercises 1–8, find equations for the

(a) tangent plane and

(b) normal line at the point P_0 on the given surface.

1. $x^2 + y^2 + z^2 = 3$, $\quad P_0(1, 1, 1)$
2. $x^2 + y^2 - z^2 = 18$, $\quad P_0(3, 5, -4)$
3. $2z - x^2 = 0$, $\quad P_0(2, 0, 2)$
4. $x^2 + 2xy - y^2 + z^2 = 7$, $\quad P_0(1, -1, 3)$
5. $\cos \pi x - x^2 y + e^{xz} + yz = 4$, $\quad P_0(0, 1, 2)$
6. $x^2 - xy - y^2 - z = 0$, $\quad P_0(1, 1, -1)$
7. $x + y + z = 1$, $\quad P_0(0, 1, 0)$
8. $x^2 + y^2 - 2xy - x + 3y - z = -4$, $\quad P_0(2, -3, 18)$

In Exercises 9–12, find an equation for the plane that is tangent to the given surface at the given point.

9. $z = \ln(x^2 + y^2)$, $\quad (1, 0, 0)$
10. $z = e^{-(x^2+y^2)}$, $\quad (0, 0, 1)$
11. $z = \sqrt{y - x}$, $\quad (1, 2, 1)$
12. $z = 4x^2 + y^2$, $\quad (1, 1, 5)$

Tangent Lines to Intersecting Surfaces

In Exercises 13–18, find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.

13. Surfaces: $x + y^2 + 2z = 4, \quad x = 1$

Point: $(1, 1, 1)$

14. Surfaces: $xyz = 1, \quad x^2 + 2y^2 + 3z^2 = 6$

Point: $(1, 1, 1)$

15. Surfaces: $x^2 + 2y + 2z = 4, \quad y = 1$

Point: $(1, 1, 1/2)$

16. Surfaces: $x + y^2 + z = 2, \quad y = 1$

Point: $(1/2, 1, 1/2)$

17. Surfaces: $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0,$
$x^2 + y^2 + z^2 = 11$

Point: $(1, 1, 3)$

18. Surfaces: $x^2 + y^2 = 4, \quad x^2 + y^2 - z = 0$

Point: $\left(\sqrt{2}, \sqrt{2}, 4\right)$

Estimating Change

19. By about how much will

$$f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$$

change if the point $P(x, y, z)$ moves from $P_0(3, 4, 12)$ a distance of $ds = 0.1$ unit in the direction of $3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}$?

20. By about how much will

$$f(x, y, z) = e^x \cos yz$$

change as the point $P(x, y, z)$ moves from the origin a distance of $ds = 0.1$ unit in the direction of $2\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$?

21. By about how much will

$$g(x, y, z) = x + x\cos z - y\sin z + y$$

change if the point $P(x, y, z)$ moves from $P_0(2, -1, 0)$ a distance of $ds = 0.2$ unit toward the point $P_1(0, 1, 2)$?

22. By about how much will

$$h(x, y, z) = \cos(\pi xy) + xz^2$$

change if the point $P(x, y, z)$ moves from $P_0(-1, -1, -1)$ a distance of $ds = 0.1$ unit toward the origin?

23. Temperature change along a circle Suppose that the Celsius temperature at the point (x, y) in the xy-plane is $T(x, y) = x \sin 2y$ and that distance in the xy-plane is measured in meters. A particle is moving *clockwise* around the circle of radius 1 m centered at the origin at the constant rate of 2 m/sec.

a. How fast is the temperature experienced by the particle changing in degrees Celsius per meter at the point $P\left(1/2, \sqrt{3}/2\right)$?

b. How fast is the temperature experienced by the particle changing in degrees Celsius per second at P?

24. Changing temperature along a space curve The Celsius temperature in a region in space is given by $T(x, y, z) = 2x^2 - xyz$. A particle is moving in this region and its position at time t is given by $x = 2t^2$, $y = 3t$, $z = -t^2$, where time is measured in seconds and distance in meters.

a. How fast is the temperature experienced by the particle changing in degrees Celsius per meter when the particle is at the point $P(8, 6, -4)$?

b. How fast is the temperature experienced by the particle changing in degrees Celsius per second at P?

Finding Linearizations

In Exercises 25–30, find the linearization $L(x, y)$ of the function at each point.

25. $f(x, y) = x^2 + y^2 + 1$ at **a.** $(0, 0)$, **b.** $(1, 1)$

26. $f(x, y) = (x + y + 2)^2$ at **a.** $(0, 0)$, **b.** $(1, 2)$

27. $f(x, y) = 3x - 4y + 5$ at **a.** $(0, 0)$, **b.** $(1, 1)$

28. $f(x, y) = x^3y^4$ at **a.** $(1, 1)$, **b.** $(0, 0)$

29. $f(x, y) = e^x \cos y$ at **a.** $(0, 0)$, **b.** $(0, \pi/2)$

30. $f(x, y) = e^{2y-x}$ at **a.** $(0, 0)$, **b.** $(1, 2)$

31. Wind chill factor Wind chill, a measure of the apparent temperature felt on exposed skin, is a function of air temperature and wind speed. The precise formula, updated by the National Weather Service in 2001 and based on modern heat transfer theory, a human face model, and skin tissue resistance, is

$$W = W(v, T) = 35.74 + 0.6215\,T - 35.75\,v^{0.16} + 0.4275\,T \cdot v^{0.16},$$

where T is air temperature in °F and v is wind speed in mph. A partial wind chill chart is given.

v (mph) \ T(°F)	30	25	20	15	10	5	0	−5	−10
5	25	19	13	7	1	−5	−11	−16	−22
10	21	15	9	3	−4	−10	−16	−22	−28
15	19	13	6	0	−7	−13	−19	−26	−32
20	17	11	4	−2	−9	−15	−22	−29	−35
25	16	9	3	−4	−11	−17	−24	−31	−37
30	15	8	1	−5	−12	−19	−26	−33	−39
35	14	7	0	−7	−14	−21	−27	−34	−41

a. Use the table to find $W(20, 25)$, $W(30, -10)$, and $W(15, 15)$.

b. Use the formula to find $W(10, -40)$, $W(50, -40)$, and $W(60, 30)$.

c. Find the linearization $L(v, T)$ of the function $W(v, T)$ at the point $(25, 5)$.

d. Use $L(v, T)$ in part (c) to estimate the following wind chill values.

i) $W(24, 6)$ **ii)** $W(27, 2)$

iii) $W(5, -10)$ (Explain why this value is much different from the value found in the table.)

32. Find the linearization $L(v, T)$ of the function $W(v, T)$ in Exercise 31 at the point $(50, -20)$. Use it to estimate the following wind chill values.

a. $W(49, -22)$

b. $W(53, -19)$

c. $W(60, -30)$

Bounding the Error in Linear Approximations

In Exercises 33–38, find the linearization $L(x, y)$ of the function $f(x, y)$ at P_0. Then find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R.

33. $f(x, y) = x^2 - 3xy + 5$ at $P_0(2, 1)$,
R: $|x - 2| \le 0.1$, $|y - 1| \le 0.1$

34. $f(x, y) = (1/2)x^2 + xy + (1/4)y^2 + 3x - 3y + 4$ at $P_0(2, 2)$,
R: $|x - 2| \le 0.1$, $|y - 2| \le 0.1$

35. $f(x, y) = 1 + y + x\cos y$ at $P_0(0, 0)$,
R: $|x| \le 0.2$, $|y| \le 0.2$
(Use $|\cos y| \le 1$ and $|\sin y| \le 1$ in estimating E.)

36. $f(x, y) = xy^2 + y\cos(x - 1)$ at $P_0(1, 2)$,
R: $|x - 1| \le 0.1$, $|y - 2| \le 0.1$

37. $f(x, y) = e^x \cos y$ at $P_0(0, 0)$,
R: $|x| \le 0.1$, $|y| \le 0.1$
(Use $e^x \le 1.11$ and $|\cos y| \le 1$ in estimating E.)

38. $f(x, y) = \ln x + \ln y$ at $P_0(1, 1)$,
R: $|x - 1| \le 0.2$, $|y - 1| \le 0.2$

Linearizations for Three Variables

Find the linearizations $L(x, y, z)$ of the functions in Exercises 39–44 at the given points.

39. $f(x, y, z) = xy + yz + xz$ at
a. $(1, 1, 1)$ **b.** $(1, 0, 0)$ **c.** $(0, 0, 0)$

40. $f(x, y, z) = x^2 + y^2 + z^2$ at
a. $(1, 1, 1)$ **b.** $(0, 1, 0)$ **c.** $(1, 0, 0)$

41. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at
a. $(1, 0, 0)$ **b.** $(1, 1, 0)$ **c.** $(1, 2, 2)$

42. $f(x, y, z) = (\sin xy)/z$ at
a. $(\pi/2, 1, 1)$ **b.** $(2, 0, 1)$

43. $f(x, y, z) = e^x + \cos(y + z)$ at
a. $(0, 0, 0)$ **b.** $\left(0, \frac{\pi}{2}, 0\right)$ **c.** $\left(0, \frac{\pi}{4}, \frac{\pi}{4}\right)$

44. $f(x, y, z) = \tan^{-1}(xyz)$ at
a. $(1, 0, 0)$ **b.** $(1, 1, 0)$ **c.** $(1, 1, 1)$

In Exercises 45–48, find the linearization $L(x, y, z)$ of the function $f(x, y, z)$ at P_0. Then find an upper bound for the magnitude of the error E in the approximation $f(x, y, z) \approx L(x, y, z)$ over the region R.

45. $f(x, y, z) = xz - 3yz + 2$ at $P_0(1, 1, 2)$,
R: $|x - 1| \le 0.01$, $|y - 1| \le 0.01$, $|z - 2| \le 0.02$

46. $f(x, y, z) = x^2 + xy + yz + (1/4)z^2$ at $P_0(1, 1, 2)$,
R: $|x - 1| \le 0.01$, $|y - 1| \le 0.01$, $|z - 2| \le 0.08$

47. $f(x, y, z) = xy + 2yz - 3xz$ at $P_0(1, 1, 0)$,
R: $|x - 1| \le 0.01$, $|y - 1| \le 0.01$, $|z| \le 0.01$

48. $f(x, y, z) = \sqrt{2}\cos x \sin(y + z)$ at $P_0(0, 0, \pi/4)$,
R: $|x| \le 0.01$, $|y| \le 0.01$, $|z - \pi/4| \le 0.01$

Estimating Error; Sensitivity to Change

49. Estimating maximum error Suppose that T is to be found from the formula $T = x(e^y + e^{-y})$, where x and y are found to be 2 and $\ln 2$ with maximum possible errors of $|dx| = 0.1$ and $|dy| = 0.02$. Estimate the maximum possible error in the computed value of T.

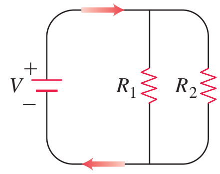
50. Variation in electrical resistance The resistance R produced by wiring resistors of R_1 and R_2 ohms in parallel (see accompanying figure) can be calculated from the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

a. Show that

$$dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2.$$

b. You have designed a two-resistor circuit, like the one shown, to have resistances of $R_1 = 100$ ohms and $R_2 = 400$ ohms, but there is always some variation in manufacturing and the resistors received by your firm will probably not have these exact values. Will the value of R be more sensitive to variation in R_1 or to variation in R_2? Give reasons for your answer.



c. In another circuit like the one shown, you plan to change R_1 from 20 to 20.1 ohms and R_2 from 25 to 24.9 ohms. By about what percentage will this change R?

51. You plan to calculate the area of a long, thin rectangle from measurements of its length and width. Which dimension should you measure more carefully? Give reasons for your answer.

52. a. Around the point $(1, 0)$, is $f(x, y) = x^2(y + 1)$ more sensitive to changes in x or to changes in y? Give reasons for your answer.

b. What ratio of dx to dy will make df equal zero at $(1, 0)$?

53. Value of a 2 × 2 determinant If $|a|$ is much greater than $|b|$, $|c|$, and $|d|$, to which of a, b, c, and d is the value of the determinant

$$f(a, b, c, d) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

most sensitive? Give reasons for your answer.

54. The Wilson lot size formula The Wilson lot size formula in economics says that the most economical quantity Q of goods (radios, shoes, brooms, whatever) for a store to order is given by the formula $Q = \sqrt{2KM/h}$, where K is the cost of placing the order, M is the number of items sold per week, and h is the weekly holding cost for each item (cost of space, utilities, security, and so on). To which of the variables K, M, and h is Q most sensitive near the point $(K_0, M_0, h_0) = (2, 20, 0.05)$? Give reasons for your answer.

Theory and Examples

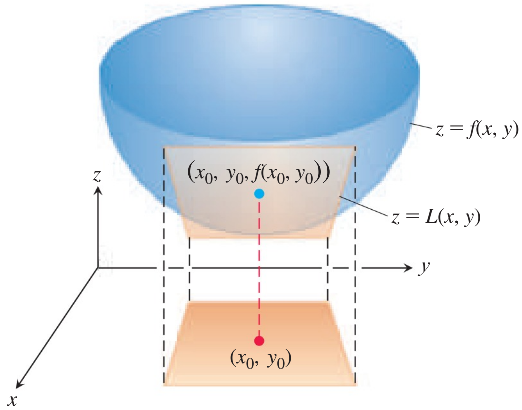
55. The linearization of $f(x, y)$ is a tangent-plane approximation Show that the tangent plane at the point $P_0(x_0, y_0, f(x_0, y_0))$ on the surface $z = f(x, y)$ defined by a differentiable function f is the plane

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - f(x_0, y_0)) = 0$$

or

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Thus, the tangent plane at P_0 is the graph of the linearization of f at P_0 (see accompanying figure).



56. Change along the involute of a circle Find the derivative of $f(x, y) = x^2 + y^2$ in the direction of the unit tangent vector of the curve

$$\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \qquad t > 0.$$

57. Tangent curves A smooth curve is *tangent* to the surface at a point of intersection if its velocity vector is orthogonal to ∇f there.

Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} + (2t - 1)\mathbf{k}$$

is tangent to the surface $x^2 + y^2 - z = 1$ when $t = 1$.

58. Normal curves A smooth curve is *normal* to a surface $f(x, y, z) = c$ at a point of intersection if the curve's velocity vector is a nonzero scalar multiple of ∇f at the point.

Show that the curve

$$\mathbf{r}(t) = \sqrt{t}\mathbf{i} + \sqrt{t}\mathbf{j} - \frac{1}{4}(t + 3)\mathbf{k}$$

is normal to the surface $x^2 + y^2 - z = 3$ when $t = 1$.

14.7 Extreme Values and Saddle Points

Continuous functions of two variables assume extreme values on closed, bounded domains (see Figures 14.41 and 14.42). We see in this section that we can narrow the search for these extreme values by examining the functions' first partial derivatives. A function of two variables can assume extreme values only at domain boundary points or at interior domain points where both first partial derivatives are zero or where one or both of the first partial derivatives fail to exist. However, the vanishing of derivatives at an interior point (a, b) does not always signal the presence of an extreme value. The surface that is the graph of the function might be shaped like a saddle right above (a, b) and cross its tangent plane there.

HISTORICAL BIOGRAPHY
Siméon-Denis Poisson
(1781–1840)

Derivative Tests for Local Extreme Values

To find the local extreme values of a function of a single variable, we look for points where the graph has a horizontal tangent line. At such points, we then look for local maxima, local minima, and points of inflection. For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a horizontal tangent *plane*. At such points, we then look for local maxima, local minima, and saddle points. We begin by defining maxima and minima.

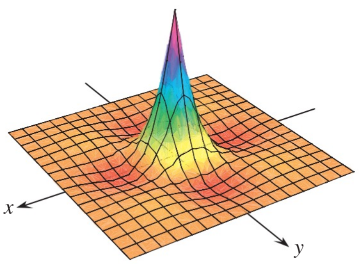


FIGURE 14.41 The function

$$z = (\cos x)(\cos y)e^{-\sqrt{x^2+y^2}}$$

has a maximum value of 1 and a minimum value of about -0.067 on the square region $|x| \leq 3\pi/2$, $|y| \leq 3\pi/2$.

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).

Local maxima correspond to mountain peaks on the surface $z = f(x, y)$ and local minima correspond to valley bottoms (Figure 14.43). At such points the tangent planes, when they exist, are horizontal. Local extrema are also called **relative extrema**.

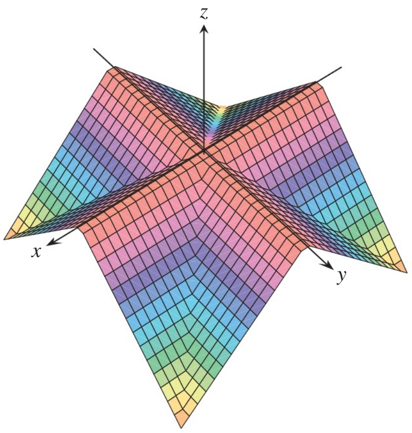


FIGURE 14.42 The "roof surface"

$$z = \frac{1}{2}\left(\left||x| - |y|\right| - |x| - |y|\right)$$

has a maximum value of 0 and a minimum value of $-a$ on the square region $|x| \le a$, $|y| \le a$.

As with functions of a single variable, the key to identifying the local extrema is the First Derivative Test, which we next state and prove.

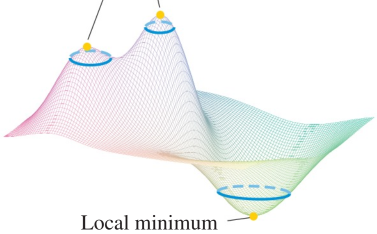


FIGURE 14.43 A local maximum occurs at a mountain peak and a local minimum occurs at a valley low point.

THEOREM 10—First Derivative Test for Local Extreme Values If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

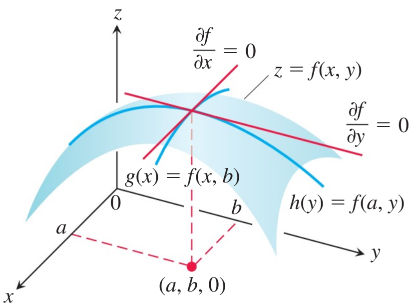


FIGURE 14.44 If a local maximum of f occurs at $x = a$, $y = b$, then the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are both zero.

Proof If f has a local extremum at (a, b), then the function $g(x) = f(x, b)$ has a local extremum at $x = a$ (Figure 14.44). Therefore, $g'(a) = 0$ (Chapter 4, Theorem 2). Now $g'(a) = f_x(a, b)$, so $f_x(a, b) = 0$. A similar argument with the function $h(y) = f(a, y)$ shows that $f_y(a, b) = 0$. ■

If we substitute the values $f_x(a, b) = 0$ and $f_y(a, b) = 0$ into the equation

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) - (z - f(a, b)) = 0$$

for the tangent plane to the surface $z = f(x, y)$ at (a, b), the equation reduces to

$$0 \cdot (x - a) + 0 \cdot (y - b) - z + f(a, b) = 0$$

or

$$z = f(a, b).$$

Thus, Theorem 10 says that the surface does indeed have a horizontal tangent plane at a local extremum, provided there is a tangent plane there.

DEFINITION An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

Theorem 10 says that the only points where a function $f(x, y)$ can assume extreme values are critical points and boundary points. As with differentiable functions of a single variable, not every critical point gives rise to a local extremum. A differentiable function of a single variable might have a point of inflection. A differentiable function of two variables might have a *saddle point*, with the graph of f crossing the tangent plane defined there.

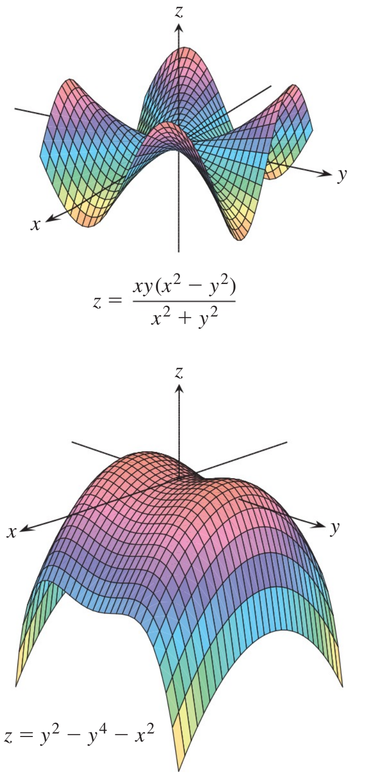


FIGURE 14.45 Saddle points at the origin.

DEFINITION A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.45).

EXAMPLE 1 Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \quad \text{and} \quad f_y = 2y - 4 = 0.$$

The only possibility is the point $(0, 2)$, where the value of f is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point $(0, 2)$ gives a local minimum (Figure 14.46). ■

EXAMPLE 2 Find the local extreme values (if any) of $f(x, y) = y^2 - x^2$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = -2x$ and $f_y = 2y$ exist everywhere. Therefore, local extrema can occur only at the origin $(0, 0)$ where $f_x = 0$ and $f_y = 0$. Along the positive x-axis, however, f has the value $f(x, 0) = -x^2 < 0$; along the positive y-axis, f has the value $f(0, y) = y^2 > 0$. Therefore, every open disk in the xy-plane centered at $(0, 0)$ contains points where the function is positive and points where it is negative. The function has a saddle point at the origin and no local extreme values (Figure 14.47a). Figure 14.47b displays the level curves (they are hyperbolas) of f, and shows the function decreasing and increasing in an alternating fashion among the four groupings of hyperbolas. ■

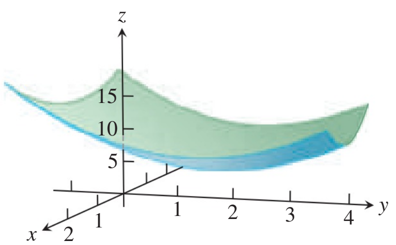


FIGURE 14.46 The graph of the function $f(x, y) = x^2 + y^2 - 4y + 9$ is a paraboloid which has a local minimum value of 5 at the point $(0, 2)$ (Example 1).

That $f_x = f_y = 0$ at an interior point (a, b) of R does not guarantee f has a local extreme value there. If f and its first and second partial derivatives are continuous on R, however, we may be able to learn more from the following theorem, proved in Section 14.9.

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - f_{xy}{}^2$ is called the **discriminant** or **Hessian** of f. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}{}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

Theorem 11 says that if the discriminant is positive at the point (a, b), then the surface curves the same way in all directions: downward if $f_{xx} < 0$, giving rise to a local maximum, and

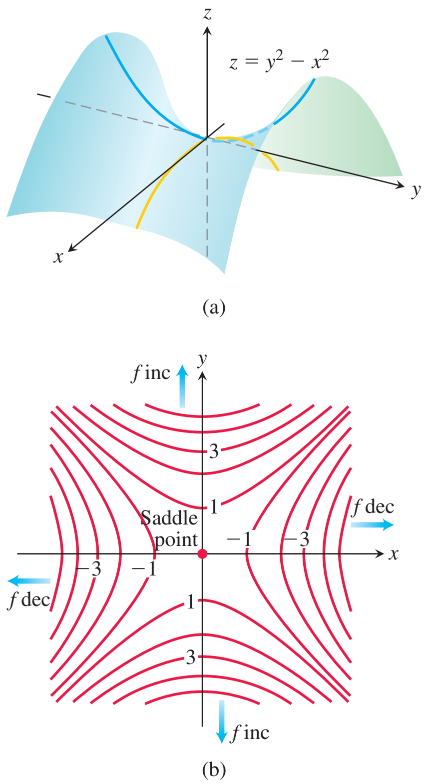


FIGURE 14.47 (a) The origin is a saddle point of the function $f(x, y) = y^2 - x^2$. There are no local extreme values (Example 2). (b) Level curves for the function f in Example 2.

upward if $f_{xx} > 0$, giving a local minimum. On the other hand, if the discriminant is negative at (a, b), then the surface curves up in some directions and down in others, so we have a saddle point.

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y, and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}{}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$. ■

EXAMPLE 4 Find the local extreme values of $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$.

Solution Since f is differentiable everywhere, it can assume extreme values only where

$$f_x = 6y - 6x = 0 \qquad \text{and} \qquad f_y = 6y - 6y^2 + 6x = 0.$$

From the first of these equations we find $x = y$, and substitution for y into the second equation then gives

$$6x - 6x^2 + 6x = 0 \qquad \text{or} \qquad 6x(2 - x) = 0.$$

The two critical points are therefore $(0, 0)$ and $(2, 2)$.

To classify the critical points, we calculate the second derivatives:

$$f_{xx} = -6, \qquad f_{yy} = 6 - 12y, \qquad f_{xy} = 6.$$

The discriminant is given by

$$f_{xx}f_{yy} - f_{xy}{}^2 = (-36 + 72y) - 36 = 72(y - 1).$$

At the critical point $(0, 0)$ we see that the value of the discriminant is the negative number -72, so the function has a saddle point at the origin. At the critical point $(2, 2)$ we see that the discriminant has the positive value 72. Combining this result with the negative value of the second partial $f_{xx} = -6$, Theorem 11 says that the critical point $(2, 2)$ gives a local maximum value of $f(2, 2) = 12 - 16 - 12 + 24 = 8$. A graph of the surface is shown in Figure 14.48. ■

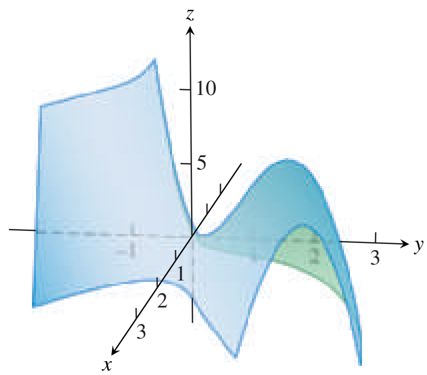


FIGURE 14.48 The surface $z = 3y^2 - 2y^3 - 3x^2 + 6xy$ has a saddle point at the origin and a local maximum at the point $(2, 2)$ (Example 4).

EXAMPLE 5 Find the critical points of the function $f(x, y) = 10xye^{-(x^2+y^2)}$ and use the Second Derivative Test to classify each point as one where a saddle, local minimum, or local maximum occurs.

Solution First we find the partial derivatives f_x and f_y and set them simultaneously to zero in seeking the critical points:

$$f_x = 10ye^{-(x^2+y^2)} - 20x^2ye^{-(x^2+y^2)} = 10y(1-2x^2)e^{-(x^2+y^2)} = 0 \Rightarrow y = 0 \text{ or } 1 - 2x^2 = 0,$$
$$f_y = 10xe^{-(x^2+y^2)} - 20xy^2e^{-(x^2+y^2)} = 10x(1-2y^2)e^{-(x^2+y^2)} = 0 \Rightarrow x = 0 \text{ or } 1 - 2y^2 = 0.$$

Since both partial derivatives are continuous everywhere, the only critical points are

$$(0, 0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \text{ and } \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

Next we calculate the second partial derivatives in order to evaluate the discriminant at each critical point:

$$f_{xx} = -20xy(1-2x^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3-2x^2)e^{-(x^2+y^2)},$$
$$f_{xy} = f_{yx} = 10(1-2x^2)e^{-(x^2+y^2)} - 20y^2(1-2x^2)e^{-(x^2+y^2)} = 10(1-2x^2)(1-2y^2)e^{-(x^2+y^2)},$$
$$f_{yy} = -20xy(1-2y^2)e^{-(x^2+y^2)} - 40xye^{-(x^2+y^2)} = -20xy(3-2y^2)e^{-(x^2+y^2)}.$$

The following table summarizes the values needed by the Second Derivative Test.

Critical Point	f_{xx}	f_{xy}	f_{yy}	Discriminant D
$(0, 0)$	0	10	0	-100
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{20}{e}$	0	$\frac{20}{e}$	$\frac{400}{e^2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{20}{e}$	0	$-\frac{20}{e}$	$\frac{400}{e^2}$

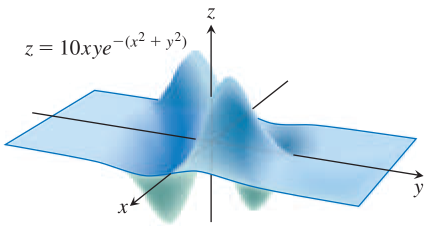


FIGURE 14.49 A graph of the function in Example 5.

From the table we find that $D < 0$ at the critical point $(0, 0)$, giving a saddle; $D > 0$ and $f_{xx} < 0$ at the critical points $\left(1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(-1/\sqrt{2}, -1/\sqrt{2}\right)$, giving local maximum values there; and $D > 0$ and $f_{xx} > 0$ at the critical points $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, -1/\sqrt{2}\right)$, each giving local minimum values. A graph of the surface is shown in Figure 14.49. ■

Absolute Maxima and Minima on Closed Bounded Regions

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps.

1. *List the interior points of* R where f may have local maxima and minima and evaluate f at these points. These are the critical points of f.
2. *List the boundary points of* R where f has local maxima and minima and evaluate f at these points. We show how to do this in the next example.
3. *Look through the lists* for the maximum and minimum values of f. These will be the absolute maximum and minimum values of f on R. Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists made in Steps 1 and 2.

EXAMPLE 6 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Solution Since f is differentiable, the only places where f can assume these values are points inside the triangle (Figure 14.50a) where $f_x = f_y = 0$ and points on the boundary.

(a) Interior points. For these we have

$$f_x = 2 - 2x = 0, \qquad f_y = 4 - 2y = 0,$$

yielding the single point $(x, y) = (1, 2)$. The value of f there is

$$f(1, 2) = 7.$$

(b) Boundary points. We take the triangle one side at a time:

i) On the segment OA, $y = 0$. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of x defined on the closed interval $0 \le x \le 9$. Its extreme values (we know from Chapter 4) may occur at the endpoints

$$\begin{aligned} x &= 0 \quad \text{where} \quad f(0, 0) = 2 \\ x &= 9 \quad \text{where} \quad f(9, 0) = 2 + 18 - 81 = -61 \end{aligned}$$

and at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, where

$$f(x, 0) = f(1, 0) = 3.$$

ii) On the segment OB, $x = 0$ and

$$f(x, y) = f(0, y) = 2 + 4y - y^2.$$

As in part i), we consider $f(0, y)$ as a function of y defined on the closed interval $[0, 9]$. Its extreme values can occur at the endpoints or at interior points where $f'(0, y) = 0$. Since $f'(0, y) = 4 - 2y$, the only interior point where $f'(0, y) = 0$ occurs at $(0, 2)$, with $f(0, 2) = 6$. So the candidates for this segment are

$$f(0, 0) = 2, \qquad f(0, 9) = -43, \qquad f(0, 2) = 6.$$

iii) We have already accounted for the values of f at the endpoints of AB, so we need only look at the interior points of the line segment AB. With $y = 9 - x$, we have

$$f(x, y) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2 = -43 + 16x - 2x^2.$$

Setting $f'(x, 9 - x) = 16 - 4x = 0$ gives

$$x = 4.$$

At this value of x,

$$y = 9 - 4 = 5 \qquad \text{and} \qquad f(x, y) = f(4, 5) = -11.$$

Summary We list all the function value candidates: $7, 2, -61, 3, -43, 6, -11$. The maximum is 7, which f assumes at $(1, 2)$. The minimum is -61, which f assumes at $(9, 0)$. See Figure 14.50b. ■

Solving extreme value problems with algebraic constraints on the variables usually requires the method of Lagrange multipliers introduced in the next section. But sometimes we can solve such problems directly, as in the next example.

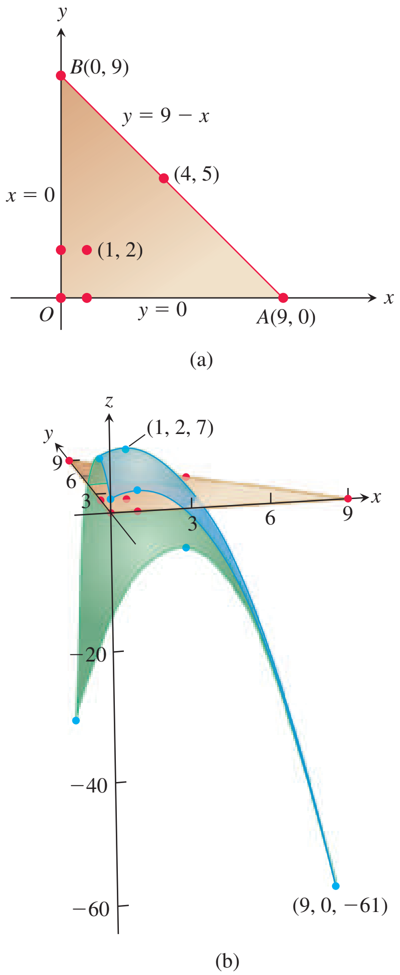


FIGURE 14.50 (a) This triangular region is the domain of the function in Example 6. (b) The graph of the function in Example 6. The blue points are the candidates for maxima or minima.

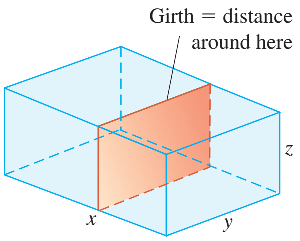


FIGURE 14.51 The box in Example 7.

EXAMPLE 7 A delivery company accepts only rectangular boxes the sum of whose length and girth (perimeter of a cross-section) does not exceed 108 in. Find the dimensions of an acceptable box of largest volume.

Solution Let x, y, and z represent the length, width, and height of the rectangular box, respectively. Then the girth is $2y + 2z$. We want to maximize the volume $V = xyz$ of the box (Figure 14.51) satisfying $x + 2y + 2z = 108$ (the largest box accepted by the delivery company). Thus, we can write the volume of the box as a function of two variables:

$$\begin{aligned} V(y, z) &= (108 - 2y - 2z)yz && V = xyz \text{ and } x = 108 - 2y - 2z \\ &= 108yz - 2y^2z - 2yz^2. \end{aligned}$$

Setting the first partial derivatives equal to zero,

$$\begin{aligned} V_y(y, z) &= 108z - 4yz - 2z^2 = (108 - 4y - 2z)z = 0 \\ V_z(y, z) &= 108y - 2y^2 - 4yz = (108 - 2y - 4z)y = 0, \end{aligned}$$

gives the critical points (0, 0), (0, 54), (54, 0), and (18, 18). The volume is zero at (0, 0), (0, 54), and (54, 0), which are not maximum values. At the point (18, 18), we apply the Second Derivative Test (Theorem 11):

$$V_{yy} = -4z, \qquad V_{zz} = -4y, \qquad V_{yz} = 108 - 4y - 4z.$$

Then

$$V_{yy}V_{zz} - V_{yz}^2 = 16yz - 16(27 - y - z)^2.$$

Thus,

$$V_{yy}(18, 18) = -4(18) < 0$$

and

$$\left[V_{yy}V_{zz} - V_{yz}^2\right]_{(18,18)} = 16(18)(18) - 16(-9)^2 > 0$$

imply that (18, 18) gives a maximum volume. The dimensions of the package are $x = 108 - 2(18) - 2(18) = 36$ in., $y = 18$ in., and $z = 18$ in. The maximum volume is $V = (36)(18)(18) = 11{,}664\ \text{in}^3$, or $6.75\ \text{ft}^3$. ■

Despite the power of Theorem 11, we urge you to remember its limitations. It does not apply to boundary points of a function's domain, where it is possible for a function to have extreme values along with nonzero derivatives. Also, it does not apply to points where either f_x or f_y fails to exist.

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i) **boundary points** of the domain of f

ii) **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local maximum**

ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at $(a, b) \Rightarrow$ **local minimum**

iii) $f_{xx}f_{yy} - f_{xy}^2 < 0$ at $(a, b) \Rightarrow$ **saddle point**

iv) $f_{xx}f_{yy} - f_{xy}^2 = 0$ at $(a, b) \Rightarrow$ **test is inconclusive**

Exercises 14.7

Finding Local Extrema

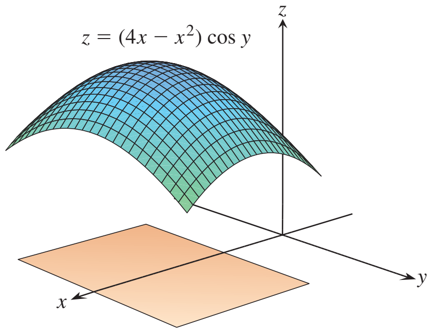
Find all the local maxima, local minima, and saddle points of the functions in Exercises 1–30.

1. $f(x, y) = x^2 + xy + y^2 + 3x - 3y + 4$
2. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
3. $f(x, y) = x^2 + xy + 3x + 2y + 5$
4. $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$
5. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$
6. $f(x, y) = x^2 - 4xy + y^2 + 6y + 2$
7. $f(x, y) = 2x^2 + 3xy + 4y^2 - 5x + 2y$
8. $f(x, y) = x^2 - 2xy + 2y^2 - 2x + 2y + 1$
9. $f(x, y) = x^2 - y^2 - 2x + 4y + 6$
10. $f(x, y) = x^2 + 2xy$
11. $f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$
12. $f(x, y) = 1 - \sqrt[3]{x^2 + y^2}$
13. $f(x, y) = x^3 - y^3 - 2xy + 6$
14. $f(x, y) = x^3 + 3xy + y^3$
15. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$
16. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$
17. $f(x, y) = x^3 + 3xy^2 - 15x + y^3 - 15y$
18. $f(x, y) = 2x^3 + 2y^3 - 9x^2 + 3y^2 - 12y$
19. $f(x, y) = 4xy - x^4 - y^4$
20. $f(x, y) = x^4 + y^4 + 4xy$
21. $f(x, y) = \dfrac{1}{x^2 + y^2 - 1}$
22. $f(x, y) = \dfrac{1}{x} + xy + \dfrac{1}{y}$
23. $f(x, y) = y \sin x$
24. $f(x, y) = e^{2x} \cos y$
25. $f(x, y) = e^{x^2+y^2-4x}$
26. $f(x, y) = e^y - ye^x$
27. $f(x, y) = e^{-y}(x^2 + y^2)$
28. $f(x, y) = e^x(x^2 - y^2)$
29. $f(x, y) = 2 \ln x + \ln y - 4x - y$
30. $f(x, y) = \ln (x + y) + x^2 - y$

Finding Absolute Extrema

In Exercises 31–38, find the absolute maxima and minima of the functions on the given domains.

31. $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular plate bounded by the lines $x = 0, y = 2, y = 2x$ in the first quadrant
32. $D(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0, y = 4, y = x$
33. $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0, y = 0, y + 2x = 2$ in the first quadrant
34. $T(x, y) = x^2 + xy + y^2 - 6x$ on the rectangular plate $0 \le x \le 5, -3 \le y \le 3$
35. $T(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5, -3 \le y \le 0$
36. $f(x, y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \le x \le 1, 0 \le y \le 1$
37. $f(x, y) = (4x - x^2) \cos y$ on the rectangular plate $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$ (see accompanying figure)



38. $f(x, y) = 4x - 8xy + 2y + 1$ on the triangular plate bounded by the lines $x = 0, y = 0, x + y = 1$ in the first quadrant
39. Find two numbers a and b with $a \le b$ such that
$$\int_a^b (6 - x - x^2)\, dx$$
has its largest value.
40. Find two numbers a and b with $a \le b$ such that
$$\int_a^b (24 - 2x - x^2)^{1/3}\, dx$$
has its largest value.
41. **Temperatures** A flat circular plate has the shape of the region $x^2 + y^2 \le 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature at the point (x, y) is
$$T(x, y) = x^2 + 2y^2 - x.$$
Find the temperatures at the hottest and coldest points on the plate.
42. Find the critical point of
$$f(x, y) = xy + 2x - \ln x^2 y$$
in the open first quadrant $(x > 0, y > 0)$ and show that f takes on a minimum there.

Theory and Examples

43. Find the maxima, minima, and saddle points of $f(x, y)$, if any, given that
    a. $f_x = 2x - 4y$ and $f_y = 2y - 4x$
    b. $f_x = 2x - 2$ and $f_y = 2y - 4$
    c. $f_x = 9x^2 - 9$ and $f_y = 2y + 4$

    Describe your reasoning in each case.
44. The discriminant $f_{xx}f_{yy} - f_{xy}{}^2$ is zero at the origin for each of the following functions, so the Second Derivative Test fails there. Determine whether the function has a maximum, a minimum, or neither at the origin by imagining what the surface $z = f(x, y)$ looks like. Describe your reasoning in each case.
    a. $f(x, y) = x^2y^2$
    b. $f(x, y) = 1 - x^2y^2$
    c. $f(x, y) = xy^2$
    d. $f(x, y) = x^3y^2$
    e. $f(x, y) = x^3y^3$
    f. $f(x, y) = x^4y^4$

45. Show that (0, 0) is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (*Hint:* Consider two cases: $k = 0$ and $k \neq 0$.)

46. For what values of the constant k does the Second Derivative Test guarantee that $f(x, y) = x^2 + kxy + y^2$ will have a saddle point at (0, 0)? A local minimum at (0, 0)? For what values of k is the Second Derivative Test inconclusive? Give reasons for your answers.

47. If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b)? Give reasons for your answer.

48. Can you conclude anything about $f(a, b)$ if f and its first and second partial derivatives are continuous throughout a disk centered at the critical point (a, b) and $f_{xx}(a, b)$ and $f_{yy}(a, b)$ differ in sign? Give reasons for your answer.

49. Among all the points on the graph of $z = 10 - x^2 - y^2$ that lie above the plane $x + 2y + 3z = 0$, find the point farthest from the plane.

50. Find the point on the graph of $z = x^2 + y^2 + 10$ nearest the plane $x + 2y - z = 0$.

51. Find the point on the plane $3x + 2y + z = 6$ that is nearest the origin.

52. Find the minimum distance from the point $(2, -1, 1)$ to the plane $x + y - z = 2$.

53. Find three numbers whose sum is 9 and whose sum of squares is a minimum.

54. Find three positive numbers whose sum is 3 and whose product is a maximum.

55. Find the maximum value of $s = xy + yz + xz$ where $x + y + z = 6$.

56. Find the minimum distance from the cone $z = \sqrt{x^2 + y^2}$ to the point $(-6, 4, 0)$.

57. Find the dimensions of the rectangular box of maximum volume that can be inscribed inside the sphere $x^2 + y^2 + z^2 = 4$.

58. Among all closed rectangular boxes of volume 27 cm^3, what is the smallest surface area?

59. You are to construct an open rectangular box from 12 ft^2 of material. What dimensions will result in a box of maximum volume?

60. Consider the function $f(x, y) = x^2 + y^2 + 2xy - x - y + 1$ over the square $0 \le x \le 1$ and $0 \le y \le 1$.

a. Show that f has an absolute minimum along the line segment $2x + 2y = 1$ in this square. What *is* the absolute minimum value?

b. Find the absolute maximum value of f over the square.

Extreme Values on Parametrized Curves To find the extreme values of a function $f(x, y)$ on a curve $x = x(t)$, $y = y(t)$, we treat f as a function of the single variable t and use the Chain Rule to find where df/dt is zero. As in any other single-variable case, the extreme values of f are then found among the values at the

a. critical points (points where df/dt is zero or fails to exist), and

b. endpoints of the parameter domain.

Find the absolute maximum and minimum values of the following functions on the given curves.

61. Functions:

a. $f(x, y) = x + y$ **b.** $g(x, y) = xy$ **c.** $h(x, y) = 2x^2 + y^2$

Curves:

i) The semicircle $x^2 + y^2 = 4, \quad y \ge 0$

ii) The quarter circle $x^2 + y^2 = 4, \quad x \ge 0, \quad y \ge 0$

Use the parametric equations $x = 2\cos t$, $y = 2\sin t$.

62. Functions:

a. $f(x, y) = 2x + 3y$

b. $g(x, y) = xy$

c. $h(x, y) = x^2 + 3y^2$

Curves:

i) The semiellipse $(x^2/9) + (y^2/4) = 1, \quad y \ge 0$

ii) The quarter ellipse $(x^2/9) + (y^2/4) = 1, \quad x \ge 0, \quad y \ge 0$

Use the parametric equations $x = 3\cos t$, $y = 2\sin t$.

63. Function: $f(x, y) = xy$

Curves:

i) The line $x = 2t, \quad y = t + 1$

ii) The line segment $x = 2t, \quad y = t + 1, \quad -1 \le t \le 0$

iii) The line segment $x = 2t, \quad y = t + 1, \quad 0 \le t \le 1$

64. Functions:

a. $f(x, y) = x^2 + y^2$

b. $g(x, y) = 1/(x^2 + y^2)$

Curves:

i) The line $x = t, \quad y = 2 - 2t$

ii) The line segment $x = t, \quad y = 2 - 2t, \quad 0 \le t \le 1$

65. Least squares and regression lines When we try to fit a line $y = mx + b$ to a set of numerical data points (x_1, y_1), $(x_2, y_2), \ldots, (x_n, y_n)$, we usually choose the line that minimizes the sum of the squares of the vertical distances from the points to the line. In theory, this means finding the values of m and b that minimize the value of the function

$$w = (mx_1 + b - y_1)^2 + \cdots + (mx_n + b - y_n)^2. \tag{1}$$

(See the accompanying figure.) Show that the values of m and b that do this are

$$m = \frac{\left(\sum x_k\right)\left(\sum y_k\right) - n\sum x_k y_k}{\left(\sum x_k\right)^2 - n\sum x_k^{\,2}}, \tag{2}$$

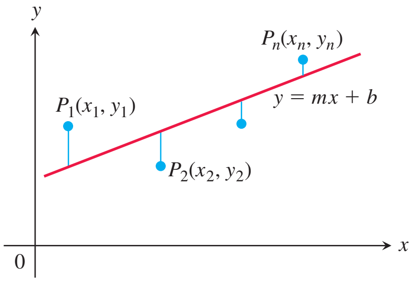
$$b = \frac{1}{n}\left(\sum y_k - m\sum x_k\right), \tag{3}$$

with all sums running from $k = 1$ to $k = n$. Many scientific calculators have these formulas built in, enabling you to find m and b with only a few keystrokes after you have entered the data.

The line $y = mx + b$ determined by these values of m and b is called the **least squares line**, **regression line**, or **trend line** for the data under study. Finding a least squares line lets you

1. summarize data with a simple expression,
2. predict values of y for other, experimentally untried values of x,
3. handle data analytically.

We demonstrated these ideas with a variety of applications in Section 1.4.



In Exercises 66–68, use Equations (2) and (3) to find the least squares line for each set of data points. Then use the linear equation you obtain to predict the value of y that would correspond to $x = 4$.

66. $(-2, 0)$, $(0, 2)$, $(2, 3)$

67. $(-1, 2)$, $(0, 1)$, $(3, -4)$

68. $(0, 0)$, $(1, 2)$, $(2, 3)$

COMPUTER EXPLORATIONS

In Exercises 69–74, you will explore functions to identify their local extrema. Use a CAS to perform the following steps:

a. Plot the function over the given rectangle.

b. Plot some level curves in the rectangle.

c. Calculate the function's first partial derivatives and use the CAS equation solver to find the critical points. How do the critical points relate to the level curves plotted in part (b)? Which critical points, if any, appear to give a saddle point? Give reasons for your answer.

d. Calculate the function's second partial derivatives and find the discriminant $f_{xx}f_{yy} - f_{xy}^{\ 2}$.

e. Using the max-min tests, classify the critical points found in part (c). Are your findings consistent with your discussion in part (c)?

69. $f(x, y) = x^2 + y^3 - 3xy, \quad -5 \le x \le 5, \quad -5 \le y \le 5$

70. $f(x, y) = x^3 - 3xy^2 + y^2, \quad -2 \le x \le 2, \quad -2 \le y \le 2$

71. $f(x, y) = x^4 + y^2 - 8x^2 - 6y + 16, \quad -3 \le x \le 3,$
$-6 \le y \le 6$

72. $f(x, y) = 2x^4 + y^4 - 2x^2 - 2y^2 + 3, \quad -3/2 \le x \le 3/2,$
$-3/2 \le y \le 3/2$

73. $f(x, y) = 5x^6 + 18x^5 - 30x^4 + 30xy^2 - 120x^3,$
$-4 \le x \le 3, \quad -2 \le y \le 2$

74. $f(x, y) = \begin{cases} x^5 \ln(x^2 + y^2), & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases},$
$-2 \le x \le 2, \quad -2 \le y \le 2$

14.8 Lagrange Multipliers

HISTORICAL BIOGRAPHY
Joseph Louis Lagrange
(1736–1813)

Sometimes we need to find the extreme values of a function whose domain is constrained to lie within some particular subset of the plane—for example, a disk, a closed triangular region, or along a curve. We saw an instance of this situation in Example 6 of the previous section. Here we explore a powerful method for finding extreme values of constrained functions: the method of *Lagrange multipliers*.

Constrained Maxima and Minima

To gain some insight, we first consider a problem where a constrained minimum can be found by eliminating a variable.

EXAMPLE 1 Find the point $p(x, y, z)$ on the plane $2x + y - z - 5 = 0$ that is closest to the origin.

Solution The problem asks us to find the minimum value of the function

$$\begin{aligned} |\overrightarrow{OP}| &= \sqrt{(x - 0)^2 + (y - 0)^2 + (z - 0)^2} \\ &= \sqrt{x^2 + y^2 + z^2} \end{aligned}$$

subject to the constraint that

$$2x + y - z - 5 = 0.$$

Since $|\overrightarrow{OP}|$ has a minimum value wherever the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

has a minimum value, we may solve the problem by finding the minimum value of $f(x, y, z)$ subject to the constraint $2x + y - z - 5 = 0$ (thus avoiding square roots). If we regard x and y as the independent variables in this equation and write z as

$$z = 2x + y - 5,$$

our problem reduces to one of finding the points (x, y) at which the function

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$$

has its minimum value or values. Since the domain of h is the entire xy-plane, the First Derivative Test of Section 14.7 tells us that any minima that h might have must occur at points where

$$h_x = 2x + 2(2x + y - 5)(2) = 0, \qquad h_y = 2y + 2(2x + y - 5) = 0.$$

This leads to

$$10x + 4y = 20, \qquad 4x + 4y = 10,$$

and the solution

$$x = \frac{5}{3}, \qquad y = \frac{5}{6}.$$

We may apply a geometric argument together with the Second Derivative Test to show that these values minimize h. The z-coordinate of the corresponding point on the plane $z = 2x + y - 5$ is

$$z = 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}.$$

Therefore, the point we seek is

$$\text{Closest point:} \qquad P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The distance from P to the origin is $5/\sqrt{6} \approx 2.04$. ■

Attempts to solve a constrained maximum or minimum problem by substitution, as we might call the method of Example 1, do not always go smoothly. This is one of the reasons for learning the new method of this section.

EXAMPLE 2 Find the points on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ that are closest to the origin.

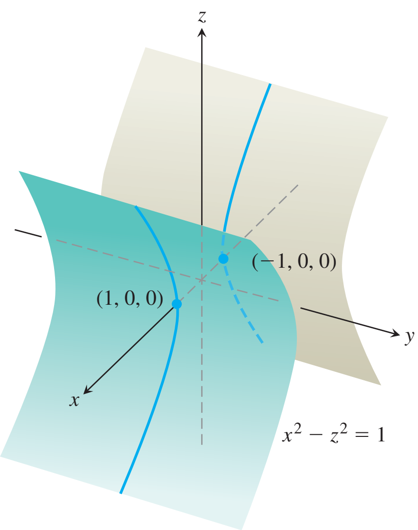


FIGURE 14.52 The hyperbolic cylinder $x^2 - z^2 - 1 = 0$ in Example 2.

Solution 1 The cylinder is shown in Figure 14.52. We seek the points on the cylinder closest to the origin. These are the points whose coordinates minimize the value of the function

$$f(x, y, z) = x^2 + y^2 + z^2 \qquad \text{Square of the distance}$$

subject to the constraint that $x^2 - z^2 - 1 = 0$. If we regard x and y as independent variables in the constraint equation, then

$$z^2 = x^2 - 1$$

and the values of $f(x, y, z) = x^2 + y^2 + z^2$ on the cylinder are given by the function

$$h(x, y) = x^2 + y^2 + (x^2 - 1) = 2x^2 + y^2 - 1.$$

To find the points on the cylinder whose coordinates minimize f, we look for the points in the xy-plane whose coordinates minimize h. The only extreme value of h occurs where

$$h_x = 4x = 0 \qquad \text{and} \qquad h_y = 2y = 0,$$

that is, at the point $(0, 0)$. But there are no points on the cylinder where both x and y are zero. What went wrong?

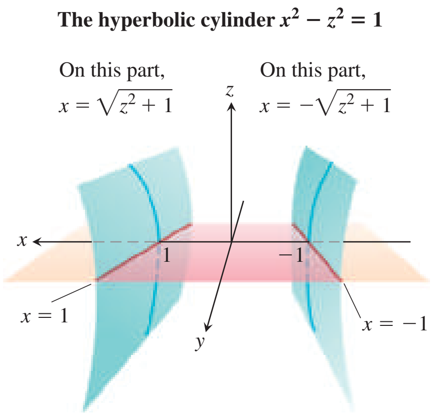


FIGURE 14.53 The region in the xy-plane from which the first two coordinates of the points (x, y, z) on the hyperbolic cylinder $x^2 - z^2 = 1$ are selected excludes the band $-1 < x < 1$ in the xy-plane (Example 2).

What happened was that the First Derivative Test found (as it should have) the point *in the domain of h* where h has a minimum value. We, on the other hand, want the points *on the cylinder* where h has a minimum value. Although the domain of h is the entire xy-plane, the domain from which we can select the first two coordinates of the points (x, y, z) on the cylinder is restricted to the "shadow" of the cylinder on the xy-plane; it does not include the band between the lines $x = -1$ and $x = 1$ (Figure 14.53).

We can avoid this problem if we treat y and z as independent variables (instead of x and y) and express x in terms of y and z as

$$x^2 = z^2 + 1.$$

With this substitution, $f(x, y, z) = x^2 + y^2 + z^2$ becomes

$$k(y, z) = (z^2 + 1) + y^2 + z^2 = 1 + y^2 + 2z^2$$

and we look for the points where k takes on its smallest value. The domain of k in the yz-plane now matches the domain from which we select the y- and z-coordinates of the points (x, y, z) on the cylinder. Hence, the points that minimize k in the plane will have corresponding points on the cylinder. The smallest values of k occur where

$$k_y = 2y = 0 \qquad \text{and} \qquad k_z = 4z = 0,$$

or where $y = z = 0$. This leads to

$$x^2 = z^2 + 1 = 1, \qquad x = \pm 1.$$

The corresponding points on the cylinder are $(\pm 1, 0, 0)$. We can see from the inequality

$$k(y, z) = 1 + y^2 + 2z^2 \geq 1$$

that the points $(\pm 1, 0, 0)$ give a minimum value for k. We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.

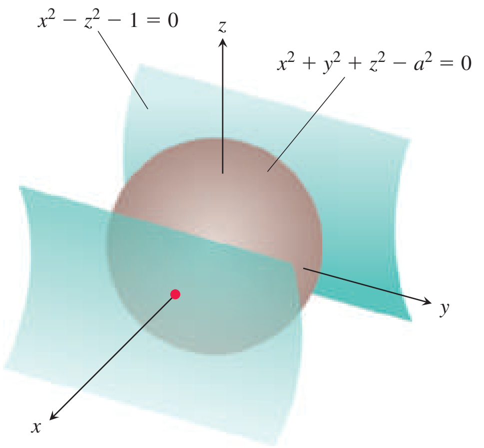


FIGURE 14.54 A sphere expanding like a soap bubble centered at the origin until it just touches the hyperbolic cylinder $x^2 - z^2 - 1 = 0$ (Example 2).

Solution 2 Another way to find the points on the cylinder closest to the origin is to imagine a small sphere centered at the origin expanding like a soap bubble until it just touches the cylinder (Figure 14.54). At each point of contact, the cylinder and sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 \qquad \text{and} \qquad g(x, y, z) = x^2 - z^2 - 1$$

equal to 0, then the gradients ∇f and ∇g will be parallel where the surfaces touch. At any point of contact, we should therefore be able to find a scalar λ ("lambda") such that

$$\nabla f = \lambda \nabla g,$$

or

$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} - 2z\mathbf{k}).$$

Thus, the coordinates x, y, and z of any point of tangency will have to satisfy the three scalar equations

$$2x = 2\lambda x, \qquad 2y = 0, \qquad 2z = -2\lambda z.$$

For what values of λ will a point (x, y, z) whose coordinates satisfy these scalar equations also lie on the surface $x^2 - z^2 - 1 = 0$? To answer this question, we use our knowledge that no point on the surface has a zero x-coordinate to conclude that $x \neq 0$. Hence, $2x = 2\lambda x$ only if

$$2 = 2\lambda, \qquad \text{or} \qquad \lambda = 1.$$

For $\lambda = 1$, the equation $2z = -2\lambda z$ becomes $2z = -2z$. If this equation is to be satisfied as well, z must be zero. Since $y = 0$ also (from the equation $2y = 0$), we conclude that the points we seek all have coordinates of the form

$$(x, 0, 0).$$

What points on the surface $x^2 - z^2 = 1$ have coordinates of this form? The answer is the points $(x, 0, 0)$ for which

$$x^2 - (0)^2 = 1, \qquad x^2 = 1, \qquad \text{or} \qquad x = \pm 1.$$

The points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$. ■

The Method of Lagrange Multipliers

In Solution 2 of Example 2, we used the **method of Lagrange multipliers**. The method says that the local extreme values of a function $f(x, y, z)$ whose variables are subject to a constraint $g(x, y, z) = 0$ are to be found on the surface $g = 0$ among the points where

$$\nabla f = \lambda \nabla g$$

for some scalar λ (called a **Lagrange multiplier**).

To explore the method further and see why it works, we first make the following observation, which we state as a theorem.

THEOREM 12—The Orthogonal Gradient Theorem Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

Proof We show that ∇f is orthogonal to the curve's tangent vector $\mathbf{r}'$ at P_0. The values of f on C are given by the composite $f(x(t), y(t), z(t))$, whose derivative with respect to t is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \nabla f \cdot \mathbf{r}'.$$

At any point P_0 where f has a local maximum or minimum relative to its values on the curve, $df/dt = 0$, so

$$\nabla f \cdot \mathbf{r}' = 0.$$ ■

By dropping the z-terms in Theorem 12, we obtain a similar result for functions of two variables.

COROLLARY At the points on a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{r}' = 0$.

Theorem 12 is the key to the method of Lagrange multipliers. Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and that P_0 is a point on the surface $g(x, y, z) = 0$ where f has a local maximum or minimum value relative to its other values on the surface. We assume also that $\nabla g \neq \mathbf{0}$ at points on the surface $g(x, y, z) = 0$. Then f takes on a local maximum or minimum at P_0 relative to its values on every differentiable curve through P_0

on the surface $g(x, y, z) = 0$. Therefore, ∇f is orthogonal to the tangent vector of every such differentiable curve through P_0. So is ∇g, moreover (because ∇g is orthogonal to the level surface $g = 0$, as we saw in Section 14.5). Therefore, at P_0, ∇f is some scalar multiple λ of ∇g.

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable z.

Some care must be used in applying this method. An extreme value may not actually exist (Exercise 45).

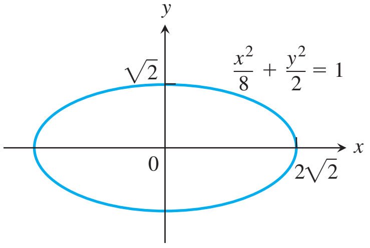


FIGURE 14.55 Example 3 shows how to find the largest and smallest values of the product xy on this ellipse.

EXAMPLE 3 Find the greatest and smallest values that the function

$$f(x, y) = xy$$

takes on the ellipse (Figure 14.55)

$$\frac{x^2}{8} + \frac{y^2}{2} = 1.$$

Solution We want to find the extreme values of $f(x, y) = xy$ subject to the constraint

$$g(x, y) = \frac{x^2}{8} + \frac{y^2}{2} - 1 = 0.$$

To do so, we first find the values of x, y, and λ for which

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y) = 0.$$

The gradient equation in Equations (1) gives

$$y\mathbf{i} + x\mathbf{j} = \frac{\lambda}{4}x\mathbf{i} + \lambda y\mathbf{j},$$

from which we find

$$y = \frac{\lambda}{4}x, \qquad x = \lambda y, \qquad \text{and} \qquad y = \frac{\lambda}{4}(\lambda y) = \frac{\lambda^2}{4}y,$$

so that $y = 0$ or $\lambda = \pm 2$. We now consider these two cases.

Case 1: If $y = 0$, then $x = y = 0$. But $(0, 0)$ is not on the ellipse. Hence, $y \neq 0$.

Case 2: If $y \neq 0$, then $\lambda = \pm 2$ and $x = \pm 2y$. Substituting this in the equation $g(x, y) = 0$ gives

$$\frac{(\pm 2y)^2}{8} + \frac{y^2}{2} = 1, \qquad 4y^2 + 4y^2 = 8 \qquad \text{and} \qquad y = \pm 1.$$

The function $f(x, y) = xy$ therefore takes on its extreme values on the ellipse at the four points $(\pm 2, 1), (\pm 2, -1)$. The extreme values are $xy = 2$ and $xy = -2$.

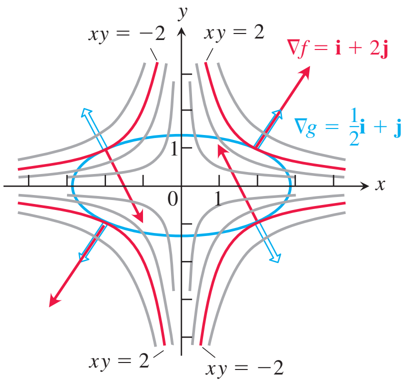


FIGURE 14.56 When subjected to the constraint $g(x, y) = x^2/8 + y^2/2 - 1 = 0$, the function $f(x, y) = xy$ takes on extreme values at the four points $(\pm 2, \pm 1)$. These are the points on the ellipse when ∇f (red) is a scalar multiple of ∇g (blue) (Example 3).

The Geometry of the Solution The level curves of the function $f(x, y) = xy$ are the hyperbolas $xy = c$ (Figure 14.56). The farther the hyperbolas lie from the origin, the larger the absolute value of f. We want to find the extreme values of $f(x, y)$, given that the point (x, y) also lies on the ellipse $x^2 + 4y^2 = 8$. Which hyperbolas intersecting the ellipse lie farthest from the origin? The hyperbolas that just graze the ellipse, the ones that are tangent to it, are farthest. At these points, any vector normal to the hyperbola is normal to the ellipse, so $\nabla f = y\mathbf{i} + x\mathbf{j}$ is a multiple $(\lambda = \pm 2)$ of $\nabla g = (x/4)\mathbf{i} + y\mathbf{j}$. At the point $(2, 1)$, for example,

$$\nabla f = \mathbf{i} + 2\mathbf{j}, \qquad \nabla g = \frac{1}{2}\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = 2\nabla g.$$

At the point $(-2, 1)$,

$$\nabla f = \mathbf{i} - 2\mathbf{j}, \qquad \nabla g = -\frac{1}{2}\mathbf{i} + \mathbf{j}, \qquad \text{and} \qquad \nabla f = -2\nabla g.$$

EXAMPLE 4 Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Solution We model this as a Lagrange multiplier problem with

$$f(x, y) = 3x + 4y, \qquad g(x, y) = x^2 + y^2 - 1$$

and look for the values of x, y, and λ that satisfy the equations

$$\nabla f = \lambda \nabla g: \quad 3\mathbf{i} + 4\mathbf{j} = 2x\lambda\mathbf{i} + 2y\lambda\mathbf{j}$$
$$g(x, y) = 0: \quad x^2 + y^2 - 1 = 0.$$

The gradient equation in Equations (1) implies that $\lambda \neq 0$ and gives

$$x = \frac{3}{2\lambda}, \qquad y = \frac{2}{\lambda}.$$

These equations tell us, among other things, that x and y have the same sign. With these values for x and y, the equation $g(x, y) = 0$ gives

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0,$$

so

$$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} = 1, \qquad 9 + 16 = 4\lambda^2, \qquad 4\lambda^2 = 25, \qquad \text{and} \qquad \lambda = \pm\frac{5}{2}.$$

Thus,

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \qquad y = \frac{2}{\lambda} = \pm\frac{4}{5},$$

and $f(x, y) = 3x + 4y$ has extreme values at $(x, y) = \pm(3/5, 4/5)$.

By calculating the value of $3x + 4y$ at the points $\pm(3/5, 4/5)$, we see that its maximum and minimum values on the circle $x^2 + y^2 = 1$ are

$$3\left(\frac{3}{5}\right) + 4\left(\frac{4}{5}\right) = \frac{25}{5} = 5 \qquad \text{and} \qquad 3\left(-\frac{3}{5}\right) + 4\left(-\frac{4}{5}\right) = -\frac{25}{5} = -5.$$

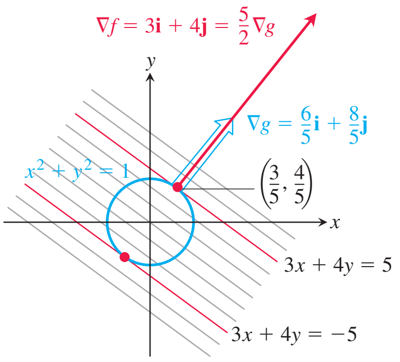


FIGURE 14.57 The function $f(x, y) = 3x + 4y$ takes on its largest value on the unit circle $g(x, y) = x^2 + y^2 - 1 = 0$ at the point $(3/5, 4/5)$ and its smallest value at the point $(-3/5, -4/5)$ (Example 4). At each of these points, ∇f is a scalar multiple of ∇g. The figure shows the gradients at the first point but not the second.

The Geometry of the Solution The level curves of $f(x, y) = 3x + 4y$ are the lines $3x + 4y = c$ (Figure 14.57). The farther the lines lie from the origin, the larger the absolute value of f. We want to find the extreme values of $f(x, y)$ given that the point (x, y) also lies on the circle $x^2 + y^2 = 1$. Which lines intersecting the circle lie farthest from the origin? The lines tangent to the circle are farthest. At the points of tangency, any vector normal to the line is normal to the circle, so the gradient $\nabla f = 3\mathbf{i} + 4\mathbf{j}$ is a multiple $(\lambda = \pm 5/2)$ of the gradient $\nabla g = 2x\mathbf{i} + 2y\mathbf{j}$. At the point $(3/5, 4/5)$, for example,

$$\nabla f = 3\mathbf{i} + 4\mathbf{j}, \qquad \nabla g = \frac{6}{5}\mathbf{i} + \frac{8}{5}\mathbf{j}, \qquad \text{and} \qquad \nabla f = \frac{5}{2}\nabla g.$$

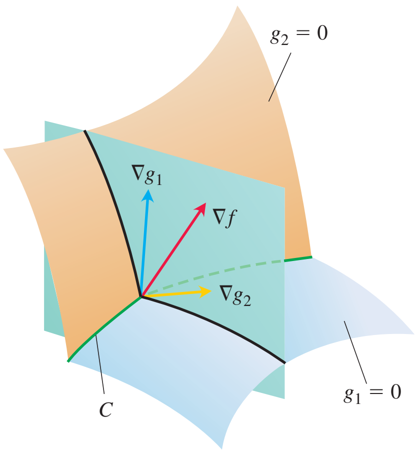


FIGURE 14.58 The vectors ∇g_1 and ∇g_2 lie in a plane perpendicular to the curve C because ∇g_1 is normal to the surface $g_1 = 0$ and ∇g_2 is normal to the surface $g_2 = 0$.

Lagrange Multipliers with Two Constraints

Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0 \quad \text{and} \quad g_2(x, y, z) = 0$$

and g_1 and g_2 are differentiable, with ∇g_1 not parallel to ∇g_2, we find the constrained local maxima and minima of f by introducing two Lagrange multipliers λ and μ (mu, pronounced "mew"). That is, we locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding the values of x, y, z, λ, and μ that simultaneously satisfy the three equations

$$\boxed{\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0 \qquad (2)}$$

Equations (2) have a nice geometric interpretation. The surfaces $g_1 = 0$ and $g_2 = 0$ (usually) intersect in a smooth curve, say C (Figure 14.58). Along this curve we seek the points where f has local maximum and minimum values relative to its other values on the curve. These are the points where ∇f is normal to C, as we saw in Theorem 12. But ∇g_1 and ∇g_2 are also normal to C at these points because C lies in the surfaces $g_1 = 0$ and $g_2 = 0$. Therefore, ∇f lies in the plane determined by ∇g_1 and ∇g_2, which means that $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$ for some λ and μ. Since the points we seek also lie in both surfaces, their coordinates must satisfy the equations $g_1(x, y, z) = 0$ and $g_2(x, y, z) = 0$, which are the remaining requirements in Equations (2).

EXAMPLE 5 The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse (Figure 14.59). Find the points on the ellipse that lie closest to and farthest from the origin.

Solution We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2$$

(the square of the distance from (x, y, z) to the origin) subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0 \qquad (3)$$
$$g_2(x, y, z) = x + y + z - 1 = 0. \qquad (4)$$

The gradient equation in Equations (2) then gives

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$$
$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = \lambda(2x\mathbf{i} + 2y\mathbf{j}) + \mu(\mathbf{i} + \mathbf{j} + \mathbf{k})$$
$$2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k} = (2\lambda x + \mu)\mathbf{i} + (2\lambda y + \mu)\mathbf{j} + \mu\mathbf{k}$$

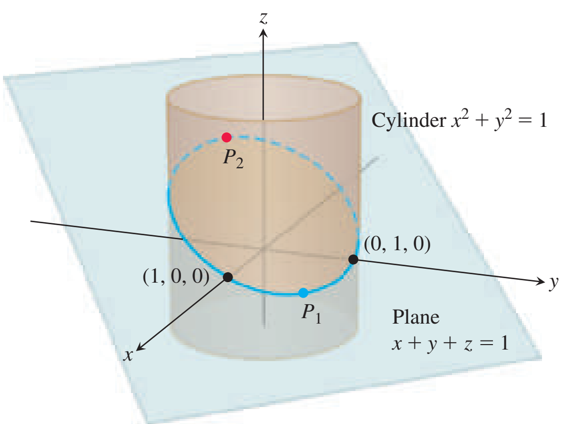


FIGURE 14.59 On the ellipse where the plane and cylinder meet, we find the points closest to and farthest from the origin (Example 5).

or

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu. \tag{5}$$

The scalar equations in Equations (5) yield

$$\begin{aligned} 2x &= 2\lambda x + 2z \Rightarrow (1 - \lambda)x = z, \\ 2y &= 2\lambda y + 2z \Rightarrow (1 - \lambda)y = z. \end{aligned} \tag{6}$$

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Equations (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points (1, 0, 0) and (0, 1, 0). This makes sense when you look at Figure 14.59.

If $x = y$, then Equations (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 & \qquad x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \quad \text{and} \quad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

Here we need to be careful, however. Although P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0). The point on the ellipse farthest from the origin is P_2. (See Figure 14.59.) ■

Exercises 14.8

Two Independent Variables with One Constraint

1. **Extrema on an ellipse** Find the points on the ellipse $x^2 + 2y^2 = 1$ where $f(x, y) = xy$ has its extreme values.

2. **Extrema on a circle** Find the extreme values of $f(x, y) = xy$ subject to the constraint $g(x, y) = x^2 + y^2 - 10 = 0$.

3. **Maximum on a line** Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$.

4. **Extrema on a line** Find the local extreme values of $f(x, y) = x^2y$ on the line $x + y = 3$.

5. **Constrained minimum** Find the points on the curve $xy^2 = 54$ nearest the origin.

6. **Constrained minimum** Find the points on the curve $x^2y = 2$ nearest the origin.

7. Use the method of Lagrange multipliers to find
   a. **Minimum on a hyperbola** The minimum value of $x + y$, subject to the constraints $xy = 16$, $x > 0$, $y > 0$
   b. **Maximum on a line** The maximum value of xy, subject to the constraint $x + y = 16$.

   Comment on the geometry of each solution.

8. **Extrema on a curve** Find the points on the curve $x^2 + xy + y^2 = 1$ in the xy-plane that are nearest to and farthest from the origin.

9. **Minimum surface area with fixed volume** Find the dimensions of the closed right circular cylindrical can of smallest surface area whose volume is 16π cm^3.

10. **Cylinder in a sphere** Find the radius and height of the open right circular cylinder of largest surface area that can be inscribed in a sphere of radius a. What *is* the largest surface area?

11. **Rectangle of greatest area in an ellipse** Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $x^2/16 + y^2/9 = 1$ with sides parallel to the coordinate axes.

12. **Rectangle of longest perimeter in an ellipse** Find the dimensions of the rectangle of largest perimeter that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ with sides parallel to the coordinate axes. What *is* the largest perimeter?

13. **Extrema on a circle** Find the maximum and minimum values of $x^2 + y^2$ subject to the constraint $x^2 - 2x + y^2 - 4y = 0$.

14. **Extrema on a circle** Find the maximum and minimum values of $3x - y + 6$ subject to the constraint $x^2 + y^2 = 4$.

15. **Ant on a metal plate** The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

16. **Cheapest storage tank** Your firm has been asked to design a storage tank for liquid petroleum gas. The customer's specifications call for a cylindrical tank with hemispherical ends, and the tank is to hold 8000 m^3 of gas. The customer also wants to use the smallest amount of material possible in building the tank. What radius and height do you recommend for the cylindrical portion of the tank?

Three Independent Variables with One Constraint

17. **Minimum distance to a point** Find the point on the plane $x + 2y + 3z = 13$ closest to the point $(1, 1, 1)$.

18. **Maximum distance to a point** Find the point on the sphere $x^2 + y^2 + z^2 = 4$ farthest from the point $(1, -1, 1)$.

19. **Minimum distance to the origin** Find the minimum distance from the surface $x^2 - y^2 - z^2 = 1$ to the origin.

20. **Minimum distance to the origin** Find the point on the surface $z = xy + 1$ nearest the origin.

21. **Minimum distance to the origin** Find the points on the surface $z^2 = xy + 4$ closest to the origin.

22. **Minimum distance to the origin** Find the point(s) on the surface $xyz = 1$ closest to the origin.

23. **Extrema on a sphere** Find the maximum and minimum values of

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

24. **Extrema on a sphere** Find the points on the sphere $x^2 + y^2 + z^2 = 25$ where $f(x, y, z) = x + 2y + 3z$ has its maximum and minimum values.

25. **Minimizing a sum of squares** Find three real numbers whose sum is 9 and the sum of whose squares is as small as possible.

26. **Maximizing a product** Find the largest product the positive numbers x, y, and z can have if $x + y + z^2 = 16$.

27. **Rectangular box of largest volume in a sphere** Find the dimensions of the closed rectangular box with maximum volume that can be inscribed in the unit sphere.

28. **Box with vertex on a plane** Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane $x/a + y/b + z/c = 1$, where $a > 0$, $b > 0$, and $c > 0$.

29. **Hottest point on a space probe** A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the probe's surface.

30. **Extreme temperatures on a sphere** Suppose that the Celsius temperature at the point (x, y, z) on the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$. Locate the highest and lowest temperatures on the sphere.

31. **Cobb-Douglas production function** During the 1920s, Charles Cobb and Paul Douglas modeled total production output P (of a firm, industry, or entire economy) as a function of labor hours involved x and capital invested y (which includes the monetary worth of all buildings and equipment). The Cobb-Douglas production function is given by

$$P(x, y) = kx^{\alpha}y^{1-\alpha},$$

where k and α are constants representative of a particular firm or economy.

   a. Show that a doubling of both labor and capital results in a doubling of production P.

   b. Suppose a particular firm has the production function for $k = 120$ and $\alpha = 3/4$. Assume that each unit of labor costs \$250 and each unit of capital costs \$400, and that the total expenses for all costs cannot exceed \$100,000. Find the maximum production level for the firm.

32. (*Continuation of Exercise 31.*) If the cost of a unit of labor is c_1 and the cost of a unit of capital is c_2, and if the firm can spend only B dollars as its total budget, then production P is constrained by $c_1x + c_2y = B$. Show that the maximum production level subject to the constraint occurs at the point

$$x = \frac{\alpha B}{c_1} \quad \text{and} \quad y = \frac{(1 - \alpha)B}{c_2}.$$

33. **Maximizing a utility function: an example from economics** In economics, the usefulness or *utility* of amounts x and y of two capital goods G_1 and G_2 is sometimes measured by a function $U(x, y)$. For example, G_1 and G_2 might be two chemicals a pharmaceutical company needs to have on hand and $U(x, y)$ the gain from manufacturing a product whose synthesis requires different amounts of the chemicals depending on the process used. If G_1 costs a dollars per kilogram, G_2 costs b dollars per kilogram, and the total amount allocated for the purchase of G_1 and G_2 together is c dollars, then the company's managers want to maximize $U(x, y)$ given that $ax + by = c$. Thus, they need to solve a typical Lagrange multiplier problem.

    Suppose that

$$U(x, y) = xy + 2x$$

and that the equation $ax + by = c$ simplifies to

$$2x + y = 30.$$

Find the maximum value of U and the corresponding values of x and y subject to this latter constraint.

34. **Blood types** Human blood types are classified by three gene forms A, B, and O. Blood types AA, BB, and OO are *homozygous*, and blood types AB, AO, and BO are *heterozygous.* If p, q, and r represent the proportions of the three gene forms to the population, respectively, then the *Hardy-Weinberg Law* asserts that the proportion Q of heterozygous persons in any specific population is modeled by

$$Q(p, q, r) = 2(pq + pr + qr),$$

subject to $p + q + r = 1$. Find the maximum value of Q.

35. **Length of a beam** In Section 4.5, Exercise 39, we posed a problem of finding the length L of the shortest beam that can reach over a wall of height h to a tall building located k units from the wall. Use Lagrange multipliers to show that

$$L = (h^{2/3} + k^{2/3})^{3/2}.$$

36. **Locating a radio telescope** You are in charge of erecting a radio telescope on a newly discovered planet. To minimize interference, you want to place it where the magnetic field of the planet is weakest. The planet is spherical, with a radius of 6 units. Based on a coordinate system whose origin is at the center of the planet, the strength of the magnetic field is given by $M(x, y, z) = 6x - y^2 + xz + 60$. Where should you locate the radio telescope?

Extreme Values Subject to Two Constraints

37. Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

38. Minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

39. Minimum distance to the origin Find the point closest to the origin on the line of intersection of the planes $y + 2z = 12$ and $x + y = 6$.

40. Maximum value on line of intersection Find the maximum value that $f(x, y, z) = x^2 + 2y - z^2$ can have on the line of intersection of the planes $2x - y = 0$ and $y + z = 0$.

41. Extrema on a curve of intersection Find the extreme values of $f(x, y, z) = x^2yz + 1$ on the intersection of the plane $z = 1$ with the sphere $x^2 + y^2 + z^2 = 10$.

42. a. Maximum on line of intersection Find the maximum value of $w = xyz$ on the line of intersection of the two planes $x + y + z = 40$ and $x + y - z = 0$.

b. Give a geometric argument to support your claim that you have found a maximum, and not a minimum, value of w.

43. Extrema on a circle of intersection Find the extreme values of the function $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.

44. Minimum distance to the origin Find the point closest to the origin on the curve of intersection of the plane $2y + 4z = 5$ and the cone $z^2 = 4x^2 + 4y^2$.

Theory and Examples

45. The condition $\nabla f = \lambda \nabla g$ is not sufficient Although $\nabla f = \lambda \nabla g$ is a necessary condition for the occurrence of an extreme value of $f(x, y)$ subject to the conditions $g(x, y) = 0$ and $\nabla g \neq \mathbf{0}$, it does not in itself guarantee that one exists. As a case in point, try using the method of Lagrange multipliers to find a maximum value of $f(x, y) = x + y$ subject to the constraint that $xy = 16$. The method will identify the two points $(4, 4)$ and $(-4, -4)$ as candidates for the location of extreme values. Yet the sum $(x + y)$ has no maximum value on the hyperbola $xy = 16$. The farther you go from the origin on this hyperbola in the first quadrant, the larger the sum $f(x, y) = x + y$ becomes.

46. A least squares plane The plane $z = Ax + By + C$ is to be "fitted" to the following points (x_k, y_k, z_k):

$$(0, 0, 0), \quad (0, 1, 1), \quad (1, 1, 1), \quad (1, 0, -1).$$

Find the values of A, B, and C that minimize

$$\sum_{k=1}^{4} (Ax_k + By_k + C - z_k)^2,$$

the sum of the squares of the deviations.

47. a. Maximum on a sphere Show that the maximum value of $a^2b^2c^2$ on a sphere of radius r centered at the origin of a Cartesian abc-coordinate system is $(r^2/3)^3$.

b. Geometric and arithmetic means Using part (a), show that for nonnegative numbers a, b, and c,

$$(abc)^{1/3} \leq \frac{a + b + c}{3};$$

that is, the *geometric mean* of three nonnegative numbers is less than or equal to their *arithmetic mean*.

48. Sum of products Let $a_1, a_2, \ldots, a_n$ be n positive numbers. Find the maximum of $\Sigma_{i=1}^{n} a_i x_i$ subject to the constraint $\Sigma_{i=1}^{n} x_i^{\,2} = 1$.

COMPUTER EXPLORATIONS

In Exercises 49–54, use a CAS to perform the following steps implementing the method of Lagrange multipliers for finding constrained extrema:

a. Form the function $h = f - \lambda_1 g_1 - \lambda_2 g_2$, where f is the function to optimize subject to the constraints $g_1 = 0$ and $g_2 = 0$.

b. Determine all the first partial derivatives of h, including the partials with respect to λ_1 and λ_2, and set them equal to 0.

c. Solve the system of equations found in part (b) for all the unknowns, including λ_1 and λ_2.

d. Evaluate f at each of the solution points found in part (c) and select the extreme value subject to the constraints asked for in the exercise.

49. Minimize $f(x, y, z) = xy + yz$ subject to the constraints $x^2 + y^2 - 2 = 0$ and $x^2 + z^2 - 2 = 0$.

50. Minimize $f(x, y, z) = xyz$ subject to the constraints $x^2 + y^2 - 1 = 0$ and $x - z = 0$.

51. Maximize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $2y + 4z - 5 = 0$ and $4x^2 + 4y^2 - z^2 = 0$.

52. Minimize $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints $x^2 - xy + y^2 - z^2 - 1 = 0$ and $x^2 + y^2 - 1 = 0$.

53. Minimize $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ subject to the constraints $2x - y + z - w - 1 = 0$ and $x + y - z + w - 1 = 0$.

54. Determine the distance from the line $y = x + 1$ to the parabola $y^2 = x$. (*Hint:* Let (x, y) be a point on the line and (w, z) a point on the parabola. You want to minimize $(x - w)^2 + (y - z)^2$.)

14.9 Taylor's Formula for Two Variables

In this section we use Taylor's formula to derive the Second Derivative Test for local extreme values (Section 14.7) and the error formula for linearizations of functions of two independent variables (Section 14.6). The use of Taylor's formula in these derivations leads to an extension of the formula that provides polynomial approximations of all orders for functions of two independent variables.

Derivation of the Second Derivative Test

Let $f(x, y)$ have continuous partial derivatives in an open region R containing a point $P(a, b)$ where $f_x = f_y = 0$ (Figure 14.60). Let h and k be increments small enough to put the

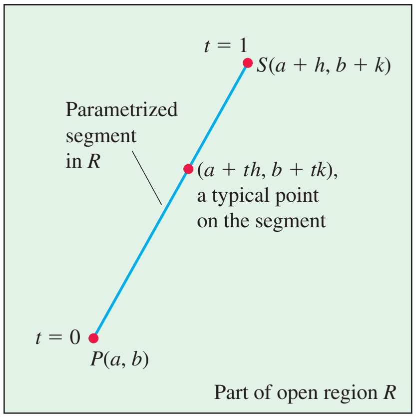


FIGURE 14.60 We begin the derivation of the Second Derivative Test at $P(a, b)$ by parametrizing a typical line segment from P to a point S nearby.

point $S(a + h, b + k)$ and the line segment joining it to P inside R. We parametrize the segment PS as

$$x = a + th, \qquad y = b + tk, \qquad 0 \le t \le 1.$$

If $F(t) = f(a + th, b + tk)$, the Chain Rule gives

$$F'(t) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = hf_x + kf_y.$$

Since f_x and f_y are differentiable (they have continuous partial derivatives), F' is a differentiable function of t and

$$\begin{aligned} F'' &= \frac{\partial F'}{\partial x}\frac{dx}{dt} + \frac{\partial F'}{\partial y}\frac{dy}{dt} = \frac{\partial}{\partial x}(hf_x + kf_y)\cdot h + \frac{\partial}{\partial y}(hf_x + kf_y)\cdot k \\ &= h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}. \qquad f_{xy} = f_{yx} \end{aligned}$$

Since F and F' are continuous on $[0, 1]$ and F' is differentiable on $(0, 1)$, we can apply Taylor's formula with $n = 2$ and $a = 0$ to obtain

$$\begin{aligned} F(1) &= F(0) + F'(0)(1 - 0) + F''(c)\frac{(1-0)^2}{2} \\ &= F(0) + F'(0) + \frac{1}{2}F''(c) \end{aligned} \tag{1}$$

for some c between 0 and 1. Writing Equation (1) in terms of f gives

$$\begin{aligned} f(a + h, b + k) &= f(a, b) + hf_x(a, b) + kf_y(a, b) \\ &\quad + \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,\, b+ck)}. \end{aligned} \tag{2}$$

Since $f_x(a, b) = f_y(a, b) = 0$, this reduces to

$$f(a + h, b + k) - f(a, b) = \frac{1}{2}\left(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy}\right)\Big|_{(a+ch,\, b+ck)}. \tag{3}$$

The presence of an extremum of f at (a, b) is determined by the sign of $f(a + h, b + k) - f(a, b)$. By Equation (3), this is the same as the sign of

$$Q(c) = (h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})|_{(a+ch,\, b+ck)}.$$

Now, if $Q(0) \neq 0$, the sign of $Q(c)$ will be the same as the sign of $Q(0)$ for sufficiently small values of h and k. We can predict the sign of

$$Q(0) = h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b) \tag{4}$$

from the signs of f_{xx} and $f_{xx}f_{yy} - f_{xy}{}^2$ at (a, b). Multiply both sides of Equation (4) by f_{xx} and rearrange the right-hand side to get

$$f_{xx}Q(0) = (hf_{xx} + kf_{xy})^2 + (f_{xx}f_{yy} - f_{xy}{}^2)k^2. \tag{5}$$

From Equation (5) we see that

1. If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b), then $Q(0) < 0$ for all sufficiently small nonzero values of h and k, and f has a *local maximum* value at (a, b).
2. If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b), then $Q(0) > 0$ for all sufficiently small nonzero values of h and k, and f has a *local minimum* value at (a, b).

3. If $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at (a, b), there are combinations of arbitrarily small nonzero values of h and k for which $Q(0) > 0$, and other values for which $Q(0) < 0$. Arbitrarily close to the point $P_0(a, b, f(a, b))$ on the surface $z = f(x, y)$ there are points above P_0 and points below P_0, so f has a *saddle point* at (a, b).
4. If $f_{xx}f_{yy} - f_{xy}{}^2 = 0$, another test is needed. The possibility that $Q(0)$ equals zero prevents us from drawing conclusions about the sign of $Q(c)$.

The Error Formula for Linear Approximations

We want to show that the difference $E(x, y)$ between the values of a function $f(x, y)$ and its linearization $L(x, y)$ at (x_0, y_0) satisfies the inequality

$$|E(x, y)| \le \frac{1}{2}M(|x - x_0| + |y - y_0|)^2.$$

The function f is assumed to have continuous second partial derivatives throughout an open set containing a closed rectangular region R centered at (x_0, y_0). The number M is an upper bound for $|f_{xx}|$, $|f_{yy}|$, and $|f_{xy}|$ on R.

The inequality we want comes from Equation (2). We substitute x_0 and y_0 for a and b, and $x - x_0$ and $y - y_0$ for h and k, respectively, and rearrange the result as

$$f(x, y) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}_{\text{linearization } L(x,\, y)}$$

$$+ \underbrace{\frac{1}{2}\Big((x - x_0)^2 f_{xx} + 2(x - x_0)(y - y_0)f_{xy} + (y - y_0)^2 f_{yy}\Big)\Big|_{(x_0 + c(x - x_0),\, y_0 + c(y - y_0))}}_{\text{error } E(x,\, y)}.$$

This equation reveals that

$$|E| \le \frac{1}{2}\Big(|x - x_0|^2|f_{xx}| + 2|x - x_0||y - y_0||f_{xy}| + |y - y_0|^2|f_{yy}|\Big).$$

Hence, if M is an upper bound for the values of $|f_{xx}|$, $|f_{xy}|$, and $|f_{yy}|$ on R,

$$\begin{aligned}|E| &\le \frac{1}{2}\Big(|x - x_0|^2 M + 2|x - x_0||y - y_0|M + |y - y_0|^2 M\Big)\\ &= \frac{1}{2}M(|x - x_0| + |y - y_0|)^2.\end{aligned}$$

Taylor's Formula for Functions of Two Variables

The formulas derived earlier for F' and F'' can be obtained by applying to $f(x, y)$ the operators

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) \quad \text{and} \quad \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 = h^2\frac{\partial^2}{\partial x^2} + 2hk\frac{\partial^2}{\partial x\,\partial y} + k^2\frac{\partial^2}{\partial y^2}.$$

These are the first two instances of a more general formula,

$$F^{(n)}(t) = \frac{d^n}{dt^n}F(t) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y), \tag{6}$$

which says that applying d^n/dt^n to $F(t)$ gives the same result as applying the operator

$$\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n$$

to $f(x, y)$ after expanding it by the Binomial Theorem.

If partial derivatives of f through order $n + 1$ are continuous throughout a rectangular region centered at (a, b), we may extend the Taylor formula for $F(t)$ to

$$F(t) = F(0) + F'(0)t + \frac{F''(0)}{2!}t^2 + \cdots + \frac{F^{(n)}(0)}{n!}t^{(n)} + \text{ remainder,}$$

and take $t = 1$ to obtain

$$F(1) = F(0) + F'(0) + \frac{F''(0)}{2!} + \cdots + \frac{F^{(n)}(0)}{n!} + \text{ remainder.}$$

When we replace the first n derivatives on the right of this last series by their equivalent expressions from Equation (6) evaluated at $t = 0$ and add the appropriate remainder term, we arrive at the following formula.

Taylor's Formula for $f(x, y)$ at the Point (a, b)

Suppose $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$\begin{aligned} f(a + h, b + k) = {} & f(a, b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)} \\ & + \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\,\Bigg|_{(a,b)} \\ & + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\,\Bigg|_{(a+ch,\, b+ck)}. \end{aligned} \tag{7}$$

The first n derivative terms are evaluated at (a, b). The last term is evaluated at some point $(a + ch, b + ck)$ on the line segment joining (a, b) and $(a + h, b + k)$.

If $(a, b) = (0, 0)$ and we treat h and k as independent variables (denoting them now by x and y), then Equation (7) assumes the following form.

Taylor's Formula for $f(x, y)$ at the Origin

$$\begin{aligned} f(x, y) = {} & f(0, 0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) \\ & + \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1}y\frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right) \\ & + \frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Bigg|_{(cx,\, cy)} \end{aligned} \tag{8}$$

The first n derivative terms are evaluated at $(0, 0)$. The last term is evaluated at a point on the line segment joining the origin and (x, y).

Taylor's formula provides polynomial approximations of two-variable functions. The first n derivative terms give the polynomial; the last term gives the approximation error. The first three terms of Taylor's formula give the function's linearization. To improve on the linearization, we add higher-power terms.

EXAMPLE 1 Find a quadratic approximation to $f(x, y) = \sin x \sin y$ near the origin. How accurate is the approximation if $|x| \le 0.1$ and $|y| \le 0.1$?

Solution We take $n = 2$ in Equation (8):

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$
$$+ \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})_{(cx,\, cy)}.$$

Calculating the values of the partial derivatives,

$$\begin{aligned} f(0, 0) &= \sin x \sin y\big|_{(0,0)} = 0, & f_{xx}(0, 0) &= -\sin x \sin y\big|_{(0,0)} = 0, \\ f_x(0, 0) &= \cos x \sin y\big|_{(0,0)} = 0, & f_{xy}(0, 0) &= \cos x \cos y\big|_{(0,0)} = 1, \\ f_y(0, 0) &= \sin x \cos y\big|_{(0,0)} = 0, & f_{yy}(0, 0) &= -\sin x \sin y\big|_{(0,0)} = 0, \end{aligned}$$

we have the result

$$\sin x \sin y \approx 0 + 0 + 0 + \frac{1}{2}(x^2(0) + 2xy(1) + y^2(0)), \quad \text{or} \quad \sin x \sin y \approx xy.$$

The error in the approximation is

$$E(x, y) = \frac{1}{6}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})\Big|_{(cx,\, cy)}.$$

The third derivatives never exceed 1 in absolute value because they are products of sines and cosines. Also, $|x| \le 0.1$ and $|y| \le 0.1$. Hence

$$|E(x, y)| \le \frac{1}{6}((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3) = \frac{8}{6}(0.1)^3 \le 0.00134$$

(rounded up). The error will not exceed 0.00134 if $|x| \le 0.1$ and $|y| \le 0.1$.

Exercises 14.9

Finding Quadratic and Cubic Approximations

In Exercises 1–10, use Taylor's formula for $f(x, y)$ at the origin to find quadratic and cubic approximations of f near the origin.

1. $f(x, y) = xe^y$
2. $f(x, y) = e^x \cos y$
3. $f(x, y) = y \sin x$
4. $f(x, y) = \sin x \cos y$
5. $f(x, y) = e^x \ln(1 + y)$
6. $f(x, y) = \ln(2x + y + 1)$
7. $f(x, y) = \sin(x^2 + y^2)$
8. $f(x, y) = \cos(x^2 + y^2)$
9. $f(x, y) = \dfrac{1}{1 - x - y}$
10. $f(x, y) = \dfrac{1}{1 - x - y + xy}$
11. Use Taylor's formula to find a quadratic approximation of $f(x, y) = \cos x \cos y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.
12. Use Taylor's formula to find a quadratic approximation of $e^x \sin y$ at the origin. Estimate the error in the approximation if $|x| \le 0.1$ and $|y| \le 0.1$.

14.10 Partial Derivatives with Constrained Variables

In finding partial derivatives of functions like $w = f(x, y)$, we have assumed x and y to be independent. In many applications, however, this is not the case. For example, the internal energy U of a gas may be expressed as a function $U = f(P, V, T)$ of pressure P, volume V, and temperature T. If the individual molecules of the gas do not interact, however, P, V, and T obey (and are constrained by) the ideal gas law

$$PV = nRT \qquad (n \text{ and } R \text{ constant}),$$

and fail to be independent. In this section we learn how to find partial derivatives in situations like this, which occur in economics, engineering, and physics.

Decide Which Variables Are Dependent and Which Are Independent

If the variables in a function $w = f(x, y, z)$ are constrained by a relation like the one imposed on x, y, and z by the equation $z = x^2 + y^2$, the geometric meanings and the numerical values of the partial derivatives of f will depend on which variables are chosen to be dependent and which are chosen to be independent. To see how this choice can affect the outcome, we consider the calculation of $\partial w/\partial x$ when $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

EXAMPLE 1 Find $\partial w/\partial x$ if $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$.

Solution We are given two equations in the four unknowns x, y, z, and w. Like many such systems, this one can be solved for two of the unknowns (the dependent variables) in terms of the others (the independent variables). In being asked for $\partial w/\partial x$, we are told that w is to be a dependent variable and x an independent variable. The possible choices for the other variables come down to

Dependent	*Independent*
w, z	x, y
w, y	x, z

In either case, we can express w explicitly in terms of the selected independent variables. We do this by using the second equation $z = x^2 + y^2$ to eliminate the remaining dependent variable in the first equation.

In the first case, the remaining dependent variable is z. We eliminate it from the first equation by replacing it by $x^2 + y^2$. The resulting expression for w is

$$\begin{aligned} w &= x^2 + y^2 + z^2 = x^2 + y^2 + (x^2 + y^2)^2 \\ &= x^2 + y^2 + x^4 + 2x^2y^2 + y^4 \end{aligned}$$

and

$$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2. \tag{1}$$

This is the formula for $\partial w/\partial x$ when x and y are the independent variables.

In the second case, where the independent variables are x and z and the remaining dependent variable is y, we eliminate the dependent variable y in the expression for w by replacing y^2 in the second equation by $z - x^2$. This gives

$$w = x^2 + y^2 + z^2 = x^2 + (z - x^2) + z^2 = z + z^2$$

and

$$\frac{\partial w}{\partial x} = 0. \tag{2}$$

This is the formula for $\partial w/\partial x$ when x and z are the independent variables.

The formulas for $\partial w/\partial x$ in Equations (1) and (2) are genuinely different. We cannot change either formula into the other by using the relation $z = x^2 + y^2$. There is not just one $\partial w/\partial x$, there are two, and we see that the original instruction to find $\partial w/\partial x$ was incomplete. *Which* $\partial w/\partial x$? we ask.

The geometric interpretations of Equations (1) and (2) help to explain why the equations differ. The function $w = x^2 + y^2 + z^2$ measures the square of the distance from the point (x, y, z) to the origin. The condition $z = x^2 + y^2$ says that the point (x, y, z) lies on the paraboloid of revolution shown in Figure 14.61. What does it mean to calculate $\partial w/\partial x$ at a point $P(x, y, z)$ that can move only on this surface? What is the value of $\partial w/\partial x$ when the coordinates of P are, say, $(1, 0, 1)$?

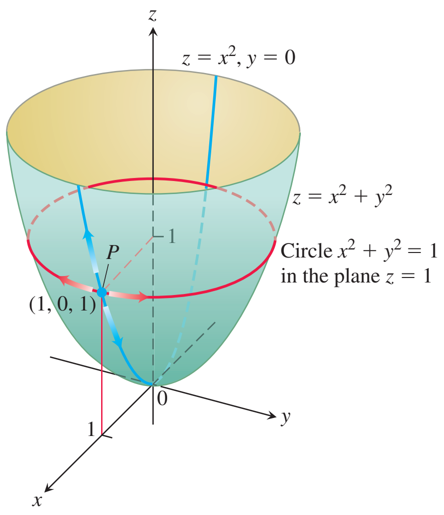


FIGURE 14.61 If P is constrained to lie on the paraboloid $z = x^2 + y^2$, the value of the partial derivative of $w = x^2 + y^2 + z^2$ with respect to x at P depends on the direction of motion (Example 1). (1) As x changes, with $y = 0$, P moves up or down the surface on the parabola $z = x^2$ in the xz-plane with $\partial w/\partial x = 2x + 4x^3$. (2) As x changes, with $z = 1$, P moves the circle $x^2 + y^2 = 1, z = 1$, and $\partial w/\partial x = 0$.

If we take x and y to be independent, then we find $\partial w/\partial x$ by holding y fixed (at $y = 0$ in this case) and letting x vary. Hence, P moves along the parabola $z = x^2$ in the xz-plane. As P moves on this parabola, w, which is the square of the distance from P to the origin, changes. We calculate $\partial w/\partial x$ in this case (our first solution above) to be

$$\frac{\partial w}{\partial x} = 2x + 4x^3 + 4xy^2.$$

At the point $P(1, 0, 1)$, the value of this derivative is

$$\frac{\partial w}{\partial x} = 2 + 4 + 0 = 6.$$

If we take x and z to be independent, then we find $\partial w/\partial x$ by holding z fixed while x varies. Since the z-coordinate of P is 1, varying x moves P along a circle in the plane $z = 1$. As P moves along this circle, its distance from the origin remains constant, and w, being the square of this distance, does not change. That is,

$$\frac{\partial w}{\partial x} = 0,$$

as we found in our second solution. ■

How to Find $\partial w/\partial x$ When the Variables in $w = f(x, y, z)$ Are Constrained by Another Equation

As we saw in Example 1, a typical routine for finding $\partial w/\partial x$ when the variables in the function $w = f(x, y, z)$ are related by another equation has three steps. These steps apply to finding $\partial w/\partial y$ and $\partial w/\partial z$ as well.

1. *Decide* which variables are to be dependent and which are to be independent. (In practice, the decision is based on the physical or theoretical context of our work. In the exercises at the end of this section, we say which variables are which.)
2. *Eliminate* the other dependent variable(s) in the expression for w.
3. *Differentiate* as usual.

If we cannot carry out Step 2 after deciding which variables are dependent, we differentiate the equations as they are and try to solve for $\partial w/\partial x$ afterward. The next example shows how this is done.

EXAMPLE 2 Find $\partial w/\partial x$ at the point $(x, y, z) = (2, -1, 1)$ if

$$w = x^2 + y^2 + z^2, \qquad z^3 - xy + yz + y^3 = 1,$$

and x and y are the independent variables.

Solution It is not convenient to eliminate z in the expression for w. We therefore differentiate both equations implicitly with respect to x, treating x and y as independent variables and w and z as dependent variables. This gives

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \tag{3}$$

and

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} + 0 = 0. \tag{4}$$

These equations may now be combined to express $\partial w/\partial x$ in terms of x, y, and z. We solve Equation (4) for $\partial z/\partial x$ to get

$$\frac{\partial z}{\partial x} = \frac{y}{y + 3z^2}$$

and substitute into Equation (3) to get

$$\frac{\partial w}{\partial x} = 2x + \frac{2yz}{y + 3z^2}.$$

The value of this derivative at $(x, y, z) = (2, -1, 1)$ is

$$\left(\frac{\partial w}{\partial x}\right)_{(2,-1,1)} = 2(2) + \frac{2(-1)(1)}{-1 + 3(1)^2} = 4 + \frac{-2}{2} = 3.$$

Notation

HISTORICAL BIOGRAPHY
Sonya Kovalevsky
(1850–1891)

To show what variables are assumed to be independent in calculating a derivative, we can use the following notation:

$$\left(\frac{\partial w}{\partial x}\right)_y \qquad \partial w/\partial x \text{ with } x \text{ and } y \text{ independent}$$

$$\left(\frac{\partial f}{\partial y}\right)_{x,t} \qquad \partial f/\partial y \text{ with } y, x, \text{ and } t \text{ independent}$$

EXAMPLE 3 Find $(\partial w/\partial x)_{y,z}$ if $w = x^2 + y - z + \sin t$ and $x + y = t$.

Solution With x, y, z independent, we have

$$t = x + y, \qquad w = x^2 + y - z + \sin(x + y)$$

$$\left(\frac{\partial w}{\partial x}\right)_{y,z} = 2x + 0 - 0 + \cos(x + y)\frac{\partial}{\partial x}(x + y)$$

$$= 2x + \cos(x + y).$$

Arrow Diagrams

In solving problems like the one in Example 3, it often helps to start with an arrow diagram that shows how the variables and functions are related. If

$$w = x^2 + y - z + \sin t \qquad \text{and} \qquad x + y = t$$

and we are asked to find $\partial w/\partial x$ when x, y, and z are independent, the appropriate diagram is one like this:

$$\underset{\text{Independent variables}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} \rightarrow \underset{\text{Intermediate variables}}{\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}} \rightarrow \underset{\text{Dependent variable}}{w} \qquad (5)$$

To avoid confusion between the independent and intermediate variables with the same symbolic names in the diagram, it is helpful to rename the intermediate variables (so they are seen as *functions* of the independent variables). Thus, let $u = x$, $v = y$, and $s = z$ denote the renamed intermediate variables. With this notation, the arrow diagram becomes

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} u \\ v \\ s \\ t \end{pmatrix} \to w \qquad (6)$$

Independent variables	Intermediate variables and relations	Dependent variable
	$u = x$ $v = y$ $s = z$ $t = x + y$	

The diagram shows the independent variables on the left, the intermediate variables and their relation to the independent variables in the middle, and the dependent variable on the right. The function w now becomes

$$w = u^2 + v - s + \sin t,$$

where

$$u = x, \qquad v = y, \qquad s = z, \qquad \text{and} \qquad t = x + y.$$

To find $\partial w/\partial x$, we apply the four-variable form of the Chain Rule to w, guided by the arrow diagram in Equation (6):

$$\begin{aligned}\frac{\partial w}{\partial x} &= \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x} \\ &= (2u)(1) + (1)(0) + (-1)(0) + (\cos t)(1) \\ &= 2u + \cos t \\ &= 2x + \cos(x + y). \end{aligned}$$

Substituting the original independent variables $u = x$ and $t = x + y$

Exercises 14.10

Finding Partial Derivatives with Constrained Variables

In Exercises 1–3, begin by drawing a diagram that shows the relations among the variables.

1. If $w = x^2 + y^2 + z^2$ and $z = x^2 + y^2$, find

a. $\left(\frac{\partial w}{\partial y}\right)_z$ **b.** $\left(\frac{\partial w}{\partial z}\right)_x$ **c.** $\left(\frac{\partial w}{\partial z}\right)_y$.

2. If $w = x^2 + y - z + \sin t$ and $x + y = t$, find

a. $\left(\frac{\partial w}{\partial y}\right)_{x,z}$ **b.** $\left(\frac{\partial w}{\partial y}\right)_{z,t}$ **c.** $\left(\frac{\partial w}{\partial z}\right)_{x,y}$

d. $\left(\frac{\partial w}{\partial z}\right)_{y,t}$ **e.** $\left(\frac{\partial w}{\partial t}\right)_{x,z}$ **f.** $\left(\frac{\partial w}{\partial t}\right)_{y,z}$.

3. Let $U = f(P, V, T)$ be the internal energy of a gas that obeys the ideal gas law $PV = nRT$ (n and R constant). Find

a. $\left(\frac{\partial U}{\partial P}\right)_V$ **b.** $\left(\frac{\partial U}{\partial T}\right)_V$.

4. Find

a. $\left(\frac{\partial w}{\partial x}\right)_y$ **b.** $\left(\frac{\partial w}{\partial z}\right)_y$

at the point $(x, y, z) = (0, 1, \pi)$ if

$$w = x^2 + y^2 + z^2 \qquad \text{and} \qquad y \sin z + z \sin x = 0.$$

5. Find

a. $\left(\frac{\partial w}{\partial y}\right)_x$ **b.** $\left(\frac{\partial w}{\partial y}\right)_z$

at the point $(w, x, y, z) = (4, 2, 1, -1)$ if

$$w = x^2y^2 + yz - z^3 \qquad \text{and} \qquad x^2 + y^2 + z^2 = 6.$$

6. Find $(\partial u/\partial y)_x$ at the point $(u, v) = \left(\sqrt{2}, 1\right)$ if $x = u^2 + v^2$ and $y = uv$.

7. Suppose that $x^2 + y^2 = r^2$ and $x = r\cos\theta$, as in polar coordinates. Find

$$\left(\frac{\partial x}{\partial r}\right)_\theta \quad \text{and} \quad \left(\frac{\partial r}{\partial x}\right)_y.$$

8. Suppose that

$$w = x^2 - y^2 + 4z + t \quad \text{and} \quad x + 2z + t = 25.$$

Show that the equations

$$\frac{\partial w}{\partial x} = 2x - 1 \quad \text{and} \quad \frac{\partial w}{\partial x} = 2x - 2$$

each give $\partial w/\partial x$, depending on which variables are chosen to be dependent and which variables are chosen to be independent. Identify the independent variables in each case.

Theory and Examples

9. Establish the fact, widely used in hydrodynamics, that if $f(x, y, z) = 0$, then

$$\left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1.$$

(*Hint:* Express all the derivatives in terms of the formal partial derivatives $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.)

10. If $z = x + f(u)$, where $u = xy$, show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = x.$$

11. Suppose that the equation $g(x, y, z) = 0$ determines z as a differentiable function of the independent variables x and y and that $g_z \neq 0$. Show that

$$\left(\frac{\partial z}{\partial y}\right)_x = -\frac{\partial g/\partial y}{\partial g/\partial z}.$$

12. Suppose that $f(x, y, z, w) = 0$ and $g(x, y, z, w) = 0$ determine z and w as differentiable functions of the independent variables x and y, and suppose that

$$\frac{\partial f}{\partial z}\frac{\partial g}{\partial w} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial z} \neq 0.$$

Show that

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial w} - \dfrac{\partial f}{\partial w}\dfrac{\partial g}{\partial x}}{\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial w} - \dfrac{\partial f}{\partial w}\dfrac{\partial g}{\partial z}}$$

and

$$\left(\frac{\partial w}{\partial y}\right)_x = -\frac{\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial z}}{\dfrac{\partial f}{\partial z}\dfrac{\partial g}{\partial w} - \dfrac{\partial f}{\partial w}\dfrac{\partial g}{\partial z}}.$$

Chapter 14 Questions to Guide Your Review

1. What is a real-valued function of two independent variables? Three independent variables? Give examples.
2. What does it mean for sets in the plane or in space to be open? Closed? Give examples. Give examples of sets that are neither open nor closed.
3. How can you display the values of a function $f(x, y)$ of two independent variables graphically? How do you do the same for a function $f(x, y, z)$ of three independent variables?
4. What does it mean for a function $f(x, y)$ to have limit L as $(x, y) \to (x_0, y_0)$? What are the basic properties of limits of functions of two independent variables?
5. When is a function of two (three) independent variables continuous at a point in its domain? Give examples of functions that are continuous at some points but not others.
6. What can be said about algebraic combinations and composites of continuous functions?
7. Explain the two-path test for nonexistence of limits.
8. How are the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ of a function $f(x, y)$ defined? How are they interpreted and calculated?
9. How does the relation between first partial derivatives and continuity of functions of two independent variables differ from the relation between first derivatives and continuity for real-valued functions of a single independent variable? Give an example.
10. What is the Mixed Derivative Theorem for mixed second-order partial derivatives? How can it help in calculating partial derivatives of second and higher orders? Give examples.
11. What does it mean for a function $f(x, y)$ to be differentiable? What does the Increment Theorem say about differentiability?
12. How can you sometimes decide from examining f_x and f_y that a function $f(x, y)$ is differentiable? What is the relation between the differentiability of f and the continuity of f at a point?
13. What is the general Chain Rule? What form does it take for functions of two independent variables? Three independent variables? Functions defined on surfaces? How do you diagram these different forms? Give examples. What pattern enables one to remember all the different forms?
14. What is the derivative of a function $f(x, y)$ at a point P_0 in the direction of a unit vector $\mathbf{u}$? What rate does it describe? What geometric interpretation does it have? Give examples.

15. What is the gradient vector of a differentiable function $f(x, y)$? How is it related to the function's directional derivatives? State the analogous results for functions of three independent variables.
16. How do you find the tangent line at a point on a level curve of a differentiable function $f(x, y)$? How do you find the tangent plane and normal line at a point on a level surface of a differentiable function $f(x, y, z)$? Give examples.
17. How can you use directional derivatives to estimate change?
18. How do you linearize a function $f(x, y)$ of two independent variables at a point (x_0, y_0)? Why might you want to do this? How do you linearize a function of three independent variables?
19. What can you say about the accuracy of linear approximations of functions of two (three) independent variables?
20. If (x, y) moves from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, how can you estimate the resulting change in the value of a differentiable function $f(x, y)$? Give an example.
21. How do you define local maxima, local minima, and saddle points for a differentiable function $f(x, y)$? Give examples.
22. What derivative tests are available for determining the local extreme values of a function $f(x, y)$? How do they enable you to narrow your search for these values? Give examples.
23. How do you find the extrema of a continuous function $f(x, y)$ on a closed bounded region of the xy-plane? Give an example.
24. Describe the method of Lagrange multipliers and give examples.
25. How does Taylor's formula for a function $f(x, y)$ generate polynomial approximations and error estimates?
26. If $w = f(x, y, z)$, where the variables x, y, and z are constrained by an equation $g(x, y, z) = 0$, what is the meaning of the notation $(\partial w/\partial x)_y$? How can an arrow diagram help you calculate this partial derivative with constrained variables? Give examples.

Chapter 14 Practice Exercises

Domain, Range, and Level Curves

In Exercises 1–4, find the domain and range of the given function and identify its level curves. Sketch a typical level curve.

1. $f(x, y) = 9x^2 + y^2$
2. $f(x, y) = e^{x+y}$
3. $g(x, y) = 1/xy$
4. $g(x, y) = \sqrt{x^2 - y}$

In Exercises 5–8, find the domain and range of the given function and identify its level surfaces. Sketch a typical level surface.

5. $f(x, y, z) = x^2 + y^2 - z$
6. $g(x, y, z) = x^2 + 4y^2 + 9z^2$
7. $h(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2}$
8. $k(x, y, z) = \dfrac{1}{x^2 + y^2 + z^2 + 1}$

Evaluating Limits

Find the limits in Exercises 9–14.

9. $\displaystyle\lim_{(x,y)\to(\pi,\, \ln 2)} e^y \cos x$
10. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2 + y}{x + \cos y}$
11. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y}{x^2 - y^2}$
12. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^3y^3 - 1}{xy - 1}$
13. $\displaystyle\lim_{P\to(1,-1,e)} \ln|x + y + z|$
14. $\displaystyle\lim_{P\to(1,-1,-1)} \tan^{-1}(x + y + z)$

By considering different paths of approach, show that the limits in Exercises 15 and 16 do not exist.

15. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y\neq x^2}} \frac{y}{x^2 - y}$
16. $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ xy\neq 0}} \frac{x^2 + y^2}{xy}$

17. **Continuous extension** Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$ for $(x, y) \neq (0, 0)$. Is it possible to define $f(0, 0)$ in a way that makes f continuous at the origin? Why?
18. **Continuous extension** Let
$$f(x, y) = \begin{cases} \dfrac{\sin(x - y)}{|x| + |y|}, & |x| + |y| \neq 0 \\ 0, & (x, y) = (0, 0). \end{cases}$$
Is f continuous at the origin? Why?

Partial Derivatives

In Exercises 19–24, find the partial derivative of the function with respect to each variable.

19. $g(r, \theta) = r\cos\theta + r\sin\theta$
20. $f(x, y) = \dfrac{1}{2}\ln(x^2 + y^2) + \tan^{-1}\dfrac{y}{x}$
21. $f(R_1, R_2, R_3) = \dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3}$
22. $h(x, y, z) = \sin(2\pi x + y - 3z)$
23. $P(n, R, T, V) = \dfrac{nRT}{V}$ (the ideal gas law)
24. $f(r, l, T, w) = \dfrac{1}{2rl}\sqrt{\dfrac{T}{\pi w}}$

Second-Order Partials

Find the second-order partial derivatives of the functions in Exercises 25–28.

25. $g(x, y) = y + \dfrac{x}{y}$
26. $g(x, y) = e^x + y\sin x$
27. $f(x, y) = x + xy - 5x^3 + \ln(x^2 + 1)$
28. $f(x, y) = y^2 - 3xy + \cos y + 7e^y$

Chain Rule Calculations

29. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi)$, $x = e^t$, and $y = \ln(t + 1)$.
30. Find dw/dt at $t = 1$ if $w = xe^y + y\sin z - \cos z$, $x = 2\sqrt{t}$, $y = t - 1 + \ln t$, and $z = \pi t$.
31. Find $\partial w/\partial r$ and $\partial w/\partial s$ when $r = \pi$ and $s = 0$ if $w = \sin(2x - y)$, $x = r + \sin s$, $y = rs$.
32. Find $\partial w/\partial u$ and $\partial w/\partial v$ when $u = v = 0$ if $w = \ln\sqrt{1 + x^2} - \tan^{-1}x$ and $x = 2e^u\cos v$.
33. Find the value of the derivative of $f(x, y, z) = xy + yz + xz$ with respect to t on the curve $x = \cos t$, $y = \sin t$, $z = \cos 2t$ at $t = 1$.

34. Show that if $w = f(s)$ is any differentiable function of s and if $s = y + 5x$, then

$$\frac{\partial w}{\partial x} - 5\frac{\partial w}{\partial y} = 0.$$

Implicit Differentiation

Assuming that the equations in Exercises 35 and 36 define y as a differentiable function of x, find the value of dy/dx at point P.

35. $1 - x - y^2 - \sin xy = 0, \quad P(0, 1)$

36. $2xy + e^{x+y} - 2 = 0, \quad P(0, \ln 2)$

Directional Derivatives

In Exercises 37–40, find the directions in which f increases and decreases most rapidly at P_0 and find the derivative of f in each direction. Also, find the derivative of f at P_0 in the direction of the vector $\mathbf{v}$.

37. $f(x, y) = \cos x \cos y, \quad P_0(\pi/4, \pi/4), \quad \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

38. $f(x, y) = x^2e^{-2y}, \quad P_0(1, 0), \quad \mathbf{v} = \mathbf{i} + \mathbf{j}$

39. $f(x, y, z) = \ln(2x + 3y + 6z), \quad P_0(-1, -1, 1),$
$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$

40. $f(x, y, z) = x^2 + 3xy - z^2 + 2y + z + 4, \quad P_0(0, 0, 0),$
$\mathbf{v} = \mathbf{i} + \mathbf{j} + \mathbf{k}$

41. **Derivative in velocity direction** Find the derivative of $f(x, y, z) = xyz$ in the direction of the velocity vector of the helix

$$\mathbf{r}(t) = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + 3t\mathbf{k}$$

at $t = \pi/3$.

42. **Maximum directional derivative** What is the largest value that the directional derivative of $f(x, y, z) = xyz$ can have at the point $(1, 1, 1)$?

43. **Directional derivatives with given values** At the point $(1, 2)$, the function $f(x, y)$ has a derivative of 2 in the direction toward $(2, 2)$ and a derivative of -2 in the direction toward $(1, 1)$.

a. Find $f_x(1, 2)$ and $f_y(1, 2)$.

b. Find the derivative of f at $(1, 2)$ in the direction toward the point $(4, 6)$.

44. Which of the following statements are true if $f(x, y)$ is differentiable at (x_0, y_0)? Give reasons for your answers.

a. If $\mathbf{u}$ is a unit vector, the derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is $(f_x(x_0, y_0)\mathbf{i} + f_y(x_0, y_0)\mathbf{j}) \cdot \mathbf{u}$.

b. The derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is a vector.

c. The directional derivative of f at (x_0, y_0) has its greatest value in the direction of ∇f.

d. At (x_0, y_0), vector ∇f is normal to the curve $f(x, y) = f(x_0, y_0)$.

Gradients, Tangent Planes, and Normal Lines

In Exercises 45 and 46, sketch the surface $f(x, y, z) = c$ together with ∇f at the given points.

45. $x^2 + y + z^2 = 0; \quad (0, -1, \pm 1), \quad (0, 0, 0)$

46. $y^2 + z^2 = 4; \quad (2, \pm 2, 0), \quad (2, 0, \pm 2)$

In Exercises 47 and 48, find an equation for the plane tangent to the level surface $f(x, y, z) = c$ at the point P_0. Also, find parametric equations for the line that is normal to the surface at P_0.

47. $x^2 - y - 5z = 0, \quad P_0(2, -1, 1)$

48. $x^2 + y^2 + z = 4, \quad P_0(1, 1, 2)$

In Exercises 49 and 50, find an equation for the plane tangent to the surface $z = f(x, y)$ at the given point.

49. $z = \ln(x^2 + y^2), \quad (0, 1, 0)$

50. $z = 1/(x^2 + y^2), \quad (1, 1, 1/2)$

In Exercises 51 and 52, find equations for the lines that are tangent and normal to the level curve $f(x, y) = c$ at the point P_0. Then sketch the lines and level curve together with ∇f at P_0.

51. $y - \sin x = 1, \quad P_0(\pi, 1)$

52. $\dfrac{y^2}{2} - \dfrac{x^2}{2} = \dfrac{3}{2}, \quad P_0(1, 2)$

Tangent Lines to Curves

In Exercises 53 and 54, find parametric equations for the line that is tangent to the curve of intersection of the surfaces at the given point.

53. Surfaces: $x^2 + 2y + 2z = 4, \quad y = 1$
Point: $(1, 1, 1/2)$

54. Surfaces: $x + y^2 + z = 2, \quad y = 1$
Point: $(1/2, 1, 1/2)$

Linearizations

In Exercises 55 and 56, find the linearization $L(x, y)$ of the function $f(x, y)$ at the point P_0. Then find an upper bound for the magnitude of the error E in the approximation $f(x, y) \approx L(x, y)$ over the rectangle R.

55. $f(x, y) = \sin x \cos y, \quad P_0(\pi/4, \pi/4)$

R: $\left|x - \dfrac{\pi}{4}\right| \le 0.1, \quad \left|y - \dfrac{\pi}{4}\right| \le 0.1$

56. $f(x, y) = xy - 3y^2 + 2, \quad P_0(1, 1)$
R: $|x - 1| \le 0.1, \quad |y - 1| \le 0.2$

Find the linearizations of the functions in Exercises 57 and 58 at the given points.

57. $f(x, y, z) = xy + 2yz - 3xz$ at $(1, 0, 0)$ and $(1, 1, 0)$

58. $f(x, y, z) = \sqrt{2}\cos x \sin(y + z)$ at $(0, 0, \pi/4)$ and $(\pi/4, \pi/4, 0)$

Estimates and Sensitivity to Change

59. **Measuring the volume of a pipeline** You plan to calculate the volume inside a stretch of pipeline that is about 36 in. in diameter and 1 mile long. With which measurement should you be more careful, the length or the diameter? Why?

60. **Sensitivity to change** Is $f(x, y) = x^2 - xy + y^2 - 3$ more sensitive to changes in x or to changes in y when it is near the point $(1, 2)$? How do you know?

61. **Change in an electrical circuit** Suppose that the current I (amperes) in an electrical circuit is related to the voltage V (volts) and the resistance R (ohms) by the equation $I = V/R$. If the voltage drops from 24 to 23 volts and the resistance drops from 100 to 80 ohms, will I increase or decrease? By about how much? Is the change in I more sensitive to change in the voltage or to change in the resistance? How do you know?

62. **Maximum error in estimating the area of an ellipse** If $a = 10$ cm and $b = 16$ cm to the nearest millimeter, what should you expect the maximum percentage error to be in the calculated area $A = \pi ab$ of the ellipse $x^2/a^2 + y^2/b^2 = 1$?

63. **Error in estimating a product** Let $y = uv$ and $z = u + v$, where u and v are positive independent variables.

a. If u is measured with an error of 2% and v with an error of 3%, about what is the percentage error in the calculated value of y?

b. Show that the percentage error in the calculated value of z is less than the percentage error in the value of y.

64. Cardiac index To make different people comparable in studies of cardiac output, researchers divide the measured cardiac output by the body surface area to find the *cardiac index C*:

$$C = \frac{\text{cardiac output}}{\text{body surface area}}.$$

The body surface area B of a person with weight w and height h is approximated by the formula

$$B = 71.84w^{0.425}h^{0.725},$$

which gives B in square centimeters when w is measured in kilograms and h in centimeters. You are about to calculate the cardiac index of a person 180 cm tall, weighing 70 kg, with cardiac output of 7 L/min. Which will have a greater effect on the calculation, a 1-kg error in measuring the weight or a 1-cm error in measuring the height?

Local Extrema

Test the functions in Exercises 65–70 for local maxima and minima and saddle points. Find each function's value at these points.

65. $f(x, y) = x^2 - xy + y^2 + 2x + 2y - 4$

66. $f(x, y) = 5x^2 + 4xy - 2y^2 + 4x - 4y$

67. $f(x, y) = 2x^3 + 3xy + 2y^3$

68. $f(x, y) = x^3 + y^3 - 3xy + 15$

69. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$

70. $f(x, y) = x^4 - 8x^2 + 3y^2 - 6y$

Absolute Extrema

In Exercises 71–78, find the absolute maximum and minimum values of f on the region R.

71. $f(x, y) = x^2 + xy + y^2 - 3x + 3y$

R: The triangular region cut from the first quadrant by the line $x + y = 4$

72. $f(x, y) = x^2 - y^2 - 2x + 4y + 1$

R: The rectangular region in the first quadrant bounded by the coordinate axes and the lines $x = 4$ and $y = 2$

73. $f(x, y) = y^2 - xy - 3y + 2x$

R: The square region enclosed by the lines $x = \pm 2$ and $y = \pm 2$

74. $f(x, y) = 2x + 2y - x^2 - y^2$

R: The square region bounded by the coordinate axes and the lines $x = 2$, $y = 2$ in the first quadrant

75. $f(x, y) = x^2 - y^2 - 2x + 4y$

R: The triangular region bounded below by the x-axis, above by the line $y = x + 2$, and on the right by the line $x = 2$

76. $f(x, y) = 4xy - x^4 - y^4 + 16$

R: The triangular region bounded below by the line $y = -2$, above by the line $y = x$, and on the right by the line $x = 2$

77. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2$

R: The square region enclosed by the lines $x = \pm 1$ and $y = \pm 1$

78. $f(x, y) = x^3 + 3xy + y^3 + 1$

R: The square region enclosed by the lines $x = \pm 1$ and $y = \pm 1$

Lagrange Multipliers

79. Extrema on a circle Find the extreme values of $f(x, y) = x^3 + y^2$ on the circle $x^2 + y^2 = 1$.

80. Extrema on a circle Find the extreme values of $f(x, y) = xy$ on the circle $x^2 + y^2 = 1$.

81. Extrema in a disk Find the extreme values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disk $x^2 + y^2 \le 1$.

82. Extrema in a disk Find the extreme values of $f(x, y) = x^2 + y^2 - 3x - xy$ on the disk $x^2 + y^2 \le 9$.

83. Extrema on a sphere Find the extreme values of $f(x, y, z) = x - y + z$ on the unit sphere $x^2 + y^2 + z^2 = 1$.

84. Minimum distance to origin Find the points on the surface $x^2 - zy = 4$ closest to the origin.

85. Minimizing cost of a box A closed rectangular box is to have volume $V\ \text{cm}^3$. The cost of the material used in the box is a cents/cm^2 for top and bottom, b cents/cm^2 for front and back, and c cents/cm^2 for the remaining sides. What dimensions minimize the total cost of materials?

86. Least volume Find the plane $x/a + y/b + z/c = 1$ that passes through the point (2, 1, 2) and cuts off the least volume from the first octant.

87. Extrema on curve of intersecting surfaces Find the extreme values of $f(x, y, z) = x(y + z)$ on the curve of intersection of the right circular cylinder $x^2 + y^2 = 1$ and the hyperbolic cylinder $xz = 1$.

88. Minimum distance to origin on curve of intersecting plane and cone Find the point closest to the origin on the curve of intersection of the plane $x + y + z = 1$ and the cone $z^2 = 2x^2 + 2y^2$.

Theory and Examples

89. Let $w = f(r, \theta)$, $r = \sqrt{x^2 + y^2}$, and $\theta = \tan^{-1}(y/x)$. Find $\partial w/\partial x$ and $\partial w/\partial y$ and express your answers in terms of r and θ.

90. Let $z = f(u, v)$, $u = ax + by$, and $v = ax - by$. Express z_x and z_y in terms of f_u, f_v, and the constants a and b.

91. If a and b are constants, $w = u^3 + \tanh u + \cos u$, and $u = ax + by$, show that

$$a\frac{\partial w}{\partial y} = b\frac{\partial w}{\partial x}.$$

92. Using the Chain Rule If $w = \ln(x^2 + y^2 + 2z)$, $x = r + s$, $y = r - s$, and $z = 2rs$, find w_r and w_s by the Chain Rule. Then check your answer another way.

93. Angle between vectors The equations $e^u \cos v - x = 0$ and $e^u \sin v - y = 0$ define u and v as differentiable functions of x and y. Show that the angle between the vectors

$$\frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} \quad \text{and} \quad \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j}$$

is constant.

94. Polar coordinates and second derivatives Introducing polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ changes $f(x, y)$ to $g(r, \theta)$. Find the value of $\partial^2 g/\partial\theta^2$ at the point $(r, \theta) = (2, \pi/2)$, given that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 1$$

at that point.

95. Normal line parallel to a plane Find the points on the surface

$$(y + z)^2 + (z - x)^2 = 16$$

where the normal line is parallel to the yz-plane.

96. Tangent plane parallel to *xy*-plane Find the points on the surface

$$xy + yz + zx - x - z^2 = 0$$

where the tangent plane is parallel to the xy-plane.

97. When gradient is parallel to position vector Suppose that $\nabla f(x, y, z)$ is always parallel to the position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $f(0, 0, a) = f(0, 0, -a)$ for any a.

98. One-sided directional derivative in all directions, but no gradient The *one-sided directional derivative of f at $P(x_0, y_0, z_0)$ in the direction* $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ is the number

$$\lim_{s\to 0^+} \frac{f(x_0 + su_1, y_0 + su_2, z_0 + su_3) - f(x_0, y_0, z_0)}{s}.$$

Show that the one-sided directional derivative of

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

at the origin equals 1 in any direction but that f has no gradient vector at the origin.

99. Normal line through origin Show that the line normal to the surface $xy + z = 2$ at the point $(1, 1, 1)$ passes through the origin.

100. Tangent plane and normal line

a. Sketch the surface $x^2 - y^2 + z^2 = 4$.

b. Find a vector normal to the surface at $(2, -3, 3)$. Add the vector to your sketch.

c. Find equations for the tangent plane and normal line at $(2, -3, 3)$.

Partial Derivatives with Constrained Variables

In Exercises 101 and 102, begin by drawing a diagram that shows the relations among the variables.

101. If $w = x^2 e^{yz}$ and $z = x^2 - y^2$ find

a. $\left(\dfrac{\partial w}{\partial y}\right)_z$ **b.** $\left(\dfrac{\partial w}{\partial z}\right)_x$ **c.** $\left(\dfrac{\partial w}{\partial z}\right)_y$.

102. Let $U = f(P, V, T)$ be the internal energy of a gas that obeys the ideal gas law $PV = nRT$ (n and R constant). Find

a. $\left(\dfrac{\partial U}{\partial T}\right)_P$ **b.** $\left(\dfrac{\partial U}{\partial V}\right)_T$.

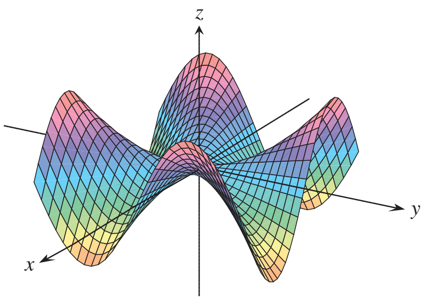
Chapter 14 Additional and Advanced Exercises

Partial Derivatives

1. Function with saddle at the origin If you did Exercise 60 in Section 14.2, you know that the function

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(see the accompanying figure) is continuous at $(0, 0)$. Find $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$.



2. Finding a function from second partials Find a function $w = f(x, y)$ whose first partial derivatives are $\partial w/\partial x = 1 + e^x\cos y$ and $\partial w/\partial y = 2y - e^x\sin y$ and whose value at the point $(\ln 2, 0)$ is $\ln 2$.

3. A proof of Leibniz's Rule Leibniz's Rule says that if f is continuous on $[a, b]$ and if $u(x)$ and $v(x)$ are differentiable functions of x whose values lie in $[a, b]$, then

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}.$$

Prove the rule by setting

$$g(u, v) = \int_u^v f(t)\,dt, \qquad u = u(x), \qquad v = v(x)$$

and calculating dg/dx with the Chain Rule.

4. Finding a function with constrained second partials Suppose that f is a twice-differentiable function of r, that $r = \sqrt{x^2 + y^2 + z^2}$, and that

$$f_{xx} + f_{yy} + f_{zz} = 0.$$

Show that for some constants a and b,

$$f(r) = \frac{a}{r} + b.$$

5. Homogeneous functions A function $f(x, y)$ is *homogeneous of degree n* (n a nonnegative integer) if $f(tx, ty) = t^n f(x, y)$ for all t, x, and y. For such a function (sufficiently differentiable), prove that

a. $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$

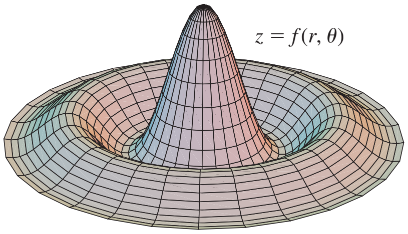
b. $x^2\left(\dfrac{\partial^2 f}{\partial x^2}\right) + 2xy\left(\dfrac{\partial^2 f}{\partial x \partial y}\right) + y^2\left(\dfrac{\partial^2 f}{\partial y^2}\right) = n(n-1)f.$

6. Surface in polar coordinates Let

$$f(r, \theta) = \begin{cases} \dfrac{\sin 6r}{6r}, & r \neq 0 \\ 1, & r = 0, \end{cases}$$

where r and θ are polar coordinates. Find

a. $\lim_{r \to 0} f(r, \theta)$ **b.** $f_r(0, 0)$ **c.** $f_\theta(r, \theta),\quad r \neq 0.$



Gradients and Tangents

7. Properties of position vectors Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and let $r = |\mathbf{r}|$.

a. Show that $\nabla r = \mathbf{r}/r$.

b. Show that $\nabla(r^n) = nr^{n-2}\mathbf{r}$.

c. Find a function whose gradient equals $\mathbf{r}$.

d. Show that $\mathbf{r} \cdot d\mathbf{r} = r\,dr$.

e. Show that $\nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}$ for any constant vector $\mathbf{A}$.

8. Gradient orthogonal to tangent Suppose that a differentiable function $f(x, y)$ has the constant value c along the differentiable curve $x = g(t)$, $y = h(t)$; that is,

$$f(g(t), h(t)) = c$$

for all values of t. Differentiate both sides of this equation with respect to t to show that ∇f is orthogonal to the curve's tangent vector at every point on the curve.

9. Curve tangent to a surface Show that the curve

$$\mathbf{r}(t) = (\ln t)\mathbf{i} + (t \ln t)\mathbf{j} + t\mathbf{k}$$

is tangent to the surface

$$xz^2 - yz + \cos xy = 1$$

at (0, 0, 1).

10. Curve tangent to a surface Show that the curve

$$\mathbf{r}(t) = \left(\frac{t^3}{4} - 2\right)\mathbf{i} + \left(\frac{4}{t} - 3\right)\mathbf{j} + \cos(t - 2)\mathbf{k}$$

is tangent to the surface

$$x^3 + y^3 + z^3 - xyz = 0$$

at $(0, -1, 1)$.

Extreme Values

11. Extrema on a surface Show that the only possible maxima and minima of z on the surface $z = x^3 + y^3 - 9xy + 27$ occur at (0, 0) and (3, 3). Show that neither a maximum nor a minimum occurs at (0, 0). Determine whether z has a maximum or a minimum at (3, 3).

12. Maximum in closed first quadrant Find the maximum value of $f(x, y) = 6xye^{-(2x+3y)}$ in the closed first quadrant (includes the nonnegative axes).

13. Minimum volume cut from first octant Find the minimum volume for a region bounded by the planes $x = 0$, $y = 0$, $z = 0$ and a plane tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at a point in the first octant.

14. Minimum distance from a line to a parabola in *xy*-plane By minimizing the function $f(x, y, u, v) = (x - u)^2 + (y - v)^2$ subject to the constraints $y = x + 1$ and $u = v^2$, find the minimum distance in the xy-plane from the line $y = x + 1$ to the parabola $y^2 = x$.

Theory and Examples

15. Boundedness of first partials implies continuity Prove the following theorem: If $f(x, y)$ is defined in an open region R of the xy-plane and if f_x and f_y are bounded on R, then $f(x, y)$ is continuous on R. (The assumption of boundedness is essential.)

16. Suppose that $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ is a smooth curve in the domain of a differentiable function $f(x, y, z)$. Describe the relation between df/dt, ∇f, and $\mathbf{v} = d\mathbf{r}/dt$. What can be said about ∇f and $\mathbf{v}$ at interior points of the curve where f has extreme values relative to its other values on the curve? Give reasons for your answer.

17. Finding functions from partial derivatives Suppose that f and g are functions of x and y such that

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial x} = \frac{\partial g}{\partial y},$$

and suppose that

$$\frac{\partial f}{\partial x} = 0, \quad f(1, 2) = g(1, 2) = 5, \quad \text{and} \quad f(0, 0) = 4.$$

Find $f(x, y)$ and $g(x, y)$.

18. Rate of change of the rate of change We know that if $f(x, y)$ is a function of two variables and if $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$ is the rate of change of $f(x, y)$ at (x, y) in the direction of $\mathbf{u}$. Give a similar formula for the rate of change *of the rate of change of* $f(x, y)$ at (x, y) in the direction $\mathbf{u}$.

19. Path of a heat-seeking particle A heat-seeking particle has the property that at any point (x, y) in the plane it moves in the direction of maximum temperature increase. If the temperature at (x, y) is $T(x, y) = -e^{-2y}\cos x$, find an equation $y = f(x)$ for the path of a heat-seeking particle at the point $(\pi/4, 0)$.

20. Velocity after a ricochet A particle traveling in a straight line with constant velocity $\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ passes through the point (0, 0, 30) and hits the surface $z = 2x^2 + 3y^2$. The particle ricochets off the surface, the angle of reflection being equal to the angle of incidence. Assuming no loss of speed, what is the velocity of the particle after the ricochet? Simplify your answer.

21. Directional derivatives tangent to a surface Let S be the surface that is the graph of $f(x, y) = 10 - x^2 - y^2$. Suppose that the temperature in space at each point (x, y, z) is $T(x, y, z) = x^2y + y^2z + 4x + 14y + z$.

a. Among all the possible directions tangential to the surface S at the point $(0, 0, 10)$, which direction will make the rate of change of temperature at $(0, 0, 10)$ a maximum?

b. Which direction tangential to S at the point $(1, 1, 8)$ will make the rate of change of temperature a maximum?

22. Drilling another borehole On a flat surface of land, geologists drilled a borehole straight down and hit a mineral deposit at 1000 ft. They drilled a second borehole 100 ft to the north of the first and hit the mineral deposit at 950 ft. A third borehole 100 ft east of the first borehole struck the mineral deposit at 1025 ft. The geologists have reasons to believe that the mineral deposit is in the shape of a dome, and for the sake of economy, they would like to find where the deposit is closest to the surface. Assuming the surface to be the xy-plane, in what direction from the first borehole would you suggest the geologists drill their fourth borehole?

The one-dimensional heat equation If $w(x, t)$ represents the temperature at position x at time t in a uniform wire with perfectly insulated sides, then the partial derivatives w_{xx} and w_t satisfy a differential equation of the form

$$w_{xx} = \frac{1}{c^2} w_t .$$

This equation is called the *one-dimensional heat equation*. The value of the positive constant c^2 is determined by the material from which the wire is made.

23. Find all solutions of the one-dimensional heat equation of the form $w = e^{rt} \sin \pi x$, where r is a constant.

24. Find all solutions of the one-dimensional heat equation that have the form $w = e^{rt} \sin kx$ and satisfy the conditions that $w(0, t) = 0$ and $w(L, t) = 0$. What happens to these solutions as $t \to \infty$?

Chapter 14 Technology Application Projects

Mathematica/Maple Modules:

Plotting Surfaces

Efficiently generate plots of surfaces, contours, and level curves.

Exploring the Mathematics Behind Skateboarding: Analysis of the Directional Derivative

The path of a skateboarder is introduced, first on a level plane, then on a ramp, and finally on a paraboloid. Compute, plot, and analyze the directional derivative in terms of the skateboarder.

Looking for Patterns and Applying the Method of Least Squares to Real Data

Fit a line to a set of numerical data points by choosing the line that minimizes the sum of the squares of the vertical distances from the points to the line.

Lagrange Goes Skateboarding: How High Does He Go?

Revisit and analyze the skateboarders' adventures for maximum and minimum heights from both a graphical and analytic perspective using Lagrange multipliers.

15

Multiple Integrals

OVERVIEW In this chapter we define the *double integral* of a function of two variables $f(x, y)$ over a region in the plane as the limit of approximating Riemann sums. Just as a single integral represents signed area, so does a double integral represent signed volume. Double integrals can be evaluated using the Fundamental Theorem of Calculus studied in Section 5.4, but now the evaluations are done twice by integrating with respect to each of the variables x and y in turn. Double integrals can be used to find areas of more general regions in the plane than those encountered in Chapter 5. Moreover, just as the Substitution Rule could simplify finding single integrals, we can sometimes use polar coordinates to simplify computing a double integral. We study more general substitutions for evaluating double integrals as well.

We also define *triple integrals* for a function of three variables $f(x, y, z)$ over a region in space. Triple integrals can be used to find volumes of still more general regions in space, and their evaluation is like that of double integrals with yet a third evaluation. *Cylindrical* or *spherical coordinates* can sometimes be used to simplify the calculation of a triple integral, and we investigate those techniques. Double and triple integrals have a number of additional applications, such as calculating the average value of a multivariable function, and finding moments and centers of mass for more general regions than those encountered before.

15.1 Double and Iterated Integrals over Rectangles

In Chapter 5 we defined the definite integral of a continuous function $f(x)$ over an interval $[a, b]$ as a limit of Riemann sums. In this section we extend this idea to define the *double integral* of a continuous function of two variables $f(x, y)$ over a bounded rectangle R in the plane. The Riemann sums for the integral of a single-variable function $f(x)$ are obtained by partitioning a finite interval into thin subintervals, multiplying the width of each subinterval by the value of f at a point c_k inside that subinterval, and then adding together all the products. A similar method of partitioning, multiplying, and summing is used to construct double integrals as limits of approximating Riemann sums.

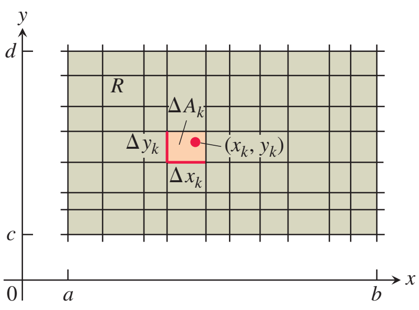


FIGURE 15.1 Rectangular grid partitioning the region R into small rectangles of area $\Delta A_k = \Delta x_k \, \Delta y_k$.

Double Integrals

We begin our investigation of double integrals by considering the simplest type of planar region, a rectangle. We consider a function $f(x, y)$ defined on a rectangular region R,

$$R: \quad a \le x \le b, \quad c \le y \le d.$$

We subdivide R into small rectangles using a network of lines parallel to the x- and y-axes (Figure 15.1). The lines divide R into n rectangular pieces, where the number of such pieces n gets large as the width and height of each piece gets small. These rectangles form a **partition** of R. A small rectangular piece of width Δx and height Δy has area $\Delta A = \Delta x \Delta y$. If we number the small pieces partitioning R in some order, then their areas are given by numbers $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$, where ΔA_k is the area of the kth small rectangle.

To form a Riemann sum over R, we choose a point (x_k, y_k) in the kth small rectangle, multiply the value of f at that point by the area ΔA_k, and add together the products:

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

Depending on how we pick (x_k, y_k) in the kth small rectangle, we may get different values for S_n.

We are interested in what happens to these Riemann sums as the widths and heights of all the small rectangles in the partition of R approach zero. The **norm** of a partition P, written $\|P\|$, is the largest width or height of any rectangle in the partition. If $\|P\| = 0.1$ then all the rectangles in the partition of R have width at most 0.1 and height at most 0.1. Sometimes the Riemann sums converge as the norm of P goes to zero, written $\|P\| \to 0$. The resulting limit is then written as

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

As $\|P\| \to 0$ and the rectangles get narrow and short, their number n increases, so we can also write this limit as

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k,$$

with the understanding that $\|P\| \to 0$, and hence $\Delta A_k \to 0$, as $n \to \infty$.

Many choices are involved in a limit of this kind. The collection of small rectangles is determined by the grid of vertical and horizontal lines that determine a rectangular partition of R. In each of the resulting small rectangles there is a choice of an arbitrary point (x_k, y_k) at which f is evaluated. These choices together determine a single Riemann sum. To form a limit, we repeat the whole process again and again, choosing partitions whose rectangle widths and heights both go to zero and whose number goes to infinity.

When a limit of the sums S_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be **integrable** and the limit is called the **double integral** of f over R, written as

$$\iint_R f(x, y)\,dA \qquad \text{or} \qquad \iint_R f(x, y)\,dx\,dy.$$

It can be shown that if $f(x, y)$ is a continuous function throughout R, then f is integrable, as in the single-variable case discussed in Chapter 5. Many discontinuous functions are also integrable, including functions that are discontinuous only on a finite number of points or smooth curves. We leave the proof of these facts to a more advanced text.

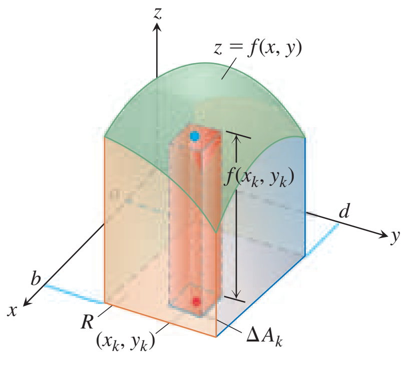


FIGURE 15.2 Approximating solids with rectangular boxes leads us to define the volumes of more general solids as double integrals. The volume of the solid shown here is the double integral of $f(x, y)$ over the base region R.

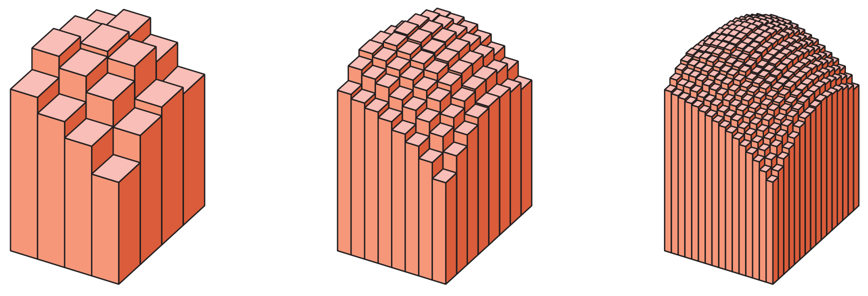
Double Integrals as Volumes

When $f(x, y)$ is a positive function over a rectangular region R in the xy-plane, we may interpret the double integral of f over R as the volume of the 3-dimensional solid region over the xy-plane bounded below by R and above by the surface $z = f(x, y)$ (Figure 15.2). Each term $f(x_k, y_k)\Delta A_k$ in the sum $S_n = \sum f(x_k, y_k)\Delta A_k$ is the volume of a vertical rectangular box that approximates the volume of the portion of the solid that stands directly above the base ΔA_k. The sum S_n thus approximates what we want to call the total volume of the solid. We *define* this volume to be

$$\text{Volume} = \lim_{n \to \infty} S_n = \iint_R f(x, y)\,dA,$$

where $\Delta A_k \to 0$ as $n \to \infty$.

As you might expect, this more general method of calculating volume agrees with the methods in Chapter 6, but we do not prove this here. Figure 15.3 shows Riemann sum approximations to the volume becoming more accurate as the number n of boxes increases.



(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

FIGURE 15.3 As n increases, the Riemann sum approximations approach the total volume of the solid shown in Figure 15.2.

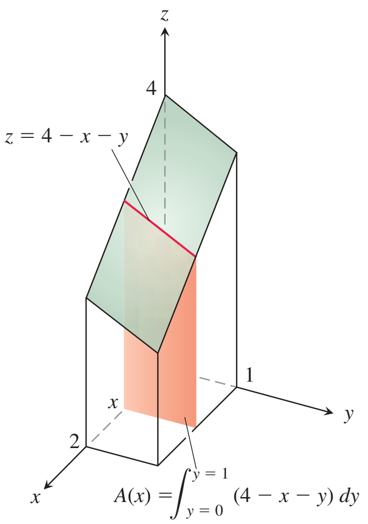


FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

Fubini's Theorem for Calculating Double Integrals

Suppose that we wish to calculate the volume under the plane $z = 4 - x - y$ over the rectangular region R: $0 \leq x \leq 2, 0 \leq y \leq 1$ in the xy-plane. If we apply the method of slicing from Section 6.1, with slices perpendicular to the x-axis (Figure 15.4), then the volume is

$$\int_{x=0}^{x=2} A(x)\,dx, \tag{1}$$

where $A(x)$ is the cross-sectional area at x. For each value of x, we may calculate $A(x)$ as the integral

$$A(x) = \int_{y=0}^{y=1} (4 - x - y)\,dy, \tag{2}$$

which is the area under the curve $z = 4 - x - y$ in the plane of the cross-section at x. In calculating $A(x)$, x is held fixed and the integration takes place with respect to y. Combining Equations (1) and (2), we see that the volume of the entire solid is

$$\begin{aligned}
\text{Volume} &= \int_{x=0}^{x=2} A(x)\,dx = \int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4 - x - y)\,dy \right) dx \\
&= \int_{x=0}^{x=2} \left[4y - xy - \frac{y^2}{2} \right]_{y=0}^{y=1} dx = \int_{x=0}^{x=2} \left(\frac{7}{2} - x \right) dx \\
&= \left[\frac{7}{2}x - \frac{x^2}{2} \right]_0^2 = 5.
\end{aligned}$$

If we just wanted to write a formula for the volume, without carrying out any of the integrations, we could write

$$\text{Volume} = \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx. \tag{3}$$

The expression on the right, called an **iterated** or **repeated integral**, says that the volume is obtained by integrating $4 - x - y$ with respect to y from $y = 0$ to $y = 1$, holding x fixed, and then integrating the resulting expression in x with respect to x from $x = 0$ to $x = 2$. The limits of integration 0 and 1 are associated with y, so they are placed on the integral closest to dy. The other limits of integration, 0 and 2, are associated with the variable x, so they are placed on the outside integral symbol that is paired with dx.

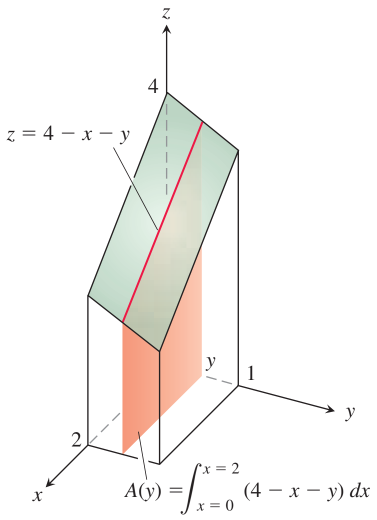


FIGURE 15.5 To obtain the cross-sectional area $A(y)$, we hold y fixed and integrate with respect to x.

What would have happened if we had calculated the volume by slicing with planes perpendicular to the y-axis (Figure 15.5)? As a function of y, the typical cross-sectional area is

$$A(y) = \int_{x=0}^{x=2} (4 - x - y)\, dx = \left[4x - \frac{x^2}{2} - xy\right]_{x=0}^{x=2} = 6 - 2y. \tag{4}$$

The volume of the entire solid is therefore

$$\text{Volume} = \int_{y=0}^{y=1} A(y)\, dy = \int_{y=0}^{y=1} (6 - 2y)\, dy = \left[6y - y^2\right]_0^1 = 5,$$

in agreement with our earlier calculation.

Again, we may give a formula for the volume as an iterated integral by writing

$$\text{Volume} = \int_0^1 \int_0^2 (4 - x - y)\, dx\, dy.$$

The expression on the right says we can find the volume by integrating $4 - x - y$ with respect to x from $x = 0$ to $x = 2$ as in Equation (4) and integrating the result with respect to y from $y = 0$ to $y = 1$. In this iterated integral, the order of integration is first x and then y, the reverse of the order in Equation (3).

What do these two volume calculations with iterated integrals have to do with the double integral

$$\iint_R (4 - x - y)\, dA$$

over the rectangle R: $0 \le x \le 2$, $0 \le y \le 1$? The answer is that both iterated integrals give the value of the double integral. This is what we would reasonably expect, since the double integral measures the volume of the same region as the two iterated integrals. A theorem published in 1907 by Guido Fubini says that the double integral of any continuous function over a rectangle can be calculated as an iterated integral in either order of integration. (Fubini proved his theorem in greater generality, but this is what it says in our setting.)

HISTORICAL BIOGRAPHY
Guido Fubini
(1879–1943)

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

Fubini's Theorem says that double integrals over rectangles can be calculated as iterated integrals. Thus, we can evaluate a double integral by integrating with respect to one variable at a time using the Fundamental Theorem of Calculus.

Fubini's Theorem also says that we may calculate the double integral by integrating in *either* order, a genuine convenience. When we calculate a volume by slicing, we may use either planes perpendicular to the x-axis or planes perpendicular to the y-axis.

EXAMPLE 1 Calculate $\iint_R f(x, y)\, dA$ for

$$f(x, y) = 100 - 6x^2 y \quad \text{and} \quad R:\ 0 \le x \le 2,\ -1 \le y \le 1.$$

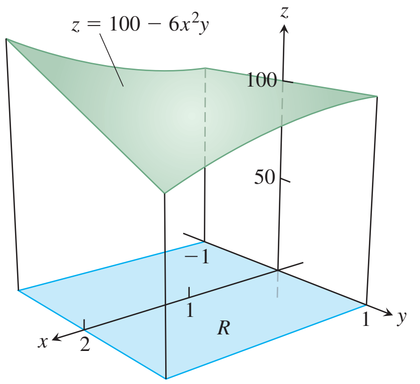


FIGURE 15.6 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 1).

Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\,dA = \int_{-1}^{1}\int_{0}^{2}(100 - 6x^2y)\,dx\,dy = \int_{-1}^{1}\Big[100x - 2x^3y\Big]_{x=0}^{x=2}\,dy$$

$$= \int_{-1}^{1}(200 - 16y)\,dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$

Reversing the order of integration gives the same answer:

$$\int_{0}^{2}\int_{-1}^{1}(100 - 6x^2y)\,dy\,dx = \int_{0}^{2}\Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1}\,dx$$

$$= \int_{0}^{2}\big[(100 - 3x^2) - (-100 - 3x^2)\big]\,dx$$

$$= \int_{0}^{2} 200\,dx = 400. \quad \blacksquare$$

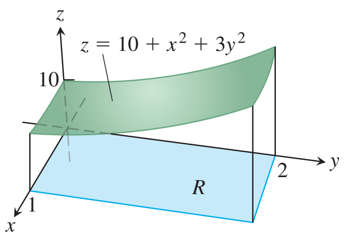


FIGURE 15.7 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 2).

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1$, $0 \le y \le 2$.

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\,dA = \int_{0}^{1}\int_{0}^{2}(10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_{0}^{1}\Big[10y + x^2y + y^3\Big]_{y=0}^{y=2}\,dx$$

$$= \int_{0}^{1}(20 + 2x^2 + 8)\,dx = \Big[20x + \frac{2}{3}x^3 + 8x\Big]_{0}^{1} = \frac{86}{3}. \quad \blacksquare$$

Exercises 15.1

Evaluating Iterated Integrals

In Exercises 1–14, evaluate the iterated integral.

1. $\displaystyle\int_{1}^{2}\int_{0}^{4} 2xy\,dy\,dx$

2. $\displaystyle\int_{0}^{2}\int_{-1}^{1}(x - y)\,dy\,dx$

3. $\displaystyle\int_{-1}^{0}\int_{-1}^{1}(x + y + 1)\,dx\,dy$

4. $\displaystyle\int_{0}^{1}\int_{0}^{1}\left(1 - \frac{x^2 + y^2}{2}\right)dx\,dy$

5. $\displaystyle\int_{0}^{3}\int_{0}^{2}(4 - y^2)\,dy\,dx$

6. $\displaystyle\int_{0}^{3}\int_{-2}^{0}(x^2y - 2xy)\,dy\,dx$

7. $\displaystyle\int_{0}^{1}\int_{0}^{1}\frac{y}{1 + xy}\,dx\,dy$

8. $\displaystyle\int_{1}^{4}\int_{0}^{4}\left(\frac{x}{2} + \sqrt{y}\right)dx\,dy$

9. $\displaystyle\int_{0}^{\ln 2}\int_{1}^{\ln 5} e^{2x+y}\,dy\,dx$

10. $\displaystyle\int_{0}^{1}\int_{1}^{2} xye^x\,dy\,dx$

11. $\displaystyle\int_{-1}^{2}\int_{0}^{\pi/2} y\sin x\,dx\,dy$

12. $\displaystyle\int_{\pi}^{2\pi}\int_{0}^{\pi}(\sin x + \cos y)\,dx\,dy$

13. $\displaystyle\int_{1}^{4}\int_{1}^{e}\frac{\ln x}{xy}\,dx\,dy$

14. $\displaystyle\int_{-1}^{2}\int_{1}^{2} x\ln y\,dy\,dx$

Evaluating Double Integrals over Rectangles

In Exercises 15–22, evaluate the double integral over the given region R.

15. $\displaystyle\iint_R (6y^2 - 2x)\,dA$, $\quad R$: $0 \le x \le 1$, $0 \le y \le 2$

16. $\displaystyle\iint_R \left(\frac{\sqrt{x}}{y^2}\right)dA$, $\quad R$: $0 \le x \le 4$, $1 \le y \le 2$

17. $\displaystyle\iint_R xy\cos y\,dA$, $\quad R$: $-1 \le x \le 1$, $0 \le y \le \pi$

18. $\displaystyle\iint_R y\sin(x + y)\,dA$, $\quad R$: $-\pi \le x \le 0$, $0 \le y \le \pi$

19. $\displaystyle\iint_R e^{x-y}\,dA, \qquad R:\quad 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

20. $\displaystyle\iint_R xye^{xy^2}\,dA, \qquad R:\quad 0 \le x \le 2, \quad 0 \le y \le 1$

21. $\displaystyle\iint_R \frac{xy^3}{x^2+1}\,dA, \qquad R:\quad 0 \le x \le 1, \quad 0 \le y \le 2$

22. $\displaystyle\iint_R \frac{y}{x^2y^2+1}\,dA, \qquad R:\quad 0 \le x \le 1, \quad 0 \le y \le 1$

In Exercises 23 and 24, integrate f over the given region.

23. **Square** $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

24. **Rectangle** $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

25. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.

26. Find the volume of the region bounded above by the elliptical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: 0 \le x \le 2, 0 \le y \le 2$.

27. Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square $R: 0 \le x \le 1$, $0 \le y \le 1$.

28. Find the volume of the region bounded above by the plane $z = y/2$ and below by the rectangle $R: 0 \le x \le 4, 0 \le y \le 2$.

29. Find the volume of the region bounded above by the surface $z = 2 \sin x \cos y$ and below by the rectangle $R: 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

30. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \le x \le 1$, $0 \le y \le 2$.

31. Find a value of the constant k so that $\displaystyle\int_1^2\int_0^3 kx^2y\,dx\,dy = 1.$

32. Evaluate $\displaystyle\int_{-1}^1\int_0^{\pi/2} x \sin\sqrt{y}\,dy\,dx.$

33. Use Fubini's Theorem to evaluate

$$\int_0^2\int_0^1 \frac{x}{1+xy}\,dx\,dy.$$

34. Use Fubini's Theorem to evaluate

$$\int_0^1\int_0^3 xe^{xy}\,dx\,dy.$$

T 35. Use a software application to compute the integrals

a. $\displaystyle\int_0^1\int_0^2 \frac{y-x}{(x+y)^3}\,dx\,dy$

b. $\displaystyle\int_0^2\int_0^1 \frac{y-x}{(x+y)^3}\,dy\,dx$

Explain why your results do not contradict Fubini's Theorem.

36. If $f(x, y)$ is continuous over $R: a \le x \le b, c \le y \le d$ and

$$F(x, y) = \int_a^x\int_c^y f(u, v)\,dv\,du$$

on the interior of R, find the second partial derivatives F_{xy} and F_{yx}.

15.2 Double Integrals over General Regions

In this section we define and evaluate double integrals over bounded regions in the plane which are more general than rectangles. These double integrals are also evaluated as iterated integrals, with the main practical problem being that of determining the limits of integration. Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the limits of integration often involve variables, not just constants.

Double Integrals over Bounded, Nonrectangular Regions

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, such as the one in Figure 15.8, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R. This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R, since its boundary is curved, and some of the small rectangles in the grid lie partly outside R. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.

Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$

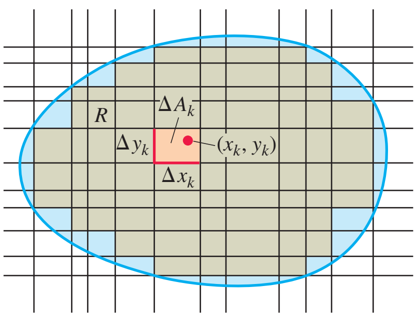


FIGURE 15.8 A rectangular grid partitioning a bounded, nonrectangular region into rectangular cells.

As the norm of the partition forming S_n goes to zero, $\|P\| \to 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity. If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is called the **double integral** of $f(x, y)$ over R:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \iint_R f(x, y)\,dA.$$

The nature of the boundary of R introduces issues not found in integrals over an interval. When R has a curved boundary, the n rectangles of a partition lie inside R but do not cover all of R. In order for a partition to approximate R well, the parts of R covered by small rectangles lying partly outside R must become negligible as the norm of the partition approaches zero. This property of being nearly filled in by a partition of small norm is satisfied by all the regions that we will encounter. There is no problem with boundaries made from polygons, circles, ellipses, and from continuous graphs over an interval, joined end to end. A curve with a "fractal" type of shape would be problematic, but such curves arise rarely in most applications. A careful discussion of which type of regions R can be used for computing double integrals is left to a more advanced text.

Volumes

If $f(x, y)$ is positive and continuous over R, we define the volume of the solid region between R and the surface $z = f(x, y)$ to be $\iint_R f(x, y)\,dA$, as before (Figure 15.9).

If R is a region like the one shown in the xy-plane in Figure 15.10, bounded "above" and "below" by the curves $y = g_2(x)$ and $y = g_1(x)$ and on the sides by the lines $x = a$, $x = b$, we may again calculate the volume by the method of slicing. We first calculate the cross-sectional area

$$A(x) = \int_{y=g_1(x)}^{y=g_2(x)} f(x, y)\,dy$$

and then integrate $A(x)$ from $x = a$ to $x = b$ to get the volume as an iterated integral:

$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx. \tag{1}$$

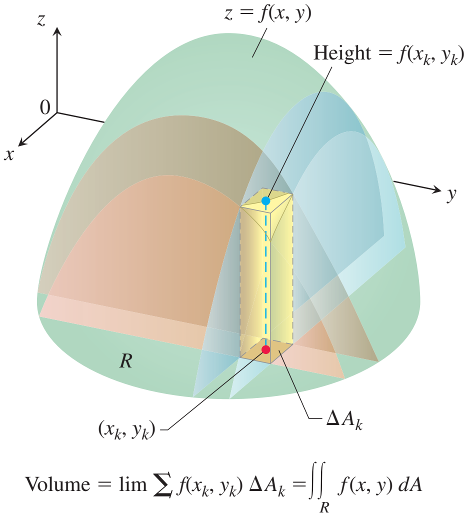


FIGURE 15.9 We define the volumes of solids with curved bases as a limit of approximating rectangular boxes.

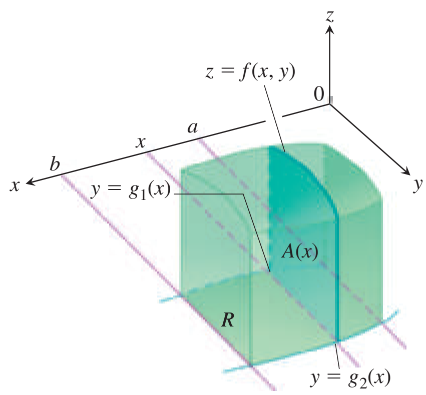


FIGURE 15.10 The area of the vertical slice shown here is $A(x)$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$:

$$\int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\,dy\,dx.$$

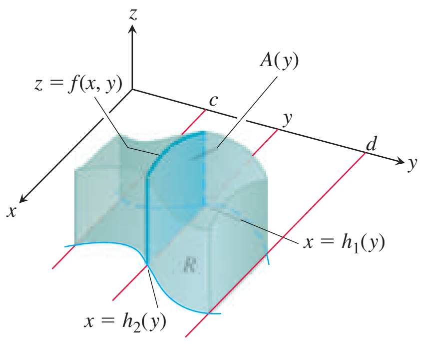


FIGURE 15.11 The volume of the solid shown here is

$$\int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

For a given solid, Theorem 2 says we can calculate the volume as in Figure 15.10, or in the way shown here. Both calculations have the same result.

Similarly, if R is a region like the one shown in Figure 15.11, bounded by the curves $x = h_2(y)$ and $x = h_1(y)$ and the lines $y = c$ and $y = d$, then the volume calculated by slicing is given by the iterated integral

$$\text{Volume} = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy. \tag{2}$$

That the iterated integrals in Equations (1) and (2) both give the volume that we defined to be the double integral of f over R is a consequence of the following stronger form of Fubini's Theorem.

THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2}y^2\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be. ■

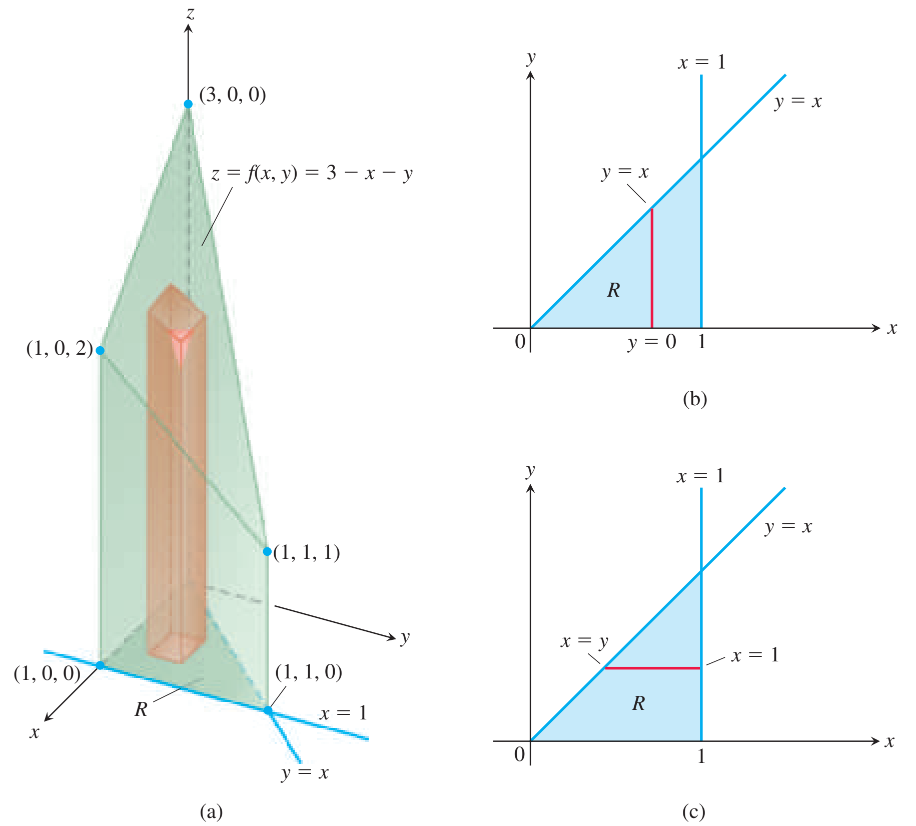


FIGURE 15.12 (a) Prism with a triangular base in the *xy*-plane. The volume of this prism is defined as a double integral over *R*. To evaluate it as an iterated integral, we may integrate first with respect to *y* and then with respect to *x*, or the other way around (Example 1). (b) Integration limits of

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x} f(x, y)\, dy\, dx.$$

If we integrate first with respect to *y*, we integrate along a vertical line through *R* and then integrate from left to right to include all the vertical lines in *R*. (c) Integration limits of

$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} f(x, y)\, dx\, dy.$$

If we integrate first with respect to *x*, we integrate along a horizontal line through *R* and then integrate from bottom to top to include all the horizontal lines in *R*.

Although Fubini's Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other. The next example shows how this can happen.

EXAMPLE 2 Calculate

$$\iint_R \frac{\sin x}{x}\, dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$, and the line $x = 1$.

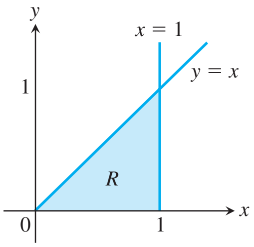


FIGURE 15.13 The region of integration in Example 2.

Solution The region of integration is shown in Figure 15.13. If we integrate first with respect to y and then with respect to x, we find

$$\int_0^1 \left(\int_0^x \frac{\sin x}{x}\, dy \right) dx = \int_0^1 \left(y \frac{\sin x}{x} \Big]_{y=0}^{y=x} \right) dx = \int_0^1 \sin x \, dx$$

$$= -\cos(1) + 1 \approx 0.46.$$

If we reverse the order of integration and attempt to calculate

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy,$$

we run into a problem because $\int ((\sin x)/x)\, dx$ cannot be expressed in terms of elementary functions (there is no simple antiderivative).

There is no general rule for predicting which order of integration will be the good one in circumstances like these. If the order you first choose doesn't work, try the other. Sometimes neither order will work, and then we need to use numerical approximations. ■

Finding Limits of Integration

We now give a procedure for finding limits of integration that applies for many regions in the plane. Regions that are more complicated, and for which this procedure fails, can often be split up into pieces on which the procedure works.

Using Vertical Cross-Sections When faced with evaluating $\iint_R f(x, y)\, dA$, integrating first with respect to y and then with respect to x, do the following three steps:

1. *Sketch.* Sketch the region of integration and label the bounding curves (Figure 15.14a).
2. *Find the y-limits of integration.* Imagine a vertical line L cutting through R in the direction of increasing y. Mark the y-values where L enters and leaves. These are the y-limits of integration and are usually functions of x (instead of constants) (Figure 15.14b).
3. *Find the x-limits of integration.* Choose x-limits that include all the vertical lines through R. The integral shown here (see Figure 15.14c) is

$$\iint_R f(x, y)\, dA = \int_{x=0}^{x=1} \int_{y=1-x}^{y=\sqrt{1-x^2}} f(x, y)\, dy\, dx.$$

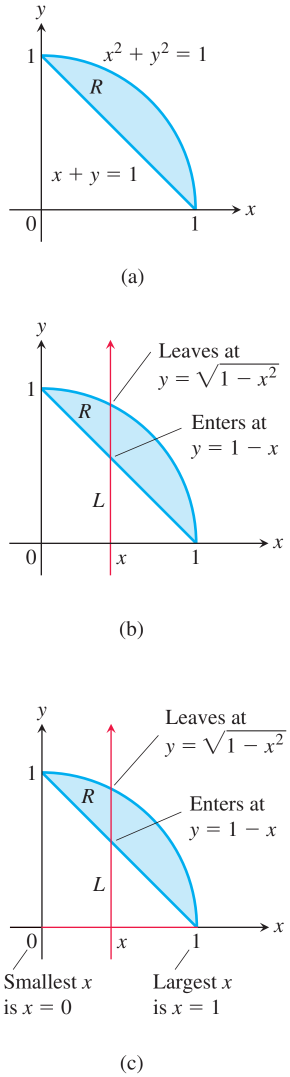


FIGURE 15.14 Finding the limits of integration when integrating first with respect to y and then with respect to x.

Using Horizontal Cross-Sections To evaluate the same double integral as an iterated integral with the order of integration reversed, use horizontal lines instead of vertical lines in Steps 2 and 3 (see Figure 15.15). The integral is

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{1-y}^{\sqrt{1-y^2}} f(x, y)\, dx\, dy.$$

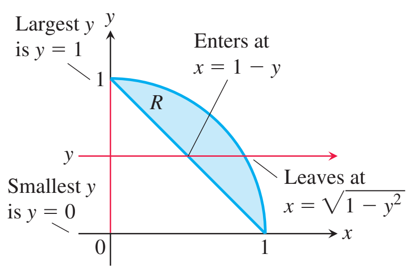


FIGURE 15.15 Finding the limits of integration when integrating first with respect to x and then with respect to y.

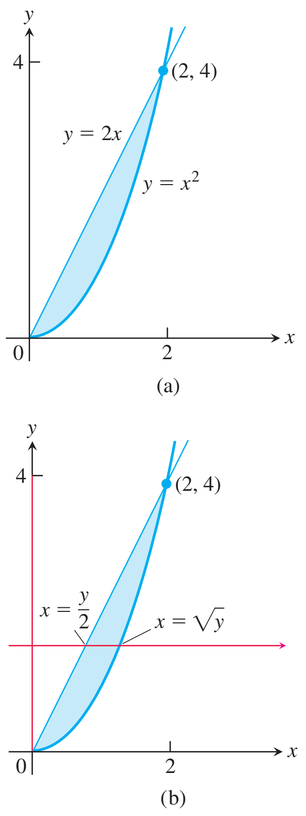


FIGURE 15.16 Region of integration for Example 3.

EXAMPLE 3 Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x + 2)\, dy\, dx$$

and write an equivalent integral with the order of integration reversed.

Solution The region of integration is given by the inequalities $x^2 \leq y \leq 2x$ and $0 \leq x \leq 2$. It is therefore the region bounded by the curves $y = x^2$ and $y = 2x$ between $x = 0$ and $x = 2$ (Figure 15.16a).

To find limits for integrating in the reverse order, we imagine a horizontal line passing from left to right through the region. It enters at $x = y/2$ and leaves at $x = \sqrt{y}$. To include all such lines, we let y run from $y = 0$ to $y = 4$ (Figure 15.16b). The integral is

$$\int_0^4 \int_{y/2}^{\sqrt{y}} (4x + 2)\, dx\, dy.$$

The common value of these integrals is 8. ■

Properties of Double Integrals

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications.

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\displaystyle\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $\displaystyle\iint_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

(b) $\displaystyle\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\displaystyle\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2

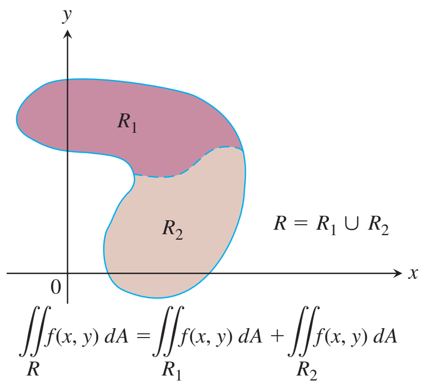


FIGURE 15.17 The Additivity Property for rectangular regions holds for regions bounded by smooth curves.

Property 4 assumes that the region of integration R is decomposed into nonoverlapping regions R_1 and R_2 with boundaries consisting of a finite number of line segments or smooth curves. Figure 15.17 illustrates an example of this property.

The idea behind these properties is that integrals behave like sums. If the function $f(x, y)$ is replaced by its constant multiple $cf(x, y)$, then a Riemann sum for f

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

is replaced by a Riemann sum for cf

$$\sum_{k=1}^{n} cf(x_k, y_k)\,\Delta A_k = c\sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = cS_n.$$

Taking limits as $n \to \infty$ shows that $c\lim_{n\to\infty} S_n = c\iint_R f\,dA$ and $\lim_{n\to\infty} cS_n = \iint_R cf\,dA$ are equal. It follows that the Constant Multiple Property carries over from sums to double integrals.

The other properties are also easy to verify for Riemann sums, and carry over to double integrals for the same reason. While this discussion gives the idea, an actual proof that these properties hold requires a more careful analysis of how Riemann sums converge.

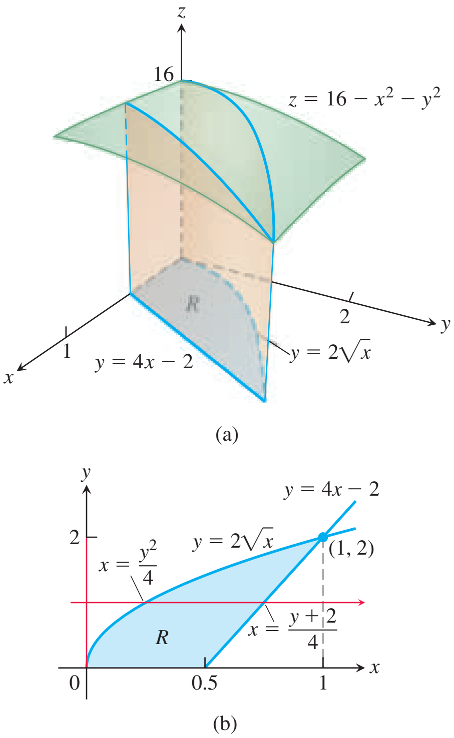


FIGURE 15.18 (a) The solid "wedgelike" region whose volume is found in Example 4. (b) The region of integration R showing the order $dx\,dy$.

EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\,dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\,dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

$$\iint_R (16 - x^2 - y^2)\,dA$$

$$= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\,dx\,dy$$

$$= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx$$

$$= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3\cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3\cdot 64} + \frac{y^4}{4}\right] dy$$

$$= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4.$$

Our development of the double integral has focused on its representation of the volume of the solid region between R and the surface $z = f(x, y)$ of a positive continuous function. Just as we saw with signed area in the case of single integrals, when $f(x_k, y_k)$ is negative, then the product $f(x_k, y_k)\Delta A_k$ is the negative of the volume of the rectangular box shown in Figure 15.9 that was used to form the approximating Riemann sum. So for an arbitrary continuous function f defined over R, the limit of any Riemann sum represents the *signed* volume (not the total volume) of the solid region between R and the surface. The double integral has other interpretations as well, and in the next section we will see how it is used to calculate the area of a general region in the plane.

Exercises 15.2

Sketching Regions of Integration

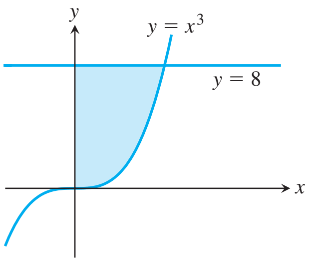
In Exercises 1–8, sketch the described regions of integration.

1. $0 \le x \le 3, \quad 0 \le y \le 2x$
2. $-1 \le x \le 2, \quad x - 1 \le y \le x^2$
3. $-2 \le y \le 2, \quad y^2 \le x \le 4$
4. $0 \le y \le 1, \quad y \le x \le 2y$
5. $0 \le x \le 1, \quad e^x \le y \le e$
6. $1 \le x \le e^2, \quad 0 \le y \le \ln x$
7. $0 \le y \le 1, \quad 0 \le x \le \sin^{-1} y$
8. $0 \le y \le 8, \quad \frac{1}{4}y \le x \le y^{1/3}$

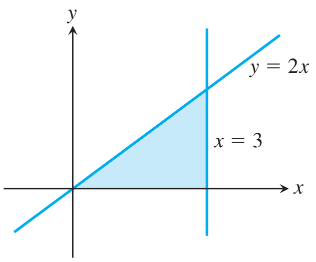
Finding Limits of Integration

In Exercises 9–18, write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

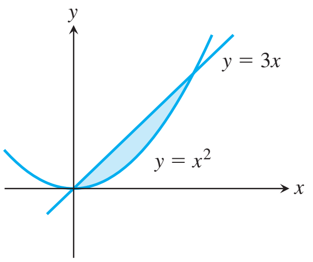
9.



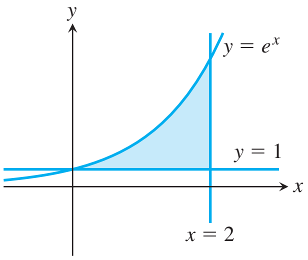
10.



11.



12.



13. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$
14. Bounded by $y = \tan x$, $x = 0$, and $y = 1$
15. Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$
16. Bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$
17. Bounded by $y = 3 - 2x$, $y = x$, and $x = 0$
18. Bounded by $y = x^2$ and $y = x + 2$

Finding Regions of Integration and Double Integrals

In Exercises 19–24, sketch the region of integration and evaluate the integral.

19. $\int_0^{\pi}\int_0^{x} x \sin y \, dy \, dx$

20. $\int_0^{\pi}\int_0^{\sin x} y \, dy \, dx$

21. $\int_1^{\ln 8}\int_0^{\ln y} e^{x+y} \, dx \, dy$

22. $\int_1^{2}\int_y^{y^2} dx \, dy$

23. $\int_0^{1}\int_0^{y^2} 3y^3 e^{xy} \, dx \, dy$

24. $\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy \, dx$

In Exercises 25–28, integrate f over the given region.

25. **Quadrilateral** $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, and $x = 2$
26. **Triangle** $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$
27. **Triangle** $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$
28. **Curved region** $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 29–32 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

29. $\int_{-2}^{0}\int_{v}^{-v} 2 \, dp \, dv$ (the pv-plane)

30. $\int_0^{1}\int_0^{\sqrt{1-s^2}} 8t \, dt \, ds$ (the st-plane)

31. $\int_{-\pi/3}^{\pi/3}\int_0^{\sec t} 3 \cos t \, du \, dt$ (the tu-plane)

32. $\int_0^{3/2}\int_1^{4-2u} \frac{4 - 2u}{v^2} \, dv \, du$ (the uv-plane)

Reversing the Order of Integration

In Exercises 33–46, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

33. $\int_0^{1}\int_2^{4-2x} dy \, dx$

34. $\int_0^{2}\int_{y-2}^{0} dx \, dy$

35. $\int_0^{1}\int_y^{\sqrt{y}} dx \, dy$

36. $\int_0^{1}\int_{1-x}^{1-x^2} dy \, dx$

37. $\int_0^{1}\int_1^{e^x} dy \, dx$

38. $\int_0^{\ln 2}\int_{e^y}^{2} dx \, dy$

39. $\int_0^{3/2}\int_0^{9-4x^2} 16x \, dy \, dx$

40. $\int_0^{2}\int_0^{4-y^2} y \, dx \, dy$

41. $\int_0^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx \, dy$

42. $\int_0^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x \, dy \, dx$

43. $\int_1^{e}\int_0^{\ln x} xy \, dy \, dx$

44. $\int_0^{\pi/6}\int_{\sin x}^{1/2} xy^2 \, dy \, dx$

45. $\int_0^{3}\int_1^{e^y} (x + y) \, dx \, dy$

46. $\int_0^{\sqrt{3}}\int_0^{\tan^{-1} y} \sqrt{xy} \, dx \, dy$

In Exercises 47–56, sketch the region of integration, reverse the order of integration, and evaluate the integral.

47. $\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$

48. $\int_0^2\int_x^2 2y^2 \sin xy\,dy\,dx$

49. $\int_0^1\int_y^1 x^2 e^{xy}\,dx\,dy$

50. $\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$

51. $\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2}\,dx\,dy$

52. $\int_0^3\int_{\sqrt{x/3}}^1 e^{y^3}\,dy\,dx$

53. $\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos\left(16\pi x^5\right)\,dx\,dy$

54. $\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\,dx}{y^4+1}$

55. Square region $\iint_R (y - 2x^2)\,dA$ where R is the region bounded by the square $|x| + |y| = 1$

56. Triangular region $\iint_R xy\,dA$ where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$

Volume Beneath a Surface $z = f(x, y)$

57. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

58. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

59. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

60. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

61. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

62. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

63. Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

64. Find the volume of the solid cut from the square column $|x| + |y| \le 1$ by the planes $z = 0$ and $3x + z = 3$.

65. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$ and $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$.

66. Find the volume of the solid bounded on the front and back by the planes $x = \pm\pi/3$, on the sides by the cylinders $y = \pm\sec x$, above by the cylinder $z = 1 + y^2$, and below by the xy-plane.

In Exercises 67 and 68, sketch the region of integration and the solid whose volume is given by the double integral.

67. $\int_0^3\int_0^{2-2x/3} \left(1 - \frac{1}{3}x - \frac{1}{2}y\right)dy\,dx$

68. $\int_0^4\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \sqrt{25 - x^2 - y^2}\,dx\,dy$

Integrals over Unbounded Regions

Improper double integrals can often be computed similarly to improper integrals of one variable. The first iteration of the following improper integrals is conducted just as if they were proper integrals. One then evaluates an improper integral of a single variable by taking appropriate limits, as in Section 8.8. Evaluate the improper integrals in Exercises 69–72 as iterated integrals.

69. $\int_1^{\infty}\int_{e^{-x}}^1 \frac{1}{x^3 y}\,dy\,dx$

70. $\int_{-1}^1\int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y + 1)\,dy\,dx$

71. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(y^2+1)}\,dx\,dy$

72. $\int_0^{\infty}\int_0^{\infty} xe^{-(x+2y)}\,dx\,dy$

Approximating Integrals with Finite Sums

In Exercises 73 and 74, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle, use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x, y)\,dA \approx \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

73. $f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

74. $f(x, y) = x + 2y$ over the region R inside the circle $(x - 2)^2 + (y - 3)^2 = 1$ using the partition $x = 1, 3/2, 2, 5/2, 3$ and $y = 2, 5/2, 3, 7/2, 4$ with (x_k, y_k) the center (centroid) in the kth subrectangle (provided the subrectangle lies within R)

Theory and Examples

75. Circular sector Integrate $f(x, y) = \sqrt{4 - x^2}$ over the smaller sector cut from the disk $x^2 + y^2 \le 4$ by the rays $\theta = \pi/6$ and $\theta = \pi/2$.

76. Unbounded region Integrate $f(x, y) = 1/\left[(x^2 - x)(y - 1)^{2/3}\right]$ over the infinite rectangle $2 \le x < \infty$, $0 \le y \le 2$.

77. Noncircular cylinder A solid right (noncircular) cylinder has its base R in the xy-plane and is bounded above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

$$V = \int_0^1\int_0^y (x^2 + y^2)\,dx\,dy + \int_1^2\int_0^{2-y} (x^2 + y^2)\,dx\,dy.$$

Sketch the base region R and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

78. Converting to a double integral Evaluate the integral

$$\int_0^2 (\tan^{-1} \pi x - \tan^{-1} x)\, dx.$$

(*Hint:* Write the integrand as an integral.)

79. Maximizing a double integral What region R in the xy-plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\, dA?$$

Give reasons for your answer.

80. Minimizing a double integral What region R in the xy-plane minimizes the value of

$$\iint_R (x^2 + y^2 - 9)\, dA?$$

Give reasons for your answer.

81. Is it possible to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy-plane and get different answers depending on the order of integration? Give reasons for your answer.

82. How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy-plane enclosed by the triangle with vertices (0, 1), (2, 0), and (1, 2)? Give reasons for your answer.

83. Unbounded region Prove that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\, dx\, dy = \lim_{b\to\infty} \int_{-b}^{b}\int_{-b}^{b} e^{-x^2-y^2}\, dx\, dy$$
$$= 4\left(\int_0^{\infty} e^{-x^2}\, dx\right)^2.$$

84. Improper double integral Evaluate the improper integral

$$\int_0^1\int_0^3 \frac{x^2}{(y-1)^{2/3}}\, dy\, dx.$$

COMPUTER EXPLORATIONS

Use a CAS double-integral evaluator to estimate the values of the integrals in Exercises 85–88.

85. $\displaystyle\int_1^3\int_1^x \frac{1}{xy}\, dy\, dx$

86. $\displaystyle\int_0^1\int_0^1 e^{-(x^2+y^2)}\, dy\, dx$

87. $\displaystyle\int_0^1\int_0^1 \tan^{-1} xy\, dy\, dx$

88. $\displaystyle\int_{-1}^1\int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\, dy\, dx$

Use a CAS double-integral evaluator to find the integrals in Exercises 89–94. Then reverse the order of integration and evaluate, again with a CAS.

89. $\displaystyle\int_0^1\int_{2y}^4 e^{x^2}\, dx\, dy$

90. $\displaystyle\int_0^3\int_{x^2}^9 x\cos(y^2)\, dy\, dx$

91. $\displaystyle\int_0^2\int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\, dx\, dy$

92. $\displaystyle\int_0^2\int_0^{4-y^2} e^{xy}\, dx\, dy$

93. $\displaystyle\int_1^2\int_0^{x^2} \frac{1}{x+y}\, dy\, dx$

94. $\displaystyle\int_1^2\int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$

15.3 Area by Double Integration

In this section we show how to use double integrals to calculate the areas of bounded regions in the plane, and to find the average value of a function of two variables.

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \sum_{k=1}^{n} \Delta A_k. \tag{1}$$

This is simply the sum of the areas of the small rectangles in the partition of R, and approximates what we would like to call the area of R. As the norm of a partition of R approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of R becomes increasingly complete (Figure 15.8). We define the area of R to be the limit

$$\lim_{||P||\to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA. \tag{2}$$

> **DEFINITION** The **area** of a closed, bounded plane region R is
> $$A = \iint_R dA.$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply. To evaluate the integral in the definition of area, we integrate the constant function $f(x, y) = 1$ over R.

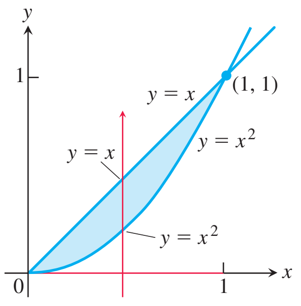


FIGURE 15.19 The region in Example 1.

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution We sketch the region (Figure 15.19), noting where the two curves intersect at the origin and $(1, 1)$, and calculate the area as

$$\begin{aligned} A &= \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \Big[y \Big]_{x^2}^{x} dx \\ &= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}. \end{aligned}$$

Notice that the single-variable integral $\int_0^1 (x - x^2)\, dx$, obtained from evaluating the inside iterated integral, is the integral for the area between these two curves using the method of Section 5.6. ■

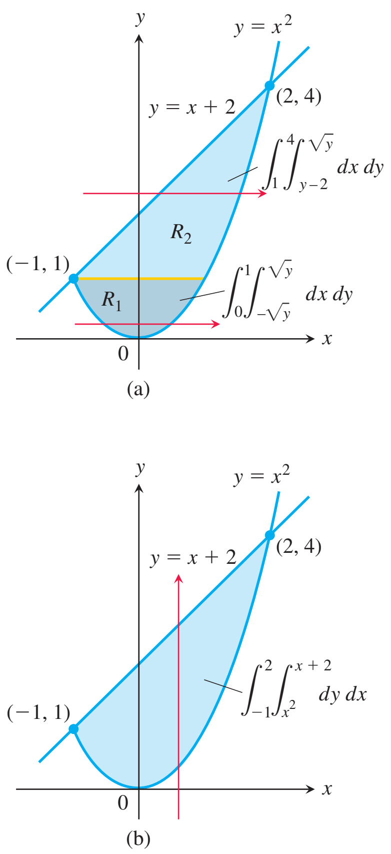


FIGURE 15.20 Calculating this area takes (a) two double integrals if the first integration is with respect to x, but (b) only one if the first integration is with respect to y (Example 2).

EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Figure 15.20a, we may calculate the area as

$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$

On the other hand, reversing the order of integration (Figure 15.20b) gives

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$

This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}. \quad ■$$

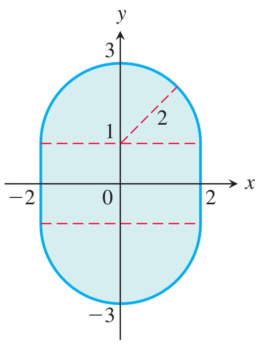


FIGURE 15.21 The playing field described by the region R in Example 3.

EXAMPLE 3 Find the area of the playing field described by $R: -2 \le x \le 2,\ -1 - \sqrt{4 - x^2} \le y \le 1 + \sqrt{4 - x^2}$, using

(a) Fubini's Theorem **(b)** Simple geometry.

Solution The region R is shown in Figure 15.21.

(a) From the symmetries observed in the figure, we see that the area of R is 4 times its area in the first quadrant. Using Fubini's Theorem, we have

$$\begin{aligned} A &= \iint_R dA = 4\int_0^2 \int_0^{1+\sqrt{4-x^2}} dy\, dx \\ &= 4\int_0^2 \left(1 + \sqrt{4 - x^2}\right) dx \\ &= 4\left[x + \frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 \qquad \text{Integral Table Formula 45} \\ &= 4\left(2 + 0 + 2\cdot\frac{\pi}{2} - 0\right) = 8 + 4\pi. \end{aligned}$$

(b) The region R consists of a rectangle mounted on two sides by half disks of radius 2. The area can be computed by summing the area of the 4×2 rectangle and the area of a circle of radius 2, so

$$A = 8 + \pi 2^2 = 8 + 4\pi.$$

Average Value

The average value of an integrable function of one variable on a closed interval is the integral of the function over the interval divided by the length of the interval. For an integrable function of two variables defined on a bounded region in the plane, the average value is the integral over the region divided by the area of the region. This can be visualized by thinking of the function as giving the height at one instant of some water sloshing around in a tank whose vertical walls lie over the boundary of the region. The average height of the water in the tank can be found by letting the water settle down to a constant height. The height is then equal to the volume of water in the tank divided by the area of R. We are led to define the average value of an integrable function f over a region R as follows:

$$\textbf{Average value of } f \text{ over } R = \frac{1}{\text{area of } R}\iint_R f\, dA. \tag{3}$$

If f is the temperature of a thin plate covering R, then the double integral of f over R divided by the area of R is the plate's average temperature. If $f(x, y)$ is the distance from the point (x, y) to a fixed point P, then the average value of f over R is the average distance of points in R from P.

EXAMPLE 4 Find the average value of $f(x, y) = x\cos xy$ over the rectangle $R: 0 \le x \le \pi,\ 0 \le y \le 1$.

Solution The value of the integral of f over R is

$$\int_0^\pi \int_0^1 x \cos xy\, dy\, dx = \int_0^\pi \left[\sin xy\right]_{y=0}^{y=1} dx \qquad \int x\cos xy\, dy = \sin xy + C$$

$$= \int_0^\pi (\sin x - 0)\, dx = -\cos x\Big]_0^\pi = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$.

Exercises 15.3

Area by Double Integrals

In Exercises 1–12, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line $x + y = 2$
2. The lines $x = 0$, $y = 2x$, and $y = 4$
3. The parabola $x = -y^2$ and the line $y = x + 2$
4. The parabola $x = y - y^2$ and the line $y = -x$
5. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
6. The curves $y = \ln x$ and $y = 2\ln x$ and the line $x = e$, in the first quadrant
7. The parabolas $x = y^2$ and $x = 2y - y^2$
8. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$
9. The lines $y = x$, $y = x/3$, and $y = 2$
10. The lines $y = 1 - x$ and $y = 2$ and the curve $y = e^x$
11. The lines $y = 2x$, $y = x/2$, and $y = 3 - x$
12. The lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Identifying the Region of Integration

The integrals and sums of integrals in Exercises 13–18 give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

13. $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dx\, dy$

14. $\displaystyle\int_0^3 \int_{-x}^{x(2-x)} dy\, dx$

15. $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$

16. $\displaystyle\int_{-1}^2 \int_{y^2}^{y+2} dx\, dy$

17. $\displaystyle\int_{-1}^0 \int_{-2x}^{1-x} dy\, dx + \int_0^2 \int_{-x/2}^{1-x} dy\, dx$

18. $\displaystyle\int_0^2 \int_{x^2-4}^{0} dy\, dx + \int_0^4 \int_0^{\sqrt{x}} dy\, dx$

Finding Average Values

19. Find the average value of $f(x, y) = \sin(x + y)$ over
    a. the rectangle $0 \le x \le \pi$, $0 \le y \le \pi$.
    b. the rectangle $0 \le x \le \pi$, $0 \le y \le \pi/2$.
20. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \le x \le 1$, $0 \le y \le 1$, or the average value of f over the quarter circle $x^2 + y^2 \le 1$ in the first quadrant? Calculate them to find out.
21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.
22. Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \le x \le 2\ln 2$, $\ln 2 \le y \le 2\ln 2$.

Theory and Examples

23. **Geometric area** Find the area of the region
$$R: 0 \le x \le 2,\ 2 - x \le y \le \sqrt{4 - x^2},$$
using **(a)** Fubini's Theorem, **(b)** simple geometry.
24. **Geometric area** Find the area of the circular washer with outer radius 2 and inner radius 1, using **(a)** Fubini's Theorem, **(b)** simple geometry.
25. **Bacterium population** If $f(x, y) = (10{,}000e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacterium on the xy-plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \le x \le 5$ and $-2 \le y \le 0$.
26. **Regional population** If $f(x, y) = 100\,(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.
27. **Average temperature in Texas** According to the *Texas Almanac*, Texas has 254 counties and a National Weather Service station in each county. Assume that at time t_0, each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation of the average temperature in Texas at time t_0. Your answer should involve information that you would expect to be readily available in the *Texas Almanac*.
28. If $y = f(x)$ is a nonnegative continuous function over the closed interval $a \le x \le b$, show that the double integral definition of area for the closed plane region bounded by the graph of f, the vertical lines $x = a$ and $x = b$, and the x-axis agrees with the definition for area beneath the curve in Section 5.3.
29. Suppose $f(x, y)$ is continuous over a region R in the plane and that the area $A(R)$ of the region is defined. If there are constants m and M such that $m \le f(x, y) \le M$ for all $(x, y) \in R$, prove that
$$mA(R) \le \iint_R f(x, y)\, dA \le MA(R).$$
30. Suppose $f(x, y)$ is continuous and nonnegative over a region R in the plane with a defined area $A(R)$. If $\iint_R f(x, y)\, dA = 0$, prove that $f(x, y) = 0$ at every point $(x, y) \in R$.

15.4 Double Integrals in Polar Form

Double integrals are sometimes easier to evaluate if we change to polar coordinates. This section shows how to accomplish the change and how to evaluate double integrals over regions whose boundaries are given by polar equations.

Integrals in Polar Coordinates

When we defined the double integral of a function over a region R in the xy-plane, we began by cutting R into rectangles whose sides were parallel to the coordinate axes. These were the natural shapes to use because their sides have either constant x-values or constant y-values. In polar coordinates, the natural shape is a "polar rectangle" whose sides have constant r- and θ-values. To avoid ambiguities when describing the region of integration with polar coordinates, we use polar coordinate points (r, θ) where $r \geq 0$.

Suppose that a function $f(r, \theta)$ is defined over a region R that is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and by the continuous curves $r = g_1(\theta)$ and $r = g_2(\theta)$. Suppose also that $0 \leq g_1(\theta) \leq g_2(\theta) \leq a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \leq r \leq a$ and $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$. See Figure 15.22.

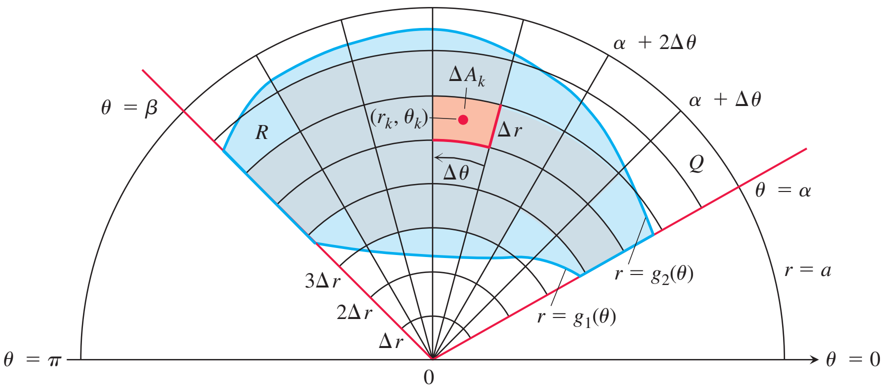


FIGURE 15.22 The region R: $g_1(\theta) \leq r \leq g_2(\theta)$, $\alpha \leq \theta \leq \beta$, is contained in the fan-shaped region Q: $0 \leq r \leq a$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$. The partition of Q by circular arcs and rays induces a partition of R.

We cover Q by a grid of circular arcs and rays. The arcs are cut from circles centered at the origin, with radii $\Delta r, 2\Delta r, \ldots, m\Delta r$, where $\Delta r = a/m$. The rays are given by

$$\theta = \alpha, \qquad \theta = \alpha + \Delta\theta, \qquad \theta = \alpha + 2\Delta\theta, \qquad \ldots, \qquad \theta = \alpha + m'\Delta\theta = \beta,$$

where $\Delta\theta = (\beta - \alpha)/m'$. The arcs and rays partition Q into small patches called "polar rectangles."

We number the polar rectangles that lie inside R (the order does not matter), calling their areas $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$. We let (r_k, θ_k) be any point in the polar rectangle whose area is ΔA_k. We then form the sum

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, \Delta A_k.$$

If f is continuous throughout R, this sum will approach a limit as we refine the grid to make Δr and $\Delta\theta$ go to zero. The limit is called the double integral of f over R. In symbols,

$$\lim_{n \to \infty} S_n = \iint_R f(r, \theta)\, dA.$$

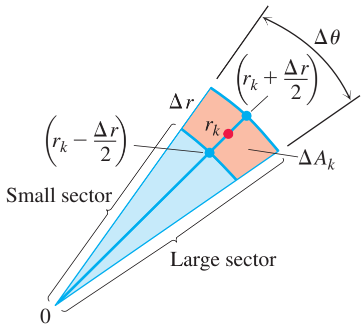


FIGURE 15.23 The observation that

$$\Delta A_k = \begin{pmatrix}\text{area of}\\ \text{large sector}\end{pmatrix} - \begin{pmatrix}\text{area of}\\ \text{small sector}\end{pmatrix}$$

leads to the formula $\Delta A_k = r_k\,\Delta r\,\Delta\theta$.

To evaluate this limit, we first have to write the sum S_n in a way that expresses ΔA_k in terms of Δr and $\Delta\theta$. For convenience we choose r_k to be the average of the radii of the inner and outer arcs bounding the kth polar rectangle ΔA_k. The radius of the inner arc bounding ΔA_k is then $r_k - (\Delta r/2)$ (Figure 15.23). The radius of the outer arc is $r_k + (\Delta r/2)$.

The area of a wedge-shaped sector of a circle having radius r and angle θ is

$$A = \frac{1}{2}\theta \cdot r^2,$$

as can be seen by multiplying πr^2, the area of the circle, by $\theta/2\pi$, the fraction of the circle's area contained in the wedge. So the areas of the circular sectors subtended by these arcs at the origin are

$$\text{Inner radius:}\quad \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta$$

$$\text{Outer radius:}\quad \frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta.$$

Therefore,

$$\begin{aligned}\Delta A_k &= \text{area of large sector} - \text{area of small sector}\\ &= \frac{\Delta\theta}{2}\left[\left(r_k + \frac{\Delta r}{2}\right)^2 - \left(r_k - \frac{\Delta r}{2}\right)^2\right] = \frac{\Delta\theta}{2}(2r_k\,\Delta r) = r_k\,\Delta r\,\Delta\theta.\end{aligned}$$

Combining this result with the sum defining S_n gives

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\, r_k\,\Delta r\,\Delta\theta.$$

As $n \to \infty$ and the values of Δr and $\Delta\theta$ approach zero, these sums converge to the double integral

$$\lim_{n\to\infty} S_n = \iint_R f(r, \theta)\, r\, dr\, d\theta.$$

A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, r\, dr\, d\theta.$$

Finding Limits of Integration

The procedure for finding limits of integration in rectangular coordinates also works for polar coordinates. To evaluate $\iint_R f(r, \theta)\, dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

1. *Sketch*. Sketch the region and label the bounding curves (Figure 15.24a).
2. *Find the r-limits of integration*. Imagine a ray L from the origin cutting through R in the direction of increasing r. Mark the r-values where L enters and leaves R. These are the r-limits of integration. They usually depend on the angle θ that L makes with the positive x-axis (Figure 15.24b).
3. *Find the θ-limits of integration*. Find the smallest and largest θ-values that bound R. These are the θ-limits of integration (Figure 15.24c). The polar iterated integral is

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r, \theta)\, r\, dr\, d\theta.$$

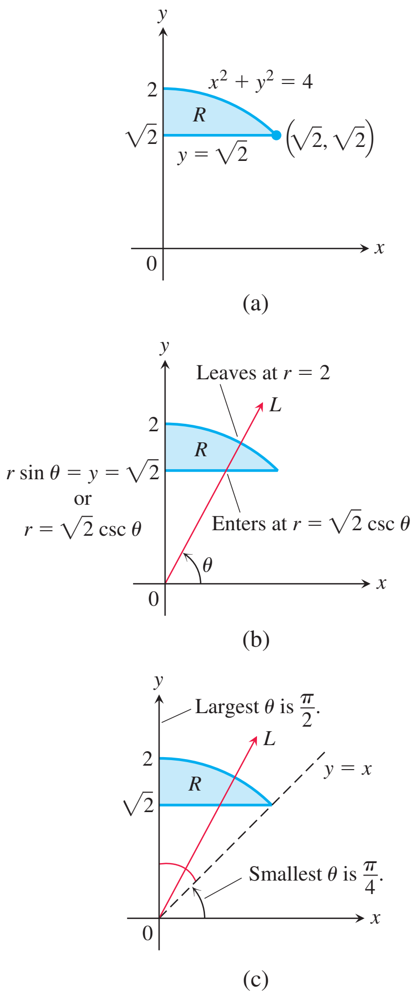


FIGURE 15.24 Finding the limits of integration in polar coordinates.

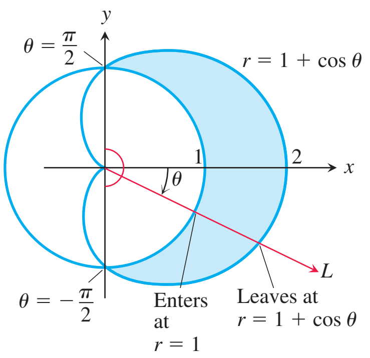


FIGURE 15.25 Finding the limits of integration in polar coordinates for the region in Example 1.

EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution

1. We first sketch the region and label the bounding curves (Figure 15.25).
2. Next we find the *r-limits of integration*. A typical ray from the origin enters R where $r = 1$ and leaves where $r = 1 + \cos\theta$.
3. Finally we find the *θ-limits of integration*. The rays from the origin that intersect R run from $\theta = -\pi/2$ to $\theta = \pi/2$. The integral is

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$

If $f(r, \theta)$ is the constant function whose value is 1, then the integral of f over R is the area of R.

Area Differential in Polar Coordinates

$$dA = r\, dr\, d\theta$$

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\, dr\, d\theta.$$

This formula for area is consistent with all earlier formulas, although we do not prove this fact.

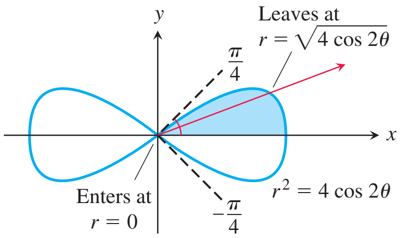


FIGURE 15.26 To integrate over the shaded region, we run r from 0 to $\sqrt{4\cos 2\theta}$ and θ from 0 to $\pi/4$ (Example 2).

EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Solution We graph the lemniscate to determine the limits of integration (Figure 15.26) and see from the symmetry of the region that the total area is 4 times the first-quadrant portion.

$$A = 4\int_0^{\pi/4} \int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta = 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

Changing Cartesian Integrals into Polar Integrals

The procedure for changing a Cartesian integral $\iint_R f(x, y)\, dx\, dy$ into a polar integral has two steps. First substitute $x = r\cos\theta$ and $y = r\sin\theta$, and replace $dx\, dy$ by $r\, dr\, d\theta$ in the Cartesian integral. Then supply polar limits of integration for the boundary of R. The Cartesian integral then becomes

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta,$$

where G denotes the same region of integration now described in polar coordinates. This is like the substitution method in Chapter 5 except that there are now two variables to substitute for instead of one. Notice that the area differential $dx\, dy$ is not replaced by $dr\, d\theta$ but by $r\, dr\, d\theta$. A more general discussion of changes of variables (substitutions) in multiple integrals is given in Section 15.8.

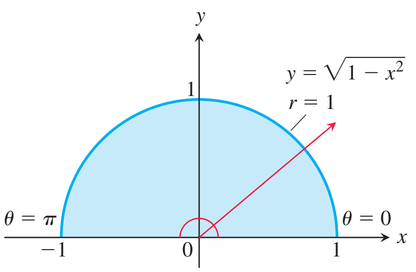


FIGURE 15.27 The semicircular region in Example 3 is the region

$$0 \le r \le 1, \qquad 0 \le \theta \le \pi.$$

EXAMPLE 3 Evaluate

$$\iint_R e^{x^2+y^2}\, dy\, dx,$$

where R is the semicircular region bounded by the x-axis and the curve $y = \sqrt{1 - x^2}$ (Figure 15.27).

Solution In Cartesian coordinates, the integral in question is a nonelementary integral and there is no direct way to integrate $e^{x^2+y^2}$ with respect to either x or y. Yet this integral and others like it are important in mathematics—in statistics, for example—and we need to find a way to evaluate it. Polar coordinates save the day. Substituting $x = r\cos\theta$, $y = r\sin\theta$ and replacing $dy\, dx$ by $r\, dr\, d\theta$ enables us to evaluate the integral as

$$\iint_R e^{x^2+y^2}\, dy\, dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\, dr\, d\theta = \int_0^{\pi}\left[\frac{1}{2}e^{r^2}\right]_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(e - 1)\, d\theta = \frac{\pi}{2}(e - 1).$$

The r in the $r\, dr\, d\theta$ was just what we needed to integrate e^{r^2}. Without it, we would have been unable to find an antiderivative for the first (innermost) iterated integral. ■

EXAMPLE 4 Evaluate the integral

$$\int_0^1\int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

Solution Integration with respect to y gives

$$\int_0^1 \left(x^2\sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3}\right) dx,$$

an integral difficult to evaluate without tables.

Things go better if we change the original integral to polar coordinates. The region of integration in Cartesian coordinates is given by the inequalities $0 \le y \le \sqrt{1 - x^2}$ and $0 \le x \le 1$, which correspond to the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant. (See Figure 15.27, first quadrant.) Substituting the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $0 \le \theta \le \pi/2$, and $0 \le r \le 1$, and replacing $dx\, dy$ by $r\, dr\, d\theta$ in the double integral, we get

$$\int_0^1\int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2}\int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2}\frac{1}{4}\, d\theta = \frac{\pi}{8}.$$

Why is the polar coordinate transformation so effective here? One reason is that $x^2 + y^2$ simplifies to r^2. Another is that the limits of integration become constants. ■

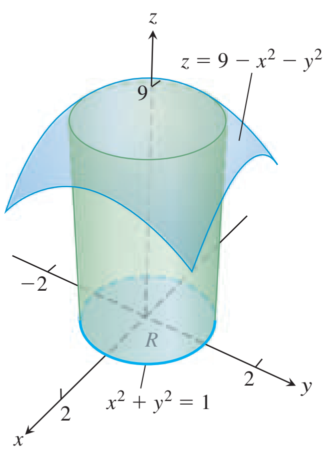


FIGURE 15.28 The solid region in Example 5.

EXAMPLE 5 Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the xy-plane.

Solution The region of integration R is the unit circle $x^2 + y^2 = 1$, which is described in polar coordinates by $r = 1$, $0 \le \theta \le 2\pi$. The solid region is shown in Figure 15.28. The volume is given by the double integral

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 (9r - r^3)\, dr\, d\theta$$
$$= \int_0^{2\pi}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$
$$= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}. \quad ■$$

EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

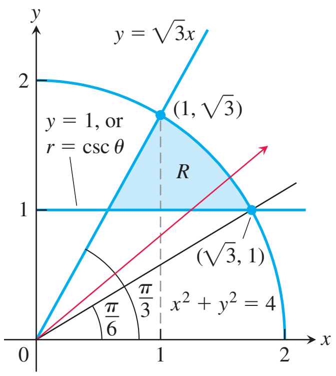


FIGURE 15.29 The region R in Example 6.

Solution A sketch of the region R is shown in Figure 15.29. First we note that the line $y = \sqrt{3}x$ has slope $\sqrt{3} = \tan\theta$, so $\theta = \pi/3$. Next we observe that the line $y = 1$ intersects the circle $x^2 + y^2 = 4$ when $x^2 + 1 = 4$, or $x = \sqrt{3}$. Moreover, the radial line from the origin through the point $(\sqrt{3}, 1)$ has slope $1/\sqrt{3} = \tan\theta$, giving its angle of inclination as $\theta = \pi/6$. This information is shown in Figure 15.29.

Now, for the region R, as θ varies from $\pi/6$ to $\pi/3$, the polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$. Substituting $r\sin\theta$ for y in the equation for the horizontal line, we have $r\sin\theta = 1$, or $r = \csc\theta$, which is the polar equation of the line. The polar equation for the circle is $r = 2$. So in polar coordinates, for $\pi/6 \le \theta \le \pi/3$, r varies from $r = \csc\theta$ to $r = 2$. It follows that the iterated integral for the area then gives

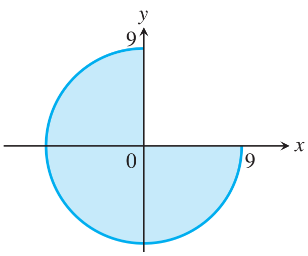
$$\iint_R dA = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\, dr\, d\theta$$
$$= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta$$
$$= \int_{\pi/6}^{\pi/3}\frac{1}{2}\left[4 - \csc^2\theta\right] d\theta$$
$$= \frac{1}{2}\Big[4\theta + \cot\theta\Big]_{\pi/6}^{\pi/3}$$
$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3}\right) = \frac{\pi - \sqrt{3}}{3}. \quad ■$$

Exercises 15.4

Regions in Polar Coordinates

In Exercises 1–8, describe the given region in polar coordinates.

1.



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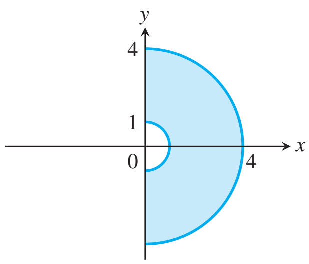

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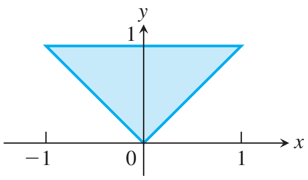
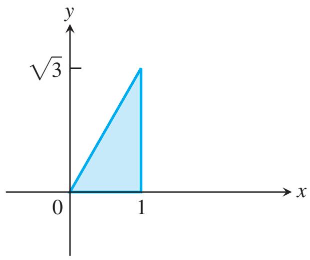


4.



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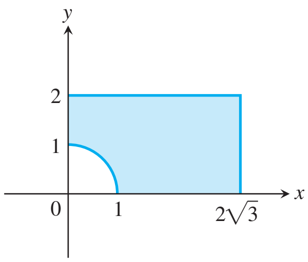

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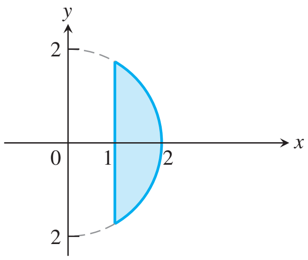


7. The region enclosed by the circle $x^2 + y^2 = 2x$

8. The region enclosed by the semicircle $x^2 + y^2 = 2y, y \geq 0$

Evaluating Polar Integrals

In Exercises 9–22, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

9. $\displaystyle\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\,dx$ **10.** $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2 + y^2)\,dx\,dy$

11. $\displaystyle\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} (x^2 + y^2)\,dx\,dy$

12. $\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\,dx$ **13.** $\displaystyle\int_{0}^{6}\int_{0}^{y} x\,dx\,dy$

14. $\displaystyle\int_{0}^{2}\int_{0}^{x} y\,dy\,dx$ **15.** $\displaystyle\int_{1}^{\sqrt{3}}\int_{1}^{x} dy\,dx$

16. $\displaystyle\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dx\,dy$ **17.** $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1 + \sqrt{x^2 + y^2}}\,dy\,dx$

18. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1 + x^2 + y^2)^2}\,dy\,dx$

19. $\displaystyle\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\,dx\,dy$

20. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\,dx\,dy$

21. $\displaystyle\int_{0}^{1}\int_{x}^{\sqrt{2-x^2}} (x + 2y)\,dy\,dx$

22. $\displaystyle\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \frac{1}{(x^2 + y^2)^2}\,dy\,dx$

In Exercises 23–26, sketch the region of integration and convert each polar integral or sum of integrals to a Cartesian integral or sum of integrals. Do not evaluate the integrals.

23. $\displaystyle\int_{0}^{\pi/2}\int_{0}^{1} r^3 \sin\theta \cos\theta\,dr\,d\theta$

24. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{1}^{\csc\theta} r^2 \cos\theta\,dr\,d\theta$

25. $\displaystyle\int_{0}^{\pi/4}\int_{0}^{2\sec\theta} r^5 \sin^2\theta\,dr\,d\theta$

26. $\displaystyle\int_{0}^{\tan^{-1}\frac{4}{3}}\int_{0}^{3\sec\theta} r^7\,dr\,d\theta + \int_{\tan^{-1}\frac{4}{3}}^{\pi/2}\int_{0}^{4\csc\theta} r^7\,dr\,d\theta$

Area in Polar Coordinates

27. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.

28. Cardioid overlapping a circle Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

29. One leaf of a rose Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

30. Snail shell Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3, 0 \leq \theta \leq 2\pi$. The region looks like a snail shell.

31. Cardioid in the first quadrant Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

32. Overlapping cardioids Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

Average Values

In polar coordinates, the **average value** of a function over a region R (Section 15.3) is given by

$$\frac{1}{\text{Area}(R)} \iint_R f(r, \theta)\,r\,dr\,d\theta.$$

33. Average height of a hemisphere Find the average height of the hemispherical surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy-plane.

34. Average height of a cone Find the average height of the (single) cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy-plane.

35. Average distance from interior of disk to center Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \leq a^2$ to the origin.

36. Average distance squared from a point in a disk to a point in its boundary Find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \leq 1$ to the boundary point $A(1, 0)$.

Theory and Examples

37. Converting to a polar integral Integrate $f(x, y) = [\ln(x^2 + y^2)]/\sqrt{x^2 + y^2}$ over the region $1 \leq x^2 + y^2 \leq e$.

38. Converting to a polar integral Integrate $f(x, y) = [\ln(x^2 + y^2)]/(x^2 + y^2)$ over the region $1 \leq x^2 + y^2 \leq e^2$.

39. Volume of noncircular right cylinder The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.

40. Volume of noncircular right cylinder The region enclosed by the lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

41. Converting to polar integrals

a. The usual way to evaluate the improper integral $I = \int_0^\infty e^{-x^2}\,dx$ is first to calculate its square:

$$I^2 = \left(\int_0^\infty e^{-x^2}\,dx\right)\left(\int_0^\infty e^{-y^2}\,dy\right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I.

b. Evaluate

$$\lim_{x\to\infty} \operatorname{erf}(x) = \lim_{x\to\infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

42. Converting to a polar integral Evaluate the integral

$$\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2}\,dx\,dy.$$

43. Existence Integrate the function $f(x, y) = 1/(1 - x^2 - y^2)$ over the disk $x^2 + y^2 \le 3/4$. Does the integral of $f(x, y)$ over the disk $x^2 + y^2 \le 1$ exist? Give reasons for your answer.

44. Area formula in polar coordinates Use the double integral in polar coordinates to derive the formula

$$A = \int_\alpha^\beta \frac{1}{2} r^2\,d\theta$$

for the area of the fan-shaped region between the origin and polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$.

45. Average distance to a given point inside a disk Let P_0 be a point inside a circle of radius a and let h denote the distance from P_0 to the center of the circle. Let d denote the distance from an arbitrary point P to P_0. Find the average value of d^2 over the region enclosed by the circle. (*Hint:* Simplify your work by placing the center of the circle at the origin and P_0 on the x-axis.)

46. Area Suppose that the area of a region in the polar coordinate plane is

$$A = \int_{\pi/4}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} r\,dr\,d\theta.$$

Sketch the region and find its area.

47. Evaluate the integral $\iint_R \sqrt{x^2 + y^2}\,dA$, where R is the region inside the upper semicircle of radius 2 centered at the origin, but outside the circle $x^2 + (y - 1)^2 = 1$.

48. Evaluate the integral $\iint_R (x^2 + y^2)^{-2}\,dA$, where R is the region inside the circle $x^2 + y^2 = 2$ for $x \le -1$.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS to change the Cartesian integrals into an equivalent polar integral and evaluate the polar integral. Perform the following steps in each exercise.

a. Plot the Cartesian region of integration in the xy-plane.

b. Change each boundary curve of the Cartesian region in part (a) to its polar representation by solving its Cartesian equation for r and θ.

c. Using the results in part (b), plot the polar region of integration in the $r\theta$-plane.

d. Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in part (c) and evaluate the polar integral using the CAS integration utility.

49. $\displaystyle\int_0^1 \int_x^1 \frac{y}{x^2 + y^2}\,dy\,dx$

50. $\displaystyle\int_0^1 \int_0^{x/2} \frac{x}{x^2 + y^2}\,dy\,dx$

51. $\displaystyle\int_0^1 \int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2 + y^2}}\,dx\,dy$

52. $\displaystyle\int_0^1 \int_y^{2-y} \sqrt{x + y}\,dx\,dy$

15.5 Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes and the average value of a function over a three-dimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as we will see in Chapter 16.

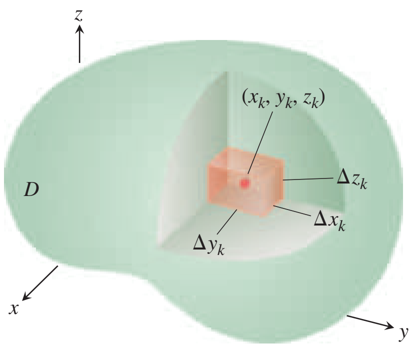


FIGURE 15.30 Partitioning a solid with rectangular cells of volume ΔV_k.

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes (Figure 15.30). We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\,\Delta V_k. \tag{1}$$

We are interested in what happens as D is partitioned by smaller and smaller cells, so that Δx_k, Δy_k, Δz_k, and the norm of the partition $\|P\|$, the largest value among Δx_k, Δy_k, Δz_k, all approach zero. When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is **integrable** over D. As before, it can be shown that when F is continuous and the bounding surface of D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable. As $\|P\| \to 0$ and the number of cells n goes to ∞, the sums S_n approach a limit. We call this limit the **triple integral of F over D** and write

$$\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)\, dV \qquad \text{or} \qquad \lim_{\|P\|\to 0} S_n = \iiint_D F(x, y, z)\, dx\, dy\, dz.$$

The regions D over which continuous functions are integrable are those having "reasonably smooth" boundaries.

Volume of a Region in Space

If F is the constant function whose value is 1, then the sums in Equation (1) reduce to

$$S_n = \sum F(x_k, y_k, z_k)\, \Delta V_k = \sum 1 \cdot \Delta V_k = \sum \Delta V_k .$$

As Δx_k, Δy_k, and Δz_k approach zero, the cells ΔV_k become smaller and more numerous and fill up more and more of D. We therefore define the volume of D to be the triple integral

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k = \iiint_D dV.$$

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

This definition is in agreement with our previous definitions of volume, although we omit the verification of this fact. As we see in a moment, this integral enables us to calculate the volumes of solids enclosed by curved surfaces. These are more general solids than the ones encountered before (Chapter 6 and Section 15.2).

Finding Limits of Integration in the Order *dz dy dx*

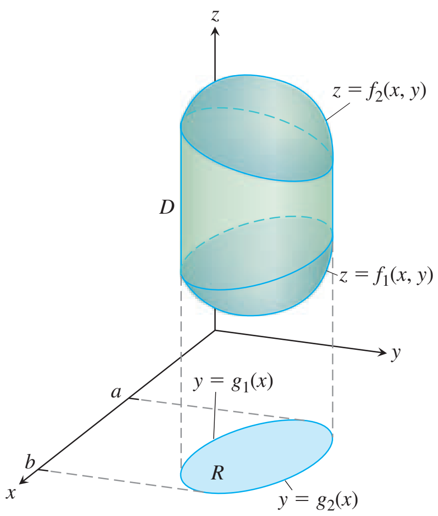
We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem (Section 15.2) to evaluate it by three repeated single integrations. As with double integrals, there is a geometric procedure for finding the limits of integration for these iterated integrals.

To evaluate

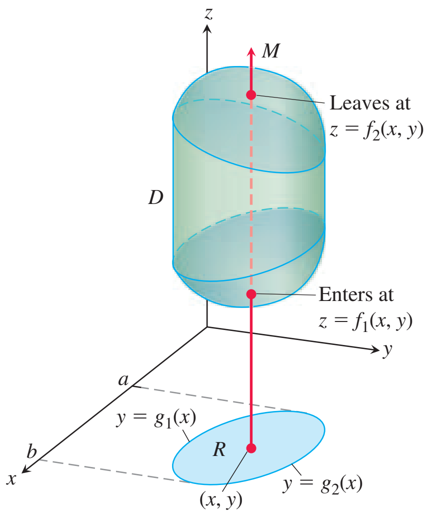
$$\iiint_D F(x, y, z)\, dV$$

over a region D, integrate first with respect to z, then with respect to y, and finally with respect to x. (You might choose a different order of integration, but the procedure is similar, as we illustrate in Example 2.)

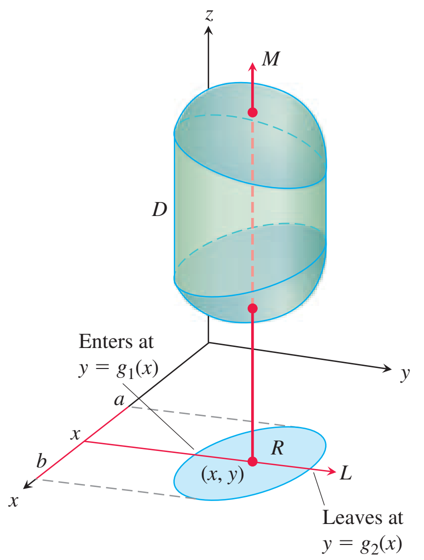
1. *Sketch.* Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. *Find the z-limits of integration.* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



3. *Find the y-limits of integration.* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. *Find the x-limits of integration.* Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Follow similar procedures if you change the order of integration. The "shadow" of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.

The preceding procedure applies whenever a solid region D is bounded above and below by a surface, and when the "shadow" region R is bounded by a lower and upper curve. It does not apply to regions with complicated holes through them, although sometimes such regions can be subdivided into simpler regions for which the procedure does apply.

EXAMPLE 1 Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution The volume is

$$V = \iiint_D dz\, dy\, dx,$$

the integral of $F(x, y, z) = 1$ over D. To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces (Figure 15.31) intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4$, $z > 0$. The boundary of the region R, the projection of D onto the xy-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$. The "upper" boundary of R is the curve $y = \sqrt{(4 - x^2)/2}$. The lower boundary is the curve $y = -\sqrt{(4 - x^2)/2}$.

Now we find the z-limits of integration. The line M passing through a typical point (x, y) in R parallel to the z-axis enters D at $z = x^2 + 3y^2$ and leaves at $z = 8 - x^2 - y^2$.

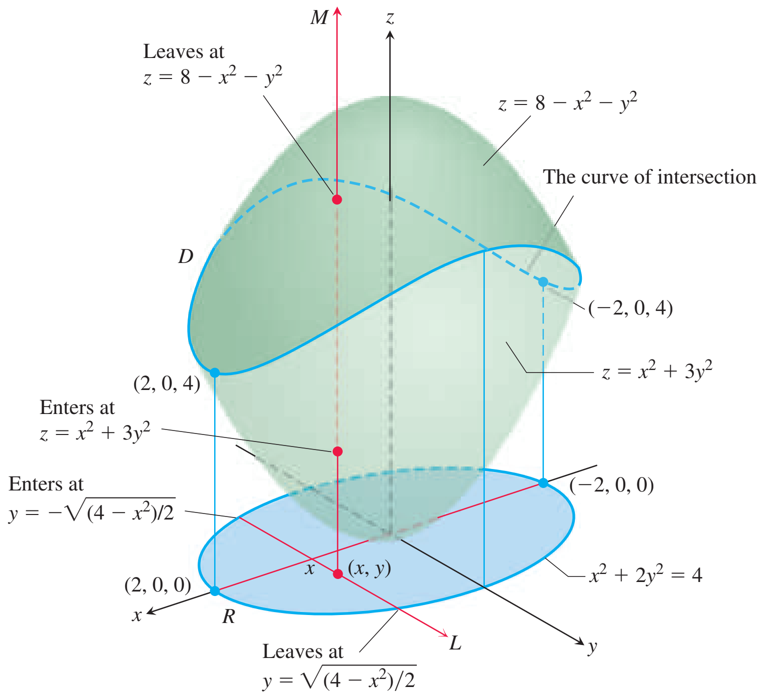


FIGURE 15.31 The volume of the region enclosed by two paraboloids, calculated in Example 1.

Next we find the y-limits of integration. The line L through (x, y) parallel to the y-axis enters R at $y = -\sqrt{(4 - x^2)/2}$ and leaves at $y = \sqrt{(4 - x^2)/2}$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = -2$ at $(-2, 0, 0)$ to $x = 2$ at $(2, 0, 0)$. The volume of D is

$$\begin{aligned}
V &= \iiint_D dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dy\,dx \\
&= \int_{-2}^{2}\int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\,dy\,dx \\
&= \int_{-2}^{2}\left[(8 - 2x^2)y - \frac{4}{3}y^3\right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx \\
&= \int_{-2}^{2}\left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right) dx \\
&= \int_{-2}^{2}\left[8\left(\frac{4 - x^2}{2}\right)^{3/2} - \frac{8}{3}\left(\frac{4 - x^2}{2}\right)^{3/2}\right] dx = \frac{4\sqrt{2}}{3}\int_{-2}^{2}(4 - x^2)^{3/2}\,dx \\
&= 8\pi\sqrt{2}. \qquad \text{After integration with the substitution } x = 2\sin u
\end{aligned}$$

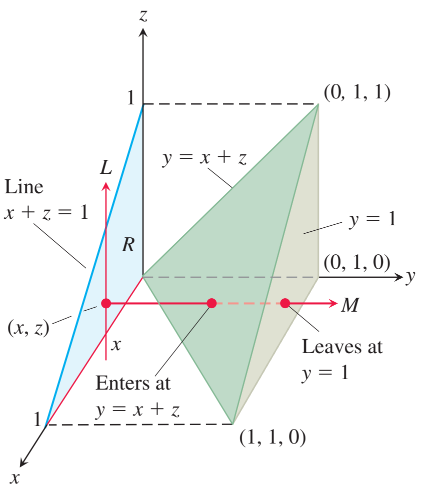


FIGURE 15.32 Finding the limits of integration for evaluating the triple integral of a function defined over the tetrahedron D (Examples 2 and 3).

In the next example, we project D onto the xz-plane instead of the xy-plane, to show how to use a different order of integration.

EXAMPLE 2 Set up the limits of integration for evaluating the triple integral of a function $F(x, y, z)$ over the tetrahedron D with vertices $(0, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, and $(0, 1, 1)$. Use the order of integration $dy\,dz\,dx$.

Solution We sketch D along with its "shadow" R in the xz-plane (Figure 15.32). The upper (right-hand) bounding surface of D lies in the plane $y = 1$. The lower (left-hand) bounding surface lies in the plane $y = x + z$. The upper boundary of R is the line $z = 1 - x$. The lower boundary is the line $z = 0$.

First we find the y-limits of integration. The line through a typical point (x, z) in R parallel to the y-axis enters D at $y = x + z$ and leaves at $y = 1$.

Next we find the z-limits of integration. The line L through (x, z) parallel to the z-axis enters R at $z = 0$ and leaves at $z = 1 - x$.

Finally we find the x-limits of integration. As L sweeps across R, the value of x varies from $x = 0$ to $x = 1$. The integral is

$$\int_0^1\int_0^{1-x}\int_{x+z}^{1} F(x, y, z)\,dy\,dz\,dx.$$

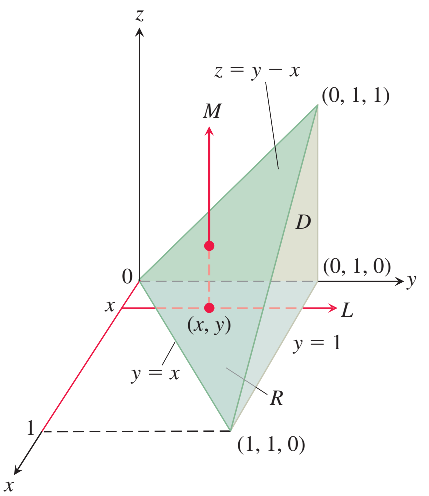


FIGURE 15.33 The tetrahedron in Example 3 showing how the limits of integration are found for the order $dz\,dy\,dx$.

EXAMPLE 3 Integrate $F(x, y, z) = 1$ over the tetrahedron D in Example 2 in the order $dz\,dy\,dx$, and then integrate in the order $dy\,dz\,dx$.

Solution First we find the z-limits of integration. A line M parallel to the z-axis through a typical point (x, y) in the xy-plane "shadow" enters the tetrahedron at $z = 0$ and exits through the upper plane where $z = y - x$ (Figure 15.33).

Next we find the y-limits of integration. On the xy-plane, where $z = 0$, the sloped side of the tetrahedron crosses the plane along the line $y = x$. A line L through (x, y) parallel to the y-axis enters the shadow in the xy-plane at $y = x$ and exits at $y = 1$ (Figure 15.33).

Finally we find the x-limits of integration. As the line L parallel to the y-axis in the previous step sweeps out the shadow, the value of x varies from $x = 0$ to $x = 1$ at the point (1, 1, 0) (see Figure 15.33). The integral is

$$\int_0^1 \int_x^1 \int_0^{y-x} F(x, y, z)\, dz\, dy\, dx.$$

For example, if $F(x, y, z) = 1$, we would find the volume of the tetrahedron to be

$$\begin{aligned}
V &= \int_0^1 \int_x^1 \int_0^{y-x} dz\, dy\, dx \\
&= \int_0^1 \int_x^1 (y - x)\, dy\, dx \\
&= \int_0^1 \left[\frac{1}{2}y^2 - xy\right]_{y=x}^{y=1} dx \\
&= \int_0^1 \left(\frac{1}{2} - x + \frac{1}{2}x^2\right) dx \\
&= \left[\frac{1}{2}x - \frac{1}{2}x^2 + \frac{1}{6}x^3\right]_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

We get the same result by integrating with the order $dy\, dz\, dx$. From Example 2,

$$\begin{aligned}
V &= \int_0^1 \int_0^{1-x} \int_{x+z}^1 dy\, dz\, dx \\
&= \int_0^1 \int_0^{1-x} (1 - x - z)\, dz\, dx \\
&= \int_0^1 \left[(1 - x)z - \frac{1}{2}z^2\right]_{z=0}^{z=1-x} dx \\
&= \int_0^1 \left[(1 - x)^2 - \frac{1}{2}(1 - x)^2\right] dx \\
&= \frac{1}{2}\int_0^1 (1 - x)^2\, dx \\
&= -\frac{1}{6}(1 - x)^3\Big]_0^1 = \frac{1}{6}.
\end{aligned}$$

■

Average Value of a Function in Space

The average value of a function F over a region D in space is defined by the formula

$$\textbf{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\, dV. \qquad (2)$$

For example, if $F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, then the average value of F over D is the average distance of points in D from the origin. If $F(x, y, z)$ is the temperature at (x, y, z) on a solid that occupies a region D in space, then the average value of F over D is the average temperature of the solid.

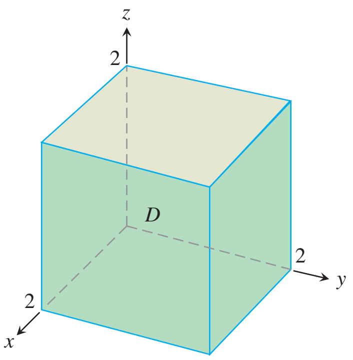


FIGURE 15.34 The region of integration in Example 4.

EXAMPLE 4 Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution We sketch the cube with enough detail to show the limits of integration (Figure 15.34). We then use Equation (2) to calculate the average value of F over the cube.

The volume of the region D is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \int_0^2\int_0^2\left[\frac{x^2}{2}yz\right]_{x=0}^{x=2} dy\,dz = \int_0^2\int_0^2 2yz\,dy\,dz$$

$$= \int_0^2\left[y^2z\right]_{y=0}^{y=2} dz = \int_0^2 4z\,dz = \left[2z^2\right]_0^2 = 8.$$

With these values, Equation (2) gives

$$\begin{array}{c}\text{Average value of}\\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}}\iiint_{\text{cube}} xyz\,dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order $dx\,dy\,dz$, but any of the other five possible orders would have done as well. ■

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals. Simply replace the double integrals in the four properties given in Section 15.2, page 880, with triple integrals.

Exercises 15.5

Triple Integrals in Different Iteration Orders

1. Evaluate the integral in Example 2 taking $F(x, y, z) = 1$ to find the volume of the tetrahedron in the order $dz\,dx\,dy$.

2. **Volume of rectangular solid** Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$. Evaluate one of the integrals.

3. **Volume of tetrahedron** Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.

4. **Volume of solid** Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.

5. **Volume enclosed by paraboloids** Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.

6. **Volume inside paraboloid beneath a plane** Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\,dx\,dy$ and $dz\,dy\,dx$ that give the volume of D. Do not evaluate either integral.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\displaystyle\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\,dx$

8. $\displaystyle\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

9. $\displaystyle\int_1^e\int_1^{e^2}\int_1^{e^3} \frac{1}{xyz}\,dx\,dy\,dz$

10. $\displaystyle\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx$

11. $\displaystyle\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\,dx\,dy\,dz$

12. $\displaystyle\int_{-1}^1\int_0^1\int_0^2 (x + y + z)\,dy\,dx\,dz$

13. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

14. $\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy$

15. $\displaystyle\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx$

16. $\displaystyle\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

17. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{\pi} \cos(u + v + w)\,du\,dv\,dw$ (uvw-space)

18. $\displaystyle\int_0^1\int_1^{\sqrt{e}}\int_1^e se^s\ln r\,\frac{(\ln t)^2}{t}\,dt\,dr\,ds$ (rst-space)

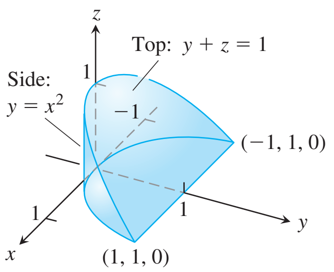
19. $\displaystyle\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv \quad (tvx\text{-space})$

20. $\displaystyle\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1}\,dp\,dq\,dr \quad (pqr\text{-space})$

Finding Equivalent Iterated Integrals

21. Here is the region of integration of the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz\,dy\,dx.$$



Rewrite the integral as an equivalent iterated integral in the order

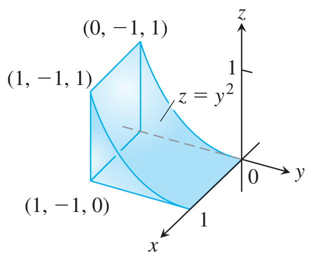
a. $dy\,dz\,dx$ **b.** $dy\,dx\,dz$

c. $dx\,dy\,dz$ **d.** $dx\,dz\,dy$

e. $dz\,dx\,dy$.

22. Here is the region of integration of the integral

$$\int_0^1\int_{-1}^{0}\int_0^{y^2} dz\,dy\,dx.$$



Rewrite the integral as an equivalent iterated integral in the order

a. $dy\,dz\,dx$ **b.** $dy\,dx\,dz$

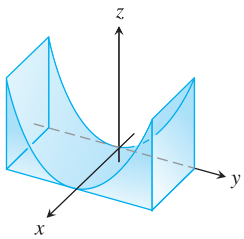
c. $dx\,dy\,dz$ **d.** $dx\,dz\,dy$

e. $dz\,dx\,dy$.

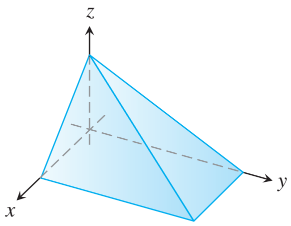
Finding Volumes Using Triple Integrals

Find the volumes of the regions in Exercises 23–36.

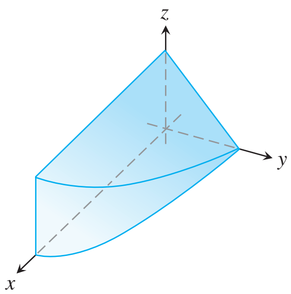
23. The region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0$, $x = 1$, $y = -1$, $y = 1$



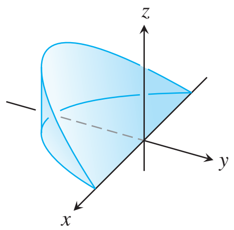
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1$, $y + 2z = 2$



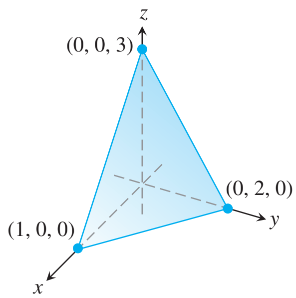
25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



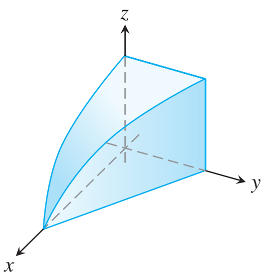
26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$



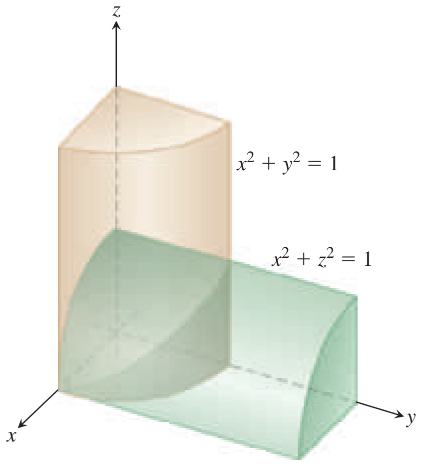
27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through (1, 0, 0), (0, 2, 0), and (0, 0, 3)



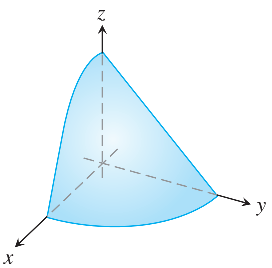
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \le x \le 1$



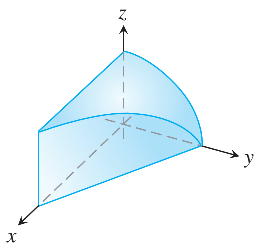
29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure



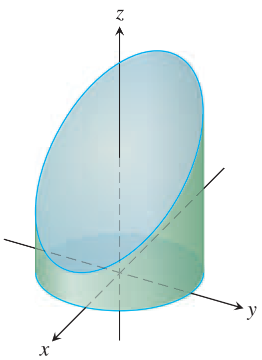
30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$



31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$



32. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



33. The region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant

34. The finite region bounded by the planes $z = x$, $x + z = 8$, $z = y$, $y = 8$, and $z = 0$

35. The region cut from the solid elliptical cylinder $x^2 + 4y^2 \le 4$ by the xy-plane and the plane $z = x + 2$

36. The region bounded in back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy-plane

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

38. $F(x, y, z) = x + y - z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$

39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\displaystyle\int_0^4\int_0^1\int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$

42. $\displaystyle\int_0^1\int_0^1\int_{x^2}^1 12xze^{zy^2}\,dy\,dx\,dz$

43. $\displaystyle\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin \pi y^2}{y^2}\,dx\,dy\,dz$

44. $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx$

Theory and Examples

45. Finding an upper limit of an iterated integral Solve for a:

$$\int_0^1\int_0^{4-a-x^2}\int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15}.$$

46. Ellipsoid For what value of c is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to 8π?

47. Minimizing a triple integral What domain D in space minimizes the value of the integral

$$\iiint_D (4x^2 + 4y^2 + z^2 - 4)\,dV\,?$$

Give reasons for your answer.

48. Maximizing a triple integral What domain D in space maximizes the value of the integral

$$\iiint_D (1 - x^2 - y^2 - z^2)\,dV\,?$$

Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS integration utility to evaluate the triple integral of the given function over the specified solid region.

49. $F(x, y, z) = x^2y^2z$ over the solid cylinder bounded by $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$

50. $F(x, y, z) = |xyz|$ over the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$

51. $F(x, y, z) = \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$ over the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$

52. $F(x, y, z) = x^4 + y^2 + z^2$ over the solid sphere $x^2 + y^2 + z^2 \le 1$

15.6 Moments and Centers of Mass

This section shows how to calculate the masses and moments of two- and three-dimensional objects in Cartesian coordinates. Section 15.7 gives the calculations for cylindrical and spherical coordinates. The definitions and ideas are similar to the single-variable case we studied in Section 6.6, but now we can consider more realistic situations.

Masses and First Moments

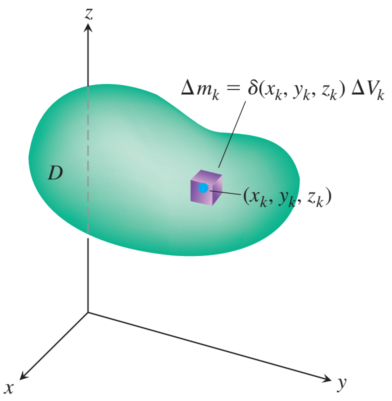


FIGURE 15.35 To define an object's mass, we first imagine it to be partitioned into a finite number of mass elements Δm_k.

If $\delta(x, y, z)$ is the density (mass per unit volume) of an object occupying a region D in space, the integral of δ over D gives the **mass** of the object. To see why, imagine partitioning the object into n mass elements like the one in Figure 15.35. The object's mass is the limit

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\,\Delta V_k = \iiint_D \delta(x, y, z)\, dV.$$

The *first moment* of a solid region D about a coordinate plane is defined as the triple integral over D of the distance from a point (x, y, z) in D to the plane multiplied by the density of the solid at that point. For instance, the first moment about the yz-plane is the integral

$$M_{yz} = \iiint_D x\delta(x, y, z)\, dV.$$

The *center of mass* is found from the first moments. For instance, the x-coordinate of the center of mass is $\bar{x} = M_{yz}/M$.

For a two-dimensional object, such as a thin, flat plate, we calculate first moments about the coordinate axes by simply dropping the z-coordinate. So the first moment about the y-axis is the double integral over the region R forming the plate of the distance from the axis multiplied by the density, or

$$M_y = \iint_R x\delta(x, y)\, dA.$$

Table 15.1 summarizes the formulas.

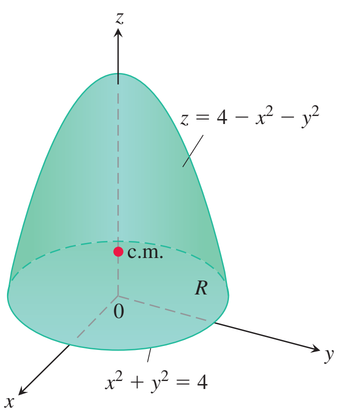


FIGURE 15.36 Finding the center of mass of a solid (Example 1).

EXAMPLE 1 Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.36).

TABLE 15.1 Mass and first moment formulas

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta\, dV$ $\quad\delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\, dV, \qquad M_{xz} = \iiint_D y\,\delta\, dV, \qquad M_{xy} = \iiint_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta\, dA$ $\quad\delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\,\delta\, dA, \qquad M_x = \iint_R y\,\delta\, dA$

Center of mass: $\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$

Solution By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate

$$\begin{aligned}
M_{xy} &= \iint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\, dz\, dy\, dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\, dy\, dx \\
&= \frac{\delta}{2} \iint_R (4 - x^2 - y^2)^2\, dy\, dx \\
&= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2\, r\, dr\, d\theta \qquad \text{Polar coordinates simplify the integration.} \\
&= \frac{\delta}{2} \int_0^{2\pi} \left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3} \int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.
\end{aligned}$$

A similar calculation gives the mass

$$M = \iint_R \int_0^{4-x^2-y^2} \delta\, dz\, dy\, dx = 8\pi\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$. ■

When the density of a solid object or plate is constant (as in Example 1), the center of mass is called the **centroid** of the object. To find a centroid, we set δ equal to 1 and proceed to find $\bar{x}$, $\bar{y}$, and $\bar{z}$ as before, by dividing first moments by masses. These calculations are also valid for two-dimensional objects.

EXAMPLE 2 Find the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.

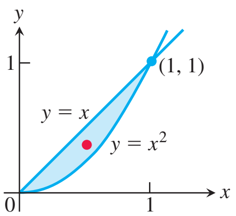


FIGURE 15.37 The centroid of this region is found in Example 2.

Solution We sketch the region and include enough detail to determine the limits of integration (Figure 15.37). We then set δ equal to 1 and evaluate the appropriate formulas from Table 15.1:

$$M = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \int_0^1 \left[y \right]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx = \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \int_0^1 \left[xy \right]_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

From these values of M, M_x, and M_y, we find

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point $(1/2, 2/5)$.

Moments of Inertia

An object's first moments (Table 15.1) tell us about balance and about the torque the object experiences about different axes in a gravitational field. If the object is a rotating shaft, however, we are more likely to be interested in how much energy is stored in the shaft or about how much energy is generated by a shaft rotating at a particular angular velocity. This is where the second moment or moment of inertia comes in.

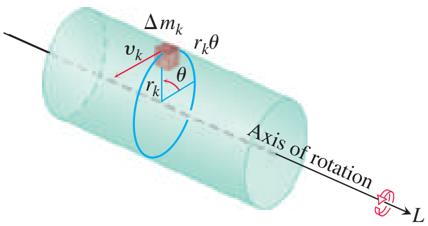


FIGURE 15.38 To find an integral for the amount of energy stored in a rotating shaft, we first imagine the shaft to be partitioned into small blocks. Each block has its own kinetic energy. We add the contributions of the individual blocks to find the kinetic energy of the shaft.

Think of partitioning the shaft into small blocks of mass Δm_k and let r_k denote the distance from the kth block's center of mass to the axis of rotation (Figure 15.38). If the shaft rotates at a constant angular velocity of $\omega = d\theta/dt$ radians per second, the block's center of mass will trace its orbit at a linear speed of

$$v_k = \frac{d}{dt}(r_k \theta) = r_k \frac{d\theta}{dt} = r_k \omega.$$

The block's kinetic energy will be approximately

$$\frac{1}{2} \Delta m_k v_k^2 = \frac{1}{2} \Delta m_k (r_k \omega)^2 = \frac{1}{2} \omega^2 r_k^2\, \Delta m_k.$$

The kinetic energy of the shaft will be approximately

$$\sum \frac{1}{2} \omega^2 r_k^2\, \Delta m_k.$$

The integral approached by these sums as the shaft is partitioned into smaller and smaller blocks gives the shaft's kinetic energy:

$$\text{KE}_{\text{shaft}} = \int \frac{1}{2} \omega^2 r^2\, dm = \frac{1}{2} \omega^2 \int r^2\, dm. \tag{1}$$

The factor

$$I = \int r^2\, dm$$

is the *moment of inertia* of the shaft about its axis of rotation, and we see from Equation (1) that the shaft's kinetic energy is

$$\text{KE}_{\text{shaft}} = \frac{1}{2} I \omega^2.$$

The moment of inertia of a shaft resembles in some ways the inertial mass of a locomotive. To start a locomotive with mass m moving at a linear velocity v, we need to provide a kinetic energy of $\mathrm{KE} = (1/2)mv^2$. To stop the locomotive we have to remove this amount of energy. To start a shaft with moment of inertia I rotating at an angular velocity ω, we need to provide a kinetic energy of $\mathrm{KE} = (1/2)I\omega^2$. To stop the shaft we have to take this amount of energy back out. The shaft's moment of inertia is analogous to the locomotive's mass. What makes the locomotive hard to start or stop is its mass. What makes the shaft hard to start or stop is its moment of inertia. The moment of inertia depends not only on the mass of the shaft but also on its distribution. Mass that is farther away from the axis of rotation contributes more to the moment of inertia.

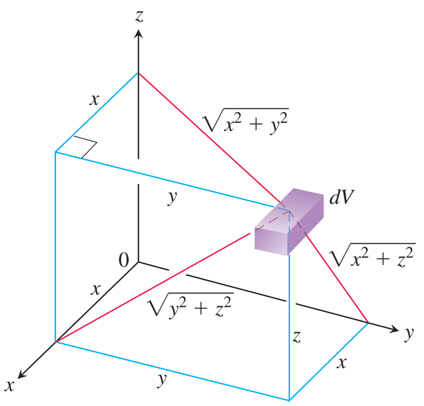


FIGURE 15.39 Distances from dV to the coordinate planes and axes.

We now derive a formula for the moment of inertia for a solid in space. If $r(x, y, z)$ is the distance from the point (x, y, z) in D to a line L, then the moment of inertia of the mass $\Delta m_k = \delta(x_k, y_k, z_k)\Delta V_k$ about the line L (as in Figure 15.38) is approximately $\Delta I_k = r^2(x_k, y_k, z_k)\Delta m_k$. **The moment of inertia about L** of the entire object is

$$I_L = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta I_k = \lim_{n\to\infty} \sum_{k=1}^{n} r^2(x_k, y_k, z_k)\,\delta(x_k, y_k, z_k)\,\Delta V_k = \iiint_D r^2\delta\, dV.$$

If L is the x-axis, then $r^2 = y^2 + z^2$ (Figure 15.39) and

$$I_x = \iiint_D (y^2 + z^2)\,\delta(x, y, z)\,dV.$$

Similarly, if L is the y-axis or z-axis we have

$$I_y = \iiint_D (x^2 + z^2)\,\delta(x, y, z)\,dV \qquad \text{and} \qquad I_z = \iiint_D (x^2 + y^2)\,\delta(x, y, z)\,dV.$$

Table 15.2 summarizes the formulas for these moments of inertia (second moments because they invoke the *squares* of the distances). It shows the definition of the *polar moment* about the origin as well.

EXAMPLE 3 Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Figure 15.40.

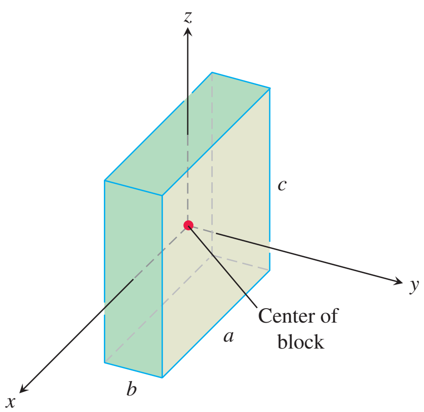


FIGURE 15.40 Finding I_x, I_y, and I_z for the block shown here. The origin lies at the center of the block (Example 3).

Solution The formula for I_x gives

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2} (y^2 + z^2)\,\delta\,dx\,dy\,dz.$$

We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x, y, and z since δ is constant. The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$\begin{aligned}
I_x &= 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2} (y^2 + z^2)\,\delta\,dx\,dy\,dz = 4a\delta\int_0^{c/2}\int_0^{b/2} (y^2 + z^2)\,dy\,dz \\
&= 4a\delta\int_0^{c/2}\left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz \\
&= 4a\delta\int_0^{c/2}\left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz \\
&= 4a\delta\left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2). \qquad M = abc\delta
\end{aligned}$$

TABLE 15.2 Moments of inertia (second moments) formulas

THREE-DIMENSIONAL SOLID

About the x-axis:	$I_x = \iiint (y^2 + z^2)\,\delta\, dV$	$\delta = \delta(x, y, z)$
About the y-axis:	$I_y = \iiint (x^2 + z^2)\,\delta\, dV$	
About the z-axis:	$I_z = \iiint (x^2 + y^2)\,\delta\, dV$	
About a line L:	$I_L = \iiint r^2(x, y, z)\,\delta\, dV$	$r(x, y, z) =$ distance from the point (x, y, z) to line L

TWO-DIMENSIONAL PLATE

About the x-axis:	$I_x = \iint y^2\,\delta\, dA$	$\delta = \delta(x, y)$
About the y-axis:	$I_y = \iint x^2\,\delta\, dA$	
About a line L:	$I_L = \iint r^2(x, y)\,\delta\, dA$	$r(x, y) =$ distance from (x, y) to L
About the origin (polar moment):	$I_0 = \iint (x^2 + y^2)\,\delta\, dA = I_x + I_y$	

Similarly,

$$I_y = \frac{M}{12}(a^2 + c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2 + b^2).$$

EXAMPLE 4 A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\delta(x, y) = 6x + 6y + 6$. Find the plate's moments of inertia about the coordinate axes and the origin.

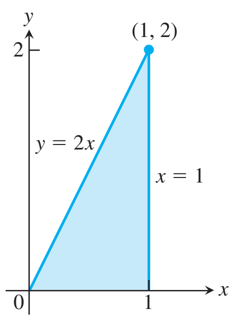


FIGURE 15.41 The triangular region covered by the plate in Example 4.

Solution We sketch the plate and put in enough detail to determine the limits of integration for the integrals we have to evaluate (Figure 15.41). The moment of inertia about the x-axis is

$$\begin{aligned} I_x &= \int_0^1 \int_0^{2x} y^2 \delta(x, y)\, dy\, dx = \int_0^1 \int_0^{2x} (6xy^2 + 6y^3 + 6y^2)\, dy\, dx \\ &= \int_0^1 \left[2xy^3 + \frac{3}{2}y^4 + 2y^3 \right]_{y=0}^{y=2x} dx = \int_0^1 (40x^4 + 16x^3)\, dx \\ &= \left[8x^5 + 4x^4 \right]_0^1 = 12. \end{aligned}$$

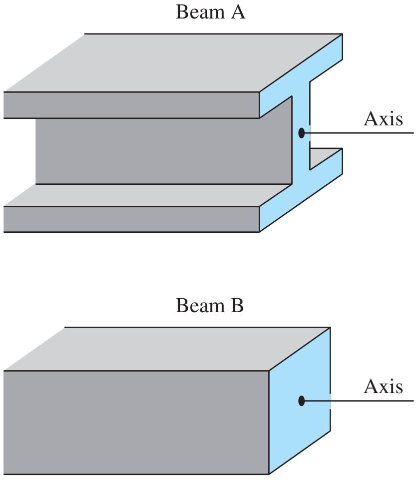


FIGURE 15.42 The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

Similarly, the moment of inertia about the y-axis is

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x, y)\, dy\, dx = \frac{39}{5}.$$

Notice that we integrate y^2 times density in calculating I_x and x^2 times density to find I_y.

Since we know I_x and I_y, we do not need to evaluate an integral to find I_0; we can use the equation $I_0 = I_x + I_y$ from Table 15.2 instead:

$$I_0 = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}.$$

The moment of inertia also plays a role in determining how much a horizontal metal beam will bend under a load. The stiffness of the beam is a constant times I, the moment of inertia of a typical cross-section of the beam about the beam's longitudinal axis. The greater the value of I, the stiffer the beam and the less it will bend under a given load. That is why we use I-beams instead of beams whose cross-sections are square. The flanges at the top and bottom of the beam hold most of the beam's mass away from the longitudinal axis to increase the value of I (Figure 15.42).

Exercises 15.6

Plates of Constant Density

1. **Finding a center of mass** Find the center of mass of a thin plate of density $\delta = 3$ bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.
2. **Finding moments of inertia** Find the moments of inertia about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant.
3. **Finding a centroid** Find the centroid of the region in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.
4. **Finding a centroid** Find the centroid of the triangular region cut from the first quadrant by the line $x + y = 3$.
5. **Finding a centroid** Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.
6. **Finding a centroid** Find the centroid of the region between the x-axis and the arch $y = \sin x$, $0 \le x \le \pi$.
7. **Finding moments of inertia** Find the moment of inertia about the x-axis of a thin plate of density $\delta = 1$ bounded by the circle $x^2 + y^2 = 4$. Then use your result to find I_y and I_0 for the plate.
8. **Finding a moment of inertia** Find the moment of inertia with respect to the y-axis of a thin sheet of constant density $\delta = 1$ bounded by the curve $y = (\sin^2 x)/x^2$ and the interval $\pi \le x \le 2\pi$ of the x-axis.
9. **The centroid of an infinite region** Find the centroid of the infinite region in the second quadrant enclosed by the coordinate axes and the curve $y = e^x$. (Use improper integrals in the mass-moment formulas.)
10. **The first moment of an infinite plate** Find the first moment about the y-axis of a thin plate of density $\delta(x, y) = 1$ covering the infinite region under the curve $y = e^{-x^2/2}$ in the first quadrant.

Plates with Varying Density

11. **Finding a moment of inertia** Find the moment of inertia about the x-axis of a thin plate bounded by the parabola $x = y - y^2$ and the line $x + y = 0$ if $\delta(x, y) = x + y$.
12. **Finding mass** Find the mass of a thin plate occupying the smaller region cut from the ellipse $x^2 + 4y^2 = 12$ by the parabola $x = 4y^2$ if $\delta(x, y) = 5x$.
13. **Finding a center of mass** Find the center of mass of a thin triangular plate bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ if $\delta(x, y) = 6x + 3y + 3$.
14. **Finding a center of mass and moment of inertia** Find the center of mass and moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.
15. **Center of mass, moment of inertia** Find the center of mass and the moment of inertia about the y-axis of a thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if $\delta(x, y) = x + y + 1$.
16. **Center of mass, moment of inertia** Find the center of mass and the moment of inertia about the y-axis of a thin plate bounded by the line $y = 1$ and the parabola $y = x^2$ if the density is $\delta(x, y) = y + 1$.
17. **Center of mass, moment of inertia** Find the center of mass and the moment of inertia about the y-axis of a thin plate bounded by the x-axis, the lines $x = \pm 1$, and the parabola $y = x^2$ if $\delta(x, y) = 7y + 1$.

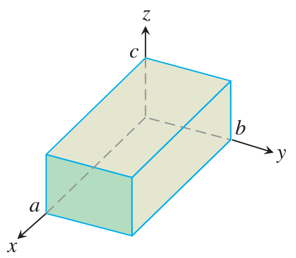
18. Center of mass, moment of inertia Find the center of mass and the moment of inertia about the x-axis of a thin rectangular plate bounded by the lines $x = 0$, $x = 20$, $y = -1$, and $y = 1$ if $\delta(x, y) = 1 + (x/20)$.

19. Center of mass, moments of inertia Find the center of mass, the moment of inertia about the coordinate axes, and the polar moment of inertia of a thin triangular plate bounded by the lines $y = x$, $y = -x$, and $y = 1$ if $\delta(x, y) = y + 1$.

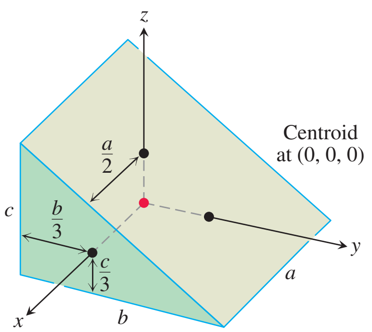
20. Center of mass, moments of inertia Repeat Exercise 19 for $\delta(x, y) = 3x^2 + 1$.

Solids with Constant Density

21. Moments of inertia Find the moments of inertia of the rectangular solid shown here with respect to its edges by calculating I_x, I_y, and I_z.



22. Moments of inertia The coordinate axes in the figure run through the centroid of a solid wedge parallel to the labeled edges. Find I_x, I_y, and I_z if $a = b = 6$ and $c = 4$.



23. Center of mass and moments of inertia A solid "trough" of constant density is bounded below by the surface $z = 4y^2$, above by the plane $z = 4$, and on the ends by the planes $x = 1$ and $x = -1$. Find the center of mass and the moments of inertia with respect to the three axes.

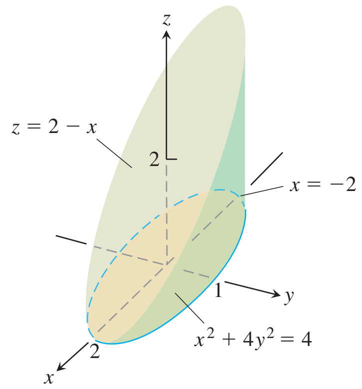
24. Center of mass A solid of constant density is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$, and above by the plane $z = 2 - x$ (see the accompanying figure).

a. Find $\bar{x}$ and $\bar{y}$.

b. Evaluate the integral

$$M_{xy} = \int_{-2}^{2} \int_{-(1/2)\sqrt{4-x^2}}^{(1/2)\sqrt{4-x^2}} \int_{0}^{2-x} z \, dz \, dy \, dx$$

using integral tables to carry out the final integration with respect to x. Then divide M_{xy} by M to verify that $\bar{z} = 5/4$.



25. a. Center of mass Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.

b. Find the plane $z = c$ that divides the solid into two parts of equal volume. This plane does not pass through the center of mass.

26. Moments A solid cube, 2 units on a side, is bounded by the planes $x = \pm 1$, $z = \pm 1$, $y = 3$, and $y = 5$. Find the center of mass and the moments of inertia about the coordinate axes.

27. Moment of inertia about a line A wedge like the one in Exercise 22 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line $L: z = 0, y = 6$ is $r^2 = (y - 6)^2 + z^2$. Then calculate the moment of inertia of the wedge about L.

28. Moment of inertia about a line A wedge like the one in Exercise 22 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line $L: x = 4, y = 0$ is $r^2 = (x - 4)^2 + y^2$. Then calculate the moment of inertia of the wedge about L.

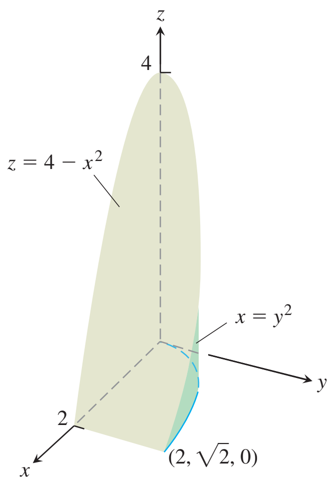
Solids with Varying Density

In Exercises 29 and 30, find

a. the mass of the solid. **b.** the center of mass.

29. A solid region in the first octant is bounded by the coordinate planes and the plane $x + y + z = 2$. The density of the solid is $\delta(x, y, z) = 2x$.

30. A solid in the first octant is bounded by the planes $y = 0$ and $z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ (see the accompanying figure). Its density function is $\delta(x, y, z) = kxy$, k a constant.



In Exercises 31 and 32, find

a. the mass of the solid.

b. the center of mass.

c. the moments of inertia about the coordinate axes.

31. A solid cube in the first octant is bounded by the coordinate planes and by the planes $x = 1$, $y = 1$, and $z = 1$. The density of the cube is $\delta(x, y, z) = x + y + z + 1$.

32. A wedge like the one in Exercise 22 has dimensions $a = 2$, $b = 6$, and $c = 3$. The density is $\delta(x, y, z) = x + 1$. Notice that if the density is constant, the center of mass will be $(0, 0, 0)$.

33. Mass Find the mass of the solid bounded by the planes $x + z = 1$, $x - z = -1$, $y = 0$, and the surface $y = \sqrt{z}$. The density of the solid is $\delta(x, y, z) = 2y + 5$.

34. Mass Find the mass of the solid region bounded by the parabolic surfaces $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$ if the density of the solid is $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

Theory and Examples

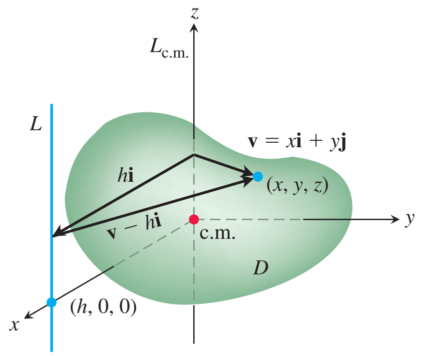
The Parallel Axis Theorem Let $L_{\text{c.m.}}$ be a line through the center of mass of a body of mass m and let L be a parallel line h units away from $L_{\text{c.m.}}$. The *Parallel Axis Theorem* says that the moments of inertia $I_{\text{c.m.}}$ and I_L of the body about $L_{\text{c.m.}}$ and L satisfy the equation

$$I_L = I_{\text{c.m.}} + mh^2. \tag{2}$$

As in the two-dimensional case, the theorem gives a quick way to calculate one moment when the other moment and the mass are known.

35. Proof of the Parallel Axis Theorem

a. Show that the first moment of a body in space about any plane through the body's center of mass is zero. (*Hint:* Place the body's center of mass at the origin and let the plane be the yz-plane. What does the formula $\bar{x} = M_{yz}/M$ then tell you?)



b. To prove the Parallel Axis Theorem, place the body with its center of mass at the origin, with the line $L_{\text{c.m.}}$ along the z-axis and the line L perpendicular to the xy-plane at the point $(h, 0, 0)$. Let D be the region of space occupied by the body. Then, in the notation of the figure,

$$I_L = \iiint_D |\mathbf{v} - h\mathbf{i}|^2 \, dm.$$

Expand the integrand in this integral and complete the proof.

36. The moment of inertia about a diameter of a solid sphere of constant density and radius a is $(2/5)ma^2$, where m is the mass of the sphere. Find the moment of inertia about a line tangent to the sphere.

37. The moment of inertia of the solid in Exercise 21 about the z-axis is $I_z = abc(a^2 + b^2)/3$.

a. Use Equation (2) to find the moment of inertia of the solid about the line parallel to the z-axis through the solid's center of mass.

b. Use Equation (2) and the result in part (a) to find the moment of inertia of the solid about the line $x = 0$, $y = 2b$.

38. If $a = b = 6$ and $c = 4$, the moment of inertia of the solid wedge in Exercise 22 about the x-axis is $I_x = 208$. Find the moment of inertia of the wedge about the line $y = 4$, $z = -4/3$ (the edge of the wedge's narrow end).

15.7 Triple Integrals in Cylindrical and Spherical Coordinates

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane studied in Section 15.4.

Integration in Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z), as shown in Figure 15.43. Here we require $r \geq 0$.

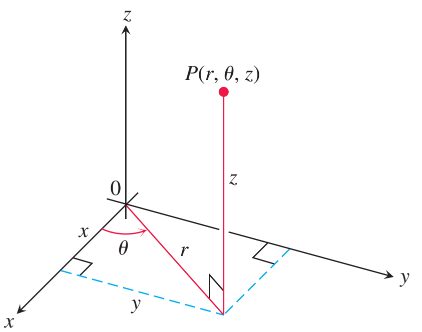


FIGURE 15.43 The cylindrical coordinates of a point in space are r, θ, and z.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which $r \geq 0$,

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

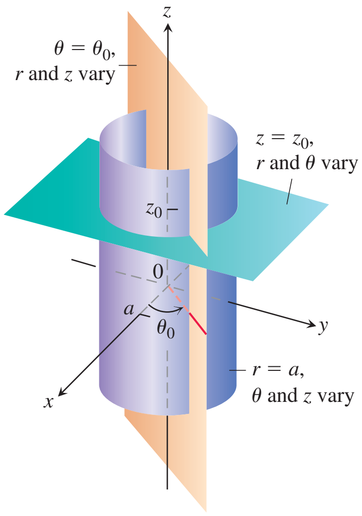


FIGURE 15.44 Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

The values of x, y, r, and θ in rectangular and cylindrical coordinates are related by the usual equations.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

In cylindrical coordinates, the equation $r = a$ describes not just a circle in the xy-plane but an entire cylinder about the z-axis (Figure 15.44). The z-axis is given by $r = 0$. The equation $\theta = \theta_0$ describes the plane that contains the z-axis and makes an angle θ_0 with the positive x-axis. And, just as in rectangular coordinates, the equation $z = z_0$ describes a plane perpendicular to the z-axis.

Cylindrical coordinates are good for describing cylinders whose axes run along the z-axis and planes that either contain the z-axis or lie perpendicular to the z-axis. Surfaces like these have equations of constant coordinate value:

$$r = 4 \qquad \text{Cylinder, radius 4, axis the } z\text{-axis}$$
$$\theta = \frac{\pi}{3} \qquad \text{Plane containing the } z\text{-axis}$$
$$z = 2. \qquad \text{Plane perpendicular to the } z\text{-axis}$$

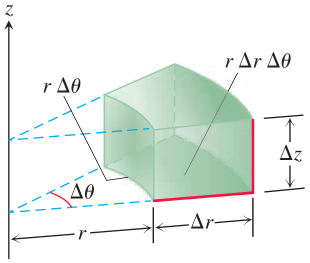


FIGURE 15.45 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

When computing triple integrals over a region D in cylindrical coordinates, we partition the region into n small cylindrical wedges, rather than into rectangular boxes. In the kth cylindrical wedge, r, θ and z change by Δr_k, $\Delta\theta_k$, and Δz_k, and the largest of these numbers among all the cylindrical wedges is called the **norm** of the partition. We define the triple integral as a limit of Riemann sums using these wedges. The volume of such a cylindrical wedge ΔV_k is obtained by taking the area ΔA_k of its base in the $r\theta$-plane and multiplying by the height Δz (Figure 15.45).

For a point (r_k, θ_k, z_k) in the center of the kth wedge, we calculated in polar coordinates that $\Delta A_k = r_k\, \Delta r_k\, \Delta\theta_k$. So $\Delta V_k = \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k$ and a Riemann sum for f over D has the form

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k, z_k)\, \Delta z_k\, r_k\, \Delta r_k\, \Delta\theta_k .$$

The triple integral of a function f over D is obtained by taking a limit of such Riemann sums with partitions whose norms approach zero:

$$\lim_{n\to\infty} S_n = \iiint_D f\, dV = \iiint_D f\, dz\, r\, dr\, d\theta .$$

Volume Differential in Cylindrical Coordinates

$$dV = dz\, r\, dr\, d\theta$$

Triple integrals in cylindrical coordinates are then evaluated as iterated integrals, as in the following example. Although the definition of cylindrical coordinates makes sense without any restrictions on θ, in most situations when integrating, we will need to restrict θ to an interval of length 2π. So we impose the requirement that $\alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$.

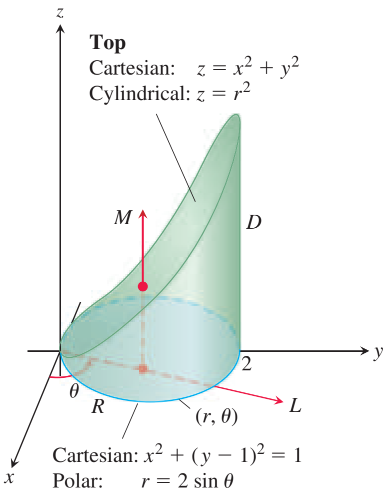


FIGURE 15.46 Finding the limits of integration for evaluating an integral in cylindrical coordinates (Example 1).

EXAMPLE 1 Find the limits of integration in cylindrical coordinates for integrating a function $f(r, \theta, z)$ over the region D bounded below by the plane $z = 0$, laterally by the circular cylinder $x^2 + (y - 1)^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

Solution The base of D is also the region's projection R on the xy-plane. The boundary of R is the circle $x^2 + (y - 1)^2 = 1$. Its polar coordinate equation is

$$\begin{aligned} x^2 + (y - 1)^2 &= 1 \\ x^2 + y^2 - 2y + 1 &= 1 \\ r^2 - 2r \sin \theta &= 0 \\ r &= 2 \sin \theta. \end{aligned}$$

The region is sketched in Figure 15.46.

We find the limits of integration, starting with the z-limits. A line M through a typical point (r, θ) in R parallel to the z-axis enters D at $z = 0$ and leaves at $z = x^2 + y^2 = r^2$.

Next we find the r-limits of integration. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2 \sin \theta$.

Finally we find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = \pi$. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_0^{\pi} \int_0^{2 \sin \theta} \int_0^{r^2} f(r, \theta, z)\, dz\, r\, dr\, d\theta. \qquad \blacksquare$$

Example 1 illustrates a good procedure for finding limits of integration in cylindrical coordinates. The procedure is summarized as follows.

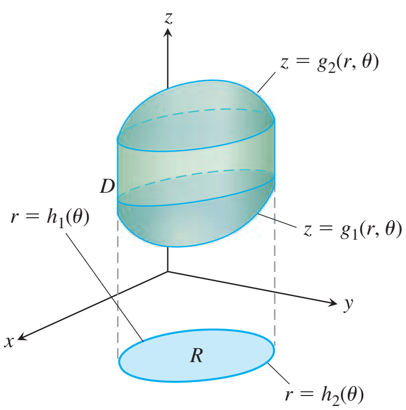
How to Integrate in Cylindrical Coordinates

To evaluate

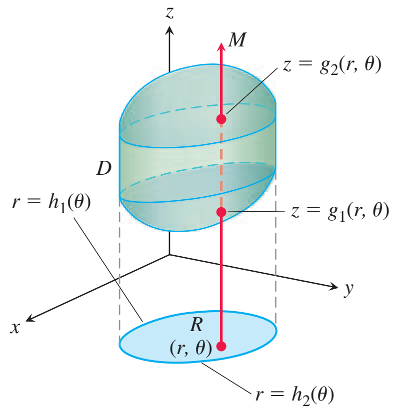
$$\iiint_D f(r, \theta, z)\, dV$$

over a region D in space in cylindrical coordinates, integrating first with respect to z, then with respect to r, and finally with respect to θ, take the following steps.

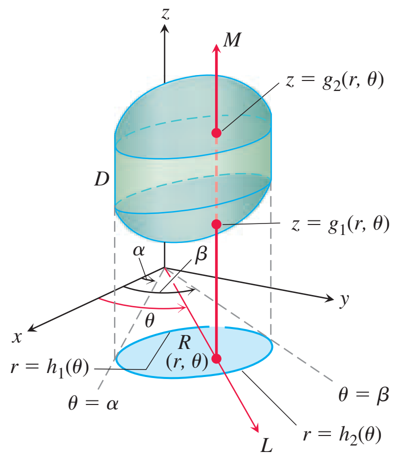
1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.



2. *Find the z-limits of integration.* Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.



3. *Find the r-limits of integration.* Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



4. *Find the θ-limits of integration.* As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

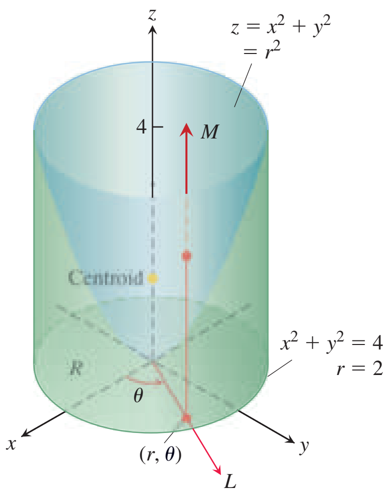


FIGURE 15.47 Example 2 shows how to find the centroid of this solid.

EXAMPLE 2 Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

Solution We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$ (Figure 15.47). Its base R is the disk $0 \leq r \leq 2$ in the xy-plane.

The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

The z-limits. A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.

The r-limits. A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.

The θ-limits. As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$. The value of M_{xy} is

$$\begin{aligned} M_{xy} &= \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 \frac{r^5}{2}dr\,d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}d\theta = \frac{32\pi}{3}. \end{aligned}$$

The value of M is

$$\begin{aligned} M &= \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[z\right]_0^{r^2} r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi. \end{aligned}$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies on the z-axis, outside the solid.

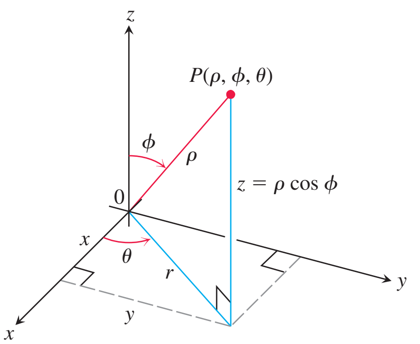


FIGURE 15.48 The spherical coordinates ρ, ϕ, and θ and their relation to x, y, z, and r.

Spherical Coordinates and Integration

Spherical coordinates locate points in space with two angles and one distance, as shown in Figure 15.48. The first coordinate, $\rho = |\overrightarrow{OP}|$, is the point's distance from the origin and is never negative. The second coordinate, ϕ, is the angle $\overrightarrow{OP}$ makes with the positive z-axis. It is required to lie in the interval $[0, \pi]$. The third coordinate is the angle θ as measured in cylindrical coordinates.

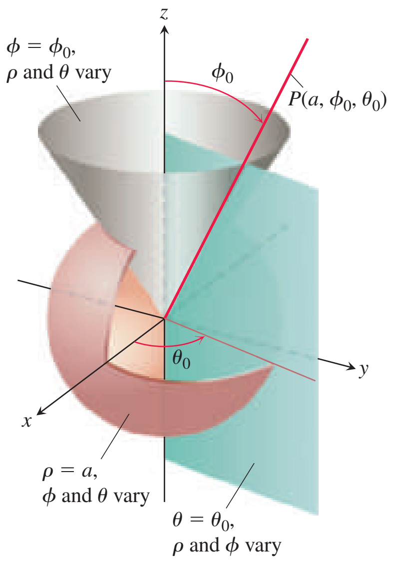


FIGURE 15.49 Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

> **DEFINITION** **Spherical coordinates** represent a point P in space by ordered triples (ρ, ϕ, θ) in which
>
> 1. ρ is the distance from P to the origin $(\rho \geq 0)$.
> 2. ϕ is the angle $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
> 3. θ is the angle from cylindrical coordinates.

On maps of the earth, θ is related to the meridian of a point on the earth and ϕ to its latitude, while ρ is related to elevation above the earth's surface.

The equation $\rho = a$ describes the sphere of radius a centered at the origin (Figure 15.49). The equation $\phi = \phi_0$ describes a single cone whose vertex lies at the origin and whose axis lies along the z-axis. (We broaden our interpretation to include the xy-plane as the cone $\phi = \pi/2$.) If ϕ_0 is greater than $\pi/2$, the cone $\phi = \phi_0$ opens downward. The equation $\theta = \theta_0$ describes the half-plane that contains the z-axis and makes an angle θ_0 with the positive x-axis.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$
\begin{aligned}
r &= \rho \sin \phi, & x &= r\cos\theta = \rho \sin\phi\cos\theta,\\
z &= \rho \cos \phi, & y &= r\sin\theta = \rho \sin\phi\sin\theta,\\
\rho &= \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2}. & &
\end{aligned}
\tag{1}
$$

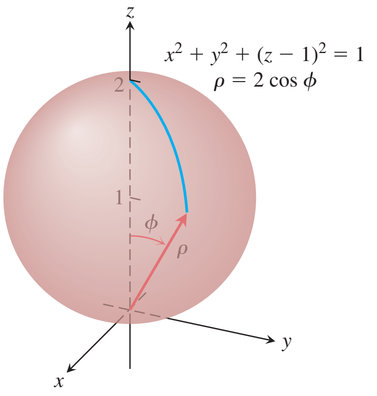


FIGURE 15.50 The sphere in Example 3.

EXAMPLE 3 Find a spherical coordinate equation for the sphere $x^2 + y^2 + (z-1)^2 = 1$.

Solution We use Equations (1) to substitute for x, y, and z:

$$
\begin{aligned}
x^2 + y^2 + (z-1)^2 &= 1\\
\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 &= 1 && \text{Eqs. (1)}\\
\rho^2\sin^2\phi\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 &= 1\\
\rho^2\underbrace{(\sin^2\phi + \cos^2\phi)}_{1} &= 2\rho\cos\phi\\
\rho^2 &= 2\rho\cos\phi\\
\rho &= 2\cos\phi. && \rho > 0
\end{aligned}
$$

The angle ϕ varies from 0 at the north pole of the sphere to $\pi/2$ at the south pole; the angle θ does not appear in the expression for ρ, reflecting the symmetry about the z-axis (see Figure 15.50). ■

EXAMPLE 4 Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.

Solution 1 *Use geometry*. The cone is symmetric with respect to the z-axis and cuts the first quadrant of the yz-plane along the line $z = y$. The angle between the cone and the positive z-axis is therefore $\pi/4$ radians. The cone consists of the points whose spherical coordinates have ϕ equal to $\pi/4$, so its equation is $\phi = \pi/4$. (See Figure 15.51.)

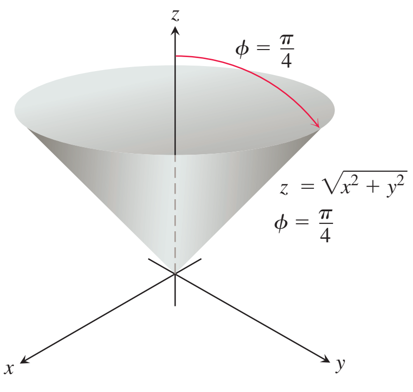


FIGURE 15.51 The cone in Example 4.

Solution 2 *Use algebra*. If we use Equations (1) to substitute for x, y, and z we obtain the same result:

$$
\begin{aligned}
z &= \sqrt{x^2+y^2}\\
\rho\cos\phi &= \sqrt{\rho^2\sin^2\phi} && \text{Example 3}\\
\rho\cos\phi &= \rho\sin\phi && \rho > 0,\ \sin\phi \ge 0\\
\cos\phi &= \sin\phi\\
\phi &= \frac{\pi}{4}. && 0 \le \phi \le \pi
\end{aligned}
$$

■

Spherical coordinates are useful for describing spheres centered at the origin, half-planes hinged along the z-axis, and cones whose vertices lie at the origin and whose axes lie along the z-axis. Surfaces like these have equations of constant coordinate value:

$$
\begin{aligned}
\rho &= 4 && \text{Sphere, radius 4, center at origin}\\
\phi &= \frac{\pi}{3} && \text{Cone opening up from the origin, making an angle of } \pi/3 \text{ radians with the positive z-axis}\\
\theta &= \frac{\pi}{3}. && \text{Half-plane, hinged along the } z\text{-axis, making an angle of } \pi/3 \text{ radians with the positive x-axis}
\end{aligned}
$$

When computing triple integrals over a region D in spherical coordinates, we partition the region into n spherical wedges. The size of the kth spherical wedge, which contains a

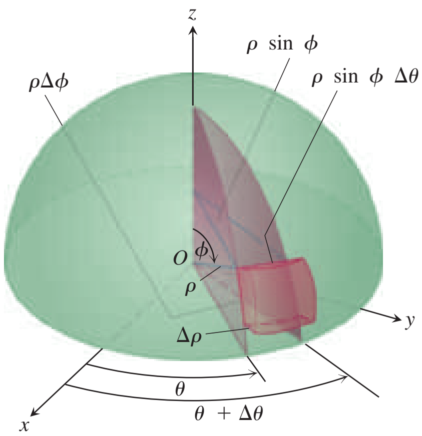


FIGURE 15.52 In spherical coordinates we use the volume of a spherical wedge, which closely approximates that of a cube.

point $(\rho_k, \phi_k, \theta_k)$, is given by the changes $\Delta\rho_k$, $\Delta\phi_k$, and $\Delta\theta_k$ in ρ, ϕ, and θ. Such a spherical wedge has one edge a circular arc of length $\rho_k\,\Delta\phi_k$, another edge a circular arc of length $\rho_k \sin\phi_k\,\Delta\theta_k$, and thickness $\Delta\rho_k$. The spherical wedge closely approximates a cube of these dimensions when $\Delta\rho_k$, $\Delta\phi_k$, and $\Delta\theta_k$ are all small (Figure 15.52). It can be shown that the volume of this spherical wedge ΔV_k is $\Delta V_k = \rho_k{}^2 \sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k$ for $(\rho_k, \phi_k, \theta_k)$, a point chosen inside the wedge.

The corresponding Riemann sum for a function $f(\rho, \phi, \theta)$ is

$$S_n = \sum_{k=1}^{n} f(\rho_k, \phi_k, \theta_k)\,\rho_k{}^2 \sin\phi_k\,\Delta\rho_k\,\Delta\phi_k\,\Delta\theta_k.$$

As the norm of a partition approaches zero, and the spherical wedges get smaller, the Riemann sums have a limit when f is continuous:

$$\boxed{\lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta)\,dV = \iiint_D f(\rho, \phi, \theta)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta.}$$

Volume Differential in Spherical Coordinates

$$dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

To evaluate integrals in spherical coordinates, we usually integrate first with respect to ρ. The procedure for finding the limits of integration is as follows. We restrict our attention to integrating over domains that are solids of revolution about the z-axis (or portions thereof) and for which the limits for θ and ϕ are constant. As with cylindrical coordinates, we restrict θ in the form $\alpha \le \theta \le \beta$ and $0 \le \beta - \alpha \le 2\pi$.

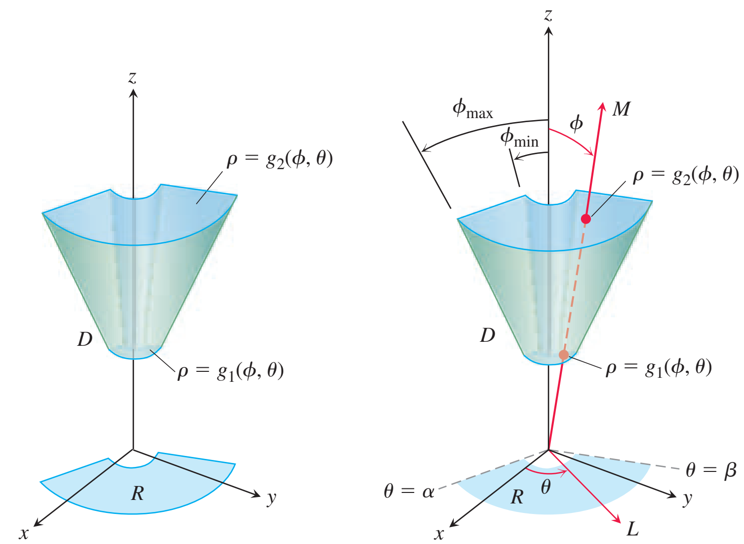
How to Integrate in Spherical Coordinates

To evaluate

$$\iiint_D f(\rho, \phi, \theta)\,dV$$

over a region D in space in spherical coordinates, integrating first with respect to ρ, then with respect to ϕ, and finally with respect to θ, take the following steps.

1. *Sketch.* Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.



2. *Find the ρ-limits of integration*. Draw a ray M from the origin through D, making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration shown in the above figure.
3. *Find the ϕ-limits of integration*. For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the ϕ-limits of integration.
4. *Find the θ-limits of integration*. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

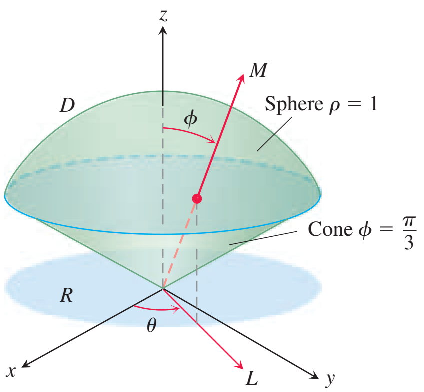


FIGURE 15.53 The ice cream cone in Example 5.

EXAMPLE 5 Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$.

Solution The volume is $V = \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$, the integral of $f(\rho, \phi, \theta) = 1$ over D.

To find the limits of integration for evaluating the integral, we begin by sketching D and its projection R on the xy-plane (Figure 15.53).

The ρ-limits of integration. We draw a ray M from the origin through D, making an angle ϕ with the positive z-axis. We also draw L, the projection of M on the xy-plane, along with the angle θ that L makes with the positive x-axis. Ray M enters D at $\rho = 0$ and leaves at $\rho = 1$.

The ϕ-limits of integration. The cone $\phi = \pi/3$ makes an angle of $\pi/3$ with the positive z-axis. For any given θ, the angle ϕ can run from $\phi = 0$ to $\phi = \pi/3$.

The θ-limits of integration. The ray L sweeps over R as θ runs from 0 to 2π. The volume is

$$\begin{aligned} V &= \iiint_D \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}. \end{aligned}$$

EXAMPLE 6 A solid of constant density $\delta = 1$ occupies the region D in Example 5. Find the solid's moment of inertia about the z-axis.

Solution In rectangular coordinates, the moment is

$$I_z = \iiint_D (x^2 + y^2)\, dV.$$

In spherical coordinates, $x^2 + y^2 = (\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \rho^2 \sin^2\phi$. Hence,

$$I_z = \iiint_D (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \iiint_D \rho^4 \sin^3\phi\, d\rho\, d\phi\, d\theta.$$

For the region D in Example 5, this becomes

$$I_z = \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^4 \sin^3\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi/3}\left[\frac{\rho^5}{5}\right]_0^1 \sin^3\phi \, d\phi \, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi/3}(1-\cos^2\phi)\sin\phi \, d\phi \, d\theta = \frac{1}{5}\int_0^{2\pi}\left[-\cos\phi + \frac{\cos^3\phi}{3}\right]_0^{\pi/3} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\left(-\frac{1}{2} + 1 + \frac{1}{24} - \frac{1}{3}\right) d\theta = \frac{1}{5}\int_0^{2\pi}\frac{5}{24}d\theta = \frac{1}{24}(2\pi) = \frac{\pi}{12}.$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

In the next section we offer a more general procedure for determining dV in cylindrical and spherical coordinates. The results, of course, will be the same.

Exercises 15.7

Evaluating Integrals in Cylindrical Coordinates

Evaluate the cylindrical coordinate integrals in Exercises 1–6.

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta$

2. $\displaystyle\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\,r\,dr\,d\theta$

3. $\displaystyle\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\,r\,dr\,d\theta$

4. $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta$

5. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta$

6. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}(r^2\sin^2\theta + z^2)\,dz\,r\,dr\,d\theta$

Changing the Order of Integration in Cylindrical Coordinates

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in Exercises 7–10.

7. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\,dr\,dz\,d\theta$

8. $\displaystyle\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\,dr\,d\theta\,dz$

9. $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}(r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz$

10. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta + 1)\,r\,d\theta\,dz\,dr$

11. Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

    a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$

12. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

    a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$

Finding Iterated Integrals in Cylindrical Coordinates

13. Give the limits of integration for evaluating the integral

$$\iiint f(r, \theta, z)\,dz\,r\,dr\,d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos\theta$, and on top by the paraboloid $z = 3r^2$.

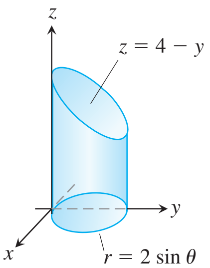
14. Convert the integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x}(x^2 + y^2)\, dz\, dx\, dy$$

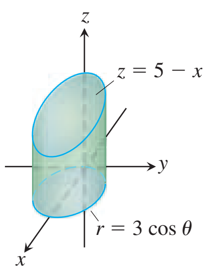
to an equivalent integral in cylindrical coordinates and evaluate the result.

In Exercises 15–20, set up the iterated integral for evaluating $\iiint_D f(r, \theta, z)\, dz\, r\, dr\, d\theta$ over the given region D.

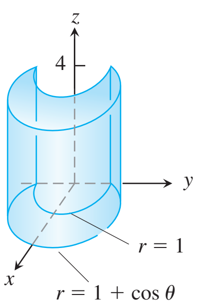
15. D is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$.



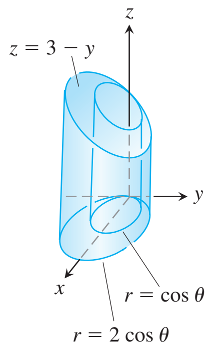
16. D is the right circular cylinder whose base is the circle $r = 3\cos\theta$ and whose top lies in the plane $z = 5 - x$.



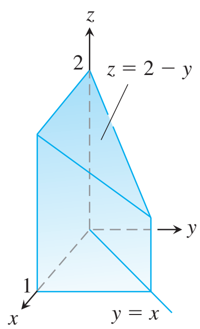
17. D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



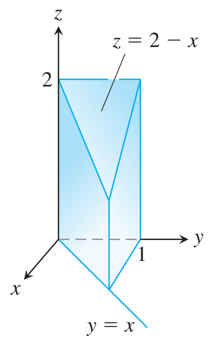
18. D is the solid right cylinder whose base is the region between the circles $r = \cos\theta$ and $r = 2\cos\theta$ and whose top lies in the plane $z = 3 - y$.



19. D is the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 2 - y$.



20. D is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$.



Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $\displaystyle\int_{0}^{\pi}\int_{0}^{\pi}\int_{0}^{2\sin\phi}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

22. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/4}\int_{0}^{2}(\rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

23. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{(1-\cos\phi)/2}\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

24. $\displaystyle\int_{0}^{3\pi/2}\int_{0}^{\pi}\int_{0}^{1}5\rho^3\sin^3\phi\, d\rho\, d\phi\, d\theta$

25. $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/3}\int_{\sec\phi}^{2}3\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

26. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Changing the Order of Integration in Spherical Coordinates

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders give the same value and are occasionally easier to evaluate. Evaluate the integrals in Exercises 27–30.

27. $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3\sin 2\phi\,d\phi\,d\theta\,d\rho$

28. $\displaystyle\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2\sin\phi\,d\theta\,d\rho\,d\phi$

29. $\displaystyle\int_0^{1}\int_0^{\pi}\int_0^{\pi/4} 12\rho\sin^3\phi\,d\phi\,d\theta\,d\rho$

30. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi$

31. Let D be the region in Exercise 11. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho\,d\phi\,d\theta$ **b.** $d\phi\,d\rho\,d\theta$

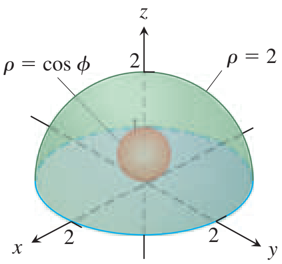
32. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho\,d\phi\,d\theta$ **b.** $d\phi\,d\rho\,d\theta$

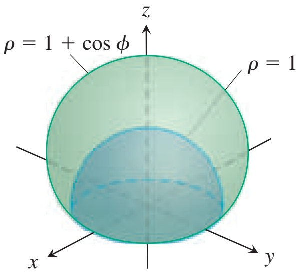
Finding Iterated Integrals in Spherical Coordinates

In Exercises 33–38, **(a)** find the spherical coordinate limits for the integral that calculates the volume of the given solid and then **(b)** evaluate the integral.

33. The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, z \geq 0$



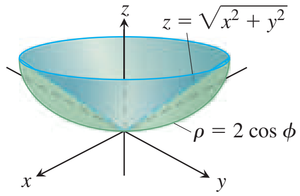
34. The solid bounded below by the hemisphere $\rho = 1, z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos\phi$



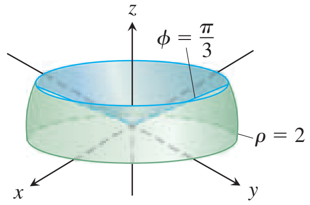
35. The solid enclosed by the cardioid of revolution $\rho = 1 - \cos\phi$

36. The upper portion cut from the solid in Exercise 35 by the xy-plane

37. The solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$



38. The solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$



Finding Triple Integrals

39. Set up triple integrals for the volume of the sphere $\rho = 2$ in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates.

40. Let D be the region in the first octant that is bounded below by the cone $\phi = \pi/4$ and above by the sphere $\rho = 3$. Express the volume of D as an iterated triple integral in **(a)** cylindrical and **(b)** spherical coordinates. Then **(c)** find V.

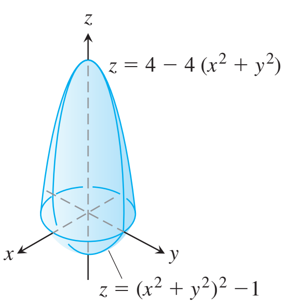
41. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates. Then **(d)** find the volume by evaluating one of the three triple integrals.

42. Express the moment of inertia I_z of the solid hemisphere $x^2 + y^2 + z^2 \leq 1, z \geq 0$, as an iterated integral in **(a)** cylindrical and **(b)** spherical coordinates. Then **(c)** find I_z.

Volumes

Find the volumes of the solids in Exercises 43–48.

43.



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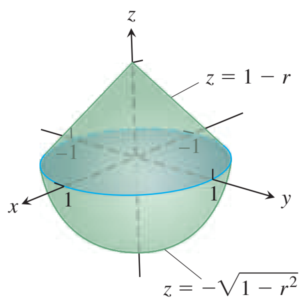

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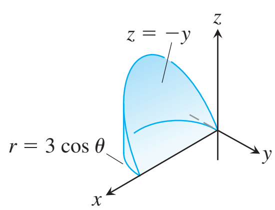

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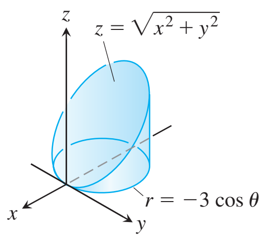

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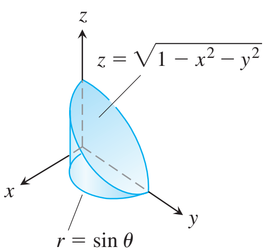

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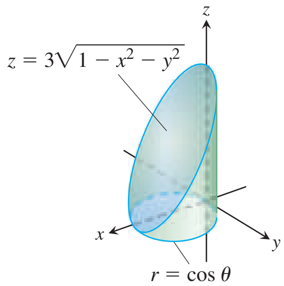


49. Sphere and cones Find the volume of the portion of the solid sphere $\rho \le a$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.

50. Sphere and half-planes Find the volume of the region cut from the solid sphere $\rho \le a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

51. Sphere and plane Find the volume of the smaller region cut from the solid sphere $\rho \le 2$ by the plane $z = 1$.

52. Cone and planes Find the volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

53. Cylinder and paraboloid Find the volume of the region bounded below by the plane $z = 0$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

54. Cylinder and paraboloids Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.

55. Cylinder and cones Find the volume of the solid cut from the thick-walled cylinder $1 \le x^2 + y^2 \le 2$ by the cones $z = \pm\sqrt{x^2 + y^2}$.

56. Sphere and cylinder Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

57. Cylinder and planes Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

58. Cylinder and planes Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + y + z = 4$.

59. Region trapped by paraboloids Find the volume of the region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.

60. Paraboloid and cylinder Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane, and lying *outside* the cylinder $x^2 + y^2 = 1$.

61. Cylinder and sphere Find the volume of the region cut from the solid cylinder $x^2 + y^2 \le 1$ by the sphere $x^2 + y^2 + z^2 = 4$.

62. Sphere and paraboloid Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Average Values

63. Find the average value of the function $f(r, \theta, z) = r$ over the region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.

64. Find the average value of the function $f(r, \theta, z) = r$ over the solid ball bounded by the sphere $r^2 + z^2 = 1$. (This is the sphere $x^2 + y^2 + z^2 = 1$.)

65. Find the average value of the function $f(\rho, \phi, \theta) = \rho$ over the solid ball $\rho \le 1$.

66. Find the average value of the function $f(\rho, \phi, \theta) = \rho\cos\phi$ over the solid upper ball $\rho \le 1$, $0 \le \phi \le \pi/2$.

Masses, Moments, and Centroids

67. Center of mass A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r$, $r \ge 0$, and on the sides by the cylinder $r = 1$. Find the center of mass.

68. Centroid Find the centroid of the region in the first octant that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the plane $z = 0$, and on the sides by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0$ and $y = 0$.

69. Centroid Find the centroid of the solid in Exercise 38.

70. Centroid Find the centroid of the solid bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/4$.

71. Centroid Find the centroid of the region that is bounded above by the surface $z = \sqrt{r}$, on the sides by the cylinder $r = 4$, and below by the xy-plane.

72. Centroid Find the centroid of the region cut from the solid ball $r^2 + z^2 \le 1$ by the half-planes $\theta = -\pi/3$, $r \ge 0$, and $\theta = \pi/3$, $r \ge 0$.

73. Moment of inertia of solid cone Find the moment of inertia of a right circular cone of base radius 1 and height 1 about an axis through the vertex parallel to the base. (Take $\delta = 1$.)

74. Moment of inertia of solid sphere Find the moment of inertia of a solid sphere of radius a about a diameter. (Take $\delta = 1$.)

75. Moment of inertia of solid cone Find the moment of inertia of a right circular cone of base radius a and height h about its axis. (*Hint:* Place the cone with its vertex at the origin and its axis along the z-axis.)

76. Variable density A solid is bounded on the top by the paraboloid $z = r^2$, on the bottom by the plane $z = 0$, and on the sides by the cylinder $r = 1$. Find the center of mass and the moment of inertia about the z-axis if the density is

a. $\delta(r, \theta, z) = z$ **b.** $\delta(r, \theta, z) = r$.

77. Variable density A solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the center of mass and the moment of inertia about the z-axis if the density is

a. $\delta(r, \theta, z) = z$ **b.** $\delta(r, \theta, z) = z^2$.

78. Variable density A solid ball is bounded by the sphere $\rho = a$. Find the moment of inertia about the z-axis if the density is

a. $\delta(\rho, \phi, \theta) = \rho^2$ **b.** $\delta(\rho, \phi, \theta) = r = \rho\sin\phi$.

79. Centroid of solid semiellipsoid Show that the centroid of the solid semiellipsoid of revolution $(r^2/a^2) + (z^2/h^2) \le 1$, $z \ge 0$, lies on the z-axis three-eighths of the way from the base to the top. The special case $h = a$ gives a solid hemisphere. Thus, the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base to the top.

80. Centroid of solid cone Show that the centroid of a solid right circular cone is one-fourth of the way from the base to the vertex. (In general, the centroid of a solid cone or pyramid is one-fourth of the way from the centroid of the base to the vertex.)

81. Density of center of a planet A planet is in the shape of a sphere of radius R and total mass M with spherically symmetric density distribution that increases linearly as one approaches its center.

What is the density at the center of this planet if the density at its edge (surface) is taken to be zero?

82. Mass of planet's atmosphere A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

Theory and Examples

83. Vertical planes in cylindrical coordinates

a. Show that planes perpendicular to the x-axis have equations of the form $r = a \sec\theta$ in cylindrical coordinates.

b. Show that planes perpendicular to the y-axis have equations of the form $r = b \csc\theta$.

84. (*Continuation of Exercise 83.*) Find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c$, $c \neq 0$.

85. Symmetry What symmetry will you find in a surface that has an equation of the form $r = f(z)$ in cylindrical coordinates? Give reasons for your answer.

86. Symmetry What symmetry will you find in a surface that has an equation of the form $\rho = f(\phi)$ in spherical coordinates? Give reasons for your answer.

15.8 Substitutions in Multiple Integrals

This section introduces the ideas involved in coordinate transformations to evaluate multiple integrals by substitution. The method replaces complicated integrals by ones that are easier to evaluate. Substitutions accomplish this by simplifying the integrand, the limits of integration, or both. A thorough discussion of multivariable transformations and substitutions is best left to a more advanced course, but our introduction here shows how the substitutions just studied reflect the general idea derived for single integral calculus.

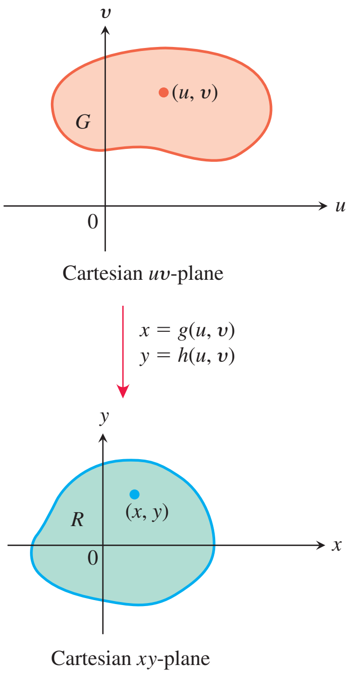


FIGURE 15.54 The equations $x = g(u, v)$ and $y = h(u, v)$ allow us to change an integral over a region R in the xy-plane into an integral over a region G in the uv-plane.

Substitutions in Double Integrals

The polar coordinate substitution of Section 15.4 is a special case of a more general substitution method for double integrals, a method that pictures changes in variables as transformations of regions.

Suppose that a region G in the uv-plane is transformed into the region R in the xy-plane by equations of the form

$$x = g(u, v), \qquad y = h(u, v),$$

as suggested in Figure 15.54. We assume the transformation is one-to-one on the interior of G. We call R the **image** of G under the transformation, and G the **preimage** of R. Any function $f(x, y)$ defined on R can be thought of as a function $f(g(u, v), h(u, v))$ defined on G as well. How is the integral of $f(x, y)$ over R related to the integral of $f(g(u, v), h(u, v))$ over G?

To gain some insight into the question, let's look again at the single variable case. To be consistent with how we are using them now, we interchange the variables x and u used in the substitution method for single integrals in Chapter 5, so the equation is stated here as

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f(g(u))\,g'(u)\,du. \qquad x = g(u),\ dx = g'(u)\,du$$

To propose an analogue for substitution in a double integral $\iint_R f(x, y)\,dx\,dy$, we need a derivative factor like $g'(u)$ as a multiplier that transforms the area element $du\,dv$ in the region G to its corresponding area element $dx\,dy$ in the region R. Let's denote this factor by J. In continuing with our analogy, it is reasonable to assume that J is a function of both variables u and v, just as g' is a function of the single variable u. Moreover, J should register instantaneous change, so partial derivatives are going to be involved in its expression. Since four partial derivatives are associated with the transforming equations $x = g(u, v)$ and $y = h(u, v)$, it is also reasonable to assume that the factor $J(u, v)$ we seek includes them all. These features are captured in the following definition constructed from the partial derivatives, and named after the German mathematician Carl Jacobi.

HISTORICAL BIOGRAPHY
Carl Gustav Jacob Jacobi
(1804–1851)

DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v)$, $y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. \tag{1}$$

The Jacobian can also be denoted by

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

to help us remember how the determinant in Equation (1) is constructed from the partial derivatives of x and y. The array of partial derivatives in Equation (1) behaves just like the derivative g' in the single variable situation. The Jacobian measures how much the transformation is expanding or contracting the area around the point (u, v). Effectively, the factor $|J|$ converts the area of the differential rectangle $du\, dv$ in G to match with its corresponding differential area $dx\, dy$ in R. We note that, in general, the value of the scaling factor $|J|$ depends on the point (u, v) in G. Our examples to follow will show how it scales the differential area $du\, dv$ for specific transformations.

With the definition of the Jacobian determinant, we can now answer our original question concerning the relationship of the integral of $f(x, y)$ over the region R to the integral of $f(g(u, v), h(u, v))$ over G.

Differential Area Change Substituting $x = g(u, v), y = h(u, v)$

$$dx\, dy = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv$$

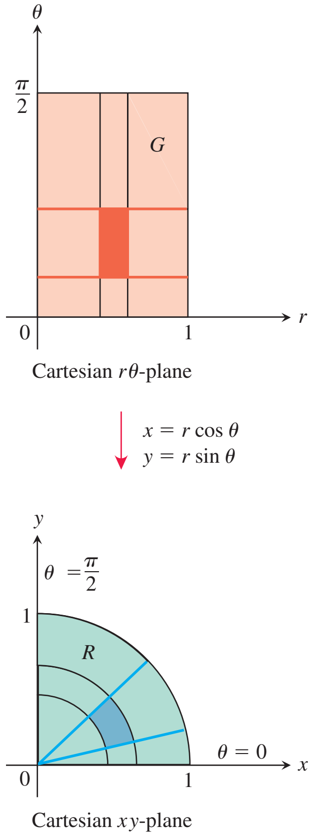


FIGURE 15.55 The equations $x = r\cos\theta$, $y = r\sin\theta$ transform G into R. The Jacobian factor r, calculated in Example 1, scales the differential rectangle $dr\, d\theta$ in G to match with the differential area element $dx\, dy$ in R.

THEOREM 3—Substitution for Double Integrals Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v)$, $y = h(u, v)$, assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv. \tag{2}$$

The derivation of Equation (2) is intricate and properly belongs to a course in advanced calculus, so we do not derive it here. We now present examples illustrating the substitution method defined by the equation.

EXAMPLE 1 Find the Jacobian for the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$, and use Equation (2) to write the Cartesian integral $\iint_R f(x, y)\, dx\, dy$ as a polar integral.

Solution Figure 15.55 shows how the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the rectangle G: $0 \le r \le 1$, $0 \le \theta \le \pi/2$, into the quarter circle R bounded by $x^2 + y^2 = 1$ in the first quadrant of the xy-plane.

For polar coordinates, we have r and θ in place of u and v. With $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian is

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\[2mm] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Since we assume $r \ge 0$ when integrating in polar coordinates, $|J(r,\theta)| = |r| = r$, so that Equation (2) gives

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta. \tag{3}$$

This is the same formula we derived independently using a geometric argument for polar area in Section 15.4.

Notice that the integral on the right-hand side of Equation (3) is not the integral of $f(r\cos\theta, r\sin\theta)$ over a region in the polar coordinate plane. It is the integral of the product of $f(r\cos\theta, r\sin\theta)$ and r over a region G in the *Cartesian* $r\theta$-plane. ■

Here is an example of a substitution in which the image of a rectangle under the coordinate transformation is a trapezoid. Transformations like this one are called **linear transformations** and their Jacobians are constant throughout G.

EXAMPLE 2 Evaluate

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x-y}{2}\,dx\,dy$$

by applying the transformation

$$u = \frac{2x-y}{2}, \qquad v = \frac{y}{2} \tag{4}$$

and integrating over an appropriate region in the uv-plane.

Solution We sketch the region R of integration in the xy-plane and identify its boundaries (Figure 15.56).

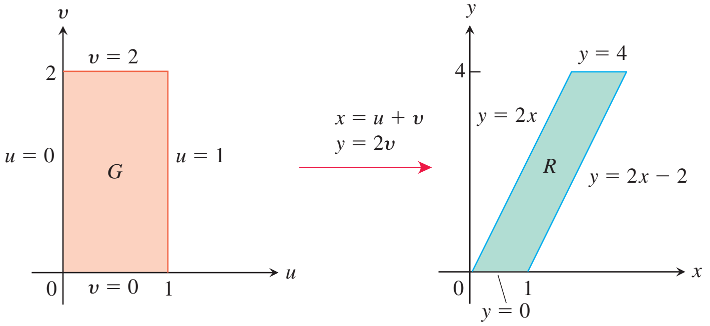


FIGURE 15.56 The equations $x = u + v$ and $y = 2v$ transform G into R. Reversing the transformation by the equations $u = (2x - y)/2$ and $v = y/2$ transforms R into G (Example 2).

To apply Equation (2), we need to find the corresponding uv-region G and the Jacobian of the transformation. To find them, we first solve Equations (4) for x and y in terms of u and v. From those equations it is easy to find algebraically that

$$x = u + v, \qquad y = 2v. \tag{5}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of R (Figure 15.56).

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$

From Equations (5) the Jacobian of the transformation is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial}{\partial u}(u + v) & \dfrac{\partial}{\partial v}(u + v) \\ \dfrac{\partial}{\partial u}(2v) & \dfrac{\partial}{\partial v}(2v) \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2.$$

We now have everything we need to apply Equation (2):

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy = \int_{v=0}^{v=2} \int_{u=0}^{u=1} u\, |J(u, v)|\, du\, dv$$

$$= \int_0^2 \int_0^1 (u)(2)\, du\, dv = \int_0^2 \Big[u^2\Big]_0^1 dv = \int_0^2 dv = 2. \blacksquare$$

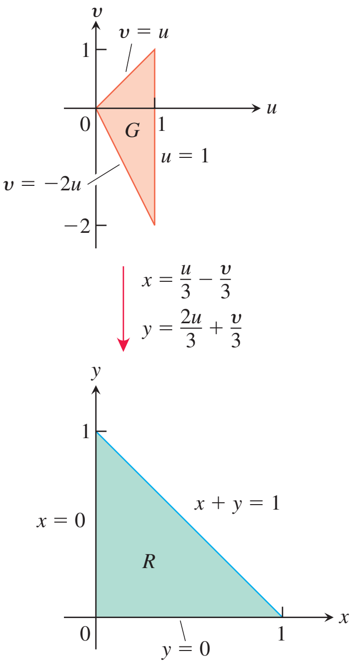


FIGURE 15.57 The equations $x = (u/3) - (v/3)$ and $y = (2u/3) + (v/3)$ transform G into R. Reversing the transformation by the equations $u = x + y$ and $v = y - 2x$ transforms R into G (Example 3).

EXAMPLE 3 Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x + y}\,(y - 2x)^2\, dy\, dx.$$

Solution We sketch the region R of integration in the xy-plane and identify its boundaries (Figure 15.57). The integrand suggests the transformation $u = x + y$ and $v = y - 2x$. Routine algebra produces x and y as functions of u and v:

$$x = \frac{u}{3} - \frac{v}{3}, \qquad y = \frac{2u}{3} + \frac{v}{3}. \tag{6}$$

From Equations (6), we can find the boundaries of the uv-region G (Figure 15.57).

xy*-equations for the boundary of *R	**Corresponding *uv*-equations for the boundary of *G***	**Simplified *uv*-equations**
$x + y = 1$	$\left(\dfrac{u}{3} - \dfrac{v}{3}\right) + \left(\dfrac{2u}{3} + \dfrac{v}{3}\right) = 1$	$u = 1$
$x = 0$	$\dfrac{u}{3} - \dfrac{v}{3} = 0$	$v = u$
$y = 0$	$\dfrac{2u}{3} + \dfrac{v}{3} = 0$	$v = -2u$

The Jacobian of the transformation in Equations (6) is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3} & -\dfrac{1}{3} \\[2ex] \dfrac{2}{3} & \dfrac{1}{3} \end{vmatrix} = \frac{1}{3}.$$

Applying Equation (2), we evaluate the integral:

$$\begin{aligned}
\int_0^1 \int_0^{1-x} \sqrt{x+y}\,(y-2x)^2\,dy\,dx &= \int_{u=0}^{u=1} \int_{v=-2u}^{v=u} u^{1/2} v^2 \,|J(u, v)|\, dv\,du \\
&= \int_0^1 \int_{-2u}^{u} u^{1/2} v^2 \left(\frac{1}{3}\right) dv\,du = \frac{1}{3} \int_0^1 u^{1/2} \left[\frac{1}{3} v^3\right]_{v=-2u}^{v=u} du \\
&= \frac{1}{9} \int_0^1 u^{1/2} (u^3 + 8u^3)\,du = \int_0^1 u^{7/2}\,du = \frac{2}{9} u^{9/2} \bigg]_0^1 = \frac{2}{9}.
\end{aligned}$$

In the next example we illustrate a nonlinear transformation of coordinates resulting from simplifying the form of the integrand. Like the polar coordinates' transformation, nonlinear transformations can map a straight-line boundary of a region into a curved boundary (or vice versa with the inverse transformation). In general, nonlinear transformations are more complex to analyze than linear ones, and a complete treatment is left to a more advanced course.

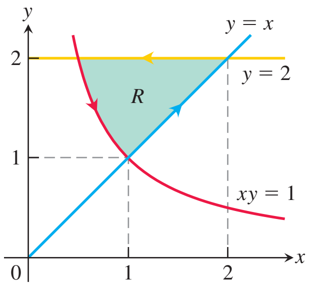


FIGURE 15.58 The region of integration R in Example 4.

EXAMPLE 4 Evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\,dx\,dy.$$

Solution The square root terms in the integrand suggest that we might simplify the integration by substituting $u = \sqrt{xy}$ and $v = \sqrt{y/x}$. Squaring these equations, we readily have $u^2 = xy$ and $v^2 = y/x$, which imply that $u^2v^2 = y^2$ and $u^2/v^2 = x^2$. So we obtain the transformation (in the same ordering of the variables as discussed before)

$$x = \frac{u}{v} \qquad \text{and} \qquad y = uv,$$

with $u > 0$ and $v > 0$. Let's first see what happens to the integrand itself under this transformation. The Jacobian of the transformation is not constant and we find

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{v} & \dfrac{-u}{v^2} \\[2ex] v & u \end{vmatrix} = \frac{2u}{v}.$$

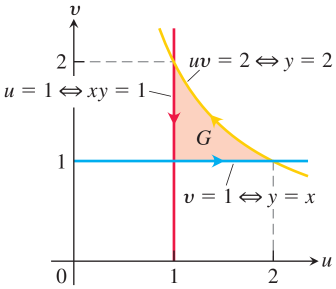


FIGURE 15.59 The boundaries of the region G correspond to those of region R in Figure 15.58. Notice as we move counterclockwise around the region R, we also move counterclockwise around the region G. The inverse transformation equations $u = \sqrt{xy}$, $v = \sqrt{y/x}$ produce the region G from the region R.

If G is the region of integration in the uv-plane, then by Equation (2) the transformed double integral under the substitution is

$$\iint_R \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\,dx\,dy = \iint_G v e^u \frac{2u}{v}\,du\,dv = \iint_G 2ue^u\,du\,dv.$$

The transformed integrand function is easier to integrate than the original one, so we proceed to determine the limits of integration for the transformed integral.

The region of integration R of the original integral in the xy-plane is shown in Figure 15.58. From the substitution equations $u = \sqrt{xy}$ and $v = \sqrt{y/x}$, we see that the image of the left-hand boundary $xy = 1$ for R is the vertical line segment $u = 1$, $2 \geq v \geq 1$, in G (see Figure 15.59). Likewise, the right-hand boundary $y = x$ of R maps to the horizontal line segment $v = 1$, $1 \leq u \leq 2$, in G. Finally, the horizontal top boundary $y = 2$ of R

maps to $uv = 2$, $1 \leq v \leq 2$, in G. As we move counterclockwise around the boundary of the region R, we also move counterclockwise around the boundary of G, as shown in Figure 15.59. Knowing the region of integration G in the uv-plane, we can now write equivalent iterated integrals:

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy = \int_1^2 \int_1^{2/u} 2ue^u\, dv\, du. \qquad \text{Note the order of integration.}$$

We now evaluate the transformed integral on the right-hand side,

$$\begin{aligned}
\int_1^2 \int_1^{2/u} 2ue^u\, dv\, du &= 2\int_1^2 \Big[vue^u \Big]_{v=1}^{v=2/u} du \\
&= 2\int_1^2 (2e^u - ue^u)\, du \\
&= 2\int_1^2 (2 - u)e^u\, du \\
&= 2\Big[(2 - u)e^u + e^u \Big]_{u=1}^{u=2} \qquad \text{Integrate by parts.} \\
&= 2(e^2 - (e + e)) = 2e(e - 2).
\end{aligned}$$

Substitutions in Triple Integrals

The cylindrical and spherical coordinate substitutions in Section 15.7 are special cases of a substitution method that pictures changes of variables in triple integrals as transformations of three-dimensional regions. The method is like the method for double integrals given by Equation (2) except that now we work in three dimensions instead of two.

Suppose that a region G in uvw-space is transformed one-to-one into the region D in xyz-space by differentiable equations of the form

$$x = g(u, v, w), \qquad y = h(u, v, w), \qquad z = k(u, v, w),$$

as suggested in Figure 15.60. Then any function $F(x, y, z)$ defined on D can be thought of as a function

$$F(g(u, v, w), h(u, v, w), k(u, v, w)) = H(u, v, w)$$

defined on G. If g, h, and k have continuous first partial derivatives, then the integral of $F(x, y, z)$ over D is related to the integral of $H(u, v, w)$ over G by the equation

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(u, v, w)\, |J(u, v, w)|\, du\, dv\, dw. \tag{7}$$

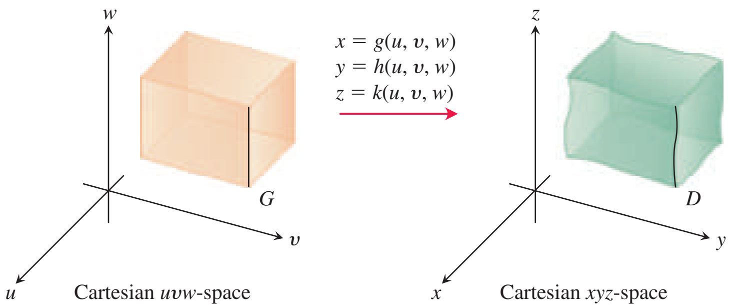


FIGURE 15.60 The equations $x = g(u, v, w)$, $y = h(u, v, w)$, and $z = k(u, v, w)$ allow us to change an integral over a region D in Cartesian xyz-space into an integral over a region G in Cartesian uvw-space using Equation (7).

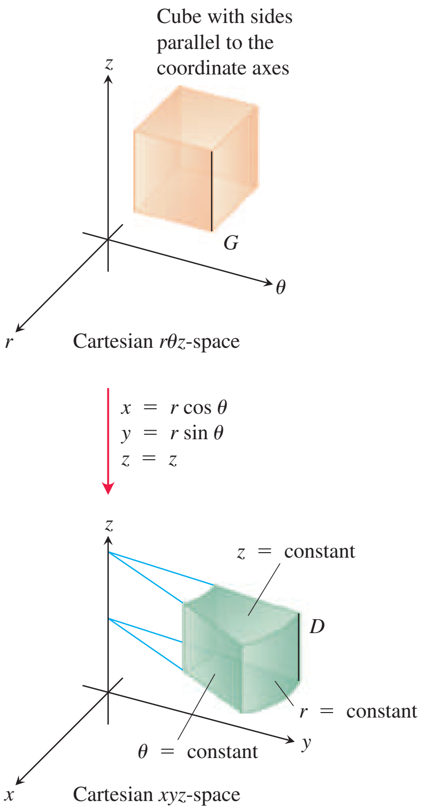


FIGURE 15.61 The equations $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$ transform the cube G into a cylindrical wedge D.

The factor $J(u, v, w)$, whose absolute value appears in this equation, is the **Jacobian determinant**

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

This determinant measures how much the volume near a point in G is being expanded or contracted by the transformation from (u, v, w) to (x, y, z) coordinates. As in the two-dimensional case, the derivation of the change-of-variable formula in Equation (7) is omitted.

For cylindrical coordinates, r, θ, and z take the place of u, v, and w. The transformation from Cartesian $r\theta z$-space to Cartesian xyz-space is given by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

(Figure 15.61). The Jacobian of the transformation is

$$J(r, \theta, z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\,dx\,dy\,dz = \iiint_G H(r, \theta, z)\,|r|\,dr\,d\theta\,dz.$$

We can drop the absolute value signs because $r \geq 0$.

For spherical coordinates, ρ, ϕ, and θ take the place of u, v, and w. The transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is given by

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi$$

(Figure 15.62). The Jacobian of the transformation (see Exercise 23) is

$$J(\rho, \phi, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2\sin\phi.$$

The corresponding version of Equation (7) is

$$\iiint_D F(x, y, z)\,dx\,dy\,dz = \iiint_G H(\rho, \phi, \theta)\,|\rho^2\sin\phi|\,d\rho\,d\phi\,d\theta.$$

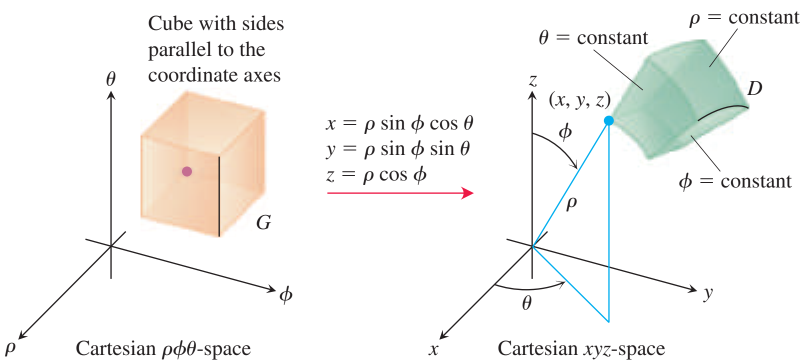


FIGURE 15.62 The equations $x = \rho \sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$ transform the cube G into the spherical wedge D.

We can drop the absolute value signs because $\sin\phi$ is never negative for $0 \le \phi \le \pi$. Note that this is the same result we obtained in Section 15.7.

Here is an example of another substitution. Although we could evaluate the integral in this example directly, we have chosen it to illustrate the substitution method in a simple (and fairly intuitive) setting.

EXAMPLE 5 Evaluate

$$\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dx\, dy\, dz$$

by applying the transformation

$$u = (2x - y)/2, \qquad v = y/2, \qquad w = z/3 \tag{8}$$

and integrating over an appropriate region in uvw-space.

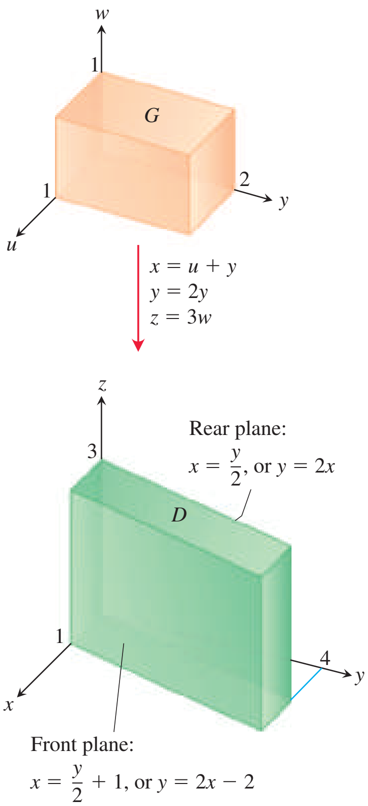


FIGURE 15.63 The equations $x = u + v$, $y = 2v$, and $z = 3w$ transform G into D. Reversing the transformation by the equations $u = (2x - y)/2$, $v = y/2$, and $w = z/3$ transforms D into G (Example 5).

Solution We sketch the region D of integration in xyz-space and identify its boundaries (Figure 15.63). In this case, the bounding surfaces are planes.

To apply Equation (7), we need to find the corresponding uvw-region G and the Jacobian of the transformation. To find them, we first solve Equations (8) for x, y, and z in terms of u, v, and w. Routine algebra gives

$$x = u + v, \qquad y = 2v, \qquad z = 3w. \tag{9}$$

We then find the boundaries of G by substituting these expressions into the equations for the boundaries of D:

xyz*-equations for the boundary of *D	**Corresponding *uvw*-equations for the boundary of *G***	**Simplified *uvw*-equations**
$x = y/2$	$u + v = 2v/2 = v$	$u = 0$
$x = (y/2) + 1$	$u + v = (2v/2) + 1 = v + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned}
&\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3} \right) dx\, dy\, dz \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w) \left| J(u, v, w) \right| du\, dv\, dw \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w)(6)\, du\, dv\, dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv\, dw \\
&= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\, dw = 6 \int_0^1 \left[\frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w)\, dw \\
&= 6 \left[w + w^2 \right]_0^1 = 6(2) = 12.
\end{aligned}$$

Exercises 15.8

Jacobians and Transformed Regions in the Plane

1. a. Solve the system

$$u = x - y, \qquad v = 2x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy-plane. Sketch the transformed region in the uv-plane.

2. a. Solve the system

$$u = x + 2y, \qquad v = x - y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x + 2y$, $v = x - y$ of the triangular region in the xy-plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv-plane.

3. a. Solve the system

$$u = 3x + 2y, \qquad v = x + 4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

4. a. Solve the system

$$u = 2x - 3y, \qquad v = -x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R in the xy-plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv-plane.

Substitutions in Double Integrals

5. Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy$$

from Example 1 directly by integration with respect to x and y to confirm that its value is 2.

6. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

7. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

8. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x - y)\, dx\, dy.$$

9. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

10. **a.** Find the Jacobian of the transformation $x = u$, $y = uv$ and sketch the region G: $1 \le u \le 2$, $1 \le uv \le 2$, in the uv-plane.

b. Then use Equation (2) to transform the integral

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx$$

into an integral over G, and evaluate both integrals.

11. **Polar moment of inertia of an elliptical plate** A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > 0$, $b > 0$, in the xy-plane. Find the first moment of the plate about the origin. (*Hint:* Use the transformation $x = ar\cos\theta$, $y = br\sin\theta$.)

12. **The area of an ellipse** The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G: $u^2 + v^2 \le 1$ in the uv-plane. Find the area this way.

13. Use the transformation in Exercise 2 to evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

14. Use the transformation $x = u + (1/2)v$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

15. Use the transformation $x = u/v$, $y = uv$ to evaluate the integral sum

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\, dx\, dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\, dx\, dy.$$

16. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\, dy\, dx.$$

(*Hint:* Show that the image of the triangular region G with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Substitutions in Triple Integrals

17. Evaluate the integral in Example 5 by integrating with respect to x, y, and z.

18. **Volume of an ellipsoid** Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw-space.)

19. Evaluate

$$\iiint |xyz|\, dx\, dy\, dz$$

over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then integrate over an appropriate region in uvw-space.)

20. Let D be the region in xyz-space defined by the inequalities

$$1 \le x \le 2, \quad 0 \le xy \le 2, \quad 0 \le z \le 1.$$

Evaluate

$$\iiint_D (x^2 y + 3xyz)\, dx\, dy\, dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

Theory and Examples

21. Find the Jacobian $\partial(x, y)/\partial(u, v)$ of the transformation

a. $x = u\cos v, \quad y = u\sin v$

b. $x = u\sin v, \quad y = u\cos v.$

22. Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ of the transformation

a. $x = u\cos v, \quad y = u\sin v, \quad z = w$

b. $x = 2u - 1, \quad y = 3v - 4, \quad z = (1/2)(w - 4).$

23. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is $\rho^2 \sin \phi$.

24. Substitutions in single integrals How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

25. Centroid of a solid semiellipsoid Assuming the result that the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base toward the top, show, by transforming the appropriate integrals, that the center of mass of a solid semiellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1$, $z \ge 0$, lies on the z-axis three-eighths of the way from the base toward the top. (You can do this without evaluating any of the integrals.)

26. Cylindrical shells In Section 6.2, we learned how to find the volume of a solid of revolution using the shell method; namely, if the region between the curve $y = f(x)$ and the x-axis from a to b $(0 < a < b)$ is revolved about the y-axis, the volume of the resulting solid is $\int_a^b 2\pi x f(x)\,dx$. Prove that finding volumes by using triple integrals gives the same result. (*Hint:* Use cylindrical coordinates with the roles of y and z changed.)

27. Inverse transform The equations $x = g(u, v)$, $y = h(u, v)$ in Figure 15.54 transform the region G in the uv-plane into the region R in the xy-plane. Since the substitution transformation is one-to-one with continuous first partial derivatives, it has an inverse transformation and there are equations $u = \alpha(x, y)$, $v = \beta(x, y)$ with continuous first partial derivatives transforming R back into G. Moreover, the Jacobian determinants of the transformations are related reciprocally by

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(\frac{\partial(u, v)}{\partial(x, y)}\right)^{-1}. \tag{10}$$

Equation (10) is proved in advanced calculus. Use it to find the area of the region R in the first quadrant of the xy-plane bounded by the lines $y = 2x$, $2y = x$, and the curves $xy = 2$, $2xy = 1$ for $u = xy$ and $v = y/x$.

28. (*Continuation of Exercise 27.*) For the region R described in Exercise 27, evaluate the integral $\iint_R y^2\,dA$.

Chapter 15 Questions to Guide Your Review

1. Define the double integral of a function of two variables over a bounded region in the coordinate plane.
2. How are double integrals evaluated as iterated integrals? Does the order of integration matter? How are the limits of integration determined? Give examples.
3. How are double integrals used to calculate areas and average values. Give examples.
4. How can you change a double integral in rectangular coordinates into a double integral in polar coordinates? Why might it be worthwhile to do so? Give an example.
5. Define the triple integral of a function $f(x, y, z)$ over a bounded region in space.
6. How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? Give an example.
7. How are double and triple integrals in rectangular coordinates used to calculate volumes, average values, masses, moments, and centers of mass? Give examples.
8. How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one of these coordinate systems to working in rectangular coordinates?
9. How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? Give examples.
10. How are substitutions in double integrals pictured as transformations of two-dimensional regions? Give a sample calculation.
11. How are substitutions in triple integrals pictured as transformations of three-dimensional regions? Give a sample calculation.

Chapter 15 Practice Exercises

Evaluating Double Iterated Integrals

In Exercises 1–4, sketch the region of integration and evaluate the double integral.

1. $\displaystyle\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy$

2. $\displaystyle\int_0^1\int_0^{x^3} e^{y/x}\,dy\,dx$

3. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt$

4. $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy$

In Exercises 5–8, sketch the region of integration and write an equivalent integral with the order of integration reversed. Then evaluate both integrals.

5. $\displaystyle\int_0^4\int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy$

6. $\displaystyle\int_0^1\int_{x^2}^{x} \sqrt{x}\,dy\,dx$

7. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy$

8. $\displaystyle\int_0^2\int_0^{4-x^2} 2x\,dy\,dx$

Evaluate the integrals in Exercises 9–12.

9. $\displaystyle\int_0^1\int_{2y}^2 4\cos\left(x^2\right) dx\, dy$ **10.** $\displaystyle\int_0^2\int_{y/2}^1 e^{x^2} dx\, dy$

11. $\displaystyle\int_0^8\int_{\sqrt[3]{x}}^2 \frac{dy\, dx}{y^4+1}$ **12.** $\displaystyle\int_0^1\int_{\sqrt[3]{y}}^1 \frac{2\pi\sin\pi x^2}{x^2} dx\, dy$

Areas and Volumes Using Double Integrals

13. Area between line and parabola Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the xy-plane.

14. Area bounded by lines and parabola Find the area of the "triangular" region in the xy-plane that is bounded on the right by the parabola $y = x^2$, on the left by the line $x + y = 2$, and above by the line $y = 4$.

15. Volume of the region under a paraboloid Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x, x = 0$, and $x + y = 2$ in the xy-plane.

16. Volume of the region under a parabolic cylinder Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy-plane.

Average Values

Find the average value of $f(x, y) = xy$ over the regions in Exercises 17 and 18.

17. The square bounded by the lines $x = 1$, $y = 1$ in the first quadrant

18. The quarter circle $x^2 + y^2 \le 1$ in the first quadrant

Polar Coordinates

Evaluate the integrals in Exercises 19 and 20 by changing to polar coordinates.

19. $\displaystyle\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\, dy\, dx}{(1+x^2+y^2)^2}$

20. $\displaystyle\int_{-1}^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln\left(x^2+y^2+1\right) dx\, dy$

21. Integrating over a lemniscate Integrate the function $f(x, y) = 1/(1 + x^2 + y^2)^2$ over the region enclosed by one loop of the lemniscate $(x^2 + y^2)^2 - (x^2 - y^2) = 0$.

22. Integrate $f(x, y) = 1/(1 + x^2 + y^2)^2$ over

a. Triangular region The triangle with vertices (0, 0), (1, 0), and $\left(1, \sqrt{3}\right)$.

b. First quadrant The first quadrant of the xy-plane.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 23–26.

23. $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(x+y+z)\, dx\, dy\, dz$

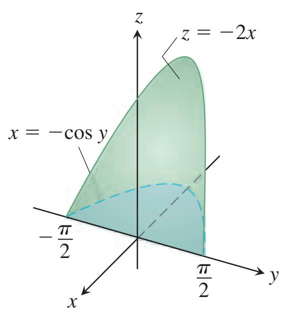
24. $\displaystyle\int_{\ln 6}^{\ln 7}\int_0^{\ln 2}\int_{\ln 4}^{\ln 5} e^{(x+y+z)}\, dz\, dy\, dx$

25. $\displaystyle\int_0^1\int_0^{x^2}\int_0^{x+y} (2x - y - z)\, dz\, dy\, dx$

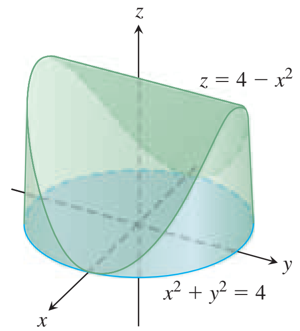
26. $\displaystyle\int_1^e\int_1^x\int_0^z \frac{2y}{z^3} dy\, dz\, dx$

Volumes and Average Values Using Triple Integrals

27. Volume Find the volume of the wedge-shaped region enclosed on the side by the cylinder $x = -\cos y, -\pi/2 \le y \le \pi/2$, on the top by the plane $z = -2x$, and below by the xy-plane.



28. Volume Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the xy-plane.



29. Average value Find the average value of $f(x, y, z) = 30xz\sqrt{x^2 + y}$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 3$, $z = 1$.

30. Average value Find the average value of ρ over the solid sphere $\rho \le a$ (spherical coordinates).

Cylindrical and Spherical Coordinates

31. Cylindrical to rectangular coordinates Convert

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3\, dz\, r\, dr\, d\theta, \qquad r \ge 0$$

to **(a)** rectangular coordinates with the order of integration $dz\, dx\, dy$ and **(b)** spherical coordinates. Then **(c)** evaluate one of the integrals.

32. Rectangular to cylindrical coordinates **(a)** Convert to cylindrical coordinates. Then **(b)** evaluate the new integral.

$$\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-(x^2+y^2)}^{(x^2+y^2)} 21xy^2\, dz\, dy\, dx$$

33. Rectangular to spherical coordinates **(a)** Convert to spherical coordinates. Then **(b)** evaluate the new integral.

$$\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 dz\, dy\, dx$$

34. Rectangular, cylindrical, and spherical coordinates Write an iterated triple integral for the integral of $f(x, y, z) = 6 + 4y$ over the region in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$, and the coordinate planes in **(a)** rectangular coordinates, **(b)** cylindrical coordinates, and **(c)** spherical coordinates. Then **(d)** find the integral of f by evaluating one of the triple integrals.

35. Cylindrical to rectangular coordinates Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2\,dz\,dr\,d\theta.$$

Arrange the order of integration to be z first, then y, then x.

36. Rectangular to cylindrical coordinates The volume of a solid is

$$\int_0^2\int_0^{\sqrt{2x-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx.$$

a. Describe the solid by giving equations for the surfaces that form its boundary.

b. Convert the integral to cylindrical coordinates but do not evaluate the integral.

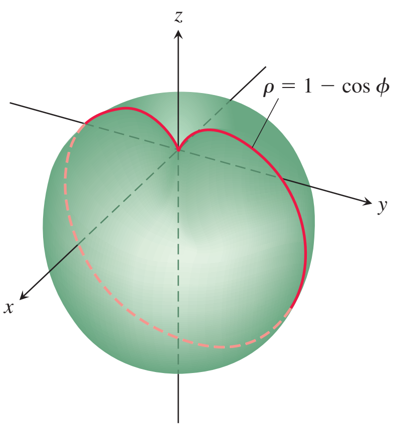
37. Spherical versus cylindrical coordinates Triple integrals involving spherical shapes do not always require spherical coordinates for convenient evaluation. Some calculations may be accomplished more easily with cylindrical coordinates. As a case in point, find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the plane $z = 2$ by using **(a)** cylindrical coordinates and **(b)** spherical coordinates.

Masses and Moments

38. Finding I_z in spherical coordinates Find the moment of inertia about the z-axis of a solid of constant density $\delta = 1$ that is bounded above by the sphere $\rho = 2$ and below by the cone $\phi = \pi/3$ (spherical coordinates).

39. Moment of inertia of a "thick" sphere Find the moment of inertia of a solid of constant density δ bounded by two concentric spheres of radii a and b $(a < b)$ about a diameter.

40. Moment of inertia of an apple Find the moment of inertia about the z-axis of a solid of density $\delta = 1$ enclosed by the spherical coordinate surface $\rho = 1 - \cos\phi$. The solid is the red curve rotated about the z-axis in the accompanying figure.



41. Centroid Find the centroid of the "triangular" region bounded by the lines $x = 2$, $y = 2$ and the hyperbola $xy = 2$ in the xy-plane.

42. Centroid Find the centroid of the region between the parabola $x + y^2 - 2y = 0$ and the line $x + 2y = 0$ in the xy-plane.

43. Polar moment Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$ bounded by the y-axis and the lines $y = 2x$ and $y = 4$ in the xy-plane.

44. Polar moment Find the polar moment of inertia about the center of a thin rectangular sheet of constant density $\delta = 1$ bounded by the lines

a. $x = \pm 2,\quad y = \pm 1$ in the xy-plane

b. $x = \pm a,\quad y = \pm b$ in the xy-plane.

(*Hint:* Find I_x. Then use the formula for I_x to find I_y, and add the two to find I_0.)

45. Inertial moment Find the moment of inertia about the x-axis of a thin plate of constant density δ covering the triangle with vertices $(0, 0)$, $(3, 0)$, and $(3, 2)$ in the xy-plane.

46. Plate with variable density Find the center of mass and the moments of inertia about the coordinate axes of a thin plate bounded by the line $y = x$ and the parabola $y = x^2$ in the xy-plane if the density is $\delta(x, y) = x + 1$.

47. Plate with variable density Find the mass and first moments about the coordinate axes of a thin square plate bounded by the lines $x = \pm 1$, $y = \pm 1$ in the xy-plane if the density is $\delta(x, y) = x^2 + y^2 + 1/3$.

48. Triangles with same inertial moment Find the moment of inertia about the x-axis of a thin triangular plate of constant density δ whose base lies along the interval $[0, b]$ on the x-axis and whose vertex lies on the line $y = h$ above the x-axis. As you will see, it does not matter where on the line this vertex lies. All such triangles have the same moment of inertia about the x-axis.

49. Centroid Find the centroid of the region in the polar coordinate plane defined by the inequalities $0 \le r \le 3, -\pi/3 \le \theta \le \pi/3$.

50. Centroid Find the centroid of the region in the first quadrant bounded by the rays $\theta = 0$ and $\theta = \pi/2$ and the circles $r = 1$ and $r = 3$.

51. a. Centroid Find the centroid of the region in the polar coordinate plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

b. Sketch the region and show the centroid in your sketch.

52. a. Centroid Find the centroid of the plane region defined by the polar coordinate inequalities $0 \le r \le a$, $-\alpha \le \theta \le \alpha$ $(0 < \alpha \le \pi)$. How does the centroid move as $\alpha \to \pi^-$?

b. Sketch the region for $\alpha = 5\pi/6$ and show the centroid in your sketch.

Substitutions

53. Show that if $u = x - y$ and $v = y$, then for any continuous f

$$\int_0^\infty\int_0^x e^{-sx}f(x-y, y)\,dy\,dx = \int_0^\infty\int_0^\infty e^{-s(u+v)}f(u, v)\,du\,dv.$$

54. What relationship must hold between the constants a, b, and c to make

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(ax^2+2bxy+cy^2)}\,dx\,dy = 1?$$

(*Hint:* Let $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$, where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$. Then $ax^2 + 2bxy + cy^2 = s^2 + t^2$.)

Chapter 15 Additional and Advanced Exercises

Volumes

1. Sand pile: double and triple integrals The base of a sand pile covers the region in the xy-plane that is bounded by the parabola $x^2 + y = 6$ and the line $y = x$. The height of the sand above the point (x, y) is x^2. Express the volume of sand as **(a)** a double integral, **(b)** a triple integral. Then **(c)** find the volume.

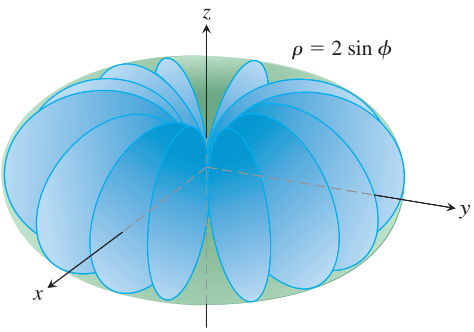
2. Water in a hemispherical bowl A hemispherical bowl of radius 5 cm is filled with water to within 3 cm of the top. Find the volume of water in the bowl.

3. Solid cylindrical region between two planes Find the volume of the portion of the solid cylinder $x^2 + y^2 \le 1$ that lies between the planes $z = 0$ and $x + y + z = 2$.

4. Sphere and paraboloid Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

5. Two paraboloids Find the volume of the region bounded above by the paraboloid $z = 3 - x^2 - y^2$ and below by the paraboloid $z = 2x^2 + 2y^2$.

6. Spherical coordinates Find the volume of the region enclosed by the spherical coordinate surface $\rho = 2 \sin \phi$ (see accompanying figure).



7. Hole in sphere A circular cylindrical hole is bored through a solid sphere, the axis of the hole being a diameter of the sphere. The volume of the remaining solid is

$$V = 2\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r\, dr\, dz\, d\theta.$$

a. Find the radius of the hole and the radius of the sphere.

b. Evaluate the integral.

8. Sphere and cylinder Find the volume of material cut from the solid sphere $r^2 + z^2 \le 9$ by the cylinder $r = 3 \sin \theta$.

9. Two paraboloids Find the volume of the region enclosed by the surfaces $z = x^2 + y^2$ and $z = (x^2 + y^2 + 1)/2$.

10. Cylinder and surface $z = xy$ Find the volume of the region in the first octant that lies between the cylinders $r = 1$ and $r = 2$ and that is bounded below by the xy-plane and above by the surface $z = xy$.

Changing the Order of Integration

11. Evaluate the integral

$$\int_0^{\infty} \frac{e^{-ax} - e^{-bx}}{x}\, dx.$$

(*Hint:* Use the relation

$$\frac{e^{-ax} - e^{-bx}}{x} = \int_a^b e^{-xy}\, dy$$

to form a double integral and evaluate the integral by changing the order of integration.)

12. a. Polar coordinates Show, by changing to polar coordinates, that

$$\int_0^{a \sin \beta}\int_{y \cot \beta}^{\sqrt{a^2-y^2}} \ln(x^2 + y^2)\, dx\, dy = a^2\beta\left(\ln a - \frac{1}{2}\right),$$

where $a > 0$ and $0 < \beta < \pi/2$.

b. Rewrite the Cartesian integral with the order of integration reversed.

13. Reducing a double to a single integral By changing the order of integration, show that the following double integral can be reduced to a single integral:

$$\int_0^x\int_0^u e^{m(x-t)} f(t)\, dt\, du = \int_0^x (x - t)e^{m(x-t)} f(t)\, dt.$$

Similarly, it can be shown that

$$\int_0^x\int_0^v\int_0^u e^{m(x-t)} f(t)\, dt\, du\, dv = \int_0^x \frac{(x - t)^2}{2} e^{m(x-t)} f(t)\, dt.$$

14. Transforming a double integral to obtain constant limits Sometimes a multiple integral with variable limits can be changed into one with constant limits. By changing the order of integration, show that

$$\int_0^1 f(x)\left(\int_0^x g(x - y)f(y)\, dy\right) dx$$
$$= \int_0^1 f(y)\left(\int_y^1 g(x - y)f(x)\, dx\right) dy$$
$$= \frac{1}{2}\int_0^1\int_0^1 g(|x - y|)f(x)f(y)\, dx\, dy.$$

Masses and Moments

15. Minimizing polar inertia A thin plate of constant density is to occupy the triangular region in the first quadrant of the xy-plane having vertices $(0, 0)$, $(a, 0)$, and $(a, 1/a)$. What value of a will minimize the plate's polar moment of inertia about the origin?

16. Polar inertia of triangular plate Find the polar moment of inertia about the origin of a thin triangular plate of constant density $\delta = 3$ bounded by the y-axis and the lines $y = 2x$ and $y = 4$ in the xy-plane.

17. Mass and polar inertia of a counterweight The counterweight of a flywheel of constant density 1 has the form of the smaller segment cut from a circle of radius a by a chord at a distance b from the center ($b < a$). Find the mass of the counterweight and its polar moment of inertia about the center of the wheel.

18. Centroid of a boomerang Find the centroid of the boomerang-shaped region between the parabolas $y^2 = -4(x - 1)$ and $y^2 = -2(x - 2)$ in the xy-plane.

Theory and Examples

19. Evaluate

$$\int_0^a \int_0^b e^{\max(b^2x^2,\, a^2y^2)}\, dy\, dx,$$

where a and b are positive numbers and

$$\max(b^2x^2, a^2y^2) = \begin{cases} b^2x^2 & \text{if } b^2x^2 \ge a^2y^2 \\ a^2y^2 & \text{if } b^2x^2 < a^2y^2. \end{cases}$$

20. Show that

$$\iint \frac{\partial^2 F(x, y)}{\partial x\, \partial y}\, dx\, dy$$

over the rectangle $x_0 \le x \le x_1$, $y_0 \le y \le y_1$, is

$$F(x_1, y_1) - F(x_0, y_1) - F(x_1, y_0) + F(x_0, y_0).$$

21. Suppose that $f(x, y)$ can be written as a product $f(x, y) = F(x)G(y)$ of a function of x and a function of y. Then the integral of f over the rectangle R: $a \le x \le b$, $c \le y \le d$ can be evaluated as a product as well, by the formula

$$\iint_R f(x, y)\, dA = \left(\int_a^b F(x)\, dx\right)\left(\int_c^d G(y)\, dy\right). \tag{1}$$

The argument is that

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_c^d \left(\int_a^b F(x)G(y)\, dx\right) dy && \text{(i)} \\ &= \int_c^d \left(G(y) \int_a^b F(x)\, dx\right) dy && \text{(ii)} \\ &= \int_c^d \left(\int_a^b F(x)\, dx\right) G(y)\, dy && \text{(iii)} \\ &= \left(\int_a^b F(x)\, dx\right) \int_c^d G(y)\, dy. && \text{(iv)} \end{aligned}$$

a. Give reasons for steps (i) through (iv).

When it applies, Equation (1) can be a time-saver. Use it to evaluate the following integrals.

b. $\displaystyle\int_0^{\ln 2} \int_0^{\pi/2} e^x \cos y\, dy\, dx$ **c.** $\displaystyle\int_1^2 \int_{-1}^1 \frac{x}{y^2}\, dx\, dy$

22. Let $D_{\mathbf{u}}f$ denote the derivative of $f(x, y) = (x^2 + y^2)/2$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$.

a. Finding average value Find the average value of $D_{\mathbf{u}}f$ over the triangular region cut from the first quadrant by the line $x + y = 1$.

b. Average value and centroid Show in general that the average value of $D_{\mathbf{u}}f$ over a region in the xy-plane is the value of $D_{\mathbf{u}}f$ at the centroid of the region.

23. The value of $\Gamma(1/2)$ The gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt,$$

extends the factorial function from the nonnegative integers to other real values. Of particular interest in the theory of differential equations is the number

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{(1/2)-1} e^{-t}\, dt = \int_0^\infty \frac{e^{-t}}{\sqrt{t}} dt. \tag{2}$$

a. If you have not yet done Exercise 41 in Section 15.4, do it now to show that

$$I = \int_0^\infty e^{-y^2}\, dy = \frac{\sqrt{\pi}}{2}.$$

b. Substitute $y = \sqrt{t}$ in Equation (2) to show that $\Gamma(1/2) = 2I = \sqrt{\pi}$.

24. Total electrical charge over circular plate The electrical charge distribution on a circular plate of radius R meters is $\sigma(r, \theta) = kr(1 - \sin\theta)$ coulomb/m^2 (k a constant). Integrate σ over the plate to find the total charge Q.

25. A parabolic rain gauge A bowl is in the shape of the graph of $z = x^2 + y^2$ from $z = 0$ to $z = 10$ in. You plan to calibrate the bowl to make it into a rain gauge. What height in the bowl would correspond to 1 in. of rain? 3 in. of rain?

26. Water in a satellite dish A parabolic satellite dish is 2 m wide and 1/2 m deep. Its axis of symmetry is tilted 30 degrees from the vertical.

a. Set up, but do not evaluate, a triple integral in rectangular coordinates that gives the amount of water the satellite dish will hold. (*Hint:* Put your coordinate system so that the satellite dish is in "standard position" and the plane of the water level is slanted.) (*Caution:* The limits of integration are not "nice.")

b. What would be the smallest tilt of the satellite dish so that it holds no water?

27. An infinite half-cylinder Let D be the interior of the infinite right circular half-cylinder of radius 1 with its single-end face suspended 1 unit above the origin and its axis the ray from (0, 0, 1) to ∞. Use cylindrical coordinates to evaluate

$$\iiint_D z(r^2 + z^2)^{-5/2}\, dV.$$

28. Hypervolume We have learned that $\int_a^b 1\, dx$ is the length of the interval $[a, b]$ on the number line (one-dimensional space), $\iint_R 1\, dA$ is the area of region R in the xy-plane (two-dimensional space), and $\iiint_D 1\, dV$ is the volume of the region D in three-dimensional space (xyz-space). We could continue: If Q is a region in 4-space ($xyzw$-space), then $\iiiint_Q 1\, dV$ is the "hypervolume" of Q. Use your generalizing abilities and a Cartesian coordinate system of 4-space to find the hypervolume inside the unit 4-dimensional sphere $x^2 + y^2 + z^2 + w^2 = 1$.

Chapter 15 Technology Application Projects

Mathematica/Maple Modules:

Take Your Chances: Try the Monte Carlo Technique for Numerical Integration in Three Dimensions
Use the Monte Carlo technique to integrate numerically in three dimensions.

Means and Moments and Exploring New Plotting Techniques, Part II
Use the method of moments in a form that makes use of geometric symmetry as well as multiple integration.

16

Integrals and Vector Fields

OVERVIEW In this chapter we extend the theory of integration over coordinate lines and planes to general curves and surfaces in space. The resulting theory of line and surface integrals gives powerful mathematical tools for science and engineering. Line integrals are used to find the work done by a force in moving an object along a path, and to find the mass of a curved wire with variable density. Surface integrals are used to find the rate of flow of a fluid across a surface. We present the fundamental theorems of vector integral calculus, and discuss their mathematical consequences and physical applications. In the final analysis, the key theorems are shown as generalized interpretations of the Fundamental Theorem of Calculus.

16.1 Line Integrals

To calculate the total mass of a wire lying along a curve in space, or to find the work done by a variable force acting along such a curve, we need a more general notion of integral than was defined in Chapter 5. We need to integrate over a curve C rather than over an interval $[a, b]$. These more general integrals are called *line integrals* (although *path* integrals might be more descriptive). We make our definitions for space curves, with curves in the xy-plane being the special case with z-coordinate identically zero.

Suppose that $f(x, y, z)$ is a real-valued function we wish to integrate over the curve C lying within the domain of f and parametrized by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$. The values of f along the curve are given by the composite function $f(g(t), h(t), k(t))$. We are going to integrate this composite with respect to arc length from $t = a$ to $t = b$. To begin, we first partition the curve C into a finite number n of subarcs (Figure 16.1). The typical subarc has length Δs_k. In each subarc we choose a point (x_k, y_k, z_k) and form the sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k,$$

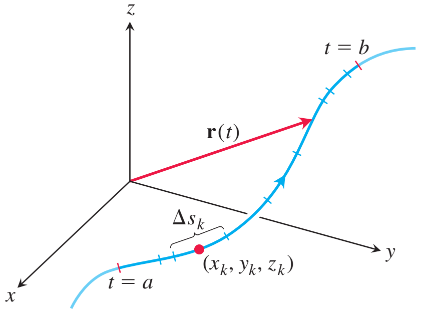


FIGURE 16.1 The curve $\mathbf{r}(t)$ partitioned into small arcs from $t = a$ to $t = b$. The length of a typical subarc is Δs_k.

which is similar to a Riemann sum. Depending on how we partition the curve C and pick (x_k, y_k, z_k) in the kth subarc, we may get different values for S_n. If f is continuous and the functions g, h, and k have continuous first derivatives, then these sums approach a limit as n increases and the lengths Δs_k approach zero. This limit gives the following definition, similar to that for a single integral. In the definition, we assume that the partition satisfies $\Delta s_k \to 0$ as $n \to \infty$.

DEFINITION If f is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, then the **line integral of f over C** is

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta s_k, \tag{1}$$

provided this limit exists.

If the curve C is smooth for $a \le t \le b$ (so $\mathbf{v} = d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$) and the function f is continuous on C, then the limit in Equation (1) can be shown to exist. We can then apply the Fundamental Theorem of Calculus to differentiate the arc length equation,

$$s(t) = \int_a^t |\mathbf{v}(\tau)|\, d\tau,$$

Eq. (3) of Section 13.3 with $t_0 = a$

to express ds in Equation (1) as $ds = |\mathbf{v}(t)|\, dt$ and evaluate the integral of f over C as

$$\frac{ds}{dt} = |\mathbf{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\,|\mathbf{v}(t)|\, dt. \tag{2}$$

Notice that the integral on the right side of Equation (2) is just an ordinary (single) definite integral, as defined in Chapter 5, where we are integrating with respect to the parameter t. The formula evaluates the line integral on the left side correctly no matter what parametrization is used, as long as the parametrization is smooth. Note that the parameter t defines a direction along the path. The starting point on C is the position $\mathbf{r}(a)$ and movement along the path is in the direction of increasing t (see Figure 16.1).

How to Evaluate a Line Integral

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b.$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\,|\mathbf{v}(t)|\, dt.$$

$$f(\mathbf{r}(t)) = f(g(t), h(t), k(t))$$

If f has the constant value 1, then the integral of f over C gives the length of C from $t = a$ to $t = b$ in Figure 16.1. We also write $f(\mathbf{r}(t))$ for the evaluation $f(g(t), h(t), k(t))$ along the curve $\mathbf{r}$.

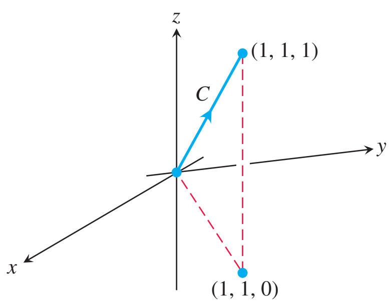


FIGURE 16.2 The integration path in Example 1.

EXAMPLE 1 Integrate $f(x, y, z) = x - 3y^2 + z$ over the line segment C joining the origin to the point (1, 1, 1) (Figure 16.2).

Solution We choose the simplest parametrization we can think of:

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 1.$$

The components have continuous first derivatives and $|\mathbf{v}(t)| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$ is never 0, so the parametrization is smooth. The integral of f over C is

$$\int_C f(x, y, z)\, ds = \int_0^1 f(t, t, t)\left(\sqrt{3}\right) dt \qquad \text{Eq. (2), } ds = |\mathbf{v}(t)|\, dt = \sqrt{3}\, dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt$$

$$= \sqrt{3}\int_0^1 (2t - 3t^2)\, dt = \sqrt{3}\left[t^2 - t^3\right]_0^1 = 0.$$

Additivity

Line integrals have the useful property that if a piecewise smooth curve C is made by joining a finite number of smooth curves $C_1, C_2, \ldots, C_n$ end to end (Section 13.1), then the integral of a function over C is the sum of the integrals over the curves that make it up:

$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \cdots + \int_{C_n} f\, ds. \tag{3}$$

EXAMPLE 2 Figure 16.3 shows another path from the origin to (1, 1, 1), the union of line segments C_1 and C_2. Integrate $f(x, y, z) = x - 3y^2 + z$ over $C_1 \cup C_2$.

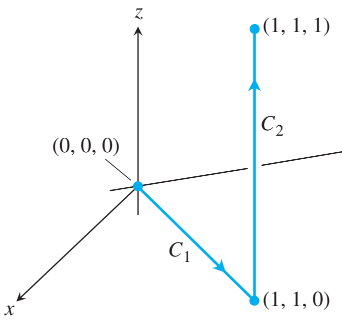


FIGURE 16.3 The path of integration in Example 2.

Solution We choose the simplest parametrizations for C_1 and C_2 we can find, calculating the lengths of the velocity vectors as we go along:

$$C_1: \quad \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$C_2: \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1; \quad |\mathbf{v}| = \sqrt{0^2 + 0^2 + 1^2} = 1.$$

With these parametrizations we find that

$$\int_{C_1 \cup C_2} f(x, y, z)\, ds = \int_{C_1} f(x, y, z)\, ds + \int_{C_2} f(x, y, z)\, ds \qquad \text{Eq. (3)}$$

$$= \int_0^1 f(t, t, 0)\sqrt{2}\, dt + \int_0^1 f(1, 1, t)(1)\, dt \qquad \text{Eq. (2)}$$

$$= \int_0^1 (t - 3t^2 + 0)\sqrt{2}\, dt + \int_0^1 (1 - 3 + t)(1)\, dt$$

$$= \sqrt{2}\left[\frac{t^2}{2} - t^3\right]_0^1 + \left[\frac{t^2}{2} - 2t\right]_0^1 = -\frac{\sqrt{2}}{2} - \frac{3}{2}.$$

Notice three things about the integrations in Examples 1 and 2. First, as soon as the components of the appropriate curve were substituted into the formula for f, the integration became a standard integration with respect to t. Second, the integral of f over $C_1 \cup C_2$ was obtained by integrating f over each section of the path and adding the results. Third, the integrals of f over C and $C_1 \cup C_2$ had different values. We investigate this third observation in Section 16.3.

> The value of the line integral along a path joining two points can change if you change the path between them.

EXAMPLE 3 Find the line integral of $f(x, y, z) = 2xy + \sqrt{z}$ over the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi$.

Solution For the helix we find, $\mathbf{v}(t) = \mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$ and $|\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$. Evaluating the function f along the path, we obtain

$$f(\mathbf{r}(t)) = f(\cos t, \sin t, t) = 2\cos t \sin t + \sqrt{t} = \sin 2t + \sqrt{t}.$$

The line integral is given by

$$\begin{aligned}\int_C f(x, y, z)\, ds &= \int_0^{\pi} \left(\sin 2t + \sqrt{t}\right)\sqrt{2}\, dt \\ &= \sqrt{2}\left[-\frac{1}{2}\cos 2t + \frac{2}{3}t^{3/2}\right]_0^{\pi} \\ &= \frac{2\sqrt{2}}{3}\pi^{3/2} \approx 5.25.\end{aligned}$$

Mass and Moment Calculations

We treat coil springs and wires as masses distributed along smooth curves in space. The distribution is described by a continuous density function $\delta(x, y, z)$ representing mass per unit length. When a curve C is parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, then x, y, and z are functions of the parameter t, the density is the function $\delta(x(t), y(t), z(t))$, and the arc length differential is given by

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

(See Section 13.3.) The spring's or wire's mass, center of mass, and moments are then calculated with the formulas in Table 16.1, with the integrations in terms of the parameter t over the interval $[a, b]$. For example, the formula for mass becomes

$$M = \int_a^b \delta(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

These formulas also apply to thin rods, and their derivations are similar to those in Section 6.6. Notice how alike the formulas are to those in Tables 15.1 and 15.2 for double and triple integrals. The double integrals for planar regions, and the triple integrals for solids, become line integrals for coil springs, wires, and thin rods.

Notice that the element of mass dm is equal to $\delta\, ds$ in the table rather than $\delta\, dV$ as in Table 15.1, and that the integrals are taken over the curve C.

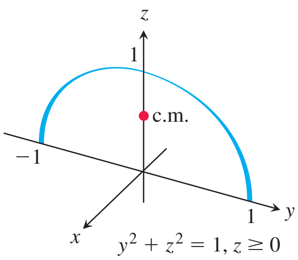


FIGURE 16.4 Example 4 shows how to find the center of mass of a circular arch of variable density.

EXAMPLE 4 A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2 + z^2 = 1$, $z \ge 0$, in the yz-plane (Figure 16.4). Find the center of the arch's mass if the density at the point (x, y, z) on the arch is $\delta(x, y, z) = 2 - z$.

Solution We know that $\bar{x} = 0$ and $\bar{y} = 0$ because the arch lies in the yz-plane with its mass distributed symmetrically about the z-axis. To find $\bar{z}$, we parametrize the circle as

$$\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \qquad 0 \le t \le \pi.$$

TABLE 16.1 Mass and moment formulas for coil springs, wires, and thin rods lying along a smooth curve C in space

Mass: $M = \int_C \delta\, ds$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z)

First moments about the coordinate planes:

$$M_{yz} = \int_C x\,\delta\, ds, \qquad M_{xz} = \int_C y\,\delta\, ds, \qquad M_{xy} = \int_C z\,\delta\, ds$$

Coordinates of the center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about axes and other lines:

$$I_x = \int_C (y^2 + z^2)\,\delta\, ds, \qquad I_y = \int_C (x^2 + z^2)\,\delta\, ds, \qquad I_z = \int_C (x^2 + y^2)\,\delta\, ds,$$

$$I_L = \int_C r^2\delta\, ds \qquad r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L$$

For this parametrization,

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2} = 1,$$

so $ds = |\mathbf{v}|\, dt = dt.$

The formulas in Table 16.1 then give

$$M = \int_C \delta\, ds = \int_C (2 - z)\, ds = \int_0^{\pi} (2 - \sin t)\, dt = 2\pi - 2$$

$$M_{xy} = \int_C z\delta\, ds = \int_C z(2 - z)\, ds = \int_0^{\pi} (\sin t)(2 - \sin t)\, dt$$

$$= \int_0^{\pi} (2 \sin t - \sin^2 t)\, dt = \frac{8 - \pi}{2} \qquad \text{Routine integration}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2} = \frac{8 - \pi}{4\pi - 4} \approx 0.57.$$

With $\bar{z}$ to the nearest hundredth, the center of mass is (0, 0, 0.57).

Line Integrals in the Plane

There is an interesting geometric interpretation for line integrals in the plane. If C is a smooth curve in the xy-plane parametrized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \le t \le b$, we generate a cylindrical surface by moving a straight line along C orthogonal to the plane, holding the line parallel to the z-axis, as in Section 12.6. If $z = f(x, y)$ is a nonnegative continuous function over a region in the plane containing the curve C, then the graph of f is a surface that lies above the plane. The cylinder cuts through this surface, forming a curve on it that lies above the curve C and follows its winding nature. The part of the cylindrical surface that lies beneath the surface curve and above the xy-plane is like a "winding wall" or "fence" standing on the curve C and orthogonal to the plane. At any point (x, y) along the curve, the height of the wall is $f(x, y)$. We show the wall in Figure 16.5, where the "top" of

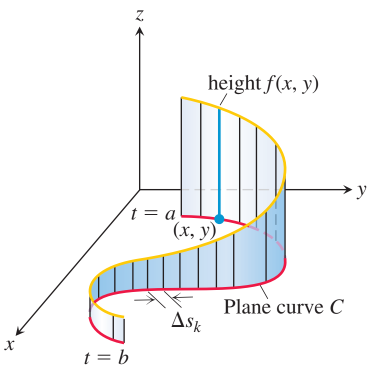


FIGURE 16.5 The line integral $\int_C f\, ds$ gives the area of the portion of the cylindrical surface or "wall" beneath $z = f(x, y) \ge 0$.

the wall is the curve lying on the surface $z = f(x, y)$. (We do not display the surface formed by the graph of f in the figure, only the curve on it that is cut out by the cylinder.) From the definition

$$\int_C f\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\,\Delta s_k,$$

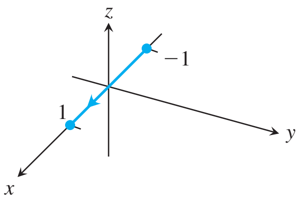
where $\Delta s_k \to 0$ as $n \to \infty$, we see that the line integral $\int_C f\,ds$ is the area of the wall shown in the figure.

Exercises 16.1

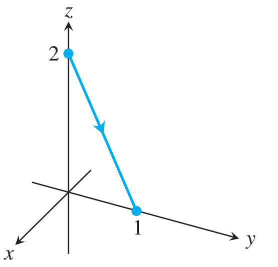
Graphs of Vector Equations

Match the vector equations in Exercises 1–8 with the graphs (a)–(h) given here.

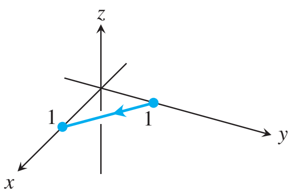
a.



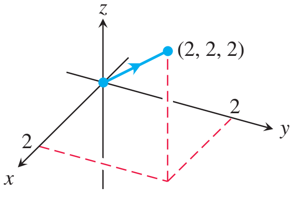
b.



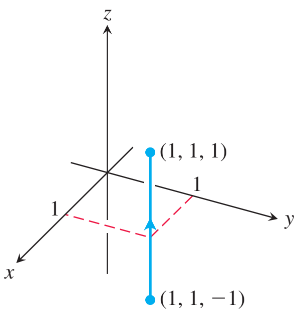
c.



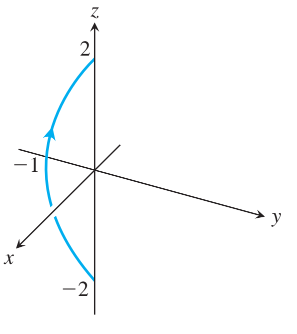
d.



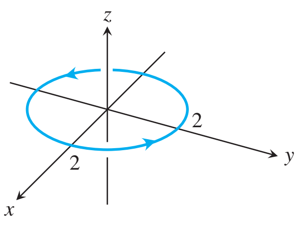
e.



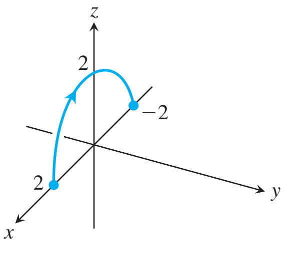
f.



g.



h.

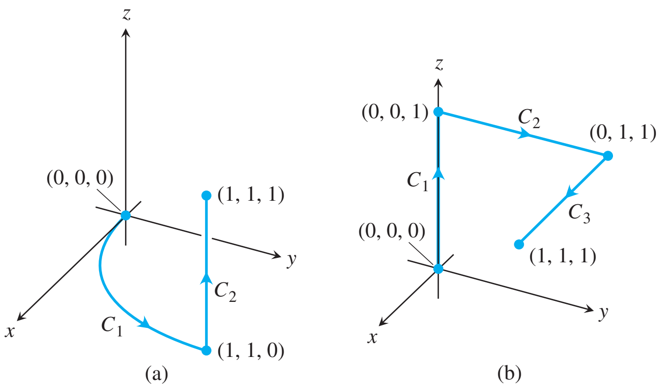


1. $\mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \le t \le 1$
2. $\mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad -1 \le t \le 1$
3. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$
4. $\mathbf{r}(t) = t\mathbf{i}, \quad -1 \le t \le 1$
5. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2$
6. $\mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k}, \quad 0 \le t \le 1$
7. $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}, \quad -1 \le t \le 1$
8. $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{k}, \quad 0 \le t \le \pi$

Evaluating Line Integrals over Space Curves

9. Evaluate $\int_C (x + y)\,ds$ where C is the straight-line segment $x = t$, $y = (1 - t)$, $z = 0$, from $(0, 1, 0)$ to $(1, 0, 0)$.
10. Evaluate $\int_C (x - y + z - 2)\,ds$ where C is the straight-line segment $x = t$, $y = (1 - t)$, $z = 1$, from $(0, 1, 1)$ to $(1, 0, 1)$.
11. Evaluate $\int_C (xy + y + z)\,ds$ along the curve $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + (2 - 2t)\mathbf{k}$, $0 \le t \le 1$.
12. Evaluate $\int_C \sqrt{x^2 + y^2}\,ds$ along the curve $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}$, $-2\pi \le t \le 2\pi$.
13. Find the line integral of $f(x, y, z) = x + y + z$ over the straight-line segment from $(1, 2, 3)$ to $(0, -1, 1)$.
14. Find the line integral of $f(x, y, z) = \sqrt{3}/(x^2 + y^2 + z^2)$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $1 \le t \le \infty$.
15. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ (see accompanying figure) given by

$$C_1: \quad \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \le t \le 1$$
$$C_2: \quad \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$$



The paths of integration for Exercises 15 and 16.

16. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from (0, 0, 0) to (1, 1, 1) (see accompanying figure) given by

$$C_1: \quad \mathbf{r}(t) = t\mathbf{k}, \quad 0 \le t \le 1$$
$$C_2: \quad \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$$
$$C_3: \quad \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$$

17. Integrate $f(x, y, z) = (x + y + z)/(x^2 + y^2 + z^2)$ over the path $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 < a \le t \le b$.

18. Integrate $f(x, y, z) = -\sqrt{x^2 + z^2}$ over the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

Line Integrals over Plane Curves

19. Evaluate $\int_C x\, ds$, where C is

a. the straight-line segment $x = t$, $y = t/2$, from (0, 0) to (4, 2).

b. the parabolic curve $x = t$, $y = t^2$, from (0, 0) to (2, 4).

20. Evaluate $\int_C \sqrt{x + 2y}\, ds$, where C is

a. the straight-line segment $x = t$, $y = 4t$, from (0, 0) to (1, 4).

b. $C_1 \cup C_2$; C_1 is the line segment from (0, 0) to (1, 0) and C_2 is the line segment from (1, 0) to (1, 2).

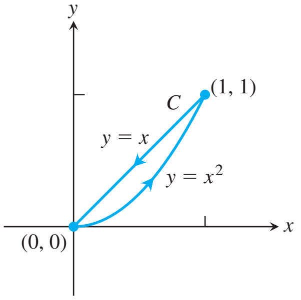
21. Find the line integral of $f(x, y) = ye^{x^2}$ along the curve $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j}$, $-1 \le t \le 2$.

22. Find the line integral of $f(x, y) = x - y + 3$ along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

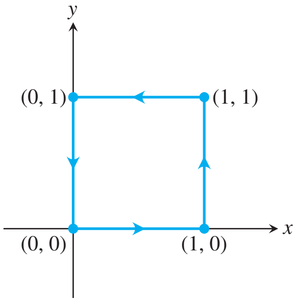
23. Evaluate $\displaystyle\int_C \frac{x^2}{y^{4/3}}\, ds$, where C is the curve $x = t^2$, $y = t^3$, for $1 \le t \le 2$.

24. Find the line integral of $f(x, y) = \sqrt{y}/x$ along the curve $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $1/2 \le t \le 1$.

25. Evaluate $\int_C \left(x + \sqrt{y}\right) ds$ where C is given in the accompanying figure.



26. Evaluate $\displaystyle\int_C \frac{1}{x^2 + y^2 + 1}\, ds$ where C is given in the accompanying figure.



In Exercises 27–30, integrate f over the given curve.

27. $f(x, y) = x^3/y$, $\quad C: \quad y = x^2/2, \quad 0 \le x \le 2$

28. $f(x, y) = \left(x + y^2\right)/\sqrt{1 + x^2}$, $\quad C: \quad y = x^2/2$ from (1, 1/2) to (0, 0)

29. $f(x, y) = x + y$, $\quad C: \quad x^2 + y^2 = 4$ in the first quadrant from (2, 0) to (0, 2)

30. $f(x, y) = x^2 - y$, $\quad C: \quad x^2 + y^2 = 4$ in the first quadrant from (0, 2) to $\left(\sqrt{2}, \sqrt{2}\right)$

31. Find the area of one side of the "winding wall" standing orthogonally on the curve $y = x^2$, $0 \le x \le 2$, and beneath the curve on the surface $f(x, y) = x + \sqrt{y}$.

32. Find the area of one side of the "wall" standing orthogonally on the curve $2x + 3y = 6$, $0 \le x \le 6$, and beneath the curve on the surface $f(x, y) = 4 + 3x + 2y$.

Masses and Moments

33. Mass of a wire Find the mass of a wire that lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}$, $0 \le t \le 1$, if the density is $\delta = (3/2)t$.

34. Center of mass of a curved wire A wire of density $\delta(x, y, z) = 15\sqrt{y + 2}$ lies along the curve $\mathbf{r}(t) = (t^2 - 1)\mathbf{j} + 2t\mathbf{k}$, $-1 \le t \le 1$. Find its center of mass. Then sketch the curve and center of mass together.

35. Mass of wire with variable density Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}$, $0 \le t \le 1$, if the density is **(a)** $\delta = 3t$ and **(b)** $\delta = 1$.

36. Center of mass of wire with variable density Find the center of mass of a thin wire lying along the curve $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (2/3)t^{3/2}\mathbf{k}$, $0 \le t \le 2$, if the density is $\delta = 3\sqrt{5 + t}$.

37. Moment of inertia of wire hoop A circular wire hoop of constant density δ lies along the circle $x^2 + y^2 = a^2$ in the xy-plane. Find the hoop's moment of inertia about the z-axis.

38. Inertia of a slender rod A slender rod of constant density lies along the line segment $\mathbf{r}(t) = t\mathbf{j} + (2 - 2t)\mathbf{k}$, $0 \le t \le 1$, in the yz-plane. Find the moments of inertia of the rod about the three coordinate axes.

39. Two springs of constant density A spring of constant density δ lies along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

a. Find I_z.

b. Suppose that you have another spring of constant density δ that is twice as long as the spring in part (a) and lies along the helix for $0 \le t \le 4\pi$. Do you expect I_z for the longer spring to be the same as that for the shorter one, or should it be different? Check your prediction by calculating I_z for the longer spring.

40. Wire of constant density A wire of constant density $\delta = 1$ lies along the curve

$$\mathbf{r}(t) = (t \cos t)\mathbf{i} + (t \sin t)\mathbf{j} + \left(2\sqrt{2}/3\right)t^{3/2}\mathbf{k}, \qquad 0 \le t \le 1.$$

Find $\bar{z}$ and I_z.

41. The arch in Example 4 Find I_x for the arch in Example 4.

42. **Center of mass and moments of inertia for wire with variable density** Find the center of mass and the moments of inertia about the coordinate axes of a thin wire lying along the curve

$$\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \qquad 0 \le t \le 2,$$

if the density is $\delta = 1/(t + 1)$.

COMPUTER EXPLORATIONS

In Exercises 43–46, use a CAS to perform the following steps to evaluate the line integrals.

a. Find $ds = |\mathbf{v}(t)|\,dt$ for the path $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$.

b. Express the integrand $f(g(t), h(t), k(t))|\mathbf{v}(t)|$ as a function of the parameter t.

c. Evaluate $\int_C f\,ds$ using Equation (2) in the text.

43. $f(x, y, z) = \sqrt{1 + 30x^2 + 10y}$; $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 3t^2\mathbf{k}$, $0 \le t \le 2$

44. $f(x, y, z) = \sqrt{1 + x^3 + 5y^3}$; $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}t^2\mathbf{j} + \sqrt{t}\mathbf{k}$, $0 \le t \le 2$

45. $f(x, y, z) = x\sqrt{y} - 3z^2$; $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j} + 5t\mathbf{k}$, $0 \le t \le 2\pi$

46. $f(x, y, z) = \left(1 + \frac{9}{4}z^{1/3}\right)^{1/4}$; $\mathbf{r}(t) = (\cos 2t)\mathbf{i} + (\sin 2t)\mathbf{j} + t^{5/2}\mathbf{k}$, $0 \le t \le 2\pi$

16.2 Vector Fields and Line Integrals: Work, Circulation, and Flux

Gravitational and electric forces have both a direction and a magnitude. They are represented by a vector at each point in their domain, producing a *vector field.* In this section we show how to compute the work done in moving an object through such a field by using a line integral involving the vector field. We also discuss velocity fields, such as the vector field representing the velocity of a flowing fluid in its domain. A line integral can be used to find the rate at which the fluid flows along or across a curve within the domain.

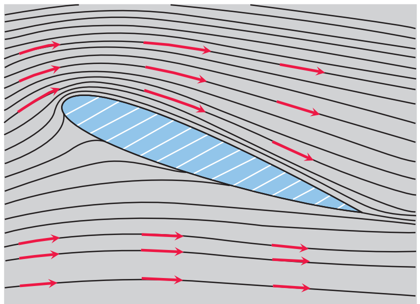


FIGURE 16.6 Velocity vectors of a flow around an airfoil in a wind tunnel.

Vector Fields

Suppose a region in the plane or in space is occupied by a moving fluid, such as air or water. The fluid is made up of a large number of particles, and at any instant of time, a particle has a velocity $\mathbf{v}$. At different points of the region at a given (same) time, these velocities can vary. We can think of a velocity vector being attached to each point of the fluid representing the velocity of a particle at that point. Such a fluid flow is an example of a *vector field.* Figure 16.6 shows a velocity vector field obtained from air flowing around an airfoil in a wind tunnel. Figure 16.7 shows a vector field of velocity vectors along the streamlines of water moving through a contracting channel. Vector fields are also associated with forces such as gravitational attraction (Figure 16.8), and with magnetic fields, electric fields, and there are also purely mathematical fields.

Generally, a **vector field** is a function that assigns a vector to each point in its domain. A vector field on a three-dimensional domain in space might have a formula like

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}.$$

The field is **continuous** if the **component functions** M, N, and P are continuous; it is **differentiable** if each of the component functions is differentiable. The formula for a field of two-dimensional vectors could look like

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}.$$

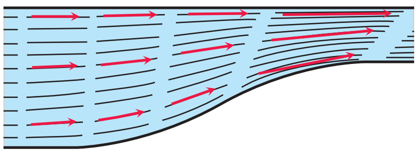


FIGURE 16.7 Streamlines in a contracting channel. The water speeds up as the channel narrows and the velocity vectors increase in length.

We encountered another type of vector field in Chapter 13. The tangent vectors $\mathbf{T}$ and normal vectors $\mathbf{N}$ for a curve in space both form vector fields along the curve. Along a curve $\mathbf{r}(t)$ they might have a component formula similar to the velocity field expression

$$\mathbf{v}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

If we attach the gradient vector ∇f of a scalar function $f(x, y, z)$ to each point of a level surface of the function, we obtain a three-dimensional field on the surface. If we attach the velocity vector to each point of a flowing fluid, we have a three-dimensional field defined on a region in space. These and other fields are illustrated in Figures 16.6–16.15. To sketch the fields, we picked a representative selection of domain points and drew the vectors attached to them. The arrows are drawn with their tails, not their heads, attached to the points where the vector functions are evaluated.

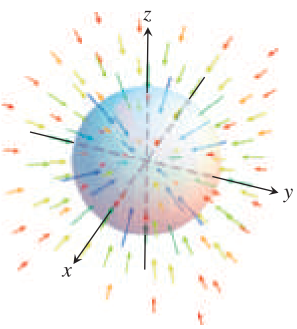


FIGURE 16.8 Vectors in a gravitational field point toward the center of mass that gives the source of the field.

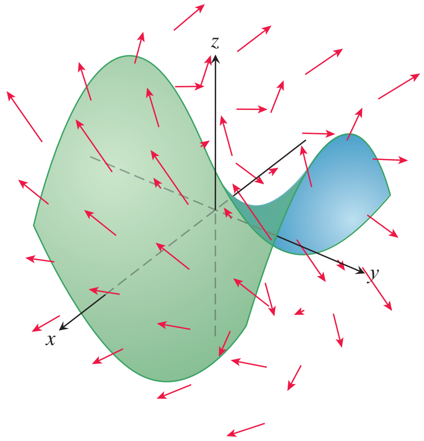


FIGURE 16.9 A surface, like a mesh net or parachute, in a vector field representing water or wind flow velocity vectors. The arrows show the direction and their lengths indicate speed.

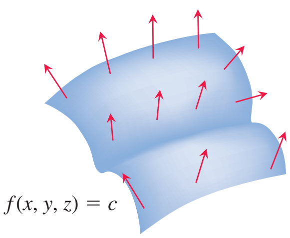


FIGURE 16.10 The field of gradient vectors ∇f on a surface $f(x, y, z) = c$.

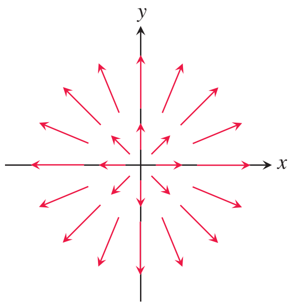


FIGURE 16.11 The radial field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ of position vectors of points in the plane. Notice the convention that an arrow is drawn with its tail, not its head, at the point where $\mathbf{F}$ is evaluated.

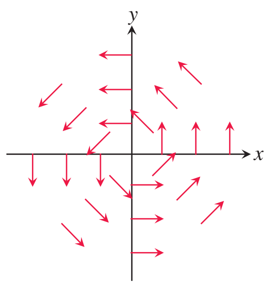


FIGURE 16.12 A "spin" field of rotating unit vectors

$$\mathbf{F} = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2)^{1/2}$$

in the plane. The field is not defined at the origin.

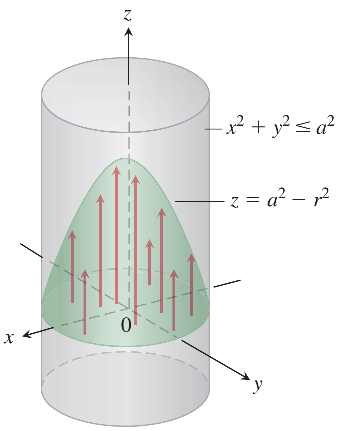


FIGURE 16.13 The flow of fluid in a long cylindrical pipe. The vectors $\mathbf{v} = (a^2 - r^2)\mathbf{k}$ inside the cylinder that have their bases in the xy-plane have their tips on the paraboloid $z = a^2 - r^2$.

Gradient Fields

The gradient vector of a differentiable scalar-valued function at a point gives the direction of greatest increase of the function. An important type of vector field is formed by all the gradient vectors of the function (see Section 14.5). We define the **gradient field** of a differentiable function $f(x, y, z)$ to be the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

At each point (x, y, z), the gradient field gives a vector pointing in the direction of greatest increase of f, with magnitude being the value of the directional derivative in that direction. The gradient field is not always a force field or a velocity field.

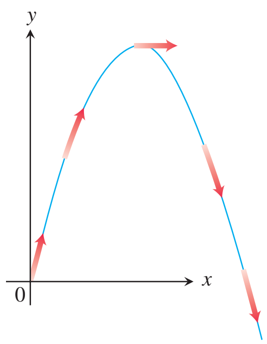


FIGURE 16.14 The velocity vectors $\mathbf{v}(t)$ of a projectile's motion make a vector field along the trajectory.

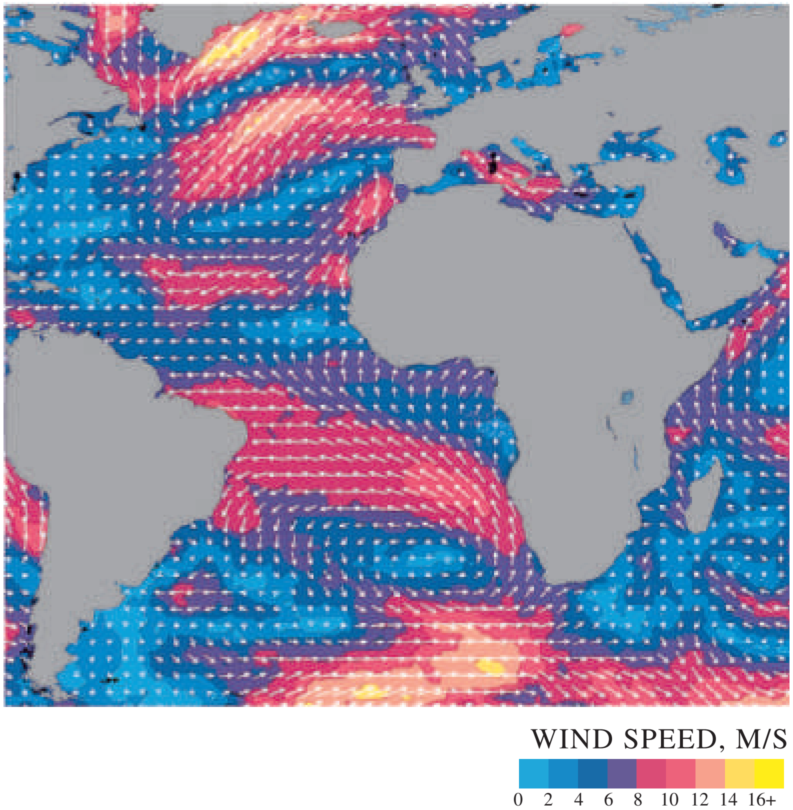


FIGURE 16.15 NASA's *Seasat* used radar to take 350,000 wind measurements over the world's oceans. The arrows show wind direction; their length and the color contouring indicate speed. Notice the heavy storm south of Greenland.

EXAMPLE 1 Suppose that the temperature T at each point (x, y, z) in a region of space is given by

$$T = 100 - x^2 - y^2 - z^2,$$

and that $\mathbf{F}(x, y, z)$ is defined to be the gradient of T. Find the vector field $\mathbf{F}$.

Solution The gradient field $\mathbf{F}$ is the field $\mathbf{F} = \nabla T = -2x\mathbf{i} - 2y\mathbf{j} - 2z\mathbf{k}$. At each point in space, the vector field $\mathbf{F}$ gives the direction for which the increase in temperature is greatest. ■

Line Integrals of Vector Fields

In Section 16.1 we defined the line integral of a scalar function $f(x, y, z)$ over a path C. We turn our attention now to the idea of a line integral of a vector field $\mathbf{F}$ along the curve C. Such line integrals have important applications in studying fluid flows, and electrical or gravitational fields.

Assume that the vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ has continuous components, and that the curve C has a smooth parametrization $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$. As discussed in Section 16.1, the parametrization $\mathbf{r}(t)$ defines a direction (or orientation) along C which we call the **forward direction**. At each point along the path C, the tangent vector $\mathbf{T} = d\mathbf{r}/ds = \mathbf{v}/|\mathbf{v}|$ is a unit vector tangent to the path and pointing in this forward direction. (The vector $\mathbf{v} = d\mathbf{r}/dt$ is the velocity vector tangent to C at the point, as discussed in Sections 13.1 and 13.3.) Intuitively, the line integral of the vector field is the line integral of the scalar tangential component of **F** along C. This tangential component is given by the dot product

$$\mathbf{F}\cdot\mathbf{T} = \mathbf{F}\cdot\frac{d\mathbf{r}}{ds},$$

so we have the following formal definition, where $f = \mathbf{F}\cdot\mathbf{T}$ in Equation (1) of Section 16.1.

DEFINITION Let **F** be a vector field with continuous components defined along a smooth curve C parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along** C is

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right) ds = \int_C \mathbf{F}\cdot d\mathbf{r}.$$

We evaluate line integrals of vector fields in a way similar to how we evaluate line integrals of scalar functions (Section 16.1).

Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ Along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

1. Express the vector field **F** in terms of the parametrized curve C as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of **r** into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of **F**.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter t, $a \le t \le b$, to obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

EXAMPLE 2 Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = z\mathbf{i} + xy\mathbf{j} - y^2\mathbf{k}$ along the curve C given by $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \sqrt{t}\,\mathbf{k}$, $0 \le t \le 1$.

Solution We have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{t}\,\mathbf{i} + t^3\mathbf{j} - t^2\mathbf{k} \qquad z = \sqrt{t},\ xy = t^3,\ -y^2 = -t^2$$

and

$$\frac{d\mathbf{r}}{dt} = 2t\mathbf{i} + \mathbf{j} + \frac{1}{2\sqrt{t}}\mathbf{k}.$$

Thus,

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_0^1 \left(2t^{3/2} + t^3 - \frac{1}{2}t^{3/2}\right) dt \\ &= \left[\left(\frac{3}{2}\right)\left(\frac{2}{5}t^{5/2}\right) + \frac{1}{4}t^4\right]_0^1 = \frac{17}{20}.\end{aligned}$$

■

Line Integrals with Respect to *dx*, *dy*, or *dz*

When analyzing forces or flows, it is often useful to consider each component direction separately. In such situations we want a line integral of a scalar function with respect to one of the coordinates, such as $\int_C M\,dx$. This integral is not the same as the arc length line integral $\int_C M\,ds$ we defined in Section 16.1. To define the integral $\int_C M\,dx$ for the scalar function $M(x, y, z)$, we specify a vector field $\mathbf{F} = M(x, y, z)\mathbf{i}$ having a component only in the x-direction, and none in the y- or z-direction. Then, over the curve C parametrized by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ for $a \le t \le b$, we have $x = g(t)$, $dx = g'(t)\,dt$, and

$$\mathbf{F}\cdot d\mathbf{r} = \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt = M(x, y, z)g'(t)\,dt = M(x, y, z)\,dx.$$

From the definition of the line integral of $\mathbf{F}$ along C, we define

$$\int_C M(x, y, z)\,dx = \int_C \mathbf{F}\cdot d\mathbf{r}, \quad \text{where} \quad \mathbf{F} = M(x, y, z)\mathbf{i}.$$

In the same way, by defining $\mathbf{F} = N(x, y, z)\mathbf{j}$ with a component only in the y-direction, or as $\mathbf{F} = P(x, y, z)\mathbf{k}$ with a component only in the z-direction, we can obtain the line integrals $\int_C N\,dy$ and $\int_C P\,dz$. Expressing everything in terms of the parameter t along the curve C, we have the following formulas for these three integrals:

$$\int_C M(x, y, z)\,dx = \int_a^b M(g(t), h(t), k(t))\,g'(t)\,dt \qquad (1)$$

$$\int_C N(x, y, z)\,dy = \int_a^b N(g(t), h(t), k(t))\,h'(t)\,dt \qquad (2)$$

$$\int_C P(x, y, z)\,dz = \int_a^b P(g(t), h(t), k(t))\,k'(t)\,dt \qquad (3)$$

It often happens that these line integrals occur in combination, and we abbreviate the notation by writing

$$\int_C M(x, y, z)\,dx + \int_C N(x, y, z)\,dy + \int_C P(x, y, z)\,dz = \int_C M\,dx + N\,dy + P\,dz.$$

EXAMPLE 3 Evaluate the line integral $\int_C -y\,dx + z\,dy + 2x\,dz$, where C is the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 2\pi$.

Solution We express everything in terms of the parameter t, so $x = \cos t$, $y = \sin t$, $z = t$, and $dx = -\sin t\,dt$, $dy = \cos t\,dt$, $dz = dt$. Then,

$$\begin{aligned}
\int_C -y\,dx + z\,dy + 2x\,dz &= \int_0^{2\pi} \left[(-\sin t)(-\sin t) + t\cos t + 2\cos t\right] dt \\
&= \int_0^{2\pi} \left[2\cos t + t\cos t + \sin^2 t\right] dt \\
&= \left[2\sin t + (t\sin t + \cos t) + \left(\frac{t}{2} - \frac{\sin 2t}{4}\right)\right]_0^{2\pi} \\
&= \left[0 + (0 + 1) + (\pi - 0)\right] - \left[0 + (0 + 1) + (0 - 0)\right] \\
&= \pi.
\end{aligned}$$

Work Done by a Force over a Curve in Space

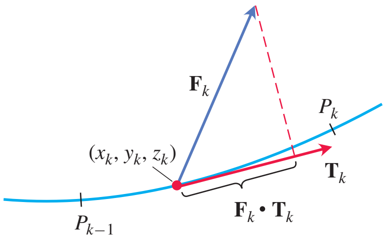


FIGURE 16.16 The work done along the subarc shown here is approximately $\mathbf{F}_k \cdot \mathbf{T}_k\, \Delta s_k$, where $\mathbf{F}_k = \mathbf{F}(x_k, y_k, z_k)$ and $\mathbf{T}_k = \mathbf{T}(x_k, y_k, z_k)$.

Suppose that the vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ represents a force throughout a region in space (it might be the force of gravity or an electromagnetic force of some kind) and that

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b,$$

is a smooth curve in the region. The formula for the work done by the force in moving an object along the curve is motivated by the same kind of reasoning we used in Chapter 6 to derive the ordinary single integral for the work done by a continuous force of magnitude $F(x)$ directed along an interval of the x-axis. For a curve C in space, we define the work done by a continuous force field $\mathbf{F}$ to move an object along C from a point A to another point B as follows.

We divide C into n subarcs $P_{k-1}P_k$ with lengths Δs_k, starting at A and ending at B. We choose any point (x_k, y_k, z_k) in the subarc $P_{k-1}P_k$ and let $\mathbf{T}(x_k, y_k, z_k)$ be the unit tangent vector at the chosen point. The work W_k done to move the object along the subarc $P_{k-1}P_k$ is approximated by the tangential component of the force $\mathbf{F}(x_k, y_k, z_k)$ times the arclength Δs_k approximating the distance the object moves along the subarc (see Figure 16.16). The total work done in moving the object from point A to point B is then approximated by summing the work done along each of the subarcs, so

$$W \approx \sum_{k=1}^{n} W_k \approx \sum_{k=1}^{n} \mathbf{F}(x_k, y_k, z_k) \cdot \mathbf{T}(x_k, y_k, z_k)\, \Delta s_k.$$

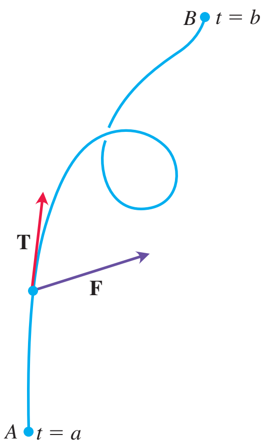


FIGURE 16.17 The work done by a force $\mathbf{F}$ is the line integral of the scalar component $\mathbf{F} \cdot \mathbf{T}$ over the smooth curve from A to B.

For any subdivision of C into n subarcs, and for any choice of the points (x_k, y_k, z_k) within each subarc, as $n \to \infty$ and $\Delta s_k \to 0$, these sums approach the line integral

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds.$$

This is just the line integral of $\mathbf{F}$ along C, which defines the total work done.

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{4}$$

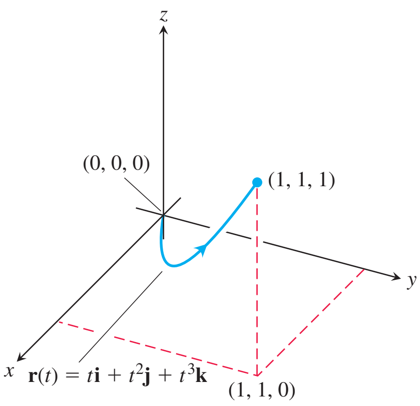


FIGURE 16.18 The curve in Example 4.

The sign of the number we calculate with this integral depends on the direction in which the curve is traversed. If we reverse the direction of motion, then we reverse the direction of $\mathbf{T}$ in Figure 16.17 and change the sign of $\mathbf{F} \cdot \mathbf{T}$ and its integral.

Using the notations we have presented, we can express the work integral in a variety of ways, depending upon what seems most suitable or convenient for a particular discussion. Table 16.2 shows five ways we can write the work integral in Equation (4). In the table, the field components M, N, and P are functions of the intermediate variables x, y, and z, which in turn are functions of the independent variable t along the curve C in the vector field. So along the curve, $x = g(t)$, $y = h(t)$, and $z = k(t)$ with $dx = g'(t)\,dt$, $dy = h'(t)\,dt$, and $dz = k'(t)\,dt$.

EXAMPLE 4 Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.18).

TABLE 16.2 Different ways to write the work integral for $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ over the curve $C: \mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k},\ a \le t \le b$

$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$	The definition
$= \int_C \mathbf{F} \cdot d\mathbf{r}$	Vector differential form
$= \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt$	Parametric vector evaluation
$= \int_a^b \left(Mg'(t) + Nh'(t) + Pk'(t)\right) dt$	Parametric scalar evaluation
$= \int_C M\, dx + N\, dy + P\, dz$	Scalar differential form

Solution First we evaluate $\mathbf{F}$ on the curve $\mathbf{r}(t)$:

$$\begin{aligned}\mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= \underbrace{(t^2 - t^2)}_{0}\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}.\end{aligned}$$

Substitute $x = t$, $y = t^2$, $z = t^3$.

Then we find $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

Finally, we find $\mathbf{F} \cdot d\mathbf{r}/dt$ and integrate from $t = 0$ to $t = 1$:

$$\begin{aligned}\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= \left[(t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}\right] \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\ &= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8.\end{aligned}$$

So,

$$\begin{aligned}\text{Work} &= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt \\ &= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9\right]_0^1 = \frac{29}{60}.\end{aligned}$$

EXAMPLE 5 Find the work done by the force field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in moving an object along the curve C parametrized by $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + t^2\mathbf{j} + \sin(\pi t)\mathbf{k}$, $0 \le t \le 1$.

Solution We begin by writing $\mathbf{F}$ along C as a function of t,

$$\mathbf{F}(\mathbf{r}(t)) = \cos(\pi t)\mathbf{i} + t^2\mathbf{j} + \sin(\pi t)\mathbf{k}.$$

Next we compute $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = -\pi \sin(\pi t)\mathbf{i} + 2t\mathbf{j} + \pi \cos(\pi t)\mathbf{k}.$$

We then calculate the dot product,

$$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = -\pi \sin(\pi t)\cos(\pi t) + 2t^3 + \pi \sin(\pi t)\cos(\pi t) = 2t^3.$$

The work done is the line integral

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 2t^3\, dt = \frac{t^4}{2}\bigg]_0^1 = \frac{1}{2}.$$

Flow Integrals and Circulation for Velocity Fields

Suppose that $\mathbf{F}$ represents the velocity field of a fluid flowing through a region in space (a tidal basin or the turbine chamber of a hydroelectric generator, for example). Under these circumstances, the integral of $\mathbf{F} \cdot \mathbf{T}$ along a curve in the region gives the fluid's flow along, or *circulation* around, the curve. For instance, the vector field in Figure 16.11 gives zero circulation around the unit circle in the plane. By contrast, the vector field in Figure 16.12 gives a nonzero circulation around the unit circle.

DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{5}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

The direction we travel along C matters. If we reverse the direction, then $\mathbf{T}$ is replaced by $-\mathbf{T}$ and the sign of the integral changes. We evaluate flow integrals the same way we evaluate work integrals.

EXAMPLE 6 A fluid's velocity field is $\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow along the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi/2$.

Solution We evaluate $\mathbf{F}$ on the curve,

$$\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k} = (\cos t)\mathbf{i} + t\mathbf{j} + (\sin t)\mathbf{k} \qquad \text{Substitute } x = \cos t,\ z = t,\ y = \sin t.$$

and then find $d\mathbf{r}/dt$:

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

Then we integrate $\mathbf{F} \cdot (d\mathbf{r}/dt)$ from $t = 0$ to $t = \dfrac{\pi}{2}$:

$$\begin{aligned} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1) \\ &= -\sin t \cos t + t \cos t + \sin t. \end{aligned}$$

So,

$$\begin{aligned} \text{Flow} &= \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\, dt \\ &= \left[\frac{\cos^2 t}{2} + t \sin t\right]_0^{\pi/2} = \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2}. \end{aligned}$$

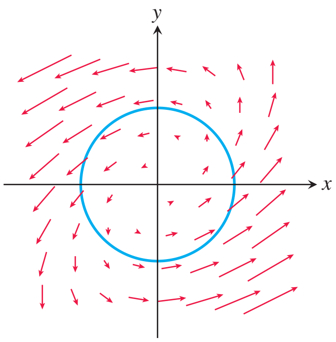


FIGURE 16.19 The vector field $\mathbf{F}$ and curve $\mathbf{r}(t)$ in Example 7.

EXAMPLE 7 Find the circulation of the field $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ around the circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$ (Figure 16.19).

Solution On the circle, $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Then

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t \cos t + \underbrace{\sin^2 t + \cos^2 t}_{1}$$

gives

$$\begin{aligned}\text{Circulation} &= \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} (1 - \sin t \cos t)\, dt \\ &= \left[t - \frac{\sin^2 t}{2} \right]_0^{2\pi} = 2\pi.\end{aligned}$$

As Figure 16.19 suggests, a fluid with this velocity field is circulating *counterclockwise* around the circle, so the circulation is positive. ■

Flux Across a Simple Closed Plane Curve

A curve in the xy-plane is **simple** if it does not cross itself (Figure 16.20). When a curve starts and ends at the same point, it is a **closed curve** or **loop**. To find the rate at which a fluid is entering or leaving a region enclosed by a smooth simple closed curve C in the xy-plane, we calculate the line integral over C of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of the fluid's velocity field in the direction of the curve's outward-pointing normal vector. We use only the normal component of $\mathbf{F}$, while ignoring the tangential component, because the normal component leads to the flow across C. The value of this integral is the *flux* of $\mathbf{F}$ across C. *Flux* is Latin for *flow*, but many flux calculations involve no motion at all. If $\mathbf{F}$ were an electric field or a magnetic field, for instance, the integral of $\mathbf{F} \cdot \mathbf{n}$ is still called the flux of the field across C.

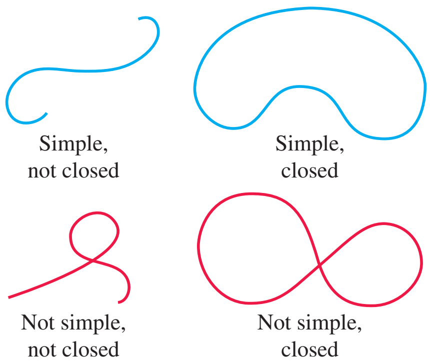


FIGURE 16.20 Distinguishing curves that are simple or closed. Closed curves are also called loops.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across C} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. \tag{6}$$

Notice the difference between flux and circulation. The flux of $\mathbf{F}$ across C is the line integral with respect to arc length of $\mathbf{F} \cdot \mathbf{n}$, the scalar component of $\mathbf{F}$ in the direction of the outward normal. The circulation of $\mathbf{F}$ around C is the line integral with respect to arc length of $\mathbf{F} \cdot \mathbf{T}$, the scalar component of $\mathbf{F}$ in the direction of the unit tangent vector. Flux is the integral of the normal component of $\mathbf{F}$; circulation is the integral of the tangential component of $\mathbf{F}$. In Section 16.6 we define flux across a surface.

To evaluate the integral for flux in Equation (6), we begin with a smooth parametrization

$$x = g(t), \qquad y = h(t), \qquad a \le t \le b,$$

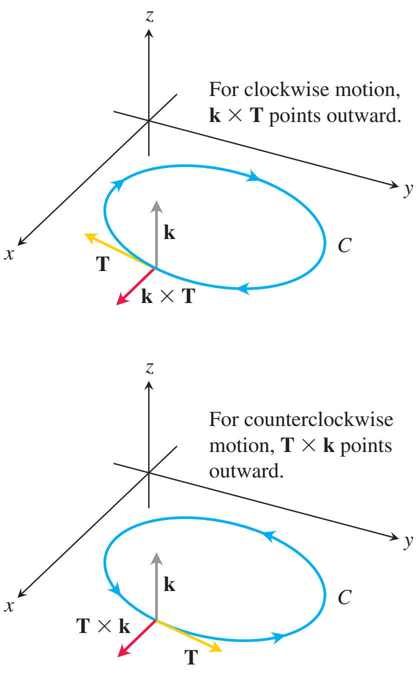


FIGURE 16.21 To find an outward unit normal vector for a smooth simple curve C in the xy-plane that is traversed counterclockwise as t increases, we take $\mathbf{n} = \mathbf{T} \times \mathbf{k}$. For clockwise motion, we take $\mathbf{n} = \mathbf{k} \times \mathbf{T}$.

that traces the curve C exactly once as t increases from a to b. We can find the outward unit normal vector $\mathbf{n}$ by crossing the curve's unit tangent vector $\mathbf{T}$ with the vector $\mathbf{k}$. But which order do we choose, $\mathbf{T} \times \mathbf{k}$ or $\mathbf{k} \times \mathbf{T}$? Which one points outward? It depends on which way C is traversed as t increases. If the motion is clockwise, $\mathbf{k} \times \mathbf{T}$ points outward; if the motion is counterclockwise, $\mathbf{T} \times \mathbf{k}$ points outward (Figure 16.21). The usual choice is $\mathbf{n} = \mathbf{T} \times \mathbf{k}$, the choice that assumes counterclockwise motion. Thus, although the value of the integral in Equation (6) does not depend on which way C is traversed, the formulas we are about to derive for computing $\mathbf{n}$ and evaluating the integral assume counterclockwise motion.

In terms of components,

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) \times \mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

If $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then

$$\mathbf{F} \cdot \mathbf{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}.$$

Hence,

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\, dy - N\, dx.$$

We put a directed circle $\circlearrowleft$ on the last integral as a reminder that the integration around the closed curve C is to be in the counterclockwise direction. To evaluate this integral, we express M, dy, N, and dx in terms of the parameter t and integrate from $t = a$ to $t = b$. We do not need to know $\mathbf{n}$ or ds explicitly to find the flux.

Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\, dy - N\, dx \tag{7}$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.

EXAMPLE 8 Find the flux of $\mathbf{F} = (x - y)\mathbf{i} + x\mathbf{j}$ across the circle $x^2 + y^2 = 1$ in the xy-plane. (The vector field and curve were shown previously in Figure 16.19.)

Solution The parametrization $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, traces the circle counterclockwise exactly once. We can therefore use this parametrization in Equation (7). With

$$\begin{aligned} M &= x - y = \cos t - \sin t, & dy &= d(\sin t) = \cos t\, dt \\ N &= x = \cos t, & dx &= d(\cos t) = -\sin t\, dt, \end{aligned}$$

we find

$$\begin{aligned} \text{Flux} &= \oint_C M\, dy - N\, dx = \int_0^{2\pi} (\cos^2 t - \sin t \cos t + \cos t \sin t)\, dt \qquad \text{Eq. (7)} \\ &= \int_0^{2\pi} \cos^2 t\, dt = \int_0^{2\pi} \frac{1 + \cos 2t}{2} dt = \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi. \end{aligned}$$

The flux of $\mathbf{F}$ across the circle is π. Since the answer is positive, the net flow across the curve is outward. A net inward flow would have given a negative flux. ■

Exercises 16.2

Vector Fields

Find the gradient fields of the functions in Exercises 1–4.

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$
2. $f(x, y, z) = \ln \sqrt{x^2 + y^2 + z^2}$
3. $g(x, y, z) = e^z - \ln(x^2 + y^2)$
4. $g(x, y, z) = xy + yz + xz$
5. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the property that $\mathbf{F}$ points toward the origin with magnitude inversely proportional to the square of the distance from (x, y) to the origin. (The field is not defined at $(0, 0)$.)
6. Give a formula $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ for the vector field in the plane that has the properties that $\mathbf{F} = \mathbf{0}$ at $(0, 0)$ and that at any other point (a, b), $\mathbf{F}$ is tangent to the circle $x^2 + y^2 = a^2 + b^2$ and points in the clockwise direction with magnitude $|\mathbf{F}| = \sqrt{a^2 + b^2}$.

Line Integrals of Vector Fields

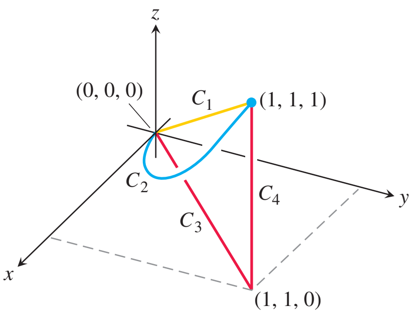
In Exercises 7–12, find the line integrals of $\mathbf{F}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

a. The straight-line path C_1: $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $\quad 0 \le t \le 1$

b. The curved path C_2: $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $\quad 0 \le t \le 1$

c. The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$

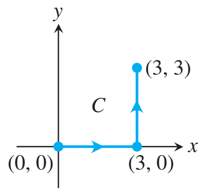
7. $\mathbf{F} = 3y\mathbf{i} + 2x\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = [1/(x^2 + 1)]\mathbf{j}$
9. $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$
10. $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$
11. $\mathbf{F} = (3x^2 - 3x)\mathbf{i} + 3z\mathbf{j} + \mathbf{k}$
12. $\mathbf{F} = (y + z)\mathbf{i} + (z + x)\mathbf{j} + (x + y)\mathbf{k}$



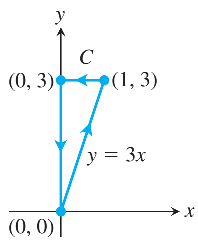
Line Integrals with Respect to *x*, *y*, and *z*

In Exercises 13–16, find the line integrals along the given path C.

13. $\int_C (x - y)\, dx$, where C: $x = t, y = 2t + 1$, for $0 \le t \le 3$
14. $\int_C \frac{x}{y}\, dy$, where C: $x = t, y = t^2$, for $1 \le t \le 2$
15. $\int_C (x^2 + y^2)\, dy$, where C is given in the accompanying figure



16. $\int_C \sqrt{x + y}\, dx$, where C is given in the accompanying figure



17. Along the curve $\mathbf{r}(t) = t\mathbf{i} - \mathbf{j} + t^2\mathbf{k}$, $0 \le t \le 1$, evaluate each of the following integrals.

a. $\int_C (x + y - z)\, dx$ $\quad$ b. $\int_C (x + y - z)\, dy$

c. $\int_C (x + y - z)\, dz$

18. Along the curve $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} - (\cos t)\mathbf{k}$, $0 \le t \le \pi$, evaluate each of the following integrals.

a. $\int_C xz\, dx$ $\quad$ b. $\int_C xz\, dy$ $\quad$ c. $\int_C xyz\, dz$

Work

In Exercises 19–22, find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

19. $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$
$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}$, $\quad 0 \le t \le 1$
20. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x + y)\mathbf{k}$
$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}$, $\quad 0 \le t \le 2\pi$
21. $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}$, $\quad 0 \le t \le 2\pi$
22. $\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}$, $\quad 0 \le t \le 2\pi$

Line Integrals in the Plane

23. Evaluate $\int_C xy\, dx + (x + y)\, dy$ along the curve $y = x^2$ from $(-1, 1)$ to $(2, 4)$.
24. Evaluate $\int_C (x - y)\, dx + (x + y)\, dy$ counterclockwise around the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.
25. Evaluate $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$ for the vector field $\mathbf{F} = x^2\mathbf{i} - y\mathbf{j}$ along the curve $x = y^2$ from $(4, 2)$ to $(1, -1)$.
26. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

Work, Circulation, and Flux in the Plane

27. Work Find the work done by the force $\mathbf{F} = xy\mathbf{i} + (y - x)\mathbf{j}$ over the straight line from $(1, 1)$ to $(2, 3)$.

28. Work Find the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2, 0)$ to itself.

29. Circulation and flux Find the circulation and flux of the fields

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = -y\mathbf{i} + x\mathbf{j}$$

around and across each of the following curves.

a. The circle $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

b. The ellipse $\mathbf{r}(t) = (\cos t)\mathbf{i} + (4 \sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

30. Flux across a circle Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

In Exercises 31–34, find the circulation and flux of the field $\mathbf{F}$ around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}, 0 \le t \le \pi$, followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i}, -a \le t \le a$.

31. $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

32. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$

33. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

34. $\mathbf{F} = -y^2\mathbf{i} + x^2\mathbf{j}$

35. Flow integrals Find the flow of the velocity field $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$ along each of the following paths from $(1, 0)$ to $(-1, 0)$ in the xy-plane.

a. The upper half of the circle $x^2 + y^2 = 1$

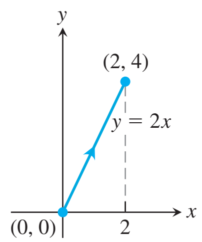
b. The line segment from $(1, 0)$ to $(-1, 0)$

c. The line segment from $(1, 0)$ to $(0, -1)$ followed by the line segment from $(0, -1)$ to $(-1, 0)$

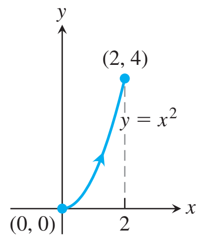
36. Flux across a triangle Find the flux of the field $\mathbf{F}$ in Exercise 35 outward across the triangle with vertices $(1, 0), (0, 1), (-1, 0)$.

37. Find the flow of the velocity field $\mathbf{F} = y^2\mathbf{i} + 2xy\mathbf{j}$ along each of the following paths from $(0, 0)$ to $(2, 4)$.

a.



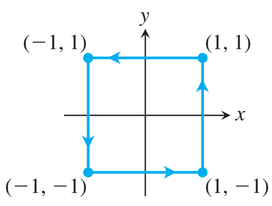
b.



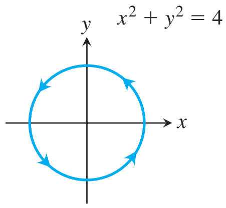
c. Use any path from $(0, 0)$ to $(2, 4)$ different from parts (a) and (b).

38. Find the circulation of the field $\mathbf{F} = y\mathbf{i} + (x + 2y)\mathbf{j}$ around each of the following closed paths.

a.



b.



c. Use any closed path different from parts (a) and (b).

Vector Fields in the Plane

39. Spin field Draw the spin field

$$\mathbf{F} = -\frac{y}{\sqrt{x^2 + y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2 + y^2}}\mathbf{j}$$

(see Figure 16.12) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 4$.

40. Radial field Draw the radial field

$$\mathbf{F} = x\mathbf{i} + y\mathbf{j}$$

(see Figure 16.11) along with its horizontal and vertical components at a representative assortment of points on the circle $x^2 + y^2 = 1$.

41. A field of tangent vectors

a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the xy-plane with the property that at any point $(a, b) \ne (0, 0)$, $\mathbf{G}$ is a vector of magnitude $\sqrt{a^2 + b^2}$ tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the counterclockwise direction. (The field is undefined at $(0, 0)$.)

b. How is $\mathbf{G}$ related to the spin field $\mathbf{F}$ in Figure 16.12?

42. A field of tangent vectors

a. Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the xy-plane with the property that at any point $(a, b) \ne (0, 0)$, $\mathbf{G}$ is a unit vector tangent to the circle $x^2 + y^2 = a^2 + b^2$ and pointing in the clockwise direction.

b. How is $\mathbf{G}$ related to the spin field $\mathbf{F}$ in Figure 16.12?

43. Unit vectors pointing toward the origin Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy-plane with the property that at each point $(x, y) \ne (0, 0)$, $\mathbf{F}$ is a unit vector pointing toward the origin. (The field is undefined at $(0, 0)$.)

44. Two "central" fields Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy-plane with the property that at each point $(x, y) \ne (0, 0)$, $\mathbf{F}$ points toward the origin and $|\mathbf{F}|$ is **(a)** the distance from (x, y) to the origin, **(b)** inversely proportional to the distance from (x, y) to the origin. (The field is undefined at $(0, 0)$.)

45. Work and area Suppose that $f(t)$ is differentiable and positive for $a \le t \le b$. Let C be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}, a \le t \le b$, and $\mathbf{F} = y\mathbf{i}$. Is there any relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the t-axis, the graph of f, and the lines $t = a$ and $t = b$? Give reasons for your answer.

46. Work done by a radial force with constant magnitude A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to

$(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = k\left[\left(b^2+(f(b))^2\right)^{1/2} - \left(a^2+(f(a))^2\right)^{1/2}\right].$$

Flow Integrals in Space

In Exercises 47–50, $\mathbf{F}$ is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing t.

47. $\mathbf{F} = -4xy\mathbf{i} + 8y\mathbf{j} + 2\mathbf{k}$

$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 2$

48. $\mathbf{F} = x^2\mathbf{i} + yz\mathbf{j} + y^2\mathbf{k}$

$\mathbf{r}(t) = 3t\mathbf{j} + 4t\mathbf{k}, \quad 0 \le t \le 1$

49. $\mathbf{F} = (x - z)\mathbf{i} + x\mathbf{k}$

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{k}, \quad 0 \le t \le \pi$

50. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + 2\mathbf{k}$

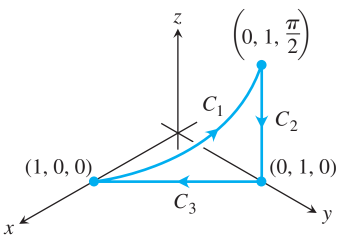
$\mathbf{r}(t) = (-2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + 2t\mathbf{k}, \quad 0 \le t \le 2\pi$

51. Circulation Find the circulation of $\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t.

C_1: $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi/2$

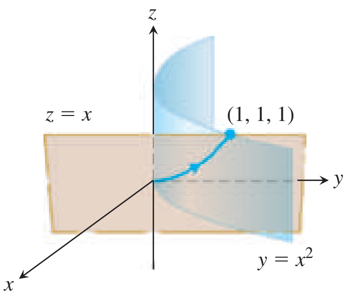
C_2: $\mathbf{r}(t) = \mathbf{j} + (\pi/2)(1 - t)\mathbf{k}, \quad 0 \le t \le 1$

C_3: $\mathbf{r}(t) = t\mathbf{i} + (1 - t)\mathbf{j}, \quad 0 \le t \le 1$



52. Zero circulation Let C be the ellipse in which the plane $2x + 3y - z = 0$ meets the cylinder $x^2 + y^2 = 12$. Show, without evaluating either line integral directly, that the circulation of the field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ around C in either direction is zero.

53. Flow along a curve The field $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ is the velocity field of a flow in space. Find the flow from $(0, 0, 0)$ to $(1, 1, 1)$ along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. (*Hint:* Use $t = x$ as the parameter.)



54. Flow of a gradient field Find the flow of the field $\mathbf{F} = \nabla(xy^2z^3)$:

a. Once around the curve C in Exercise 52, clockwise as viewed from above

b. Along the line segment from $(1, 1, 1)$ to $(2, 1, -1)$.

COMPUTER EXPLORATIONS

In Exercises 55–60, use a CAS to perform the following steps for finding the work done by force $\mathbf{F}$ over the given path:

a. Find $d\mathbf{r}$ for the path $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$.

b. Evaluate the force $\mathbf{F}$ along the path.

c. Evaluate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$.

55. $\mathbf{F} = xy^6\mathbf{i} + 3x(xy^5 + 2)\mathbf{j}$; $\quad \mathbf{r}(t) = (2\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$

56. $\mathbf{F} = \dfrac{3}{1+x^2}\mathbf{i} + \dfrac{2}{1+y^2}\mathbf{j}$; $\quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le \pi$

57. $\mathbf{F} = (y + yz\cos xyz)\mathbf{i} + (x^2 + xz\cos xyz)\mathbf{j} + (z + xy\cos xyz)\mathbf{k}$; $\quad \mathbf{r}(t) = (2\cos t)\mathbf{i} + (3\sin t)\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$

58. $\mathbf{F} = 2xy\mathbf{i} - y^2\mathbf{j} + ze^x\mathbf{k}$; $\quad \mathbf{r}(t) = -t\mathbf{i} + \sqrt{t}\mathbf{j} + 3t\mathbf{k}$, $1 \le t \le 4$

59. $\mathbf{F} = (2y + \sin x)\mathbf{i} + (z^2 + (1/3)\cos y)\mathbf{j} + x^4\mathbf{k}$; $\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (\sin 2t)\mathbf{k}$, $\quad -\pi/2 \le t \le \pi/2$

60. $\mathbf{F} = (x^2y)\mathbf{i} + \frac{1}{3}x^3\mathbf{j} + xy\mathbf{k}$; $\quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (2\sin^2 t - 1)\mathbf{k}$, $\quad 0 \le t \le 2\pi$

16.3 Path Independence, Conservative Fields, and Potential Functions

A **gravitational field G** is a vector field that represents the effect of gravity at a point in space due to the presence of a massive object. The gravitational force on a body of mass m placed in the field is given by $\mathbf{F} = m\mathbf{G}$. Similarly, an **electric field E** is a vector field in space that represents the effect of electric forces on a charged particle placed within it. The force on a body of charge q placed in the field is given by $\mathbf{F} = q\mathbf{E}$. In gravitational and electric fields, the amount of work it takes to move a mass or charge from one point to another depends on the initial and final positions of the object—not on which path is taken between these positions. In this section we study vector fields with this property and the calculation of work integrals associated with them.

Path Independence

If A and B are two points in an open region D in space, the line integral of $\mathbf{F}$ along C from A to B for a field $\mathbf{F}$ defined on D usually depends on the path C taken, as we saw in Section 16.1. For some special fields, however, the integral's value is the same for all paths from A to B.

> **DEFINITIONS** Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

The word *conservative* comes from physics, where it refers to fields in which the principle of conservation of energy holds. When a line integral is independent of the path C from point A to point B, we sometimes represent the integral by the symbol $\int_A^B$ rather than the usual line integral symbol $\int_C$. This substitution helps us remember the path-independence property.

Under differentiability conditions normally met in practice, we will show that a field $\mathbf{F}$ is conservative if and only if it is the gradient field of a scalar function f—that is, if and only if $\mathbf{F} = \nabla f$ for some f. The function f then has a special name.

> **DEFINITION** If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for F**.

A gravitational potential is a scalar function whose gradient field is a gravitational field, an electric potential is a scalar function whose gradient field is an electric field, and so on. As we will see, once we have found a potential function f for a field $\mathbf{F}$, we can evaluate all the line integrals in the domain of $\mathbf{F}$ over any path between A and B by

$$\int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^B \nabla f \cdot d\mathbf{r} = f(B) - f(A). \tag{1}$$

If you think of ∇f for functions of several variables as analogous to the derivative f' for functions of a single variable, then you see that Equation (1) is the vector calculus rendition of the Fundamental Theorem of Calculus formula (also called the Net Change Theorem)

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

Conservative fields have other important properties. For example, saying that $\mathbf{F}$ is conservative on D is equivalent to saying that the integral of $\mathbf{F}$ around every closed path in D is zero. Certain conditions on the curves, fields, and domains must be satisfied for Equation (1) to be valid. We discuss these conditions next.

Assumptions on Curves, Vector Fields, and Domains

In order for the computations and results we derive below to be valid, we must assume certain properties for the curves, surfaces, domains, and vector fields we consider. We give these assumptions in the statements of theorems, and they also apply to the examples and exercises unless otherwise stated.

The curves we consider are **piecewise smooth**. Such curves are made up of finitely many smooth pieces connected end to end, as discussed in Section 13.1. We will treat vector fields $\mathbf{F}$ whose components have continuous first partial derivatives.

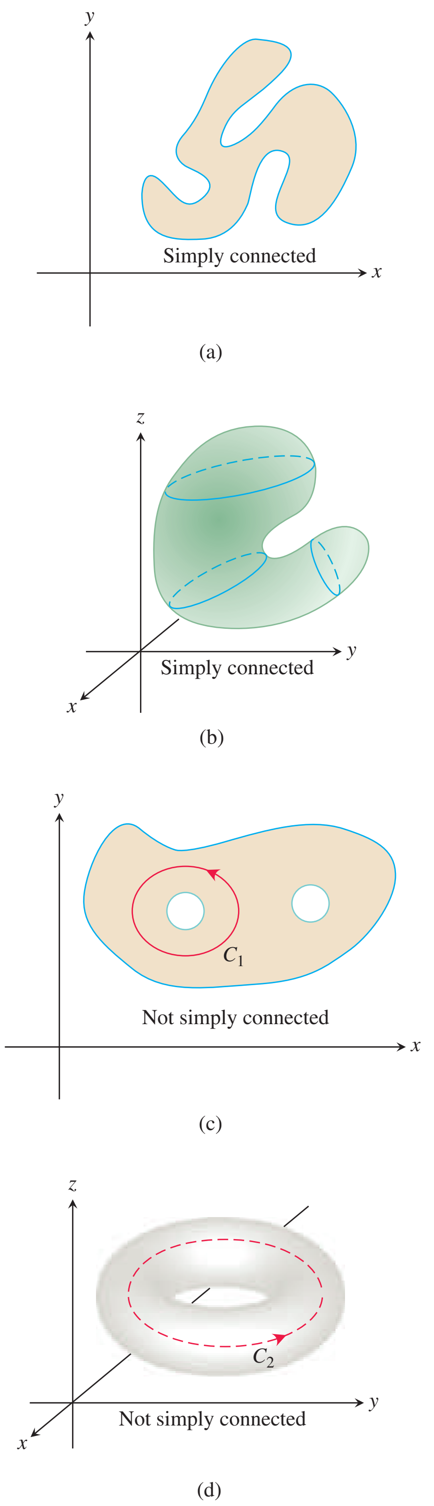


FIGURE 16.22 Four connected regions. In (a) and (b), the regions are simply connected. In (c) and (d), the regions are not simply connected because the curves C_1 and C_2 cannot be contracted to a point inside the regions containing them.

The domains D we consider are **connected**. For an open region, this means that any two points in D can be joined by a smooth curve that lies in the region. Some results also require D to be **simply connected**, which means that every loop in D can be contracted to a point in D without ever leaving D. The plane with a disk removed is a two-dimensional region that is *not* simply connected; a loop in the plane that goes around the disk cannot be contracted to a point without going into the "hole" left by the removed disk (see Figure 16.22c). Similarly, if we remove a line from space, the remaining region D is *not* simply connected. A curve encircling the line cannot be shrunk to a point while remaining inside D.

Connectivity and simple connectivity are not the same, and neither property implies the other. Think of connected regions as being in "one piece" and simply connected regions as not having any "loop-catching holes." All of space itself is both connected and simply connected. Figure 16.22 illustrates some of these properties.

Caution Some of the results in this chapter can fail to hold if applied to situations where the conditions we've imposed do not hold. In particular, the component test for conservative fields, given later in this section, is not valid on domains that are not simply connected (see Example 5). We do not always require that a domain be simply connected, so the condition will be stated when needed.

Line Integrals in Conservative Fields

Gradient fields $\mathbf{F}$ are obtained by differentiating a scalar function f. A theorem analogous to the Fundamental Theorem of Calculus gives a way to evaluate the line integrals of gradient fields.

THEOREM 1—Fundamental Theorem of Line Integrals Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

Like the Fundamental Theorem, Theorem 1 gives a way to evaluate line integrals without having to take limits of Riemann sums or finding the line integral by the procedure used in Section 16.2. Before proving Theorem 1, we give an example.

EXAMPLE 1 Suppose the force field $\mathbf{F} = \nabla f$ is the gradient of the function

$$f(x, y, z) = -\frac{1}{x^2 + y^2 + z^2}.$$

Find the work done by $\mathbf{F}$ in moving an object along a smooth curve C joining (1, 0, 0) to (0, 0, 2) that does not pass through the origin.

Solution An application of Theorem 1 shows that the work done by $\mathbf{F}$ along any smooth curve C joining the two points and not passing through the origin is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 0, 2) - f(1, 0, 0) = -\frac{1}{4} - (-1) = \frac{3}{4}. \qquad \blacksquare$$

The gravitational force due to a planet, and the electric force associated with a charged particle, can both be modeled by the field $\mathbf{F}$ given in Example 1 up to a constant that depends on the units of measurement.

Proof of Theorem 1 Suppose that A and B are two points in region D and that C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, is a smooth curve in D joining A to B. In Section 14.5 we found that the derivative of a scalar function f along a path C is the dot product $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$, so we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} && \mathbf{F} = \nabla f \\
&= \int_{t=a}^{t=b} \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt && \text{Eq. (7) of Section 14.5} \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a)) && \text{Net Change Theorem} \\
&= f(B) - f(A). && \mathbf{r}(a) = A,\ \mathbf{r}(b) = B \quad \blacksquare
\end{aligned}$$

So we see from Theorem 1 that the line integral of a gradient field $\mathbf{F} = \nabla f$ is straightforward to compute once we know the function f. Many important vector fields arising in applications are indeed gradient fields. The next result, which follows from Theorem 1, shows that any conservative field is of this type.

THEOREM 2—Conservative Fields are Gradient Fields Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

Theorem 2 says that $\mathbf{F} = \nabla f$ if and only if for any two points A and B in the region D, the value of line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining A to B in D.

Proof of Theorem 2 If $\mathbf{F}$ is a gradient field, then $\mathbf{F} = \nabla f$ for a differentiable function f, and Theorem 1 shows that $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$. The value of the line integral does not depend on C, but only on its endpoints A and B. So the line integral is path independent and $\mathbf{F}$ satisfies the definition of a conservative field.

On the other hand, suppose that $\mathbf{F}$ is a conservative vector field. We want to find a function f on D satisfying $\nabla f = \mathbf{F}$. First, pick a point A in D and set $f(A) = 0$. For any other point B in D define $f(B)$ to equal $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is *any* smooth path in D from A to B. The value of $f(B)$ does not depend on the choice of C, since $\mathbf{F}$ is conservative. To show that $\nabla f = \mathbf{F}$ we need to demonstrate that $\partial f/\partial x = M$, $\partial f/\partial y = N$, and $\partial f/\partial z = P$.

Suppose that B has coordinates (x, y, z). By definition, the value of the function f at a nearby point B_0 located at (x_0, y, z) is $\int_{C_0} \mathbf{F} \cdot d\mathbf{r}$, where C_0 is any path from A to B_0. We take a path $C = C_0 \cup L$ from A to B formed by first traveling along C_0 to arrive at B_0 and then traveling along the line segment L from B_0 to B (Figure 16.23). When B_0 is close to B, the segment L lies in D and, since the value $f(B)$ is independent of the path from A to B,

$$f(x, y, z) = \int_{C_0} \mathbf{F} \cdot d\mathbf{r} + \int_L \mathbf{F} \cdot d\mathbf{r}.$$

Differentiating, we have

$$\frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial x}\left(\int_{C_0} \mathbf{F} \cdot d\mathbf{r} + \int_L \mathbf{F} \cdot d\mathbf{r}\right).$$

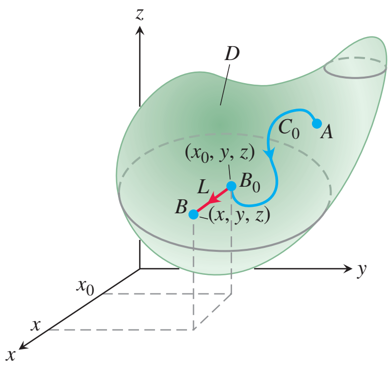


FIGURE 16.23 The function $f(x, y, z)$ in the proof of Theorem 2 is computed by a line integral $\int_{C_0} \mathbf{F} \cdot d\mathbf{r} = f(B_0)$ from A to B_0, plus a line integral $\int_L \mathbf{F} \cdot d\mathbf{r}$ along a line segment L parallel to the x-axis and joining B_0 to B located at (x, y, z). The value of f at A is $f(A) = 0$.

Only the last term on the right depends on x, so

$$\frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial x} \int_L \mathbf{F} \cdot d\mathbf{r}.$$

Now parametrize L as $\mathbf{r}(t) = t\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $x_0 \le t \le x$. Then $d\mathbf{r}/dt = \mathbf{i}$, $\mathbf{F} \cdot d\mathbf{r}/dt = M$, and $\int_L \mathbf{F} \cdot d\mathbf{r} = \int_{x_0}^{x} M(t, y, z)\, dt$. Differentiating then gives

$$\frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial x} \int_{x_0}^{x} M(t, y, z)\, dt = M(x, y, z)$$

by the Fundamental Theorem of Calculus. The partial derivatives $\partial f / \partial y = N$ and $\partial f / \partial z = P$ follow similarly, showing that $\mathbf{F} = \nabla f$. ■

EXAMPLE 2 Find the work done by the conservative field

$$\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} = \nabla f, \quad \text{where} \quad f(x, y, z) = xyz,$$

along any smooth curve C joining the point $A(-1, 3, 9)$ to $B(1, 6, -4)$.

Solution With $f(x, y, z) = xyz$, we have

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_A^B \nabla f \cdot d\mathbf{r} && \mathbf{F} = \nabla f \text{ and path independence} \\
&= f(B) - f(A) && \text{Theorem 1} \\
&= xyz\big|_{(1,6,-4)} - xyz\big|_{(-1,3,9)} \\
&= (1)(6)(-4) - (-1)(3)(9) \\
&= -24 + 27 = 3.
\end{aligned}$$ ■

A very useful property of line integrals in conservative fields comes into play when the path of integration is a closed curve, or loop. We often use the notation $\oint_C$ for integration around a closed path (discussed with more detail in the next section).

THEOREM 3—Loop Property of Conservative Fields The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.

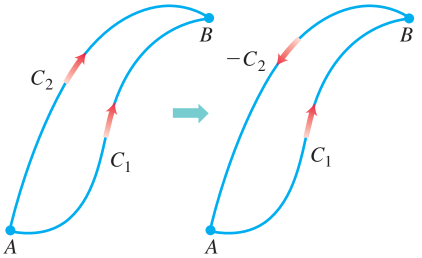


FIGURE 16.24 If we have two paths from A to B, one of them can be reversed to make a loop.

Proof that Part 1 ⇒ Part 2 We want to show that for any two points A and B in D, the integral of $\mathbf{F} \cdot d\mathbf{r}$ has the same value over any two paths C_1 and C_2 from A to B. We reverse the direction on C_2 to make a path $-C_2$ from B to A (Figure 16.24). Together, C_1 and $-C_2$ make a closed loop C, and by assumption,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Thus, the integrals over C_1 and C_2 give the same value. Note that the definition of $\mathbf{F} \cdot d\mathbf{r}$ shows that changing the direction along a curve reverses the sign of the line integral.

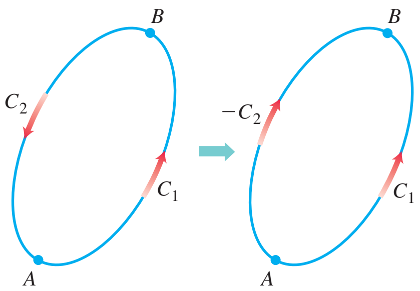


FIGURE 16.25 If A and B lie on a loop, we can reverse part of the loop to make two paths from A to B.

Proof that Part 2 $\Rightarrow$ Part 1 We want to show that the integral of $\mathbf{F} \cdot d\mathbf{r}$ is zero over any closed loop C. We pick two points A and B on C and use them to break C into two pieces: C_1 from A to B followed by C_2 from B back to A (Figure 16.25). Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_A^B \mathbf{F} \cdot d\mathbf{r} - \int_A^B \mathbf{F} \cdot d\mathbf{r} = 0. \qquad \blacksquare$$

The following diagram summarizes the results of Theorems 2 and 3.

$$\mathbf{F} = \nabla f \text{ on D} \quad \overset{\text{Theorem 2}}{\Leftrightarrow} \quad \mathbf{F} \text{ conservative on D} \quad \overset{\text{Theorem 3}}{\Leftrightarrow} \quad \oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ over any loop in D}$$

Two questions arise:

1. How do we know whether a given vector field $\mathbf{F}$ is conservative?
2. If $\mathbf{F}$ is in fact conservative, how do we find a potential function f (so that $\mathbf{F} = \nabla f$)?

Finding Potentials for Conservative Fields

The test for a vector field being conservative involves the equivalence of certain first partial derivatives of the field components.

> **Component Test for Conservative Fields**
>
> Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on an open simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if
>
> $$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

Proof that Equations (2) hold if F is conservative There is a potential function f such that

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Hence,

$$\begin{aligned}\frac{\partial P}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial^2 f}{\partial y\, \partial z} \\ &= \frac{\partial^2 f}{\partial z\, \partial y} && \text{Mixed Derivative Theorem, Section 14.3} \\ &= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial z}.\end{aligned}$$

The others in Equations (2) are proved similarly. $\blacksquare$

The second half of the proof, that Equations (2) imply that $\mathbf{F}$ is conservative, is a consequence of Stokes' Theorem, taken up in Section 16.7, and requires our assumption that the domain of $\mathbf{F}$ be simply connected.

Once we know that $\mathbf{F}$ is conservative, we usually want to find a potential function for $\mathbf{F}$. This requires solving the equation $\nabla f = \mathbf{F}$ or

$$\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$$

for f. We accomplish this by integrating the three equations

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N, \qquad \frac{\partial f}{\partial z} = P,$$

as illustrated in the next example.

EXAMPLE 3 Show that $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ is conservative over its natural domain and find a potential function for it.

Solution The natural domain of $\mathbf{F}$ is all of space, which is open and simply connected. We apply the test in Equations (2) to

$$M = e^x \cos y + yz, \qquad N = xz - e^x \sin y, \qquad P = xy + z$$

and calculate

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

The partial derivatives are continuous, so these equalities tell us that $\mathbf{F}$ is conservative, so there is a function f with $\nabla f = \mathbf{F}$ (Theorem 2).

We find f by integrating the equations

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z. \tag{3}$$

We integrate the first equation with respect to x, holding y and z fixed, to get

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

We write the constant of integration as a function of y and z because its value may depend on y and z, though not on x. We then calculate $\partial f/\partial y$ from this equation and match it with the expression for $\partial f/\partial y$ in Equations (3). This gives

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y,$$

so $\partial g/\partial y = 0$. Therefore, g is a function of z alone, and

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

We now calculate $\partial f/\partial z$ from this equation and match it to the formula for $\partial f/\partial z$ in Equations (3). This gives

$$xy + \frac{dh}{dz} = xy + z, \qquad \text{or} \qquad \frac{dh}{dz} = z,$$

so

$$h(z) = \frac{z^2}{2} + C.$$

Hence,

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

We found infinitely many potential functions of **F**, one for each value of C. ■

EXAMPLE 4 Show that $\mathbf{F} = (2x - 3)\mathbf{i} - z\mathbf{j} + (\cos z)\mathbf{k}$ is not conservative.

Solution We apply the Component Test in Equations (2) and find immediately that

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(\cos z) = 0, \qquad \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(-z) = -1.$$

The two are unequal, so **F** is not conservative. No further testing is required. ■

EXAMPLE 5 Show that the vector field

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + 0\mathbf{k}$$

satisfies the equations in the Component Test, but is not conservative over its natural domain. Explain why this is possible.

Solution We have $M = -y/(x^2 + y^2)$, $N = x/(x^2 + y^2)$, and $P = 0$. If we apply the Component Test, we find

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial P}{\partial x} = 0 = \frac{\partial M}{\partial z}, \qquad \text{and} \qquad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}.$$

So it may appear that the field **F** passes the Component Test. However, the test assumes that the domain of **F** is simply connected, which is not the case here. Since $x^2 + y^2$ cannot equal zero, the natural domain is the complement of the z-axis and contains loops that cannot be contracted to a point. One such loop is the unit circle C in the xy-plane. The circle is parametrized by $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}$, $0 \le t \le 2\pi$. This loop wraps around the z-axis and cannot be contracted to a point while staying within the complement of the z-axis.

To show that **F** is not conservative, we compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around the loop C. First we write the field in terms of the parameter t:

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} = \frac{-\sin t}{\sin^2 t + \cos^2 t}\mathbf{i} + \frac{\cos t}{\sin^2 t + \cos^2 t}\mathbf{j} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Next we find $d\mathbf{r}/dt = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and then calculate the line integral as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 2\pi.$$

Since the line integral of **F** around the loop C is not zero, the field **F** is not conservative, by Theorem 3. The field **F** is displayed in Figure 16.28d in the next section. ■

Example 5 shows that the Component Test does not apply when the domain of the field is not simply connected. However, if we change the domain in the example so that it is restricted to the ball of radius 1 centered at the point (2, 2, 2), or to any similar ball-shaped region which does not contain a piece of the z-axis, then this new domain D *is* simply connected. Now the partial derivative Equations (2), as well as all the assumptions of the Component Test, are satisfied. In this new situation, the field **F** in Example 5 is conservative on D.

Just as we must be careful with a function when determining if it satisfies a property throughout its domain (like continuity or the Intermediate Value Property), so must we also be careful with a vector field in determining the properties it may or may not have over its assigned domain.

Exact Differential Forms

It is often convenient to express work and circulation integrals in the differential form

$$\int_C M\,dx + N\,dy + P\,dz$$

discussed in Section 16.2. Such line integrals are relatively easy to evaluate if $M\,dx + N\,dy + P\,dz$ is the total differential of a function f and C is any path joining the two points from A to B. For then

$$\begin{aligned}\int_C M\,dx + N\,dy + P\,dz &= \int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= \int_A^B \nabla f \cdot d\mathbf{r} && \nabla f \text{ is conservative.} \\ &= f(B) - f(A). && \text{Theorem 1}\end{aligned}$$

Thus,

$$\int_A^B df = f(B) - f(A),$$

just as with differentiable functions of a single variable.

DEFINITIONS Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Notice that if $M\,dx + N\,dy + P\,dz = df$ on D, then $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is the gradient field of f on D. Conversely, if $\mathbf{F} = \nabla f$, then the form $M\,dx + N\,dy + P\,dz$ is exact. The test for the form's being exact is therefore the same as the test for $\mathbf{F}$ being conservative.

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

EXAMPLE 6 Show that $y\,dx + x\,dy + 4\,dz$ is exact and evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

over any path from $(1, 1, 1)$ to $(2, 3, -1)$.

Solution We let $M = y, N = x, P = 4$ and apply the Test for Exactness:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.$$

These equalities tell us that $y\,dx + x\,dy + 4\,dz$ is exact, so

$$y\,dx + x\,dy + 4\,dz = df$$

for some function f, and the integral's value is $f(2, 3, -1) - f(1, 1, 1)$.

We find f up to a constant by integrating the equations

$$\frac{\partial f}{\partial x} = y, \qquad \frac{\partial f}{\partial y} = x, \qquad \frac{\partial f}{\partial z} = 4. \tag{4}$$

From the first equation we get

$$f(x, y, z) = xy + g(y, z).$$

The second equation tells us that

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x, \qquad \text{or} \qquad \frac{\partial g}{\partial y} = 0.$$

Hence, g is a function of z alone, and

$$f(x, y, z) = xy + h(z).$$

The third of Equations (4) tells us that

$$\frac{\partial f}{\partial z} = 0 + \frac{dh}{dz} = 4, \qquad \text{or} \qquad h(z) = 4z + C.$$

Therefore,

$$f(x, y, z) = xy + 4z + C.$$

The value of the line integral is independent of the path taken from $(1, 1, 1)$ to $(2, 3, -1)$, and equals

$$f(2, 3, -1) - f(1, 1, 1) = 2 + C - (5 + C) = -3. \qquad \blacksquare$$

Exercises 16.3

Testing for Conservative Fields

Which fields in Exercises 1–6 are conservative, and which are not?

1. $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$
2. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
3. $\mathbf{F} = y\mathbf{i} + (x + z)\mathbf{j} - y\mathbf{k}$
4. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$
5. $\mathbf{F} = (z + y)\mathbf{i} + z\mathbf{j} + (y + x)\mathbf{k}$
6. $\mathbf{F} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j} + z\mathbf{k}$

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$
8. $\mathbf{F} = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$
9. $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$
10. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$
11. $\mathbf{F} = (\ln x + \sec^2(x + y))\mathbf{i} + \left(\sec^2(x + y) + \dfrac{y}{y^2 + z^2}\right)\mathbf{j} + \dfrac{z}{y^2 + z^2}\mathbf{k}$
12. $\mathbf{F} = \dfrac{y}{1 + x^2y^2}\mathbf{i} + \left(\dfrac{x}{1 + x^2y^2} + \dfrac{z}{\sqrt{1 - y^2z^2}}\right)\mathbf{j} + \left(\dfrac{y}{\sqrt{1 - y^2z^2}} + \dfrac{1}{z}\right)\mathbf{k}$

Exact Differential Forms

In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.

13. $\displaystyle\int_{(0,0,0)}^{(2,3,-6)} 2x\,dx + 2y\,dy + 2z\,dz$

14. $\displaystyle\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$

15. $\displaystyle\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz$

16. $\displaystyle\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1+z^2}dz$

17. $\displaystyle\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x\,dx + \cos y \sin x\,dy + dz$

Finding Potential Functions to Evaluate Line Integrals

Although they are not defined on all of space R^3, the fields associated with Exercises 18–22 are conservative. Find a potential function for each field and evaluate the integrals as in Example 6.

18. $\displaystyle\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}dz$

19. $\displaystyle\int_{(1,1,1)}^{(1,2,3)} 3x^2\,dx + \frac{z^2}{y}dy + 2z\ln y\,dz$

20. $\displaystyle\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

21. $\displaystyle\int_{(1,1,1)}^{(2,2,2)} \frac{1}{y}dx + \left(\frac{1}{z} - \frac{x}{y^2}\right)dy - \frac{y}{z^2}dz$

22. $\displaystyle\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$

Applications and Examples

23. Revisiting Example 6 Evaluate the integral

$$\int_{(1,1,1)}^{(2,3,-1)} y\,dx + x\,dy + 4\,dz$$

from Example 6 by finding parametric equations for the line segment from (1, 1, 1) to (2, 3, −1) and evaluating the line integral of $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + 4\mathbf{k}$ along the segment. Since $\mathbf{F}$ is conservative, the integral is independent of the path.

24. Evaluate

$$\int_C x^2\,dx + yz\,dy + (y^2/2)\,dz$$

along the line segment C joining (0, 0, 0) to (0, 3, 4).

Independence of path Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from A to B.

25. $\displaystyle\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$ **26.** $\displaystyle\int_A^B \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$

In Exercises 27 and 28, find a potential function for $\mathbf{F}$.

27. $\mathbf{F} = \dfrac{2x}{y}\mathbf{i} + \left(\dfrac{1-x^2}{y^2}\right)\mathbf{j}, \qquad \{(x, y): y > 0\}$

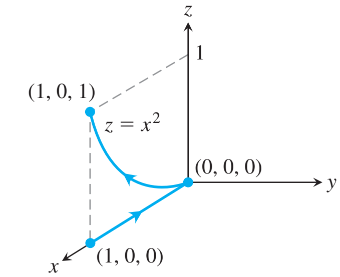
28. $\mathbf{F} = (e^x \ln y)\mathbf{i} + \left(\dfrac{e^x}{y} + \sin z\right)\mathbf{j} + (y\cos z)\mathbf{k}$

29. Work along different paths Find the work done by $\mathbf{F} = (x^2 + y)\mathbf{i} + (y^2 + x)\mathbf{j} + ze^z\mathbf{k}$ over the following paths from (1, 0, 0) to (1, 0, 1).

a. The line segment $x = 1, y = 0, 0 \le z \le 1$

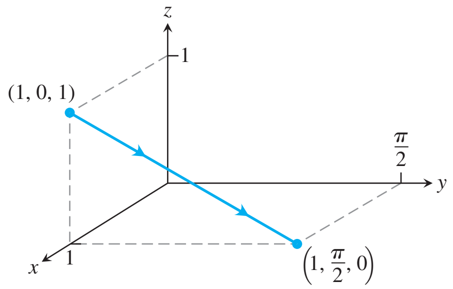
b. The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/2\pi)\mathbf{k}, 0 \le t \le 2\pi$

c. The x-axis from (1, 0, 0) to (0, 0, 0) followed by the parabola $z = x^2, y = 0$ from (0, 0, 0) to (1, 0, 1)

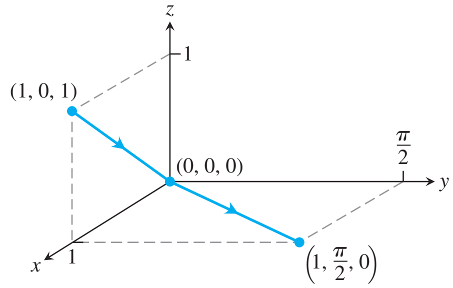


30. Work along different paths Find the work done by $\mathbf{F} = e^{yz}\mathbf{i} + (xze^{yz} + z\cos y)\mathbf{j} + (xye^{yz} + \sin y)\mathbf{k}$ over the following paths from (1, 0, 1) to (1, π/2, 0).

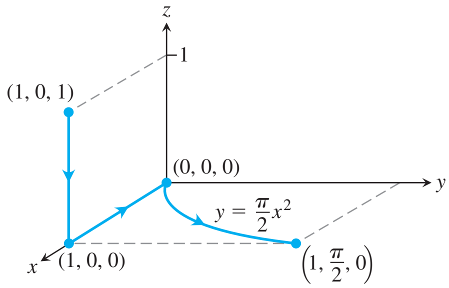
a. The line segment $x = 1, y = \pi t/2, z = 1 - t, 0 \le t \le 1$



b. The line segment from (1, 0, 1) to the origin followed by the line segment from the origin to (1, π/2, 0)



c. The line segment from (1, 0, 1) to (1, 0, 0), followed by the x-axis from (1, 0, 0) to the origin, followed by the parabola $y = \pi x^2/2, z = 0$ from there to (1, π/2, 0)



31. Evaluating a work integral two ways Let $\mathbf{F} = \nabla(x^3y^2)$ and let C be the path in the xy-plane from $(-1, 1)$ to $(1, 1)$ that consists of the line segment from $(-1, 1)$ to $(0, 0)$ followed by the line segment from $(0, 0)$ to $(1, 1)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ in two ways.

a. Find parametrizations for the segments that make up C and evaluate the integral.

b. Use $f(x, y) = x^3y^2$ as a potential function for $\mathbf{F}$.

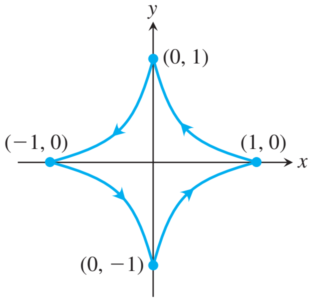
32. Integral along different paths Evaluate the line integral $\int_C 2x \cos y \, dx - x^2 \sin y \, dy$ along the following paths C in the xy-plane.

a. The parabola $y = (x - 1)^2$ from $(1, 0)$ to $(0, 1)$

b. The line segment from $(-1, \pi)$ to $(1, 0)$

c. The x-axis from $(-1, 0)$ to $(1, 0)$

d. The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}$, $0 \le t \le 2\pi$, counterclockwise from $(1, 0)$ back to $(1, 0)$



33. a. Exact differential form How are the constants a, b, and c related if the following differential form is exact?

$$(ay^2 + 2czx)\,dx + y(bx + cz)\,dy + (ay^2 + cx^2)\,dz$$

b. Gradient field For what values of b and c will

$$\mathbf{F} = (y^2 + 2czx)\mathbf{i} + y(bx + cz)\mathbf{j} + (y^2 + cx^2)\mathbf{k}$$

be a gradient field?

34. Gradient of a line integral Suppose that $\mathbf{F} = \nabla f$ is a conservative vector field and

$$g(x, y, z) = \int_{(0,0,0)}^{(x,y,z)} \mathbf{F} \cdot d\mathbf{r}.$$

Show that $\nabla g = \mathbf{F}$.

35. Path of least work You have been asked to find the path along which a force field $\mathbf{F}$ will perform the least work in moving a particle between two locations. A quick calculation on your part shows $\mathbf{F}$ to be conservative. How should you respond? Give reasons for your answer.

36. A revealing experiment By experiment, you find that a force field $\mathbf{F}$ performs only half as much work in moving an object along path C_1 from A to B as it does in moving the object along path C_2 from A to B. What can you conclude about $\mathbf{F}$? Give reasons for your answer.

37. Work by a constant force Show that the work done by a constant force field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in moving a particle along any path from A to B is $W = \mathbf{F} \cdot \overrightarrow{AB}$.

38. Gravitational field

a. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

(G, m, and M are constants).

b. Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

16.4 Green's Theorem in the Plane

If $\mathbf{F}$ is a conservative field, then we know $\mathbf{F} = \nabla f$ for a differentiable function f, and we can calculate the line integral of $\mathbf{F}$ over any path C joining point A to B as $\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$. In this section we derive a method for computing a work or flux integral over a *closed* curve C in the plane when the field $\mathbf{F}$ is *not* conservative. This method comes from Green's Theorem, which allows us to convert the line integral into a double integral over the region enclosed by C.

The discussion is given in terms of velocity fields of fluid flows (a fluid is a liquid or a gas) because they are easy to visualize. However, Green's Theorem applies to any vector field, independent of any particular interpretation of the field, provided the assumptions of the theorem are satisfied. We introduce two new ideas for Green's Theorem: *circulation density* around an axis perpendicular to the plane and *divergence* (or *flux density*).

Spin Around an Axis: The k-Component of Curl

Suppose that $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is the velocity field of a fluid flowing in the plane and that the first partial derivatives of M and N are continuous at each point of a region R. Let (x, y) be a point in R and let A be a small rectangle with one corner at (x, y) that, along with its interior, lies entirely in R. The sides of the rectangle, parallel to the coordinate axes, have lengths of Δx and Δy. Assume that the components M and N do not

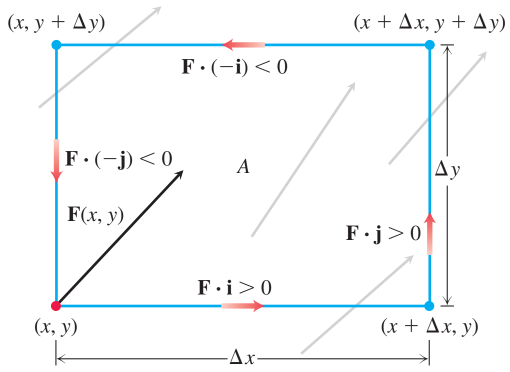


FIGURE 16.26 The rate at which a fluid flows along the bottom edge of a rectangular region A in the direction $\mathbf{i}$ is approximately $\mathbf{F}(x, y) \cdot \mathbf{i}\, \Delta x$, which is positive for the vector field $\mathbf{F}$ shown here. To approximate the rate of circulation at the point (x, y), we calculate the (approximate) flow rates along each edge in the directions of the red arrows, sum these rates, and then divide the sum by the area of A. Taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$ gives the rate of the circulation per unit area.

change sign throughout a small region containing the rectangle A. The first idea we use to convey Green's Theorem quantifies the rate at which a floating paddle wheel, with axis perpendicular to the plane, spins at a point in a fluid flowing in a plane region. This idea gives some sense of how the fluid is circulating around axes located at different points and perpendicular to the region. Physicists sometimes refer to this as the *circulation density* of a vector field $\mathbf{F}$ at a point. To obtain it, we consider the velocity field

$$\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$$

and the rectangle A in Figure 16.26 (where we assume both components of $\mathbf{F}$ are positive).

The circulation rate of $\mathbf{F}$ around the boundary of A is the sum of flow rates along the sides in the tangential direction. For the bottom edge, the flow rate is approximately

$$\mathbf{F}(x, y) \cdot \mathbf{i}\, \Delta x = M(x, y)\Delta x.$$

This is the scalar component of the velocity $\mathbf{F}(x, y)$ in the tangent direction $\mathbf{i}$ times the length of the segment. The flow rates may be positive or negative depending on the components of $\mathbf{F}$. We approximate the net circulation rate around the rectangular boundary of A by summing the flow rates along the four edges as defined by the following dot products.

Top:	$\mathbf{F}(x, y + \Delta y) \cdot (-\mathbf{i})\, \Delta x = -M(x, y + \Delta y)\Delta x$
Bottom:	$\mathbf{F}(x, y) \cdot \mathbf{i}\, \Delta x = M(x, y)\Delta x$
Right:	$\mathbf{F}(x + \Delta x, y) \cdot \mathbf{j}\, \Delta y = N(x + \Delta x, y)\Delta y$
Left:	$\mathbf{F}(x, y) \cdot (-\mathbf{j})\, \Delta y = -N(x, y)\Delta y$

We sum opposite pairs to get

$$\text{Top and bottom:} \qquad -(M(x, y + \Delta y) - M(x, y))\Delta x \approx -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

$$\text{Right and left:} \qquad (N(x + \Delta x, y) - N(x, y))\Delta y \approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y.$$

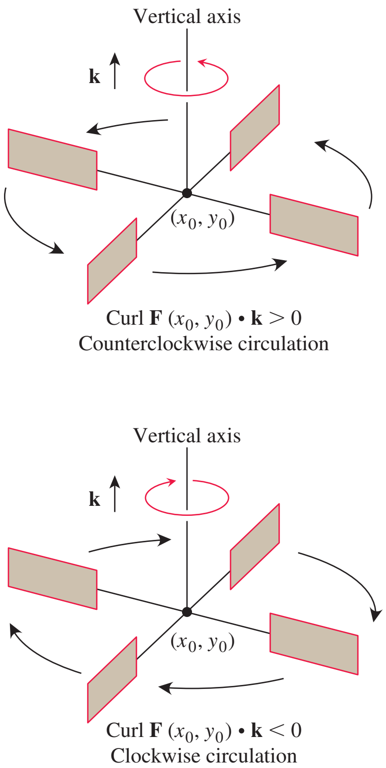


FIGURE 16.27 In the flow of an incompressible fluid over a plane region, the **k**-component of the curl measures the rate of the fluid's rotation at a point. The **k**-component of the curl is positive at points where the rotation is counterclockwise and negative where the rotation is clockwise.

Adding these last two equations gives the net circulation rate relative to the counterclockwise orientation,

$$\text{Circulation rate around rectangle} \approx \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x\, \Delta y.$$

We now divide by $\Delta x\, \Delta y$ to estimate the circulation rate per unit area or *circulation density* for the rectangle:

$$\frac{\text{Circulation around rectangle}}{\text{rectangle area}} \approx \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

We let Δx and Δy approach zero to define the *circulation density* of **F** at the point (x, y).

If we see a counterclockwise rotation looking downward onto the xy-plane from the tip of the unit **k** vector, then the circulation density is positive (Figure 16.27). The value of the circulation density is the **k**-component of a more general circulation vector field we define in Section 16.7, called the *curl* of the vector field **F**. For Green's Theorem, we need only this **k**-component, obtained by taking the dot product of curl **F** with **k**.

> **DEFINITION** The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression
>
> $$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{1}$$
>
> This expression is also called **the k-component of the curl**, denoted by $(\text{curl}\ \mathbf{F}) \cdot \mathbf{k}$.

If water is moving about a region in the xy-plane in a thin layer, then the **k**-component of the curl at a point (x_0, y_0) gives a way to measure how fast and in what direction a small paddle wheel spins if it is put into the water at (x_0, y_0) with its axis perpendicular to the plane, parallel to **k** (Figure 16.27). Looking downward onto the xy-plane, it spins counterclockwise when $(\text{curl}\ \mathbf{F}) \cdot \mathbf{k}$ is positive and clockwise when the **k**-component is negative.

EXAMPLE 1 The following vector fields represent the velocity of a gas flowing in the xy-plane. Find the circulation density of each vector field and interpret its physical meaning Figure 16.28 displays the vector fields.

(a) *Uniform expansion or compression:* $\mathbf{F}(x, y) = cx\mathbf{i} + cy\mathbf{j}$

(b) *Uniform rotation:* $\mathbf{F}(x, y) = -cy\mathbf{i} + cx\mathbf{j}$

(c) *Shearing flow:* $\mathbf{F}(x, y) = y\mathbf{i}$

(d) *Whirlpool effect:* $\mathbf{F}(x, y) = \dfrac{-y}{x^2 + y^2}\mathbf{i} + \dfrac{x}{x^2 + y^2}\mathbf{j}$

Solution

(a) *Uniform expansion:* $(\text{curl}\ \mathbf{F}) \cdot \mathbf{k} = \dfrac{\partial}{\partial k}(cy) - \dfrac{\partial}{\partial y}(cx) = 0$. The gas is not circulating at very small scales.

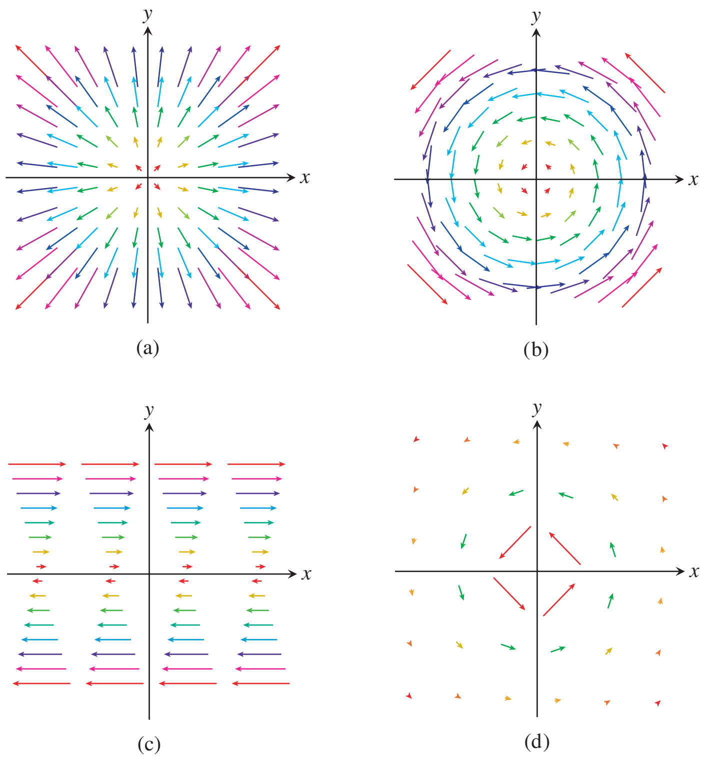


FIGURE 16.28 Velocity fields of a gas flowing in the plane (Example 1).

(b) *Rotation:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}(cx) - \frac{\partial}{\partial y}(-cy) = 2c$. The constant circulation density indicates rotation at every point. If $c > 0$, the rotation is counterclockwise; if $c < 0$, the rotation is clockwise.

(c) *Shear:* $(\text{curl } \mathbf{F}) \cdot \mathbf{k} = -\frac{\partial}{\partial y}(y) = -1$. The circulation density is constant and negative, so a paddle wheel floating in water undergoing such a shearing flow spins clockwise. The rate of rotation is the same at each point. The average effect of the fluid flow is to push fluid clockwise around each of small circles shown in Figure 16.29.

(d) *Whirlpool:*

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} = 0.$$

The circulation density is 0 at every point away from the origin (where the vector field is undefined and the whirlpool effect is taking place), and the gas is not circulating at any point for which the vector field is defined. ■

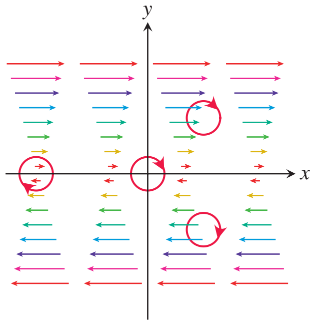


FIGURE 16.29 A shearing flow pushes the fluid clockwise around each point (Example 1c).

One form of Green's Theorem tells us how circulation density can be used to calculate the line integral for flow in the xy-plane. (The flow integral was defined in Section 16.2.) A second form of the theorem tells us how we can calculate the flux integral from *flux density*. We define this idea next, and then we present both versions of the theorem.

Divergence

Consider again the velocity field $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in a domain containing the rectangle A, as shown in Figure 16.30. As before, we assume the field components do not change sign throughout a small region containing the rectangle A. Our interest now is to determine the rate at which the fluid leaves A.

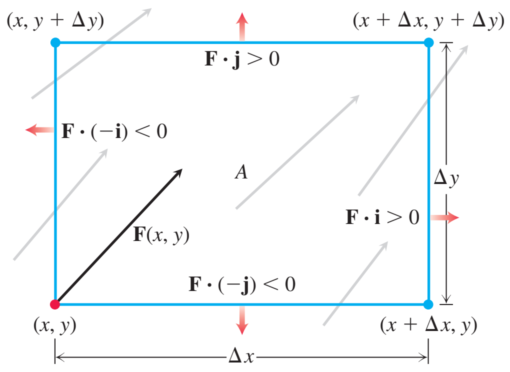


FIGURE 16.30 The rate at which the fluid leaves the rectangular region A across the bottom edge in the direction of the outward normal $-\mathbf{j}$ is approximately $\mathbf{F}(x, y) \cdot (-\mathbf{j})\,\Delta x$, which is negative for the vector field $\mathbf{F}$ shown here. To approximate the flow rate at the point (x, y), we calculate the (approximate) flow rates across each edge in the directions of the red arrows, sum these rates, and then divide the sum by the area of A. Taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$ gives the flow rate per unit area.

The rate at which fluid leaves the rectangle across the bottom edge is approximately (Figure 16.30)

$$\mathbf{F}(x, y) \cdot (-\mathbf{j})\,\Delta x = -N(x, y)\Delta x.$$

This is the scalar component of the velocity at (x, y) in the direction of the outward normal times the length of the segment. If the velocity is in meters per second, for example, the flow rate will be in meters per second times meters or square meters per second. The rates at which the fluid crosses the other three sides in the directions of their outward normals can be estimated in a similar way. The flow rates may be positive or negative depending on the signs of the components of $\mathbf{F}$. We approximate the net flow rate across the rectangular boundary of A by summing the flow rates across the four edges as defined by the following dot products.

Fluid Flow Rates:

Top: $\mathbf{F}(x, y + \Delta y) \cdot \mathbf{j}\,\Delta x = N(x, y + \Delta y)\Delta x$

Bottom: $\mathbf{F}(x, y) \cdot (-\mathbf{j})\,\Delta x = -N(x, y)\Delta x$

Right: $\mathbf{F}(x + \Delta x, y) \cdot \mathbf{i}\,\Delta y = M(x + \Delta x, y)\Delta y$

Left: $\mathbf{F}(x, y) \cdot (-\mathbf{i})\,\Delta y = -M(x, y)\Delta y$

Summing opposite pairs gives

Top and bottom: $$(N(x, y + \Delta y) - N(x, y))\Delta x \approx \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

Right and left: $$(M(x + \Delta x, y) - M(x, y))\Delta y \approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y.$$

Adding these last two equations gives the net effect of the flow rates, or the

$$\text{Flux across rectangle boundary} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y.$$

We now divide by $\Delta x \Delta y$ to estimate the total flux per unit area or *flux density* for the rectangle:

$$\frac{\text{Flux across rectangle boundary}}{\text{rectangle area}} \approx \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right).$$

Finally, we let Δx and Δy approach zero to define the flux density of $\mathbf{F}$ at the point (x, y). In mathematics, we call the flux density the *divergence* of $\mathbf{F}$. The symbol for it is div $\mathbf{F}$, pronounced "divergence of **F**" or "div **F**."

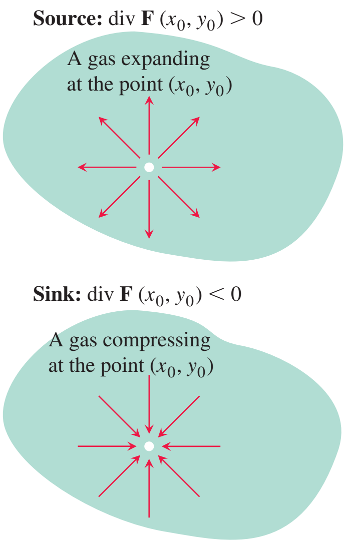


FIGURE 16.31 If a gas is expanding at a point (x_0, y_0), the lines of flow have positive divergence; if the gas is compressing, the divergence is negative.

DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div}\,\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{2}$$

A gas is compressible, unlike a liquid, and the divergence of its velocity field measures to what extent it is expanding or compressing at each point. Intuitively, if a gas is expanding at the point (x_0, y_0), the lines of flow would diverge there (hence the name) and, since the gas would be flowing out of a small rectangle about (x_0, y_0), the divergence of $\mathbf{F}$ at (x_0, y_0) would be positive. If the gas were compressing instead of expanding, the divergence would be negative (Figure 16.31).

EXAMPLE 2 Find the divergence, and interpret what it means, for each vector field in Example 1 representing the velocity of a gas flowing in the xy-plane.

Solution

(a) div $\mathbf{F} = \dfrac{\partial}{\partial x}(cx) + \dfrac{\partial}{\partial y}(cy) = 2c$: If $c > 0$, the gas is undergoing uniform expansion; if $c < 0$, it is undergoing uniform compression.

(b) div $\mathbf{F} = \dfrac{\partial}{\partial x}(-cy) + \dfrac{\partial}{\partial y}(cx) = 0$: The gas is neither expanding nor compressing.

(c) div $\mathbf{F} = \dfrac{\partial}{\partial x}(y) = 0$: The gas is neither expanding nor compressing.

(d) div $\mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{-y}{x^2 + y^2}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2 + y^2}\right) = \dfrac{2xy}{(x^2 + y^2)^2} - \dfrac{2xy}{(x^2 + y^2)^2} = 0$: Again, the divergence is zero at all points in the domain of the velocity field. ■

Cases (b), (c), and (d) of Figure 16.28 are plausible models for the two-dimensional flow of a liquid. In fluid dynamics, when the velocity field of a flowing liquid always has divergence equal to zero, as in those cases, the liquid is said to be **incompressible**.

Two Forms for Green's Theorem

We introduced the notation $\oint_C$ in Section 16.3 for integration around a closed curve. We elaborate further on the notation here. A simple closed curve C can be traversed in two possible directions. (Recall that a curve is simple if it does not cross itself.) The curve is traversed counterclockwise, and said to be *positively oriented*, if the region it encloses is always to the left of an object as it moves along the path. Otherwise it is traversed clockwise and *negatively oriented*. The line integral of a vector field $\mathbf{F}$ along C reverses sign if we change the orientation. We use the notation

$$\oint_C \mathbf{F}(x, y) \cdot d\mathbf{r}$$

for the line integral when the simple closed curve C is traversed counterclockwise, with its positive orientation.

In one form, Green's Theorem says that the counterclockwise circulation of a vector field around a simple closed curve is the double integral of the $\mathbf{k}$-component of the curl of the field over the region enclosed by the curve. Recall the defining Equation (5) for circulation in Section 16.2.

THEOREM 4—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C M\, dx + N\, dy}_{\text{Counterclockwise circulation}} = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy}_{\text{Curl integral}} \tag{3}$$

A second form of Green's Theorem says that the outward flux of a vector field across a simple closed curve in the plane equals the double integral of the divergence of the field over the region enclosed by the curve. Recall the formulas for flux in Equations (6) and (7) in Section 16.2.

THEOREM 5—Green's Theorem (Flux-Divergence or Normal Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy}_{\text{Divergence integral}} \tag{4}$$

The two forms of Green's Theorem are equivalent. Applying Equation (3) to the field $\mathbf{G}_1 = -N\mathbf{i} + M\mathbf{j}$ gives Equation (4), and applying Equation (4) to $\mathbf{G}_2 = N\mathbf{i} - M\mathbf{j}$ gives Equation (3).

Both forms of Green's Theorem can be viewed as two-dimensional generalizations of the Net Change Theorem in Section 5.4. The counterclockwise circulation of $\mathbf{F}$ around C, defined by the line integral on the left-hand side of Equation (3), is the integral of its rate of change (circulation density) over the region R enclosed by C, which is the double integral on the right-hand side of Equation (3). Likewise, the outward flux of $\mathbf{F}$ across C, defined by the line integral on the left-hand side of Equation (4), is the integral of its rate of change (flux density) over the region R enclosed by C, which is the double integral on the right-hand side of Equation (4).

EXAMPLE 3 Verify both forms of Green's Theorem for the vector field

$$\mathbf{F}(x, y) = (x - y)\mathbf{i} + x\mathbf{j}$$

and the region R bounded by the unit circle

$$C: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$$

Solution Evaluating $\mathbf{F}(\mathbf{r}(t))$ and computing the partial derivatives of the components of $\mathbf{F}$, we have

$$\begin{aligned} &M = \cos t - \sin t, && dx = d(\cos t) = -\sin t\,dt, \\ &N = \cos t, && dy = d(\sin t) = \cos t\,dt, \end{aligned}$$

$$\frac{\partial M}{\partial x} = 1, \qquad \frac{\partial M}{\partial y} = -1, \qquad \frac{\partial N}{\partial x} = 1, \qquad \frac{\partial N}{\partial y} = 0.$$

The two sides of Equation (3) are

$$\begin{aligned} \oint_C M\,dx + N\,dy &= \int_{t=0}^{t=2\pi} (\cos t - \sin t)(-\sin t\,dt) + (\cos t)(\cos t\,dt) \\ &= \int_0^{2\pi} (-\sin t \cos t + 1)\,dt = 2\pi \end{aligned}$$

$$\begin{aligned} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy &= \iint_R (1 - (-1))\,dx\,dy \\ &= 2\iint_R dx\,dy = 2(\text{area inside the unit circle}) = 2\pi. \end{aligned}$$

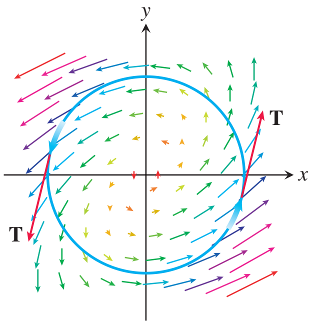


FIGURE 16.32 The vector field in Example 3 has a counterclockwise circulation of 2π around the unit circle.

Figure 16.32 displays the vector field and circulation around C.

The two sides of Equation (4) are

$$\begin{aligned} \oint_C M\,dy - N\,dx &= \int_{t=0}^{t=2\pi} (\cos t - \sin t)(\cos t\,dt) - (\cos t)(-\sin t\,dt) \\ &= \int_0^{2\pi} \cos^2 t\,dt = \pi \end{aligned}$$

$$\begin{aligned} \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy &= \iint_R (1 + 0)\,dx\,dy \\ &= \iint_R dx\,dy = \pi. \end{aligned}$$

■

Using Green's Theorem to Evaluate Line Integrals

If we construct a closed curve C by piecing together a number of different curves end to end, the process of evaluating a line integral over C can be lengthy because there are so many different integrals to evaluate. If C bounds a region R to which Green's Theorem applies, however, we can use Green's Theorem to change the line integral around C into one double integral over R.

EXAMPLE 4 Evaluate the line integral

$$\oint_C xy\,dy - y^2\,dx,$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution We can use either form of Green's Theorem to change the line integral into a double integral over the square, where C is the square's boundary and R is its interior.

1. *With the Tangential Form* Equation (3): Taking $M = -y^2$ and $N = xy$ gives the result:

$$\oint_C -y^2\,dx + xy\,dy = \iint_R (y - (-2y))\,dx\,dy = \int_0^1\int_0^1 3y\,dx\,dy$$

$$= \int_0^1 \left[3xy\right]_{x=0}^{x=1} dy = \int_0^1 3y\,dy = \frac{3}{2}y^2\bigg]_0^1 = \frac{3}{2}.$$

2. *With the Normal Form* Equation (4): Taking $M = xy$, $N = y^2$, gives the same result:

$$\oint_C xy\,dy - y^2\,dx = \iint_R (y + 2y)\,dx\,dy = \frac{3}{2}.$$

EXAMPLE 5 Calculate the outward flux of the vector field $\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's Theorem, we can change the line integral to one double integral. With $M = 2e^{xy}$, $N = y^3$, C the square, and R the square's interior, we have

$$\text{Flux} = \oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \qquad \text{Green's Theorem, Eq. (4)}$$

$$= \int_{-1}^1\int_{-1}^1 (2ye^{xy} + 3y^2)\,dx\,dy = \int_{-1}^1 \left[2e^{xy} + 3xy^2\right]_{x=-1}^{x=1} dy$$

$$= \int_{-1}^1 (2e^y + 6y^2 - 2e^{-y})\,dy = \left[2e^y + 2y^3 + 2e^{-y}\right]_{-1}^1 = 4.$$

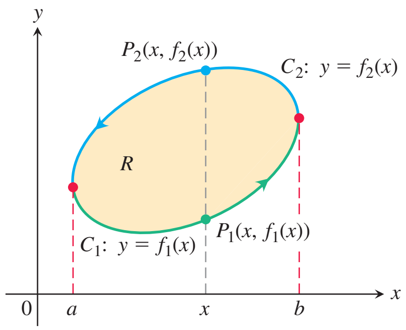


FIGURE 16.33 The boundary curve C is made up of C_1, the graph of $y = f_1(x)$, and C_2, the graph of $y = f_2(x)$.

Proof of Green's Theorem for Special Regions

Let C be a smooth simple closed curve in the xy-plane with the property that lines parallel to the axes cut it at no more than two points. Let R be the region enclosed by C and suppose that M, N, and their first partial derivatives are continuous at every point of some open region containing C and R. We want to prove the circulation-curl form of Green's Theorem,

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \tag{5}$$

Figure 16.33 shows C made up of two directed parts:

$$C_1:\quad y = f_1(x),\quad a \le x \le b, \qquad C_2:\quad y = f_2(x),\quad b \ge x \ge a.$$

For any x between a and b, we can integrate $\partial M/\partial y$ with respect to y from $y = f_1(x)$ to $y = f_2(x)$ and obtain

$$\int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy = M(x, y)\Big]_{y=f_1(x)}^{y=f_2(x)} = M(x, f_2(x)) - M(x, f_1(x)).$$

We can then integrate this with respect to x from a to b:

$$\begin{aligned}
\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}\,dy\,dx &= \int_a^b \left[M(x, f_2(x)) - M(x, f_1(x))\right] dx \\
&= -\int_b^a M(x, f_2(x))\,dx - \int_a^b M(x, f_1(x))\,dx \\
&= -\int_{C_2} M\,dx - \int_{C_1} M\,dx \\
&= -\oint_C M\,dx.
\end{aligned}$$

Therefore, reversing the order of the equations, we have

$$\oint_C M\,dx = \iint_R \left(-\frac{\partial M}{\partial y}\right) dx\,dy. \tag{6}$$

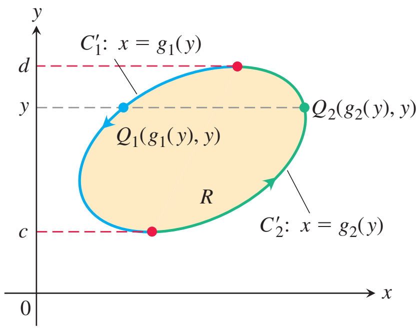


FIGURE 16.34 The boundary curve C is made up of C_1', the graph of $x = g_1(y)$, and C_2', the graph of $x = g_2(y)$.

Equation (6) is half the result we need for Equation (5). We derive the other half by integrating $\partial N/\partial x$ first with respect to x and then with respect to y, as suggested by Figure 16.34. This shows the curve C of Figure 16.33 decomposed into the two directed parts C_1': $x = g_1(y)$, $d \ge y \ge c$ and C_2': $x = g_2(y)$, $c \le y \le d$. The result of this double integration is

$$\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy. \tag{7}$$

Summing Equations (6) and (7) gives Equation (5). This concludes the proof. ■

Green's Theorem also holds for more general regions, such as those shown in Figure 16.35, but we will not prove this result here. Notice that the region in Figure 16.35(c) is not simply connected. The curves C_1 and C_h on its boundary are oriented so that the region R is always on the left-hand side as the curves are traversed in the directions shown, and cancelation occurs over common boundary arcs traversed in opposite directions. With this convention, Green's Theorem is valid for regions that are not simply connected.

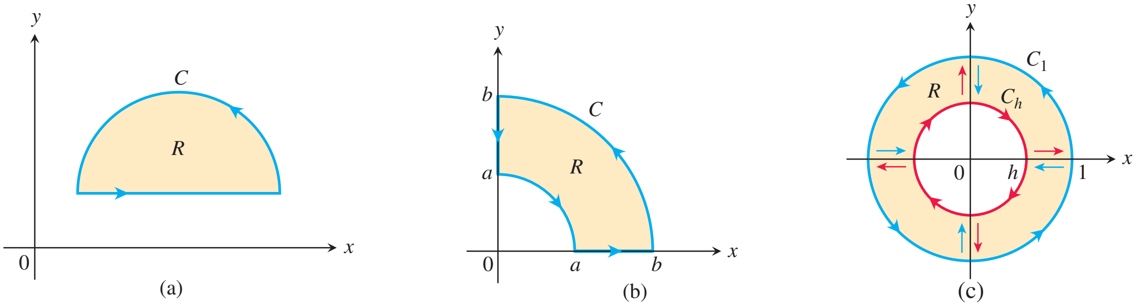


FIGURE 16.35 Other regions to which Green's Theorem applies. In (c) the axes convert the region into four simply connected regions, and we sum the line integrals along the oriented boundaries.

Exercises 16.4

Verifying Green's Theorem

In Exercises 1–4, verify the conclusion of Green's Theorem by evaluating both sides of Equations (3) and (4) for the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. Take the domains of integration in each case to be the disk R: $x^2 + y^2 \le a^2$ and its bounding circle C: $\mathbf{r} = (a \cos t)\mathbf{i} + (a \sin t)\mathbf{j}$, $0 \le t \le 2\pi$.

1. $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$

2. $\mathbf{F} = y\mathbf{i}$

3. $\mathbf{F} = 2x\mathbf{i} - 3y\mathbf{j}$

4. $\mathbf{F} = -x^2y\mathbf{i} + xy^2\mathbf{j}$

Circulation and Flux

In Exercises 5–14, use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

5. $\mathbf{F} = (x - y)\mathbf{i} + (y - x)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

6. $\mathbf{F} = (x^2 + 4y)\mathbf{i} + (x + y^2)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

7. $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$

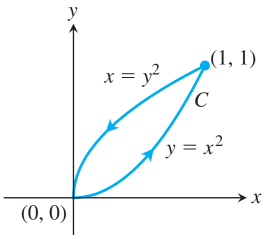
C: The triangle bounded by $y = 0, x = 3$, and $y = x$

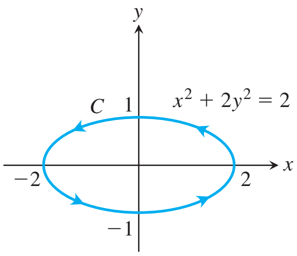
8. $\mathbf{F} = (x + y)\mathbf{i} - (x^2 + y^2)\mathbf{j}$

C: The triangle bounded by $y = 0, x = 1$, and $y = x$

9. $\mathbf{F} = (xy + y^2)\mathbf{i} + (x - y)\mathbf{j}$

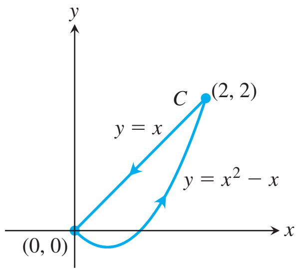
10. $\mathbf{F} = (x + 3y)\mathbf{i} + (2x - y)\mathbf{j}$

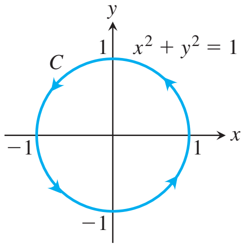




11. $\mathbf{F} = x^3y^2\mathbf{i} + \frac{1}{2}x^4y\,\mathbf{j}$

12. $\mathbf{F} = \dfrac{x}{1 + y^2}\mathbf{i} + \left(\tan^{-1} y\right)\mathbf{j}$





13. $\mathbf{F} = (x + e^x \sin y)\mathbf{i} + (x + e^x \cos y)\mathbf{j}$

C: The right-hand loop of the lemniscate $r^2 = \cos 2\theta$

14. $\mathbf{F} = \left(\tan^{-1} \dfrac{y}{x}\right)\mathbf{i} + \ln(x^2 + y^2)\mathbf{j}$

C: The boundary of the region defined by the polar coordinate inequalities $1 \le r \le 2, 0 \le \theta \le \pi$

15. Find the counterclockwise circulation and outward flux of the field $\mathbf{F} = xy\mathbf{i} + y^2\mathbf{j}$ around and over the boundary of the region enclosed by the curves $y = x^2$ and $y = x$ in the first quadrant.

16. Find the counterclockwise circulation and the outward flux of the field $\mathbf{F} = (-\sin y)\mathbf{i} + (x \cos y)\mathbf{j}$ around and over the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

17. Find the outward flux of the field

$$\mathbf{F} = \left(3xy - \frac{x}{1 + y^2}\right)\mathbf{i} + (e^x + \tan^{-1} y)\mathbf{j}$$

across the cardioid $r = a(1 + \cos \theta)$, $a > 0$.

18. Find the counterclockwise circulation of $\mathbf{F} = (y + e^x \ln y)\mathbf{i} + (e^x/y)\mathbf{j}$ around the boundary of the region that is bounded above by the curve $y = 3 - x^2$ and below by the curve $y = x^4 + 1$.

Work

In Exercises 19 and 20, find the work done by $\mathbf{F}$ in moving a particle once counterclockwise around the given curve.

19. $\mathbf{F} = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$

C: The boundary of the "triangular" region in the first quadrant enclosed by the x-axis, the line $x = 1$, and the curve $y = x^3$

20. $\mathbf{F} = (4x - 2y)\mathbf{i} + (2x - 4y)\mathbf{j}$

C: The circle $(x - 2)^2 + (y - 2)^2 = 4$

Using Green's Theorem

Apply Green's Theorem to evaluate the integrals in Exercises 21–24.

21. $\displaystyle\oint_C (y^2\,dx + x^2\,dy)$

C: The triangle bounded by $x = 0, x + y = 1, y = 0$

22. $\displaystyle\oint_C (3y\,dx + 2x\,dy)$

C: The boundary of $0 \le x \le \pi, 0 \le y \le \sin x$

23. $\displaystyle\oint_C (6y + x)\,dx + (y + 2x)\,dy$

C: The circle $(x - 2)^2 + (y - 3)^2 = 4$

24. $\displaystyle\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy$

C: Any simple closed curve in the plane for which Green's Theorem holds

Calculating Area with Green's Theorem If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

> **Green's Theorem Area Formula**
>
> $$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx$$

The reason is that by Equation (4), run backward,

$$\begin{aligned}\text{Area of } R &= \iint_R dy\,dx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dy\,dx \\ &= \oint_C \frac{1}{2}x\,dy - \frac{1}{2}y\,dx.\end{aligned}$$

Use the Green's Theorem area formula given above to find the areas of the regions enclosed by the curves in Exercises 25–28.

25. The circle $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

26. The ellipse $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$

27. The astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, \quad 0 \le t \le 2\pi$

28. One arch of the cycloid $x = t - \sin t, \quad y = 1 - \cos t$

29. Let C be the boundary of a region on which Green's Theorem holds. Use Green's Theorem to calculate

a. $\displaystyle\oint_C f(x)\,dx + g(y)\,dy$

b. $\displaystyle\oint_C ky\,dx + hx\,dy$ (k and h constants).

30. Integral dependent only on area Show that the value of

$$\oint_C xy^2\,dx + (x^2y + 2x)\,dy$$

around any square depends only on the area of the square and not on its location in the plane.

31. Evaluate the integral

$$\oint_C 4x^3y\,dx + x^4\,dy$$

for any closed path C.

32. Evaluate the integral

$$\oint_C -y^3\,dy + x^3\,dx$$

for any closed path C.

33. Area as a line integral Show that if R is a region in the plane bounded by a piecewise smooth, simple closed curve C, then

$$\text{Area of } R = \oint_C x\,dy = -\oint_C y\,dx.$$

34. Definite integral as a line integral Suppose that a nonnegative function $y = f(x)$ has a continuous first derivative on $[a, b]$. Let C be the boundary of the region in the xy-plane that is bounded below by the x-axis, above by the graph of f, and on the sides by the lines $x = a$ and $x = b$. Show that

$$\int_a^b f(x)\,dx = -\oint_C y\,dx.$$

35. Area and the centroid Let A be the area and $\bar{x}$ the x-coordinate of the centroid of a region R that is bounded by a piecewise smooth, simple closed curve C in the xy-plane. Show that

$$\frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\bar{x}.$$

36. Moment of inertia Let I_y be the moment of inertia about the y-axis of the region in Exercise 35. Show that

$$\frac{1}{3}\oint_C x^3\,dy = -\oint_C x^2y\,dx = \frac{1}{4}\oint_C x^3\,dy - x^2y\,dx = I_y.$$

37. Green's Theorem and Laplace's equation Assuming that all the necessary derivatives exist and are continuous, show that if $f(x, y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

then

$$\oint_C \frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy = 0$$

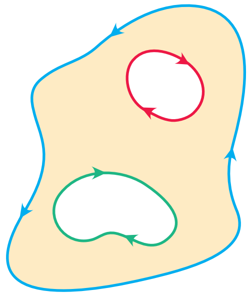
for all closed curves C to which Green's Theorem applies. (The converse is also true: If the line integral is always zero, then f satisfies the Laplace equation.)

38. Maximizing work Among all smooth, simple closed curves in the plane, oriented counterclockwise, find the one along which the work done by

$$\mathbf{F} = \left(\frac{1}{4}x^2y + \frac{1}{3}y^3\right)\mathbf{i} + x\mathbf{j}$$

is greatest. (*Hint:* Where is (curl $\mathbf{F}$) $\cdot$ $\mathbf{k}$ positive?)

39. Regions with many holes Green's Theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along (see accompanying figure).



a. Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \mathbf{n}\, ds.$$

b. Let K be an arbitrary smooth, simple closed curve in the plane that does not pass through (0, 0). Use Green's Theorem to show that

$$\oint_K \nabla f \cdot \mathbf{n}\, ds$$

has two possible values, depending on whether (0, 0) lies inside K or outside K.

40. Bendixson's criterion The *streamlines* of a planar fluid flow are the smooth curves traced by the fluid's individual particles. The vectors $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ of the flow's velocity field are the tangent vectors of the streamlines. Show that if the flow takes place over a simply connected region R (no holes or missing points) and that if $M_x + N_y \neq 0$ throughout R, then none of the streamlines in R is closed. In other words, no particle of fluid ever has a closed trajectory in R. The criterion $M_x + N_y \neq 0$ is called **Bendixson's criterion** for the nonexistence of closed trajectories.

41. Establish Equation (7) to finish the proof of the special case of Green's Theorem.

42. Curl component of conservative fields Can anything be said about the curl component of a conservative two-dimensional vector field? Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 43–46, use a CAS and Green's Theorem to find the counterclockwise circulation of the field $\mathbf{F}$ around the simple closed curve C. Perform the following CAS steps.

a. Plot C in the xy-plane.

b. Determine the integrand $(\partial N/\partial x) - (\partial M/\partial y)$ for the tangential form of Green's Theorem.

c. Determine the (double integral) limits of integration from your plot in part (a) and evaluate the curl integral for the circulation.

43. $\mathbf{F} = (2x - y)\mathbf{i} + (x + 3y)\mathbf{j}$, C: The ellipse $x^2 + 4y^2 = 4$

44. $\mathbf{F} = (2x^3 - y^3)\mathbf{i} + (x^3 + y^3)\mathbf{j}$, C: The ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

45. $\mathbf{F} = x^{-1}e^y\mathbf{i} + (e^y \ln x + 2x)\mathbf{j}$,

C: The boundary of the region defined by $y = 1 + x^4$ (below) and $y = 2$ (above)

46. $\mathbf{F} = xe^y\mathbf{i} + (4x^2 \ln y)\mathbf{j}$,

C: The triangle with vertices (0, 0), (2, 0), and (0, 4)

16.5 Surfaces and Area

We have defined curves in the plane in three different ways:

Explicit form:	$y = f(x)$
Implicit form:	$F(x, y) = 0$
Parametric vector form:	$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}, \quad a \le t \le b.$

We have analogous definitions of surfaces in space:

Explicit form:	$z = f(x, y)$
Implicit form:	$F(x, y, z) = 0.$

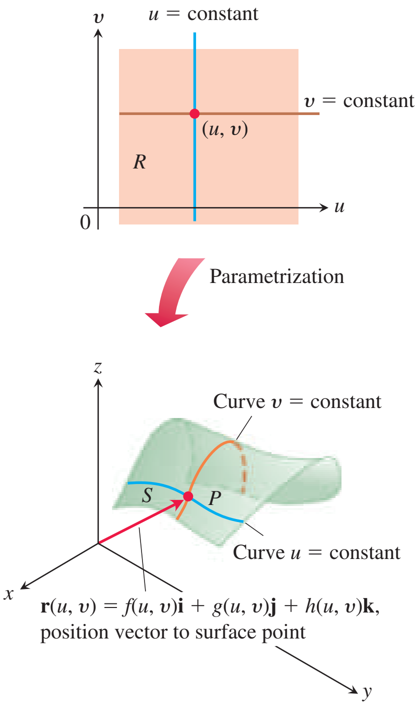


FIGURE 16.36 A parametrized surface S expressed as a vector function of two variables defined on a region R.

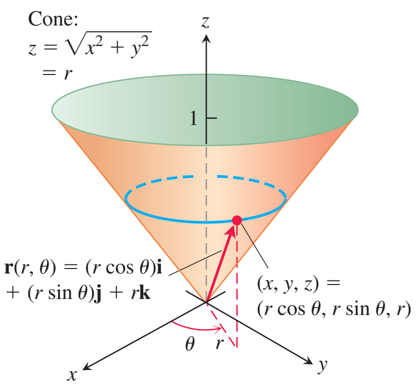


FIGURE 16.37 The cone in Example 1 can be parametrized using cylindrical coordinates.

There is also a parametric form for surfaces that gives the position of a point on the surface as a vector function of two variables. We discuss this new form in this section and apply the form to obtain the area of a surface as a double integral. Double integral formulas for areas of surfaces given in implicit and explicit forms are then obtained as special cases of the more general parametric formula.

Parametrizations of Surfaces

Suppose

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k} \tag{1}$$

is a continuous vector function that is defined on a region R in the uv-plane and one-to-one on the interior of R (Figure 16.36). We call the range of $\mathbf{r}$ the **surface** S defined or traced by $\mathbf{r}$. Equation (1) together with the domain R constitutes a **parametrization** of the surface. The variables u and v are the **parameters**, and R is the **parameter domain**. To simplify our discussion, we take R to be a rectangle defined by inequalities of the form $a \le u \le b$, $c \le v \le d$. The requirement that $\mathbf{r}$ be one-to-one on the interior of R ensures that S does not cross itself. Notice that Equation (1) is the vector equivalent of *three* parametric equations:

$$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v).$$

EXAMPLE 1 Find a parametrization of the cone

$$z = \sqrt{x^2 + y^2}, \qquad 0 \le z \le 1.$$

Solution Here, cylindrical coordinates provide a parametrization. A typical point (x, y, z) on the cone (Figure 16.37) has $x = r\cos\theta$, $y = r\sin\theta$, and $z = \sqrt{x^2 + y^2} = r$, with $0 \le r \le 1$ and $0 \le \theta \le 2\pi$. Taking $u = r$ and $v = \theta$ in Equation (1) gives the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

The parametrization is one-to-one on the interior of the domain R, though not on the boundary tip of its cone where $r = 0$. ■

EXAMPLE 2 Find a parametrization of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution Spherical coordinates provide what we need. A typical point (x, y, z) on the sphere (Figure 16.38) has $x = a\sin\phi\cos\theta$, $y = a\sin\phi\sin\theta$, and $z = a\cos\phi$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Taking $u = \phi$ and $v = \theta$ in Equation (1) gives the parametrization

$$\mathbf{r}(\phi, \theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k},$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

Again, the parametrization is one-to-one on the interior of the domain R, though not on its boundary "poles" where $\phi = 0$ or $\phi = \pi$. ■

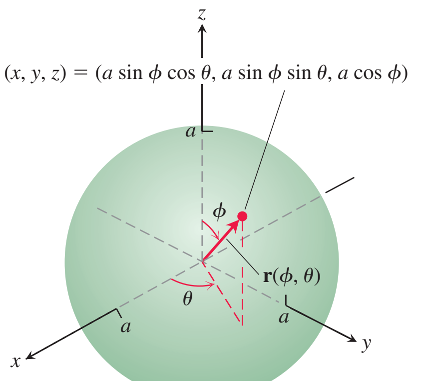


FIGURE 16.38 The sphere in Example 2 can be parametrized using spherical coordinates.

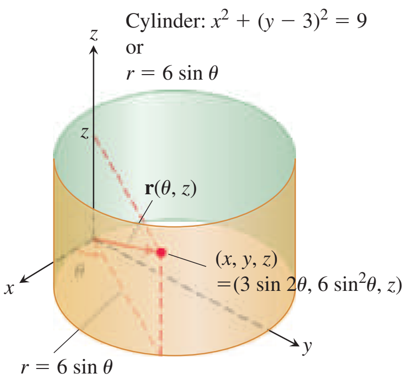


FIGURE 16.39 The cylinder in Example 3 can be parametrized using cylindrical coordinates.

EXAMPLE 3 Find a parametrization of the cylinder

$$x^2 + (y-3)^2 = 9, \qquad 0 \le z \le 5.$$

Solution In cylindrical coordinates, a point (x, y, z) has $x = r\cos\theta$, $y = r\sin\theta$, and $z = z$. For points on the cylinder $x^2 + (y-3)^2 = 9$ (Figure 16.39), the equation is the same as the polar equation for the cylinder's base in the xy-plane:

$$\begin{aligned} x^2 + \left(y^2 - 6y + 9\right) &= 9 \\ r^2 - 6r\sin\theta &= 0 \end{aligned} \qquad \begin{aligned} &x^2 + y^2 = r^2, \\ &y = r\sin\theta \end{aligned}$$

or

$$r = 6\sin\theta, \qquad 0 \le \theta \le \pi.$$

A typical point on the cylinder therefore has

$$\begin{aligned} x &= r\cos\theta = 6\sin\theta\cos\theta = 3\sin 2\theta \\ y &= r\sin\theta = 6\sin^2\theta \\ z &= z. \end{aligned}$$

Taking $u = \theta$ and $v = z$ in Equation (1) gives the one-to-one parametrization

$$\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}, \quad 0 \le \theta \le \pi, \quad 0 \le z \le 5. \qquad \blacksquare$$

Surface Area

Our goal is to find a double integral for calculating the area of a curved surface S based on the parametrization

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \quad a \le u \le b, \quad c \le v \le d.$$

We need S to be smooth for the construction we are about to carry out. The definition of smoothness involves the partial derivatives of $\mathbf{r}$ with respect to u and v:

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial f}{\partial u}\mathbf{i} + \frac{\partial g}{\partial u}\mathbf{j} + \frac{\partial h}{\partial u}\mathbf{k} \\ \mathbf{r}_v &= \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial f}{\partial v}\mathbf{i} + \frac{\partial g}{\partial v}\mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}. \end{aligned}$$

DEFINITION A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.

The condition that $\mathbf{r}_u \times \mathbf{r}_v$ is never the zero vector in the definition of smoothness means that the two vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and never lie along the same line, so they always determine a plane tangent to the surface. We relax this condition on the boundary of the domain, but this does not affect the area computations.

Now consider a small rectangle ΔA_{uv} in R with sides on the lines $u = u_0$, $u = u_0 + \Delta u$, $v = v_0$, and $v = v_0 + \Delta v$ (Figure 16.40). Each side of ΔA_{uv} maps to a curve on the surface S, and together these four curves bound a "curved patch element" $\Delta\sigma_{uv}$. In the notation of the figure, the side $v = v_0$ maps to curve C_1, the side $u = u_0$ maps to C_2, and their common vertex (u_0, v_0) maps to P_0.

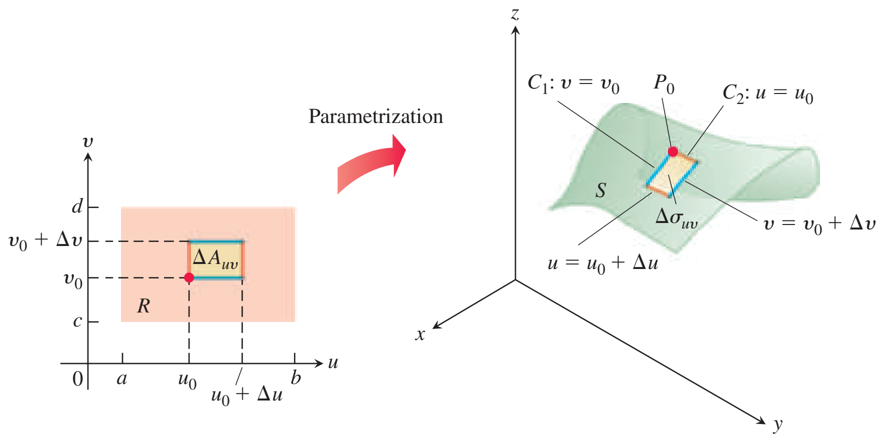


FIGURE 16.40 A rectangular area element ΔA_{uv} in the uv-plane maps onto a curved patch element $\Delta\sigma_{uv}$ on S.

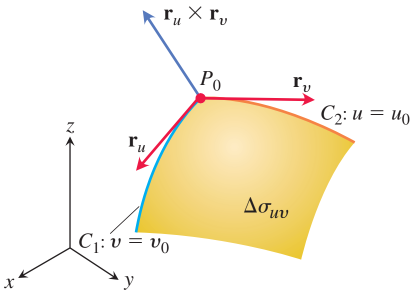


FIGURE 16.41 A magnified view of a surface patch element $\Delta\sigma_{uv}$.

Figure 16.41 shows an enlarged view of $\Delta\sigma_{uv}$. The partial derivative vector $\mathbf{r}_u(u_0, v_0)$ is tangent to C_1 at P_0. Likewise, $\mathbf{r}_v(u_0, v_0)$ is tangent to C_2 at P_0. The cross product $\mathbf{r}_u \times \mathbf{r}_v$ is normal to the surface at P_0. (Here is where we begin to use the assumption that S is smooth. We want to be sure that $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$.)

We next approximate the surface patch element $\Delta\sigma_{uv}$ by the parallelogram on the tangent plane whose sides are determined by the vectors $\Delta u\mathbf{r}_u$ and $\Delta v\mathbf{r}_v$ (Figure 16.42). The area of this parallelogram is

$$|\Delta u\mathbf{r}_u \times \Delta v\mathbf{r}_v| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v. \tag{2}$$

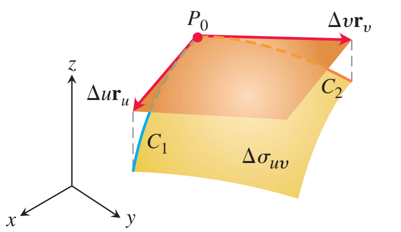


FIGURE 16.42 The area of the parallelogram determined by the vectors $\Delta u\mathbf{r}_u$ and $\Delta v\mathbf{r}_v$ approximates the area of the surface patch element $\Delta\sigma_{uv}$.

A partition of the region R in the uv-plane by rectangular regions ΔA_{uv} induces a partition of the surface S into surface patch elements $\Delta\sigma_{uv}$. We approximate the area of each surface patch element $\Delta\sigma_{uv}$ by the parallelogram area in Equation (2) and sum these areas together to obtain an approximation of the surface area of S:

$$\sum_n |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v. \tag{3}$$

As Δu and Δv approach zero independently, the number of area elements n tends to ∞ and the continuity of $\mathbf{r}_u$ and $\mathbf{r}_v$ guarantees that the sum in Equation (3) approaches the double integral $\int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$. This double integral over the region R defines the area of the surface S.

DEFINITION The **area** of the smooth surface

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{4}$$

We can abbreviate the integral in Equation (4) by writing $d\sigma$ for $|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$. The surface area differential $d\sigma$ is analogous to the arc length differential ds in Section 13.3.

Surface Area Differential for a Parametrized Surface

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv \qquad \iint_S d\sigma \tag{5}$$

Surface area differential — Differential formula for surface area

EXAMPLE 4 Find the surface area of the cone in Example 1 (Figure 16.37).

Solution In Example 1, we found the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

To apply Equation (4), we first find $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\begin{aligned}\mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \\ &= -(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{r}\mathbf{k}.\end{aligned}$$

Thus, $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$. The area of the cone is

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta|\,dr\,d\theta \qquad \text{Eq. (4) with } u = r, v = \theta \\ &= \int_0^{2\pi}\int_0^1 \sqrt{2}\,r\,dr\,d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2}\,d\theta = \frac{\sqrt{2}}{2}(2\pi) = \pi\sqrt{2} \text{ units squared.} \end{aligned}$$

■

EXAMPLE 5 Find the surface area of a sphere of radius a.

Solution We use the parametrization from Example 2:

$$\mathbf{r}(\phi, \theta) = (a\sin\phi\cos\theta)\mathbf{i} + (a\sin\phi\sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k},$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

For $\mathbf{r}_\phi \times \mathbf{r}_\theta$, we get

$$\begin{aligned}\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} \\ &= (a^2\sin^2\phi\cos\theta)\mathbf{i} + (a^2\sin^2\phi\sin\theta)\mathbf{j} + (a^2\sin\phi\cos\phi)\mathbf{k}.\end{aligned}$$

Thus,

$$\begin{aligned}|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi} \\ &= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} = \sqrt{a^4\sin^2\phi(\sin^2\phi + \cos^2\phi)} \\ &= a^2\sqrt{\sin^2\phi} = a^2\sin\phi,\end{aligned}$$

since $\sin\phi \geq 0$ for $0 \leq \phi \leq \pi$. Therefore, the area of the sphere is

$$\begin{aligned} A &= \int_0^{2\pi}\int_0^{\pi} a^2 \sin\phi \, d\phi \, d\theta \\ &= \int_0^{2\pi}\Big[-a^2\cos\phi\Big]_0^{\pi} d\theta = \int_0^{2\pi} 2a^2 \, d\theta = 4\pi a^2 \quad \text{units squared.} \end{aligned}$$

This agrees with the well-known formula for the surface area of a sphere. ■

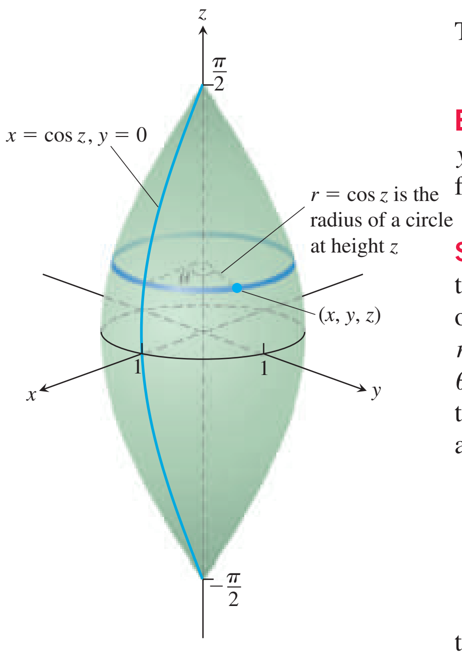


FIGURE 16.43 The "football" surface in Example 6 obtained by rotating the curve $x = \cos z$ about the z-axis.

EXAMPLE 6 Let S be the "football" surface formed by rotating the curve $x = \cos z$, $y = 0, -\pi/2 \leq z \leq \pi/2$ around the z-axis (see Figure 16.43). Find a parametrization for S and compute its surface area.

Solution Example 2 suggests finding a parametrization of S based on its rotation around the z-axis. If we rotate a point $(x, 0, z)$ on the curve $x = \cos z$, $y = 0$ about the z-axis, we obtain a circle at height z above the xy-plane that is centered on the z-axis and has radius $r = \cos z$ (see Figure 16.43). The point sweeps out the circle through an angle of rotation θ, $0 \leq \theta \leq 2\pi$. We let (x, y, z) be an arbitrary point on this circle, and define the parameters $u = z$ and $v = \theta$. Then we have $x = r\cos\theta = \cos u \cos v$, $y = r\sin\theta = \cos u \sin v$, and $z = u$ giving a parametrization for S as

$$\mathbf{r}(u, v) = \cos u \cos v\,\mathbf{i} + \cos u \sin v\,\mathbf{j} + u\mathbf{k}, \quad -\frac{\pi}{2} \leq u \leq \frac{\pi}{2}, \quad 0 \leq v \leq 2\pi.$$

Next we use Equation (5) to find the surface area of S. Differentiation of the parametrization gives

$$\mathbf{r}_u = -\sin u \cos v\,\mathbf{i} - \sin u \sin v\,\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{r}_v = -\cos u \sin v\,\mathbf{i} + \cos u \cos v\,\mathbf{j}.$$

Computing the cross product we have

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix} \\ &= -\cos u \cos v\,\mathbf{i} - \cos u \sin v\,\mathbf{j} - (\sin u \cos u \cos^2 v + \cos u \sin u \sin^2 v)\mathbf{k}. \end{aligned}$$

Taking the magnitude of the cross product gives

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{\cos^2 u\,(\cos^2 v + \sin^2 v) + \sin^2 u \cos^2 u} \\ &= \sqrt{\cos^2 u\,(1 + \sin^2 u)} \\ &= \cos u\,\sqrt{1 + \sin^2 u}. \qquad \cos u \geq 0 \text{ for } -\frac{\pi}{2} \leq u \leq \frac{\pi}{2} \end{aligned}$$

From Equation (4) the surface area is given by the integral

$$A = \int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \cos u\,\sqrt{1 + \sin^2 u}\,du\,dv.$$

To evaluate the integral, we substitute $w = \sin u$ and $dw = \cos u\, du$, $-1 \le w \le 1$. Since the surface S is symmetric across the xy-plane, we need only integrate with respect to w from 0 to 1, and multiply the result by 2. In summary, we have

$$\begin{aligned}
A &= 2\int_0^{2\pi}\int_0^1 \sqrt{1+w^2}\, dw\, dv \\
&= 2\int_0^{2\pi}\left[\frac{w}{2}\sqrt{1+w^2} + \frac{1}{2}\ln\left(w + \sqrt{1+w^2}\right)\right]_0^1 dv \qquad \text{Integral Table Formula 35} \\
&= \int_0^{2\pi} 2\left[\frac{1}{2}\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)\right] dv \\
&= 2\pi\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right].
\end{aligned}$$

Implicit Surfaces

Surfaces are often presented as level sets of a function, described by an equation such as

$$F(x, y, z) = c,$$

for some constant c. Such a level surface does not come with an explicit parametrization, and is called an *implicitly defined surface.* Implicit surfaces arise, for example, as equipotential surfaces in electric or gravitational fields. Figure 16.44 shows a piece of such a surface. It may be difficult to find explicit formulas for the functions f, g, and h that describe the surface in the form $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$. We now show how to compute the surface area differential $d\sigma$ for implicit surfaces.

Figure 16.44 shows a piece of an implicit surface S that lies above its "shadow" region R in the plane beneath it. The surface is defined by the equation $F(x, y, z) = c$ and $\mathbf{p}$ is a unit vector normal to the plane region R. We assume that the surface is **smooth** (F is differentiable and ∇F is nonzero and continuous on S) and that $\nabla F \cdot \mathbf{p} \neq 0$, so the surface never folds back over itself.

Assume that the normal vector $\mathbf{p}$ is the unit vector $\mathbf{k}$, so the region R in Figure 16.44 lies in the xy-plane. By assumption, we then have $\nabla F \cdot \mathbf{p} = \nabla F \cdot \mathbf{k} = F_z \neq 0$ on S. The Implicit Function Theorem (see Section 14.4) implies that S is then the graph of a differentiable function $z = h(x, y)$, although the function $h(x, y)$ is not explicitly known. Define the parameters u and v by $u = x$ and $v = y$. Then $z = h(u, v)$ and

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + h(u, v)\mathbf{k} \tag{6}$$

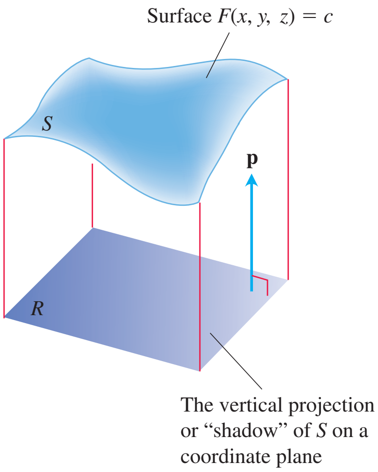


FIGURE 16.44 As we soon see, the area of a surface S in space can be calculated by evaluating a related double integral over the vertical projection or "shadow" of S on a coordinate plane. The unit vector $\mathbf{p}$ is normal to the plane.

gives a parametrization of the surface S. We use Equation (4) to find the area of S.

Calculating the partial derivatives of $\mathbf{r}$, we find

$$\mathbf{r}_u = \mathbf{i} + \frac{\partial h}{\partial u}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = \mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

Applying the Chain Rule for implicit differentiation (see Equation (2) in Section 14.4) to $F(x, y, z) = c$, where $x = u$, $y = v$, and $z = h(u, v)$, we obtain the partial derivatives

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \qquad \text{and} \qquad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}.$$

Substitution of these derivatives into the derivatives of $\mathbf{r}$ gives

$$\mathbf{r}_u = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}.$$

From a routine calculation of the cross product we find

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} && F_z \neq 0 \\
&= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\
&= \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot \mathbf{k}} \\
&= \frac{\nabla F}{\nabla F \cdot \mathbf{p}}. && \mathbf{p} = \mathbf{k}
\end{aligned}$$

Therefore, the surface area differential is given by

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dx\, dy. \qquad u = x \text{ and } v = y$$

We obtain similar calculations if instead the vector $\mathbf{p} = \mathbf{j}$ is normal to the xz-plane when $F_y \neq 0$ on S, or if $\mathbf{p} = \mathbf{i}$ is normal to the yz-plane when $F_x \neq 0$ on S. Combining these results with Equation (4) then gives the following general formula.

Formula for the Surface Area of an Implicit Surface

The area of the surface $F(x, y, z) = c$ over a closed and bounded plane region R is

$$\text{Surface area} = \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \tag{7}$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$, or $\mathbf{k}$ is normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

Thus, the area is the double integral over R of the magnitude of ∇F divided by the magnitude of the scalar component of ∇F normal to R.

We reached Equation (7) under the assumption that $\nabla F \cdot \mathbf{p} \neq 0$ throughout R and that ∇F is continuous. Whenever the integral exists, however, we define its value to be the area of the portion of the surface $F(x, y, z) = c$ that lies over R. (Recall that the projection is assumed to be one-to-one.)

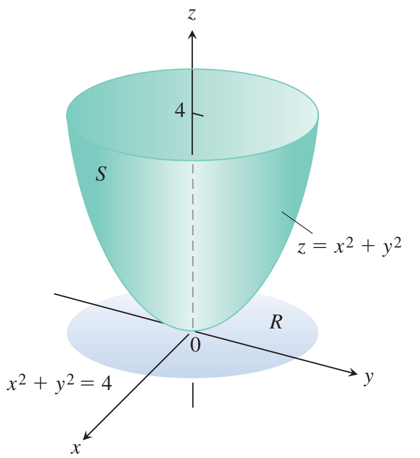


FIGURE 16.45 The area of this parabolic surface is calculated in Example 7.

EXAMPLE 7 Find the area of the surface cut from the bottom of the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 4$.

Solution We sketch the surface S and the region R below it in the xy-plane (Figure 16.45). The surface S is part of the level surface $F(x, y, z) = x^2 + y^2 - z = 0$, and R is the disk $x^2 + y^2 \leq 4$ in the xy-plane. To get a unit vector normal to the plane of R, we can take $\mathbf{p} = \mathbf{k}$.

At any point (x, y, z) on the surface, we have

$$\begin{aligned}
F(x, y, z) &= x^2 + y^2 - z \\
\nabla F &= 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k} \\
|\nabla F| &= \sqrt{(2x)^2 + (2y)^2 + (-1)^2} \\
&= \sqrt{4x^2 + 4y^2 + 1} \\
|\nabla F \cdot \mathbf{p}| &= |\nabla F \cdot \mathbf{k}| = |-1| = 1.
\end{aligned}$$

In the region R, $dA = dx\,dy$. Therefore,

$$\begin{aligned}
\text{Surface area} &= \iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\,dA && \text{Eq. (7)}\\
&= \iint_{x^2+y^2\le 4} \sqrt{4x^2+4y^2+1}\,dx\,dy\\
&= \int_0^{2\pi}\int_0^2 \sqrt{4r^2+1}\,r\,dr\,d\theta && \text{Polar coordinates}\\
&= \int_0^{2\pi}\left[\frac{1}{12}(4r^2+1)^{3/2}\right]_0^2 d\theta\\
&= \int_0^{2\pi}\frac{1}{12}(17^{3/2}-1)\,d\theta = \frac{\pi}{6}\left(17\sqrt{17}-1\right).
\end{aligned}$$

Example 7 illustrates how to find the surface area for a function $z = f(x, y)$ over a region R in the xy-plane. Actually, the surface area differential can be obtained in two ways, and we show this in the next example.

EXAMPLE 8 Derive the surface area differential $d\sigma$ of the surface $z = f(x, y)$ over a region R in the xy-plane **(a)** parametrically using Equation (5), and **(b)** implicitly, as in Equation (7).

Solution

(a) We parametrize the surface by taking $x = u$, $y = v$, and $z = f(x, y)$ over R. This gives the parametrization

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + f(u, v)\mathbf{k}.$$

Computing the partial derivatives gives $\mathbf{r}_u = \mathbf{i} + f_u\mathbf{k}$, $\mathbf{r}_v = \mathbf{j} + f_v\mathbf{k}$ and

$$\mathbf{r}_u \times \mathbf{r}_v = -f_u\mathbf{i} - f_v\mathbf{j} + \mathbf{k}. \qquad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_u \\ 0 & 1 & f_v \end{vmatrix}$$

Then $|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \sqrt{f_u^{\,2} + f_v^{\,2} + 1}\,du\,dv$. Substituting for u and v then gives the surface area differential

$$d\sigma = \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\,dx\,dy.$$

(b) We define the implicit function $F(x, y, z) = f(x, y) - z$. Since (x, y) belongs to the region R, the unit normal to the plane of R is $\mathbf{p} = \mathbf{k}$. Then $\nabla F = f_x\mathbf{i} + f_y\mathbf{j} - \mathbf{k}$ so that $|\nabla F \cdot \mathbf{p}| = |-1| = 1$, $|\nabla F| = \sqrt{f_x^{\,2} + f_y^{\,2} + 1}$, and $|\nabla F| \,/\, |\nabla F \cdot \mathbf{p}| = |\nabla F|$. The surface area differential is again given by

$$d\sigma = \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\,dx\,dy.$$

The surface area differential derived in Example 8 gives the following formula for calculating the surface area of the graph of a function defined explicitly as $z = f(x, y)$.

Formula for the Surface Area of a Graph $z = f(x, y)$

For a graph $z = f(x, y)$ over a region R in the xy-plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\,dx\,dy. \tag{8}$$

Exercises 16.5

Finding Parametrizations

In Exercises 1–16, find a parametrization of the surface. (There are many correct ways to do these, so your answers may not be the same as those in the back of the book.)

1. The paraboloid $z = x^2 + y^2$, $z \le 4$
2. The paraboloid $z = 9 - x^2 - y^2$, $z \ge 0$
3. **Cone frustum** The first-octant portion of the cone $z = \sqrt{x^2 + y^2}/2$ between the planes $z = 0$ and $z = 3$
4. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$
5. **Spherical cap** The cap cut from the sphere $x^2 + y^2 + z^2 = 9$ by the cone $z = \sqrt{x^2 + y^2}$
6. **Spherical cap** The portion of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant between the xy-plane and the cone $z = \sqrt{x^2 + y^2}$
7. **Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 3$ between the planes $z = \sqrt{3}/2$ and $z = -\sqrt{3}/2$
8. **Spherical cap** The upper portion cut from the sphere $x^2 + y^2 + z^2 = 8$ by the plane $z = -2$
9. **Parabolic cylinder between planes** The surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 2$, and $z = 0$
10. **Parabolic cylinder between planes** The surface cut from the parabolic cylinder $y = x^2$ by the planes $z = 0$, $z = 3$, and $y = 2$
11. **Circular cylinder band** The portion of the cylinder $y^2 + z^2 = 9$ between the planes $x = 0$ and $x = 3$
12. **Circular cylinder band** The portion of the cylinder $x^2 + z^2 = 4$ above the xy-plane between the planes $y = -2$ and $y = 2$
13. **Tilted plane inside cylinder** The portion of the plane $x + y + z = 1$
    a. Inside the cylinder $x^2 + y^2 = 9$
    b. Inside the cylinder $y^2 + z^2 = 9$
14. **Tilted plane inside cylinder** The portion of the plane $x - y + 2z = 2$
    a. Inside the cylinder $x^2 + z^2 = 3$
    b. Inside the cylinder $y^2 + z^2 = 2$
15. **Circular cylinder band** The portion of the cylinder $(x - 2)^2 + z^2 = 4$ between the planes $y = 0$ and $y = 3$
16. **Circular cylinder band** The portion of the cylinder $y^2 + (z - 5)^2 = 25$ between the planes $x = 0$ and $x = 10$

Surface Area of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the book. They should have the same values, however.)

17. **Tilted plane inside cylinder** The portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$
18. **Plane inside cylinder** The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$
19. **Cone frustum** The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$
20. **Cone frustum** The portion of the cone $z = \sqrt{x^2 + y^2}/3$ between the planes $z = 1$ and $z = 4/3$
21. **Circular cylinder band** The portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$
22. **Circular cylinder band** The portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$
23. **Parabolic cap** The cap cut from the paraboloid $z = 2 - x^2 - y^2$ by the cone $z = \sqrt{x^2 + y^2}$
24. **Parabolic band** The portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$
25. **Sawed-off sphere** The lower portion cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$
26. **Spherical band** The portion of the sphere $x^2 + y^2 + z^2 = 4$ between the planes $z = -1$ and $z = \sqrt{3}$

Planes Tangent to Parametrized Surfaces

The tangent plane at a point $P_0(f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$, the cross product of the tangent vectors $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ at P_0. In Exercises 27–30, find an equation for the plane tangent to the surface at P_0. Then find a Cartesian equation for the surface and sketch the surface and tangent plane together.

27. **Cone** The cone $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $r \ge 0$, $0 \le \theta \le 2\pi$ at the point $P_0(\sqrt{2}, \sqrt{2}, 2)$ corresponding to $(r, \theta) = (2, \pi/4)$
28. **Hemisphere** The hemisphere surface $\mathbf{r}(\phi, \theta) = (4\sin\phi\cos\theta)\mathbf{i} + (4\sin\phi\sin\theta)\mathbf{j} + (4\cos\phi)\mathbf{k}$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$, at the point $P_0(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$ corresponding to $(\phi, \theta) = (\pi/6, \pi/4)$
29. **Circular cylinder** The circular cylinder $\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}$, $0 \le \theta \le \pi$, at the point $P_0(3\sqrt{3}/2, 9/2, 0)$ corresponding to $(\theta, z) = (\pi/3, 0)$ (See Example 3.)
30. **Parabolic cylinder** The parabolic cylinder surface $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, $-\infty < x < \infty$, $-\infty < y < \infty$, at the point $P_0(1, 2, -1)$ corresponding to $(x, y) = (1, 2)$

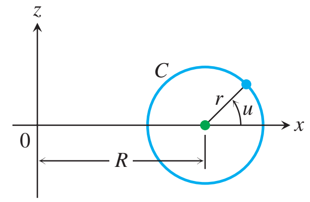
More Parametrizations of Surfaces

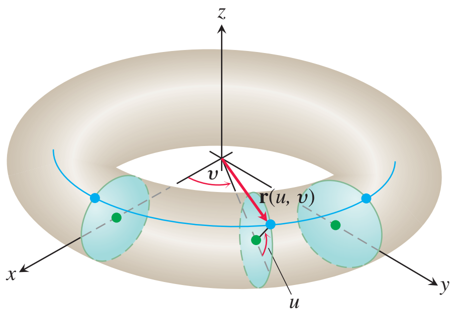
31. a. A *torus of revolution* (doughnut) is obtained by rotating a circle C in the xz-plane about the z-axis in space. (See the accompanying figure.) If C has radius $r > 0$ and center $(R, 0, 0)$, show that a parametrization of the torus is

$$\mathbf{r}(u, v) = ((R + r\cos u)\cos v)\mathbf{i} + ((R + r\cos u)\sin v)\mathbf{j} + (r\sin u)\mathbf{k},$$

where $0 \le u \le 2\pi$ and $0 \le v \le 2\pi$ are the angles in the figure.

b. Show that the surface area of the torus is $A = 4\pi^2 Rr$.



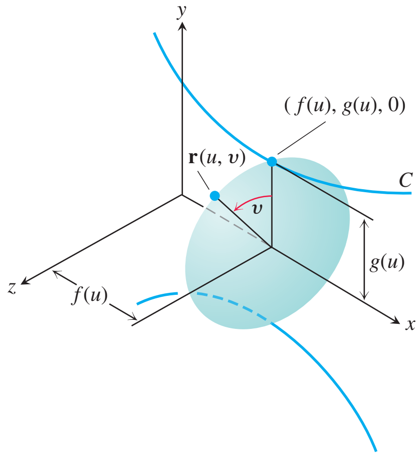


32. Parametrization of a surface of revolution Suppose that the parametrized curve C: $(f(u), g(u))$ is revolved about the x-axis, where $g(u) > 0$ for $a \le u \le b$.

a. Show that

$$\mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u)\cos v)\mathbf{j} + (g(u)\sin v)\mathbf{k}$$

is a parametrization of the resulting surface of revolution, where $0 \le v \le 2\pi$ is the angle from the xy-plane to the point $\mathbf{r}(u, v)$ on the surface. (See the accompanying figure.) Notice that $f(u)$ measures distance *along* the axis of revolution and $g(u)$ measures distance *from* the axis of revolution.



b. Find a parametrization for the surface obtained by revolving the curve $x = y^2, y \ge 0$, about the x-axis.

33. a. Parametrization of an ellipsoid The parametrization $x = a\cos\theta$, $y = b\sin\theta$, $0 \le \theta \le 2\pi$ gives the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Using the angles θ and ϕ in spherical coordinates, show that

$$\mathbf{r}(\theta, \phi) = (a\cos\theta\cos\phi)\mathbf{i} + (b\sin\theta\cos\phi)\mathbf{j} + (c\sin\phi)\mathbf{k}$$

is a parametrization of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$.

b. Write an integral for the surface area of the ellipsoid, but do not evaluate the integral.

34. Hyperboloid of one sheet

a. Find a parametrization for the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ in terms of the angle θ associated with the circle $x^2 + y^2 = r^2$ and the hyperbolic parameter u associated with the hyperbolic function $r^2 - z^2 = 1$. (*Hint*: $\cosh^2 u - \sinh^2 u = 1$.)

b. Generalize the result in part (a) to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.

35. (*Continuation of Exercise 34.*) Find a Cartesian equation for the plane tangent to the hyperboloid $x^2 + y^2 - z^2 = 25$ at the point $(x_0, y_0, 0)$, where $x_0{}^2 + y_0{}^2 = 25$.

36. Hyperboloid of two sheets Find a parametrization of the hyperboloid of two sheets $(z^2/c^2) - (x^2/a^2) - (y^2/b^2) = 1$.

Surface Area for Implicit and Explicit Forms

37. Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

38. Find the area of the band cut from the paraboloid $x^2 + y^2 - z = 0$ by the planes $z = 2$ and $z = 6$.

39. Find the area of the region cut from the plane $x + 2y + 2z = 5$ by the cylinder whose walls are $x = y^2$ and $x = 2 - y^2$.

40. Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}$, $y = 0$, and $y = x$ in the xy-plane.

41. Find the area of the surface $x^2 - 2y - 2z = 0$ that lies above the triangle bounded by the lines $x = 2$, $y = 0$, and $y = 3x$ in the xy-plane.

42. Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

43. Find the area of the ellipse cut from the plane $z = cx$ (c a constant) by the cylinder $x^2 + y^2 = 1$.

44. Find the area of the upper portion of the cylinder $x^2 + z^2 = 1$ that lies between the planes $x = \pm 1/2$ and $y = \pm 1/2$.

45. Find the area of the portion of the paraboloid $x = 4 - y^2 - z^2$ that lies above the ring $1 \le y^2 + z^2 \le 4$ in the yz-plane.

46. Find the area of the surface cut from the paraboloid $x^2 + y + z^2 = 2$ by the plane $y = 0$.

47. Find the area of the surface $x^2 - 2\ln x + \sqrt{15}y - z = 0$ above the square R: $1 \le x \le 2$, $0 \le y \le 1$, in the xy-plane.

48. Find the area of the surface $2x^{3/2} + 2y^{3/2} - 3z = 0$ above the square R: $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane.

Find the area of the surfaces in Exercises 49–54.

49. The surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 3$

50. The surface cut from the "nose" of the paraboloid $x = 1 - y^2 - z^2$ by the yz-plane

51. The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies over the region between the circle $x^2 + y^2 = 1$ and the ellipse $9x^2 + 4y^2 = 36$ in the xy-plane. (*Hint:* Use formulas from geometry to find the area of the region.)

52. The triangle cut from the plane $2x + 6y + 3z = 6$ by the bounding planes of the first octant. Calculate the area three ways, using different explicit forms.

53. The surface in the first octant cut from the cylinder $y = (2/3)z^{3/2}$ by the planes $x = 1$ and $y = 16/3$

54. The portion of the plane $y + z = 4$ that lies above the region cut from the first quadrant of the xz-plane by the parabola $x = 4 - z^2$

55. Use the parametrization

$$\mathbf{r}(x, z) = x\mathbf{i} + f(x, z)\mathbf{j} + z\mathbf{k}$$

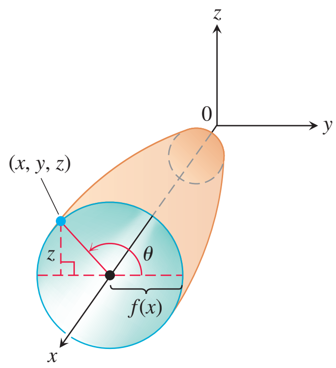
and Equation (5) to derive a formula for $d\sigma$ associated with the explicit form $y = f(x, z)$.

56. Let S be the surface obtained by rotating the smooth curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$.

a. Show that the vector function

$$\mathbf{r}(x, \theta) = x\mathbf{i} + f(x)\cos\theta\mathbf{j} + f(x)\sin\theta\mathbf{k}$$

is a parametrization of S, where θ is the angle of rotation around the x-axis (see the accompanying figure).



b. Use Equation (4) to show that the surface area of this surface of revolution is given by

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

16.6 Surface Integrals

To compute the mass of a surface, the flow of a liquid across a curved membrane, or the total electrical charge on a surface, we need to integrate a function over a curved surface in space. Such a *surface integral* is the two-dimensional extension of the line integral concept used to integrate over a one-dimensional curve. Like line integrals, surface integrals arise in two forms. One form occurs when we integrate a scalar function over a surface, such as integrating a mass density function defined on a surface to find its total mass. This form corresponds to line integrals of scalar functions defined in Section 16.1, and we used it to find the mass of a thin wire. The second form is for surface integrals of vector fields, analogous to the line integrals for vector fields defined in Section 16.2. An example of this form occurs when we want to measure the net flow of a fluid across a surface submerged in the fluid (just as we previously defined the flux of **F** across a curve). In this section we investigate these ideas and some of their applications.

Surface Integrals

Suppose that the function $G(x, y, z)$ gives the *mass density* (mass per unit area) at each point on a surface S. Then we can calculate the total mass of S as an integral in the following way.

Assume, as in Section 16.5, that the surface S is defined parametrically on a region R in the uv-plane,

$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad (u, v) \in R.$$

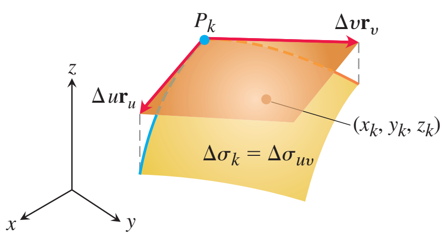


FIGURE 16.46 The area of the patch $\Delta\sigma_k$ is approximated by the area of the tangent parallelogram determined by the vectors $\Delta u\,\mathbf{r}_u$ and $\Delta v\,\mathbf{r}_v$. The point (x_k, y_k, z_k) lies on the surface patch, beneath the parallelogram shown here.

In Figure 16.46, we see how a subdivision of R (considered as a rectangle for simplicity) divides the surface S into corresponding curved surface elements, or patches, of area

$$\Delta\sigma_{uv} \approx |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv.$$

As we did for the subdivisions when defining double integrals in Section 15.2, we number the surface element patches in some order with their areas given by $\Delta\sigma_1, \Delta\sigma_2, \ldots, \Delta\sigma_n$. To form a Riemann sum over S, we choose a point (x_k, y_k, z_k) in the kth patch, multiply the value of the function G at that point by the area $\Delta\sigma_k$, and add together the products:

$$\sum_{k=1}^{n} G(x_k, y_k, z_k)\,\Delta\sigma_k.$$

Depending on how we pick (x_k, y_k, z_k) in the kth patch, we may get different values for this Riemann sum. Then we take the limit as the number of surface patches increases, their areas shrink to zero, and both $\Delta u \to 0$ and $\Delta v \to 0$. This limit, whenever it exists independent of all choices made, defines the **surface integral of G over the surface S** as

$$\iint_S G(x, y, z)\, d\sigma = \lim_{n\to\infty} \sum_{k=1}^{n} G(x_k, y_k, z_k)\, \Delta\sigma_k. \tag{1}$$

Notice the analogy with the definition of the double integral (Section 15.2) and with the line integral (Section 16.1). If S is a piecewise smooth surface, and G is continuous over S, then the surface integral defined by Equation (1) can be shown to exist.

The formula for evaluating the surface integral depends on the manner in which S is described, parametrically, implicitly or explicitly, as discussed in Section 16.5.

Formulas for a Surface Integral of a Scalar Function

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{2}$$

2. For a surface S given **implicitly** by $F(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z) \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \tag{3}$$

where $\mathbf{p}$ is a unit vector normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y))\sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy. \tag{4}$$

The surface integral in Equation (1) takes on different meanings in different applications. If G has the constant value 1, the integral gives the area of S. If G gives the mass density of a thin shell of material modeled by S, the integral gives the mass of the shell. If G gives the charge density of a thin shell, then the integral gives the total charge.

EXAMPLE 1 Integrate $G(x, y, z) = x^2$ over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$.

Solution Using Equation (2) and the calculations from Example 4 in Section 16.5, we have $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r$ and

$$\begin{aligned}
\iint_S x^2\, d\sigma &= \int_0^{2\pi}\int_0^1 (r^2\cos^2\theta)(\sqrt{2}r)\, dr\, d\theta \qquad x = r\cos\theta \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 r^3\cos^2\theta\, dr\, d\theta \\
&= \frac{\sqrt{2}}{4}\int_0^{2\pi}\cos^2\theta\, d\theta = \frac{\sqrt{2}}{4}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \frac{\pi\sqrt{2}}{4}.
\end{aligned}$$

Surface integrals behave like other double integrals, the integral of the sum of two functions being the sum of their integrals and so on. The domain Additivity Property takes the form

$$\iint_S G\, d\sigma = \iint_{S_1} G\, d\sigma + \iint_{S_2} G\, d\sigma + \cdots + \iint_{S_n} G\, d\sigma.$$

When S is partitioned by smooth curves into a finite number of smooth patches with nonoverlapping interiors (i.e., if S is piecewise smooth), then the integral over S is the sum of the integrals over the patches. Thus, the integral of a function over the surface of a cube is the sum of the integrals over the faces of the cube. We integrate over a turtle shell of welded plates by integrating over one plate at a time and adding the results.

EXAMPLE 2 Integrate $G(x, y, z) = xyz$ over the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ (Figure 16.47).

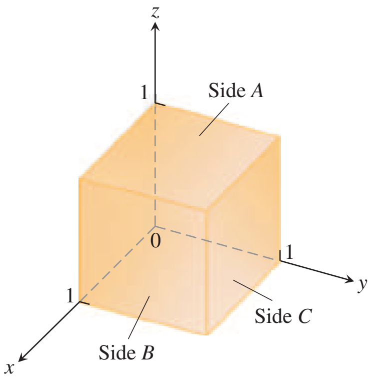


FIGURE 16.47 The cube in Example 2.

Solution We integrate xyz over each of the six sides and add the results. Since $xyz = 0$ on the sides that lie in the coordinate planes, the integral over the surface of the cube reduces to

$$\iint_{\substack{\text{Cube}\\ \text{surface}}} xyz\, d\sigma = \iint_{\text{Side } A} xyz\, d\sigma + \iint_{\text{Side } B} xyz\, d\sigma + \iint_{\text{Side } C} xyz\, d\sigma.$$

Side A is the surface $f(x, y, z) = z = 1$ over the square region R_{xy}: $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane. For this surface and region,

$$\mathbf{p} = \mathbf{k}, \qquad \nabla f = \mathbf{k}, \qquad |\nabla f| = 1, \qquad |\nabla f \cdot \mathbf{p}| = |\mathbf{k}\cdot\mathbf{k}| = 1$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|}dA = \frac{1}{1}dx\, dy = dx\, dy$$

$$xyz = xy(1) = xy$$

and

$$\iint_{\text{Side } A} xyz\, d\sigma = \iint_{R_{xy}} xy\, dx\, dy = \int_0^1\int_0^1 xy\, dx\, dy = \int_0^1 \frac{y}{2}dy = \frac{1}{4}.$$

Symmetry tells us that the integrals of xyz over sides B and C are also $1/4$. Hence,

$$\iint_{\substack{\text{Cube}\\\text{surface}}} xyz\, d\sigma = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$

EXAMPLE 3 Integrate $G(x, y, z) = \sqrt{1 - x^2 - y^2}$ over the "football" surface S formed by rotating the curve $x = \cos z$, $y = 0$, $-\pi/2 \le z \le \pi/2$, around the z-axis.

Solution The surface is displayed in Figure 16.43, and in Example 6 of Section 16.5 we found the parametrization

$$x = \cos u \cos v, \quad y = \cos u \sin v, \quad z = u, \quad -\frac{\pi}{2} \le u \le \frac{\pi}{2} \quad \text{and} \quad 0 \le v \le 2\pi,$$

where v represents the angle of rotation from the xz-plane about the z-axis. Substituting this parametrization into the expression for G gives

$$\sqrt{1 - x^2 - y^2} = \sqrt{1 - (\cos^2 u)(\cos^2 v + \sin^2 v)} = \sqrt{1 - \cos^2 u} = |\sin u|.$$

The surface area differential for the parametrization was found to be (Example 6, Section 16.5)

$$d\sigma = \cos u \sqrt{1 + \sin^2 u}\, du\, dv.$$

These calculations give the surface integral

$$\begin{aligned}
\iint_S \sqrt{1 - x^2 - y^2}\, d\sigma &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} |\sin u| \cos u \sqrt{1 + \sin^2 u}\, du\, dv \\
&= 2\int_0^{2\pi} \int_0^{\pi/2} \sin u \cos u \sqrt{1 + \sin^2 u}\, du\, dv \\
&= \int_0^{2\pi} \int_1^2 \sqrt{w}\, dw\, dv \qquad \begin{array}{l} w = 1 + \sin^2 u, \\ dw = 2 \sin u \cos u\, du \\ \text{When } u = 0,\ w = 1. \\ \text{When } u = \pi/2,\ w = 2. \end{array} \\
&= 2\pi \cdot \frac{2}{3} w^{3/2} \Big]_1^2 = \frac{4\pi}{3}\left(2\sqrt{2} - 1\right).
\end{aligned}$$

EXAMPLE 4 Evaluate $\iint_S \sqrt{x(1 + 2z)}\, d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region R: $x \ge 0$, $y \ge 0$, $x + y \le 1$ in the xy-plane (Figure 16.48).

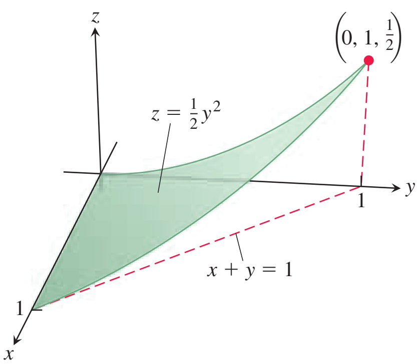


FIGURE 16.48 The surface S in Example 4.

Solution The function G on the surface S is given by

$$G(x, y, z) = \sqrt{x(1 + 2z)} = \sqrt{x}\sqrt{1 + y^2}.$$

With $z = f(x, y) = y^2/2$, we use Equation (4) to evaluate the surface integral:

$$d\sigma = \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dx\, dy = \sqrt{0 + y^2 + 1}\, dx\, dy$$

and

$$
\begin{aligned}
\iint_S G(x, y, z)\, d\sigma &= \iint_R \left(\sqrt{x}\sqrt{1 + y^2}\right)\sqrt{1 + y^2}\, dx\, dy \\
&= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1 + y^2)\, dy\, dx \\
&= \int_0^1 \sqrt{x}\left[(1 - x) + \frac{1}{3}(1 - x)^3\right] dx \qquad \text{Integrate and evaluate.} \\
&= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2}\right) dx \qquad \text{Routine algebra} \\
&= \left[\frac{8}{9}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{27}x^{9/2}\right]_0^1 \\
&= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945} \approx 0.30.
\end{aligned}
$$

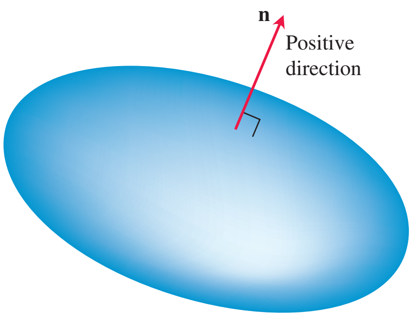


FIGURE 16.49 Smooth closed surfaces in space are orientable. The outward unit normal vector defines the positive direction at each point.

Orientation of a Surface

The curve C in a line integral inherits a natural orientation from its parametrization $\mathbf{r}(t)$ because the parameter belongs to an interval $a \le t \le b$ directed by the real line. The unit tangent vector $\mathbf{T}$ along C points in this forward direction. For a surface S, the parametrization $\mathbf{r}(u, v)$ gives a vector $\mathbf{r}_u \times \mathbf{r}_v$ that is normal to the surface, but if S has two "sides," then at each point the negative $-(\mathbf{r}_u \times \mathbf{r}_v)$ is also normal to the surface, so we need to choose which direction to use. For example, if you look at the sphere in Figure 16.38, at any point on the sphere there is a normal vector pointing inward toward the center of the sphere and another opposite normal pointing outward. When we specify which of these normals we are going to use consistently across the entire surface, the surface is given an *orientation*. A smooth surface S is **orientable** (or **two-sided**) if it is possible to define a field of unit normal vectors $\mathbf{n}$ on S which varies continuously with position. Any patch or subportion of an orientable surface is orientable. Spheres and other smooth closed surfaces in space (smooth surfaces that enclose solids) are orientable. By convention, we usually choose $\mathbf{n}$ on a closed surface to point outward.

Once $\mathbf{n}$ has been chosen, we say that we have **oriented** the surface, and we call the surface together with its normal field an **oriented surface**. The vector $\mathbf{n}$ at any point is called the **positive direction** at that point (Figure 16.49).

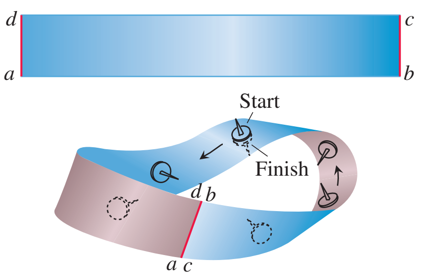


FIGURE 16.50 To make a Möbius band, take a rectangular strip of paper *abcd*, give the end *bc* a single twist, and paste the ends of the strip together to match *a* with *c* and *b* with *d*. The Möbius band is a nonorientable or one-sided surface.

The Möbius band in Figure 16.50 is not orientable. No matter where you start to construct a continuous unit normal field (shown as the shaft of a thumbtack in the figure), moving the vector continuously around the surface in the manner shown will return it to the starting point with a direction opposite to the one it had when it started out. The vector at that point cannot point both ways and yet it must if the field is to be continuous. We conclude that no such field exists.

Surface Integrals of Vector Fields

In Section 16.2 we defined the line integral of a vector field along a path C as $\int_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent vector to the path pointing in the forward oriented direction. By analogy we now have the following corresponding definition for surface integrals.

DEFINITION Let **F** be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors **n** orienting S. Then the **surface integral of F over S** is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma. \tag{5}$$

The surface integral of **F** is also called the **flux** of the vector field across the oriented surface S (analogous to the definition of flux of a vector field in the xy-plane across a closed curve in the plane, as defined in Section 16.2). The expression $\mathbf{F} \cdot \mathbf{n}\, d\sigma$ in the integral (5) is also written as $\mathbf{F} \cdot d\boldsymbol{\sigma}$, which corresponds to the notation $\mathbf{F} \cdot d\mathbf{r}$ used for $\mathbf{F} \cdot \mathbf{T}\, ds$ in line integrals for vector fields. If **F** is the velocity field of a three-dimensional fluid flow, then the flux of **F** across S is the net rate at which fluid is crossing S per unit time in the chosen positive direction **n** defined by the orientation of S. Fluid flows are discussed in more detail in Section 16.7, so here we focus on several examples calculating surface integrals of vector fields.

EXAMPLE 5 Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the parabolic cylinder $y = x^2$, $0 \le x \le 1$, $0 \le z \le 4$, in the direction **n** indicated in Figure 16.51.

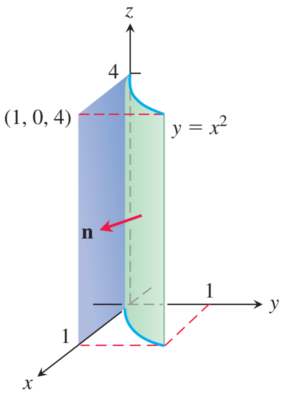


FIGURE 16.51 Finding the flux through the surface of a parabolic cylinder (Example 5).

Solution On the surface we have $x = x$, $y = x^2$, and $z = z$, so we automatically have the parametrization $\mathbf{r}(x, z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$, $0 \le x \le 1$, $0 \le z \le 4$. The cross product of tangent vectors is

$$\mathbf{r}_x \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\mathbf{i} - \mathbf{j}.$$

The unit normal vectors pointing outward from the surface as indicated in Figure 16.51 are

$$\mathbf{n} = \frac{\mathbf{r}_x \times \mathbf{r}_z}{|\mathbf{r}_x \times \mathbf{r}_z|} = \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2 + 1}}.$$

On the surface, $y = x^2$, so the vector field there is

$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

Thus,

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{4x^2 + 1}}\big((x^2z)(2x) + (x)(-1) + (-z^2)(0)\big) \\ &= \frac{2x^3z - x}{\sqrt{4x^2 + 1}}. \end{aligned}$$

The flux of **F** outward through the surface is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}|\mathbf{r}_x \times \mathbf{r}_z|\,dx\,dz$$

$$= \int_0^4\int_0^1 \frac{2x^3z - x}{\sqrt{4x^2+1}}\sqrt{4x^2+1}\,dx\,dz$$

$$= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz = \int_0^4 \left[\frac{1}{2}x^4z - \frac{1}{2}x^2\right]_{x=0}^{x=1} dz$$

$$= \int_0^4 \frac{1}{2}(z-1)\,dz = \frac{1}{4}(z-1)^2\Big]_0^4$$

$$= \frac{1}{4}(9) - \frac{1}{4}(1) = 2.$$

There is a simple formula for the flux of **F** across a parametrized surface $\mathbf{r}(u, v)$. Since

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv,$$

with the orientation

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

Flux Across a Parametrized Surface

$$\text{Flux} = \iint_R \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv$$

it follows that

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_R \mathbf{F}\cdot\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}|\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \iint_R \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv.$$

This integral for flux simplifies the computation in Example 5. Since

$$\mathbf{F}\cdot(\mathbf{r}_x \times \mathbf{r}_z) = (x^2z)(2x) + (x)(-1) = 2x^3z - x,$$

we obtain directly

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz = 2$$

in Example 5.

If S is part of a level surface $g(x, y, z) = c$, then **n** may be taken to be one of the two fields

$$\mathbf{n} = \pm\frac{\nabla g}{|\nabla g|}, \tag{6}$$

depending on which one gives the preferred direction. The corresponding flux is

$$\text{Flux} = \iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$$

$$= \iint_R \left(\mathbf{F}\cdot\frac{\pm\nabla g}{|\nabla g|}\right)\frac{|\nabla g|}{|\nabla g\cdot\mathbf{p}|}\,dA \qquad \text{Eqs. (6) and (3)}$$

$$= \iint_R \mathbf{F}\cdot\frac{\pm\nabla g}{|\nabla g\cdot\mathbf{p}|}\,dA. \tag{7}$$

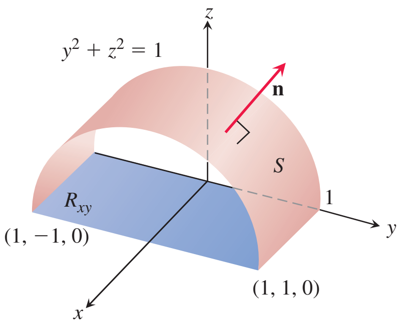


FIGURE 16.52 Calculating the flux of a vector field outward through the surface S. The area of the shadow region R_{xy} is 2 (Example 6).

EXAMPLE 6 Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1$, $z \geq 0$, by the planes $x = 0$ and $x = 1$.

Solution The outward normal field on S (Figure 16.52) may be calculated from the gradient of $g(x, y, z) = y^2 + z^2$ to be

$$\mathbf{n} = +\frac{\nabla g}{|\nabla g|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4y^2 + 4z^2}} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{1}} = y\mathbf{j} + z\mathbf{k}.$$

With $\mathbf{p} = \mathbf{k}$, we also have

$$d\sigma = \frac{|\nabla g|}{|\nabla g \cdot \mathbf{k}|}\,dA = \frac{2}{|2z|}\,dA = \frac{1}{z}\,dA.$$

We can drop the absolute value bars because $z \geq 0$ on S.

The value of $\mathbf{F} \cdot \mathbf{n}$ on the surface is

$$\begin{aligned}
\mathbf{F} \cdot \mathbf{n} &= (yz\mathbf{j} + z^2\mathbf{k}) \cdot (y\mathbf{j} + z\mathbf{k}) \\
&= y^2z + z^3 = z(y^2 + z^2) \\
&= z. && y^2 + z^2 = 1 \text{ on } S
\end{aligned}$$

The surface projects onto the shadow region R_{xy}, which is the rectangle in the xy-plane shown in Figure 16.52. Therefore, the flux of $\mathbf{F}$ outward through S is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{R_{xy}} (z)\left(\frac{1}{z}\,dA\right) = \iint_{R_{xy}} dA = \text{area}(R_{xy}) = 2.$$

Moments and Masses of Thin Shells

Thin shells of material like bowls, metal drums, and domes are modeled with surfaces. Their moments and masses are calculated with the formulas in Table 16.3. The derivations are similar to those in Section 6.6. The formulas are like those for line integrals in Table 16.1, Section 16.1.

TABLE 16.3 Mass and moment formulas for very thin shells

Mass: $M = \iint_S \delta\, d\sigma$ $\quad \delta = \delta(x, y, z) =$ density at (x, y, z) as mass per unit area

First moments about the coordinate planes:

$$M_{yz} = \iint_S x\,\delta\, d\sigma, \qquad M_{xz} = \iint_S y\,\delta\, d\sigma, \qquad M_{xy} = \iint_S z\,\delta\, d\sigma$$

Coordinates of center of mass:

$$\bar{x} = M_{yz}/M, \qquad \bar{y} = M_{xz}/M, \qquad \bar{z} = M_{xy}/M$$

Moments of inertia about coordinate axes:

$$I_x = \iint_S (y^2 + z^2)\,\delta\, d\sigma, \quad I_y = \iint_S (x^2 + z^2)\,\delta\, d\sigma, \quad I_z = \iint_S (x^2 + y^2)\,\delta\, d\sigma,$$

$I_L = \iint_S r^2\delta\, d\sigma$ $\quad r(x, y, z) =$ distance from point (x, y, z) to line L

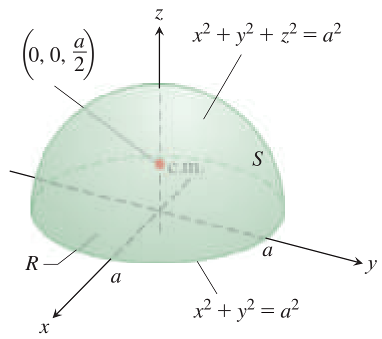


FIGURE 16.53 The center of mass of a thin hemispherical shell of constant density lies on the axis of symmetry halfway from the base to the top (Example 7).

EXAMPLE 7 Find the center of mass of a thin hemispherical shell of radius a and constant density δ.

Solution We model the shell with the hemisphere

$$f(x, y, z) = x^2 + y^2 + z^2 = a^2, \qquad z \geq 0$$

(Figure 16.53). The symmetry of the surface about the z-axis tells us that $\bar{x} = \bar{y} = 0$. It remains only to find $\bar{z}$ from the formula $\bar{z} = M_{xy}/M$.

The mass of the shell is

$$M = \iint_S \delta \, d\sigma = \delta \iint_S d\sigma = (\delta)(\text{area of } S) = 2\pi a^2 \delta. \qquad \delta = \text{constant}$$

To evaluate the integral for M_{xy}, we take $\mathbf{p} = \mathbf{k}$ and calculate

$$|\nabla f| = |2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}| = 2\sqrt{x^2 + y^2 + z^2} = 2a$$

$$|\nabla f \cdot \mathbf{p}| = |\nabla f \cdot \mathbf{k}| = |2z| = 2z$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \mathbf{p}|} dA = \frac{a}{z} dA.$$

Then

$$M_{xy} = \iint_S z\delta \, d\sigma = \delta \iint_R z \frac{a}{z} dA = \delta a \iint_R dA = \delta a(\pi a^2) = \delta \pi a^3$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi a^3 \delta}{2\pi a^2 \delta} = \frac{a}{2}.$$

The shell's center of mass is the point $(0, 0, a/2)$. ■

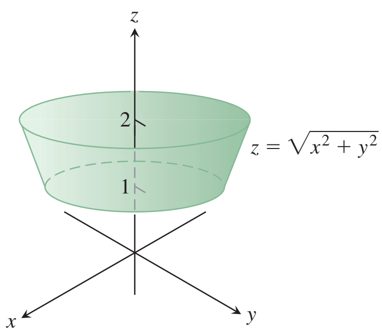


FIGURE 16.54 The cone frustum formed when the cone $z = \sqrt{x^2 + y^2}$ is cut by the planes $z = 1$ and $z = 2$ (Example 8).

EXAMPLE 8 Find the center of mass of a thin shell of density $\delta = 1/z^2$ cut from the cone $z = \sqrt{x^2 + y^2}$ by the planes $z = 1$ and $z = 2$ (Figure 16.54).

Solution The symmetry of the surface about the z-axis tells us that $\bar{x} = \bar{y} = 0$. We find $\bar{z} = M_{xy}/M$. Working as in Example 4 of Section 16.5, we have

$$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r\mathbf{k}, \qquad 1 \leq r \leq 2, \quad 0 \leq \theta \leq 2\pi,$$

and

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{2}r.$$

Therefore,

$$M = \iint_S \delta \, d\sigma = \int_0^{2\pi} \int_1^2 \frac{1}{r^2} \sqrt{2} r \, dr \, d\theta$$

$$= \sqrt{2} \int_0^{2\pi} \left[\ln r\right]_1^2 d\theta = \sqrt{2} \int_0^{2\pi} \ln 2 \, d\theta$$

$$= 2\pi\sqrt{2} \ln 2,$$

$$\begin{aligned} M_{xy} &= \iint_S \delta z\, d\sigma = \int_0^{2\pi}\int_1^2 \frac{1}{r^2} r\sqrt{2}r\, dr\, d\theta \\ &= \sqrt{2}\int_0^{2\pi}\int_1^2 dr\, d\theta \\ &= \sqrt{2}\int_0^{2\pi} d\theta = 2\pi\sqrt{2}, \\ \bar{z} &= \frac{M_{xy}}{M} = \frac{2\pi\sqrt{2}}{2\pi\sqrt{2}\ln 2} = \frac{1}{\ln 2}. \end{aligned}$$

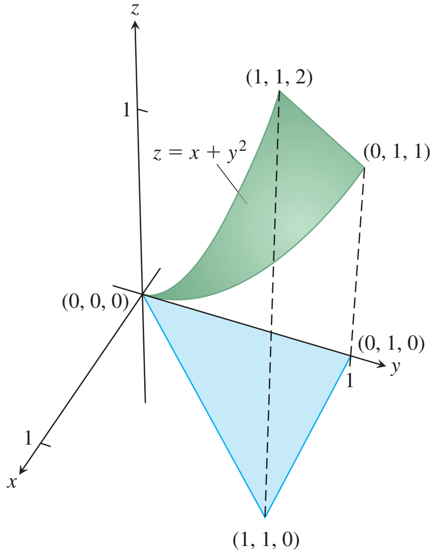
The shell's center of mass is the point $(0, 0, 1/\ln 2)$. ■

Exercises 16.6

Surface Integrals of Scalar Functions

In Exercises 1–8, integrate the given function over the given surface.

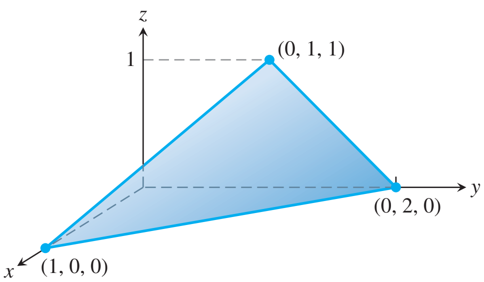
1. **Parabolic cylinder** $G(x, y, z) = x$, over the parabolic cylinder $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$
2. **Circular cylinder** $G(x, y, z) = z$, over the cylindrical surface $y^2 + z^2 = 4$, $z \ge 0$, $1 \le x \le 4$
3. **Sphere** $G(x, y, z) = x^2$, over the unit sphere $x^2 + y^2 + z^2 = 1$
4. **Hemisphere** $G(x, y, z) = z^2$, over the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \ge 0$
5. **Portion of plane** $F(x, y, z) = z$, over the portion of the plane $x + y + z = 4$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$, in the xy-plane
6. **Cone** $F(x, y, z) = z - x$, over the cone $z = \sqrt{x^2 + y^2}$, $0 \le z \le 1$
7. **Parabolic dome** $H(x, y, z) = x^2\sqrt{5 - 4z}$, over the parabolic dome $z = 1 - x^2 - y^2$, $z \ge 0$
8. **Spherical cap** $H(x, y, z) = yz$, over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$
9. Integrate $G(x, y, z) = x + y + z$ over the surface of the cube cut from the first octant by the planes $x = a$, $y = a$, $z = a$.
10. Integrate $G(x, y, z) = y + z$ over the surface of the wedge in the first octant bounded by the coordinate planes and the planes $x = 2$ and $y + z = 1$.
11. Integrate $G(x, y, z) = xyz$ over the surface of the rectangular solid cut from the first octant by the planes $x = a$, $y = b$, and $z = c$.
12. Integrate $G(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a$, $y = \pm b$, and $z = \pm c$.
13. Integrate $G(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.
14. Integrate $G(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.
15. Integrate $G(x, y, z) = z - x$ over the portion of the graph of $z = x + y^2$ above the triangle in the xy-plane having vertices $(0, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$. (See accompanying figure.)



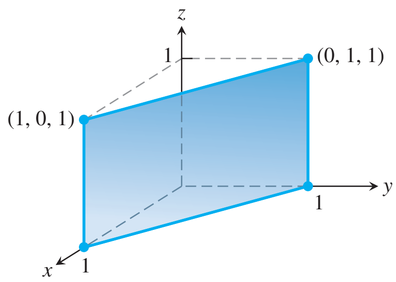
16. Integrate $G(x, y, z) = x$ over the surface given by

$$z = x^2 + y \quad \text{for} \quad 0 \le x \le 1, \quad -1 \le y \le 1.$$

17. Integrate $G(x, y, z) = xyz$ over the triangular surface with vertices $(1, 0, 0)$, $(0, 2, 0)$, and $(0, 1, 1)$.



18. Integrate $G(x, y, z) = x - y - z$ over the portion of the plane $x + y = 1$ in the first octant between $z = 0$ and $z = 1$ (see the accompanying figure on the next page).



Finding Flux or Surface Integrals of Vector Fields

In Exercises 19–28, use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ across the surface in the specified direction.

19. Parabolic cylinder $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0, x = 1$, and $z = 0$

20. Parabolic cylinder $\mathbf{F} = x^2\mathbf{j} - xz\mathbf{k}$ outward (normal away from the yz-plane) through the surface cut from the parabolic cylinder $y = x^2, -1 \le x \le 1$, by the planes $z = 0$ and $z = 2$

21. Sphere $\mathbf{F} = z\mathbf{k}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin

22. Sphere $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the sphere $x^2 + y^2 + z^2 = a^2$ in the direction away from the origin

23. Plane $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ upward across the portion of the plane $x + y + z = 2a$ that lies above the square $0 \le x \le a$, $0 \le y \le a$, in the xy-plane

24. Cylinder $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = a$

25. Cone $\mathbf{F} = xy\mathbf{i} - z\mathbf{k}$ outward (normal away from the z-axis) through the cone $z = \sqrt{x^2 + y^2}, 0 \le z \le 1$

26. Cone $\mathbf{F} = y^2\mathbf{i} + xz\mathbf{j} - \mathbf{k}$ outward (normal away from the z-axis) through the cone $z = 2\sqrt{x^2 + y^2}, 0 \le z \le 2$

27. Cone frustum $\mathbf{F} = -x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$ outward (normal away from the z-axis) through the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$

28. Paraboloid $\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (normal away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$

In Exercises 29 and 30, find the surface integral of the field $\mathbf{F}$ over the portion of the given surface in the specified direction.

29. $\mathbf{F}(x, y, z) = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

S: rectangular surface $z = 0, \quad 0 \le x \le 2, \quad 0 \le y \le 3$, direction $\mathbf{k}$

30. $\mathbf{F}(x, y, z) = yx^2\mathbf{i} - 2\mathbf{j} + xz\mathbf{k}$

S: rectangular surface $y = 0, \quad -1 \le x \le 2, \quad 2 \le z \le 7$, direction $-\mathbf{j}$

In Exercises 31–36, use Equation (7) to find the surface integral of the field $\mathbf{F}$ over the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.

31. $\mathbf{F}(x, y, z) = z\mathbf{k}$

32. $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

33. $\mathbf{F}(x, y, z) = y\mathbf{i} - x\mathbf{j} + \mathbf{k}$

34. $\mathbf{F}(x, y, z) = zx\mathbf{i} + zy\mathbf{j} + z^2\mathbf{k}$

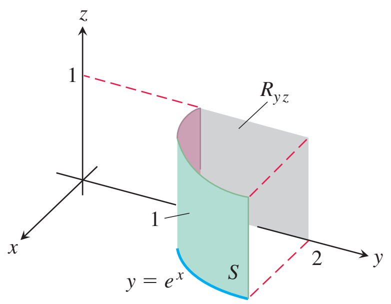
35. $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

36. $\mathbf{F}(x, y, z) = \dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}$

37. Find the flux of the field $\mathbf{F}(x, y, z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0, x = 1$, and $z = 0$.

38. Find the flux of the field $\mathbf{F}(x, y, z) = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (away from the z-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

39. Let S be the portion of the cylinder $y = e^x$ in the first octant that projects parallel to the x-axis onto the rectangle R_{yz}: $1 \le y \le 2$, $0 \le z \le 1$ in the yz-plane (see the accompanying figure). Let $\mathbf{n}$ be the unit vector normal to S that points away from the yz-plane. Find the flux of the field $\mathbf{F}(x, y, z) = -2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$ across S in the direction of $\mathbf{n}$.



40. Let S be the portion of the cylinder $y = \ln x$ in the first octant whose projection parallel to the y-axis onto the xz-plane is the rectangle R_{xz}: $1 \le x \le e, 0 \le z \le 1$. Let $\mathbf{n}$ be the unit vector normal to S that points away from the xz-plane. Find the flux of $\mathbf{F} = 2y\mathbf{j} + z\mathbf{k}$ through S in the direction of $\mathbf{n}$.

41. Find the outward flux of the field $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ across the surface of the cube cut from the first octant by the planes $x = a, y = a, z = a$.

42. Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \le 25$ by the plane $z = 3$.

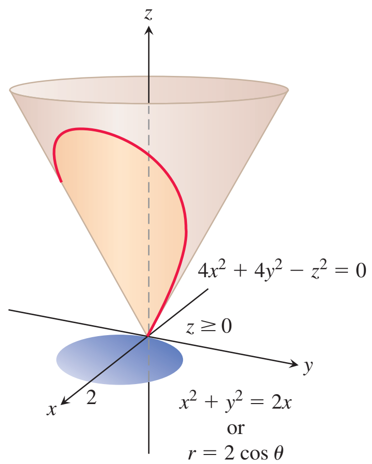
Moments and Masses

43. Centroid Find the centroid of the portion of the sphere $x^2 + y^2 + z^2 = a^2$ that lies in the first octant.

44. Centroid Find the centroid of the surface cut from the cylinder $y^2 + z^2 = 9, z \ge 0$, by the planes $x = 0$ and $x = 3$ (resembles the surface in Example 6).

45. Thin shell of constant density Find the center of mass and the moment of inertia about the z-axis of a thin shell of constant density δ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.

46. Conical surface of constant density Find the moment of inertia about the z-axis of a thin shell of constant density δ cut from the cone $4x^2 + 4y^2 - z^2 = 0, z \ge 0$, by the circular cylinder $x^2 + y^2 = 2x$ (see the accompanying figure).



47. Spherical shells

a. Find the moment of inertia about a diameter of a thin spherical shell of radius a and constant density δ. (Work with a hemispherical shell and double the result.)

b. Use the Parallel Axis Theorem (Exercises 15.6) and the result in part (a) to find the moment of inertia about a line tangent to the shell.

48. Conical Surface Find the centroid of the lateral surface of a solid cone of base radius a and height h (cone surface minus the base).

16.7 Stokes' Theorem

To calculate the counterclockwise circulation of a two-dimensional vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ around a simple closed curve in the plane, Green's Theorem says we can compute the double integral over the region enclosed by the curve of the scalar quantity $(\partial N/\partial x - \partial M/\partial y)$. This expression is the **k**-component of a *curl vector* field, which we define in this section, and it measures the rate of rotation of $\mathbf{F}$ at each point in the region around an axis parallel to $\mathbf{k}$. For a vector field on three-dimensional space, the rotation at each point is around an axis that is parallel to the curl vector at that point. When a closed curve C in space is the boundary of an oriented surface, we will see that the circulation of $\mathbf{F}$ around C is equal to the surface integral of the curl vector field. This result extends Green's Theorem from regions in the plane to general surfaces in space having a smooth boundary curve.

The Curl Vector Field

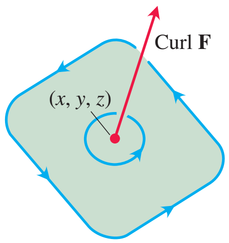


FIGURE 16.55 The circulation vector at a point (x, y, z) in a plane in a three-dimensional fluid flow. Notice its right-hand relation to the rotating particles in the fluid.

Suppose that $\mathbf{F}$ is the velocity field of a fluid flowing in space. Particles near the point (x, y, z) in the fluid tend to rotate around an axis through (x, y, z) that is parallel to a certain vector we are about to define. This vector points in the direction for which the rotation is counterclockwise when viewed looking down onto the plane of the circulation from the tip of the arrow representing the vector. This is the direction your right-hand thumb points when your fingers curl around the axis of rotation in the way consistent with the rotating motion of the particles in the fluid (see Figure 16.55). The length of the vector measures the rate of rotation. The vector is called the **curl vector**, and for the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ it is defined to be

$$\text{curl}\,\mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}. \tag{1}$$

This information is a consequence of Stokes' Theorem, the generalization to space of the circulation-curl form of Green's Theorem and the subject of this section.

Notice that $(\text{curl}\,\mathbf{F})\cdot\mathbf{k} = (\partial N/\partial x - \partial M/\partial y)$ is consistent with our definition in Section 16.4 when $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$. The formula for curl $\mathbf{F}$ in Equation (1) is often expressed with the symbolic operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{2}$$

to compute the curl of $\mathbf{F}$ as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$

The symbol ∇ is pronounced "del," and we often use this cross product notation to write the curl symbolically as "del cross $\mathbf{F}$."

$$\boxed{\text{curl } \mathbf{F} = \nabla \times \mathbf{F} \qquad (3)}$$

EXAMPLE 1 Find the curl of $\mathbf{F} = (x^2 - z)\mathbf{i} + xe^z\mathbf{j} + xy\mathbf{k}$.

Solution We use Equation (3) and the determinant form, so

$$\begin{aligned} \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - z & xe^z & xy \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xe^z)\right)\mathbf{i} - \left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(x^2 - z)\right)\mathbf{j} \\ &\quad + \left(\frac{\partial}{\partial x}(xe^z) - \frac{\partial}{\partial y}(x^2 - z)\right)\mathbf{k} \\ &= (x - xe^z)\mathbf{i} - (y + 1)\mathbf{j} + (e^z - 0)\mathbf{k} \\ &= x(1 - e^z)\mathbf{i} - (y + 1)\mathbf{j} + e^z\mathbf{k}. \end{aligned}$$

As we will see, the operator ∇ has a number of other applications. For instance, when applied to a scalar function $f(x, y, z)$, it gives the gradient of f:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

It is sometimes read as "del f" as well as "grad f."

Stokes' Theorem

Stokes' Theorem generalizes Green's Theorem to three dimensions. The circulation-curl form of Green's Theorem relates the counterclockwise circulation of a vector field around a simple closed curve C in the xy-plane to a double integral over the plane region R enclosed by C. Stokes' Theorem relates the circulation of a vector field around the boundary C of an oriented surface S in space (Figure 16.56) to a surface integral over the surface S. We require that the surface be **piecewise smooth**, which means that it is a finite union of smooth surfaces joining along smooth curves.

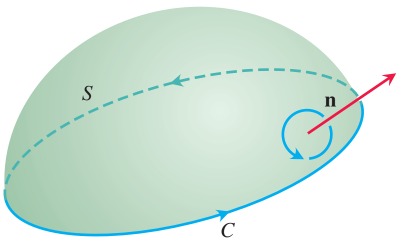


FIGURE 16.56 The orientation of the bounding curve C gives it a right-handed relation to the normal field $\mathbf{n}$. If the thumb of a right hand points along $\mathbf{n}$, the fingers curl in the direction of C.

THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S:

$$\underbrace{\oint_C \mathbf{F}\cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\, d\sigma}_{\text{Curl integral}} \tag{4}$$

Notice from Equation (4) that if two different oriented surfaces S_1 and S_2 have the same boundary C, their curl integrals are equal:

$$\iint_{S_1} \nabla \times \mathbf{F}\cdot\mathbf{n}_1\, d\sigma = \iint_{S_2} \nabla \times \mathbf{F}\cdot\mathbf{n}_2\, d\sigma.$$

Both curl integrals equal the counterclockwise circulation integral on the left side of Equation (4) as long as the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ correctly orient the surfaces. So the curl integral is independent of the surface and depends only on circulation along the boundary curve. This independence of surface resembles the path independence for the flow integral of a conservative velocity field along a curve, where the value of the flow integral depends only on the endpoints (that is, the boundary points) of the path. The curl field $\nabla \times \mathbf{F}$ is analogous to the gradient field ∇f of a scalar function f.

If C is a curve in the xy-plane, oriented counterclockwise, and R is the region in the xy-plane bounded by C, then $d\sigma = dx\,dy$ and

$$(\nabla \times \mathbf{F})\cdot\mathbf{n} = (\nabla \times \mathbf{F})\cdot\mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Under these conditions, Stokes' equation becomes

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy,$$

which is the circulation-curl form of the equation in Green's Theorem. Conversely, by reversing these steps we can rewrite the circulation-curl form of Green's Theorem for two-dimensional fields in del notation as

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F}\cdot\mathbf{k}\, dA. \tag{5}$$

See Figure 16.57.

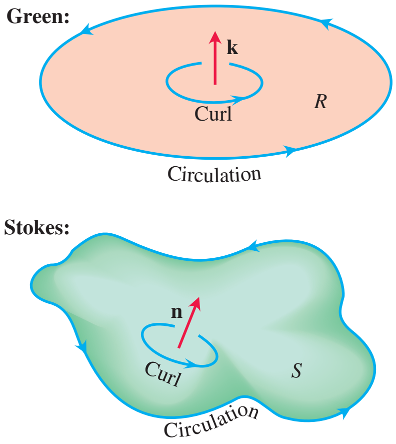


FIGURE 16.57 Comparison of Green's Theorem and Stokes' Theorem.

EXAMPLE 2 Evaluate Equation (4) for the hemisphere S: $x^2 + y^2 + z^2 = 9$, $z \geq 0$, its bounding circle C: $x^2 + y^2 = 9$, $z = 0$, and the field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$.

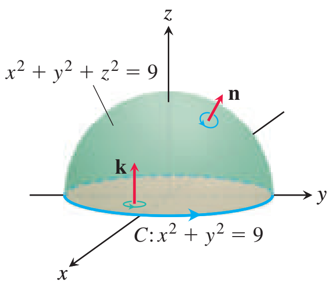


FIGURE 16.58 A hemisphere and a disk, each with boundary C (Examples 2 and 3).

Solution The hemisphere looks much like the surface in Figure 16.56 with the bounding circle C in the xy-plane (see Figure 16.58). We calculate the counterclockwise circulation around C (as viewed from above) using the parametrization $\mathbf{r}(\theta) = (3\cos\theta)\mathbf{i} + (3\sin\theta)\mathbf{j}$, $0 \le \theta \le 2\pi$:

$$\begin{aligned} d\mathbf{r} &= (-3\sin\theta\, d\theta)\mathbf{i} + (3\cos\theta\, d\theta)\mathbf{j} \\ \mathbf{F} &= y\mathbf{i} - x\mathbf{j} = (3\sin\theta)\mathbf{i} - (3\cos\theta)\mathbf{j} \\ \mathbf{F}\cdot d\mathbf{r} &= -9\sin^2\theta\, d\theta - 9\cos^2\theta\, d\theta = -9\, d\theta \\ \oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} -9\, d\theta = -18\pi. \end{aligned}$$

For the curl integral of $\mathbf{F}$, we have

$$\begin{aligned} \nabla\times\mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \\ &= (0-0)\mathbf{i} + (0-0)\mathbf{j} + (-1-1)\mathbf{k} = -2\mathbf{k} \\ \mathbf{n} &= \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{3} && \text{Outer unit normal} \\ d\sigma &= \frac{3}{z}\, dA && \text{Section 16.6, Example 7, with } a = 3 \\ \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma &= -\frac{2z}{3}\frac{3}{z}\, dA = -2\, dA \end{aligned}$$

and

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_{x^2+y^2\le 9} -2\, dA = -18\pi.$$

The circulation around the circle equals the integral of the curl over the hemisphere, as it should from Stokes' Theorem. ■

The surface integral in Stokes' Theorem can be computed using any surface having boundary curve C, provided the surface is properly oriented and lies within the domain of the field $\mathbf{F}$. The next example illustrates this fact for the circulation around the curve C in Example 2.

EXAMPLE 3 Calculate the circulation around the bounding circle C in Example 2 using the disk of radius 3 centered at the origin in the xy-plane as the surface S (instead of the hemisphere). See Figure 16.58.

Solution As in Example 2, $\nabla\times\mathbf{F} = -2\mathbf{k}$. For the surface being the described disk in the xy-plane, we have the normal vector $\mathbf{n} = \mathbf{k}$ so that

$$\nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma = -2\mathbf{k}\cdot\mathbf{k}\, dA = -2\, dA$$

and

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma = \iint_{x^2+y^2\le 9} -2\, dA = -18\pi,$$

a simpler calculation than before. ■

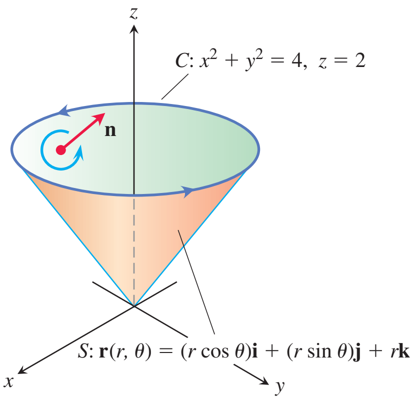


FIGURE 16.59 The curve C and cone S in Example 4.

EXAMPLE 4 Find the circulation of the field $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2+y^2}$, counterclockwise as viewed from above (Figure 16.59).

Solution Stokes' Theorem enables us to find the circulation by integrating over the surface of the cone. Traversing C in the counterclockwise direction viewed from above corresponds to taking the *inner* normal $\mathbf{n}$ to the cone, the normal with a positive $\mathbf{k}$-component.

We parametrize the cone as

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$$

We then have

$$\begin{aligned}
\mathbf{n} &= \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} = \frac{-(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}}{r\sqrt{2}} && \text{Section 16.5, Example 4}\\
&= \frac{1}{\sqrt{2}}\big(-(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} + \mathbf{k}\big)\\
d\sigma &= r\sqrt{2}\,dr\,d\theta && \text{Section 16.5, Example 4}\\
\nabla \times \mathbf{F} &= -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k} && \text{Routine calculation}\\
&= -4\mathbf{i} - 2r\cos\theta\mathbf{j} + \mathbf{k}. && x = r\cos\theta
\end{aligned}$$

Accordingly,

$$\begin{aligned}
\nabla \times \mathbf{F}\cdot\mathbf{n} &= \frac{1}{\sqrt{2}}\big(4\cos\theta + 2r\cos\theta\sin\theta + 1\big)\\
&= \frac{1}{\sqrt{2}}\big(4\cos\theta + r\sin 2\theta + 1\big)
\end{aligned}$$

and the circulation is

$$\begin{aligned}
\oint_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma && \text{Stokes' Theorem, Eq. (4)}\\
&= \int_0^{2\pi}\int_0^2 \frac{1}{\sqrt{2}}\big(4\cos\theta + r\sin 2\theta + 1\big)\big(r\sqrt{2}\,dr\,d\theta\big) = 4\pi.
\end{aligned}$$

■

EXAMPLE 5 The cone used in Example 4 is not the easiest surface to use for calculating the circulation around the bounding circle C lying in the plane $z = 3$. If instead we use the flat disk of radius 3 centered on the z-axis and lying in the plane $z = 3$, then the normal vector to the surface S is $\mathbf{n} = \mathbf{k}$. Just as in the computation for Example 4, we still have $\nabla \times \mathbf{F} = -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k}$. However, now we get $\nabla \times \mathbf{F}\cdot\mathbf{n} = 1$, so that

$$\iint_S \nabla \times \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_{x^2+y^2\le 4} 1\,dA = 4\pi. \qquad \text{The shadow is the disk of radius 2 in the } xy\text{-plane.}$$

This result agrees with the circulation value found in Example 4. ■

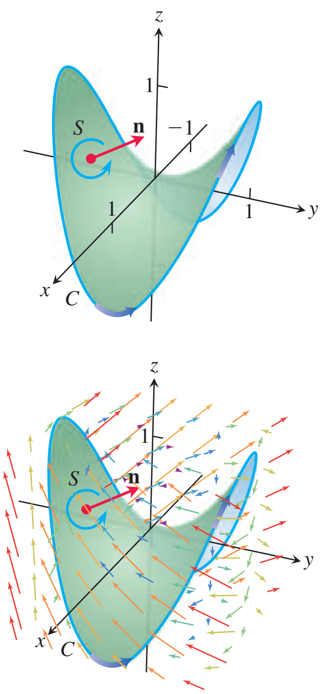


FIGURE 16.60 The surface and vector field for Example 6.

EXAMPLE 6 Find a parametrization for the surface S formed by the part of the hyperbolic paraboloid $z = y^2 - x^2$ lying inside the cylinder of radius one around the z-axis and for the boundary curve C of S. (See Figure 16.60.) Then verify Stokes' Theorem for S using the normal having positive $\mathbf{k}$-component and the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$.

Solution As the unit circle is traversed counterclockwise in the xy-plane, the z-coordinate of the surface with the curve C as boundary is given by $y^2 - x^2$. A parametrization of C is given by

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin^2 t - \cos^2 t)\mathbf{k}, \quad 0 \le t \le 2\pi$$

with

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (4 \sin t \cos t)\mathbf{k}, \ \ 0 \le t \le 2\pi.$$

Along the curve $\mathbf{r}(t)$ the formula for the vector field $\mathbf{F}$ is

$$\mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j} + (\cos^2 t)\mathbf{k}.$$

The counterclockwise circulation along C is the value of the line integral

$$\begin{aligned}\int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t + 4 \sin t \cos^3 t\right) dt \\ &= \int_0^{2\pi} \left(4 \sin t \cos^3 t - 1\right) dt \\ &= \left[-\cos^4 t - t\right]_0^{2\pi} = -2\pi.\end{aligned}$$

We now compute the same quantity by integrating $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over the surface S. We use polar coordinates and parametrize S by noting that above the point (r, θ) in the plane, the z–coordinate of S is $y^2 - x^2 = r^2 \sin^2 \theta - r^2 \cos^2 \theta$. A parametrization of S is

$$\mathbf{r}(r, \theta) = (r \cos \theta)\mathbf{i} + (r \sin \theta)\mathbf{j} + r^2(\sin^2 \theta - \cos^2 \theta)\mathbf{k}, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

We next compute $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & x^2 \end{vmatrix} = -2x\mathbf{j} - 2\mathbf{k} = -(2r \cos \theta)\mathbf{j} - 2\mathbf{k}$$

and

$$\begin{aligned}\mathbf{r}_r &= (\cos \theta)\mathbf{i} + (\sin \theta)\mathbf{j} + 2r(\sin^2 \theta - \cos^2 \theta)\mathbf{k} \\ \mathbf{r}_\theta &= (-r \sin \theta)\mathbf{i} + (r \cos \theta)\mathbf{j} + 4r^2(\sin \theta \cos \theta)\mathbf{k} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \theta & \sin \theta & 2r(\sin^2 \theta - \cos^2 \theta) \\ -r \sin \theta & r \cos \theta & 4r^2(\sin \theta \cos \theta) \end{vmatrix} \\ &= 2r^2(2 \sin^2 \theta \cos \theta - \sin^2 \theta \cos \theta + \cos^3 \theta)\mathbf{i} \\ &\quad -2r^2(2 \sin \theta \cos^2 \theta + \sin^3 \theta + \sin \theta \cos^2 \theta)\mathbf{j} + r\mathbf{k}.\end{aligned}$$

We now obtain

$$\begin{aligned}\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \int_0^{2\pi} \int_0^1 \nabla \times \mathbf{F} \cdot \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 \nabla \times \mathbf{F} \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 \left[4r^3\left(2 \sin \theta \cos^3 \theta + \sin^3 \theta \cos \theta + \sin \theta \cos^3 \theta\right) - 2r\right] dr\, d\theta \\ &= \int_0^{2\pi} \left[r^4\left(3 \sin \theta \cos^3 \theta + \sin^3 \theta \cos \theta\right) - r^2\right]_{r=0}^{r=1} d\theta \qquad \text{Integrate.} \\ &= \int_0^{2\pi} \left(3 \sin \theta \cos^3 \theta + \sin^3 \theta \cos \theta - 1\right) d\theta \qquad \text{Evaluate.} \\ &= \left[-\frac{3}{4} \cos^4 \theta + \frac{1}{4} \sin^4 \theta - \theta\right]_0^{2\pi} \\ &= \left(-\frac{3}{4} + 0 - 2\pi + \frac{3}{4} - 0 + 0\right) = -2\pi.\end{aligned}$$

So the surface integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S equals the counterclockwise circulation of $\mathbf{F}$ along C, as asserted by Stokes' Theorem. ■

EXAMPLE 7 Calculate the circulation of the vector field

$$\mathbf{F} = (x^2 + z)\mathbf{i} + (y^2 + 2x)\mathbf{j} + (z^2 - y)\mathbf{k}$$

along the curve of intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the cone $z = \sqrt{x^2 + y^2}$ traversed in the counterclockwise direction around the z-axis when viewed from above.

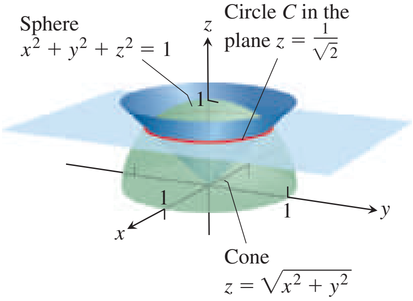


FIGURE 16.61 Circulation curve C in Example 7.

Solution The sphere and cone intersect when $1 = (x^2 + y^2) + z^2 = z^2 + z^2 = 2z^2$, or $z = 1/\sqrt{2}$ (see Figure 16.61). We apply Stokes' Theorem to the curve of intersection $x^2 + y^2 = 1/2$ considered as the boundary of the enclosed disk in the plane $z = 1/\sqrt{2}$. The normal vector to the surface is then $\mathbf{n} = \mathbf{k}$. We calculate the curl vector as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 + z & y^2 + 2x & z^2 - y \end{vmatrix} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}, \qquad \text{Routine calculation}$$

so that $\nabla \times \mathbf{F} \cdot \mathbf{k} = 2$. The circulation around the disk is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{k}\, d\sigma \\ &= \iint_S 2\, d\sigma = 2 \cdot \text{area of disk} = 2 \cdot \pi \left(\frac{1}{\sqrt{2}}\right)^2 = \pi. \end{aligned}$$ ■

Paddle Wheel Interpretation of $\nabla \times \mathbf{F}$

Suppose that $\mathbf{F}$ is the velocity field of a fluid moving in a region R in space containing the closed curve C. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is the circulation of the fluid around C. By Stokes' Theorem, the circulation is equal to the flux of $\nabla \times \mathbf{F}$ through any suitably oriented surface S with boundary C:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Suppose we fix a point Q in the region R and a direction $\mathbf{u}$ at Q. Take C to be a circle of radius ρ, with center at Q, whose plane is normal to $\mathbf{u}$. If $\nabla \times \mathbf{F}$ is continuous at Q, the average value of the $\mathbf{u}$-component of $\nabla \times \mathbf{F}$ over the circular disk S bounded by C approaches the $\mathbf{u}$-component of $\nabla \times \mathbf{F}$ at Q as the radius $\rho \to 0$:

$$(\nabla \times \mathbf{F} \cdot \mathbf{u})_Q = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{u}\, d\sigma.$$

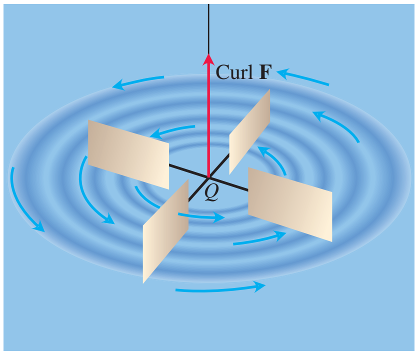


FIGURE 16.62 A small paddle wheel in a fluid spins fastest at point Q when its axle points in the direction of curl **F**.

If we apply Stokes' Theorem and replace the surface integral by a line integral over C, we get

$$(\nabla \times \mathbf{F} \cdot \mathbf{u})_Q = \lim_{\rho \to 0} \frac{1}{\pi \rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r}. \tag{6}$$

The left-hand side of Equation (6) has its maximum value when $\mathbf{u}$ is the direction of $\nabla \times \mathbf{F}$. When ρ is small, the limit on the right-hand side of Equation (6) is approximately

$$\frac{1}{\pi \rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

which is the circulation around C divided by the area of the disk (circulation density). Suppose that a small paddle wheel of radius ρ is introduced into the fluid at Q, with its axle directed along $\mathbf{u}$ (Figure 16.62). The circulation of the fluid around C affects the rate of spin of the paddle wheel. The wheel spins fastest when the circulation integral is maximized; therefore it spins fastest when the axle of the paddle wheel points in the direction of $\nabla \times \mathbf{F}$.

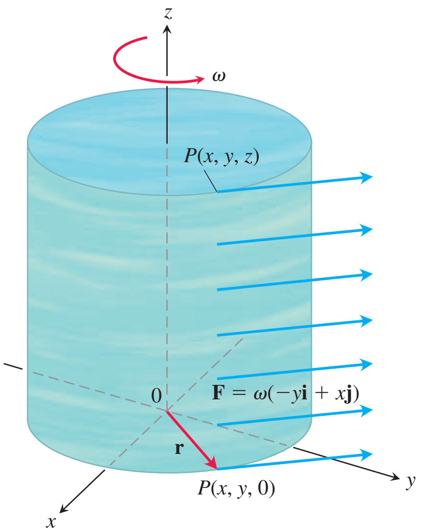


FIGURE 16.63 A steady rotational flow parallel to the xy-plane, with constant angular velocity ω in the positive (counterclockwise) direction (Example 8).

EXAMPLE 8 A fluid of constant density rotates around the z-axis with velocity $\mathbf{F} = \omega(-y\mathbf{i} + x\mathbf{j})$, where ω is a positive constant called the *angular velocity* of the rotation (Figure 16.63). Find $\nabla \times \mathbf{F}$ and relate it to the circulation density.

Solution With $\mathbf{F} = -\omega y\mathbf{i} + \omega x\mathbf{j}$, we find the curl

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k} \\ &= (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (\omega - (-\omega))\mathbf{k} = 2\omega\mathbf{k}.\end{aligned}$$

By Stokes' Theorem, the circulation of $\mathbf{F}$ around a circle C of radius ρ bounding a disk S in a plane normal to $\nabla \times \mathbf{F}$, say the xy-plane, is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 2\omega\mathbf{k} \cdot \mathbf{k}\, dx\, dy = (2\omega)(\pi\rho^2).$$

Thus solving this last equation for 2ω, we have

$$(\nabla \times \mathbf{F}) \cdot \mathbf{k} = 2\omega = \frac{1}{\pi\rho^2} \oint_C \mathbf{F} \cdot d\mathbf{r},$$

consistent with Equation (6) when $\mathbf{u} = \mathbf{k}$. ■

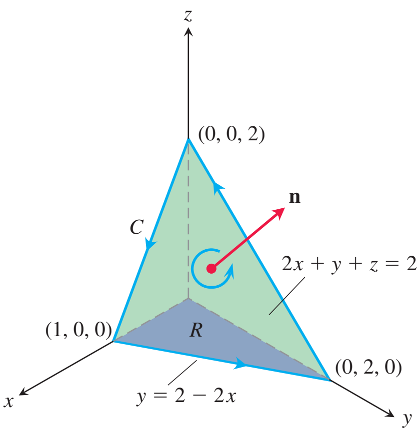


FIGURE 16.64 The planar surface in Example 9.

EXAMPLE 9 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, if $\mathbf{F} = xz\mathbf{i} + xy\mathbf{j} + 3xz\mathbf{k}$ and C is the boundary of the portion of the plane $2x + y + z = 2$ in the first octant, traversed counterclockwise as viewed from above (Figure 16.64).

Solution The plane is the level surface $f(x, y, z) = 2$ of the function $f(x, y, z) = 2x + y + z$. The unit normal vector

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{(2\mathbf{i} + \mathbf{j} + \mathbf{k})}{|2\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{1}{\sqrt{6}}\left(2\mathbf{i} + \mathbf{j} + \mathbf{k}\right)$$

is consistent with the counterclockwise motion around C. To apply Stokes' Theorem, we find

$$\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xy & 3xz \end{vmatrix} = (x - 3z)\mathbf{j} + y\mathbf{k}.$$

On the plane, z equals $2 - 2x - y$, so

$$\nabla\times\mathbf{F} = (x - 3(2 - 2x - y))\mathbf{j} + y\mathbf{k} = (7x + 3y - 6)\mathbf{j} + y\mathbf{k}$$

and

$$\nabla\times\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{6}}\left(7x + 3y - 6 + y\right) = \frac{1}{\sqrt{6}}\left(7x + 4y - 6\right).$$

The surface area element is

$$d\sigma = \frac{|\nabla f|}{|\nabla f\cdot\mathbf{k}|}\,dA = \frac{\sqrt{6}}{1}\,dx\,dy.$$

The circulation is

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma \qquad \text{Stokes' Theorem, Eq. (4)}$$

$$= \int_0^1\int_0^{2-2x} \frac{1}{\sqrt{6}}\left(7x + 4y - 6\right)\sqrt{6}\,dy\,dx$$

$$= \int_0^1\int_0^{2-2x} (7x + 4y - 6)\,dy\,dx = -1.$$

EXAMPLE 10 Let the surface S be the elliptical paraboloid $z = x^2 + 4y^2$ lying beneath the plane $z = 1$ (Figure 16.65). We define the orientation of S by taking the *inner* normal vector $\mathbf{n}$ to the surface, which is the normal having a positive $\mathbf{k}$-component. Find the flux of $\nabla\times\mathbf{F}$ across S in the direction $\mathbf{n}$ for the vector field $\mathbf{F} = y\mathbf{i} - xz\mathbf{j} + xz^2\mathbf{k}$.

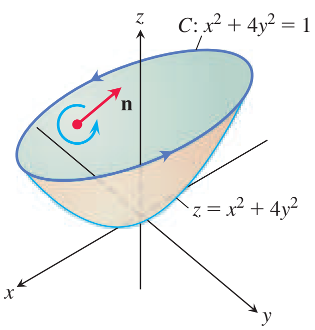


FIGURE 16.65 The portion of the elliptical paraboloid in Example 10, showing its curve of intersection C with the plane $z = 1$ and its inner normal orientation by $\mathbf{n}$.

Solution We use Stokes' Theorem to calculate the curl integral by finding the equivalent counterclockwise circulation of $\mathbf{F}$ around the curve of intersection C of the paraboloid $z = x^2 + 4y^2$ and the plane $z = 1$, as shown in Figure 16.65. Note that the orientation of S is consistent with traversing C in a counterclockwise direction around the z-axis. The curve C is the ellipse $x^2 + 4y^2 = 1$ in the plane $z = 1$. We can parametrize the ellipse by $x = \cos t$, $y = \frac{1}{2}\sin t$, $z = 1$ for $0 \le t \le 2\pi$, so C is given by

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + \frac{1}{2}(\sin t)\mathbf{j} + \mathbf{k}, \qquad 0 \le t \le 2\pi.$$

To compute the circulation integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$, we evaluate $\mathbf{F}$ along C and find the velocity vector $d\mathbf{r}/dt$:

$$\mathbf{F}(\mathbf{r}(t)) = \frac{1}{2}(\sin t)\mathbf{i} - (\cos t)\mathbf{j} + (\cos t)\mathbf{k}$$

and

$$\frac{d\mathbf{r}}{dt} = -(\sin t)\mathbf{i} + \frac{1}{2}(\cos t)\mathbf{j}.$$

Then,

$$\begin{aligned}\oint_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_0^{2\pi}\left(-\frac{1}{2}\sin^2 t - \frac{1}{2}\cos^2 t\right)dt \\ &= -\frac{1}{2}\int_0^{2\pi} dt = -\pi.\end{aligned}$$

Therefore, by Stokes' Theorem the flux of the curl across S in the direction $\mathbf{n}$ for the field $\mathbf{F}$ is

$$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma = -\pi.$$

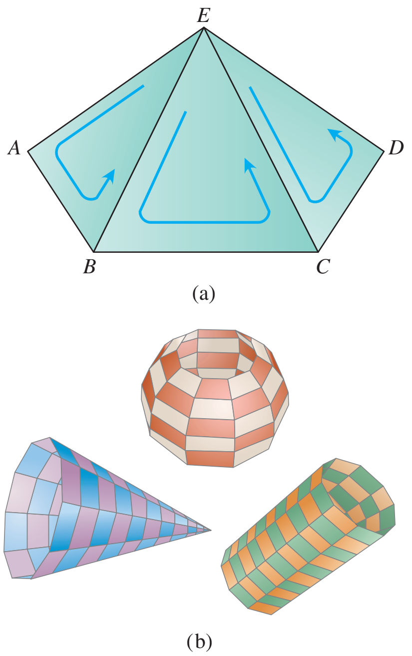


FIGURE 16.66 (a) Part of a polyhedral surface. (b) Other polyhedral surfaces.

Proof Outline of Stokes' Theorem for Polyhedral Surfaces

Let S be a polyhedral surface consisting of a finite number of plane regions or faces. (See Figure 16.66 for examples.) We apply Green's Theorem to each separate face of S. There are two types of faces:

1. Those that are surrounded on all sides by other faces.
2. Those that have one or more edges that are not adjacent to other faces.

The boundary Δ of S consists of those edges of the type 2 faces that are not adjacent to other faces. In Figure 16.66a, the triangles EAB, BCE, and CDE represent a part of S, with $ABCD$ part of the boundary Δ. Although Green's Theorem was stated for curves in the xy-plane, a generalized form applies to plane curves in space, where $\mathbf{n}$ is normal to the plane (instead of $\mathbf{k}$). In the generalized tangential form, the theorem asserts that the line integral of $\mathbf{F}$ around the curve enclosing the plane region R normal to $\mathbf{n}$ equals the double integral of $(\operatorname{curl}\mathbf{F})\cdot\mathbf{n}$ over R. Applying this generalized form to the three triangles of Figure 16.66a in turn, and adding the results, gives

$$\left(\oint_{EAB} + \oint_{BCE} + \oint_{CDE}\right)\mathbf{F}\cdot d\mathbf{r} = \left(\iint_{EAB} + \iint_{BCE} + \iint_{CDE}\right)\nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma. \tag{7}$$

The three line integrals on the left-hand side of Equation (7) combine into a single line integral taken around the periphery $ABCDE$ because the integrals along interior segments cancel in pairs. For example, the integral along segment BE in triangle ABE is opposite in sign to the integral along the same segment in triangle EBC. The same holds for segment CE. Hence, Equation (7) reduces to

$$\oint_{ABCDE} \mathbf{F}\cdot d\mathbf{r} = \iint_{ABCDE} \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

When we apply the generalized form of Green's Theorem to all the faces and add the results, we get

$$\oint_{\Delta} \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\,d\sigma.$$

This is Stokes' Theorem for the polyhedral surface S in Figure 16.66a. More general polyhedral surfaces are shown in Figure 16.66b and the proof can be extended to them. General smooth surfaces can be obtained as limits of polyhedral surfaces and a complete proof can be found in more advanced texts.

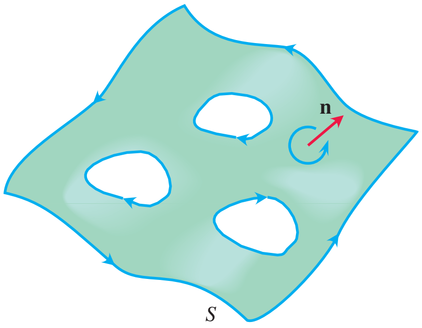


FIGURE 16.67 Stokes' Theorem also holds for oriented surfaces with holes. Consistent with the orientation of S, the outer curve is traversed counterclockwise around $\mathbf{n}$ and the inner curves surrounding the holes are traversed clockwise.

Stokes' Theorem for Surfaces with Holes

Stokes' Theorem holds for an oriented surface S that has one or more holes (Figure 16.67). The surface integral over S of the normal component of $\nabla \times \mathbf{F}$ equals the sum of the line integrals around all the boundary curves of the tangential component of $\mathbf{F}$, where the curves are to be traced in the direction induced by the orientation of S. For such surfaces the theorem is unchanged, but C is considered as a union of simple closed curves.

An Important Identity

The following identity arises frequently in mathematics and the physical sciences.

$$\text{curl grad } f = \mathbf{0} \qquad \text{or} \qquad \nabla \times \nabla f = \mathbf{0} \tag{8}$$

Forces arising in the study of electromagnetism and gravity are often associated with a potential function f. The identity (8) says that these forces have curl equal to zero. The identity (8) holds for any function $f(x, y, z)$ whose second partial derivatives are continuous. The proof goes like this:

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k}.$$

If the second partial derivatives are continuous, the mixed second derivatives in parentheses are equal (Theorem 2, Section 14.3) and the vector is zero.

Conservative Fields and Stokes' Theorem

In Section 16.3, we found that a field $\mathbf{F}$ being conservative in an open region D in space is equivalent to the integral of $\mathbf{F}$ around every closed loop in D being zero. This, in turn, is equivalent in *simply connected* open regions to saying that $\nabla \times \mathbf{F} = \mathbf{0}$ (which gives a test for determining if $\mathbf{F}$ is conservative for such regions).

THEOREM 7—Curl F = 0 Related to the Closed-Loop Property If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

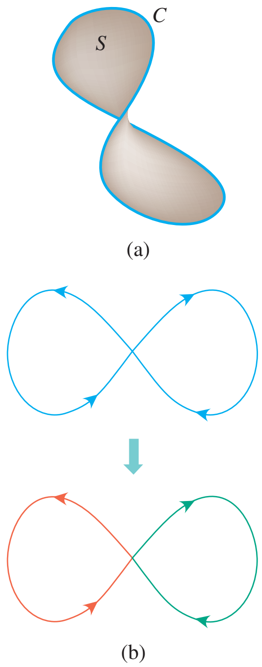


FIGURE 16.68 (a) In a simply connected open region in space, a simple closed curve C is the boundary of a smooth surface S. (b) Smooth curves that cross themselves can be divided into loops to which Stokes' Theorem applies.

Sketch of a Proof Theorem 7 can be proved in two steps. The first step is for simple closed curves (loops that do not cross themselves), like the one in Figure 16.68a. A theorem from topology, a branch of advanced mathematics, states that every smooth simple closed curve C in a simply connected open region D is the boundary of a smooth two-sided surface S that also lies in D. Hence, by Stokes' Theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

The second step is for curves that cross themselves, like the one in Figure 16.68b. The idea is to break these into simple loops spanned by orientable surfaces, apply Stokes' Theorem one loop at a time, and add the results. ■

The following diagram summarizes the results for conservative fields defined on connected, simply connected open regions. For such regions, the four statements are equivalent to each other.

$$\begin{array}{ccc}
\mathbf{F} \text{ conservative on } D & \overset{\text{Theorem 2, Section 16.3}}{\Longleftrightarrow} & \mathbf{F} = \nabla f \text{ on } D \\
\text{Theorem 3, Section 16.3} \ \Updownarrow & & \Downarrow \ \text{Vector identity (Eq. 8) (continuous second partial derivatives)} \\
\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 \text{ over any closed path in } D & \underset{\text{Theorem 7 Domain's simple connectivity and Stokes' Theorem}}{\Longleftarrow} & \nabla \times \mathbf{F} = \mathbf{0} \text{ throughout } D
\end{array}$$

Exercises 16.7

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

1. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$

C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counterclockwise when viewed from above

2. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$

C: The circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above

3. $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$

C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

4. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above

5. $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counterclockwise when viewed from above

6. $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$

C: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above

Integral of the Curl Vector Field

7. Let $\mathbf{n}$ be the outer unit normal of the elliptical shell

$$S: \quad 4x^2 + 9y^2 + 36z^2 = 36, \qquad z \geq 0,$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + (x^2 + y^4)^{3/2} \sin e^{\sqrt{xyz}}\,\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

(*Hint:* One parametrization of the ellipse at the base of the shell is $x = 3\cos t$, $y = 2\sin t$, $0 \leq t \leq 2\pi$.)

8. Let $\mathbf{n}$ be the outer unit normal (normal away from the origin) of the parabolic shell

$$S: \quad 4x^2 + y + z^2 = 4, \qquad y \geq 0,$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2 + x}\right)\mathbf{i} + (\tan^{-1} y)\mathbf{j} + \left(x + \frac{1}{4 + z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

9. Let S be the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$, together with its top, $x^2 + y^2 \leq a^2$, $z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through S.

10. Evaluate

$$\iint_S \nabla \times (y\mathbf{i}) \cdot \mathbf{n}\, d\sigma,$$

where S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$.

11. Suppose $\mathbf{F} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = \left(y + \sqrt{z}\right)\mathbf{i} + e^{xyz}\,\mathbf{j} + \cos(xz)\,\mathbf{k}.$$

Determine the flux of $\mathbf{F}$ outward through the hemisphere $x^2 + y^2 + z^2 = 1, z \ge 0$.

12. Repeat Exercise 11 for the flux of $\mathbf{F}$ across the entire unit sphere.

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

13. $\mathbf{F} = 2z\mathbf{i} + 3x\mathbf{j} + 5y\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(4 - r^2\right)\mathbf{k}$, $0 \le r \le 2, \quad 0 \le \theta \le 2\pi$

14. $\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(9 - r^2\right)\mathbf{k}$, $0 \le r \le 3, \quad 0 \le \theta \le 2\pi$

15. $\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

16. $\mathbf{F} = (x - y)\mathbf{i} + (y - z)\mathbf{j} + (z - x)\mathbf{k}$

S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (5 - r)\mathbf{k}$, $0 \le r \le 5, \quad 0 \le \theta \le 2\pi$

17. $\mathbf{F} = 3y\mathbf{i} + (5 - 2x)\mathbf{j} + (z^2 - 2)\mathbf{k}$

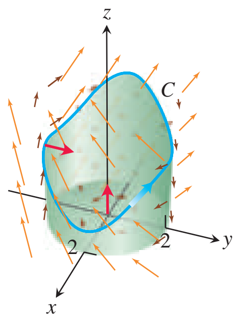
S: $\mathbf{r}(\phi, \theta) = \left(\sqrt{3}\sin\phi\cos\theta\right)\mathbf{i} + \left(\sqrt{3}\sin\phi\sin\theta\right)\mathbf{j} + \left(\sqrt{3}\cos\phi\right)\mathbf{k}, \quad 0 \le \phi \le \pi/2, \quad 0 \le \theta \le 2\pi$

18. $\mathbf{F} = y^2\mathbf{i} + z^2\mathbf{j} + x\mathbf{k}$

S: $\mathbf{r}(\phi, \theta) = (2\sin\phi\cos\theta)\mathbf{i} + (2\sin\phi\sin\theta)\mathbf{j} + (2\cos\phi)\mathbf{k}$, $0 \le \phi \le \pi/2, \quad 0 \le \theta \le 2\pi$

Theory and Examples

19. Let C be the smooth curve $\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + (3 - 2\cos^3 t)\mathbf{k}$, oriented to be traversed counterclockwise around the z-axis when viewed from above. Let S be the piecewise smooth cylindrical surface $x^2 + y^2 = 4$, below the curve for $z \ge 0$, together with the base disk in the xy-plane. Note that C lies on the cylinder S and above the xy-plane (see the accompanying figure). Verify Equation (4) in Stokes' Theorem for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$.



20. Verify Stokes' Theorem for the vector field $\mathbf{F} = 2xy\mathbf{i} + x\mathbf{j} + (y + z)\mathbf{k}$ and surface $z = 4 - x^2 - y^2, z \ge 0$, oriented with unit normal $\mathbf{n}$ pointing upward.

21. Zero circulation Use Equation (8) and Stokes' Theorem to show that the circulations of the following fields around the boundary of any smooth orientable surface in space are zero.

a. $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ **b.** $\mathbf{F} = \nabla(xy^2z^3)$

c. $\mathbf{F} = \nabla \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$ **d.** $\mathbf{F} = \nabla f$

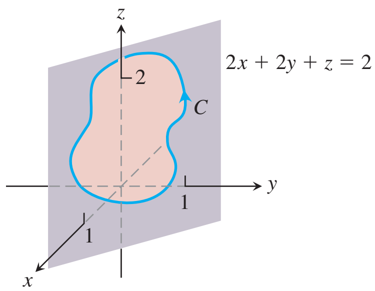
22. Zero circulation Let $f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$. Show that the clockwise circulation of the field $\mathbf{F} = \nabla f$ around the circle $x^2 + y^2 = a^2$ in the xy-plane is zero

a. by taking $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, 0 \le t \le 2\pi$, and integrating $\mathbf{F}\cdot d\mathbf{r}$ over the circle.

b. by applying Stokes' Theorem.

23. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented as shown here. Show that

$$\oint_C 2y\,dx + 3z\,dy - x\,dz$$



depends only on the area of the region enclosed by C and not on the position or shape of C.

24. Show that if $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then $\nabla \times \mathbf{F} = \mathbf{0}$.

25. Find a vector field with twice-differentiable components whose curl is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or prove that no such field exists.

26. Does Stokes' Theorem say anything special about circulation in a field whose curl is zero? Give reasons for your answer.

27. Let R be a region in the xy-plane that is bounded by a piecewise smooth simple closed curve C and suppose that the moments of inertia of R about the x- and y-axes are known to be I_x and I_y. Evaluate the integral

$$\oint_C \nabla(r^4)\cdot\mathbf{n}\,ds,$$

where $r = \sqrt{x^2 + y^2}$, in terms of I_x and I_y.

28. Zero curl, yet the field is not conservative Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F}\cdot d\mathbf{r}$$

is not zero if C is the circle $x^2 + y^2 = 1$ in the xy-plane. (Theorem 7 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the z-axis so there is no way to contract C to a point without leaving the domain of $\mathbf{F}$.)

16.8 The Divergence Theorem and a Unified Theory

The divergence form of Green's Theorem in the plane states that the net outward flux of a vector field across a simple closed curve can be calculated by integrating the divergence of the field over the region enclosed by the curve. The corresponding theorem in three dimensions, called the *Divergence Theorem*, states that the net outward flux of a vector field across a closed surface in space can be calculated by integrating the divergence of the field over the region enclosed by the surface. In this section we prove the Divergence Theorem and show how it simplifies the calculation of flux, which is the integral of the field over the closed oriented surface. We also derive Gauss's law for flux in an electric field and the continuity equation of hydrodynamics. Finally, we summarize the chapter's vector integral theorems in a single unifying principle generalizing the Fundamental Theorem of Calculus.

Divergence in Three Dimensions

The **divergence** of a vector field $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ is the scalar function

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}. \tag{1}$$

The symbol "div **F**" is read as "divergence of **F**" or "div **F**." The notation $\nabla \cdot \mathbf{F}$ is read "del dot **F**."

Div **F** has the same physical interpretation in three dimensions that it does in two. If **F** is the velocity field of a flowing gas, the value of div **F** at a point (x, y, z) is the rate at which the gas is compressing or expanding at (x, y, z). The divergence is the flux per unit volume or *flux density* at the point.

EXAMPLE 1 The following vector fields represent the velocity of a gas flowing in space. Find the divergence of each vector field and interpret its physical meaning. Figure 16.69 displays the vector fields.

(a) Expansion: $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(b) Compression: $\mathbf{F}(x, y, z) = -x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$

(c) Rotation about the z-axis: $\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j}$

(d) Shearing along parallel horizontal planes: $\mathbf{F}(x, y, z) = z\mathbf{j}$

Solution

(a) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$: The gas is undergoing constant uniform expansion at all points.

(b) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(-z) = -3$: The gas is undergoing constant uniform compression at all points.

(c) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0$: The gas is neither expanding nor compressing at any point.

(d) $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial y}(z) = 0$: Again, the divergence is zero at all points in the domain of the velocity field, so the gas is neither expanding nor compressing at any point. ■

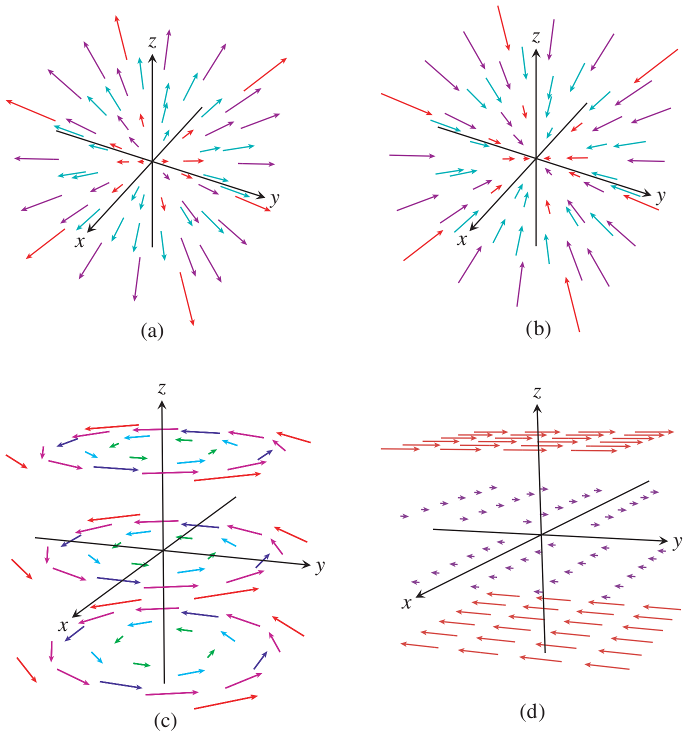


FIGURE 16.69 Velocity fields of a gas flowing in space (Example 1).

Divergence Theorem

The Divergence Theorem says that under suitable conditions, the outward flux of a vector field across a closed surface equals the triple integral of the divergence of the field over the three-dimensional region enclosed by the surface.

THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the triple integral of the divergence $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla\cdot\mathbf{F}\, dV.}_{\text{Divergence integral}} \tag{2}$$

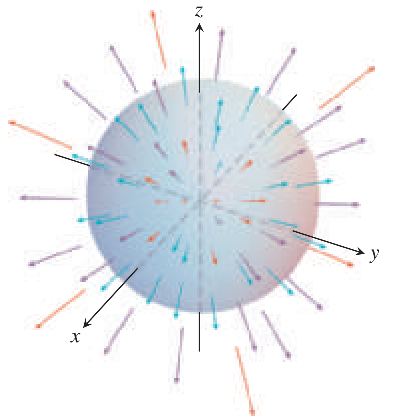


FIGURE 16.70 A uniformly expanding vector field and a sphere (Example 2).

EXAMPLE 2 Evaluate both sides of Equation (2) for the expanding vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$ (Figure 16.70).

Solution The outer unit normal to S, calculated from the gradient of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$, is

$$\mathbf{n} = \frac{2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}. \qquad x^2 + y^2 + z^2 = a^2 \text{ on } S$$

It follows that

$$\mathbf{F}\cdot\mathbf{n}\,d\sigma = \frac{x^2+y^2+z^2}{a}\,d\sigma = \frac{a^2}{a}\,d\sigma = a\,d\sigma.$$

Therefore, the outward flux is

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S a\,d\sigma = a\iint_S d\sigma = a(4\pi a^2) = 4\pi a^3. \qquad \text{Area of } S \text{ is } 4\pi a^2.$$

For the right-hand side of Equation (2), the divergence of $\mathbf{F}$ is

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

so we obtain the divergence integral,

$$\iiint_D \nabla\cdot\mathbf{F}\,dV = \iiint_D 3\,dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3. \qquad \blacksquare$$

Many vector fields of interest in applied science have zero divergence at each point. A common example is the velocity field of a circulating incompressible liquid, since it is neither expanding nor contracting. Other examples include constant vector fields $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, and velocity fields for shearing action along a fixed plane (see Example 1d). If $\mathbf{F}$ is a vector field whose divergence is zero at each point in the region D, then the integral on the right-hand side of Equation (2) equals 0. So if S is any closed surface for which the Divergence Theorem applies, then the outward flux of $\mathbf{F}$ across S is zero. We state this important application of the Divergence Theorem.

COROLLARY The outward flux across a piecewise smooth oriented closed surface S is zero for any vector field $\mathbf{F}$ having zero divergence at every point of the region enclosed by the surface.

EXAMPLE 3 Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ outward through the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$.

Solution Instead of calculating the flux as a sum of six separate integrals, one for each face of the cube, we can calculate the flux by integrating the divergence

$$\nabla\cdot\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

over the cube's interior:

$$\begin{aligned}\text{Flux} &= \iint_{\substack{\text{Cube}\\\text{surface}}} \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iiint_{\substack{\text{Cube}\\\text{interior}}} \nabla\cdot\mathbf{F}\,dV && \text{The Divergence Theorem}\\ &= \int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz = \frac{3}{2}. && \text{Routine integration} \qquad \blacksquare\end{aligned}$$

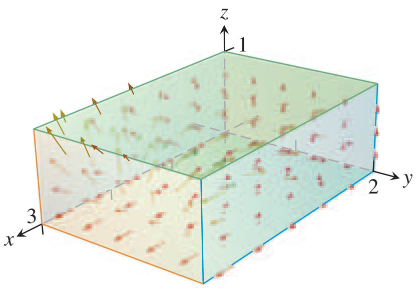


FIGURE 16.71 The integral of div $\mathbf{F}$ over this region equals the total flux across the six sides (Example 4).

EXAMPLE 4

(a) Calculate the flux of the vector field

$$\mathbf{F} = x^2\mathbf{i} + 4xyz\mathbf{j} + ze^x\mathbf{k}$$

out of the box-shaped region D: $0 \le x \le 3,\ 0 \le y \le 2,\ 0 \le z \le 1$. (See Figure 16.71.)

(b) Integrate div $\mathbf{F}$ over this region and show that the result is the same value as in part (a), as asserted by the Divergence Theorem.

Solution

(a) The region D has six sides. We calculate the flux across each side in turn. Consider the top side in the plane $z = 1$, having outward normal $\mathbf{n} = \mathbf{k}$. The flux across this side is given by $\mathbf{F}\cdot\mathbf{n} = ze^x$. Since $z = 1$ on this side, the flux at a point (x, y, z) on the top is e^x. The total outward flux across this side is given by the surface integral

$$\int_0^2\int_0^3 e^x\,dx\,dy = 2e^3 - 2. \qquad \text{Routine integration}$$

The outward flux across the other sides is computed similarly, and the results are summarized in the following table.

Side	Unit normal n	$\mathbf{F}\cdot\mathbf{n}$	Flux across side
$x = 0$	$-\mathbf{i}$	$-x^2 = 0$	0
$x = 3$	$\mathbf{i}$	$x^2 = 9$	18
$y = 0$	$-\mathbf{j}$	$-4xyz = 0$	0
$y = 2$	$\mathbf{j}$	$4xyz = 8xz$	18
$z = 0$	$-\mathbf{k}$	$-ze^x = 0$	0
$z = 1$	$\mathbf{k}$	$ze^x = e^x$	$2e^3 - 2$

The total outward flux is obtained by adding the terms for each of the six sides, giving

$$18 + 18 + 2e^3 - 2 = 34 + 2e^3.$$

(b) We first compute the divergence of $\mathbf{F}$, obtaining

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = 2x + 4xz + e^x.$$

The integral of the divergence of $\mathbf{F}$ over D is

$$\iiint_D \text{div}\,\mathbf{F}\,dV = \int_0^1\int_0^2\int_0^3 (2x + 4xz + e^x)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^2 (8 + 18z + e^3)\,dy\,dz$$

$$= \int_0^1 (16 + 36z + 2e^3)\,dz$$

$$= 34 + 2e^3.$$

As asserted by the Divergence Theorem, the integral of the divergence over D equals the outward flux across the boundary surface of D. ■

Divergence and the Curl

If $\mathbf{F}$ is a vector field on three-dimensional space, then the curl $\nabla \times \mathbf{F}$ is also a vector field on three-dimensional space. So we can calculate the divergence of $\nabla \times \mathbf{F}$ using Equation (1). The result of this calculation is always 0.

THEOREM 9 If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field with continuous second partial derivatives, then

$$\text{div}\,(\text{curl}\,\mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Proof From the definitions of the divergence and curl, we have

$$\begin{aligned}
\text{div}\,(\text{curl}\,\mathbf{F}) &= \nabla \cdot (\nabla \times \mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= \frac{\partial^2 P}{\partial x \partial y} - \frac{\partial^2 N}{\partial x \partial z} + \frac{\partial^2 M}{\partial y \partial z} - \frac{\partial^2 P}{\partial y \partial x} + \frac{\partial^2 N}{\partial z \partial x} - \frac{\partial^2 M}{\partial z \partial y} \\
&= 0,
\end{aligned}$$

because the mixed second partial derivatives cancel by the Mixed Derivative Theorem in Section 14.3. ■

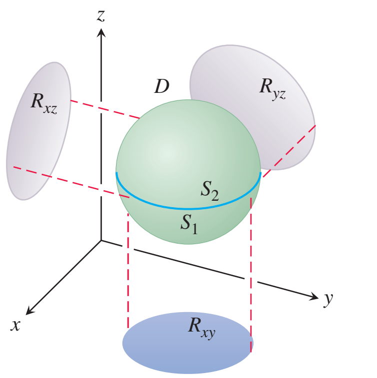


FIGURE 16.72 We prove the Divergence Theorem for the kind of three-dimensional region shown here.

Theorem 9 has some interesting applications. If a vector field $\mathbf{G} = \text{curl}\,\mathbf{F}$, then the field $\mathbf{G}$ must have divergence 0. Saying this another way, if div $\mathbf{G} \neq 0$, then $\mathbf{G}$ cannot be the curl of any vector field $\mathbf{F}$ having continuous second partial derivatives. Moreover, if $\mathbf{G} = \text{curl}\,\mathbf{F}$, then the outward flux of $\mathbf{G}$ across any closed surface S is zero by the corollary to the Divergence Theorem, provided the conditions of the theorem are satisfied. So if there is a closed surface for which the surface integral of the vector field $\mathbf{G}$ is nonzero, we can conclude that $\mathbf{G}$ is *not* the curl of some vector field $\mathbf{F}$.

Proof of the Divergence Theorem for Special Regions

To prove the Divergence Theorem, we take the components of $\mathbf{F}$ to have continuous first partial derivatives. We first assume that D is a convex region with no holes or bubbles, such as a solid ball, cube, or ellipsoid, and that S is a piecewise smooth surface. In addition, we assume that any line perpendicular to the xy-plane at an interior point of the region R_{xy} that is the projection of D on the xy-plane intersects the surface S in exactly two points, producing surfaces

$$S_1: \quad z = f_1(x, y), \qquad (x, y) \text{ in } R_{xy}$$
$$S_2: \quad z = f_2(x, y), \qquad (x, y) \text{ in } R_{xy},$$

with $f_1 \leq f_2$. We make similar assumptions about the projection of D onto the other coordinate planes. See Figure 16.72, which illustrates these assumptions.

The components of the unit normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$ are the cosines of the angles α, β, and γ that $\mathbf{n}$ makes with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ (Figure 16.73). This is true because all the vectors involved are unit vectors, giving the *direction cosines*

$$\begin{aligned}
n_1 &= \mathbf{n} \cdot \mathbf{i} = |\mathbf{n}||\mathbf{i}| \cos \alpha = \cos \alpha \\
n_2 &= \mathbf{n} \cdot \mathbf{j} = |\mathbf{n}||\mathbf{j}| \cos \beta = \cos \beta \\
n_3 &= \mathbf{n} \cdot \mathbf{k} = |\mathbf{n}||\mathbf{k}| \cos \gamma = \cos \gamma.
\end{aligned}$$

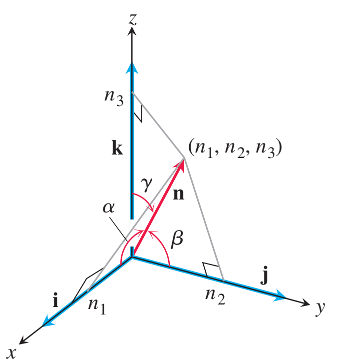


FIGURE 16.73 The components of $\mathbf{n}$ are the cosines of the angles α, β, and γ that it makes with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Thus, the unit normal vector is given by

$$\mathbf{n} = (\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}$$

and

$$\mathbf{F}\cdot\mathbf{n} = M\cos\alpha + N\cos\beta + P\cos\gamma.$$

In component form, the Divergence Theorem states that

$$\iint_S \underbrace{(M\cos\alpha + N\cos\beta + P\cos\gamma)}_{\mathbf{F}\cdot\mathbf{n}}\, d\sigma = \iiint_D \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right)}_{\operatorname{div}\mathbf{F}} dx\, dy\, dz.$$

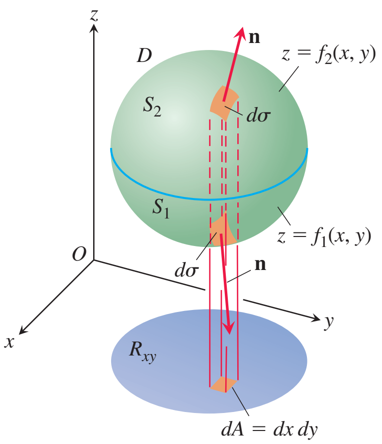


FIGURE 16.74 The region D enclosed by the surfaces S_1 and S_2 projects vertically onto R_{xy} in the xy-plane.

We prove the theorem by establishing the following three equations:

$$\iint_S M\cos\alpha\, d\sigma = \iiint_D \frac{\partial M}{\partial x}\, dx\, dy\, dz \tag{3}$$

$$\iint_S N\cos\beta\, d\sigma = \iiint_D \frac{\partial N}{\partial y}\, dx\, dy\, dz \tag{4}$$

$$\iint_S P\cos\gamma\, d\sigma = \iiint_D \frac{\partial P}{\partial z}\, dx\, dy\, dz \tag{5}$$

Proof of Equation (5) We prove Equation (5) by converting the surface integral on the left to a double integral over the projection R_{xy} of D on the xy-plane (Figure 16.74). The surface S consists of an upper part S_2 whose equation is $z = f_2(x, y)$ and a lower part S_1 whose equation is $z = f_1(x, y)$. On S_2, the outer normal $\mathbf{n}$ has a positive $\mathbf{k}$-component and

$$\cos\gamma\, d\sigma = dx\, dy \qquad \text{because} \qquad d\sigma = \frac{dA}{|\cos\gamma|} = \frac{dx\, dy}{\cos\gamma}.$$

See Figure 16.75. On S_1, the outer normal $\mathbf{n}$ has a negative $\mathbf{k}$-component and

$$\cos\gamma\, d\sigma = -dx\, dy.$$

Therefore,

$$\begin{aligned}
\iint_S P\cos\gamma\, d\sigma &= \iint_{S_2} P\cos\gamma\, d\sigma + \iint_{S_1} P\cos\gamma\, d\sigma \\
&= \iint_{R_{xy}} P(x, y, f_2(x, y))\, dx\, dy - \iint_{R_{xy}} P(x, y, f_1(x, y))\, dx\, dy \\
&= \iint_{R_{xy}} \left[P(x, y, f_2(x, y)) - P(x, y, f_1(x, y)) \right] dx\, dy \\
&= \iint_{R_{xy}} \left[\int_{f_1(x, y)}^{f_2(x, y)} \frac{\partial P}{\partial z}\, dz \right] dx\, dy = \iiint_D \frac{\partial P}{\partial z}\, dz\, dx\, dy.
\end{aligned}$$

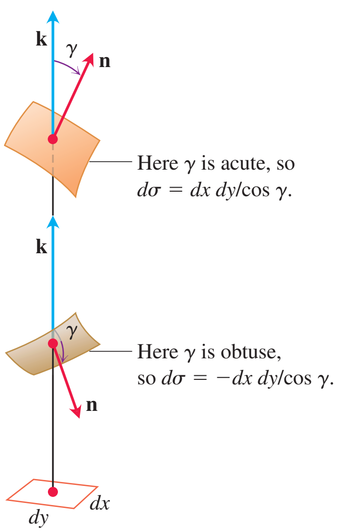


FIGURE 16.75 An enlarged view of the area patches in Figure 16.74. The relations $d\sigma = \pm dx\, dy/\cos\gamma$ come from Eq. (7) in Section 16.5 with $F = \mathbf{F}\cdot\mathbf{n}$.

This proves Equation (5). The proofs for Equations (3) and (4) follow the same pattern; or just permute x, y, z; M, N, P; α, β, γ, in order, and get those results from Equation (5). This proves the Divergence Theorem for these special regions. ■

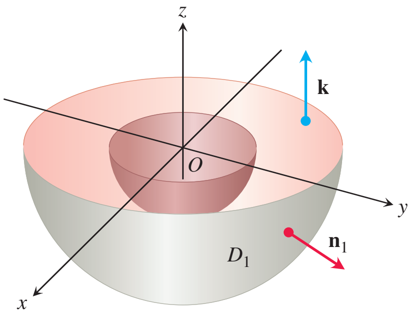


FIGURE 16.76 The lower half of the solid region between two concentric spheres.

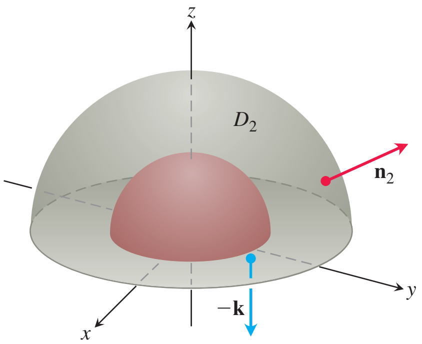


FIGURE 16.77 The upper half of the solid region between two concentric spheres.

Divergence Theorem for Other Regions

The Divergence Theorem can be extended to regions that can be partitioned into a finite number of simple regions of the type just discussed and to regions that can be defined as limits of simpler regions in certain ways. For an example of one step in such a splitting process, suppose that D is the region between two concentric spheres and that $\mathbf{F}$ has continuously differentiable components throughout D and on the bounding surfaces. Split D by an equatorial plane and apply the Divergence Theorem to each half separately. The bottom half, D_1, is shown in Figure 16.76. The surface S_1 that bounds D_1 consists of an outer hemisphere, a plane washer-shaped base, and an inner hemisphere. The Divergence Theorem says that

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n}_1 \, d\sigma_1 = \iiint_{D_1} \nabla \cdot \mathbf{F} \, dV_1. \tag{6}$$

The unit normal $\mathbf{n}_1$ that points outward from D_1 points away from the origin along the outer surface, equals $\mathbf{k}$ along the flat base, and points toward the origin along the inner surface. Next apply the Divergence Theorem to D_2, and its surface S_2 (Figure 16.77):

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, d\sigma_2 = \iiint_{D_2} \nabla \cdot \mathbf{F} \, dV_2. \tag{7}$$

As we follow $\mathbf{n}_2$ over S_2, pointing outward from D_2, we see that $\mathbf{n}_2$ equals $-\mathbf{k}$ along the washer-shaped base in the xy-plane, points away from the origin on the outer sphere, and points toward the origin on the inner sphere. When we add Equations (6) and (7), the integrals over the flat base cancel because of the opposite signs of $\mathbf{n}_1$ and $\mathbf{n}_2$. We thus arrive at the result

$$\iint_{S} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_{D} \nabla \cdot \mathbf{F} \, dV,$$

with D the region between the spheres, S the boundary of D consisting of two spheres, and $\mathbf{n}$ the unit normal to S directed outward from D.

EXAMPLE 5 Find the net outward flux of the field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2} \tag{8}$$

across the boundary of the region D: $0 < b^2 \le x^2 + y^2 + z^2 \le a^2$ (Figure 16.78).

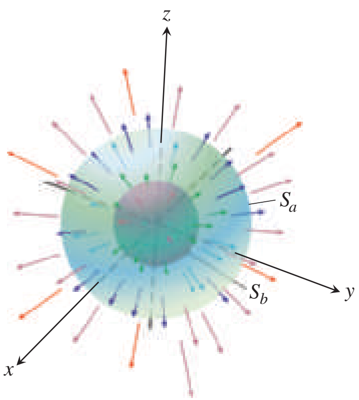


FIGURE 16.78 Two concentric spheres in an expanding vector field. The outer sphere is S_a and surrounds the inner sphere S_b.

Solution The flux can be calculated by integrating $\nabla \cdot \mathbf{F}$ over D. Note that $\rho \neq 0$ in D. We have

$$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\rho}$$

and

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(x\rho^{-3}) = \rho^{-3} - 3x\rho^{-4}\frac{\partial \rho}{\partial x} = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}.$$

Similarly,

$$\frac{\partial N}{\partial y} = \frac{1}{\rho^3} - \frac{3y^2}{\rho^5} \qquad \text{and} \qquad \frac{\partial P}{\partial z} = \frac{1}{\rho^3} - \frac{3z^2}{\rho^5}.$$

Hence,

$$\text{div}\,\mathbf{F} = \frac{3}{\rho^3} - \frac{3}{\rho^5}(x^2 + y^2 + z^2) = \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0.$$

So the net outward flux of $\mathbf{F}$ across the boundary of D is zero by the corollary to the Divergence Theorem. There is more to learn about this vector field $\mathbf{F}$, though. The flux leaving D across the inner sphere S_b is the negative of the flux leaving D across the outer sphere S_a (because the sum of these fluxes is zero). Hence, the flux of $\mathbf{F}$ across S_b in the direction away from the origin equals the flux of $\mathbf{F}$ across S_a in the direction away from the origin. Thus, the flux of $\mathbf{F}$ across a sphere centered at the origin is independent of the radius of the sphere. What is this flux?

To find it, we evaluate the flux integral directly for an arbitrary sphere S_a. The outward unit normal on the sphere of radius a is

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

Hence, on the sphere,

$$\mathbf{F} \cdot \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a^3} \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} = \frac{x^2 + y^2 + z^2}{a^4} = \frac{a^2}{a^4} = \frac{1}{a^2}$$

and

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{1}{a^2} \iint_{S_a} d\sigma = \frac{1}{a^2}(4\pi a^2) = 4\pi.$$

The outward flux of $\mathbf{F}$ in Equation (8) across any sphere centered at the origin is 4π. This result does not contradict the Divergence Theorem because $\mathbf{F}$ is not continuous at the origin. ■

Gauss's Law: One of the Four Great Laws of Electromagnetic Theory

There is still more to be learned from Example 5. In electromagnetic theory, the electric field created by a point charge q located at the origin is

$$\mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{r}|^2}\left(\frac{\mathbf{r}}{|\mathbf{r}|}\right) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{|\mathbf{r}|^3} = \frac{q}{4\pi\epsilon_0}\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3},$$

where ϵ_0 is a physical constant, $\mathbf{r}$ is the position vector of the point (x, y, z), and $\rho = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. From Equation (8),

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\mathbf{F}.$$

The calculations in Example 5 show that the outward flux of $\mathbf{E}$ across any sphere centered at the origin is q/ϵ_0, but this result is not confined to spheres. The outward flux of $\mathbf{E}$ across any closed surface S that encloses the origin (and to which the Divergence Theorem applies) is also q/ϵ_0. To see why, we have only to imagine a large sphere S_a centered at the origin and enclosing the surface S (see Figure 16.79). Since

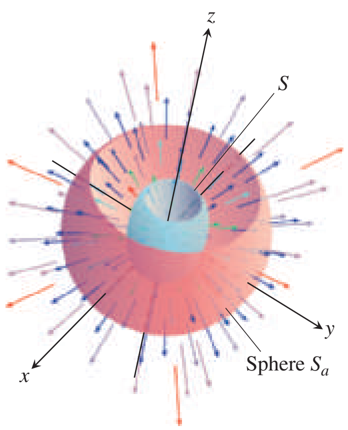


FIGURE 16.79 A sphere S_a surrounding another surface S. The tops of the surfaces are removed for visualization.

$$\nabla \cdot \mathbf{E} = \nabla \cdot \frac{q}{4\pi\epsilon_0}\mathbf{F} = \frac{q}{4\pi\epsilon_0}\nabla \cdot \mathbf{F} = 0$$

when $\rho > 0$, the triple integral of $\nabla \cdot \mathbf{E}$ over the region D between S and S_a is zero. Hence, by the Divergence Theorem,

$$\iint_{\substack{\text{Boundary}\\ \text{of } D}} \mathbf{E} \cdot \mathbf{n}\, d\sigma = 0.$$

So the flux of $\mathbf{E}$ across S in the direction away from the origin must be the same as the flux of $\mathbf{E}$ across S_a in the direction away from the origin, which is q/ϵ_0. This statement, called *Gauss's law*, also applies to charge distributions that are more general than the one assumed here, as shown in nearly any physics text. For any closed surface that encloses the origin, we have

$$\text{Gauss's law:} \quad \iint_S \mathbf{E} \cdot \mathbf{n}\, d\sigma = \frac{q}{\epsilon_0}.$$

Continuity Equation of Hydrodynamics

Let D be a region in space bounded by a closed oriented surface S. If $\mathbf{v}(x, y, z)$ is the velocity field of a fluid flowing smoothly through D, $\delta = \delta(t, x, y, z)$ is the fluid's density at (x, y, z) at time t, and $\mathbf{F} = \delta\mathbf{v}$, then the **continuity equation** of hydrodynamics states that

$$\nabla \cdot \mathbf{F} + \frac{\partial \delta}{\partial t} = 0.$$

If the functions involved have continuous first partial derivatives, the equation evolves naturally from the Divergence Theorem, as we now demonstrate.

First, the integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$$

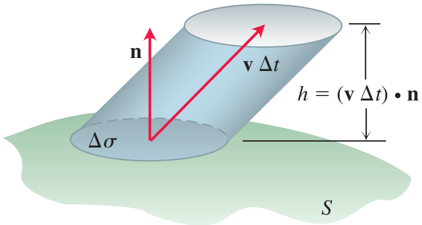


FIGURE 16.80 The fluid that flows upward through the patch $\Delta\sigma$ in a short time Δt fills a "cylinder" whose volume is approximately base $\times$ height $=$ $\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t$.

is the rate at which mass leaves D across S (leaves because $\mathbf{n}$ is the outer normal). To see why, consider a patch of area $\Delta\sigma$ on the surface (Figure 16.80). In a short time interval Δt, the volume ΔV of fluid that flows across the patch is approximately equal to the volume of a cylinder with base area $\Delta\sigma$ and height $(\mathbf{v}\Delta t) \cdot \mathbf{n}$, where $\mathbf{v}$ is a velocity vector rooted at a point of the patch:

$$\Delta V \approx \mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t.$$

The mass of this volume of fluid is about

$$\Delta m \approx \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma\, \Delta t,$$

so the rate at which mass is flowing out of D across the patch is about

$$\frac{\Delta m}{\Delta t} \approx \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma.$$

This leads to the approximation

$$\frac{\sum \Delta m}{\Delta t} \approx \sum \delta\mathbf{v} \cdot \mathbf{n}\, \Delta\sigma$$

as an estimate of the average rate at which mass flows across S. Finally, letting $\Delta\sigma \to 0$ and $\Delta t \to 0$ gives the instantaneous rate at which mass leaves D across S as

$$\frac{dm}{dt} = \iint_S \delta\mathbf{v} \cdot \mathbf{n}\, d\sigma,$$

which for our particular flow is

$$\frac{dm}{dt} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Now let B be a solid sphere centered at a point Q in the flow. The average value of $\nabla \cdot \mathbf{F}$ over B is

$$\frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F}\, dV.$$

It is a consequence of the continuity of the divergence that $\nabla \cdot \mathbf{F}$ actually takes on this value at some point P in B. Thus, by the Divergence Theorem Equation (2),

$$\begin{aligned}(\nabla \cdot \mathbf{F})_P &= \frac{1}{\text{volume of } B} \iiint_B \nabla \cdot \mathbf{F}\, dV = \frac{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}{\text{volume of } B} \\ &= \frac{\text{rate at which mass leaves } B \text{ across its surface } S}{\text{volume of } B}.\end{aligned} \tag{9}$$

The last term of the equation describes decrease in mass per unit volume.

Now let the radius of B approach zero while the center Q stays fixed. The left side of Equation (9) converges to $(\nabla \cdot \mathbf{F})_Q$, and the right side converges to $(-\partial\delta/\partial t)_Q$, since $\delta = m/V$. The equality of these two limits is the continuity equation

$$\nabla \cdot \mathbf{F} = -\frac{\partial\delta}{\partial t}.$$

The continuity equation "explains" $\nabla \cdot \mathbf{F}$: The divergence of $\mathbf{F}$ at a point is the rate at which the density of the fluid is decreasing there. The Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

now says that the net decrease in density of the fluid in region D (divergence integral) is accounted for by the mass transported across the surface S (outward flux integral). So, the theorem is a statement about conservation of mass (Exercise 31).

Unifying the Integral Theorems

If we think of a two-dimensional field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ as a three-dimensional field whose $\mathbf{k}$-component is zero, then $\nabla \cdot \mathbf{F} = (\partial M/\partial x) + (\partial N/\partial y)$ and the normal form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy = \iint_R \nabla \cdot \mathbf{F}\, dA.$$

Similarly, $\nabla \times \mathbf{F} \cdot \mathbf{k} = (\partial N/\partial x) - (\partial M/\partial y)$, so the tangential form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA.$$

With the equations of Green's Theorem now in del notation, we can see their relationships to the equations in Stokes' Theorem and the Divergence Theorem, all summarized here.

Green's Theorem and Its Generalization to Three Dimensions

Tangential form of Green's Theorem: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$

Stokes' Theorem: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

Normal form of Green's Theorem: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$

Divergence Theorem: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$

Notice how Stokes' Theorem generalizes the tangential (curl) form of Green's Theorem from a flat surface in the plane to a surface in three-dimensional space. In each case, the surface integral of curl $\mathbf{F}$ over the interior of the oriented surface equals the circulation of $\mathbf{F}$ around the boundary.

Likewise, the Divergence Theorem generalizes the normal (flux) form of Green's Theorem from a two-dimensional region in the plane to a three-dimensional region in space. In each case, the integral of $\nabla \cdot \mathbf{F}$ over the interior of the region equals the total flux of the field across the boundary enclosing the region.

FIGURE 16.81 The outward unit normals at the boundary of $[a, b]$ in one-dimensional space.

There is still more to be learned here. All these results can be thought of as forms of a *single fundamental theorem*. Think back to the Fundamental Theorem of Calculus in Section 5.4. It says that if $f(x)$ is differentiable on (a, b) and continuous on $[a, b]$, then

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a).$$

If we let $\mathbf{F} = f(x)\mathbf{i}$ throughout $[a, b]$, then $(df/dx) = \nabla \cdot \mathbf{F}$. If we define the unit vector field $\mathbf{n}$ normal to the boundary of $[a, b]$ to be $\mathbf{i}$ at b and $-\mathbf{i}$ at a (Figure 16.81), then

$$\begin{aligned} f(b) - f(a) &= f(b)\mathbf{i} \cdot (\mathbf{i}) + f(a)\mathbf{i} \cdot (-\mathbf{i}) \\ &= \mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} \\ &= \text{total outward flux of } \mathbf{F} \text{ across the boundary of } [a, b]. \end{aligned}$$

The Fundamental Theorem now says that

$$\mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} = \int_{[a,\,b]} \nabla \cdot \mathbf{F}\,dx.$$

The Fundamental Theorem of Calculus, the normal form of Green's Theorem, and the Divergence Theorem all say that the integral of the differential operator $\nabla \cdot$ operating on a field $\mathbf{F}$ over a region equals the sum of the normal field components over the boundary enclosing the region. (Here we are interpreting the line integral in Green's Theorem and the surface integral in the Divergence Theorem as "sums" over the boundary.)

Stokes' Theorem and the tangential form of Green's Theorem say that, when things are properly oriented, the surface integral of the differential operator $\nabla \times$ operating on a field equals the sum of the tangential field components over the boundary of the surface.

The beauty of these interpretations is the observance of a single unifying principle, which we might state as follows.

> **A Unifying Fundamental Theorem of Vector Integral Calculus**
>
> The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

Exercises 16.8

Calculating Divergence

In Exercises 1–4, find the divergence of the field.

1. The spin field in Figure 16.12
2. The radial field in Figure 16.11
3. The gravitational field in Figure 16.8 and Exercise 38a in Section 16.3
4. The velocity field in Figure 16.13

Calculating Flux Using the Divergence Theorem

In Exercises 5–16, use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

5. **Cube** $\mathbf{F} = (y - x)\mathbf{i} + (z - y)\mathbf{j} + (y - x)\mathbf{k}$

   D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

6. $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

   a. Cube D: The cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

   b. Cube D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

   c. Cylindrical can D: The region cut from the solid cylinder $x^2 + y^2 \le 4$ by the planes $z = 0$ and $z = 1$

7. **Cylinder and paraboloid** $\mathbf{F} = y\mathbf{i} + xy\mathbf{j} - z\mathbf{k}$

   D: The region inside the solid cylinder $x^2 + y^2 \le 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$

8. **Sphere** $\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$

   D: The solid sphere $x^2 + y^2 + z^2 \le 4$

9. **Portion of sphere** $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$

   D: The region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$

10. **Cylindrical can** $\mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2z)\mathbf{j} + 4x^2y^3\mathbf{k}$

    D: The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$

11. Wedge $\mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z^2\mathbf{k}$

D: The wedge cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$

12. Sphere $\mathbf{F} = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$

D: The solid sphere $x^2 + y^2 + z^2 \le a^2$

13. Thick sphere $\mathbf{F} = \sqrt{x^2 + y^2 + z^2}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

D: The region $1 \le x^2 + y^2 + z^2 \le 2$

14. Thick sphere $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$

D: The region $1 \le x^2 + y^2 + z^2 \le 4$

15. Thick sphere $\mathbf{F} = (5x^3 + 12xy^2)\mathbf{i} + (y^3 + e^y \sin z)\mathbf{j} + (5z^3 + e^y \cos z)\mathbf{k}$

D: The solid region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$

16. Thick cylinder $\mathbf{F} = \ln(x^2 + y^2)\mathbf{i} - \left(\frac{2z}{x}\tan^{-1}\frac{y}{x}\right)\mathbf{j} + z\sqrt{x^2 + y^2}\mathbf{k}$

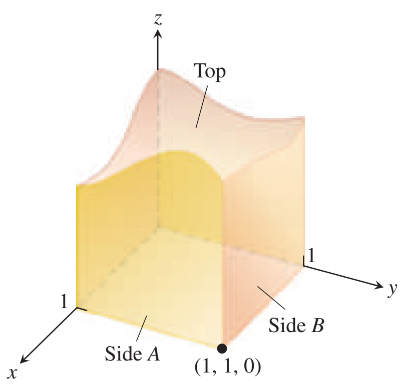
D: The thick-walled cylinder $1 \le x^2 + y^2 \le 2$, $-1 \le z \le 2$

Theory and Examples

17. a. Show that the outward flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through a smooth closed surface S is three times the volume of the region enclosed by the surface.

b. Let $\mathbf{n}$ be the outward unit normal vector field on S. Show that it is not possible for $\mathbf{F}$ to be orthogonal to $\mathbf{n}$ at every point of S.

18. The base of the closed cubelike surface shown here is the unit square in the xy-plane. The four sides lie in the planes $x = 0$, $x = 1$, $y = 0$, and $y = 1$. The top is an arbitrary smooth surface whose identity is unknown. Let $\mathbf{F} = x\mathbf{i} - 2y\mathbf{j} + (z + 3)\mathbf{k}$ and suppose the outward flux of $\mathbf{F}$ through Side A is 1 and through Side B is -3. Can you conclude anything about the outward flux through the top? Give reasons for your answer.



19. Let $\mathbf{F} = (y \cos 2x)\mathbf{i} + (y^2 \sin 2x)\mathbf{j} + (x^2y + z)\mathbf{k}$. Is there a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$? Explain your answer.

20. Outward flux of a gradient field Let S be the surface of the portion of the solid sphere $x^2 + y^2 + z^2 \le a^2$ that lies in the first octant and let $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$. Calculate

$$\iint_S \nabla f \cdot \mathbf{n}\, d\sigma.$$

($\nabla f \cdot \mathbf{n}$ is the derivative of f in the direction of outward normal $\mathbf{n}$.)

21. Let $\mathbf{F}$ be a field whose components have continuous first partial derivatives throughout a portion of space containing a region D bounded by a smooth closed surface S. If $|\mathbf{F}| \le 1$, can any bound be placed on the size of

$$\iiint_D \nabla \cdot \mathbf{F}\, dV?$$

Give reasons for your answer.

22. Maximum flux Among all rectangular solids defined by the inequalities $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le 1$, find the one for which the total flux of $\mathbf{F} = (-x^2 - 4xy)\mathbf{i} - 6yz\mathbf{j} + 12z\mathbf{k}$ outward through the six sides is greatest. What *is* the greatest flux?

23. Calculate the net outward flux of the vector field

$$\mathbf{F} = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2} + x)\mathbf{k}$$

over the surface S surrounding the region D bounded by the planes $y = 0$, $z = 0$, $z = 2 - y$ and the parabolic cylinder $z = 1 - x^2$.

24. Compute the net outward flux of the vector field $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$ across the ellipsoid $9x^2 + 4y^2 + 6z^2 = 36$.

25. Let $\mathbf{F}$ be a differentiable vector field and let $g(x, y, z)$ be a differentiable scalar function. Verify the following identities.

a. $\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$

b. $\nabla \times (g\mathbf{F}) = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$

26. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be differentiable vector fields and let a and b be arbitrary real constants. Verify the following identities.

a. $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$

b. $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

c. $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

27. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a differentiable vector field, we define the notation $\mathbf{F} \cdot \nabla$ to mean

$$M\frac{\partial}{\partial x} + N\frac{\partial}{\partial y} + P\frac{\partial}{\partial z}.$$

For differentiable vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$, verify the following identities.

a. $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$

b. $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$

28. Harmonic functions A function $f(x, y, z)$ is said to be *harmonic* in a region D in space if it satisfies the Laplace equation

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D.

a. Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that $\mathbf{n}$ is the chosen unit normal vector on S. Show that the integral over S of $\nabla f \cdot \mathbf{n}$, the derivative of f in the direction of $\mathbf{n}$, is zero.

b. Show that if f is harmonic on D, then

$$\iint_S f\,\nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV.$$

29. Green's first formula Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise smooth surface S. Show that

$$\iint_S f\nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D (f\nabla^2 g + \nabla f \cdot \nabla g)\, dV. \tag{10}$$

Equation (10) is **Green's first formula**. (*Hint:* Apply the Divergence Theorem to the field $\mathbf{F} = f\nabla g$.)

30. Green's second formula (*Continuation of Exercise 29.*) Interchange f and g in Equation (10) to obtain a similar formula. Then subtract this formula from Equation (10) to show that

$$\iint_S (f\nabla g - g\nabla f)\cdot \mathbf{n}\, d\sigma = \iiint_D (f\nabla^2 g - g\nabla^2 f)\, dV. \tag{11}$$

This equation is **Green's second formula**.

31. Conservation of mass Let $\mathbf{v}(t, x, y, z)$ be a continuously differentiable vector field over the region D in space and let $p(t, x, y, z)$ be a continuously differentiable scalar function. The variable t represents the time domain. The Law of Conservation of Mass asserts that

$$\frac{d}{dt}\iiint_D p(t, x, y, z)\, dV = -\iint_S p\mathbf{v}\cdot\mathbf{n}\, d\sigma,$$

where S is the surface enclosing D.

a. Give a physical interpretation of the conservation of mass law if $\mathbf{v}$ is a velocity flow field and p represents the density of the fluid at point (x, y, z) at time t.

b. Use the Divergence Theorem and Leibniz's Rule,

$$\frac{d}{dt}\iiint_D p(t, x, y, z)\, dV = \iiint_D \frac{\partial p}{\partial t}\, dV,$$

to show that the Law of Conservation of Mass is equivalent to the continuity equation,

$$\nabla \cdot p\mathbf{v} + \frac{\partial p}{\partial t} = 0.$$

(In the first term $\nabla \cdot p\mathbf{v}$, the variable t is held fixed, and in the second term $\partial p/\partial t$, it is assumed that the point (x, y, z) in D is held fixed.)

32. The heat diffusion equation Let $T(t, x, y, z)$ be a function with continuous second derivatives giving the temperature at time t at the point (x, y, z) of a solid occupying a region D in space. If the solid's heat capacity and mass density are denoted by the constants c and ρ, respectively, the quantity $c\rho T$ is called the solid's **heat energy per unit volume**.

a. Explain why $-\nabla T$ points in the direction of heat flow.

b. Let $-k\nabla T$ denote the **energy flux vector**. (Here the constant k is called the **conductivity**.) Assuming the Law of Conservation of Mass with $-k\nabla T = \mathbf{v}$ and $c\rho T = p$ in Exercise 31, derive the diffusion (heat) equation

$$\frac{\partial T}{\partial t} = K\nabla^2 T,$$

where $K = k/(c\rho) > 0$ is the *diffusivity* constant. (Notice that if $T(t, x)$ represents the temperature at time t at position x in a uniform conducting rod with perfectly insulated sides, then $\nabla^2 T = \partial^2 T/\partial x^2$ and the diffusion equation reduces to the one-dimensional heat equation in Chapter 14's Additional Exercises.)

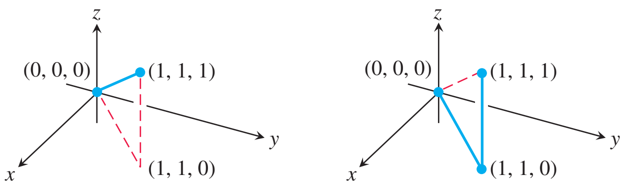
Chapter 16 Questions to Guide Your Review

1. What are line integrals of scalar functions? How are they evaluated? Give examples.
2. How can you use line integrals to find the centers of mass of springs or wires? Explain.
3. What is a vector field? What is the line integral of a vector field? What is a gradient field? Give examples.
4. What is the flow of a vector field along a curve? What is the work done by vector field moving an object along a curve? How do you calculate the work done? Give examples.
5. What is the Fundamental Theorem of line integrals? Explain how it relates to the Fundamental Theorem of Calculus.
6. Specify three properties that are special about conservative fields. How can you tell when a field is conservative?
7. What is special about path independent fields?
8. What is a potential function? Show by example how to find a potential function for a conservative field.
9. What is a differential form? What does it mean for such a form to be exact? How do you test for exactness? Give examples.
10. What is Green's Theorem? Discuss how the two forms of Green's Theorem extend the Net Change Theorem in Chapter 5.
11. How do you calculate the area of a parametrized surface in space? Of an implicitly defined surface $F(x, y, z) = 0$? Of the surface which is the graph of $z = f(x, y)$? Give examples.
12. How do you integrate a scalar function over a parametrized surface? Of surfaces that are defined implicitly or in explicit form? Give examples.
13. What is an oriented surface? What is the surface integral of a vector field in three-dimensional space over an oriented surface? How is it related to the net outward flux of the field? Give examples.
14. What is the curl of a vector field? How can you interpret it?
15. What is Stokes' Theorem? Explain how it generalizes Green's Theorem to three dimensions.
16. What is the divergence of a vector field? How can you interpret it?
17. What is the Divergence Theorem? Explain how it generalizes Green's Theorem to three dimensions.
18. How do Green's Theorem, Stokes' Theorem, and the Divergence Theorem relate to the Fundamental Theorem of Calculus for ordinary single integrals?

Chapter 16 Practice Exercises

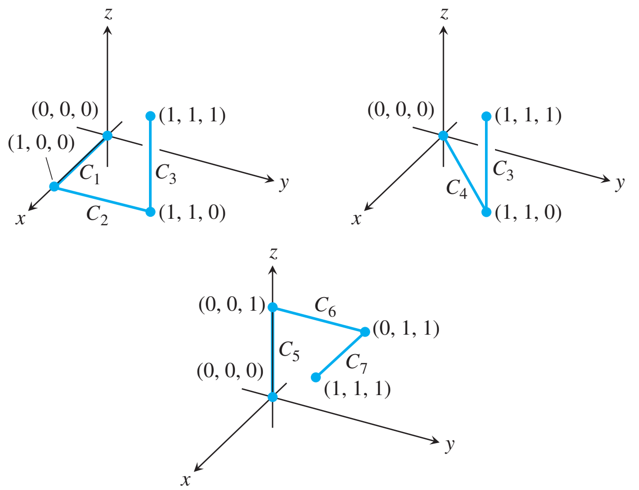
Evaluating Line Integrals

1. The accompanying figure shows two polygonal paths in space joining the origin to the point $(1, 1, 1)$. Integrate $f(x, y, z) = 2x - 3y^2 - 2z + 3$ over each path.



Path 1 Path 2

2. The accompanying figure shows three polygonal paths joining the origin to the point $(1, 1, 1)$. Integrate $f(x, y, z) = x^2 + y - z$ over each path.



3. Integrate $f(x, y, z) = \sqrt{x^2 + z^2}$ over the circle

$$\mathbf{r}(t) = (a \cos t)\mathbf{j} + (a \sin t)\mathbf{k}, \qquad 0 \le t \le 2\pi.$$

4. Integrate $f(x, y, z) = \sqrt{x^2 + y^2}$ over the involute curve

$$\mathbf{r}(t) = (\cos t + t \sin t)\mathbf{i} + (\sin t - t \cos t)\mathbf{j}, \qquad 0 \le t \le \sqrt{3}.$$

Evaluate the integrals in Exercises 5 and 6.

5. $\displaystyle\int_{(-1,1,1)}^{(4,-3,0)} \frac{dx + dy + dz}{\sqrt{x + y + z}}$ **6.** $\displaystyle\int_{(1,1,1)}^{(10,3,3)} dx - \sqrt{\frac{z}{y}}\,dy - \sqrt{\frac{y}{z}}\,dz$

7. Integrate $\mathbf{F} = -(y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$ around the circle cut from the sphere $x^2 + y^2 + z^2 = 5$ by the plane $z = -1$, clockwise as viewed from above.

8. Integrate $\mathbf{F} = 3x^2y\mathbf{i} + (x^3 + 1)\mathbf{j} + 9z^2\mathbf{k}$ around the circle cut from the sphere $x^2 + y^2 + z^2 = 9$ by the plane $x = 2$.

Evaluate the integrals in Exercises 9 and 10.

9. $\displaystyle\int_C 8x \sin y\, dx - 8y \cos x\, dy$

C is the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

10. $\displaystyle\int_C y^2\, dx + x^2\, dy$

C is the circle $x^2 + y^2 = 4$.

Finding and Evaluating Surface Integrals

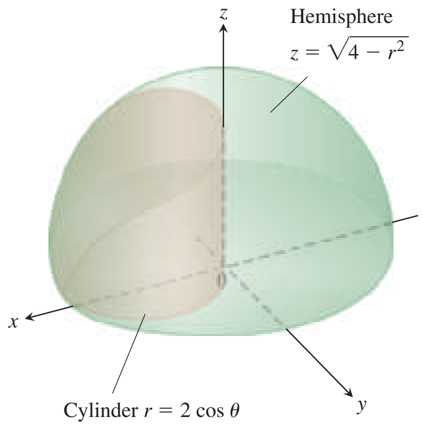
11. Area of an elliptical region Find the area of the elliptical region cut from the plane $x + y + z = 1$ by the cylinder $x^2 + y^2 = 1$.

12. Area of a parabolic cap Find the area of the cap cut from the paraboloid $y^2 + z^2 = 3x$ by the plane $x = 1$.

13. Area of a spherical cap Find the area of the cap cut from the top of the sphere $x^2 + y^2 + z^2 = 1$ by the plane $z = \sqrt{2}/2$.

14. a. Hemisphere cut by cylinder Find the area of the surface cut from the hemisphere $x^2 + y^2 + z^2 = 4$, $z \ge 0$, by the cylinder $x^2 + y^2 = 2x$.

b. Find the area of the portion of the cylinder that lies inside the hemisphere. (*Hint:* Project onto the xz-plane. Or evaluate the integral $\int h\, ds$, where h is the altitude of the cylinder and ds is the element of arc length on the circle $x^2 + y^2 = 2x$ in the xy-plane.)



15. Area of a triangle Find the area of the triangle in which the plane $(x/a) + (y/b) + (z/c) = 1$ $(a, b, c > 0)$ intersects the first octant. Check your answer with an appropriate vector calculation.

16. Parabolic cylinder cut by planes Integrate

a. $g(x, y, z) = \dfrac{yz}{\sqrt{4y^2 + 1}}$ **b.** $g(x, y, z) = \dfrac{z}{\sqrt{4y^2 + 1}}$

over the surface cut from the parabolic cylinder $y^2 - z = 1$ by the planes $x = 0$, $x = 3$, and $z = 0$.

17. Circular cylinder cut by planes Integrate $g(x, y, z) = x^4y(y^2 + z^2)$ over the portion of the cylinder $y^2 + z^2 = 25$ that lies in the first octant between the planes $x = 0$ and $x = 1$ and above the plane $z = 3$.

18. Area of Wyoming The state of Wyoming is bounded by the meridians 111°3′ and 104°3′ west longitude and by the circles 41° and 45° north latitude. Assuming that Earth is a sphere of radius $R = 3959$ mi, find the area of Wyoming.

Parametrized Surfaces

Find parametrizations for the surfaces in Exercises 19–24. (There are many ways to do these, so your answers may not be the same as those in the back of the book.)

19. Spherical band The portion of the sphere $x^2 + y^2 + z^2 = 36$ between the planes $z = -3$ and $z = 3\sqrt{3}$

20. Parabolic cap The portion of the paraboloid $z = -(x^2 + y^2)/2$ above the plane $z = -2$

21. Cone The cone $z = 1 + \sqrt{x^2 + y^2}$, $z \le 3$

22. Plane above square The portion of the plane $4x + 2y + 4z = 12$ that lies above the square $0 \le x \le 2$, $0 \le y \le 2$ in the first quadrant

23. Portion of paraboloid The portion of the paraboloid $y = 2(x^2 + z^2)$, $y \le 2$, that lies above the xy-plane

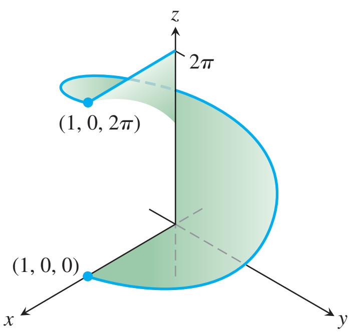
24. Portion of hemisphere The portion of the hemisphere $x^2 + y^2 + z^2 = 10$, $y \ge 0$, in the first octant

25. Surface area Find the area of the surface

$$\mathbf{r}(u, v) = (u + v)\mathbf{i} + (u - v)\mathbf{j} + v\mathbf{k},$$
$$0 \le u \le 1,\ 0 \le v \le 1.$$

26. Surface integral Integrate $f(x, y, z) = xy - z^2$ over the surface in Exercise 25.

27. Area of a helicoid Find the surface area of the helicoid $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \theta\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$, in the accompanying figure.



28. Surface integral Evaluate the integral $\iint_S \sqrt{x^2 + y^2 + 1}\, d\sigma$, where S is the helicoid in Exercise 27.

Conservative Fields

Which of the fields in Exercises 29–32 are conservative, and which are not?

29. $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

30. $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/(x^2 + y^2 + z^2)^{3/2}$

31. $\mathbf{F} = xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$

32. $\mathbf{F} = (\mathbf{i} + z\mathbf{j} + y\mathbf{k})/(x + yz)$

Find potential functions for the fields in Exercises 33 and 34.

33. $\mathbf{F} = 2\mathbf{i} + (2y + z)\mathbf{j} + (y + 1)\mathbf{k}$

34. $\mathbf{F} = (z\cos xz)\mathbf{i} + e^y\mathbf{j} + (x\cos xz)\mathbf{k}$

Work and Circulation

In Exercises 35 and 36, find the work done by each field along the paths from $(0, 0, 0)$ to $(1, 1, 1)$ in Exercise 1.

35. $\mathbf{F} = 2xy\mathbf{i} + \mathbf{j} + x^2\mathbf{k}$ **36.** $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j} + \mathbf{k}$

37. Finding work in two ways Find the work done by

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{(x^2 + y^2)^{3/2}}$$

over the plane curve $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j}$ from the point $(1, 0)$ to the point $(e^{2\pi}, 0)$ in two ways:

a. By using the parametrization of the curve to evaluate the work integral.

b. By evaluating a potential function for $\mathbf{F}$.

38. Flow along different paths Find the flow of the field $\mathbf{F} = \nabla(x^2ze^y)$

a. once around the ellipse C in which the plane $x + y + z = 1$ intersects the cylinder $x^2 + z^2 = 25$, clockwise as viewed from the positive y-axis.

b. along the curved boundary of the helicoid in Exercise 27 from $(1, 0, 0)$ to $(1, 0, 2\pi)$.

In Exercises 39 and 40, use the curl integral in Stokes' Theorem to find the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

39. Circulation around an ellipse $\mathbf{F} = y^2\mathbf{i} - y\mathbf{j} + 3z^2\mathbf{k}$

C: The ellipse in which the plane $2x + 6y - 3z = 6$ meets the cylinder $x^2 + y^2 = 1$, counterclockwise as viewed from above

40. Circulation around a circle $\mathbf{F} = (x^2 + y)\mathbf{i} + (x + y)\mathbf{j} + (4y^2 - z)\mathbf{k}$

C: The circle in which the plane $z = -y$ meets the sphere $x^2 + y^2 + z^2 = 4$, counterclockwise as viewed from above

Masses and Moments

41. Wire with different densities Find the mass of a thin wire lying along the curve $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + \sqrt{2}t\mathbf{j} + (4 - t^2)\mathbf{k}$, $0 \le t \le 1$, if the density at t is **(a)** $\delta = 3t$ and **(b)** $\delta = 1$.

42. Wire with variable density Find the center of mass of a thin wire lying along the curve $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + (2/3)t^{3/2}\mathbf{k}$, $0 \le t \le 2$, if the density at t is $\delta = 3\sqrt{5 + t}$.

43. Wire with variable density Find the center of mass and the moments of inertia about the coordinate axes of a thin wire lying along the curve

$$\mathbf{r}(t) = t\mathbf{i} + \frac{2\sqrt{2}}{3}t^{3/2}\mathbf{j} + \frac{t^2}{2}\mathbf{k}, \qquad 0 \le t \le 2,$$

if the density at t is $\delta = 1/(t + 1)$.

44. Center of mass of an arch A slender metal arch lies along the semicircle $y = \sqrt{a^2 - x^2}$ in the xy-plane. The density at the point (x, y) on the arch is $\delta(x, y) = 2a - y$. Find the center of mass.

45. Wire with constant density A wire of constant density $\delta = 1$ lies along the curve $\mathbf{r}(t) = (e^t\cos t)\mathbf{i} + (e^t\sin t)\mathbf{j} + e^t\mathbf{k}$, $0 \le t \le \ln 2$. Find $\bar{z}$ and I_z.

46. Helical wire with constant density Find the mass and center of mass of a wire of constant density δ that lies along the helix $\mathbf{r}(t) = (2\sin t)\mathbf{i} + (2\cos t)\mathbf{j} + 3t\mathbf{k}$, $0 \le t \le 2\pi$.

47. Inertia and center of mass of a shell Find I_z and the center of mass of a thin shell of density $\delta(x, y, z) = z$ cut from the upper portion of the sphere $x^2 + y^2 + z^2 = 25$ by the plane $z = 3$.

48. Moment of inertia of a cube Find the moment of inertia about the z-axis of the surface of the cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$ if the density is $\delta = 1$.

Flux Across a Plane Curve or Surface

Use Green's Theorem to find the counterclockwise circulation and outward flux for the fields and curves in Exercises 49 and 50.

49. Square $\mathbf{F} = (2xy + x)\mathbf{i} + (xy - y)\mathbf{j}$

C: The square bounded by $x = 0, x = 1, y = 0, y = 1$

50. Triangle $\mathbf{F} = (y - 6x^2)\mathbf{i} + (x + y^2)\mathbf{j}$

C: The triangle made by the lines $y = 0, y = x$, and $x = 1$

51. Zero line integral Show that

$$\oint_C \ln x \sin y \, dy - \frac{\cos y}{x} dx = 0$$

for any closed curve C to which Green's Theorem applies.

52. a. Outward flux and area Show that the outward flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ across any closed curve to which Green's Theorem applies is twice the area of the region enclosed by the curve.

b. Let $\mathbf{n}$ be the outward unit normal vector to a closed curve to which Green's Theorem applies. Show that it is not possible for $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ to be orthogonal to $\mathbf{n}$ at every point of C.

In Exercises 53–56, find the outward flux of $\mathbf{F}$ across the boundary of D.

53. Cube $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$

D: The cube cut from the first octant by the planes $x = 1, y = 1, z = 1$

54. Spherical cap $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$

D: The entire surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \le 25$ by the plane $z = 3$

55. Spherical cap $\mathbf{F} = -2x\mathbf{i} - 3y\mathbf{j} + z\mathbf{k}$

D: The upper region cut from the solid sphere $x^2 + y^2 + z^2 \le 2$ by the paraboloid $z = x^2 + y^2$

56. Cone and cylinder $\mathbf{F} = (6x + y)\mathbf{i} - (x + z)\mathbf{j} + 4yz\mathbf{k}$

D: The region in the first octant bounded by the cone $z = \sqrt{x^2 + y^2}$, the cylinder $x^2 + y^2 = 1$, and the coordinate planes

57. Hemisphere, cylinder, and plane Let S be the surface that is bounded on the left by the hemisphere $x^2 + y^2 + z^2 = a^2, y \le 0$, in the middle by the cylinder $x^2 + z^2 = a^2, 0 \le y \le a$, and on the right by the plane $y = a$. Find the flux of $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$ outward across S.

58. Cylinder and planes Find the outward flux of the field $\mathbf{F} = 3xz^2\mathbf{i} + y\mathbf{j} - z^3\mathbf{k}$ across the surface of the solid in the first octant that is bounded by the cylinder $x^2 + 4y^2 = 16$ and the planes $y = 2z, x = 0$, and $z = 0$.

59. Cylindrical can Use the Divergence Theorem to find the flux of $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ outward through the surface of the region enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 1$ and $z = -1$.

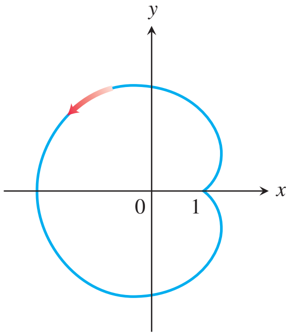
60. Hemisphere Find the flux of $\mathbf{F} = (3z + 1)\mathbf{k}$ upward across the hemisphere $x^2 + y^2 + z^2 = a^2, z \ge 0$ **(a)** with the Divergence Theorem and **(b)** by evaluating the flux integral directly.

Chapter 16 Additional and Advanced Exercises

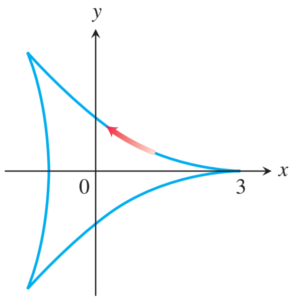
Finding Areas with Green's Theorem

Use the Green's Theorem area formula in Exercises 16.4 to find the areas of the regions enclosed by the curves in Exercises 1–4.

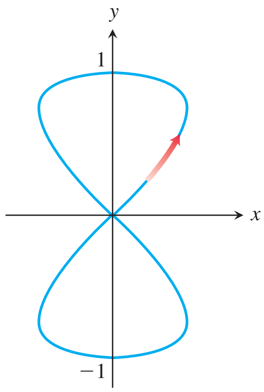
1. The limaçon $x = 2\cos t - \cos 2t$, $y = 2\sin t - \sin 2t$, $0 \le t \le 2\pi$



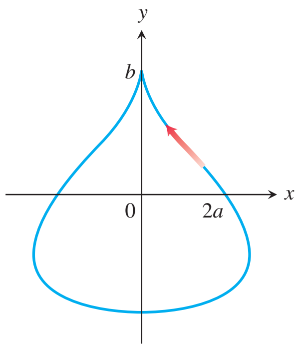
2. The deltoid $x = 2\cos t + \cos 2t$, $y = 2\sin t - \sin 2t$, $0 \le t \le 2\pi$



3. The eight curve $x = (1/2)\sin 2t$, $y = \sin t$, $0 \le t \le \pi$ (one loop)



4. The teardrop $x = 2a\cos t - a\sin 2t$, $y = b\sin t$, $0 \le t \le 2\pi$



Theory and Applications

5. **a.** Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ at only one point and such that curl $\mathbf{F}$ is nonzero everywhere. Be sure to identify the point and compute the curl.

b. Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ on precisely one line and such that curl $\mathbf{F}$ is nonzero everywhere. Be sure to identify the line and compute the curl.

c. Give an example of a vector field $\mathbf{F}(x, y, z)$ that has value $\mathbf{0}$ on a surface and such that curl $\mathbf{F}$ is nonzero everywhere. Be sure to identify the surface and compute the curl.

6. Find all points (a, b, c) on the sphere $x^2 + y^2 + z^2 = R^2$ where the vector field $\mathbf{F} = yz^2\mathbf{i} + xz^2\mathbf{j} + 2xyz\mathbf{k}$ is normal to the surface and $\mathbf{F}(a, b, c) \neq \mathbf{0}$.

7. Find the mass of a spherical shell of radius R such that at each point (x, y, z) on the surface the mass density $\delta(x, y, z)$ is its distance to some fixed point (a, b, c) of the surface.

8. Find the mass of a helicoid

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \theta\mathbf{k},$$

$0 \le r \le 1, 0 \le \theta \le 2\pi$, if the density function is $\delta(x, y, z) = 2\sqrt{x^2 + y^2}$. See Practice Exercise 27 for a figure.

9. Among all rectangular regions $0 \le x \le a, 0 \le y \le b$, find the one for which the total outward flux of $\mathbf{F} = (x^2 + 4xy)\mathbf{i} - 6y\mathbf{j}$ across the four sides is least. What *is* the least flux?

10. Find an equation for the plane through the origin such that the circulation of the flow field $\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ around the circle of intersection of the plane with the sphere $x^2 + y^2 + z^2 = 4$ is a maximum.

11. A string lies along the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(0, 2)$ in the first quadrant. The density of the string is $\rho(x, y) = xy$.

a. Partition the string into a finite number of subarcs to show that the work done by gravity to move the string straight down to the x-axis is given by

$$\text{Work} = \lim_{n\to\infty} \sum_{k=1}^{n} g\, x_k {y_k}^2 \Delta s_k = \int_C g\, xy^2\, ds,$$

where g is the gravitational constant.

b. Find the total work done by evaluating the line integral in part (a).

c. Show that the total work done equals the work required to move the string's center of mass $(\bar{x}, \bar{y})$ straight down to the x-axis.

12. A thin sheet lies along the portion of the plane $x + y + z = 1$ in the first octant. The density of the sheet is $\delta(x, y, z) = xy$.

a. Partition the sheet into a finite number of subpieces to show that the work done by gravity to move the sheet straight down to the xy-plane is given by

$$\text{Work} = \lim_{n\to\infty} \sum_{k=1}^{n} g\, x_k y_k z_k\, \Delta\sigma_k = \iint_S g\, xyz\, d\sigma,$$

where g is the gravitational constant.

b. Find the total work done by evaluating the surface integral in part (a).

c. Show that the total work done equals the work required to move the sheet's center of mass $(\bar{x}, \bar{y}, \bar{z})$ straight down to the xy-plane.

13. Archimedes' principle If an object such as a ball is placed in a liquid, it will either sink to the bottom, float, or sink a certain distance and remain suspended in the liquid. Suppose a fluid has constant weight density w and that the fluid's surface coincides with the plane $z = 4$. A spherical ball remains suspended in the fluid and occupies the region $x^2 + y^2 + (z - 2)^2 \le 1$.

a. Show that the surface integral giving the magnitude of the total force on the ball due to the fluid's pressure is

$$\text{Force} = \lim_{n\to\infty} \sum_{k=1}^{n} w(4 - z_k)\, \Delta\sigma_k = \iint_S w(4 - z)\, d\sigma.$$

b. Since the ball is not moving, it is being held up by the buoyant force of the liquid. Show that the magnitude of the buoyant force on the sphere is

$$\text{Buoyant force} = \iint_S w(z - 4)\mathbf{k}\cdot\mathbf{n}\, d\sigma,$$

where $\mathbf{n}$ is the outer unit normal at (x, y, z). This illustrates Archimedes' principle that the magnitude of the buoyant force on a submerged solid equals the weight of the displaced fluid.

c. Use the Divergence Theorem to find the magnitude of the buoyant force in part (b).

14. Fluid force on a curved surface A cone in the shape of the surface $z = \sqrt{x^2 + y^2}, 0 \le z \le 2$ is filled with a liquid of constant weight density w. Assuming the xy-plane is "ground level," show that the total force on the portion of the cone from $z = 1$ to $z = 2$ due to liquid pressure is the surface integral

$$F = \iint_S w(2 - z)\, d\sigma.$$

Evaluate the integral.

15. Faraday's law If $\mathbf{E}(t, x, y, z)$ and $\mathbf{B}(t, x, y, z)$ represent the electric and magnetic fields at point (x, y, z) at time t, a basic principle of electromagnetic theory says that $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. In this expression $\nabla \times \mathbf{E}$ is computed with t held fixed and $\partial\mathbf{B}/\partial t$ is calculated with (x, y, z) fixed. Use Stokes' Theorem to derive Faraday's law,

$$\oint_C \mathbf{E}\cdot d\mathbf{r} = -\frac{\partial}{\partial t}\iint_S \mathbf{B}\cdot\mathbf{n}\, d\sigma,$$

where C represents a wire loop through which current flows counterclockwise with respect to the surface's unit normal $\mathbf{n}$, giving rise to the voltage

$$\oint_C \mathbf{E}\cdot d\mathbf{r}$$

around C. The surface integral on the right side of the equation is called the *magnetic flux*, and S is any oriented surface with boundary C.

16. Let

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^3}\mathbf{r}$$

be the gravitational force field defined for $\mathbf{r} \neq \mathbf{0}$. Use Gauss's law in Section 16.8 to show that there is no continuously differentiable vector field $\mathbf{H}$ satisfying $\mathbf{F} = \nabla \times \mathbf{H}$.

17. If $f(x, y, z)$ and $g(x, y, z)$ are continuously differentiable scalar functions defined over the oriented surface S with boundary curve C, prove that

$$\iint_S (\nabla f \times \nabla g) \cdot \mathbf{n}\, d\sigma = \oint_C f\, \nabla g \cdot d\mathbf{r}.$$

18. Suppose that $\nabla \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{F}_2$ and $\nabla \times \mathbf{F}_1 = \nabla \times \mathbf{F}_2$ over a region D enclosed by the oriented surface S with outward unit normal $\mathbf{n}$ and that $\mathbf{F}_1 \cdot \mathbf{n} = \mathbf{F}_2 \cdot \mathbf{n}$ on S. Prove that $\mathbf{F}_1 = \mathbf{F}_2$ throughout D.

19. Prove or disprove that if $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \mathbf{0}$.

20. Let S be an oriented surface parametrized by $\mathbf{r}(u, v)$. Define the notation $d\boldsymbol{\sigma} = \mathbf{r}_u\, du \times \mathbf{r}_v\, dv$ so that $d\boldsymbol{\sigma}$ is a vector normal to the surface. Also, the magnitude $d\sigma = |d\boldsymbol{\sigma}|$ is the element of surface area (by Equation 5 in Section 16.5). Derive the identity

$$d\sigma = (EG - F^2)^{1/2}\, du\, dv$$

where

$$E = |\mathbf{r}_u|^2, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad \text{and} \quad G = |\mathbf{r}_v|^2.$$

21. Show that the volume V of a region D in space enclosed by the oriented surface S with outward normal $\mathbf{n}$ satisfies the identity

$$V = \frac{1}{3} \iint_S \mathbf{r} \cdot \mathbf{n}\, d\sigma,$$

where $\mathbf{r}$ is the position vector of the point (x, y, z) in D.

Chapter 16 Technology Application Projects

Mathematica/Maple Modules:

Work in Conservative and Nonconservative Force Fields

Explore integration over vector fields and experiment with conservative and nonconservative force functions along different paths in the field.

How Can You Visualize Green's Theorem?

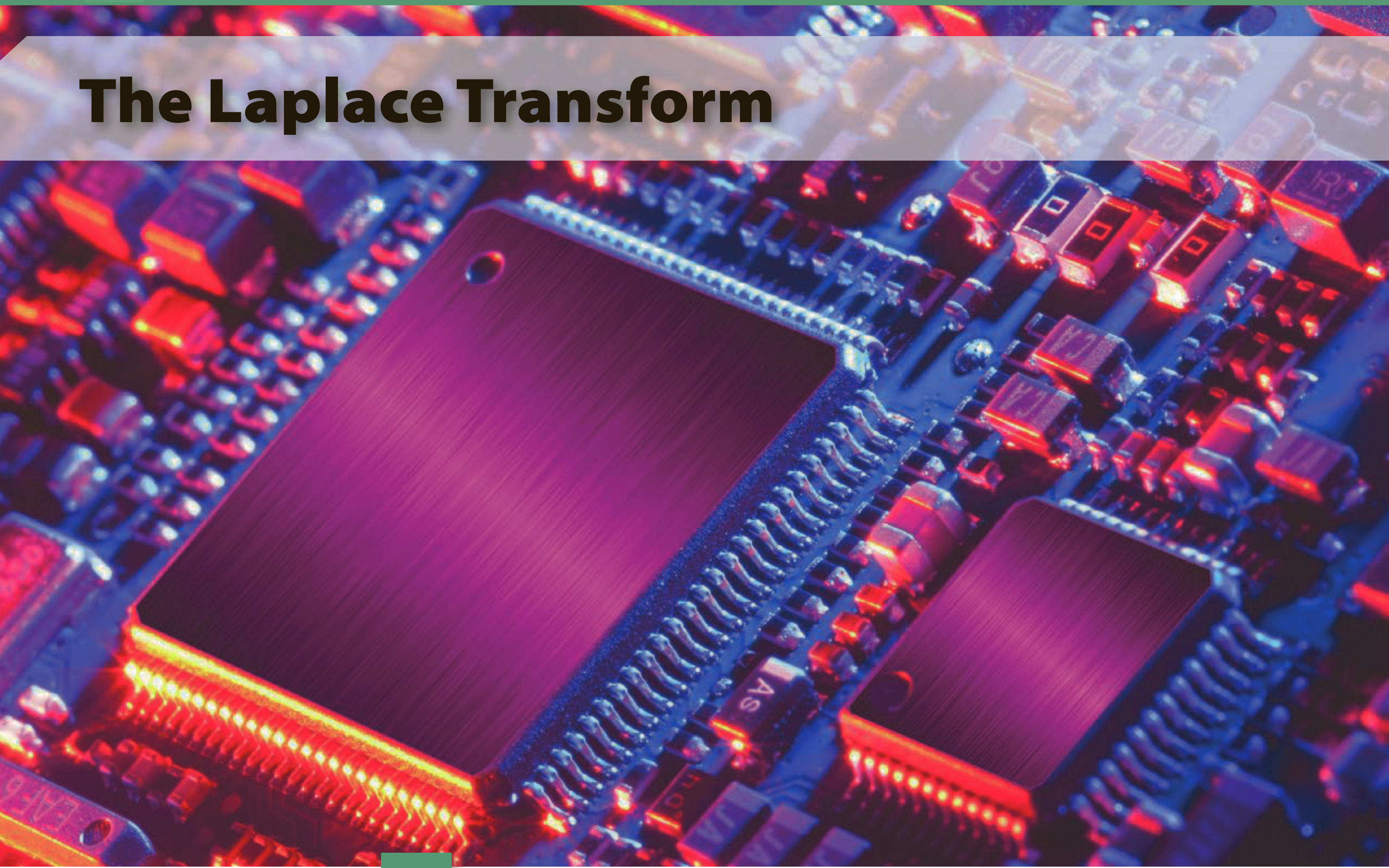
Explore integration over vector fields and use parametrizations to compute line integrals. Both forms of Green's Theorem are explored.

Visualizing and Interpreting the Divergence Theorem

Verify the Divergence Theorem by formulating and evaluating certain divergence and surface integrals.

7

The Laplace Transform



Raimundas/Shutterstock.com

In the linear mathematical models for a physical system such as a spring/mass system or a series electrical circuit, the right-hand member, or driving function, of the differential equations $mx'' + \beta x' + kx = f(t)$ or $Lq'' + Rq' + q/C = E(t)$ represents either an external force $f(t)$ or an impressed voltage $E(t)$. In Section 5.1 we solved problems in which the functions f and E were continuous. However, in practice discontinuous driving functions are not uncommon. Although we have solved piecewise-linear differential equations using the techniques of Chapters 2 and 4, the Laplace transform discussed in this chapter is an especially valuable tool that simplifies the solution of such equations.

7.1 Definition of the Laplace Transform

INTRODUCTION In elementary calculus you learned that differentiation and integration are *transforms*; this means, roughly speaking, that these operations transform a function into another function. For example, the function $f(x) = x^2$ is transformed, in turn, into a linear function and a family of cubic polynomial functions by the operations of differentiation and integration:

$$\frac{d}{dx}x^2 = 2x \qquad \text{and} \qquad \int x^2\,dx = \frac{1}{3}x^3 + c.$$

Moreover, these two transforms possess the **linearity property** that the transform of a linear combination of functions is a linear combination of the transforms. For α and β constants

$$\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha f'(x) + \beta g'(x)$$

and

$$\int [\alpha f(x) + \beta g(x)]\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

provided that each derivative and integral exists. In this section we will examine a special type of integral transform called the **Laplace transform.** In addition to possessing the linearity property the Laplace transform has many other interesting properties that make it very useful in solving linear initial-value problems.

INTEGRAL TRANSFORM If $f(x, y)$ is a function of two variables, then a definite integral of f with respect to one of the variables leads to a function of the other variable. For example, by holding y constant, we see that $\int_1^2 2xy^2\,dx = 3y^2$. Similarly, a definite integral such as $\int_a^b K(s, t)\,f(t)\,dt$ transforms a function f of the variable t into a function F of the variable s. We are particularly interested in an **integral transform,** where the interval of integration is the unbounded interval $[0, \infty)$. If $f(t)$ is defined for $t \ge 0$, then the improper integral $\int_0^\infty K(s, t)\,f(t)\,dt$ is defined as a limit:

$$\int_0^\infty K(s, t)\,f(t)\,dt = \lim_{b\to\infty} \int_0^b K(s, t)\,f(t)\,dt. \tag{1}$$

If the limit in (1) exists, then we say that the integral exists or is **convergent;** if the limit does not exist, the integral does not exist and is **divergent.** The limit in (1) will, in general, exist for only certain values of the variable s.

We will assume throughout that s is a real variable.

A DEFINITION The function $K(s, t)$ in (1) is called the **kernel** of the transform. The choice $K(s, t) = e^{-st}$ as the kernel gives us an especially important integral transform.

DEFINITION 7.1.1 Laplace Transform

Let f be a function defined for $t \ge 0$. Then the integral

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \tag{2}$$

is said to be the **Laplace transform** of f, provided that the integral converges.

The Laplace transform is named in honor of the French mathematician and astronomer **Pierre-Simon Marquis de Laplace** (1749–1827).

When the defining integral (2) converges, the result is a function of s. In general discussion we shall use a lowercase letter to denote the function being transformed and the corresponding capital letter to denote its Laplace transform—for example,

$$\mathscr{L}\{f(t)\} = F(s), \qquad \mathscr{L}\{g(t)\} = G(s), \qquad \mathscr{L}\{y(t)\} = Y(s).$$

As the next four examples show, the domain of the function $F(s)$ depends on the function $f(t)$.

EXAMPLE 1 Applying Definition 7.1.1

Evaluate $\mathscr{L}\{1\}$.

SOLUTION From (2),

$$\begin{aligned}\mathscr{L}\{1\} &= \int_0^\infty e^{-st}(1)\,dt = \lim_{b\to\infty}\int_0^b e^{-st}\,dt \\ &= \lim_{b\to\infty}\frac{-e^{-st}}{s}\bigg|_0^b = \lim_{b\to\infty}\frac{-e^{-sb}+1}{s} = \frac{1}{s}\end{aligned}$$

provided that $s > 0$. In other words, when $s > 0$, the exponent $-sb$ is negative, and $e^{-sb} \to 0$ as $b \to \infty$. The integral diverges for $s < 0$. ■

The use of the limit sign becomes somewhat tedious, so we shall adopt the notation $|_0^\infty$ as a shorthand for writing $\lim_{b\to\infty}(\ )|_0^b$. For example,

$$\mathscr{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \frac{-e^{-st}}{s}\bigg|_0^\infty = \frac{1}{s}, \qquad s > 0.$$

At the upper limit, it is understood that we mean $e^{-st} \to 0$ as $t \to \infty$ for $s > 0$.

EXAMPLE 2 Applying Definition 7.1.1

Evaluate $\mathscr{L}\{t\}$.

SOLUTION From Definition 7.1.1 we have $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\,t\,dt$. Integrating by parts and using $\lim\limits_{t\to\infty} te^{-st} = 0$, $s > 0$, along with the result from Example 1, we obtain

$$\mathscr{L}\{t\} = \frac{-te^{-st}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt = \frac{1}{s}\mathscr{L}\{1\} = \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}.$$ ■

EXAMPLE 3 Applying Definition 7.1.1

Evaluate **(a)** $\mathscr{L}\{e^{-3t}\}$ **(b)** $\mathscr{L}\{e^{5t}\}$

SOLUTION In each case we use Definition 7.1.1.

(a)

$$\begin{aligned}\mathscr{L}\{e^{-3t}\} &= \int_0^\infty e^{-3t}e^{-st}\,dt = \int_0^\infty e^{-(s+3)t}\,dt \\ &= \frac{-e^{-(s+3)t}}{s+3}\bigg|_0^\infty \\ &= \frac{1}{s+3}.\end{aligned}$$

The last result is valid for $s > -3$ because in order to have $\lim_{t\to\infty} e^{-(s+3)t} = 0$ we must require that $s + 3 > 0$ or $s > -3$.

(b)
$$\mathscr{L}\{e^{5t}\} = \int_0^\infty e^{5t}e^{-st}\,dt = \int_0^\infty e^{-(s-5)t}\,dt$$
$$= \frac{-e^{-(s-5)t}}{s-5}\bigg|_0^\infty$$
$$= \frac{1}{s-5}.$$

In contrast to part (a), this result is valid for $s > 5$ because $\lim_{t\to\infty} e^{-(s-5)t} = 0$ demands $s - 5 > 0$ or $s > 5$. ■

EXAMPLE 4 Applying Definition 7.1.1

Evaluate $\mathscr{L}\{\sin 2t\}$.

SOLUTION From Definition 7.1.1 and two applications of integration by parts we obtain

$$\mathscr{L}\{\sin 2t\} = \int_0^\infty e^{-st}\sin 2t\,dt = \frac{-e^{-st}\sin 2t}{s}\bigg|_0^\infty + \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt$$
$$= \frac{2}{s}\int_0^\infty e^{-st}\cos 2t\,dt, \qquad s > 0$$

$\lim_{t\to\infty} e^{-st}\cos 2t = 0,\ s > 0$ Laplace transform of $\sin 2t$

$$= \frac{2}{s}\left[\frac{-e^{-st}\cos 2t}{s}\bigg|_0^\infty - \frac{2}{s}\int_0^\infty e^{-st}\sin 2t\,dt\right]$$
$$= \frac{2}{s^2} - \frac{4}{s^2}\mathscr{L}\{\sin 2t\}.$$

At this point we have an equation with $\mathscr{L}\{\sin 2t\}$ on both sides of the equality. Solving for that quantity yields the result

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4}, \qquad s > 0.$$ ■

$\mathscr{L}$ IS A LINEAR TRANSFORM For a linear combination of functions we can write

$$\int_0^\infty e^{-st}[\alpha f(t) + \beta g(t)]\,dt = \alpha\int_0^\infty e^{-st}f(t)\,dt + \beta\int_0^\infty e^{-st}g(t)\,dt$$

whenever both integrals converge for $s > c$. Hence it follows that

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha\mathscr{L}\{f(t)\} + \beta\mathscr{L}\{g(t)\} = \alpha F(s) + \beta G(s). \tag{3}$$

Because of the property given in (3), $\mathscr{L}$ is said to be a **linear transform.**

EXAMPLE 5 Linearity of the Laplace Transform

In this example we use the results of the preceding examples to illustrate the linearity of the Laplace transform.

(a) From Examples 1 and 2 we have for $s > 0$,

$$\mathscr{L}\{1 + 5t\} = \mathscr{L}\{1\} + 5\mathscr{L}\{t\} = \frac{1}{s} + \frac{5}{s^2}.$$

(b) From Examples 3 and 4 we have for $s > 5$,

$$\mathscr{L}\left\{4e^{5t} - 10\sin 2t\right\} = 4\mathscr{L}\left\{e^{5t}\right\} - 10\mathscr{L}\{\sin 2t\} = \frac{4}{s-5} - \frac{20}{s^2+4}.$$

(c) From Examples 1, 2, and 3 we have for $s > 0$,

$$\mathscr{L}\{20e^{-3t} + 7t - 9\} = 20\mathscr{L}\{e^{-3t}\} + 7\mathscr{L}\{t\} - 9\mathscr{L}\{1\}$$
$$= \frac{20}{s+3} + \frac{7}{s^2} - \frac{9}{s}.$$

We state the generalization of some of the preceding examples by means of the next theorem. From this point on we shall also refrain from stating any restrictions on s; it is understood that s is sufficiently restricted to guarantee the convergence of the appropriate Laplace transform.

THEOREM 7.1.1 Transforms of Some Basic Functions

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$ **(c)** $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$ **(e)** $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$ **(g)** $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

This result in (b) of Theorem 7.1.1 can be formally justified for n a positive integer using integration by parts to first show that

$$\mathscr{L}\{t^n\} = \frac{n}{s}\,\mathscr{L}\{t^{n-1}\}.$$

Then for $n = 1, 2$, and 3, we have, respectively,

$$\mathscr{L}\{t\} = \frac{1}{s}\cdot\mathscr{L}\{1\} = \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$
$$\mathscr{L}\{t^2\} = \frac{2}{s}\cdot\mathscr{L}\{t\} = \frac{2}{s}\cdot\frac{1}{s^2} = \frac{2\cdot 1}{s^3}$$
$$\mathscr{L}\{t^3\} = \frac{3}{s}\cdot\mathscr{L}\{t^2\} = \frac{3}{s}\cdot\frac{2\cdot 1}{s^3} = \frac{3\cdot 2\cdot 1}{s^4}$$

If we carry on in this manner, you should be convinced that

$$\mathscr{L}\{t^n\} = \frac{n\cdots 3\cdot 2\cdot 1}{s^{n+1}} = \frac{n!}{s^{n+1}}.$$

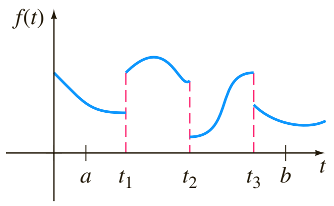


FIGURE 7.1.1 Piecewise continuous function

SUFFICIENT CONDITIONS FOR EXISTENCE OF $\mathscr{L}\{f(t)\}$ The integral that defines the Laplace transform does not have to converge. For example, neither $\mathscr{L}\{1/t\}$ nor $\mathscr{L}\{e^{t^2}\}$ exists. Sufficient conditions guaranteeing the existence of $\mathscr{L}\{f(t)\}$ are that f be piecewise continuous on $[0, \infty)$ and that f be of exponential order for $t > T$. Recall that a function f is **piecewise continuous** on $[0, \infty)$ if, in any interval $0 \le a \le t \le b$, there are at most a finite number of points t_k, $k = 1, 2, \ldots, n$ $(t_{k-1} < t_k)$ at which f has finite discontinuities and is continuous on each open interval (t_{k-1}, t_k). See Figure 7.1.1. The concept of **exponential order** is defined in the following manner.

DEFINITION 7.1.2 Exponential Order

A function f is said to be of **exponential order** if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$.

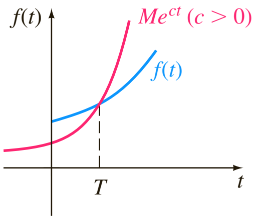


FIGURE 7.1.2 f is of exponential order

If f is an *increasing* function, then the condition $|f(t)| \le Me^{ct}$, $t > T$, simply states that the graph of f on the interval (T, ∞) does not grow faster than the graph of the exponential function Me^{ct}, where c is a positive constant. See Figure 7.1.2. The functions $f(t) = t$, $f(t) = e^{-t}$, and $f(t) = 2\cos t$ are all of exponential order because for $c = 1$, $M = 1$, $T = 0$ we have, respectively, for $t > 0$

$$|t| \le e^t, \qquad |e^{-t}| \le e^t, \qquad \text{and} \qquad |2\cos t| \le 2e^t.$$

A comparison of the graphs on the interval $[0, \infty)$ is given in Figure 7.1.3.

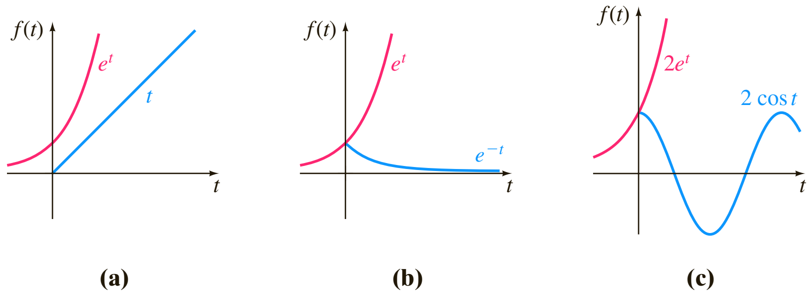


FIGURE 7.1.3 Three functions of exponential order

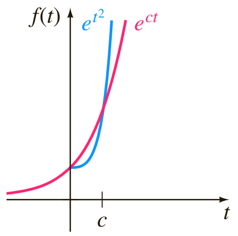


FIGURE 7.1.4 e^{t^2} is not of exponential order

A positive integral power of t is always of exponential order, since, for $c > 0$,

$$|t^n| \le Me^{ct} \qquad \text{or} \qquad \left|\frac{t^n}{e^{ct}}\right| \le M \quad \text{for } t > T$$

is equivalent to showing that $\lim_{t\to\infty} t^n/e^{ct}$ is finite for $n = 1, 2, 3, \ldots$. The result follows from n applications of L'Hôpital's rule. A function such as $f(t) = e^{t^2}$ is not of exponential order since, as shown in Figure 7.1.4, e^{t^2} grows faster than any positive linear power of e for $t > c > 0$. This can also be seen from

$$\left|\frac{e^{t^2}}{e^{ct}}\right| = e^{t^2 - ct} = e^{t(t-c)} \to \infty \quad \text{as} \quad t \to \infty$$

for *any* value of c. By the same reasoning $e^{-st}e^{t^2} \to \infty$ as $t \to \infty$ for any s and so the improper integral $\int_0^\infty e^{-st}e^{t^2}\,dt$ diverges. That is, $\mathscr{L}\{e^{t^2}\}$ does not exist.

THEOREM 7.1.2 Sufficient Conditions for Existence

If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\mathscr{L}\{f(t)\}$ exists for $s > c$.

PROOF By the additive interval property of definite integrals we can write

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt = I_1 + I_2.$$

The integral I_1 exists because it can be written as a sum of integrals over intervals on which $e^{-st} f(t)$ is continuous. Now since f is of exponential order, there exist constants c, $M > 0$, $T > 0$ so that $|f(t)| \le Me^{ct}$ for $t > T$. We can then write

$$|I_2| \le \int_T^\infty |e^{-st} f(t)|\,dt \le M\int_T^\infty e^{-st}e^{ct}\,dt = M\int_T^\infty e^{-(s-c)t}\,dt = M\frac{e^{-(s-c)T}}{s-c}$$

for $s > c$. Since $\int_T^\infty Me^{-(s-c)t}\,dt$ converges, the integral $\int_T^\infty |e^{-st} f(t)|\,dt$ converges by the comparison test for improper integrals. This, in turn, implies that I_2 exists for $s > c$. The existence of I_1 and I_2 implies that $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$ exists for $s > c$. ■

See (*i*) in the *Remarks*.

EXAMPLE 6 Transform of a Piecewise Continuous Function

Evaluate $\mathscr{L}\{f(t)\}$ where $f(t) = \begin{cases} 0, & 0 \le t < 3 \\ 2, & t \ge 3. \end{cases}$

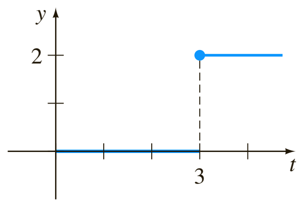


FIGURE 7.1.5 Piecewise continuous function in Example 6

SOLUTION The function f, shown in Figure 7.1.5, is piecewise continuous and of exponential order for $t > 0$. Since f is defined in two pieces, $\mathscr{L}\{f(t)\}$ is expressed as the sum of two integrals:

$$\begin{aligned}\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt &= \int_0^3 e^{-st}(0)\,dt + \int_3^\infty e^{-st}(2)\,dt \\ &= 0 + \left.\frac{2e^{-st}}{-s}\right|_3^\infty \\ &= \frac{2e^{-3s}}{s}, \qquad s > 0.\end{aligned}$$

We conclude this section with an additional bit of theory related to the types of functions of s that we will, generally, be working with. The next theorem indicates that not every arbitrary function of s is a Laplace transform of a piecewise continuous function of exponential order.

THEOREM 7.1.3 Behavior of $F(s)$ as $s \to \infty$

If f is piecewise continuous on $[0, \infty)$ and of exponential order and $F(s) = \mathscr{L}\{f(t)\}$, then $\lim_{s\to\infty} F(s) = 0$.

PROOF Since f is of exponential order, there exist constants γ, $M_1 > 0$, and $T > 0$ so that $|f(t)| \le M_1 e^{\gamma t}$ for $t > T$. Also, since f is piecewise continuous for $0 \le t \le T$, it is necessarily bounded on the interval; that is, $|f(t)| \le M_2 = M_2 e^{0t}$. If M denotes the maximum of the set $\{M_1, M_2\}$ and c denotes the maximum of $\{0, \gamma\}$, then

$$|F(s)| \le \int_0^\infty e^{-st}|f(t)|\,dt \le M\int_0^\infty e^{-st}e^{ct}\,dt = M\int_0^\infty e^{-(s-c)t}\,dt = \frac{M}{s-c}$$

for $s > c$. As $s \to \infty$, we have $|F(s)| \to 0$, and so $F(s) = \mathscr{L}\{f(t)\} \to 0$.

REMARKS

(i) Throughout this chapter we shall be concerned primarily with functions that are both piecewise continuous and of exponential order. We note, however, that these two conditions are sufficient but not necessary for the existence of a Laplace transform. The function $f(t) = t^{-1/2}$ is not piecewise continuous on the interval $[0, \infty)$, but its Laplace transform exists. The function $f(t) = 2te^{t^2}\cos e^{t^2}$ is not of exponential order, but it can be shown that its Laplace transform exists. See Problems 43 and 53 in Exercises 7.1.

(ii) As a consequence of Theorem 7.1.3 we can say that functions of s such as $F_1(s) = 1$ and $F_2(s) = s/(s + 1)$ are not the Laplace transforms of piecewise continuous functions of exponential order, since $F_1(s) \not\to 0$ and $F_2(s) \not\to 0$ as $s \to \infty$. But you should not conclude from this that $F_1(s)$ and $F_2(s)$ are *not* Laplace transforms. There are other kinds of functions.

EXERCISES 7.1

Answers to selected odd-numbered problems begin on page ANS-11.

In Problems 1–18 use Definition 7.1.1 to find $\mathscr{L}\{f(t)\}$.

1. $f(t) = \begin{cases} -1, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

2. $f(t) = \begin{cases} 4, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

3. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1 \end{cases}$

4. $f(t) = \begin{cases} 2t + 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

5. $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

6. $f(t) = \begin{cases} 0, & 0 \le t < \pi/2 \\ \cos t, & t \ge \pi/2 \end{cases}$

7.

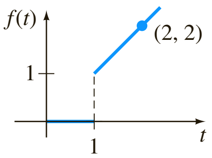


FIGURE 7.1.6 Graph for Problem 7

8.

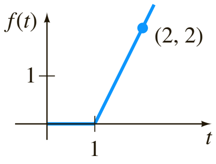


FIGURE 7.1.7 Graph for Problem 8

9.

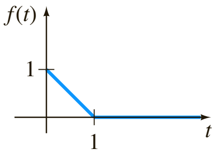


FIGURE 7.1.8 Graph for Problem 9

10.

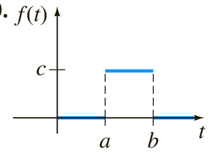


FIGURE 7.1.9 Graph for Problem 10

11. $f(t) = e^{t+7}$ **12.** $f(t) = e^{-2t-5}$

13. $f(t) = te^{4t}$ **14.** $f(t) = t^2e^{-2t}$

15. $f(t) = e^{-t}\sin t$ **16.** $f(t) = e^{t}\cos t$

17. $f(t) = t\cos t$ **18.** $f(t) = t\sin t$

In Problems 19–36 use Theorem 7.1.1 to find $\mathscr{L}\{f(t)\}$.

19. $f(t) = 2t^4$ **20.** $f(t) = t^5$

21. $f(t) = 4t - 10$ **22.** $f(t) = 7t + 3$

23. $f(t) = t^2 + 6t - 3$ **24.** $f(t) = -4t^2 + 16t + 9$

25. $f(t) = (t + 1)^3$ **26.** $f(t) = (2t - 1)^3$

27. $f(t) = 1 + e^{4t}$ **28.** $f(t) = t^2 - e^{-9t} + 5$

29. $f(t) = (1 + e^{2t})^2$ **30.** $f(t) = (e^t - e^{-t})^2$

31. $f(t) = 4t^2 - 5\sin 3t$ **32.** $f(t) = \cos 5t + \sin 2t$

33. $f(t) = \sinh kt$ **34.** $f(t) = \cosh kt$

35. $f(t) = e^t \sinh t$ **36.** $f(t) = e^{-t}\cosh t$

In Problems 37–40 find $\mathscr{L}\{f(t)\}$ by first using a trigonometric identity.

37. $f(t) = \sin 2t \cos 2t$ **38.** $f(t) = \cos^2 t$

39. $f(t) = \sin(4t + 5)$ **40.** $f(t) = 10\cos\left(t - \dfrac{\pi}{6}\right)$

41. We have encountered the **gamma function** $\Gamma(\alpha)$ in our study of Bessel functions in Section 6.4 (page 263). One definition of this function is given by the improper integral

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}\,dt, \quad \alpha > 0.$$

Use this definition to show that $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$. When $\alpha = n$ is a positive integer the last property can be used to show that $\Gamma(n + 1) = n!$. See Appendix A.

42. Use Problem 41 and the change of variable $u = st$ to obtain the generalization

$$\mathscr{L}\{t^\alpha\} = \frac{\Gamma(\alpha + 1)}{s^{\alpha+1}}, \quad \alpha > -1,$$

of the result in Theorem 7.1.1(b).

In Problems 43–46 use Problems 41 and 42 and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ to find the Laplace transform of the given function.

43. $f(t) = t^{-1/2}$ **44.** $f(t) = t^{1/2}$

45. $f(t) = t^{3/2}$ **46.** $f(t) = 2t^{1/2} + 8t^{5/2}$

Discussion Problems

47. Suppose that $\mathscr{L}\{f_1(t)\} = F_1(s)$ for $s > c_1$ and that $\mathscr{L}\{f_2(t)\} = F_2(s)$ for $s > c_2$. When does

$$\mathscr{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)?$$

48. Figure 7.1.4 suggests, but does not prove, that the function $f(t) = e^{t^2}$ is not of exponential order. How does the observation that $t^2 > \ln M + ct$, for $M > 0$ and t sufficiently large, show that $e^{t^2} > Me^{ct}$ for any c?

49. Use part (c) of Theorem 7.1.1 to show that

$$\mathscr{L}\{e^{(a+ib)t}\} = \frac{s - a + ib}{(s - a)^2 + b^2},$$

where a and b are real and $i^2 = -1$. Show how Euler's formula (page 136) can then be used to deduce the results

$$\mathscr{L}\{e^{at}\cos bt\} = \frac{s - a}{(s - a)^2 + b^2}$$

$$\mathscr{L}\{e^{at}\sin bt\} = \frac{b}{(s - a)^2 + b^2}.$$

50. Under what conditions is a linear function $f(x) = mx + b$, $m \ne 0$, a linear transform?

51. Explain why the function

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 4, & 2 < t < 5 \\ 1/(t - 5), & t > 5 \end{cases}$$

is not piecewise continuous on $[0, \infty)$.

52. Show that the function $f(t) = 1/t^2$ does not possess a Laplace transform. [*Hint*: Write $\mathscr{L}\{1/t^2\}$ as two improper integrals:

$$\mathscr{L}\{1/t^2\} = \int_0^1 \frac{e^{-st}}{t^2}\,dt + \int_1^\infty \frac{e^{-st}}{t^2}\,dt = I_1 + I_2.$$

Show that I_1 diverges.]

53. The function $f(t) = 2te^{t^2}\cos t^2$ is not of exponential order. Nevertheless, show that the Laplace transform $\mathscr{L}\{2te^{t^2}\cos e^{t^2}\}$ exists. [*Hint*: Start with integration by parts.]

54. If $\mathscr{L}\{f(t)\} = F(s)$ and $a > 0$ is a constant, show that

$$\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right).$$

This result is known as the **change of scale theorem.**

In Problems 55–58 use the given Laplace transform and the result in Problem 54 to find the indicated Laplace transform. Assume that a and k are positive constants.

55. $\mathscr{L}\{e^t\} = \dfrac{1}{s-1}$; $\mathscr{L}\{e^{at}\}$ **56.** $\mathscr{L}\{\sin t\} = \dfrac{1}{s^2+1}$; $\mathscr{L}\{\sin kt\}$

57. $\mathscr{L}\{1 - \cos t\} = \dfrac{1}{s(s^2+1)}$; $\mathscr{L}\{1 - \cos kt\}$

58. $\mathscr{L}\{\sin t \sinh t\} = \dfrac{2s}{s^4+4}$; $\mathscr{L}\{\sin kt \sinh kt\}$

7.2 Inverse Transforms and Transforms of Derivatives

INTRODUCTION In this section we take a few small steps into an investigation of how the Laplace transform can be used to solve certain types of equations for an unknown function. We begin the discussion with the concept of the inverse Laplace transform or, more precisely, the inverse of a Laplace transform $F(s)$. After some important preliminary background material on the Laplace transform of derivatives $f'(t), f''(t), \ldots,$ we then illustrate how both the Laplace transform and the inverse Laplace transform come into play in solving some simple ordinary differential equations.

7.2.1 INVERSE TRANSFORMS

THE INVERSE PROBLEM If $F(s)$ represents the Laplace transform of a function $f(t)$, that is, $\mathscr{L}\{f(t)\} = F(s)$, we then say $f(t)$ is the **inverse Laplace transform** of $F(s)$ and write $f(t) = \mathscr{L}^{-1}\{F(s)\}$. For example, from Examples 1, 2, and 3 of Section 7.1 we have, respectively,

Transform	Inverse Transform
$\mathscr{L}\{1\} = \dfrac{1}{s}$	$1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$
$\mathscr{L}\{t\} = \dfrac{1}{s^2}$	$t = \mathscr{L}^{-1}\left\{\dfrac{1}{s^2}\right\}$
$\mathscr{L}\{e^{-3t}\} = \dfrac{1}{s+3}$	$e^{-3t} = \mathscr{L}^{-1}\left\{\dfrac{1}{s+3}\right\}$

We shall see shortly that in the application of the Laplace transform to equations we are not able to determine an unknown function $f(t)$ directly; rather, we are able to solve for the Laplace transform $F(s)$ of $f(t)$; but from that knowledge we ascertain f by computing $f(t) = \mathscr{L}^{-1}\{F(s)\}$. The idea is simply this: Suppose $F(s) = \dfrac{-2s+6}{s^2+4}$ is a Laplace transform; find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$. We shall show how to solve this problem in Example 2.

For future reference the analogue of Theorem 7.1.1 for the inverse transform is presented as our next theorem.

THEOREM 7.2.1 Some Inverse Transforms

$$\textbf{(a)}\ \ 1 = \mathscr{L}^{-1}\left\{\frac{1}{s}\right\}$$

(b) $t^n = \mathscr{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\}, \quad n = 1, 2, 3, \ldots$ **(c)** $e^{at} = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2+k^2}\right\}$ **(e)** $\cos kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2-k^2}\right\}$ **(g)** $\cosh kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2-k^2}\right\}$

In evaluating inverse transforms, it often happens that a function of s under consideration does not match *exactly* the form of a Laplace transform $F(s)$ given in a table. It may be necessary to "fix up" the function of s by multiplying and dividing by an appropriate constant.

EXAMPLE 1 Applying Theorem 7.2.1

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^5}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2+7}\right\}$.

SOLUTION (a) To match the form given in part (b) of Theorem 7.2.1, we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) To match the form given in part (d) of Theorem 7.2.1, we identify $k^2 = 7$, so $k = \sqrt{7}$. We fix up the expression by multiplying and dividing by $\sqrt{7}$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

$\mathscr{L}^{-1}$ IS A LINEAR TRANSFORM The inverse Laplace transform is also a linear transform; that is, for constants α and β

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha\mathscr{L}^{-1}\{F(s)\} + \beta\mathscr{L}^{-1}\{G(s)\}, \tag{1}$$

where F and G are the transforms of some functions f and g. Like (3) of Section 7.1, (1) extends to any finite linear combination of Laplace transforms.

EXAMPLE 2 Termwise Division and Linearity

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{-2s+6}{s^2+4}\right\}$.

SOLUTION We first rewrite the given function of s as two expressions by means of termwise division and then use (1):

$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right\} = -2\,\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \quad (2)$$

termwise division; linearity and fixing up constants

$$= -2\cos 2t + 3\sin 2t. \quad \leftarrow \text{parts (e) and (d) of Theorem 7.2.1 with } k = 2$$

PARTIAL FRACTIONS Partial fractions play an important role in finding inverse Laplace transforms. The decomposition of a rational expression into component fractions can be done quickly by means of a single command on most computer algebra systems. Indeed, some CASs have packages that implement Laplace transform and inverse Laplace transform commands. But for those of you without access to such software, we will review in this and subsequent sections some of the basic algebra in the important cases in which the denominator of a Laplace transform $F(s)$ contains distinct linear factors, repeated linear factors, and quadratic polynomials with no real factors. Although we shall examine each of these cases as this chapter develops, it still might be a good idea for you to consult either a calculus text or a current precalculus text for a more comprehensive review of this theory.

The following example illustrates partial fraction decomposition in the case when the denominator of $F(s)$ is factorable into *distinct linear factors*.

EXAMPLE 3 Partial Fractions: Distinct Linear Factors

Evaluate $\mathscr{L}^{-1}\left\{\dfrac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\}$.

SOLUTION There exist unique real constants A, B, and C so that

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+4}$$
$$= \frac{A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2)}{(s-1)(s-2)(s+4)}.$$

Since the denominators are identical, the numerators are identical:

$$s^2+6s+9 = A(s-2)(s+4) + B(s-1)(s+4) + C(s-1)(s-2). \quad (3)$$

By comparing coefficients of powers of s on both sides of the equality, we know that (3) is equivalent to a system of three equations in the three unknowns A, B, and C. However, there is a shortcut for determining these unknowns. If we set $s = 1$, $s = 2$, and $s = -4$ in (3), we obtain, respectively,

$$16 = A(-1)(5), \qquad 25 = B(1)(6), \qquad \text{and} \qquad 1 = C(-5)(-6),$$

and so $A = -\frac{16}{5}$, $B = \frac{25}{6}$, and $C = \frac{1}{30}$. Hence the partial fraction decomposition is

$$\frac{s^2+6s+9}{(s-1)(s-2)(s+4)} = -\frac{16/5}{s-1} + \frac{25/6}{s-2} + \frac{1/30}{s+4}, \quad (4)$$

and thus, from the linearity of $\mathscr{L}^{-1}$ and part (c) of Theorem 7.2.1,

$$\mathscr{L}^{-1}\left\{\frac{s^2+6s+9}{(s-1)(s-2)(s+4)}\right\} = -\frac{16}{5}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{25}{6}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{30}\mathscr{L}^{-1}\left\{\frac{1}{s+4}\right\}$$
$$= -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}. \quad (5)$$

7.2.2 TRANSFORMS OF DERIVATIVES

TRANSFORM A DERIVATIVE As was pointed out in the introduction to this chapter, our immediate goal is to use the Laplace transform to solve differential equations. To that end we need to evaluate quantities such as $\mathscr{L}\{dy/dt\}$ and $\mathscr{L}\{d^2y/dt^2\}$. For example, if f' is continuous for $t \geq 0$, then integration by parts gives

$$\begin{aligned}\mathscr{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\,dt = e^{-st} f(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt \\ &= -f(0) + s\mathscr{L}\{f(t)\}\end{aligned}$$

or
$$\mathscr{L}\{f'(t)\} = sF(s) - f(0). \tag{6}$$

Here we have assumed that $e^{-st}f(t) \to 0$ as $t \to \infty$. Similarly, with the aid of (6),

$$\begin{aligned}\mathscr{L}\{f''(t)\} &= \int_0^\infty e^{-st} f''(t)\,dt = e^{-st} f'(t)\Big|_0^\infty + s\int_0^\infty e^{-st} f'(t)\,dt \\ &= -f'(0) + s\mathscr{L}\{f'(t)\} \\ &= s[sF(s) - f(0)] - f'(0) \qquad \leftarrow \text{from (6)}\end{aligned}$$

or
$$\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0). \tag{7}$$

In like manner it can be shown that

$$\mathscr{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0). \tag{8}$$

The recursive nature of the Laplace transform of the derivatives of a function f should be apparent from the results in (6), (7), and (8). The next theorem gives the Laplace transform of the nth derivative of f.

THEOREM 7.2.2 Transform of a Derivative

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0),$$

where $F(s) = \mathscr{L}\{f(t)\}$.

SOLVING LINEAR ODEs It is apparent from the general result given in Theorem 7.2.2 that $\mathscr{L}\{d^ny/dt^n\}$ depends on $Y(s) = \mathscr{L}\{y(t)\}$ and the $n - 1$ derivatives of $y(t)$ evaluated at $t = 0$. This property makes the Laplace transform ideally suited for solving linear initial-value problems in which the differential equation has *constant coefficients*. Such a differential equation is simply a linear combination of terms $y, y', y'', \ldots, y^{(n)}$:

$$a_n\frac{d^ny}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0y = g(t),$$
$$y(0) = y_0,\ y'(0) = y_1, \ldots, y^{(n-1)}(0) = y_{n-1},$$

where the a_i, $i = 0, 1, \ldots, n$ and $y_0, y_1, \ldots, y_{n-1}$ are constants. By the linearity property the Laplace transform of this linear combination is a linear combination of Laplace transforms:

$$a_n\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} + a_{n-1}\mathscr{L}\left\{\frac{d^{n-1}y}{dt^{n-1}}\right\} + \cdots + a_0\mathscr{L}\{y\} = \mathscr{L}\{g(t)\}. \tag{9}$$

From Theorem 7.2.2, (9) becomes

$$\begin{aligned} &a_n[s^nY(s) - s^{n-1}y(0) - \cdots - y^{(n-1)}(0)] \\ &\quad + a_{n-1}[s^{n-1}Y(s) - s^{n-2}y(0) - \cdots - y^{(n-2)}(0)] + \cdots + a_0Y(s) = G(s), \end{aligned} \tag{10}$$

where $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$. In other words,

The Laplace transform of a linear differential equation with constant coefficients becomes an algebraic equation in $Y(s)$.

If we solve the general transformed equation (10) for the symbol $Y(s)$, we first obtain $P(s)Y(s) = Q(s) + G(s)$ and then write

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}, \tag{11}$$

where $P(s) = a_ns^n + a_{n-1}s^{n-1} + \cdots + a_0$, $Q(s)$ is a polynomial in s of degree less than or equal to $n - 1$ consisting of the various products of the coefficients a_i, $i = 1, \ldots, n$ and the prescribed initial conditions $y_0, y_1, \ldots, y_{n-1}$, and $G(s)$ is the Laplace transform of $g(t)$.* Typically, we put the two terms in (11) over the least common denominator and then decompose the expression into two or more partial fractions. Finally, the solution $y(t)$ of the original initial-value problem is $y(t) = \mathscr{L}^{-1}\{Y(s)\}$, where the inverse transform is done term by term.

The procedure is summarized in the diagram in Figure 7.2.1.

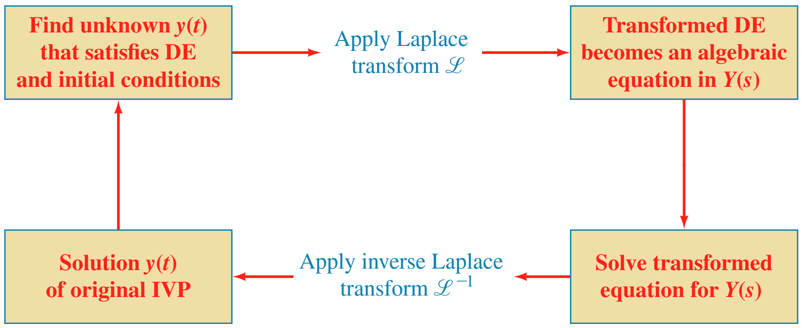


FIGURE 7.2.1 Steps in solving an IVP by the Laplace transform

The next example illustrates the foregoing method of solving DEs, as well as partial fraction decomposition in the case when the denominator of $Y(s)$ contains a *quadratic polynomial with no real factors*.

EXAMPLE 4 Solving a First-Order IVP

Use the Laplace transform to solve the initial-value problem

$$\frac{dy}{dt} + 3y = 13 \sin 2t, \quad y(0) = 6.$$

SOLUTION We first take the transform of each member of the differential equation:

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} + 3\mathscr{L}\{y\} = 13\,\mathscr{L}\{\sin 2t\}. \tag{12}$$

From (6), $\mathscr{L}\{dy/dt\} = sY(s) - y(0) = sY(s) - 6$, and from part (d) of Theorem 7.1.1, $\mathscr{L}\{\sin 2t\} = 2/(s^2 + 4)$, so (12) is the same as

$$sY(s) - 6 + 3Y(s) = \frac{26}{s^2 + 4} \quad \text{or} \quad (s + 3)Y(s) = 6 + \frac{26}{s^2 + 4}.$$

*The polynomial $P(s)$ is the same as the nth-degree auxiliary polynomial in (12) in Section 4.3 with the usual symbol m replaced by s.

Solving the last equation for $Y(s)$, we get

$$Y(s) = \frac{6}{s+3} + \frac{26}{(s+3)(s^2+4)} = \frac{6s^2+50}{(s+3)(s^2+4)}. \tag{13}$$

Since the quadratic polynomial $s^2 + 4$ does not factor using real numbers, its assumed numerator in the partial fraction decomposition is a linear polynomial in s:

$$\frac{6s^2+50}{(s+3)(s^2+4)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+4}.$$

Putting the right-hand side of the equality over a common denominator and equating numerators gives $6s^2 + 50 = A(s^2+4) + (Bs+C)(s+3)$. Setting $s = -3$ then immediately yields $A = 8$. Since the denominator has no more real zeros, we equate the coefficients of s^2 and s: $6 = A + B$ and $0 = 3B + C$. Using the value of A in the first equation gives $B = -2$, and then using this last value in the second equation gives $C = 6$. Thus

$$Y(s) = \frac{6s^2+50}{(s+3)(s^2+4)} = \frac{8}{s+3} + \frac{-2s+6}{s^2+4}.$$

We are not quite finished because the last rational expression still has to be written as two fractions. This was done by termwise division in Example 2. From (2) of that example,

$$y(t) = 8\mathscr{L}^{-1}\left\{\frac{1}{s+3}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}.$$

It follows from parts (c), (d), and (e) of Theorem 7.2.1 that the solution of the initial-value problem is $y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$. ■

EXAMPLE 5 Solving a Second-Order IVP

Solve $y'' - 3y' + 2y = e^{-4t}$, $y(0) = 1$, $y'(0) = 5$.

SOLUTION Proceeding as in Example 4, we transform the DE. We take the sum of the transforms of each term, use (6) and (7), use the given initial conditions, use (c) of Theorem 7.1.1, and then solve for $Y(s)$:

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} - 3\mathscr{L}\left\{\frac{dy}{dt}\right\} + 2\mathscr{L}\{y\} = \mathscr{L}\{e^{-4t}\}$$

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = \frac{1}{s+4}$$

$$(s^2 - 3s + 2)Y(s) = s + 2 + \frac{1}{s+4}$$

$$Y(s) = \frac{s+2}{s^2-3s+2} + \frac{1}{(s^2-3s+2)(s+4)} = \frac{s^2+6s+9}{(s-1)(s-2)(s+4)}. \tag{14}$$

The details of the partial fraction decomposition of $Y(s)$ in (14) have already been carried out in Example 3. In view of the results in (4) and (5) we have the solution of the initial-value problem

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = -\frac{16}{5}e^{t} + \frac{25}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$ ■

Examples 4 and 5 illustrate the basic procedure for using the Laplace transform to solve a linear initial-value problem, but these examples may appear to demonstrate a method that is not much better than the approach to such problems outlined in Sections 2.3 and 4.3–4.6. Don't draw any negative conclusions from only two examples. Yes, there is a lot of algebra inherent in the use of the Laplace transform, *but* observe that we do not have to use variation of parameters or worry about the cases and algebra in the method of undetermined coefficients. Moreover, since the method incorporates the prescribed initial conditions directly into the solution, there is no need for the separate operation of applying the initial conditions to the general solution $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$ of the DE to find specific constants in a particular solution of the IVP.

The Laplace transform has many operational properties. In the sections that follow we will examine some of these properties and see how they enable us to solve problems of greater complexity.

REMARKS

(i) The inverse Laplace transform of a function $F(s)$ may not be unique; in other words, it is possible that $\mathscr{L}\{f_1(t)\} = \mathscr{L}\{f_2(t)\}$ and yet $f_1 \neq f_2$. For our purposes this is not anything to be concerned about. If f_1 and f_2 are piecewise continuous on $[0, \infty)$ and of exponential order, then f_1 and f_2 are *essentially* the same. See Problem 50 in Exercises 7.2. However, if f_1 and f_2 are continuous on $[0, \infty)$ and $\mathscr{L}\{f_1(t)\} = \mathscr{L}\{f_2(t)\}$, then $f_1 = f_2$ on the interval.

(ii) This remark is for those of you who will be required to do partial fraction decompositions by hand. There is another way of determining the coefficients in a partial fraction decomposition in the special case when $\mathscr{L}\{f(t)\} = F(s)$ is a rational function of s and the denominator of F is a product of *distinct* linear factors. Let us illustrate by reexamining Example 3. Suppose we multiply both sides of the assumed decomposition

$$\frac{s^2 + 6s + 9}{(s - 1)(s - 2)(s + 4)} = \frac{A}{s - 1} + \frac{B}{s - 2} + \frac{C}{s + 4} \tag{15}$$

by, say, $s - 1$, simplify, and then set $s = 1$. Since the coefficients of B and C on the right-hand side of the equality are zero, we get

$$\left.\frac{s^2 + 6s + 9}{(s - 2)(s + 4)}\right|_{s=1} = A \qquad \text{or} \qquad A = -\frac{16}{5}.$$

Written another way,

$$\left.\frac{s^2 + 6s + 9}{\boxed{(s - 1)}(s - 2)(s + 4)}\right|_{s=1} = -\frac{16}{5} = A,$$

where we have shaded, or *covered up*, the factor that canceled when the left-hand side was multiplied by $s - 1$. Now to obtain B and C, we simply evaluate the left-hand side of (15) while covering up, in turn, $s - 2$ and $s + 4$:

$$\left.\frac{s^2 + 6s + 9}{(s - 1)\boxed{(s - 2)}(s + 4)}\right|_{s=2} = \frac{25}{6} = B$$

$$\text{and} \qquad \left.\frac{s^2 + 6s + 9}{(s - 1)(s - 2)\boxed{(s + 4)}}\right|_{s=-4} = \frac{1}{30} = C.$$

The desired decomposition (15) is given in (4). This special technique for determining coefficients is naturally known as the **cover-up method.**

(iii) In this remark we continue our introduction to the terminology of dynamical systems. Because of (9) and (10) the Laplace transform is well adapted to *linear* dynamical systems. The polynomial $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$ in (11) is the total coefficient of $Y(s)$ in (10) and is simply the left-hand side of the DE with the derivatives $d^k y/dt^k$ replaced by powers s^k, $k = 0, 1, \ldots, n$. It is usual practice to call the reciprocal of $P(s)$—namely, $W(s) = 1/P(s)$—the **transfer function** of the system and write (11) as

$$Y(s) = W(s)Q(s) + W(s)G(s). \tag{16}$$

In this manner we have separated, in an additive sense, the effects on the response that are due to the initial conditions (that is, $W(s)Q(s)$) from those due to the input function g (that is, $W(s)G(s)$). See (13) and (14). Hence the response $y(t)$ of the system is a superposition of two responses:

$$y(t) = \mathscr{L}^{-1}\{W(s)Q(s)\} + \mathscr{L}^{-1}\{W(s)G(s)\} = y_0(t) + y_1(t).$$

If the input is $g(t) = 0$, then the solution of the problem is $y_0(t) = \mathscr{L}^{-1}\{W(s)Q(s)\}$. This solution is called the **zero-input response** of the system. On the other hand, the function $y_1(t) = \mathscr{L}^{-1}\{W(s)G(s)\}$ is the output due to the input $g(t)$. Now if the initial state of the system is the zero state (all the initial conditions are zero), then $Q(s) = 0$, and so the only solution of the initial-value problem is $y_1(t)$. The latter solution is called the **zero-state response** of the system. Both $y_0(t)$ and $y_1(t)$ are particular solutions: $y_0(t)$ is a solution of the IVP consisting of the associated homogeneous equation with the given initial conditions, and $y_1(t)$ is a solution of the IVP consisting of the nonhomogeneous equation with zero initial conditions. In Example 5 we see from (14) that the transfer function is $W(s) = 1/(s^2 - 3s + 2)$, the zero-input response is

$$y_0(t) = \mathscr{L}^{-1}\left\{\frac{s+2}{(s-1)(s-2)}\right\} = -3e^t + 4e^{2t},$$

and the zero-state response is

$$y_1(t) = \mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s-2)(s+4)}\right\} = -\frac{1}{5}e^t + \frac{1}{6}e^{2t} + \frac{1}{30}e^{-4t}.$$

Verify that the sum of $y_0(t)$ and $y_1(t)$ is the solution $y(t)$ in Example 5 and that $y_0(0) = 1$, $y_0'(0) = 5$, whereas $y_1(0) = 0$, $y_1'(0) = 0$.

EXERCISES 7.2

Answers to selected odd-numbered problems begin on page ANS-11.

7.2.1 Inverse Transforms

In Problems 1–30 use appropriate algebra and Theorem 7.2.1 to find the given inverse Laplace transform.

1. $\mathscr{L}^{-1}\left\{\frac{1}{s^3}\right\}$

2. $\mathscr{L}^{-1}\left\{\frac{1}{s^4}\right\}$

3. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{48}{s^5}\right\}$

4. $\mathscr{L}^{-1}\left\{\left(\frac{2}{s} - \frac{1}{s^3}\right)^2\right\}$

5. $\mathscr{L}^{-1}\left\{\frac{(s+1)^3}{s^4}\right\}$

6. $\mathscr{L}^{-1}\left\{\frac{(s+2)^2}{s^3}\right\}$

7. $\mathscr{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s-2}\right\}$

8. $\mathscr{L}^{-1}\left\{\frac{4}{s} + \frac{6}{s^5} - \frac{1}{s+8}\right\}$

9. $\mathscr{L}^{-1}\left\{\frac{1}{4s+1}\right\}$

10. $\mathscr{L}^{-1}\left\{\frac{1}{5s-2}\right\}$

11. $\mathscr{L}^{-1}\left\{\frac{5}{s^2+49}\right\}$

12. $\mathscr{L}^{-1}\left\{\frac{10s}{s^2+16}\right\}$

13. $\mathscr{L}^{-1}\left\{\frac{4s}{4s^2+1}\right\}$

14. $\mathscr{L}^{-1}\left\{\frac{1}{4s^2+1}\right\}$

15. $\mathscr{L}^{-1}\left\{\frac{2s-6}{s^2+9}\right\}$

16. $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+2}\right\}$

17. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+3s}\right\}$

18. $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2-4s}\right\}$

19. $\mathscr{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\}$

20. $\mathscr{L}^{-1}\left\{\frac{1}{s^2+s-20}\right\}$

21. $\mathscr{L}^{-1}\left\{\dfrac{0.9s}{(s-0.1)(s+0.2)}\right\}$

22. $\mathscr{L}^{-1}\left\{\dfrac{s-3}{(s-\sqrt{3})(s+\sqrt{3})}\right\}$

23. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s-2)(s-3)(s-6)}\right\}$

24. $\mathscr{L}^{-1}\left\{\dfrac{s^2+1}{s(s-1)(s+1)(s-2)}\right\}$

25. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3+5s}\right\}$

26. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s+2)(s^2+4)}\right\}$

27. $\mathscr{L}^{-1}\left\{\dfrac{2s-4}{(s^2+s)(s^2+1)}\right\}$

28. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^4-9}\right\}$

29. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+1)(s^2+4)}\right\}$

30. $\mathscr{L}^{-1}\left\{\dfrac{6s+3}{s^4+5s^2+4}\right\}$

In Problems 31–34 find the given inverse Laplace transform by finding the Laplace transform of the indicated function f.

31. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-a)^2-b^2}\right\}$; $f(t) = e^{at}\sinh bt$

32. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s^2+a^2)}\right\}$; $f(t) = at - \sin at$

33. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s^2+a^2)(s^2+b^2)}\right\}$; $f(t) = a\sin bt - b\sin at$

34. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right\}$; $f(t) = \cos bt - \cos at$

7.2.2 Transforms of Derivatives

In Problems 35–44 use the Laplace transform to solve the given initial-value problem.

35. $\dfrac{dy}{dt} - y = 1, \quad y(0) = 0$

36. $2\dfrac{dy}{dt} + y = 0, \quad y(0) = -3$

37. $y' + 6y = e^{4t}, \quad y(0) = 2$

38. $y' - y = 2\cos 5t, \quad y(0) = 0$

39. $y'' + 5y' + 4y = 0, \quad y(0) = 1, \quad y'(0) = 0$

40. $y'' - 4y' = 6e^{3t} - 3e^{-t}, \quad y(0) = 1, \quad y'(0) = -1$

41. $y'' + y = \sqrt{2}\sin\sqrt{2}t, \quad y(0) = 10, \quad y'(0) = 0$

42. $y'' + 9y = e^t, \quad y(0) = 0, \quad y'(0) = 0$

43. $2y''' + 3y'' - 3y' - 2y = e^{-t}, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

44. $y''' + 2y'' - y' - 2y = \sin 3t, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1$

The inverse forms of the results in Problem 49 in Exercises 7.1 are

$$\mathscr{L}^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$$

$$\mathscr{L}^{-1}\left\{\frac{b}{(s-a)^2+b^2}\right\} = e^{at}\sin bt.$$

In Problems 45 and 46 use the Laplace transform and these inverses to solve the given initial-value problem.

45. $y' + y = e^{-3t}\cos 2t, \quad y(0) = 0$

46. $y'' - 2y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 3$

In Problems 47 and 48 use one of the inverse Laplace transforms found in Problems 31–34 to solve the given initial-value problem.

47. $y'' + 4y = 10\cos 5t, \quad y(0) = 0, \quad y'(0) = 0$

48. $y'' + 2y = 4t, \quad y(0) = 0, \quad y'(0) = 0$

Discussion Problems

49. (a) With a slight change in notation the transform in (6) is the same as

$$\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0).$$

With $f(t) = te^{at}$, discuss how this result in conjunction with (c) of Theorem 7.1.1 can be used to evaluate $\mathscr{L}\{te^{at}\}$.

(b) Proceed as in part (a), but this time discuss how to use (7) with $f(t) = t\sin kt$ in conjunction with (d) and (e) of Theorem 7.1.1 to evaluate $\mathscr{L}\{t\sin kt\}$.

50. Make up two functions f_1 and f_2 that have the same Laplace transform. Do not think profound thoughts.

51. Reread (*iii*) in the *Remarks* on page 293. Find the zero-input and the zero-state response for the IVP in Problem 40.

52. Suppose $f(t)$ is a function for which $f'(t)$ is piecewise continuous and of exponential order c. Use results in this section and Section 7.1 to justify

$$f(0) = \lim_{s\to\infty} sF(s),$$

where $F(s) = \mathscr{L}\{f(t)\}$. Verify this result with $f(t) = \cos kt$.

7.3 Operational Properties I

INTRODUCTION It is not convenient to use Definition 7.1.1 each time we wish to find the Laplace transform of a function $f(t)$. For example, the integration by parts involved in evaluating, say, $\mathscr{L}\{e^t t^2 \sin 3t\}$ is formidable, to say the least. In this section and the next we present several labor-saving operational properties of the Laplace transform that enable us to build up a more extensive list of transforms (see the table in Appendix C) without having to resort to the basic definition and integration.

7.3.1 TRANSLATION ON THE s-AXIS

A TRANSLATION Evaluating transforms such as $\mathscr{L}\{e^{5t}t^3\}$ and $\mathscr{L}\{e^{-2t}\cos 4t\}$ is straightforward provided that we know (and we do) $\mathscr{L}\{t^3\}$ and $\mathscr{L}\{\cos 4t\}$. In general, if we know the Laplace transform of a function f, $\mathscr{L}\{f(t)\} = F(s)$, it is possible to compute the Laplace transform of an exponential multiple of f, that is, $\mathscr{L}\{e^{at}f(t)\}$, with no additional effort other than *translating*, or *shifting*, the transform $F(s)$ to $F(s-a)$. This result is known as the **first translation theorem** or **first shifting theorem.**

THEOREM 7.3.1 First Translation Theorem

If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then

$$\mathscr{L}\{e^{at}f(t)\} = F(s-a).$$

PROOF The proof is immediate, since by Definition 7.1.1

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt = F(s-a).$$

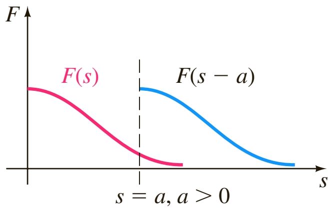


FIGURE 7.3.1 Shift on s-axis

If we consider s a real variable, then the graph of $F(s-a)$ is the graph of $F(s)$ shifted on the s-axis by the amount $|a|$. If $a > 0$, the graph of $F(s)$ is shifted a units to the right, whereas if $a < 0$, the graph is shifted $|a|$ units to the left. See Figure 7.3.1.

For emphasis it is sometimes useful to use the symbolism

$$\mathscr{L}\{e^{at}f(t)\} = \mathscr{L}\{f(t)\}|_{s\to s-a},$$

where $s \to s-a$ means that in the Laplace transform $F(s)$ of $f(t)$ we replace the symbol s wherever it appears by $s-a$.

EXAMPLE 1 Using the First Translation Theorem

Evaluate **(a)** $\mathscr{L}\{e^{5t}t^3\}$ **(b)** $\mathscr{L}\{e^{-2t}\cos 4t\}$.

SOLUTION The results follow from Theorems 7.1.1 and 7.3.1.

(a) $$\mathscr{L}\{e^{5t}t^3\} = \mathscr{L}\{t^3\}|_{s\to s-5} = \left.\frac{3!}{s^4}\right|_{s\to s-5} = \frac{6}{(s-5)^4}$$

(b) $$\mathscr{L}\{e^{-2t}\cos 4t\} = \mathscr{L}\{\cos 4t\}|_{s\to s-(-2)} = \left.\frac{s}{s^2+16}\right|_{s\to s+2} = \frac{s+2}{(s+2)^2+16}$$

INVERSE FORM OF THEOREM 7.3.1 To compute the inverse of $F(s-a)$, we must recognize $F(s)$, find $f(t)$ by taking the inverse Laplace transform of $F(s)$, and then multiply $f(t)$ by the exponential function e^{at}. This procedure can be summarized symbolically in the following manner:

$$\mathscr{L}^{-1}\{F(s-a)\} = \mathscr{L}^{-1}\{F(s)|_{s\to s-a}\} = e^{at}f(t), \tag{1}$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$.

The first part of the next example illustrates partial fraction decomposition in the case when the denominator of $Y(s)$ contains *repeated linear factors*.

EXAMPLE 2 Partial Fractions: Repeated Linear Factors

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\dfrac{2s+5}{(s-3)^2}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{s/2+5/3}{s^2+4s+6}\right\}$.

SOLUTION **(a)** A repeated linear factor is a term $(s-a)^n$, where a is a real number and n is a positive integer ≥ 2. Recall that if $(s-a)^n$ appears in the denominator of a rational expression, then the assumed decomposition contains n partial fractions with constant numerators and denominators $s-a, (s-a)^2, \ldots, (s-a)^n$. Hence with $a=3$ and $n=2$ we write

$$\frac{2s+5}{(s-3)^2}=\frac{A}{s-3}+\frac{B}{(s-3)^2}.$$

By putting the two terms on the right-hand side over a common denominator, we obtain the numerator $2s+5=A(s-3)+B$, and this identity yields $A=2$ and $B=11$. Therefore

$$\frac{2s+5}{(s-3)^2}=\frac{2}{s-3}+\frac{11}{(s-3)^2} \tag{2}$$

and

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}=2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\}+11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}. \tag{3}$$

Now $1/(s-3)^2$ is $F(s)=1/s^2$ shifted three units to the right. Since $\mathscr{L}^{-1}\{1/s^2\}=t$, it follows from (1) that

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}=\mathscr{L}^{-1}\left\{\frac{1}{s^2}\bigg|_{s\to s-3}\right\}=e^{3t}t.$$

Finally, (3) is

$$\mathscr{L}^{-1}\left\{\frac{2s+5}{(s-3)^2}\right\}=2e^{3t}+11e^{3t}t. \tag{4}$$

(b) To start, observe that the quadratic polynomial s^2+4s+6 has no real zeros and so has no real linear factors. In this situation we *complete the square*:

$$\frac{s/2+5/3}{s^2+4s+6}=\frac{s/2+5/3}{(s+2)^2+2}. \tag{5}$$

Our goal here is to recognize the expression on the right-hand side as some Laplace transform $F(s)$ in which s has been replaced throughout by $s+2$. What we are trying to do is analogous to working part (b) of Example 1 backwards. The denominator in (5) is already in the correct form—that is, s^2+2 with s replaced by $s+2$. However, we must fix up the numerator by manipulating the constants: $\frac{1}{2}s+\frac{5}{3}=\frac{1}{2}(s+2)+\frac{5}{3}-\frac{2}{2}=\frac{1}{2}(s+2)+\frac{2}{3}$.

Now by termwise division, the linearity of $\mathscr{L}^{-1}$, parts (d) and (e) of Theorem 7.2.1, and finally (1),

$$\frac{s/2+5/3}{(s+2)^2+2}=\frac{\frac{1}{2}(s+2)+\frac{2}{3}}{(s+2)^2+2}=\frac{1}{2}\frac{s+2}{(s+2)^2+2}+\frac{2}{3}\frac{1}{(s+2)^2+2}$$

$$\mathscr{L}^{-1}\left\{\frac{s/2+5/3}{s^2+4s+6}\right\}=\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\}+\frac{2}{3}\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2+2}\right\}$$

$$=\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+2}\bigg|_{s\to s+2}\right\}+\frac{2}{3\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\bigg|_{s\to s+2}\right\} \tag{6}$$

$$=\frac{1}{2}e^{-2t}\cos\sqrt{2}t+\frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t. \tag{7}$$

EXAMPLE 3 An Initial-Value Problem

Solve $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17.$

SOLUTION Before transforming the DE, note that its right-hand side is similar to the function in part (a) of Example 1. After using linearity, Theorem 7.3.1, and the initial conditions, we simplify and then solve for $Y(s) = \mathscr{L}\{f(t)\}$:

$$\begin{aligned}
\mathscr{L}\{y''\} - 6\mathscr{L}\{y'\} + 9\mathscr{L}\{y\} &= \mathscr{L}\{t^2e^{3t}\} \\
s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 9Y(s) &= \frac{2}{(s-3)^3} \\
(s^2 - 6s + 9)Y(s) &= 2s + 5 + \frac{2}{(s-3)^3} \\
(s-3)^2Y(s) &= 2s + 5 + \frac{2}{(s-3)^3} \\
Y(s) &= \frac{2s+5}{(s-3)^2} + \frac{2}{(s-3)^5}.
\end{aligned}$$

The first term on the right-hand side was already decomposed into individual partial fractions in (2) in part (a) of Example 2:

$$Y(s) = \frac{2}{s-3} + \frac{11}{(s-3)^2} + \frac{2}{(s-3)^5}.$$

Thus $$y(t) = 2\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} + 11\mathscr{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} + \frac{2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{(s-3)^5}\right\}. \tag{8}$$

From the inverse form (1) of Theorem 7.3.1, the last two terms in (8) are

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\bigg|_{s \to s-3}\right\} = te^{3t} \quad \text{and} \quad \mathscr{L}^{-1}\left\{\frac{4!}{s^5}\bigg|_{s \to s-3}\right\} = t^4e^{3t}.$$

Thus (8) is $y(t) = 2e^{3t} + 11te^{3t} + \frac{1}{12}t^4e^{3t}$. ■

EXAMPLE 4 An Initial-Value Problem

Solve $y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

SOLUTION

$$\begin{aligned}
\mathscr{L}\{y''\} + 4\mathscr{L}\{y'\} + 6\mathscr{L}\{y\} &= \mathscr{L}\{1\} + \mathscr{L}\{e^{-t}\} \\
s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 6Y(s) &= \frac{1}{s} + \frac{1}{s+1} \\
(s^2 + 4s + 6)Y(s) &= \frac{2s+1}{s(s+1)} \\
Y(s) &= \frac{2s+1}{s(s+1)(s^2+4s+6)}
\end{aligned}$$

Since the quadratic term in the denominator does not factor into real linear factors, the partial fraction decomposition for $Y(s)$ is found to be

$$Y(s) = \frac{1/6}{s} + \frac{1/3}{s+1} - \frac{s/2 + 5/3}{s^2 + 4s + 6}.$$

Moreover, in preparation for taking the inverse transform we already manipulated the last term into the necessary form in part (b) of Example 2. So in view of the results in (6) and (7) we have the solution

$$y(t) = \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s+2}{(s+2)^2+2}\right\} - \frac{2}{3\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{(s+2)^2+2}\right\}$$

$$= \frac{1}{6} + \frac{1}{3}e^{-t} - \frac{1}{2}e^{-2t}\cos\sqrt{2}t - \frac{\sqrt{2}}{3}e^{-2t}\sin\sqrt{2}t. \qquad \blacksquare$$

7.3.2 TRANSLATION ON THE *t*-AXIS

UNIT STEP FUNCTION In engineering, one frequently encounters functions that are either "off" or "on." For example, an external force acting on a mechanical system or a voltage impressed on a circuit can be turned off after a period of time. It is convenient, then, to define a special function that is the number 0 (off) up to a certain time $t = a$ and then the number 1 (on) after that time. This function is called the **unit step function** or the **Heaviside function,** named after the English polymath **Oliver Heaviside** (1850–1925).

DEFINITION 7.3.1 Unit Step Function

The **unit step function** $\mathscr{U}(t-a)$ is defined to be

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a. \end{cases}$$

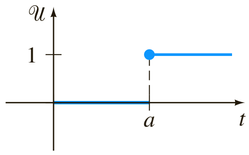


FIGURE 7.3.2 Graph of unit step function

Notice that we define $\mathscr{U}(t-a)$ only on the nonnegative t-axis, since this is all that we are concerned with in the study of the Laplace transform. In a broader sense $\mathscr{U}(t-a) = 0$ for $t < a$. The graph of $\mathscr{U}(t-a)$ is given in Figure 7.3.2. In the case when $a = 0$, we take $\mathscr{U}(t) = 1$ for $t \ge 0$.

When a function f defined for $t \ge 0$ is multiplied by $\mathscr{U}(t-a)$, the unit step function "turns off" a portion of the graph of that function. For example, consider the function $f(t) = 2t - 3$. To "turn off" the portion of the graph of f for $0 \le t < 1$, we simply form the product $(2t-3)\mathscr{U}(t-1)$. See Figure 7.3.3. In general, the graph of $f(t)\,\mathscr{U}(t-a)$ is 0 (off) for $0 \le t < a$ and is the portion of the graph of f (on) for $t \ge a$.

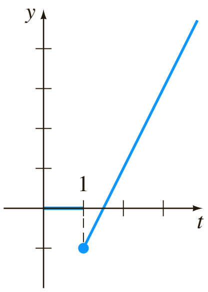


FIGURE 7.3.3 Function is $f(t) = (2t-3)\,\mathscr{U}(t-1)$

The unit step function can also be used to write piecewise-defined functions in a compact form. For example, if we consider $0 \le t < 2$, $2 \le t < 3$, and $t \ge 3$ and the corresponding values of $\mathscr{U}(t-2)$ and $\mathscr{U}(t-3)$, it should be apparent that the piecewise-defined function shown in Figure 7.3.4 is the same as $f(t) = 2 - 3\mathscr{U}(t-2) + \mathscr{U}(t-3)$. Also, a general piecewise-defined function of the type

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \tag{9}$$

is the same as

$$f(t) = g(t) - g(t)\,\mathscr{U}(t-a) + h(t)\,\mathscr{U}(t-a). \tag{10}$$

Similarly, a function of the type

$$f(t) = \begin{cases} 0, & 0 \le t < a \\ g(t), & a \le t < b \\ 0, & t \ge b \end{cases} \tag{11}$$

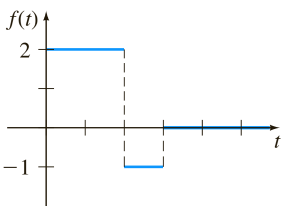


FIGURE 7.3.4 Function is $f(t) = 2 - 3\mathscr{U}(t-2) + \mathscr{U}(t-3)$

can be written

$$f(t) = g(t)[\mathscr{U}(t-a) - \mathscr{U}(t-b)]. \tag{12}$$

EXAMPLE 5 A Piecewise-Defined Function

Express $f(t) = \begin{cases} 20t, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$ in terms of unit step functions. Graph.

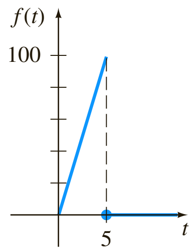


FIGURE 7.3.5 Function f in Example 5

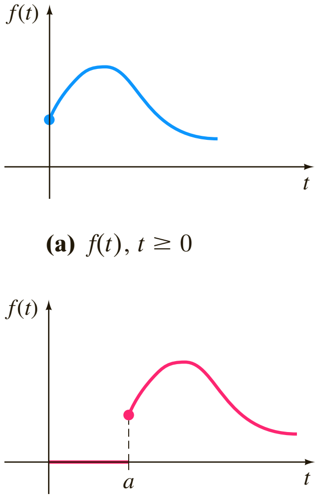
SOLUTION The graph of f is given in Figure 7.3.5. Now from (9) and (10) with $a = 5$, $g(t) = 20t$, and $h(t) = 0$ we get $f(t) = 20t - 20t\,\mathscr{U}(t - 5)$. ■

Consider a general function $y = f(t)$ defined for $t \geq 0$. The piecewise-defined function

$$f(t - a)\,\mathscr{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ f(t - a), & t \geq a \end{cases} \tag{13}$$

plays a significant role in the discussion that follows. As shown in Figure 7.3.6, for $a > 0$ the graph of the function $y = f(t - a)\,\mathscr{U}(t - a)$ coincides with the graph of $y = f(t - a)$ for $t \geq a$ (which is the *entire* graph of $y = f(t)$, $t \geq 0$ shifted a units to the right on the t-axis), but is identically zero for $0 \leq t < a$.

We saw in Theorem 7.3.1 that an exponential multiple of $f(t)$ results in a translation of the transform $F(s)$ on the s-axis. As a consequence of the next theorem we see that whenever $F(s)$ is multiplied by an exponential function e^{-as}, $a > 0$, the inverse transform of the product $e^{-as}F(s)$ is the function f shifted along the t-axis in the manner illustrated in Figure 7.3.6(b). This result, presented next in its direct transform version, is called the **second translation theorem** or **second shifting theorem.**



(a) $f(t)$, $t \geq 0$

(b) $f(t - a)\,\mathscr{U}(t - a)$

FIGURE 7.3.6 Shift on t-axis

THEOREM 7.3.2 Second Translation Theorem

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t - a)\,\mathscr{U}(t - a)\} = e^{-as}F(s).$$

PROOF By the additive interval property of integrals,

$$\int_0^\infty e^{-st} f(t - a)\,\mathscr{U}(t - a)\,dt$$

can be written as two integrals:

$$\mathscr{L}\{f(t - a)\,\mathscr{U}(t - a)\} = \int_0^a e^{-st}f(t - a)\,\underbrace{\mathscr{U}(t - a)}_{\substack{\text{zero for} \\ 0 \leq t < a}}\,dt + \int_a^\infty e^{-st}f(t - a)\,\underbrace{\mathscr{U}(t - a)}_{\substack{\text{one for} \\ t \geq a}}\,dt = \int_a^\infty e^{-st}f(t - a)\,dt.$$

Now if we let $v = t - a$, $dv = dt$ in the last integral, then

$$\mathscr{L}\{f(t - a)\,\mathscr{U}(t - a)\} = \int_0^\infty e^{-s(v+a)}f(v)\,dv = e^{-as}\int_0^\infty e^{-sv}f(v)\,dv = e^{-as}\mathscr{L}\{f(t)\}. \quad ■$$

We often wish to find the Laplace transform of just a unit step function. This can be from either Definition 7.1.1 or Theorem 7.3.2. If we identify $f(t) = 1$ in Theorem 7.3.2, then $f(t - a) = 1$, $F(s) = \mathscr{L}\{1\} = 1/s$, and so

$$\mathscr{L}\{\mathscr{U}(t - a)\} = \frac{e^{-as}}{s}. \tag{14}$$

EXAMPLE 6 Figure 7.3.4 Revisited

Find the Laplace transform of the function f in Figure 7.3.4.

SOLUTION We use f expressed in terms of the unit step function

$$f(t) = 2 - 3\mathscr{U}(t - 2) + \mathscr{U}(t - 3)$$

and the result given in (14):

$$\begin{aligned} \mathscr{L}\{f(t)\} &= 2\mathscr{L}\{1\} - 3\mathscr{L}\{\mathscr{U}(t - 2)\} + \mathscr{L}\{\mathscr{U}(t - 3)\} \\ &= \frac{2}{s} - 3\frac{e^{-2s}}{s} + \frac{e^{-3s}}{s}. \end{aligned}$$ ■

INVERSE FORM OF THEOREM 7.3.2 If $f(t) = \mathscr{L}^{-1}\{F(s)\}$, the inverse form of Theorem 7.3.2, $a > 0$, is

$$\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,\mathscr{U}(t-a). \tag{15}$$

EXAMPLE 7 Using Formula (15)

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\dfrac{1}{s-4}e^{-2s}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+9}e^{-\pi s/2}\right\}$.

SOLUTION (a) With the three identifications $a = 2$, $F(s) = 1/(s-4)$, and $\mathscr{L}^{-1}\{F(s)\} = e^{4t}$, we have from (15)

$$\mathscr{L}^{-1}\left\{\frac{1}{s-4}e^{-2s}\right\} = e^{4(t-2)}\,\mathscr{U}(t-2).$$

(b) With $a = \pi/2$, $F(s) = s/(s^2+9)$, and $\mathscr{L}^{-1}\{F(s)\} = \cos 3t$, (15) yields

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}e^{-\pi s/2}\right\} = \cos 3\left(t-\frac{\pi}{2}\right)\mathscr{U}\left(t-\frac{\pi}{2}\right).$$

The last expression can be simplified somewhat by using the addition formula for the cosine. Verify that the result is the same as $-\sin 3t\,\mathscr{U}\left(t-\dfrac{\pi}{2}\right)$. ■

ALTERNATIVE FORM OF THEOREM 7.3.2 We are frequently confronted with the problem of finding the Laplace transform of a product of a function g and a unit step function $\mathscr{U}(t-a)$ where the function g lacks the precise shifted form $f(t-a)$ in Theorem 7.3.2. To find the Laplace transform of $g(t)\mathscr{U}(t-a)$, it is possible to fix up $g(t)$ into the required form $f(t-a)$ by algebraic manipulations. For example, if we wanted to use Theorem 7.3.2 to find the Laplace transform of $t^2\mathscr{U}(t-2)$, we would have to force $g(t) = t^2$ into the form $f(t-2)$. You should work through the details and verify that $t^2 = (t-2)^2 + 4(t-2) + 4$ is an identity. Therefore

$$\mathscr{L}\{t^2\mathscr{U}(t-2)\} = \mathscr{L}\{(t-2)^2\mathscr{U}(t-2) + 4(t-2)\,\mathscr{U}(t-2) + 4\mathscr{U}(t-2)\},$$

where each term on the right-hand side can now be evaluated by Theorem 7.3.2. But since these manipulations are time consuming and often not obvious, it is simpler to devise an alternative version of Theorem 7.3.2. Using Definition 7.1.1, the definition of $\mathscr{U}(t-a)$, and the substitution $u = t-a$, we obtain

$$\mathscr{L}\{g(t)\,\mathscr{U}(t-a)\} = \int_a^\infty e^{-st}g(t)\,dt = \int_0^\infty e^{-s(u+a)}\,g(u+a)\,du.$$

That is,

$$\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}. \tag{16}$$

EXAMPLE 8 Second Translation Theorem—Alternative Form

Evaluate $\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\}$.

SOLUTION With $g(t) = \cos t$ and $a = \pi$, then $g(t+\pi) = \cos(t+\pi) = -\cos t$ by the addition formula for the cosine function. Hence by (16),

$$\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\} = -e^{-\pi s}\mathscr{L}\{\cos t\} = -\frac{s}{s^2+1}e^{-\pi s}.$$ ■

In the next two examples we solve, in turn, an initial-value problem and a boundary-value problem involving a piecewise-linear differential equation.

EXAMPLE 9 An Initial-Value Problem

Solve $y' + y = f(t), \quad y(0) = 5, \quad$ where $\quad f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 3\cos t, & t \ge \pi. \end{cases}$

SOLUTION The function f can be written as $f(t) = 3\cos t\,\mathscr{U}(t - \pi)$, so by linearity, the results of Example 7, and the usual partial fractions, we have

$$\begin{aligned} \mathscr{L}\{y'\} + \mathscr{L}\{y\} &= 3\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi)\} \\ sY(s) - y(0) + Y(s) &= -3\frac{s}{s^2+1}e^{-\pi s} \\ (s+1)Y(s) &= 5 - \frac{3s}{s^2+1}e^{-\pi s} \end{aligned}$$

$$Y(s) = \frac{5}{s+1} - \frac{3}{2}\left[-\frac{1}{s+1}e^{-\pi s} + \frac{1}{s^2+1}e^{-\pi s} + \frac{s}{s^2+1}e^{-\pi s}\right]. \tag{17}$$

Now proceeding as we did in Example 7, it follows from (15) with $a = \pi$ that the inverses of the terms inside the brackets are

$$\mathscr{L}^{-1}\left\{\frac{1}{s+1}e^{-\pi s}\right\} = e^{-(t-\pi)}\mathscr{U}(t-\pi), \qquad \mathscr{L}^{-1}\left\{\frac{1}{s^2+1}e^{-\pi s}\right\} = \sin(t-\pi)\,\mathscr{U}(t-\pi),$$

and $$\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}e^{-\pi s}\right\} = \cos(t-\pi)\,\mathscr{U}(t-\pi).$$

Thus the inverse of (17) is

$$\begin{aligned} y(t) &= 5e^{-t} + \frac{3}{2}e^{-(t-\pi)}\mathscr{U}(t-\pi) - \frac{3}{2}\sin(t-\pi)\,\mathscr{U}(t-\pi) - \frac{3}{2}\cos(t-\pi)\,\mathscr{U}(t-\pi) \\ &= 5e^{-t} + \frac{3}{2}[e^{-(t-\pi)} + \sin t + \cos t]\,\mathscr{U}(t-\pi) \qquad \leftarrow \text{trigonometric identities} \\ &= \begin{cases} 5e^{-t}, & 0 \le t < \pi \\ 5e^{-t} + \dfrac{3}{2}e^{-(t-\pi)} + \dfrac{3}{2}\sin t + \dfrac{3}{2}\cos t, & t \ge \pi. \end{cases} \end{aligned} \tag{18}$$

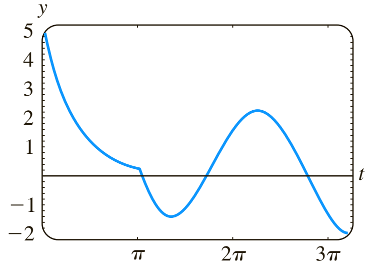


FIGURE 7.3.7 Graph of function (18) in Example 9

We obtained the graph of (18) shown in Figure 7.3.7 by using a graphing utility. ■

BEAMS In Section 5.2 we saw that the static deflection $y(x)$ of a uniform beam of length L carrying load $w(x)$ per unit length is found from the linear fourth-order differential equation

$$EI\frac{d^4y}{dx^4} = w(x), \tag{19}$$

where E is Young's modulus of elasticity and I is a moment of inertia of a cross section of the beam. The Laplace transform is particularly useful in solving (19) when $w(x)$ is piecewise-defined. However, to use the Laplace transform, we must tacitly assume that $y(x)$ and $w(x)$ are defined on $(0, \infty)$ rather than on $(0, L)$. Note, too, that the next example is a boundary-value problem rather than an initial-value problem.

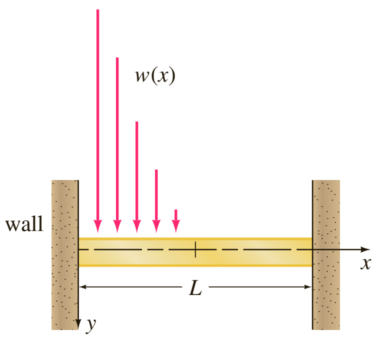


FIGURE 7.3.8 Embedded beam with variable load in Example 10

EXAMPLE 10 A Boundary-Value Problem

A beam of length L is embedded at both ends, as shown in Figure 7.3.8. Find the deflection of the beam when the load is given by

$$w(x) = \begin{cases} w_0\left(1 - \dfrac{2}{L}x\right), & 0 < x < L/2 \\ 0, & L/2 < x < L. \end{cases}$$

SOLUTION Recall that because the beam is embedded at both ends, the boundary conditions are $y(0) = 0$, $y'(0) = 0$, $y(L) = 0$, $y'(L) = 0$. Now by (10) we can express $w(x)$ in terms of the unit step function:

$$\begin{aligned} w(x) &= w_0\left(1 - \frac{2}{L}x\right) - w_0\left(1 - \frac{2}{L}x\right)\mathscr{U}\left(x - \frac{L}{2}\right) \\ &= \frac{2w_0}{L}\left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right)\mathscr{U}\left(x - \frac{L}{2}\right)\right]. \end{aligned}$$

Transforming (19) with respect to the variable x gives

$$EI(s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)) = \frac{2w_0}{L}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right]$$

or

$$s^4Y(s) - sy''(0) - y'''(0) = \frac{2w_0}{EIL}\left[\frac{L/2}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-Ls/2}\right].$$

If we let $c_1 = y''(0)$ and $c_2 = y'''(0)$, then

$$Y(s) = \frac{c_1}{s^3} + \frac{c_2}{s^4} + \frac{2w_0}{EIL}\left[\frac{L/2}{s^5} - \frac{1}{s^6} + \frac{1}{s^6}e^{-Ls/2}\right],$$

and consequently

$$\begin{aligned} y(x) &= \frac{c_1}{2!}\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} + \frac{c_2}{3!}\mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \frac{2w_0}{EIL}\left[\frac{L/2}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} - \frac{1}{5!}\mathscr{L}^{-1}\left\{\frac{5!}{s^6}\right\} + \frac{1}{5!}\mathscr{L}^{-1}\left\{\frac{5!}{s^6}e^{-Ls/2}\right\}\right] \\ &= \frac{c_1}{2}x^2 + \frac{c_2}{6}x^3 + \frac{w_0}{60\,EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5\mathscr{U}\left(x - \frac{L}{2}\right)\right]. \end{aligned}$$

Applying the conditions $y(L) = 0$ and $y'(L) = 0$ to the last result yields a system of equations for c_1 and c_2:

$$\begin{aligned} c_1\frac{L^2}{2} + c_2\frac{L^3}{6} + \frac{49w_0L^4}{1920EI} &= 0 \\ c_1 L + c_2\frac{L^2}{2} + \frac{85w_0L^3}{960EI} &= 0. \end{aligned}$$

Solving, we find $c_1 = 23w_0L^2/(960EI)$ and $c_2 = -9w_0L/(40EI)$. Thus the deflection is given by

$$y(x) = \frac{23w_0L^2}{1920EI}x^2 - \frac{3w_0L}{80EI}x^3 + \frac{w_0}{60EIL}\left[\frac{5L}{2}x^4 - x^5 + \left(x - \frac{L}{2}\right)^5\mathscr{U}\left(x - \frac{L}{2}\right)\right].$$ ■

REMARKS

Outside the discussion of the Laplace transform, the unit step function is defined on the interval $(-\infty, \infty)$, that is,

$$\mathscr{U}(t - a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$$

Using this slight modification of Definition 7.3.1, a special case of (12) when $g(t) = 1$ is sometimes called the **boxcar function** and denoted by

$$\Pi(t) = \mathscr{U}(t - a) - \mathscr{U}(t - b).$$

See Figure 7.3.9.

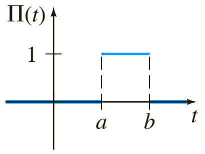


FIGURE 7.3.9 Boxcar function

EXERCISES 7.3

Answers to selected odd-numbered problems begin on page ANS-11.

7.3.1 Translation on the s-Axis

In Problems 1–20 find either $F(s)$ or $f(t)$, as indicated.

1. $\mathscr{L}\{te^{10t}\}$

2. $\mathscr{L}\{te^{-6t}\}$

3. $\mathscr{L}\{t^3e^{-2t}\}$

4. $\mathscr{L}\{t^{10}e^{-7t}\}$

5. $\mathscr{L}\{t(e^t + e^{2t})^2\}$

6. $\mathscr{L}\{e^{2t}(t-1)^2\}$

7. $\mathscr{L}\{e^t \sin 3t\}$

8. $\mathscr{L}\{e^{-2t}\cos 4t\}$

9. $\mathscr{L}\{(1 - e^t + 3e^{-4t})\cos 5t\}$

10. $\mathscr{L}\left\{e^{3t}\left(9 - 4t + 10\sin\dfrac{t}{2}\right)\right\}$

11. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s+2)^3}\right\}$

12. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s-1)^4}\right\}$

13. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 - 6s + 10}\right\}$

14. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + 2s + 5}\right\}$

15. $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 + 4s + 5}\right\}$

16. $\mathscr{L}^{-1}\left\{\dfrac{2s + 5}{s^2 + 6s + 34}\right\}$

17. $\mathscr{L}^{-1}\left\{\dfrac{s}{(s+1)^2}\right\}$

18. $\mathscr{L}^{-1}\left\{\dfrac{5s}{(s-2)^2}\right\}$

19. $\mathscr{L}^{-1}\left\{\dfrac{2s - 1}{s^2(s+1)^3}\right\}$

20. $\mathscr{L}^{-1}\left\{\dfrac{(s+1)^2}{(s+2)^4}\right\}$

In Problems 21–30 use the Laplace transform to solve the given initial-value problem.

21. $y' + 4y = e^{-4t}, \quad y(0) = 2$

22. $y' - y = 1 + te^t, \quad y(0) = 0$

23. $y'' + 2y' + y = 0, \quad y(0) = 1, \quad y'(0) = 1$

24. $y'' - 4y' + 4y = t^3e^{2t}, \quad y(0) = 0, \quad y'(0) = 0$

25. $y'' - 6y' + 9y = t, \quad y(0) = 0, \quad y'(0) = 1$

26. $y'' - 4y' + 4y = t^3, \quad y(0) = 1, \quad y'(0) = 0$

27. $y'' - 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = -3$

28. $2y'' + 20y' + 51y = 0, \quad y(0) = 2, \quad y'(0) = 0$

29. $y'' - y' = e^t \cos t, \quad y(0) = 0, \quad y'(0) = 0$

30. $y'' - 2y' + 5y = 1 + t, \quad y(0) = 0, \quad y'(0) = 4$

In Problems 31 and 32 use the Laplace transform and the procedure outlined in Example 10 to solve the given boundary-value problem.

31. $y'' + 2y' + y = 0, \quad y'(0) = 2, \quad y(1) = 2$

32. $y'' + 8y' + 20y = 0, \quad y(0) = 0, \quad y'(\pi) = 0$

33. A 4-pound weight stretches a spring 2 feet. The weight is released from rest 18 inches above the equilibrium position, and the resulting motion takes place in a medium offering a damping force numerically equal to $\frac{7}{8}$ times the instantaneous velocity. Use the Laplace transform to find the equation of motion $x(t)$.

34. Recall that the differential equation for the instantaneous charge $q(t)$ on the capacitor in an LRC-series circuit is given by

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{20}$$

See Section 5.1. Use the Laplace transform to find $q(t)$ when $L = 1$ h, $R = 20\ \Omega$, $C = 0.005$ f, $E(t) = 150$ V, $t > 0$, $q(0) = 0$, and $i(0) = 0$. What is the current $i(t)$?

35. Consider a battery of constant voltage E_0 that charges the capacitor shown in Figure 7.3.10. Divide equation (20) by L and define $2\lambda = R/L$ and $\omega^2 = 1/LC$. Use the Laplace transform to show that the solution $q(t)$ of $q'' + 2\lambda q' + \omega^2 q = E_0/L$ subject to $q(0) = 0$, $i(0) = 0$ is

$$q(t) = \begin{cases} E_0C\left[1 - e^{-\lambda t}\left(\cosh\sqrt{\lambda^2 - \omega^2}\,t \right.\right. \\ \qquad \left.\left. + \dfrac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\sinh\sqrt{\lambda^2 - \omega^2}\,t\right)\right], & \lambda > \omega, \\ E_0C[1 - e^{-\lambda t}(1 + \lambda t)], & \lambda = \omega, \\ E_0C\left[1 - e^{-\lambda t}\left(\cos\sqrt{\omega^2 - \lambda^2}\,t \right.\right. \\ \qquad \left.\left. + \dfrac{\lambda}{\sqrt{\omega^2 - \lambda^2}}\sin\sqrt{\omega^2 - \lambda^2}\,t\right)\right], & \lambda < \omega. \end{cases}$$

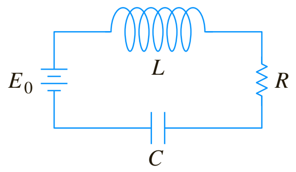


FIGURE 7.3.10 Series circuit in Problem 35

36. Use the Laplace transform to find the charge $q(t)$ in an RC series circuit when $q(0) = 0$ and $E(t) = E_0e^{-kt}$, $k > 0$. Consider two cases: $k \neq 1/RC$ and $k = 1/RC$.

7.3.2 Translation on the t-Axis

In Problems 37–48 find either $F(s)$ or $f(t)$, as indicated.

37. $\mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$

38. $\mathscr{L}\{e^{2-t}\mathscr{U}(t-2)\}$

39. $\mathscr{L}\{t\,\mathscr{U}(t-2)\}$

40. $\mathscr{L}\{(3t+1)\,\mathscr{U}(t-1)\}$

41. $\mathscr{L}\{\cos 2t\,\mathscr{U}(t-\pi)\}$

42. $\mathscr{L}\left\{\sin t\,\mathscr{U}\left(t - \dfrac{\pi}{2}\right)\right\}$

43. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^3}\right\}$

44. $\mathscr{L}^{-1}\left\{\dfrac{(1 + e^{-2s})^2}{s+2}\right\}$

45. $\mathscr{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2 + 1}\right\}$

46. $\mathscr{L}^{-1}\left\{\dfrac{se^{-\pi s/2}}{s^2 + 4}\right\}$

47. $\mathscr{L}^{-1}\left\{\dfrac{e^{-s}}{s(s+1)}\right\}$

48. $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2(s-1)}\right\}$

In Problems 49–54 match the given graph with one of the functions in (a)–(f). The graph of $f(t)$ is given in Figure 7.3.11.

(a) $f(t) - f(t)\,\mathcal{U}(t-a)$

(b) $f(t-b)\,\mathcal{U}(t-b)$

(c) $f(t)\,\mathcal{U}(t-a)$

(d) $f(t) - f(t)\,\mathcal{U}(t-b)$

(e) $f(t)\,\mathcal{U}(t-a) - f(t)\,\mathcal{U}(t-b)$

(f) $f(t-a)\,\mathcal{U}(t-a) - f(t-a)\,\mathcal{U}(t-b)$

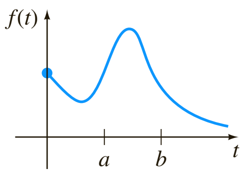


FIGURE 7.3.11 Graph for Problems 49–54

49.

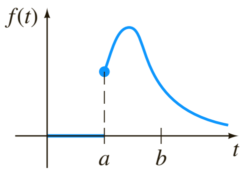


FIGURE 7.3.12 Graph for Problem 49

50.

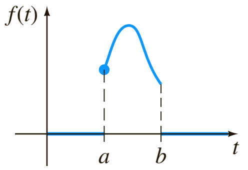


FIGURE 7.3.13 Graph for Problem 50

51.

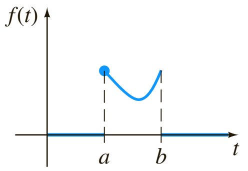


FIGURE 7.3.14 Graph for Problem 51

52.

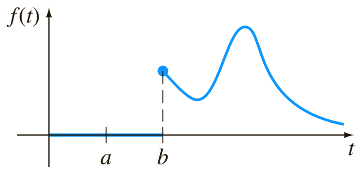


FIGURE 7.3.15 Graph for Problem 52

53.

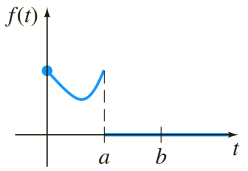


FIGURE 7.3.16 Graph for Problem 53

54.

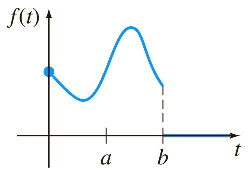


FIGURE 7.3.17 Graph for Problem 54

In Problems 55–62 write each function in terms of unit step functions. Find the Laplace transform of the given function.

55. $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ -2, & t \ge 3 \end{cases}$

56. $f(t) = \begin{cases} 1, & 0 \le t < 4 \\ 0, & 4 \le t < 5 \\ 1, & t \ge 5 \end{cases}$

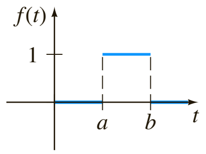
57. $f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & t \ge 1 \end{cases}$

58. $f(t) = \begin{cases} 0, & 0 \le t < 3\pi/2 \\ \sin t, & t \ge 3\pi/2 \end{cases}$

59. $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

60. $f(t) = \begin{cases} \sin t, & 0 \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$

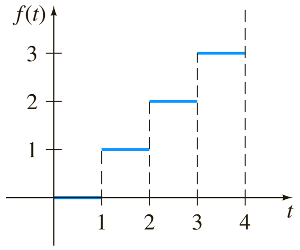
61.



rectangular pulse

FIGURE 7.3.18 Graph for Problem 61

62.



staircase function

FIGURE 7.3.19 Graph for Problem 62

In Problems 63–70 use the Laplace transform to solve the given initial-value problem.

63. $y' + y = f(t), \quad y(0) = 0$, where

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 5, & t \ge 1 \end{cases}$$

64. $y' + y = f(t), \quad y(0) = 0$, where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ -1, & t \ge 1 \end{cases}$$

65. $y' + 2y = f(t), \quad y(0) = 0$, where

$$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

66. $y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = -1$, where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$$

67. $y'' + 4y = \sin t\, \mathcal{U}(t - 2\pi), \quad y(0) = 1, \quad y'(0) = 0$

68. $y'' - 5y' + 6y = \mathcal{U}(t - 1), \quad y(0) = 0, \quad y'(0) = 1$

69. $y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1$, where

$$f(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0, & t \ge 2\pi \end{cases}$$

70. $y'' + 4y' + 3y = 1 - \mathcal{U}(t - 2) - \mathcal{U}(t - 4) + \mathcal{U}(t - 6)$,
$y(0) = 0, \quad y'(0) = 0$

71. Suppose a 32-pound weight stretches a spring 2 feet. If the weight is released from rest at the equilibrium position, find the equation of motion $x(t)$ if an impressed force $f(t) = 20t$ acts on the system for $0 \le t < 5$ and is then removed (see Example 5). Ignore any damping forces. Use a graphing utility to graph $x(t)$ on the interval [0, 10].

72. Solve Problem 71 if the impressed force $f(t) = \sin t$ acts on the system for $0 \le t < 2\pi$ and is then removed.

In Problems 73 and 74 use the Laplace transform to find the charge $q(t)$ on the capacitor in an RC-series circuit subject to the given conditions.

73. $q(0) = 0$, $R = 2.5\ \Omega$, $C = 0.08$ f, $E(t)$ given in Figure 7.3.20

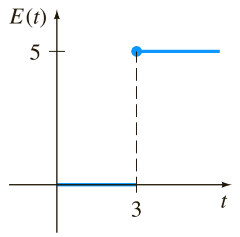


FIGURE 7.3.20 $E(t)$ in Problem 73

74. $q(0) = q_0$, $R = 10\ \Omega$, $C = 0.1$ f, $E(t)$ given in Figure 7.3.21

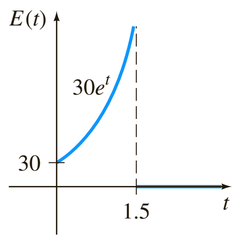


FIGURE 7.3.21 $E(t)$ in Problem 74

75. (a) Use the Laplace transform to find the current $i(t)$ in a single-loop LR-series circuit when $i(0) = 0$, $L = 1$ h, $R = 10\ \Omega$, and $E(t)$ is as given in Figure 7.3.22.

(b) Use a graphing utility to graph $i(t)$ for $0 \le t \le 6$. Use the graph to estimate i_{max} and i_{min}, the maximum and minimum values of the current.

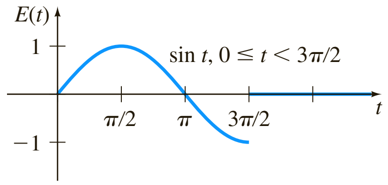


FIGURE 7.3.22 $E(t)$ in Problem 75

76. (a) Use the Laplace transform to find the charge $q(t)$ on the capacitor in an RC-series circuit when $q(0) = 0$, $R = 50\ \Omega$, $C = 0.01$ f, and $E(t)$ is as given in Figure 7.3.23.

(b) Assume that $E_0 = 100$ V. Use a graphing utility to graph $q(t)$ for $0 \le t \le 6$. Use the graph to estimate q_{max}, the maximum value of the charge.

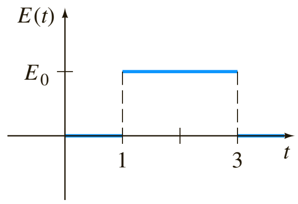


FIGURE 7.3.23 $E(t)$ in Problem 76

77. A cantilever beam is embedded at its left end and free at its right end. Use the Laplace transform to find the deflection $y(x)$ when the load is given by

$$w(x) = \begin{cases} w_0, & 0 < x < L/2 \\ 0, & L/2 \le x < L. \end{cases}$$

78. Solve Problem 77 when the load is given by

$$w(x) = \begin{cases} 0, & 0 < x < L/3 \\ w_0, & L/3 < x < 2L/3 \\ 0, & 2L/3 < x < L. \end{cases}$$

79. Find the deflection $y(x)$ of a cantilever beam embedded at its left end and free at its right end when the load is as given in Example 10.

80. A beam is embedded at its left end and simply supported at its right end. Find the deflection $y(x)$ when the load is as given in Problem 77.

Mathematical Model

81. Cake Inside an Oven Reread Example 4 in Section 3.1 on the cooling of a cake that is taken out of an oven.

(a) Devise a mathematical model for the temperature of a cake while it is *inside* the oven based on the following assumptions: At $t = 0$ the cake mixture is at the room temperature of 70°; the oven is not preheated, so at $t = 0$, when the cake mixture is placed into the oven, the temperature inside the oven is also 70°; the temperature of the oven increases linearly until $t = 4$ minutes, when the desired temperature of 300° is attained; the oven temperature is a constant 300° for $t \geq 4$.

(b) Use the Laplace transform to solve the initial-value problem in part (a).

Discussion Problems

82. Discuss how you would fix up each of the following functions so that Theorem 7.3.2 could be used directly to find the given Laplace transform. Check your answers using (16) of this section.

(a) $\mathscr{L}\{(2t + 1)\mathscr{U}(t - 1)\}$ **(b)** $\mathscr{L}\{e^t\mathscr{U}(t - 5)\}$

(c) $\mathscr{L}\{\cos t\,\mathscr{U}(t - \pi)\}$ **(d)** $\mathscr{L}\{(t^2 - 3t)\mathscr{U}(t - 2)\}$

83. (a) Assume that Theorem 7.3.1 holds when the symbol a is replaced by ki, where k is a real number and $i^2 = -1$. Show that $\mathscr{L}\{te^{kti}\}$ can be used to deduce

$$\mathscr{L}\{t\cos kt\} = \frac{s^2 - k^2}{(s^2 + k^2)^2}$$

$$\mathscr{L}\{t\sin kt\} = \frac{2ks}{(s^2 + k^2)^2}.$$

(b) Now use the Laplace transform to solve the initial-value problem $x'' + \omega^2 x = \cos \omega t$, $x(0) = 0$, $x'(0) = 0$.

7.4 Operational Properties II

INTRODUCTION In this section we develop several more operational properties of the Laplace transform. Specifically, we shall see how to find the transform of a function $f(t)$ that is multiplied by a monomial t^n, the transform of a special type of integral, and the transform of a periodic function. The last two transform properties allow us to solve some equations that we have not encountered up to this point: Volterra integral equations, integrodifferential equations, and ordinary differential equations in which the input function is a periodic piecewise-defined function.

7.4.1 DERIVATIVES OF A TRANSFORM

MULTIPLYING A FUNCTION BY t^n The Laplace transform of the product of a function $f(t)$ with t can be found by differentiating the Laplace transform of $f(t)$. To motivate this result, let us assume that $F(s) = \mathscr{L}\{f(t)\}$ exists and that it is possible to interchange the order of differentiation and integration. Then

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\,dt = -\int_0^\infty e^{-st}\,tf(t)\,dt = -\mathscr{L}\{tf(t)\};$$

that is,
$$\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\mathscr{L}\{f(t)\}.$$

We can use the last result to find the Laplace transform of $t^2f(t)$:

$$\mathscr{L}\{t^2f(t)\} = \mathscr{L}\{t\cdot tf(t)\} = -\frac{d}{ds}\mathscr{L}\{tf(t)\} = -\frac{d}{ds}\left(-\frac{d}{ds}\mathscr{L}\{f(t)\}\right) = \frac{d^2}{ds^2}\mathscr{L}\{f(t)\}.$$

The preceding two cases suggest the general result for $\mathscr{L}\{t^n f(t)\}$.

THEOREM 7.4.1 Derivatives of Transforms

If $F(s) = \mathscr{L}\{f(t)\}$ and $n = 1, 2, 3, \ldots$, then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s).$$

EXAMPLE 1 Using Theorem 7.4.1

Evaluate $\mathscr{L}\{t \sin kt\}$.

SOLUTION With $f(t) = \sin kt$, $F(s) = k/(s^2 + k^2)$, and $n = 1$, Theorem 7.4.1 gives

$$\mathscr{L}\{t \sin kt\} = -\frac{d}{ds}\mathscr{L}\{\sin kt\} = -\frac{d}{ds}\left(\frac{k}{s^2 + k^2}\right) = \frac{2ks}{(s^2 + k^2)^2}.$$

If we want to evaluate $\mathscr{L}\{t^2 \sin kt\}$ and $\mathscr{L}\{t^3 \sin kt\}$, all we need do, in turn, is take the negative of the derivative with respect to s of the result in Example 1 and then take the negative of the derivative with respect to s of $\mathscr{L}\{t^2 \sin kt\}$.

NOTE To find transforms of functions $t^n e^{at}$ we can use either Theorem 7.3.1 or Theorem 7.4.1. For example,

$$\text{Theorem 7.3.1:} \quad \mathscr{L}\{te^{3t}\} = \mathscr{L}\{t\}_{s \to s-3} = \left.\frac{1}{s^2}\right|_{s \to s-3} = \frac{1}{(s-3)^2}.$$

$$\text{Theorem 7.4.1:} \quad \mathscr{L}\{te^{3t}\} = -\frac{d}{ds}\mathscr{L}\{e^{3t}\} = -\frac{d}{ds}\frac{1}{s-3} = (s-3)^{-2} = \frac{1}{(s-3)^2}.$$

EXAMPLE 2 An Initial-Value Problem

Solve $x'' + 16x = \cos 4t, \quad x(0) = 0, \quad x'(0) = 1$.

SOLUTION The initial-value problem could describe the forced, undamped, and resonant motion of a mass on a spring. The mass starts with an initial velocity of 1 ft/s in the downward direction from the equilibrium position.

Transforming the differential equation gives

$$(s^2 + 16)X(s) = 1 + \frac{s}{s^2 + 16} \qquad \text{or} \qquad X(s) = \frac{1}{s^2 + 16} + \frac{s}{(s^2 + 16)^2}.$$

Now we just saw in Example 1 that

$$\mathscr{L}^{-1}\left\{\frac{2ks}{(s^2 + k^2)^2}\right\} = t \sin kt, \tag{1}$$

and so with the identification $k = 4$ in (1) and in part (d) of Theorem 7.2.1, we obtain

$$\begin{aligned} x(t) &= \frac{1}{4}\mathscr{L}^{-1}\left\{\frac{4}{s^2 + 16}\right\} + \frac{1}{8}\mathscr{L}^{-1}\left\{\frac{8s}{(s^2 + 16)^2}\right\} \\ &= \frac{1}{4}\sin 4t + \frac{1}{8}t \sin 4t. \end{aligned}$$

7.4.2 TRANSFORMS OF INTEGRALS

CONVOLUTION If functions f and g are piecewise continuous on the interval $[0, \infty)$, then the **convolution** of f and g, denoted by the symbol $f * g$, is a function defined by the integral

$$f * g = \int_0^t f(\tau)g(t - \tau)\,d\tau. \tag{2}$$

Because we are integrating in (2) with respect to the variable τ (the lower case Greek letter *tau*), the convolution $f * g$ is a function of t. To emphasize this fact, (2) is also written $(f * g)(t)$. As the notation $f * g$ suggests, the convolution (2) is often interpreted as a *generalized product* of two functions f and g.

EXAMPLE 3 Convolution of Two Functions

Evaluate **(a)** $e^t * \sin t$ **(b)** $\mathscr{L}\{e^t * \sin t\}$.

SOLUTION **(a)** With the identifications

$$f(t) = e^t,\ g(t) = \sin t \qquad \text{and} \qquad f(\tau) = e^\tau,\ g(t-\tau) = \sin(t-\tau),$$

it follows from (2) and integration by parts that

$$\begin{aligned} e^t * \sin t &= \int_0^t e^\tau \sin(t-\tau)\, d\tau \\ &= \frac{1}{2}[e^\tau \sin(t-\tau) + e^\tau \cos(t-\tau)]_0^t \\ &= \frac{1}{2}(-\sin t - \cos t + e^t). \end{aligned} \tag{3}$$

(b) Then from (3) and parts (c), (d), and (e) of Theorem 7.1.1 we find

$$\begin{aligned} \mathscr{L}\{e^t * \sin t\} &= -\frac{1}{2}\mathscr{L}\{\sin t\} - \frac{1}{2}\mathscr{L}\{\cos t\} + \frac{1}{2}\mathscr{L}\{e^t\} \\ &= -\frac{1}{2}\frac{1}{s^2+1} - \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s-1} \\ &= \frac{1}{(s-1)(s^2+1)}. \end{aligned}$$

It is left as an exercise to show that

$$\int_0^t f(\tau)g(t-\tau)\, d\tau = \int_0^t f(t-\tau)g(\tau)\, d\tau,$$

that is, $f * g = g * f$. In other words, the convolution of two functions is commutative.

CONVOLUTION THEOREM We have seen that if f and g are both piecewise continuous for $t \geq 0$, then the Laplace transform of a sum $f + g$ is the sum of the individual Laplace transforms. While it is *not* true that the Laplace transform of the product fg is the product of the Laplace transforms, we see in the next theorem—called the **convolution theorem**—that the Laplace transform of the generalized product $f * g$ is the product of the Laplace transforms of f and g.

THEOREM 7.4.2 Convolution Theorem

If $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)G(s).$$

PROOF Let
$$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

and
$$G(s) = \mathscr{L}\{g(t)\} = \int_0^\infty e^{-s\beta} g(\beta)\, d\beta.$$

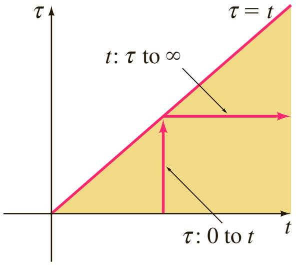


FIGURE 7.4.1 Changing order of integration from t first to τ first

Proceeding formally, we have

$$\begin{aligned} F(s)G(s) &= \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau\right)\left(\int_0^\infty e^{-s\beta} g(\beta)\, d\beta\right) \\ &= \int_0^\infty \int_0^\infty e^{-s(\tau+\beta)} f(\tau) g(\beta)\, d\tau\, d\beta \\ &= \int_0^\infty f(\tau)\, d\tau \int_0^\infty e^{-s(\tau+\beta)} g(\beta)\, d\beta. \end{aligned}$$

Holding τ fixed, we let $t = \tau + \beta$, $dt = d\beta$, so that

$$F(s)G(s) = \int_0^\infty f(\tau)\, d\tau \int_\tau^\infty e^{-st} g(t - \tau)\, dt.$$

In the $t\tau$-plane we are integrating over the shaded region in Figure 7.4.1. Since f and g are piecewise continuous on $[0, \infty)$ and of exponential order, it is possible to interchange the order of integration:

$$F(s)\, G(s) = \int_0^\infty e^{-st} dt \int_0^t f(\tau) g(t - \tau)\, d\tau = \int_0^\infty e^{-st} \left\{\int_0^t f(\tau) g(t - \tau)\, d\tau\right\} dt = \mathscr{L}\{f * g\}. \quad \blacksquare$$

Theorem 7.4.2 shows that we can find the Laplace transform of the convolution $f * g$ of two functions without actually evaluating the definite integral $\int_0^t f(\tau) g(t - \tau)\, d\tau$ as we did in (3). The next example illustrates the idea.

EXAMPLE 4 Using Theorem 7.4.2

Evaluate $\mathscr{L}\left\{\int_0^t e^\tau \sin(t - \tau)\, d\tau\right\}$.

SOLUTION This is the same as the transform $\mathscr{L}\{e^t * \sin t\}$ that we found in part (b) of Example 3. This time we use Theorem 7.4.2 that the Laplace transform of the convolution of f and g is the product of their Laplace transforms:

$$\begin{aligned} \mathscr{L}\left\{\int_0^t e^\tau \sin(t - \tau)\, d\tau\right\} &= \mathscr{L}\{e^t * \sin t\} \\ &= \mathscr{L}\{e^t\} \cdot \mathscr{L}\{\sin t\} \\ &= \frac{1}{s-1} \cdot \frac{1}{s^2+1} \\ &= \frac{1}{(s-1)(s^2+1)}. \quad \blacksquare \end{aligned}$$

INVERSE FORM OF THEOREM 7.4.2 The convolution theorem is sometimes useful in finding the inverse Laplace transform of the product of two Laplace transforms. From Theorem 7.4.2 we have

$$\mathscr{L}^{-1}\{F(s)G(s)\} = f * g. \tag{4}$$

Many of the results in the table of Laplace transforms in Appendix C can be derived using (4). For example, in the next example we obtain entry 25 of the table:

$$\mathscr{L}\{\sin kt - kt \cos kt\} = \frac{2k^3}{(s^2 + k^2)^2}. \tag{5}$$

EXAMPLE 5 Inverse Transform as a Convolution

Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{(s^2 + k^2)^2}\right\}$.

SOLUTION Let $F(s) = G(s) = \dfrac{1}{s^2 + k^2}$ so that

$$f(t) = g(t) = \frac{1}{k} \mathscr{L}^{-1}\left\{\frac{k}{s^2 + k^2}\right\} = \frac{1}{k} \sin kt.$$

In this case (4) gives

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{k^2}\int_0^t \sin k\tau \sin k(t-\tau)\,d\tau. \tag{6}$$

With the aid of the product-to-sum trigonometric identity

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

and the substitutions $A = k\tau$ and $B = k(t-\tau)$ we can carry out the integration in (6):

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} &= \frac{1}{2k^2}\int_0^t [\cos k(2\tau - t) - \cos kt]\,d\tau \\ &= \frac{1}{2k^2}\left[\frac{1}{2k}\sin k(2\tau - t) - \tau\cos kt\right]_0^t \\ &= \frac{\sin kt - kt\cos kt}{2k^3}.\end{aligned}$$

Multiplying both sides by $2k^3$ gives the inverse form of (5). ■

TRANSFORM OF AN INTEGRAL When $g(t) = 1$ and $\mathscr{L}\{g(t)\} = G(s) = 1/s$, the convolution theorem implies that the Laplace transform of the integral of f is

$$\mathscr{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}. \tag{7}$$

The inverse form of (7),

$$\int_0^t f(\tau)\,d\tau = \mathscr{L}^{-1}\left\{\frac{F(s)}{s}\right\}, \tag{8}$$

can be used in lieu of partial fractions when s^n is a factor of the denominator and $f(t) = \mathscr{L}^{-1}\{F(s)\}$ is easy to integrate. For example, we know for $f(t) = \sin t$ that $F(s) = 1/(s^2+1)$, and so by (8)

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1/(s^2+1)}{s}\right\} = \int_0^t \sin\tau\,d\tau = 1 - \cos t \\ \mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1/s(s^2+1)}{s}\right\} = \int_0^t (1-\cos\tau)\,d\tau = t - \sin t \\ \mathscr{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1/s^2(s^2+1)}{s}\right\} = \int_0^t (\tau - \sin\tau)\,d\tau = \tfrac{1}{2}t^2 - 1 + \cos t\end{aligned}$$

and so on.

VOLTERRA INTEGRAL EQUATION The convolution theorem and the result in (7) are useful in solving other types of equations in which an unknown function appears under an integral sign. In the next example we solve a **Volterra integral equation** for $f(t)$,

$$f(t) = g(t) + \int_0^t f(\tau)\,h(t-\tau)\,d\tau. \tag{9}$$

The functions $g(t)$ and $h(t)$ are known. Notice that the integral in (9) has the convolution form (2) with the symbol h playing the part of g.

EXAMPLE 6 An Integral Equation

Solve $f(t) = 3t^2 - e^{-t} - \displaystyle\int_0^t f(\tau)\,e^{t-\tau}\,d\tau$ for $f(t)$.

SOLUTION In the integral we identify $h(t-\tau) = e^{t-\tau}$ so that $h(t) = e^t$. We take the Laplace transform of each term; in particular, by Theorem 7.4.2 the transform of the integral is the product of $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{e^t\} = 1/(s-1)$:

$$F(s) = 3 \cdot \frac{2}{s^3} - \frac{1}{s+1} - F(s) \cdot \frac{1}{s-1}.$$

After solving the last equation for $F(s)$ and carrying out the partial fraction decomposition, we find

$$F(s) = \frac{6}{s^3} - \frac{6}{s^4} + \frac{1}{s} - \frac{2}{s+1}.$$

The inverse transform then gives

$$\begin{aligned} f(t) &= 3\mathscr{L}^{-1}\left\{\frac{2!}{s^3}\right\} - \mathscr{L}^{-1}\left\{\frac{3!}{s^4}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} \\ &= 3t^2 - t^3 + 1 - 2e^{-t}. \end{aligned}$$

SERIES CIRCUITS In a single-loop or series circuit, Kirchhoff's second law states that the sum of the voltage drops across an inductor, resistor, and capacitor is equal to the impressed voltage $E(t)$. Now it is known that the voltage drops across an inductor, resistor, and capacitor are, respectively,

$$L\frac{di}{dt}, \qquad R\,i(t), \quad \text{and} \quad \frac{1}{C}\int_0^t i(\tau)\,d\tau,$$

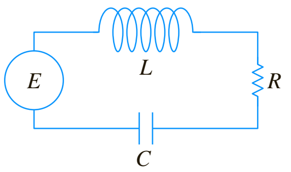


FIGURE 7.4.2 *LRC*-series circuit

where $i(t)$ is the current and L, R, and C are constants. It follows that the current in a circuit, such as that shown in Figure 7.4.2, is governed by the **integrodifferential equation**

$$L\frac{di}{dt} + R\,i(t) + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t). \tag{10}$$

EXAMPLE 7 An Integrodifferential Equation

Determine the current $i(t)$ in a single-loop *LRC*-series circuit when $L = 0.1$ h, $R = 2\ \Omega$, $C = 0.1$ f, $i(0) = 0$, and the impressed voltage is

$$E(t) = 120t - 120t\,\mathscr{U}(t-1).$$

SOLUTION With the given data equation (10) becomes

$$0.1\frac{di}{dt} + 2i + 10\int_0^t i(\tau)\,d\tau = 120t - 120t\,\mathscr{U}(t-1).$$

Now by (7), $\mathscr{L}\left\{\int_0^t i(\tau)\,d\tau\right\} = I(s)/s$, where $I(s) = \mathscr{L}\{i(t)\}$. Thus the Laplace transform of the integrodifferential equation is

$$0.1sI(s) + 2I(s) + 10\frac{I(s)}{s} = 120\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}\right]. \qquad \leftarrow \text{by (16) of Section 7.3}$$

Multiplying this equation by $10s$, using $s^2 + 20s + 100 = (s+10)^2$, and then solving for $I(s)$ gives

$$I(s) = 1200\left[\frac{1}{s(s+10)^2} - \frac{1}{s(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\right].$$

By partial fractions,

$$\begin{aligned} I(s) = 1200\bigg[&\frac{1/100}{s} - \frac{1/100}{s+10} - \frac{1/10}{(s+10)^2} - \frac{1/100}{s}e^{-s} \\ &+ \frac{1/100}{s+10}e^{-s} + \frac{1/10}{(s+10)^2}e^{-s} - \frac{1}{(s+10)^2}e^{-s}\bigg]. \end{aligned}$$

From the inverse form of the second translation theorem, (15) of Section 7.3, we finally obtain

$$i(t) = 12[1 - \mathscr{U}(t-1)] - 12[e^{-10t} - e^{-10(t-1)}\mathscr{U}(t-1)]$$
$$- 120te^{-10t} - 1080(t-1)e^{-10(t-1)}\mathscr{U}(t-1).$$

Written as a piecewise-defined function, the current is

$$i(t) = \begin{cases} 12 - 12e^{-10t} - 120te^{-10t}, & 0 \le t < 1 \\ -12e^{-10t} + 12e^{-10(t-1)} - 120te^{-10t} - 1080(t-1)e^{-10(t-1)}, & t \ge 1. \end{cases}$$

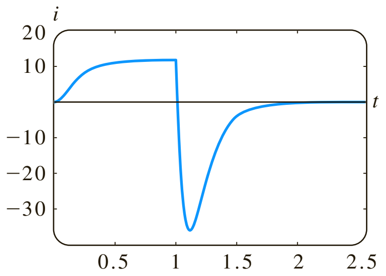


FIGURE 7.4.3 Graph of current $i(t)$ in Example 7

Using this last expression and a CAS, we graph $i(t)$ on each of the two intervals and then combine the graphs. Note in Figure 7.4.3 that even though the input $E(t)$ is discontinuous, the output or response $i(t)$ is a continuous function. ■

Optional material if Section 4.8 was covered.

POST SCRIPT—GREEN'S FUNCTIONS REDUX By applying the Laplace transform to the initial-value problem

$$y'' + ay' + by = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where a and b are constants, we find that the transform of $y(t)$ is

$$Y(s) = \frac{F(s)}{s^2 + as + b},$$

where $F(s) = \mathscr{L}\{f(t)\}$. By rewriting the foregoing transform as the product

$$Y(s) = \frac{1}{s^2 + as + b}F(s)$$

we can use the inverse form of the convolution theorem (4) to write the solution of the IVP as

$$y(t) = \int_0^t g(t - \tau) f(\tau)\, d\tau, \tag{11}$$

where $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + as + b}\right\} = g(t)$ and $\mathscr{L}^{-1}\{F(s)\} = f(t)$. On the other hand, we know from (10) of Section 4.8 that the solution of the IVP is also given by

$$y(t) = \int_0^t G(t, \tau) f(\tau)\, d\tau, \tag{12}$$

where $G(t, \tau)$ is the Green's function for the differential equation.

By comparing (11) and (12) we see that the Green's function for the differential equation is related to $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 + as + b}\right\} = g(t)$ by

$$G(t, \tau) = g(t - \tau). \tag{13}$$

For example, for the initial-value problem $y'' + 4y = f(t)$, $y(0) = 0$, $y'(0) = 0$ we find

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \tfrac{1}{2}\sin 2t = g(t).$$

In Example 4 of Section 4.8, the roles of the symbols x and t are played by t and τ in this discussion.

Thus from (13) we see that the Green's function for the DE $y'' + 4y = f(t)$ is $G(t, \tau) = g(t - \tau) = \frac{1}{2}\sin 2(t - \tau)$. See Example 4 in Section 4.8.

REMARKS

Although the Laplace transform was known for a long time prior to the twentieth century, it was not used to solve differential equations. The fact that we use the Laplace transform today to solve a variety of equations is due to Oliver Heaviside (see page 298). In 1893 Heaviside invented an operational calculus for solving differential equations encountered in electrical engineering. Heaviside was no mathematician, and his procedures for solving differential equations were formal manipulations or procedures lacking mathematical justification. Nonetheless these procedures worked. In an attempt to put his operational calculus on a sound foundation, mathematicians discovered that the rules of his calculus matched many properties of the Laplace transform. Over time, Heaviside's operation calculus disappeared to be replaced by the theory and applications of the Laplace transform.

You should verify either by substitution in the equation or by the methods of Section 2.3 that $y(t) = e^{-t}\int_0^t e^{u^2+u}\,du$ is a perfectly good solution of the linear initial-value problem $y' + y = e^{t^2}$, $y(0) = 0$. We now solve the same equation with a formal application of the Laplace transform. If we denote $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{e^{t^2}\} = F(s)$, then the transform of the equation is

$$sY(s) - y(0) + Y(s) = F(s) \quad \text{or} \quad Y(s) = \frac{F(s)}{s+1}.$$

Using $\mathscr{L}^{-1}\{F(s)\} = e^{t^2}$ and $\mathscr{L}^{-1}\left\{\dfrac{1}{s+1}\right\} = e^{-t}$ it follows from the inverse form (4) of the convolution theorem that the solution of the initial-value problem is

$$y(t) = \mathscr{L}^{-1}\left\{F(s)\cdot\frac{1}{s+1}\right\} = \int_0^t e^{\tau^2}\cdot e^{-(t-\tau)}\,d\tau = e^{-t}\int_0^t e^{\tau^2+\tau}\,d\tau.$$

With τ playing the part of u, this is the solution as first given. What's wrong here?

7.4.3 TRANSFORM OF A PERIODIC FUNCTION

PERIODIC FUNCTION If a periodic function has period T, $T > 0$, then $f(t + T) = f(t)$. The next theorem shows that the Laplace transform of a periodic function can be obtained by integration over one period.

THEOREM 7.4.3 Transform of a Periodic Function

If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt.$$

PROOF Write the Laplace transform of f as two integrals:

$$\mathscr{L}\{f(t)\} = \int_0^T e^{-st}f(t)\,dt + \int_T^\infty e^{-st}f(t)\,dt.$$

When we let $t = u + T$, the last integral becomes

$$\int_T^\infty e^{-st}f(t)\,dt = \int_0^\infty e^{-s(u+T)}f(u+T)\,du = e^{-sT}\int_0^\infty e^{-su}f(u)\,du = e^{-sT}\mathscr{L}\{f(t)\}.$$

Therefore $$\mathscr{L}\{f(t)\} = \int_0^T e^{-st} f(t)\, dt + e^{-sT}\mathscr{L}\{f(t)\}.$$

Solving the equation in the last line for $\mathscr{L}\{f(t)\}$ proves the theorem. ■

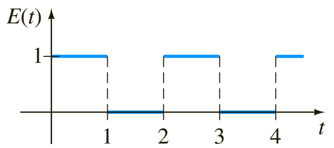


FIGURE 7.4.4 Square wave in Example 8

EXAMPLE 8 Transform of a Periodic Function

Find the Laplace transform of the periodic function shown in Figure 7.4.4.

SOLUTION The function $E(t)$ is called a square wave and has period $T = 2$. For $0 \le t < 2$, $E(t)$ can be defined by

$$E(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$$

and outside the interval by $E(t + 2) = E(t)$. Now from Theorem 7.4.3

$$\begin{aligned}\mathscr{L}\{E(t)\} &= \frac{1}{1 - e^{-2s}} \int_0^2 e^{-st} E(t)\, dt = \frac{1}{1 - e^{-2s}} \left[\int_0^1 e^{-st} \cdot 1\, dt + \int_1^2 e^{-st} \cdot 0\, dt \right] \\ &= \frac{1}{1 - e^{-2s}} \frac{1 - e^{-s}}{s} \qquad \leftarrow 1 - e^{-2s} = (1 + e^{-s})(1 - e^{-s}) \\ &= \frac{1}{s(1 + e^{-s})}. \end{aligned} \tag{14}$$ ■

EXAMPLE 9 A Periodic Impressed Voltage

The differential equation for the current $i(t)$ in a single-loop LR-series circuit is

$$L\frac{di}{dt} + Ri = E(t). \tag{15}$$

Determine the current $i(t)$ when $i(0) = 0$ and $E(t)$ is the square wave function given in Figure 7.4.4.

SOLUTION If we use the result in (14) of the preceding example, the Laplace transform of the DE is

$$LsI(s) + RI(s) = \frac{1}{s(1 + e^{-s})} \quad \text{or} \quad I(s) = \frac{1/L}{s(s + R/L)} \cdot \frac{1}{1 + e^{-s}}. \tag{16}$$

To find the inverse Laplace transform of the last function, we first make use of geometric series. With the identification $x = e^{-s}$, $s > 0$, the geometric series

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots \quad \text{becomes} \quad \frac{1}{1 + e^{-s}} = 1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots.$$

From $$\frac{1}{s(s + R/L)} = \frac{L/R}{s} - \frac{L/R}{s + R/L}$$

we can then rewrite (16) as

$$\begin{aligned} I(s) &= \frac{1}{R}\left(\frac{1}{s} - \frac{1}{s + R/L}\right)(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots) \\ &= \frac{1}{R}\left(\frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \cdots\right) - \frac{1}{R}\left(\frac{1}{s + R/L} - \frac{1}{s + R/L}e^{-s} + \frac{e^{-2s}}{s + R/L} - \frac{e^{-3s}}{s + R/L} + \cdots\right). \end{aligned}$$

By applying the form of the second translation theorem to each term of both series, we obtain

$$i(t) = \frac{1}{R}(1 - \mathscr{U}(t-1) + \mathscr{U}(t-2) - \mathscr{U}(t-3) + \cdots)$$
$$-\frac{1}{R}(e^{-Rt/L} - e^{-R(t-1)/L}\,\mathscr{U}(t-1) + e^{-R(t-2)/L}\,\mathscr{U}(t-2) - e^{-R(t-3)/L}\,\mathscr{U}(t-3) + \cdots)$$

or, equivalently,

$$i(t) = \frac{1}{R}(1 - e^{-Rt/L}) + \frac{1}{R}\sum_{n=1}^{\infty}(-1)^n(1 - e^{-R(t-n)/L})\,\mathscr{U}(t-n).$$

To interpret the solution, let us assume for the sake of illustration that $R = 1$, $L = 1$, and $0 \le t < 4$. In this case

$$i(t) = 1 - e^{-t} - (1 - e^{t-1})\,\mathscr{U}(t-1) + (1 - e^{-(t-2)})\,\mathscr{U}(t-2) - (1 - e^{-(t-3)})\,\mathscr{U}(t-3);$$

in other words,

$$i(t) = \begin{cases} 1 - e^{-t}, & 0 \le t < 1 \\ -e^{-t} + e^{-(t-1)}, & 1 \le t < 2 \\ 1 - e^{-t} + e^{-(t-1)} - e^{-(t-2)}, & 2 \le t < 3 \\ -e^{-t} + e^{-(t-1)} - e^{-(t-2)} + e^{-(t-3)}, & 3 \le t < 4. \end{cases}$$

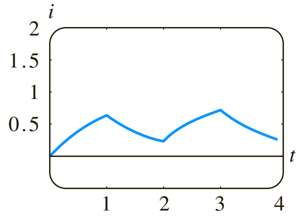


FIGURE 7.4.5 Graph of current $i(t)$ in Example 9

The graph of $i(t)$ for $0 \le t < 4$, given in Figure 7.4.5, was obtained with the help of a CAS. ■

EXERCISES 7.4

Answers to selected odd-numbered problems begin on page ANS-12.

7.4.1 Derivatives of a Transform

In Problems 1–8 use Theorem 7.4.1 to evaluate the given Laplace transform.

1. $\mathscr{L}\{te^{-10t}\}$

2. $\mathscr{L}\{t^3e^t\}$

3. $\mathscr{L}\{t\cos 2t\}$

4. $\mathscr{L}\{t\sinh 3t\}$

5. $\mathscr{L}\{t^2\sinh t\}$

6. $\mathscr{L}\{t^2\cos t\}$

7. $\mathscr{L}\{te^{2t}\sin 6t\}$

8. $\mathscr{L}\{te^{-3t}\cos 3t\}$

In Problems 9–14 use the Laplace transform to solve the given initial-value problem. Use the table of Laplace transforms in Appendix C as needed.

9. $y' + y = t\sin t, \quad y(0) = 0$

10. $y' - y = te^t\sin t, \quad y(0) = 0$

11. $y'' + 9y = \cos 3t, \quad y(0) = 2, \quad y'(0) = 5$

12. $y'' + y = \sin t, \quad y(0) = 1, \quad y'(0) = -1$

13. $y'' + 16y = f(t), \quad y(0) = 0, \quad y'(0) = 1$, where

$$f(t) = \begin{cases} \cos 4t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$$

14. $y'' + y = f(t), \quad y(0) = 1, \quad y'(0) = 0$, where

$$f(t) = \begin{cases} 1, & 0 \le t < \pi/2 \\ \sin t, & t \ge \pi/2 \end{cases}$$

In Problems 15 and 16 use a graphing utility to graph the indicated solution.

15. $y(t)$ of Problem 13 for $0 \le t < 2\pi$

16. $y(t)$ of Problem 14 for $0 \le t < 3\pi$

In some instances the Laplace transform can be used to solve linear differential equations with variable monomial coefficients. In Problems 17 and 18 use Theorem 7.4.1 to reduce the given differential equation to a linear first-order DE in the transformed function $Y(s) = \mathscr{L}\{y(t)\}$. Solve the first-order DE for $Y(s)$ and then find $y(t) = \mathscr{L}^{-1}\{Y(s)\}$.

17. $ty'' - y' = 2t^2, \quad y(0) = 0$

18. $2y'' + ty' - 2y = 10, \quad y(0) = y'(0) = 0$

7.4.2 Transforms of Integrals

In Problems 19–22 proceed as in Example 3 and find the convolution $f * g$ of the given functions. After integrating find the Laplace transform of $f * g$.

19. $f(t) = 4t, \quad g(t) = 3t^2$

20. $f(t) = t, \quad g(t) = e^{-t}$

21. $f(t) = e^{-t}, \quad g(t) = e^t$

22. $f(t) = \cos 2t, \quad g(t) = e^t$

In Problems 23–34 proceed as in Example 4 and find the Laplace transform of $f * g$ using Theorem 7.4.2. Do not evaluate the convolution integral before transforming.

23. $\mathscr{L}\{1 * t^3\}$

24. $\mathscr{L}\{t^2 * te^t\}$

25. $\mathscr{L}\{e^{-t} * e^t\cos t\}$

26. $\mathscr{L}\{e^{2t} * \sin t\}$

27. $\mathscr{L}\left\{\int_0^t e^{\tau}\, d\tau\right\}$

28. $\mathscr{L}\left\{\int_0^t \cos \tau\, d\tau\right\}$

29. $\mathscr{L}\left\{\int_0^t e^{-\tau} \cos \tau\, d\tau\right\}$

30. $\mathscr{L}\left\{\int_0^t \tau \sin \tau\, d\tau\right\}$

31. $\mathscr{L}\left\{\int_0^t \tau\, e^{t-\tau}\, d\tau\right\}$

32. $\mathscr{L}\left\{\int_0^t \sin \tau \cos (t-\tau)\, d\tau\right\}$

33. $\mathscr{L}\left\{t \int_0^t \sin \tau\, d\tau\right\}$

34. $\mathscr{L}\left\{t \int_0^t \tau\, e^{-\tau}\, d\tau\right\}$

In Problems 35–38 use (8) to evaluate the given inverse transform.

35. $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s-1)}\right\}$

36. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2(s-1)}\right\}$

37. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^3(s-1)}\right\}$

38. $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s-a)^2}\right\}$

39. The table in Appendix C does not contain an entry for

$$\mathscr{L}^{-1}\left\{\frac{8k^3 s}{(s^2+k^2)^3}\right\}.$$

(a) Use (4) along with the results in (5) to evaluate this inverse transform. Use a CAS as an aid in evaluating the convolution integral.

(b) Reexamine your answer to part (a). Could you have obtained the result in a different manner?

40. Use the Laplace transform and the results of Problem 39 to solve the initial-value problem

$$y'' + y = \sin t + t \sin t, \quad y(0) = 0, \quad y'(0) = 0.$$

Use a graphing utility to graph the solution.

In Problems 41–50 use the Laplace transform to solve the given integral equation or integrodifferential equation.

41. $f(t) + \int_0^t (t-\tau) f(\tau)\, d\tau = t$

42. $f(t) = 2t - 4\int_0^t \sin \tau\, f(t-\tau)\, d\tau$

43. $f(t) = te^t + \int_0^t \tau\, f(t-\tau)\, d\tau$

44. $f(t) + 2\int_0^t f(\tau) \cos (t-\tau)\, d\tau = 4e^{-t} + \sin t$

45. $f(t) + \int_0^t f(\tau)\, d\tau = 1$

46. $f(t) = \cos t + \int_0^t e^{-\tau} f(t-\tau)\, d\tau$

47. $f(t) = 1 + t - \dfrac{8}{3}\int_0^t (\tau - t)^3 f(\tau)\, d\tau$

48. $t - 2f(t) = \int_0^t (e^{\tau} - e^{-\tau}) f(t-\tau)\, d\tau$

49. $y'(t) = 1 - \sin t - \int_0^t y(\tau)\, d\tau, \quad y(0) = 0$

50. $\dfrac{dy}{dt} + 6y(t) + 9\int_0^t y(\tau)\, d\tau = 1, \quad y(0) = 0$

In Problems 51 and 52 solve equation (10) subject to $i(0) = 0$ with L, R, C, and $E(t)$ as given. Use a graphing utility to graph the solution for $0 \le t \le 3$.

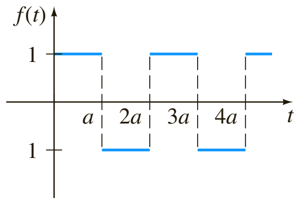
51. $L = 0.1$ h, $R = 3\ \Omega$, $C = 0.05$ f,
$E(t) = 100[\mathscr{U}(t-1) - \mathscr{U}(t-2)]$

52. $L = 0.005$ h, $R = 1\ \Omega$, $C = 0.02$ f,
$E(t) = 100[t - (t-1)\mathscr{U}(t-1)]$

7.4.3 Transform of a Periodic Function

In Problems 53–58 use Theorem 7.4.3 to find the Laplace transform of the given periodic function.

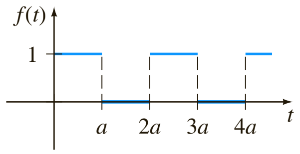
53.



meander function

FIGURE 7.4.6 Graph for Problem 53

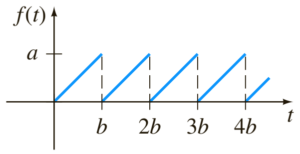
54.



square wave

FIGURE 7.4.7 Graph for Problem 54

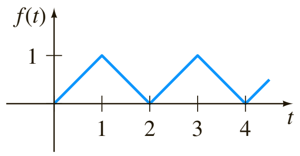
55.



sawtooth function

FIGURE 7.4.8 Graph for Problem 55

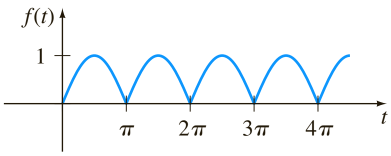
56.



triangular wave

FIGURE 7.4.9 Graph for Problem 56

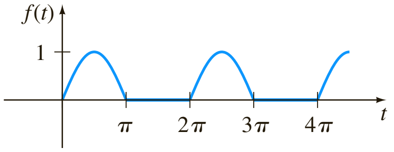
57.



full-wave rectification of sin t

FIGURE 7.4.10 Graph for Problem 57

58.



half-wave rectification of sin t

FIGURE 7.4.11 Graph for Problem 58

In Problems 59 and 60 solve equation (15) subject to $i(0) = 0$ with $E(t)$ as given. Use a graphing utility to graph the solution for $0 \le t \le 4$ in the case when $L = 1$ and $R = 1$.

59. $E(t)$ is the meander function in Problem 53 with amplitude 1 and $a = 1$.

60. $E(t)$ is the sawtooth function in Problem 55 with amplitude 1 and $b = 1$.

In Problems 61 and 62 solve the model for a driven spring/mass system with damping

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where the driving function f is as specified. Use a graphing utility to graph $x(t)$ for the indicated values of t.

61. $m = \frac{1}{2}$, $\beta = 1$, $k = 5$, f is the meander function in Problem 53 with amplitude 10, and $a = \pi$, $0 \le t \le 2\pi$.

62. $m = 1$, $\beta = 2$, $k = 1$, f is the square wave in Problem 54 with amplitude 5, and $a = \pi$, $0 \le t \le 4\pi$.

Discussion Problems

63. Discuss how Theorem 7.4.1 can be used to find

$$\mathscr{L}^{-1}\left\{\ln\frac{s-3}{s+1}\right\}.$$

64. In Section 6.4 we saw that $ty'' + y' + ty = 0$ is Bessel's equation of order $\nu = 0$. In view of (24) of that section and Table 6.4.1 a solution of the initial-value problem $ty'' + y' + ty = 0$, $y(0) = 1$, $y'(0) = 0$, is $y = J_0(t)$. Use this result and the procedure outlined in the instructions to Problems 17 and 18 to show that

$$\mathscr{L}\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}.$$

[*Hint*: You might need to use Problem 52 in Exercises 7.2.]

65. (a) Laguerre's differential equation

$$ty'' + (1-t)y' + ny = 0$$

is known to possess polynomial solutions when n is a nonnegative integer. These solutions are naturally called **Laguerre polynomials** and are denoted by $L_n(t)$. Find $y = L_n(t)$, for $n = 0, 1, 2, 3, 4$ if it is known that $L_n(0) = 1$.

(b) Show that

$$\mathscr{L}\left\{\frac{e^t}{n!}\frac{d^n}{dt^n}t^ne^{-t}\right\} = Y(s),$$

where $Y(s) = \mathscr{L}\{y\}$ and $y = L_n(t)$ is a polynomial solution of the DE in part (a). Conclude that

$$L_n(t) = \frac{e^t}{n!}\frac{d^n}{dt^n}t^ne^{-t}, \qquad n = 0, 1, 2, \ldots.$$

This last relation for generating the Laguerre polynomials is the analogue of Rodrigues' formula for the Legendre polynomials. See (36) in Section 6.4.

66. The Laplace transform $\mathscr{L}\{e^{-t^2}\}$ exists, but without finding it solve the initial-value problem $y'' + y = e^{-t^2}$, $y(0) = 0$, $y'(0) = 0$.

67. Solve the integral equation

$$f(t) = e^t + e^t\int_0^t e^{-\tau}f(\tau)\,d\tau.$$

68. (a) Show that the square wave function $E(t)$ given in Figure 7.4.4 can be written

$$E(t) = \sum_{k=0}^{\infty}(-1)^k\,\mathscr{U}(t-k).$$

(b) Obtain (14) of this section by taking the Laplace transform of each term in the series in part (a).

69. Use the Laplace transform as an aide in evaluating the improper integral $\int_0^\infty te^{-2t}\sin 4t\,dt$.

70. If we assume that $\mathscr{L}\{f(t)/t\}$ exists and $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du.$$

Use this result to find the Laplace transform of the given function. The symbols a and k are positive constants.

(a) $f(t) = \dfrac{\sin at}{t}$ **(b)** $f(t) = \dfrac{2(1-\cos kt)}{t}$

71. Transform of the Logarithm Because $f(t) = \ln t$ has an infinite discontinuity at $t = 0$ it might be assumed that $\mathscr{L}\{\ln t\}$ does not exist; however, this is incorrect. The point of this problem is to guide you through the formal steps leading to the Laplace transform of $f(t) = \ln t$, $t > 0$.

(a) Use integration by parts to show that

$$\mathscr{L}\{\ln t\} = s\mathscr{L}\{t\ln t\} - \frac{1}{s}.$$

(b) If $\mathscr{L}\{\ln t\} = Y(s)$, use Theorem 7.4.1 with $n = 1$ to show that part (a) becomes

$$s\frac{dY}{ds} + Y = -\frac{1}{s}.$$

Find an explicit solution $Y(s)$ of the foregoing differential equation.

(c) Finally, the integral definition of **Euler's constant** (sometimes called the **Euler-Mascheroni constant**) is $\gamma = -\int_0^\infty e^{-t}\ln t\,dt$, where $\gamma = 0.5772156649\ldots$. Use $Y(1) = -\gamma$ in the solution in part (b) to show that

$$\mathscr{L}\{\ln t\} = -\frac{\gamma}{s} - \frac{\ln s}{s}, \quad s > 0.$$

Computer Lab Assignments

72. In this problem you are led through the commands in *Mathematica* that enable you to obtain the symbolic Laplace transform of a differential equation and the solution of the initial-value problem by finding the inverse transform. In *Mathematica* the Laplace transform of a function $y(t)$ is obtained using **LaplaceTransform [y[t], t, s].** In line two of the syntax we replace **LaplaceTransform [y[t], t, s]** by the symbol **Y.** (*If you do not have* Mathematica, *then adapt the given procedure by finding the corresponding syntax for the CAS you have on hand.*)

Consider the initial-value problem

$$y'' + 6y' + 9y = t \sin t, \quad y(0) = 2, \quad y'(0) = -1.$$

Load the Laplace transform package. Precisely reproduce and then, in turn, execute each line in the following sequence of commands. Either copy the output by hand or print out the results.

```
diffequat = y''[t] + 6y'[t] + 9y[t] == t Sin[t]
transformdeq = LaplaceTransform [diffequat, t, s] /.
  {y[0] - > 2, y'[0] - > -1,
  LaplaceTransform [y[t], t, s] - > Y}
soln = Solve[transformdeq, Y]//Flatten
Y = Y/.soln
InverseLaplaceTransform[Y, s, t]
```

73. Appropriately modify the procedure of Problem 72 to find a solution of

$$y''' + 3y' - 4y = 0,$$
$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1.$$

74. The charge $q(t)$ on a capacitor in an LC-series circuit is given by

$$\frac{d^2q}{dt^2} + q = 1 - 4\mathscr{U}(t - \pi) + 6\mathscr{U}(t - 3\pi),$$
$$q(0) = 0, \quad q'(0) = 0.$$

Appropriately modify the procedure of Problem 72 to find $q(t)$. Graph your solution.

7.5 The Dirac Delta Function

INTRODUCTION In the last paragraph on page 284, we indicated that as an immediate consequence of Theorem 7.1.3, $F(s) = 1$ cannot be the Laplace transform of a function f that is piecewise continuous on $[0, \infty)$ and of exponential order. In the discussion that follows we are going to introduce a function that is very different from the kinds that you have studied in previous courses. We shall see that there does indeed exist a function—or, more precisely, a *generalized function*—whose Laplace transform is $F(s) = 1$.

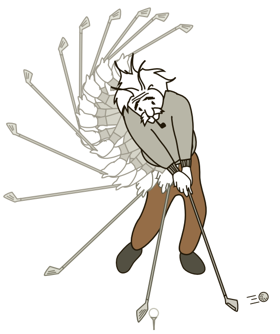
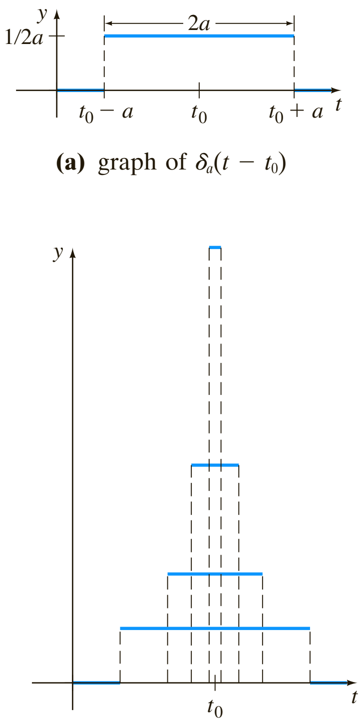


FIGURE 7.5.1 A golf club applies a force of large magnitude on the ball for a very short period of time

UNIT IMPULSE Mechanical systems are often acted on by an external force (or electromotive force in an electrical circuit) of large magnitude that acts only for a very short period of time. For example, a vibrating airplane wing could be struck by lightning, a mass on a spring could be given a sharp blow by a ball peen hammer, and a ball (baseball, golf ball, tennis ball) could be sent soaring when struck violently by some kind of club (baseball bat, golf club, tennis racket). See Figure 7.5.1. The graph of the piecewise-defined function

$$\delta_a(t - t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \dfrac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a, \end{cases} \tag{1}$$

$a > 0$, $t_0 > 0$, shown in Figure 7.5.2(a), could serve as a model for such a force. For a small value of a, $\delta_a(t - t_0)$ is essentially a constant function of large magnitude that is "on" for just a very short period of time, around t_0. The behavior of $\delta_a(t - t_0)$ as $a \to 0$ is illustrated in Figure 7.5.2(b). The function $\delta_a(t - t_0)$ is called a **unit impulse,** because it possesses the integration property $\int_0^\infty \delta_a(t - t_0)\, dt = 1$.



(a) graph of $\delta_a(t - t_0)$

(b) behavior of δ_a as $a \to 0$

FIGURE 7.5.2 Unit impulse

DIRAC DELTA FUNCTION In practice it is convenient to work with another type of unit impulse, a "function" that approximates $\delta_a(t - t_0)$ and is defined by the limit

$$\delta(t - t_0) = \lim_{a \to 0} \delta_a(t - t_0). \tag{2}$$

The latter expression, which is not a function at all, can be characterized by the two properties

$$(i)\ \delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases} \quad \text{and} \quad (ii)\ \int_0^\infty \delta(t - t_0)\,dt = 1.$$

The unit impulse $\delta(t - t_0)$ is called the **Dirac delta function.**

It is possible to obtain the Laplace transform of the Dirac delta function by the formal assumption that $\mathscr{L}\{\delta(t - t_0)\} = \lim_{a \to 0} \mathscr{L}\{\delta_a(t - t_0)\}$.

THEOREM 7.5.1 Transform of the Dirac Delta Function

For $t_0 > 0$,

$$\mathscr{L}\{\delta(t - t_0)\} = e^{-st_0}. \tag{3}$$

PROOF To begin, we can write $\delta_a(t - t_0)$ in terms of the unit step function by virtue of (11) and (12) of Section 7.3:

$$\delta_a(t - t_0) = \frac{1}{2a}\left[\mathscr{U}(t - (t_0 - a)) - \mathscr{U}(t - (t_0 + a))\right].$$

By linearity and (14) of Section 7.3 the Laplace transform of this last expression is

$$\mathscr{L}\{\delta_a(t - t_0)\} = \frac{1}{2a}\left[\frac{e^{-s(t_0 - a)}}{s} - \frac{e^{-s(t_0 + a)}}{s}\right] = e^{-st_0}\left(\frac{e^{sa} - e^{-sa}}{2sa}\right). \tag{4}$$

Since (4) has the indeterminate form $0/0$ as $a \to 0$, we apply L'Hôpital's Rule:

$$\mathscr{L}\{\delta(t - t_0)\} = \lim_{a \to 0} \mathscr{L}\{\delta_a(t - t_0)\} = e^{-st_0} \lim_{a \to 0}\left(\frac{e^{sa} - e^{-sa}}{2sa}\right) = e^{-st_0}. \qquad \blacksquare$$

Now when $t_0 = 0$, it seems plausible to conclude from (3) that

$$\mathscr{L}\{\delta(t)\} = 1.$$

The last result emphasizes the fact that $\delta(t)$ is not the usual type of function that we have been considering, since we expect from Theorem 7.1.3 that $\mathscr{L}\{f(t)\} \to 0$ as $s \to \infty$.

EXAMPLE 1 Two Initial-Value Problems

Solve $y'' + y = 4\,\delta(t - 2\pi)$ subject to

(a) $y(0) = 1, \quad y'(0) = 0$ **(b)** $y(0) = 0, \quad y'(0) = 0$.

The two initial-value problems could serve as models for describing the motion of a mass on a spring moving in a medium in which damping is negligible. At $t = 2\pi$ the mass is given a sharp blow. In (a) the mass is released from rest 1 unit below the equilibrium position. In (b) the mass is at rest in the equilibrium position.

SOLUTION **(a)** From (3) the Laplace transform of the differential equation is

$$s^2 Y(s) - s + Y(s) = 4e^{-2\pi s} \quad \text{or} \quad Y(s) = \frac{s}{s^2 + 1} + \frac{4e^{-2\pi s}}{s^2 + 1}.$$

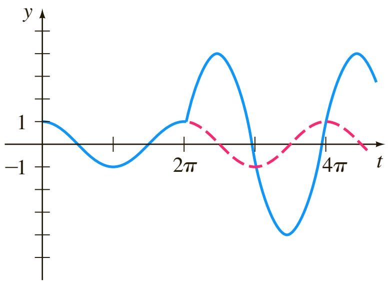


FIGURE 7.5.3 Mass is struck at $t = 2\pi$ in part (a) of Example 1

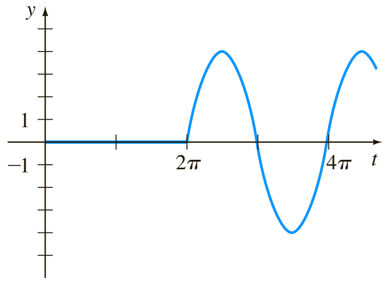


FIGURE 7.5.4 No motion until mass is struck at $t = 2\pi$ in part (b) of Example 1

Using the inverse form of the second translation theorem, (15) of Section 7.3, we find

$$y(t) = \cos t + 4 \sin (t - 2\pi)\, \mathscr{U}(t - 2\pi).$$

Since $\sin(t - 2\pi) = \sin t$, the foregoing solution can be written as

$$y(t) = \begin{cases} \cos t, & 0 \le t < 2\pi \\ \cos t + 4 \sin t, & t \ge 2\pi. \end{cases} \tag{5}$$

In Figure 7.5.3 we see from the graph of (5) that the mass is exhibiting simple harmonic motion until it is struck at $t = 2\pi$. The influence of the unit impulse is to increase the amplitude of vibration to $\sqrt{17}$ for $t > 2\pi$.

(b) In this case the transform of the equation is simply

$$Y(s) = \frac{4e^{-2\pi s}}{s^2 + 1},$$

and so

$$\begin{aligned} y(t) &= 4 \sin (t - 2\pi)\, \mathscr{U}(t - 2\pi) \\ &= \begin{cases} 0, & 0 \le t < 2\pi \\ 4 \sin t, & t \ge 2\pi. \end{cases} \end{aligned} \tag{6}$$

The graph of (6) in Figure 7.5.4 shows, as we would expect from the initial conditions that the mass exhibits no motion until it is struck at $t = 2\pi$. ■

REMARKS

(i) If $\delta(t - t_0)$ were a function in the usual sense, then property (*i*) on page 319 would imply $\int_0^\infty \delta(t - t_0)\,dt = 0$ rather than $\int_0^\infty \delta(t - t_0)\,dt = 1$. Because the Dirac delta function did not "behave" like an ordinary function, even though its users produced correct results, it was met initially with great scorn by mathematicians. However, in the 1940s Dirac's controversial function was put on a rigorous footing by the French mathematician Laurent Schwartz in his book *Théorie des distributions*, and this, in turn, led to an entirely new branch of mathematics known as the **theory of distributions** or **generalized functions.** In this theory (2) is not an accepted definition of $\delta(t - t_0)$, nor does one speak of a function whose values are either ∞ or 0. Although we shall not pursue this topic any further, suffice it to say that the Dirac delta function is best characterized by its effect on other functions. If f is a continuous function, then

$$\int_0^\infty f(t)\, \delta(t - t_0)\, dt = f(t_0) \tag{7}$$

can be taken as the *definition* of $\delta(t - t_0)$. This result is known as the **sifting property,** since $\delta(t - t_0)$ has the effect of sifting the value $f(t_0)$ out of the set of values of f on $[0, \infty)$. Note that property (*ii*) (with $f(t) = 1$) and (3) (with $f(t) = e^{-st}$) are consistent with (7).

(ii) In (*iii*) in the *Remarks* at the end of Section 7.2 we indicated that the transfer function of a general linear nth-order differential equation with constant coefficients is $W(s) = 1/P(s)$, where $P(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0$. The transfer function is the Laplace transform of function $w(t)$, called the **weight function** of a linear system. But $w(t)$ can also be characterized in terms of the discussion at hand. For simplicity let us consider a second-order linear system in which the input is a unit impulse at $t = 0$:

$$a_2y'' + a_1y' + a_0y = \delta(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Applying the Laplace transform and using $\mathscr{L}\{\delta(t)\} = 1$ shows that the transform of the response y in this case is the transfer function

$$Y(s) = \frac{1}{a_2s^2 + a_1s + a_0} = \frac{1}{P(s)} = W(s) \quad \text{and} \quad \text{so} \quad y = \mathscr{L}^{-1}\left\{\frac{1}{P(s)}\right\} = w(t).$$

From this we can see, in general, that the weight function $y = w(t)$ of an nth-order linear system is the zero-state response of the system to a unit impulse. For this reason $w(t)$ is also called the **impulse response** of the system.

EXERCISES 7.5

Answers to selected odd-numbered problems begin on page ANS-12.

In Problems 1–12 use the Laplace transform to solve the given initial-value problem.

1. $y' - 3y = \delta(t - 2), \quad y(0) = 0$

2. $y' + y = \delta(t - 1), \quad y(0) = 2$

3. $y'' + y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1$

4. $y'' + 16y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0$

5. $y'' + y = \delta\left(t - \frac{1}{2}\pi\right) + \delta\left(t - \frac{3}{2}\pi\right), \quad y(0) = 0, \quad y'(0) = 0$

6. $y'' + y = \delta(t - 2\pi) + \delta(t - 4\pi), \quad y(0) = 1, \quad y'(0) = 0$

7. $y'' + 2y' = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 1$

8. $y'' - 2y' = 1 + \delta(t - 2), \quad y(0) = 0, \quad y'(0) = 1$

9. $y'' + 4y' + 5y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 0$

10. $y'' + 2y' + y = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 0$

11. $y'' + 4y' + 13y = \delta(t - \pi) + \delta(t - 3\pi), \quad y(0) = 1, \quad y'(0) = 0$

12. $y'' - 7y' + 6y = e^t + \delta(t - 2) + \delta(t - 4), \quad y(0) = 0, \quad y'(0) = 0$

In Problems 13 and 14 use the Laplace transform to solve the given initial-value problem. Graph your solution on the interval $[0, 8\pi]$.

13. $y'' + y = \sum_{k=1}^{\infty} \delta(t - k\pi), \quad y(0) = 0, \quad y'(0) = 1$

14. $y'' + y = \sum_{k=1}^{\infty} \delta(t - 2k\pi), \quad y(0) = 0, \quad y'(0) = 1$

In Problems 15 and 16 a uniform beam of length L carries a concentrated load w_0 at $x = \frac{1}{2}L$. See Figure 7.5.5 (Problem 15) and Figure 7.5.6 (Problem 16). Use the Laplace transform to solve the differential equation

$$EI\frac{d^4y}{dx^4} = w_0\,\delta\left(x - \tfrac{1}{2}L\right), \quad 0 < x < L,$$

subject to the given boundary conditions.

15. $y(0) = 0, \quad y'(0) = 0, \quad y''(L) = 0, \quad y'''(L) = 0$

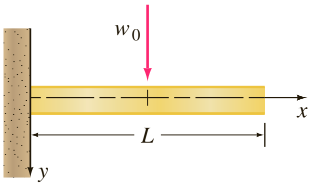


FIGURE 7.5.5 Beam embedded at its left end and free at its right end

16. $y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y'(L) = 0$

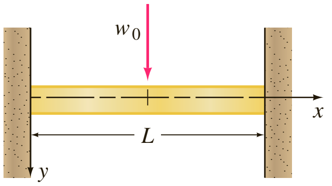


FIGURE 7.5.6 Beam embedded at both ends

Discussion Problems

17. Someone tells you that the solutions of the two IVPs

$$y'' + 2y' + 10y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$
$$y'' + 2y' + 10y = \delta(t), \qquad y(0) = 0, \quad y'(0) = 0$$

are exactly the same. Do you agree or disagree? Defend your answer.

18. Reread (*i*) in the *Remarks* at the end of this section. Then use the Laplace transform to solve the initial-value problem:

$$y'' + 4y' + 3y = e^t\delta(t - 1), \quad y(0) = 0, \quad y'(0) = 2.$$

Use a graphing utility to graph $y(t)$ for $0 \le t \le 5$.

7.6 Systems of Linear Differential Equations

INTRODUCTION When initial conditions are specified, the Laplace transform of each equation in a system of linear differential equations with constant coefficients reduces the system of DEs to a set of simultaneous algebraic equations in the transformed functions. We solve the system of algebraic equations for each of the transformed functions and then find the inverse Laplace transforms in the usual manner.

COUPLED SPRINGS Two masses m_1 and m_2 are connected to two springs A and B of negligible mass having spring constants k_1 and k_2, respectively. In turn the two springs are attached as shown in Figure 7.6.1. Let $x_1(t)$ and $x_2(t)$ denote the vertical displacements of the masses from their equilibrium positions. When the system is in motion, spring B is subject to both an elongation and a compression; hence its net elongation is $x_2 - x_1$. Therefore it follows from Hooke's law that springs A and B exert forces $-k_1x_1$ and $k_2(x_2 - x_1)$, respectively, on m_1. If no external force is impressed on the system and if no damping force is present, then the net force on m_1 is $-k_1x_1 + k_2(x_2 - x_1)$. By Newton's second law we can write

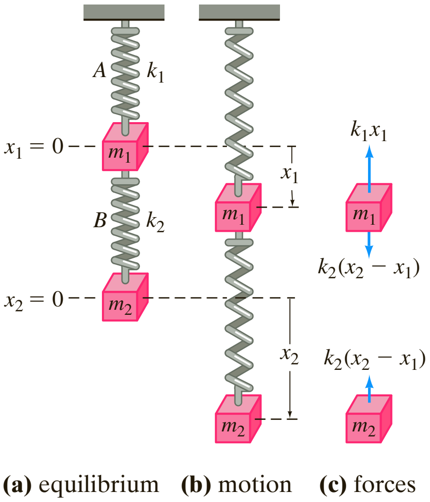


FIGURE 7.6.1 Coupled spring/mass system

$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1).$$

Similarly, the net force exerted on mass m_2 is due solely to the net elongation of B; that is, $-k_2(x_2 - x_1)$. Hence we have

$$m_2\frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1).$$

In other words, the motion of the coupled system is represented by the system of simultaneous second-order differential equations

$$\begin{aligned} m_1x_1'' &= -k_1x_1 + k_2(x_2 - x_1)\\ m_2x_2'' &= -k_2(x_2 - x_1). \end{aligned} \tag{1}$$

In the next example we solve (1) under the assumptions that $k_1 = 6$, $k_2 = 4$, $m_1 = 1$, $m_2 = 1$, and that the masses start from their equilibrium positions with opposite unit velocities.

EXAMPLE 1 Coupled Springs

Solve
$$\begin{aligned} x_1'' + 10x_1 \qquad - 4x_2 &= 0\\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned} \tag{2}$$

subject to $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 0$, $x_2'(0) = -1$.

SOLUTION The Laplace transform of each equation is

$$\begin{aligned} s^2X_1(s) - sx_1(0) - x_1'(0) + 10X_1(s) - 4X_2(s) &= 0\\ -4X_1(s) + s^2X_2(s) - sx_2(0) - x_2'(0) + 4X_2(s) &= 0, \end{aligned}$$

where $X_1(s) = \mathscr{L}\{x_1(t)\}$ and $X_2(s) = \mathscr{L}\{x_2(t)\}$. The preceding system is the same as

$$\begin{aligned} (s^2 + 10)X_1(s) - \qquad 4X_2(s) &= 1\\ -4X_1(s) + (s^2 + 4)X_2(s) &= -1. \end{aligned} \tag{3}$$

Solving (3) for $X_1(s)$ and using partial fractions on the result yields

$$X_1(s) = \frac{s^2}{(s^2+2)(s^2+12)} = -\frac{1/5}{s^2+2} + \frac{6/5}{s^2+12},$$

and therefore

$$\begin{aligned} x_1(t) &= -\frac{1}{5\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} + \frac{6}{5\sqrt{12}}\mathscr{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\} \\ &= -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t. \end{aligned}$$

Substituting the expression for $X_1(s)$ into the first equation of (3) gives

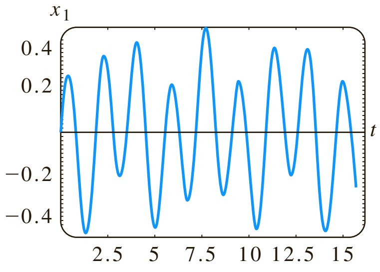
$$X_2(s) = -\frac{s^2+6}{(s^2+2)(s^2+12)} = -\frac{2/5}{s^2+2} - \frac{3/5}{s^2+12}$$

and

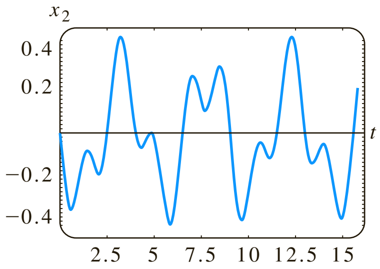
$$\begin{aligned} x_2(t) &= -\frac{2}{5\sqrt{2}}\mathscr{L}^{-1}\left\{\frac{\sqrt{2}}{s^2+2}\right\} - \frac{3}{5\sqrt{12}}\mathscr{L}^{-1}\left\{\frac{\sqrt{12}}{s^2+12}\right\} \\ &= -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t. \end{aligned}$$

Finally, the solution to the given system (2) is

$$\begin{aligned} x_1(t) &= -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t \\ x_2(t) &= -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t. \end{aligned} \tag{4}$$



(a) plot of $x_1(t)$



(b) plot of $x_2(t)$

FIGURE 7.6.2 Displacements of the two masses in Example 1

The graphs of x_1 and x_2 in Figure 7.6.2 reveal the complicated oscillatory motion of each mass. ■

NETWORKS In (18) of Section 3.3 we saw the currents $i_1(t)$ and $i_2(t)$ in the network shown in Figure 7.6.3, containing an inductor, a resistor, and a capacitor, were governed by the system of first-order differential equations

$$\begin{aligned} L\frac{di_1}{dt} + Ri_2 &= E(t) \\ RC\frac{di_2}{dt} + i_2 - i_1 &= 0. \end{aligned} \tag{5}$$

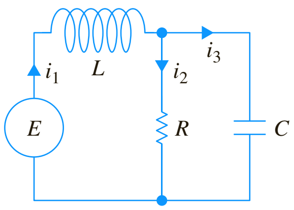


FIGURE 7.6.3 Electrical network

We solve this system by the Laplace transform in the next example.

EXAMPLE 2 An Electrical Network

Solve the system in (5) under the conditions $E(t) = 60$ V, $L = 1$ h, $R = 50\ \Omega$, $C = 10^{-4}$ f, and the currents i_1 and i_2 are initially zero.

SOLUTION We must solve

$$\begin{aligned} \frac{di_1}{dt} + 50i_2 &= 60 \\ 50(10^{-4})\frac{di_2}{dt} + i_2 - i_1 &= 0 \end{aligned}$$

subject to $i_1(0) = 0$, $i_2(0) = 0$.

Applying the Laplace transform to each equation of the system and simplifying gives

$$\begin{aligned} sI_1(s) + \quad 50I_2(s) &= \frac{60}{s} \\ -200I_1(s) + (s + 200)I_2(s) &= 0, \end{aligned}$$

where $I_1(s) = \mathscr{L}\{i_1(t)\}$ and $I_2(s) = \mathscr{L}\{i_2(t)\}$. Solving the system for I_1 and I_2 and decomposing the results into partial fractions gives

$$\begin{aligned} I_1(s) &= \frac{60s + 12{,}000}{s(s + 100)^2} = \frac{6/5}{s} - \frac{6/5}{s + 100} - \frac{60}{(s + 100)^2} \\ I_2(s) &= \frac{12{,}000}{s(s + 100)^2} = \frac{6/5}{s} - \frac{6/5}{s + 100} - \frac{120}{(s + 100)^2}. \end{aligned}$$

Taking the inverse Laplace transform, we find the currents to be

$$\begin{aligned} i_1(t) &= \frac{6}{5} - \frac{6}{5}e^{-100t} - 60te^{-100t} \\ i_2(t) &= \frac{6}{5} - \frac{6}{5}e^{-100t} - 120te^{-100t}. \end{aligned}$$

Note that both $i_1(t)$ and $i_2(t)$ in Example 2 tend toward the value $E/R = \frac{6}{5}$ as $t \to \infty$. Furthermore, since the current through the capacitor is $i_3(t) = i_1(t) - i_2(t) = 60te^{-100t}$, we observe that $i_3(t) \to 0$ as $t \to \infty$.

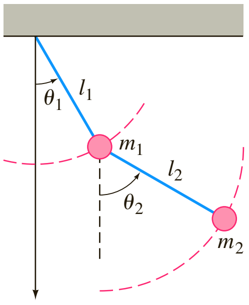


FIGURE 7.6.4 Double pendulum

DOUBLE PENDULUM Consider the double-pendulum system consisting of a pendulum attached to a pendulum shown in Figure 7.6.4. We assume that the system oscillates in a vertical plane under the influence of gravity, that the mass of each rod is negligible, and that no damping forces act on the system. Figure 7.6.4 also shows that the displacement angle θ_1 is measured (in radians) from a vertical line extending downward from the pivot of the system and that θ_2 is measured from a vertical line extending downward from the center of mass m_1. The positive direction is to the right; the negative direction is to the left. As we might expect from the analysis leading to equation (6) of Section 5.3, the system of differential equations describing the motion is nonlinear:

$$\begin{aligned} (m_1 + m_2)l_1^2\theta_1'' + m_2l_1l_2\theta_2''\cos(\theta_1 - \theta_2) + m_2l_1l_2(\theta_2')^2\sin(\theta_1 - \theta_2) + (m_1 + m_2)l_1g\sin\theta_1 &= 0 \\ m_2l_2^2\theta_2'' + m_2l_1l_2\theta_1''\cos(\theta_1 - \theta_2) - m_2l_1l_2(\theta_1')^2\sin(\theta_1 - \theta_2) + m_2l_2g\sin\theta_2 &= 0. \end{aligned} \tag{6}$$

But if the displacements $\theta_1(t)$ and $\theta_2(t)$ are assumed to be small, then the approximations $\cos(\theta_1 - \theta_2) \approx 1$, $\sin(\theta_1 - \theta_2) \approx 0$, $\sin\theta_1 \approx \theta_1$, $\sin\theta_2 \approx \theta_2$ enable us to replace system (6) by the linearization

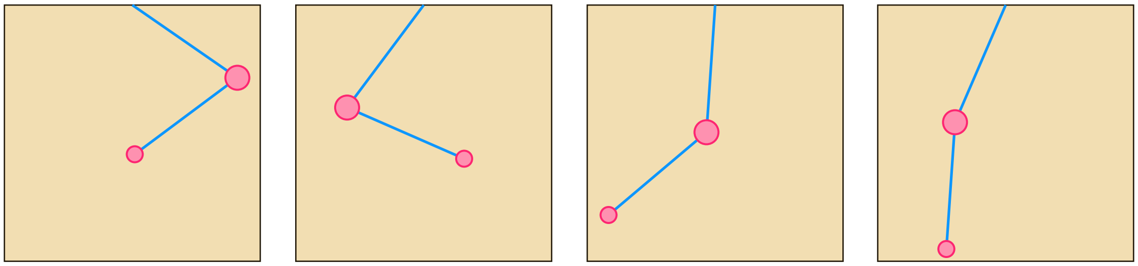
$$\begin{aligned} (m_1 + m_2)l_1^2\theta_1'' + m_2l_1l_2\theta_2'' + (m_1 + m_2)l_1g\theta_1 &= 0 \\ m_2l_2^2\theta_2'' + m_2l_1l_2\theta_1'' + m_2l_2g\theta_2 &= 0. \end{aligned} \tag{7}$$

EXAMPLE 3 Double Pendulum

It is left as an exercise to fill in the details of using the Laplace transform to solve system (7) when $m_1 = 3$, $m_2 = 1$, $l_1 = l_2 = 16$, $\theta_1(0) = 1$, $\theta_2(0) = -1$, $\theta_1'(0) = 0$, and $\theta_2'(0) = 0$. You should find that

$$\begin{aligned} \theta_1(t) &= \frac{1}{4}\cos\frac{2}{\sqrt{3}}t + \frac{3}{4}\cos 2t \\ \theta_2(t) &= \frac{1}{2}\cos\frac{2}{\sqrt{3}}t - \frac{3}{2}\cos 2t. \end{aligned} \tag{8}$$

With the aid of a CAS the positions of the two masses at $t = 0$ and at subsequent times are shown in Figure 7.6.5. See Problem 21 in Exercises 7.6.



(a) $t = 0$ **(b)** $t = 1.4$ **(c)** $t = 2.5$ **(d)** $t = 8.5$

FIGURE 7.6.5 Positions of masses on double pendulum at various times in Example 3

EXERCISES 7.6

Answers to selected odd-numbered problems begin on page ANS-12.

In Problems 1–12 use the Laplace transform to solve the given system of differential equations.

1. $\dfrac{dx}{dt} = -x + y$

$\dfrac{dy}{dt} = 2x$

$x(0) = 0, \quad y(0) = 1$

2. $\dfrac{dx}{dt} = 2y + e^t$

$\dfrac{dy}{dt} = 8x - t$

$x(0) = 1, \quad y(0) = 1$

3. $\dfrac{dx}{dt} = x - 2y$

$\dfrac{dy}{dt} = 5x - y$

$x(0) = -1, \quad y(0) = 2$

4. $\dfrac{dx}{dt} + 3x + \dfrac{dy}{dt} = 1$

$\dfrac{dx}{dt} - x + \dfrac{dy}{dt} - y = e^t$

$x(0) = 0, \quad y(0) = 0$

5. $2\dfrac{dx}{dt} + \dfrac{dy}{dt} - 2x = 1$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} - 3x - 3y = 2$

$x(0) = 0, \quad y(0) = 0$

6. $\dfrac{dx}{dt} + x - \dfrac{dy}{dt} + y = 0$

$\dfrac{dx}{dt} + \dfrac{dy}{dt} + 2y = 0$

$x(0) = 0, \quad y(0) = 1$

7. $\dfrac{d^2x}{dt^2} + x - y = 0$

$\dfrac{d^2y}{dt^2} + y - x = 0$

$x(0) = 0, \quad x'(0) = -2,$

$y(0) = 0, \quad y'(0) = 1$

8. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + \dfrac{dy}{dt} = 0$

$\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} - 4\dfrac{dx}{dt} = 0$

$x(0) = 1, \quad x'(0) = 0,$

$y(0) = -1, \quad y'(0) = 5$

9. $\dfrac{d^2x}{dt^2} + \dfrac{d^2y}{dt^2} = t^2$

$\dfrac{d^2x}{dt^2} - \dfrac{d^2y}{dt^2} = 4t$

$x(0) = 8, \quad x'(0) = 0,$

$y(0) = 0, \quad y'(0) = 0$

10. $\dfrac{dx}{dt} - 4x + \dfrac{d^3y}{dt^3} = 6\sin t$

$\dfrac{dx}{dt} + 2x - 2\dfrac{d^3y}{dt^3} = 0$

$x(0) = 0, \quad y(0) = 0,$

$y'(0) = 0, \quad y''(0) = 0$

11. $\dfrac{d^2x}{dt^2} + 3\dfrac{dy}{dt} + 3y = 0$

$\dfrac{d^2x}{dt^2} + 3y = te^{-t}$

$x(0) = 0, \quad x'(0) = 2, \quad y(0) = 0$

12. $\dfrac{dx}{dt} = 4x - 2y + 2\mathcal{U}(t-1)$

$\dfrac{dy}{dt} = 3x - y + \mathcal{U}(t-1)$

$x(0) = 0, \quad y(0) = \frac{1}{2}$

13. Solve system (1) when $k_1 = 3$, $k_2 = 2$, $m_1 = 1$, $m_2 = 1$ and $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 1$, $x_2'(0) = 0$.

14. Derive the system of differential equations describing the straight-line vertical motion of the coupled springs shown in Figure 7.6.6. Use the Laplace transform to solve the system when $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $m_1 = 1$, $m_2 = 1$ and $x_1(0) = 0$, $x_1'(0) = -1$, $x_2(0) = 0$, $x_2'(0) = 1$.

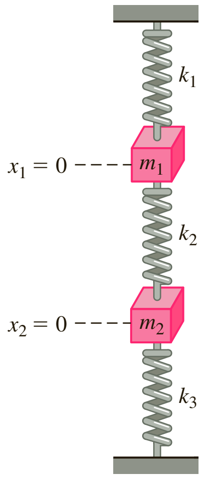


FIGURE 7.6.6 Coupled springs in Problem 14

15. (a) Show that the system of differential equations for the currents $i_2(t)$ and $i_3(t)$ in the electrical network shown in Figure 7.6.7 is

$$L_1\frac{di_2}{dt} + Ri_2 + Ri_3 = E(t)$$

$$L_2\frac{di_3}{dt} + Ri_2 + Ri_3 = E(t).$$

(b) Solve the system in part (a) if $R = 5\ \Omega$, $L_1 = 0.01$ h, $L_2 = 0.0125$ h, $E = 100$ V, $i_2(0) = 0$, and $i_3(0) = 0$.

(c) Determine the current $i_1(t)$.

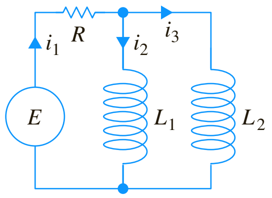


FIGURE 7.6.7 Network in Problem 15

16. (a) In Problem 12 in Exercises 3.3 you were asked to show that the currents $i_2(t)$ and $i_3(t)$ in the electrical network shown in Figure 7.6.8 satisfy

$$L\frac{di_2}{dt} + L\frac{di_3}{dt} + R_1 i_2 = E(t)$$

$$-R_1\frac{di_2}{dt} + R_2\frac{di_3}{dt} + \frac{1}{C}i_3 = 0.$$

Solve the system if $R_1 = 10\ \Omega$, $R_2 = 5\ \Omega$, $L = 1$ h, $C = 0.2$ f,

$$E(t) = \begin{cases} 120, & 0 \le t < 2 \\ 0, & t \ge 2, \end{cases}$$

$i_2(0) = 0$, and $i_3(0) = 0$.

(b) Determine the current $i_1(t)$.

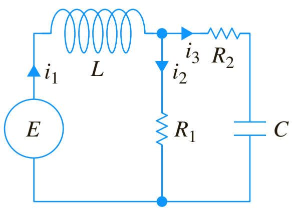


FIGURE 7.6.8 Network in Problem 16

17. Solve the system given in (17) of Section 3.3 when $R_1 = 6\ \Omega$, $R_2 = 5\ \Omega$, $L_1 = 1$ h, $L_2 = 1$ h, $E(t) = 50 \sin t$ V, $i_2(0) = 0$, and $i_3(0) = 0$.

18. Solve (5) when $E = 60$ V, $L = \frac{1}{2}$ h, $R = 50\ \Omega$, $C = 10^{-4}$ f, $i_1(0) = 0$, and $i_2(0) = 0$.

19. Solve (5) when $E = 60$ V, $L = 2$ h, $R = 50\ \Omega$, $C = 10^{-4}$ f, $i_1(0) = 0$, and $i_2(0) = 0$.

20. (a) Show that the system of differential equations for the charge on the capacitor $q(t)$ and the current $i_3(t)$ in the electrical network shown in Figure 7.6.9 is

$$R_1\frac{dq}{dt} + \frac{1}{C}q + R_1 i_3 = E(t)$$

$$L\frac{di_3}{dt} + R_2 i_3 - \frac{1}{C}q = 0.$$

(b) Find the charge on the capacitor when $L = 1$ h, $R_1 = 1\ \Omega$, $R_2 = 1\ \Omega$, $C = 1$ f,

$$E(t) = \begin{cases} 0, & 0 < t < 1 \\ 50e^{-t}, & t \ge 1, \end{cases}$$

$i_3(0) = 0$, and $q(0) = 0$.

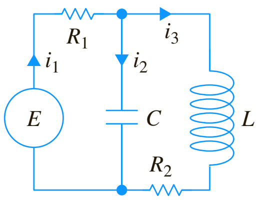


FIGURE 7.6.9 Network in Problem 20

Computer Lab Assignments

21. (a) Use the Laplace transform and the information given in Example 3 to obtain the solution (8) of the system given in (7).

(b) Use a graphing utility to graph $\theta_1(t)$ and $\theta_2(t)$ in the $t\theta$-plane. Which mass has extreme displacements of greater magnitude? Use the graphs to estimate the first time that each mass passes through its equilibrium position. Discuss whether the motion of the pendulums is periodic.

(c) Graph $\theta_1(t)$ and $\theta_2(t)$ in the $\theta_1\theta_2$-plane as parametric equations. The curve defined by these parametric equations is called a **Lissajous curve.**

(d) The positions of the masses at $t = 0$ are given in Figure 7.6.5(a). Note that we have used 1 radian $\approx 57.3°$. Use a calculator or a table application in a CAS to construct a table of values of the angles θ_1 and θ_2 for $t = 1, 2, \ldots, 10$ s. Then plot the positions of the two masses at these times.

(e) Use a CAS to find the first time that $\theta_1(t) = \theta_2(t)$ and compute the corresponding angular value. Plot the positions of the two masses at these times.

(f) Utilize the CAS to draw appropriate lines to simulate the pendulum rods, as in Figure 7.6.5. Use the animation capability of your CAS to make a "movie" of the motion of the double pendulum from $t = 0$ to $t = 10$ using a time increment of 0.1. [*Hint*: Express the coordinates $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ of the masses m_1 and m_2, respectively, in terms of $\theta_1(t)$ and $\theta_2(t)$.]

Chapter 7 In Review

Answers to selected odd-numbered problems begin on page ANS-13.

In Problems 1 and 2 use the definition of the Laplace transform to find $\mathscr{L}\{f(t)\}$.

1. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2 - t, & t \ge 1 \end{cases}$

2. $f(t) = \begin{cases} 0, & 0 \le t < 2 \\ 1, & 2 \le t < 4 \\ 0, & t \ge 4 \end{cases}$

In Problems 3–24 fill in the blanks or answer true or false.

3. If f is not piecewise continuous on $[0, \infty)$, then $\mathscr{L}\{f(t)\}$ will not exist. ________

4. The function $f(t) = (e^t)^{10}$ is not of exponential order. ________

5. $F(s) = s^2/(s^2 + 4)$ is not the Laplace transform of a function that is piecewise continuous and of exponential order. ________

6. If $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$, then $\mathscr{L}^{-1}\{F(s)G(s)\} = f(t)g(t)$. ________

7. $\mathscr{L}\{e^{-7t}\} =$ ________

8. $\mathscr{L}\{te^{-7t}\} =$ ________

9. $\mathscr{L}\{\sin 2t\} =$ ________

10. $\mathscr{L}\{e^{-3t}\sin 2t\} =$ ________

11. $\mathscr{L}\{t \sin 2t\} =$ ________

12. $\mathscr{L}\{\sin 2t\, \mathscr{U}(t - \pi)\} =$ ________

13. $\mathscr{L}^{-1}\left\{\dfrac{20}{s^6}\right\} =$ ________

14. $\mathscr{L}^{-1}\left\{\dfrac{1}{3s - 1}\right\} =$ ________

15. $\mathscr{L}^{-1}\left\{\dfrac{1}{(s - 5)^3}\right\} =$ ________

16. $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2 - 5}\right\} =$ ________

17. $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 - 10s + 29}\right\} =$ ________

18. $\mathscr{L}^{-1}\left\{\dfrac{e^{-5s}}{s^2}\right\} =$ ________

19. $\mathscr{L}^{-1}\left\{\dfrac{s + \pi}{s^2 + \pi^2}e^{-s}\right\} =$ ________

20. $\mathscr{L}^{-1}\left\{\dfrac{1}{L^2s^2 + n^2\pi^2}\right\} =$ ________

21. $\mathscr{L}\{e^{-5t}\}$ exists for $s >$ ________.

22. If $\mathscr{L}\{f(t)\} = F(s)$, then $\mathscr{L}\{te^{8t}f(t)\} =$ ________.

23. If $\mathscr{L}\{f(t)\} = F(s)$ and $k > 0$, then $\mathscr{L}\{e^{at}f(t - k)\mathscr{U}(t - k)\} =$ ________.

24. $\mathscr{L}\{\int_0^t e^{a\tau}f(\tau)\, d\tau\} =$ ________ whereas $\mathscr{L}\{e^{at}\int_0^t f(\tau)\, d\tau\} =$ ________.

In Problems 25–28 use the unit step function to find an equation for each graph in terms of the function $y = f(t)$, whose graph is given in Figure 7.R.1.

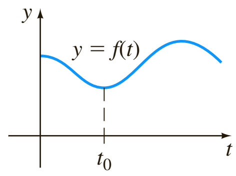


FIGURE 7.R.1 Graph for Problems 25–28

25.

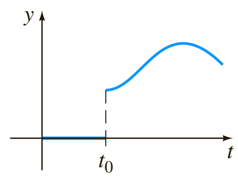


FIGURE 7.R.2 Graph for Problem 25

26.

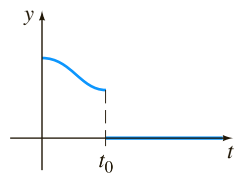


FIGURE 7.R.3 Graph for Problem 26

27.

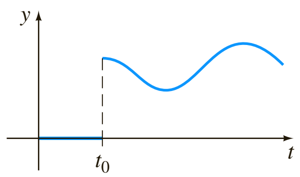


FIGURE 7.R.4 Graph for Problem 27

28.

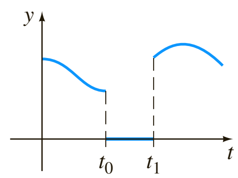


FIGURE 7.R.5 Graph for Problem 28

In Problems 29–32 express f in terms of unit step functions. Find $\mathscr{L}\{f(t)\}$ and $\mathscr{L}\{e^t f(t)\}$.

29.

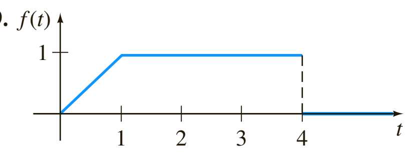


FIGURE 7.R.6 Graph for Problem 29

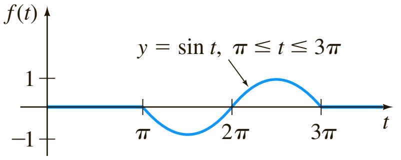
30. 

FIGURE 7.R.7 Graph for Problem 30

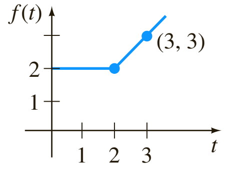
31. 

FIGURE 7.R.8 Graph for Problem 31

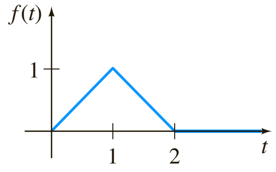
32. 

FIGURE 7.R.9 Graph for Problem 32

In Problems 33 and 34 sketch the graph of the given function. Find $\mathscr{L}\{f(t)\}$.

33. $f(t) = -1 + 2\sum_{k=1}^{\infty}(-1)^{k+1}\mathscr{U}(t-k)$

34. $f(t) = \sum_{k=0}^{\infty}(2k+1-t)[\mathscr{U}(t-2k) - \mathscr{U}(t-2k-1)]$

In Problems 35–42 use the Laplace transform to solve the given equation.

35. $y'' - 2y' + y = e^t, \quad y(0) = 0, \quad y'(0) = 5$

36. $y'' - 8y' + 20y = te^t, \quad y(0) = 0, \quad y'(0) = 0$

37. $y'' + 6y' + 5y = t - t\mathscr{U}(t-2), \quad y(0) = 1, \quad y'(0) = 0$

38. $y' - 5y = f(t)$, where

$$f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}, \quad y(0) = 1$$

39. $y' + 2y = f(t), \quad y(0) = 1$, where $f(t)$ is given in Figure 7.R.10.

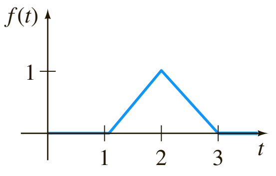


FIGURE 7.R.10 Graph for Problem 39

40. $y'' + 5y' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = 3$, where

$$f(t) = 12\sum_{k=0}^{\infty}(-1)^k\mathscr{U}(t-k).$$

41. $y'(t) = \cos t + \int_0^t y(\tau)\cos(t-\tau)\,d\tau, \quad y(0) = 1$

42. $\int_0^t f(\tau)f(t-\tau)\,d\tau = 6t^3$

In Problems 43 and 44 use the Laplace transform to solve each system.

43. $x' + y = t$
$4x + y' = 0$
$x(0) = 1, \quad y(0) = 2$

44. $x'' + y'' = e^{2t}$
$2x' + y'' = -e^{2t}$
$x(0) = 0, \quad y(0) = 0,$
$x'(0) = 0, \quad y'(0) = 0$

45. The current $i(t)$ in an RC-series circuit can be determined from the integral equation

$$Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t),$$

where $E(t)$ is the impressed voltage. Determine $i(t)$ when $R = 10\ \Omega$, $C = 0.5$ f, and $E(t) = 2(t^2 + t)$.

46. A series circuit contains an inductor, a resistor, and a capacitor for which $L = \frac{1}{2}$ h, $R = 10\ \Omega$, and $C = 0.01$ f, respectively. The voltage

$$E(t) = \begin{cases} 10, & 0 \le t < 5 \\ 0, & t \ge 5 \end{cases}$$

is applied to the circuit. Determine the instantaneous charge $q(t)$ on the capacitor for $t > 0$ if $q(0) = 0$ and $q'(0) = 0$.

47. A uniform cantilever beam of length L is embedded at its left end ($x = 0$) and free at its right end. Find the deflection $y(x)$ if the load per unit length is given by

$$w(x) = \frac{2w_0}{L}\left[\frac{L}{2} - x + \left(x - \frac{L}{2}\right)\mathscr{U}\left(x - \frac{L}{2}\right)\right].$$

48. When a uniform beam is supported by an elastic foundation, the differential equation for its deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} + ky = w(x),$$

where k is the modulus of the foundation and $-ky$ is the restoring force of the foundation that acts in the direction opposite to that of the load $w(x)$. See Figure 7.R.11. For algebraic convenience suppose that the differential equation is written as

$$\frac{d^4y}{dx^4} + 4a^4y = \frac{w(x)}{EI},$$

where $a = (k/4EI)^{1/4}$. Assume $L = \pi$ and $a = 1$. Find the deflection $y(x)$ of a beam that is supported on an elastic foundation when

(a) the beam is simply supported at both ends and a constant load w_0 is uniformly distributed along its length,

(b) the beam is embedded at both ends and $w(x)$ is a concentrated load w_0 applied at $x = \pi/2$. [*Hint*: In both parts of this problem use the table of Laplace transforms in Appendix C and the fact that $s^4 + 4 = (s^2 - 2s + 2)(s^2 + 2s + 2)$.]

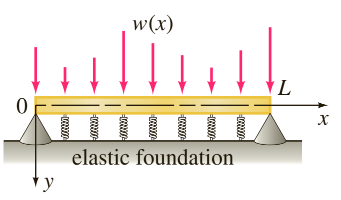


FIGURE 7.R.11 Beam on elastic foundation in Problem 48

49. (a) Suppose two identical pendulums are coupled by means of a spring with constant k. See Figure 7.R.12. Under the same assumptions made in the discussion preceding Example 3 in Section 7.6, it can be shown that when the displacement angles $\theta_1(t)$ and $\theta_2(t)$ are small, the system of linear differential equations describing the motion is

$$\theta_1'' + \frac{g}{l}\theta_1 = -\frac{k}{m}(\theta_1 - \theta_2)$$

$$\theta_2'' + \frac{g}{l}\theta_2 = \frac{k}{m}(\theta_1 - \theta_2).$$

Use the Laplace transform to solve the system when $\theta_1(0) = \theta_0$, $\theta_1'(0) = 0$, $\theta_2(0) = \psi_0$, $\theta_2'(0) = 0$, where θ_0 and ψ_0 are constants. For convenience let $\omega^2 = g/l$, $K = k/m$.

(b) Use the solution in part (a) to discuss the motion of the coupled pendulums in the special case when the initial conditions are $\theta_1(0) = \theta_0$, $\theta_1'(0) = 0$, $\theta_2(0) = \theta_0$, $\theta_2'(0) = 0$. When the initial conditions are $\theta_1(0) = \theta_0$, $\theta_1'(0) = 0$, $\theta_2(0) = -\theta_0$, $\theta_2'(0) = 0$.

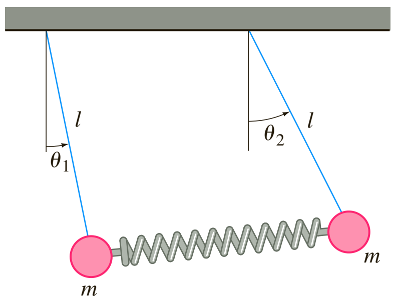


FIGURE 7.R.12 Coupled pendulums in Problem 49

50. Coulomb Friction Revisited In Problem 27 in Chapter 5 in Review we examined a spring/mass system in which a mass m slides over a dry horizontal surface whose coefficient of kinetic friction is a constant μ. The constant retarding force $f_k = \mu mg$ of the dry surface that acts opposite to the direction of motion is called Coulomb friction after the French physicist **Charles-Augustin de Coulomb** (1736–1806). You were asked to show that the piecewise-linear differential equation for the displacement $x(t)$ of the mass is given by

$$m\frac{d^2x}{dt^2} + kx = \begin{cases} f_k, & x' < 0 \text{ (motion to left)} \\ -f_k, & x' > 0 \text{ (motion to right)}. \end{cases}$$

(a) Suppose that the mass is released from rest from a point $x(0) = x_0 > 0$ and that there are no other external forces. Then the differential equations describing the motion of the mass m are

$$x'' + \omega^2 x = F, \quad 0 < t < T/2$$

$$x'' + \omega^2 x = -F, \quad T/2 < t < T$$

$$x'' + \omega^2 x = F, \quad T < t < 3T/2,$$

and so on, where $\omega^2 = k/m$, $F = f_k/m = \mu g$, $g = 32$, and $T = 2\pi/\omega$. Show that the times $0, T/2, T, 3T/2, \ldots$ correspond to $x'(t) = 0$.

(b) Explain why, in general, the initial displacement must satisfy $\omega^2|x_0| > F$.

(c) Explain why the interval $-F/\omega^2 \le x \le F/\omega^2$ is appropriately called the "dead zone" of the system.

(d) Use the Laplace transform and the concept of the meander function to solve for the displacement $x(t)$ for $t \ge 0$.

(e) Show that in the case $m = 1$, $k = 1$, $f_k = 1$, and $x_0 = 5.5$ that on the interval $[0, 2\pi)$ your solution agrees with parts (a) and (b) of Problem 28 in Chapter 5 in Review.

(f) Show that each successive oscillation is $2F/\omega^2$ shorter than the preceding one.

(g) Predict the long-term behavior of the system.

51. Range of a Projectile—No Air Resistance (a) A projectile, such as the canon ball shown in Figure 7.R.13, has weight $w = mg$ and initial velocity $\mathbf{v}_0$ that is tangent to its path of motion. If air resistance and all other forces except its weight are ignored, we saw in Problem 23 of Exercises 4.9 that motion of the projectile is described by the system of linear differential equations

$$m\frac{d^2x}{dt^2} = 0$$

$$m\frac{d^2y}{dt^2} = -mg.$$

Use the Laplace transform to solve this system subject to the initial conditions

$$x(0) = 0,\ x'(0) = v_0\cos\theta,\ y(0) = 0,\ y'(0) = v_0\sin\theta,$$

where $v_0 = |\mathbf{v}_0|$ is constant and θ is the constant angle of elevation shown in Figure 7.R.13 on page 330. The solutions $x(t)$ and $y(t)$ are parametric equations of the trajectory of the projectile.

(b) Use $x(t)$ in part (a) to eliminate the parameter t in $y(t)$. Use the resulting equation for y to show that the horizontal range R of the projectile is given by

$$R = \frac{v_0^2}{g}\sin 2\theta.$$

(c) From the formula in part (b), we see that R is a maximum when $\sin 2\theta = 1$ or when $\theta = \pi/4$. Show that the same range—less than the maximum—can be attained by firing the gun at either of two complementary angles θ and $\pi/2 - \theta$. The only difference is that the smaller angle results in a low trajectory whereas the larger angle gives a high trajectory.

(d) Suppose $g = 32$ ft/s^2, $\theta = 38°$, and $v_0 = 300$ ft/s. Use part (b) to find the horizontal range of the projectile. Find the time when the projectile hits the ground.

(e) Use the parametric equations $x(t)$ and $y(t)$ in part (a) along with the numerical data in part (d) to plot the ballistic curve of the projectile. Repeat with $\theta = 52°$ and $v_0 = 300$ ft/s. Superimpose both curves on the same coordinate system.

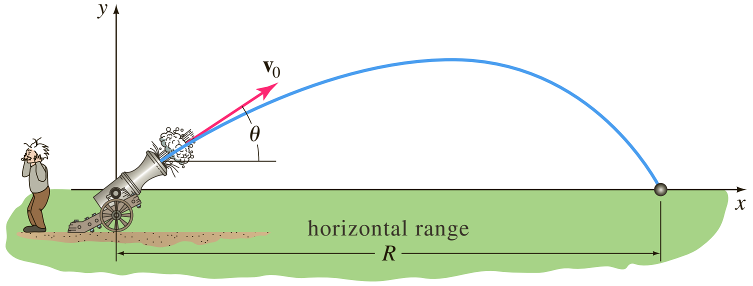


FIGURE 7.R.13 Projectile in Problem 51

52. Range of a Projectile—With Air Resistance **(a)** Now suppose that air resistance is a retarding force tangent to the path but acts opposite to the motion. If we take air resistance to be proportional to the velocity of the projectile, then we saw in Problem 24 of Exercises 4.9 that motion of the projectile is describe by the system of linear differential equations

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt}$$

$$m\frac{d^2y}{dt^2} = -mg - \beta\frac{dy}{dt},$$

where $\beta > 0$. Use the Laplace transform to solve this system subject to the initial conditions $x(0) = 0$, $x'(0) = v_0 \cos\theta$, $y(0) = 0$, $y'(0) = v_0 \sin\theta$, where $v_0 = |\mathbf{v}_0|$ and θ are constant.

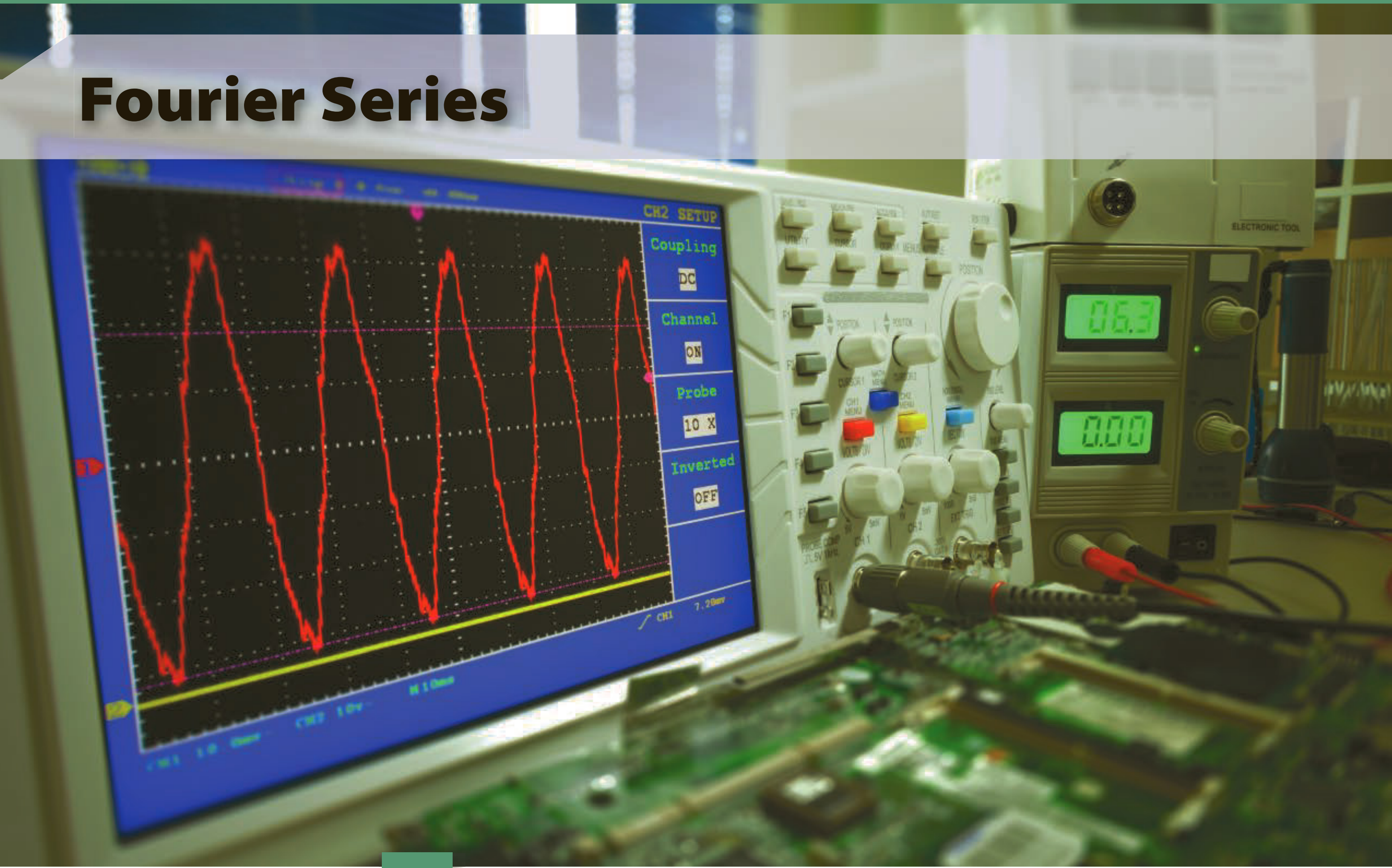
(b) Suppose $m = \frac{1}{4}$ slug, $g = 32$ ft/s^2, $\beta = 0.02$, $\theta = 38°$, and $v_0 = 300$ ft/s. Use a CAS to find the time when the projectile hits the ground and then compute its corresponding horizontal range.

(c) Repeat part (b) using the complementary angle $\theta = 52°$ and compare the range with that found in part (b). Does the property in part (c) of Problem 51 hold?

(d) Use the parametric equations $x(t)$ and $y(t)$ in part (a) along with the numerical data in part (b) to plot the ballistic curve of the projectile. Repeat with the same numerical data in part (b) but take $\theta = 52°$. Superimpose both curves on the same coordinate system. Compare these curves with those obtained in part (e) of Problem 51.

11

Fourier Series



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When you studied vectors in calculus you saw that two nonzero vectors are orthogonal when their inner (or dot) product is zero. Beyond calculus the notions of vectors, inner product, and orthogonality often lose their geometric interpretation. These concepts have been generalized; it is perfectly common to think of a function as a vector. We can then say two functions are orthogonal when their inner product is zero. We will see in this chapter that the inner product of these vectors (functions) is actually a definite integral.

The concepts of orthogonal functions and the expansion of a function in terms of an infinite set of orthogonal functions is fundamental to the material in Chapters 12 and 13.

11.1 Orthogonal Functions

INTRODUCTION The concepts of geometric vectors in two and three dimensions, orthogonal or perpendicular vectors, and the inner product of two vectors have been generalized. It is perfectly routine in mathematics to think of a function as a vector. In this section we will examine an inner product that is different from the one you studied in calculus. Using this new inner product, we define orthogonal functions and sets of orthogonal functions. Another topic in a typical calculus course is the expansion of a function f in a power series. In this section we will also see how to expand a suitable function f in terms of an infinite set of orthogonal functions.

INNER PRODUCT Recall that if $\mathbf{u}$ and $\mathbf{v}$ are two vectors in R^3 or 3-space, then the inner product $(\mathbf{u}, \mathbf{v})$ (in calculus this is called the dot product and written as $\mathbf{u} \cdot \mathbf{v}$) possesses the following properties:

(i) $(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u})$,

(ii) $(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v})$, k a scalar,

(iii) $(\mathbf{u}, \mathbf{u}) = 0$ if $\mathbf{u} = \mathbf{0}$ and $(\mathbf{u}, \mathbf{u}) > 0$ if $\mathbf{u} \neq \mathbf{0}$,

(iv) $(\mathbf{u} + \mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$.

We expect that any generalization of the inner product concept should have these same properties.

Suppose that f_1 and f_2 are functions defined on an interval $[a, b]$.* Since a *definite integral* on $[a, b]$ of the product $f_1(x)f_2(x)$ possesses the foregoing properties (i)–(iv) of an inner product whenever the integral exists, we are prompted to make the following definition.

DEFINITION 11.1.1 Inner Product of Functions

The **inner product** of two functions f_1 and f_2 on an interval $[a, b]$ is the number

$$(f_1, f_2) = \int_a^b f_1(x)f_2(x)\,dx.$$

ORTHOGONAL FUNCTIONS Motivated by the fact that two geometric vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal whenever their inner product is zero, we define **orthogonal functions** in a similar manner.

DEFINITION 11.1.2 Orthogonal Functions

Two functions f_1 and f_2 are **orthogonal** on an interval $[a, b]$ if

$$(f_1, f_2) = \int_a^b f_1(x)f_2(x)\,dx = 0. \tag{1}$$

*The interval could also be $(-\infty, \infty)$, $[0, \infty)$, and so on.

EXAMPLE 1 Orthogonal Functions

(a) The functions $f_1(x) = x^2$ and $f_2(x) = x^3$ are orthogonal on the interval $[-1, 1]$, since

$$(f_1, f_2) = \int_{-1}^{1} x^2 \cdot x^3\, dx = \int_{-1}^{1} x^5\, dx = \frac{1}{6}x^6\Big|_{-1}^{1} = \frac{1}{6}(1 - 1) = 0.$$

(b) The functions $f_1(x) = x^2$ and $f_2(x) = x^4$ are *not* orthogonal on the interval $[-1, 1]$, since

$$(f_1, f_2) = \int_{-1}^{1} x^2 \cdot x^4\, dx = \int_{-1}^{1} x^6\, dx = \frac{1}{7}x^7\Big|_{-1}^{1} = \frac{1}{7}(1 - (-1)) = \frac{2}{7} \neq 0. \quad \blacksquare$$

Unlike in vector analysis, in which the word *orthogonal* is a synonym for *perpendicular*, in this present context the term *orthogonal* and condition (1) have no geometric significance. Note that the zero function is orthogonal to every function.

ORTHOGONAL SETS We are primarily interested in infinite sets of orthogonal functions that are all defined on the same interval $[a, b]$.

DEFINITION 11.1.3 Orthogonal Set

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\phi_n(x)\, dx = 0, \quad m \neq n. \tag{2}$$

ORTHONORMAL SETS The norm, or length $\|\mathbf{u}\|$, of a vector $\mathbf{u}$ can be expressed in terms of the inner product. The expression $(\mathbf{u}, \mathbf{u}) = \|\mathbf{u}\|^2$ is called the square norm, and so the norm is $\|\mathbf{u}\| = \sqrt{(\mathbf{u}, \mathbf{u})}$. Similarly, the **square norm** of a function ϕ_n is $\|\phi_n(x)\|^2 = (\phi_n, \phi_n)$, and so the **norm,** or its generalized length, is $\|\phi_n(x)\| = \sqrt{(\phi_n, \phi_n)}$. In other words, the square norm and norm of a function ϕ_n in an orthogonal set $\{\phi_n(x)\}$ are, respectively,

$$\|\phi_n(x)\|^2 = \int_a^b \phi_n^2(x)\, dx \quad \text{and} \quad \|\phi_n(x)\| = \sqrt{\int_a^b \phi_n^2(x)\, dx}. \tag{3}$$

If $\{\phi_n(x)\}$ is an orthogonal set of functions on the interval $[a, b]$ with the additional property that $\|\phi_n(x)\| = 1$ for $n = 0, 1, 2, \ldots$, then $\{\phi_n(x)\}$ is said to be an **orthonormal set** on the interval.

EXAMPLE 2 Orthogonal Set of Functions

Show that the set $\{1, \cos x, \cos 2x, \ldots\}$ is orthogonal on the interval $[-\pi, \pi]$.

SOLUTION If we make the identification $\phi_0(x) = 1$ and $\phi_n(x) = \cos nx$, we must then show that $\int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)\, dx = 0$, $n \neq 0$, and $\int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)\, dx = 0$, $m \neq n$. We have, in the first case,

$$\begin{aligned}(\phi_0, \phi_n) &= \int_{-\pi}^{\pi} \phi_0(x)\phi_n(x)\, dx = \int_{-\pi}^{\pi} \cos nx\, dx \\ &= \frac{1}{n}\sin nx\Big|_{-\pi}^{\pi} = \frac{1}{n}[\sin n\pi - \sin(-n\pi)] = 0, \qquad n \neq 0,\end{aligned}$$

and, in the second,

$$\begin{aligned}(\phi_m, \phi_n) &= \int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)\,dx \\ &= \int_{-\pi}^{\pi} \cos mx \cos nx\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m+n)x + \cos(m-n)x]\,dx \qquad \leftarrow \text{trigonometric identity} \\ &= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_{-\pi}^{\pi} = 0, \quad m \neq n.\end{aligned}$$

EXAMPLE 3 Norms

Find the norm of each function in the orthogonal set given in Example 2.

SOLUTION For $\phi_0(x) = 1$ we have, from (3),

$$\|\phi_0(x)\|^2 = \int_{-\pi}^{\pi} dx = 2\pi,$$

so $\|\phi_0(x)\| = \sqrt{2\pi}$. For $\phi_n(x) = \cos nx$, $n > 0$, it follows that

$$\|\phi_n(x)\|^2 = \int_{-\pi}^{\pi} \cos^2 nx\,dx = \frac{1}{2}\int_{-\pi}^{\pi} [1 + \cos 2nx]\,dx = \pi.$$

Thus for $n > 0$, $\|\phi_n(x)\| = \sqrt{\pi}$.

NORMALIZATION Any orthogonal set of nonzero functions $\{\phi_n(x)\}$, $n = 0, 1, 2, \ldots$ can be made into an orthonormal set by **normalizing** each function in the set, that is, by dividing each function by its norm. The next example illustrates the idea.

EXAMPLE 4 Orthonormal Set

In Example 2 we proved that the set

$$\{1, \cos x, \cos 2x, \ldots\}$$

is orthogonal on the interval $[-\pi, \pi]$. In Example 3, we then saw that the norms of the functions in the foregoing set are

$$\|\phi_0(x)\| = \|1\| = \sqrt{2\pi} \qquad \text{and} \qquad \|\phi_n(x)\| = \|\cos nx\| = \sqrt{\pi},\ n = 1, 2, \ldots.$$

By dividing each function by its norm we obtain the set

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots\right\}$$

which is orthonormal on the interval $[-\pi, \pi]$.

VECTOR ANALOGY In the introduction to this section, we stated that our purpose for studying orthogonal functions is to be able to expand a function in terms of an infinite set $\{\phi_n(x)\}$ of orthogonal functions. To motivate this concept we shall make one more analogy between vectors and functions. Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are three mutually orthogonal nonzero vectors in R^3. Such an orthogonal set can be used as a basis for R^3; this means any three-dimensional vector $\mathbf{u}$ can be written as a linear combination

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3, \tag{4}$$

where the c_i, $i = 1, 2, 3$, are scalars called the **components** of the vector $\mathbf{u}$. Each component c_i can be expressed in terms of $\mathbf{u}$ and the corresponding vector $\mathbf{v}_i$. To see this, we take the inner product of (4) with $\mathbf{v}_1$:

$$(\mathbf{u}, \mathbf{v}_1) = c_1(\mathbf{v}_1, \mathbf{v}_1) + c_2(\mathbf{v}_2, \mathbf{v}_1) + c_3(\mathbf{v}_3, \mathbf{v}_1) = c_1\|\mathbf{v}_1\|^2 + c_2 \cdot 0 + c_3 \cdot 0.$$

Hence
$$c_1 = \frac{(\mathbf{u}, \mathbf{v}_1)}{\|\mathbf{v}_1\|^2}.$$

In like manner we find that the components c_2 and c_3 are given by

$$c_2 = \frac{(\mathbf{u}, \mathbf{v}_2)}{\|\mathbf{v}_2\|^2} \quad \text{and} \quad c_3 = \frac{(\mathbf{u}, \mathbf{v}_3)}{\|\mathbf{v}_3\|^2}.$$

Hence (4) can be expressed as

$$\mathbf{u} = \frac{(\mathbf{u}, \mathbf{v}_1)}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{(\mathbf{u}, \mathbf{v}_2)}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \frac{(\mathbf{u}, \mathbf{v}_3)}{\|\mathbf{v}_3\|^2}\mathbf{v}_3 = \sum_{n=1}^{3} \frac{(\mathbf{u}, \mathbf{v}_n)}{\|\mathbf{v}_n\|^2}\mathbf{v}_n. \tag{5}$$

ORTHOGONAL SERIES EXPANSION Suppose $\{\phi_n(x)\}$ is an infinite orthogonal set of functions on an interval $[a, b]$. We ask: If $y = f(x)$ is a function defined on the interval $[a, b]$, is it possible to determine a set of coefficients c_n, $n = 0, 1, 2, \ldots$, for which

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \cdots + c_n\phi_n(x) + \cdots ? \tag{6}$$

As in the foregoing discussion on finding components of a vector we can find the desired coefficients c_n by using the inner product. Multiplying (6) by $\phi_m(x)$ and integrating over the interval $[a, b]$ gives

$$\begin{aligned}\int_a^b f(x)\phi_m(x)\,dx &= c_0\int_a^b \phi_0(x)\phi_m(x)\,dx + c_1\int_a^b \phi_1(x)\phi_m(x)\,dx + \cdots + c_n\int_a^b \phi_n(x)\phi_m(x)\,dx + \cdots \\ &= c_0(\phi_0, \phi_m) + c_1(\phi_1, \phi_m) + \cdots + c_n(\phi_n, \phi_m) + \cdots.\end{aligned}$$

By orthogonality each term on the right-hand side of the last equation is zero *except* when $m = n$. In this case we have

$$\int_a^b f(x)\phi_n(x)\,dx = c_n\int_a^b \phi_n^2(x)\,dx.$$

It follows that the required coefficients c_n are given by

$$c_n = \frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}, \qquad n = 0, 1, 2, \ldots.$$

In other words,
$$f(x) = \sum_{n=0}^{\infty} c_n\phi_n(x), \tag{7}$$

where $$c_n = \frac{\int_a^b f(x)\,\phi_n(x)\,dx}{\|\phi_n(x)\|^2}. \tag{8}$$

With inner product notation, (7) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{\|\phi_n(x)\|^2}\,\phi_n(x). \tag{9}$$

Thus (9) is seen to be the function analogue of the vector result given in (5).

DEFINITION 11.1.4 Orthogonal Set/Weight Function

A set of real-valued functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is said to be **orthogonal with respect to a weight function** $w(x)$ on an interval $[a, b]$ if

$$\int_a^b w(x)\phi_m(x)\phi_n(x)\,dx = 0, \quad m \neq n.$$

The usual assumption is that $w(x) > 0$ on the interval of orthogonality $[a, b]$. The set $\{1, \cos x, \cos 2x, \ldots\}$ in Example 2 is orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-\pi, \pi]$.

If $\{\phi_n(x)\}$ is orthogonal with respect to a weight function $w(x)$ on the interval $[a, b]$, then multiplying (6) by $w(x)\phi_n(x)$ and integrating yields

$$c_n = \frac{\int_a^b f(x)\,w(x)\phi_n(x)\,dx}{\|\phi_n(x)\|^2}, \tag{10}$$

where $$\|\phi_n(x)\|^2 = \int_a^b w(x)\phi_n^2(x)\,dx. \tag{11}$$

The series (7) with coefficients c_n given by either (8) or (10) is said to be an **orthogonal series expansion** of f or a **generalized Fourier series.**

COMPLETE SETS The procedure outlined for determining the coefficients c_n in (8) was *formal*; that is, fundamental questions about whether or not an orthogonal series expansion of a function f such as (7) actually converges to the function were ignored. It turns out that for some specific orthogonal sets such series expansions do indeed converge to the function. In the subsequent sections of this chapter, we will state conditions on the type of functions defined on the interval $[a, b]$ of orthogonality that are sufficient to guarantee that an orthogonal series coverges to its function f. To make one last point about the kind of set $\{\phi_n(x)\}$ must be, let's go back to the vector analogy on pages 427–428. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a set of mutually orthogonal nonzero vectors in R^3, we can say that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is **complete** in R^3 because three such vectors is all we need to write any vector $\mathbf{u}$ in that space in the form (5). We could not write (5) using fewer than three vectors; a set, say, $\{\mathbf{v}_1, \mathbf{v}_2\}$, would be incomplete in R^3. As a necessary consequence of completeness of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ it is easy to see that the only vector $\mathbf{u}$ in 3-space that is orthogonal to each of the vectors $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ is the zero vector. If $\mathbf{u}$ were orthogonal to $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, then $(\mathbf{u}, \mathbf{v}_1) = 0$, $(\mathbf{u}, \mathbf{v}_2) = 0$, $(\mathbf{u}, \mathbf{v}_3) = 0$ and (5) shows $\mathbf{u} = \mathbf{0}$. Similarly, in the discussion of orthogonal series expansions, the function f as well as each of the functions in $\{\phi_n(x)\}$ are part of a larger class, or *space*, S of functions. The class S could be, say, the set of continuous functions on an interval $[a, b]$, or the set of piecewise-continuous functions on $[a, b]$. We also want the set $\{\phi_n(x)\}$ to be **complete** in S in the sense that $\{\phi_n(x)\}$ must contain sufficiently many functions so that every function f in S can be written in the form (7). As in our vector analogy, this means that the only function that is orthogonal to each member of the set $\{\phi_n(x)\}$ is the zero function. See Problem 25 in Exercises 11.1.

We assume for the remainder of the discussion in this chapter that any orthogonal set used in a series expansion of a function is complete in some class of functions S.

EXERCISES 11.1

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–6 show that the given functions are orthogonal on the indicated interval.

1. $f_1(x) = x, f_2(x) = x^2$; $[-2, 2]$

2. $f_1(x) = x^3, f_2(x) = x^2 + 1$; $[-1, 1]$

3. $f_1(x) = e^x, f_2(x) = xe^{-x} - e^{-x}$; $[0, 2]$

4. $f_1(x) = \cos x, f_2(x) = \sin^2 x$; $[0, \pi]$

5. $f_1(x) = x, f_2(x) = \cos 2x$; $[-\pi/2, \pi/2]$

6. $f_1(x) = e^x, f_2(x) = \sin x$; $[\pi/4, 5\pi/4]$

In Problems 7–12 show that the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set.

7. $\{\sin x, \sin 3x, \sin 5x, \ldots\}$; $[0, \pi/2]$

8. $\{\cos x, \cos 3x, \cos 5x, \ldots\}$; $[0, \pi/2]$

9. $\{\sin nx\}, n = 1, 2, 3, \ldots$; $[0, \pi]$

10. $\left\{\sin \frac{n\pi}{p}x\right\}, n = 1, 2, 3, \ldots$; $[0, p]$

11. $\left\{1, \cos \frac{n\pi}{p}x\right\}, n = 1, 2, 3, \ldots$; $[0, p]$

12. $\left\{1, \cos \frac{n\pi}{p}x, \sin \frac{m\pi}{p}x\right\}$, $n = 1, 2, 3, \ldots,$
$m = 1, 2, 3, \ldots$; $[-p, p]$

In Problems 13 and 14 verify by direct integration that the functions are orthogonal with respect to the indicated weight function $w(x)$ on the given interval.

13. $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$;
$w(x) = e^{-x^2}, (-\infty, \infty)$

14. $L_0(x) = 1$, $L_1(x) = -x + 1$, $L_2(x) = \frac{1}{2}x^2 - 2x + 1$;
$w(x) = e^{-x}, [0, \infty)$

15. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$. Show that $\int_a^b \phi_n(x)\,dx = 0$ for $n = 1, 2, \ldots$.

16. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$ such that $\phi_0(x) = 1$ and $\phi_1(x) = x$. Show that $\int_a^b (\alpha x + \beta)\phi_n(x)\,dx = 0$ for $n = 2, 3, \ldots$ and any constants α and β.

17. Let $\{\phi_n(x)\}$ be an orthogonal set of functions on $[a, b]$. Show that $\|\phi_m(x) + \phi_n(x)\|^2 = \|\phi_m(x)\|^2 + \|\phi_n(x)\|^2$, $m \neq n$.

18. From Problem 1 we know that $f_1(x) = x$ and $f_2(x) = x^2$ are orthogonal on the interval $[-2, 2]$. Find constants c_1 and c_2 such that $f_3(x) = x + c_1x^2 + c_2x^3$ is orthogonal to both f_1 and f_2 on the same interval.

A real-valued function is said to be **periodic** with period $T \neq 0$ if $f(x + T) = f(x)$ for all x in the domain of f. If T is the smallest positive value for which $f(x + T) = f(x)$ holds, then T is called the **fundamental period** of f. In Problems 19–24 determine the fundamental period T of the given function.

19. $f(x) = \cos 2\pi x$

20. $f(x) = \sin \frac{4}{L}x$

21. $f(x) = \sin x + \sin 2x$

22. $f(x) = \sin 2x + \cos 4x$

23. $f(x) = \sin 3x + \cos 2x$

24. $\sin^2 \pi x$

Discussion Problems

25. **(a)** In Problem 9 we saw that the set $\{\sin nx\}$, $n = 1, 2, 3, \ldots$ is orthogonal on the interval $[0, \pi]$. Show that the set is also orthogonal on the interval $[-\pi, \pi]$.

(b) Show that the set that is orthogonal on the interval $[-\pi, \pi]$ is not complete. [*Hint*: Consider $f(x) = 1$.]

26. An orthogonal set can be constructed out of any linearly independent set $\{f_0(x), f_1(x), f_2(x), \ldots\}$ of real-valued functions continuous on an interval $[a, b]$ using the **Gram-Schmidt orthogonalization process.** With the inner product $(f_n, \phi_n) = \int_a^b f_n(x)\phi_n(x)\,dx$, define the functions in the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ to be

$$\phi_0(x) = f_0(x)$$

$$\phi_1(x) = f_1(x) - \frac{(f_1, \phi_0)}{\|\phi_0\|^2}\phi_0(x)$$

$$\phi_2(x) = f_2(x) - \frac{(f_2, \phi_0)}{\|\phi_0\|^2}\phi_0(x) - \frac{(f_2, \phi_1)}{\|\phi_1\|^2}\phi_1(x)$$

$$\vdots$$

and so on.

(a) Write out $\phi_3(x)$ in the set B'.

(b) By construction, the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is orthogonal on $[a, b]$. Demonstrate that $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ are mutually orthogonal.

27. Consider the set of functions $\{1, x, x^2, x^3, \ldots\}$ defined on the interval $[-1, 1]$. Apply the Gram-Schmidt process given in Problem 26 to this set and find $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$ of the orthogonal set B'.

28. Relate the orthogonal set B' in Problem 27 with a set of polynomials found in an earlier chapter of this text.

11.2 Fourier Series

INTRODUCTION We have just seen that if $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is an orthogonal set on an interval $[a, b]$ and if f is a function defined on the same interval, then we can formally expand f in an orthogonal series

$$c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots,$$

where the coefficients c_n are determined by using the inner product concept. The orthogonal set of trigonometric functions

$$\left\{1,\ \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x,\ \sin\frac{3\pi}{p}x, \ldots\right\} \tag{1}$$

will be of particular importance later on in the solution of certain kinds of boundary-value problems involving linear partial differential equations. The set (1) is orthogonal on the interval $[-p, p]$. See Problem 12 in Exercises 11.1.

A TRIGONOMETRIC SERIES Suppose that f is a function defined on the interval $(-p, p)$ and can be expanded in an orthogonal series consisting of the trigonometric functions in the orthogonal set (1); that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right). \tag{2}$$

The coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ can be determined in exactly the same formal manner as in the general discussion of orthogonal series expansions on page 428. Before proceeding, note that we have chosen to write the coefficient of 1 in the set (1) as $\frac{1}{2}a_0$ rather than a_0. This is for convenience only; the formula of a_n will then reduce to a_0 for $n = 0$.

Now integrating both sides of (2) from $-p$ to p gives

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx + \sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\sin\frac{n\pi}{p}x\,dx\right). \tag{3}$$

Since $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$, $n \ge 1$ are orthogonal to 1 on the interval, the right side of (3) reduces to a single term:

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx = \frac{a_0}{2}x\Big|_{-p}^{p} = pa_0.$$

Solving for a_0 yields

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx. \tag{4}$$

Now we multiply (2) by $\cos(m\pi x/p)$ and integrate:

$$\begin{aligned}\int_{-p}^{p} f(x)\cos\frac{m\pi}{p}x\,dx &= \frac{a_0}{2}\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx\\ &\quad+\sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx\right).\end{aligned} \tag{5}$$

By orthogonality we have

$$\int_{-p}^{p} \cos\frac{m\pi}{p}x\,dx = 0, \quad m > 0, \qquad \int_{-p}^{p} \cos\frac{m\pi}{p}x \sin\frac{n\pi}{p}x\,dx = 0,$$

and
$$\int_{-p}^{p} \cos\frac{m\pi}{p}x \cos\frac{n\pi}{p}x\,dx = \begin{cases} 0, & m \neq n \\ p, & m = n. \end{cases}$$

Thus (5) reduces to
$$\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx = a_n p,$$

and so
$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx. \tag{6}$$

Finally, if we multiply (2) by $\sin(m\pi x/p)$, integrate, and make use of the results

$$\int_{-p}^{p} \sin\frac{m\pi}{p}x\,dx = 0, \quad m > 0, \qquad \int_{-p}^{p} \sin\frac{m\pi}{p}x \cos\frac{n\pi}{p}x\,dx = 0,$$

and
$$\int_{-p}^{p} \sin\frac{m\pi}{p}x \sin\frac{n\pi}{p}x\,dx = \begin{cases} 0, & m \neq n \\ p, & m = n, \end{cases}$$

we find that
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{7}$$

The trigonometric series (2) with coefficients a_0, a_n, and b_n defined by (4), (6), and (7), respectively, is said to be the **Fourier series** of *f*. Although the French mathematical physicist **Jean Baptiste Joseph Fourier** (1768–1830) did not invent the series that bears his name, he is at least responsible for sparking the interest of mathematicians in trigonometric series by his less than rigorous use of them in his researches on the conduction of heat. The formulas in (4), (6), and (7) that give the coefficients in a Fourier series are known as the **Euler formulas.**

DEFINITION 11.2.1 Fourier Series

The **Fourier series** of a function *f* defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right) \tag{8}$$

where
$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{9}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\,dx \tag{10}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\,dx. \tag{11}$$

CONVERGENCE OF A FOURIER SERIES In the absence of any stated conditions that guarantee the validity of the steps leading to the coefficients a_0, a_n, and b_n, the equality sign in (8) should not be taken in a strict or literal sense. Some texts use the symbol $\sim$ to signify that (8) is simply the corresponding trigonometric series with coefficients generated using *f* in formulas (9)–(11). In view of the fact that most functions in applications are of the type that guarantee convergence of the series, we shall use the equality symbol. Is it possible for a series (8) to converge at a number *x* in the interval $(-p, p)$, and yet not be equal to $f(x)$? The answer is an emphatic *Yes*.

PIECEWISE-CONTINUOUS FUNCTIONS Before stating conditions under which a Fourier series converges, we need to pause briefly to review two topics from the first semester of calculus. We shall use the symbols $f(x+)$ and $f(x-)$ to denote the one-sided limits

$$f(x+) = \lim_{\substack{h\to 0\\ h>0}} f(x+h), \quad f(x-) = \lim_{\substack{h\to 0\\ h>0}} f(x-h),$$

In Section 7.1 we defined piecewise continuity on an unbounded interval $[0, \infty)$. See Figure 7.1.1 on page 282.

called, respectively, the **right-** and **left-hand limits** of f at x. A function f is said to be **piecewise continuous** on a closed interval $[a, b]$ if there are

- a finite number of points $x_1 < x_2 < \cdots < x_n$ in $[a, b]$ at which f has a finite (or jump) discontinuity, and
- f is continuous on each open interval (x_k, x_{k+1}).

As a consequence of this definition, the one-sided limits $f(x+)$ and $f(x-)$ must exist at every x satisfying $a < x < b$. The limits $f(a+)$ and $f(b-)$ must also exist but it is not required that f be continuous or even defined at either a or b.

Our first theorem gives sufficient conditions for convergence of a Fourier series at a point x.

THEOREM 11.2.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[-p, p]$. Then for all x in the interval $(-p, p)$, the Fourier series of f converges to $f(x)$ at a point continuity. At a point of discontinuity the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ are the right- and left-hand limits of f at x, respectively.

EXAMPLE 1 Expansion in a Fourier Series

Expand
$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases} \tag{12}$$
in a Fourier series.

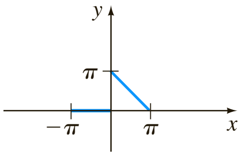


FIGURE 11.2.1 Piecewise-continuous function f in Example 1

SOLUTION The graph of f is given in Figure 11.2.1. With $p = \pi$ we have from (9) and (10) that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi - x)\,dx\right] = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}(\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right] \quad \leftarrow \text{integration by parts}$$

$$= -\frac{1}{n\pi}\,\frac{\cos nx}{n}\bigg|_0^{\pi} = \frac{1 - (-1)^n}{n^2\pi},$$

where we have used $\cos n\pi = (-1)^n$. In like manner we find from (11) that

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{n}.$$

Therefore $$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}. \tag{13}$$ ■

Note that a_n defined by (10) reduces to a_0 given by (9) when we set $n = 0$. But as Example 1 shows, this might not be the case *after* the integral for a_n is evaluated.

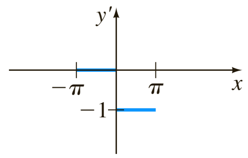


FIGURE 11.2.2 Piecewise-continuous derivative f' in Example 2

EXAMPLE 2 Example 1 Revisited

The equality in (13) of Example 1 is justified because both f and f' are piecewise continuous on the interval $[-\pi, \pi]$. See Figures 11.2.1 and 11.2.2. Because f is continuous for every x in the interval $(-\pi, \pi)$, except at $x = 0$, the series (13) will converge to the value $f(x)$. At $x = 0$ the function f is discontinuous, so the series (13) will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2}.$$ ■

PERIODIC EXTENSION Observe that each of the functions in the basic set (1) has a different fundamental period*—namely, $2p/n$, $n \geq 1$—but since a positive integer multiple of a period is also a period, we see that all the functions have in common the period $2p$. (Verify.) Hence the right-hand side of (2) is $2p$-periodic; indeed, $2p$ is the **fundamental period** of the sum. We conclude that a Fourier series not only represents the function on the interval $(-p, p)$ but also gives the **periodic extension** of f outside this interval. We can now apply Theorem 11.2.1 to the periodic extension of f, or we may assume from the outset that the given function is periodic with period $2p$; that is, $f(x + 2p) = f(x)$. When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series (8) converges to the average

$$\frac{f(p-) + f(-p+)}{2}$$

at these endpoints and to this value extended periodically to $\pm 3p$, $\pm 5p$, $\pm 7p$, and so on.

EXAMPLE 3 Example 1 Revisited

The Fourier series (13) in Example 1 converges to the periodic extension of the function (12) on the entire x-axis. See Figure 11.2.3. At $0, \pm 2\pi, \pm 4\pi, \ldots$ and at $\pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ the series converges to the values

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi-) + f(-\pi+)}{2} = 0,$$

respectively. The solid black dots in Figure 11.2.3 represent the value $\pi/2$.

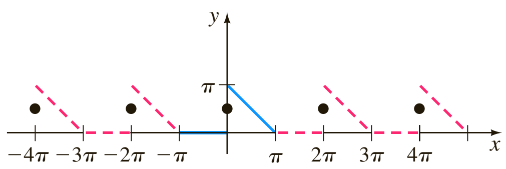


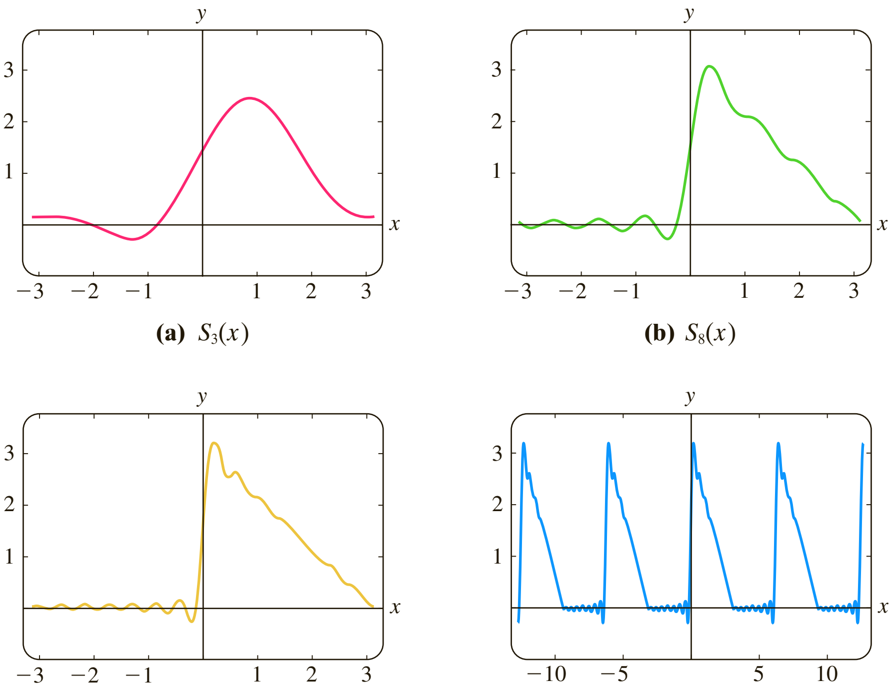
FIGURE 11.2.3 Periodic extension of function f shown in Figure 11.2.1 ■

*See Problems 19–24 in Exercises 11.1.

SEQUENCE OF PARTIAL SUMS It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. For example, the first three partial sums of (13) in Example 1 are

$$S_1(x) = \frac{\pi}{4}, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x, \quad \text{and} \quad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x.$$

In Figure 11.2.4 we have used a CAS to graph the partial sums $S_3(x)$, $S_8(x)$, and $S_{15}(x)$ of (13) on the interval $(-\pi, \pi)$. Figure 11.2.4(d) shows the periodic extension using $S_{15}(x)$ on $(-4\pi, 4\pi)$.



(a) $S_3(x)$ **(b)** $S_8(x)$ **(c)** $S_{15}(x)$ **(d)** $S_{15}(x)$

FIGURE 11.2.4 Partial sums of Fourier series (13) in Example 1

EXERCISES 11.2

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–16 find the Fourier series of f on the given interval. Give the number to which the Fourier series converges at a point of discontinuity of f.

1. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

2. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$

3. $f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

4. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

5. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

6. $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi \end{cases}$

7. $f(x) = x + \pi, \quad -\pi < x < \pi$

8. $f(x) = 3 - 2x, \quad -\pi < x < \pi$

9. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$

10. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \le x < \pi/2 \end{cases}$

11. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

12. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

13. $f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \le x < 5 \end{cases}$

14. $f(x) = \begin{cases} 2 + x, & -2 < x < 0 \\ 2, & 0 \le x < 2 \end{cases}$

15. $f(x) = e^x, \quad -\pi < x < \pi$

16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ e^x - 1, & 0 \le x < \pi \end{cases}$

In Problems 17 and 18 sketch the periodic extension of the indicated function.

17. The function f in Problem 9

18. The function f in Problem 14

19. Use the result of Problem 5 to show that

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

20. Use Problem 19 to find a series that gives the numerical value of $\pi^2/8$.

21. Use the result of Problem 7 to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

22. Use the result of Problem 9 to show that

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \cdots.$$

23. **(a)** Use the complex exponential form of the cosine and sine,

$$\cos \frac{n\pi}{p} x = \frac{e^{in\pi x/p} + e^{-in\pi x/p}}{2}$$

$$\sin \frac{n\pi}{p} x = \frac{e^{in\pi x/p} - e^{-in\pi x/p}}{2i},$$

to show that (8) can be written in the **complex form**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/p},$$

where

$$c_0 = \frac{a_0}{2}, \quad c_n = \frac{a_n - ib_n}{2}, \quad \text{and} \quad c_{-n} = \frac{a_n + ib_n}{2},$$

where $n = 1, 2, 3, \ldots$.

(b) Show that c_0, c_n, and c_{-n} of part (a) can be written as one integral

$$c_n = \frac{1}{2p} \int_{-p}^{p} f(x) e^{-in\pi x/p}\, dx, \quad n = 0, \pm 1, \pm 2, \ldots.$$

24. Use the results of Problem 23 to find the complex form of the Fourier series of $f(x) = e^{-x}$ on the interval $[-\pi, \pi]$.

11.3 Fourier Cosine and Sine Series

INTRODUCTION The effort that is expended in evaluation of the definite integrals that define the coefficients the a_0, a_n, and b_n in the expansion of a function f in a Fourier series is reduced significantly when f is either an even or an odd function. Recall that a function f is said to be

$$\textbf{even if } f(-x) = f(x) \qquad \text{and} \qquad \textbf{odd if } f(-x) = -f(x).$$

On a symmetric interval such as $(-p, p)$ the graph of an even function possesses symmetry with respect to the y-axis, whereas the graph of an odd function possesses symmetry with respect to the origin.

EVEN AND ODD FUNCTIONS It is likely that the origin of the terms *even* and *odd* derives from the fact that the graphs of polynomial functions that consist of all even powers of x are symmetric with respect to the y-axis, whereas graphs of polynomials that consist of all odd powers of x are symmetric with respect to origin. For example,

$$f(x) = x^{\overset{\text{even integer}}{\downarrow}2} \text{ is even} \quad \text{since } f(-x) = (-x)^2 = x^2 = f(x)$$

$$f(x) = x^{\overset{\text{odd integer}}{\downarrow}3} \text{ is odd} \quad \text{since } f(-x) = (-x)^3 = -x^3 = -f(x).$$

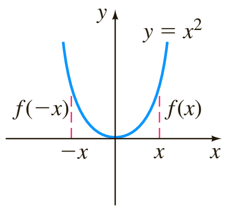


FIGURE 11.3.1 Even function; graph symmetric with respect to y-axis

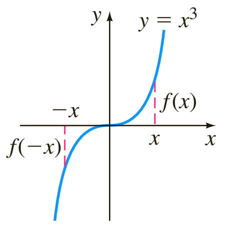


FIGURE 11.3.2 Odd function; graph symmetric with respect to origin

See Figures 11.3.1 and 11.3.2. The trigonometric cosine and sine functions are even and odd functions, respectively, since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. The exponential functions $f(x) = e^x$ and $f(x) = e^{-x}$ are neither odd nor even.

PROPERTIES The following theorem lists some properties of even and odd functions.

THEOREM 11.3.1 Properties of Even/Odd Functions

(a) The product of two even functions is even.
(b) The product of two odd functions is even.
(c) The product of an even function and an odd function is odd.
(d) The sum (difference) of two even functions is even.
(e) The sum (difference) of two odd functions is odd.
(f) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.
(g) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

PROOF OF (b) Let us suppose that f and g are odd functions. Then we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$. If we define the product of f and g as $F(x) = f(x)g(x)$, then

$$F(-x) = f(-x)\,g(-x) = (-f(x))(-g(x)) = f(x)\,g(x) = F(x).$$

This shows that the product F of two odd functions is an even function. The proofs of the remaining properties are left as exercises. See Problem 52 in Exercises 11.3. ■

COSINE AND SINE SERIES If f is an even function on $(-p, p)$, then in view of the foregoing properties the coefficients (9), (10), and (11) of Section 11.2 become

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi}{p}x}_{\text{even}}\,dx = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\frac{n\pi}{p}x}_{\text{odd}}\,dx = 0.$$

Similarly, when f is odd on the interval $(-p, p)$,

$$a_n = 0, \quad n = 0, 1, 2, \ldots, \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx.$$

We summarize the results in the following definition.

DEFINITION 11.3.1 Fourier Cosine and Sine Series

(*i*) The Fourier series of an even function f defined on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x, \tag{1}$$

(*continued*)

where
$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \tag{2}$$
$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx. \tag{3}$$

(ii) The Fourier series of an odd function f defined on the interval $(-p, p)$ is the **sine series**
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x, \tag{4}$$
where
$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx. \tag{5}$$

Because the term $\sin(n\pi x/p)$ is 0 at $x = -p$, $x = 0$, and $x = p$, a sine series (4) converges to 0 at those points regardless of whether f is defined at these points.

EXAMPLE 1 Expansion in a Sine Series

Expand $f(x) = x$, $-2 < x < 2$ in a Fourier series.

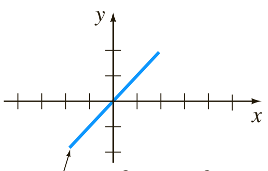


FIGURE 11.3.3 Odd function in Example 1

SOLUTION Inspection of Figure 11.3.3 shows that the given function is odd on the interval $(-2, 2)$, and so we expand f in a sine series. With the identification $2p = 4$ we have $p = 2$. Thus (5), after integration by parts, is
$$b_n = \int_0^2 x\sin\frac{n\pi}{2}x\,dx = \frac{4(-1)^{n+1}}{n\pi}.$$
Therefore
$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}x. \tag{6}$$
■

The function in Example 1 satisfies the conditions of Theorem 11.2.1. Hence the series (6) converges to the function on $(-2, 2)$ and the periodic extension (of period 4) given in Figure 11.3.4.

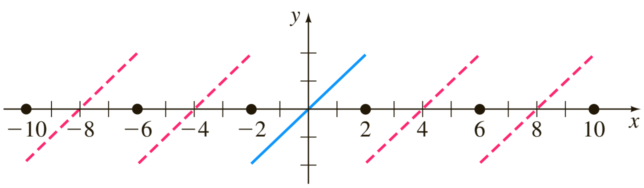


FIGURE 11.3.4 Periodic extension of function shown in Figure 11.3.3

EXAMPLE 2 Expansion in a Sine Series

The function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi, \end{cases}$ shown in Figure 11.3.5 is odd on the interval $(-\pi, \pi)$. With $p = \pi$ we have, from (5),
$$b_n = \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\frac{1-(-1)^n}{n},$$
and so
$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx. \tag{7}$$
■

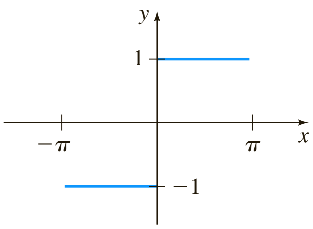


FIGURE 11.3.5 Odd function in Example 2

GIBBS PHENOMENON With the aid of a CAS we have plotted the graphs $S_1(x)$, $S_2(x)$, $S_3(x)$, and $S_{15}(x)$ of the partial sums of nonzero terms of (7) in Figure 11.3.6. As seen in Figure 11.3.6(d), the graph of $S_{15}(x)$ has pronounced spikes near the discontinuities at $x = 0$, $x = \pi$, $x = -\pi$, and so on. This "overshooting" by the partial sums S_N from the functional values near a point of discontinuity does not smooth out but remains fairly constant, even when the value N is taken to be large. This behavior of a Fourier series near a point at which f is discontinuous is known as the **Gibbs phenomenon.**

The periodic extension of f in Example 2 onto the entire x-axis is a meander function (see page 316).

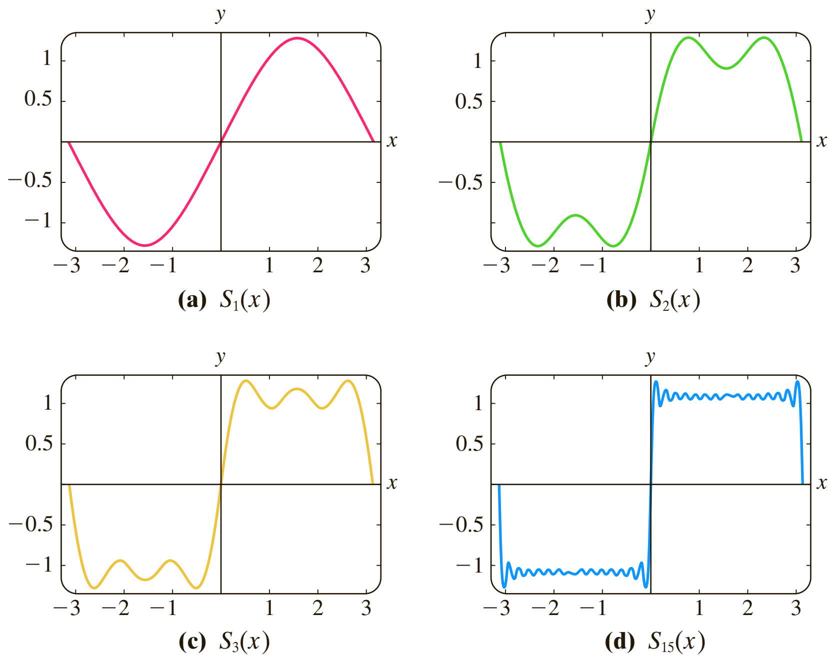


FIGURE 11.3.6 Partial sums of sine series (7)

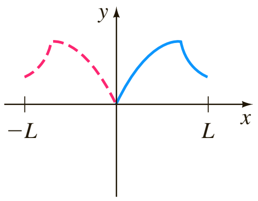


FIGURE 11.3.7 Even reflection

HALF-RANGE EXPANSIONS Throughout the preceding discussion it was understood that a function f was defined on an interval with the origin as its midpoint—that is, $(-p, p)$. However, in many instances we are interested in representing a function that is defined only for $0 < x < L$ by a trigonometric series. This can be done in many different ways by supplying an arbitrary *definition* of $f(x)$ for $-L < x < 0$. For brevity we consider the three most important cases. If $y = f(x)$ is defined on the interval $(0, L)$, then

(***i***) reflect the graph of f about the y-axis onto $(-L, 0)$; the function is now even on $(-L, L)$ (see Figure 11.3.7); or

(***ii***) reflect the graph of f through the origin onto $(-L, 0)$; the function is now odd on $(-L, L)$ (see Figure 11.3.8); or

(***iii***) define f on $(-L, 0)$ by $y = f(x + L)$ (see Figure 11.3.9).

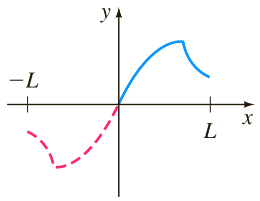


FIGURE 11.3.8 Odd reflection

Note that the coefficients of the series (1) and (4) utilize only the definition of the function on $(0, p)$ (that is, half of the interval $(-p, p)$). Hence in practice there is no actual need to make the reflections described in (*i*) and (*ii*). If f is defined for $0 < x < L$, we simply identify the half-period as the length of the interval $p = L$. The coefficient formulas (2), (3), and (5) and the corresponding series yield either an

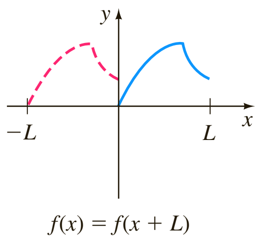


FIGURE 11.3.9 Identity reflection

even or an odd periodic extension of period $2L$ of the original function. The cosine and sine series that are obtained in this manner are known as **half-range expansions.** Finally, in case (*iii*) we are defining the function values on the interval $(-L, 0)$ to be same as the values on $(0, L)$. As in the previous two cases there is no real need to do this. It can be shown that the set of functions in (1) of Section 11.2 is orthogonal on the interval $[a, a + 2p]$ for any real number a. Choosing $a = -p$, we obtain the limits of integration in (9), (10), and (11) of that section. But for $a = 0$ the limits of integration are from $x = 0$ to $x = 2p$. Thus if f is defined on the interval $(0, L)$, we identify $2p = L$ or $p = L/2$. The resulting Fourier series will give the periodic extension of f with period L. In this manner the values to which the series converges will be the same on $(-L, 0)$ as on $(0, L)$.

EXAMPLE 3 Expansion in Three Series

Expand $f(x) = x^2$, $0 < x < L$,
(a) in a cosine series **(b)** in a sine series **(c)** in a Fourier series.

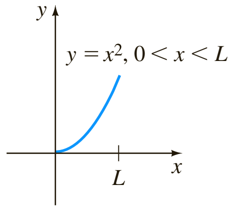


FIGURE 11.3.10 Function f in Example 3 is neither odd nor even.

SOLUTION The graph of the function is given in Figure 11.3.10.

(a) We have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \qquad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{n\pi}{L}x\,dx = \frac{4L^2(-1)^n}{n^2\pi^2},$$

where integration by parts was used twice in the evaluation of a_n. Thus

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x. \tag{8}$$

(b) In this case we must again integrate by parts twice:

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1].$$

Hence
$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x. \tag{9}$$

(c) With $p = L/2$, $1/p = 2/L$, and $n\pi/p = 2n\pi/L$ we have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2\cos\frac{2n\pi}{L}x\,dx = \frac{L^2}{n^2\pi^2},$$

and
$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{2n\pi}{L}x\,dx = -\frac{L^2}{n\pi}.$$

Therefore
$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}. \tag{10}$$

The series (8), (9), and (10) converge to the $2L$-periodic even extension of f, the $2L$-periodic odd extension of f, and the L-periodic extension of f, respectively. The graphs of these periodic extensions are shown in Figure 11.3.11.

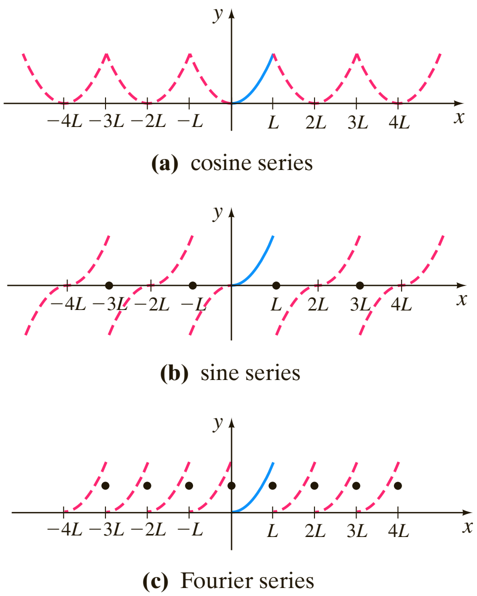


FIGURE 11.3.11 Same function on $(0, L)$ but different periodic extensions

PERIODIC DRIVING FORCE Fourier series are sometimes useful in determining a particular solution of a differential equation describing a physical system in which the input or driving force $f(t)$ is periodic. In the next example we find a periodic particular solution of the nonhomogeneous linear differential equation

$$m\frac{d^2x}{dt^2} + kx = f(t) \tag{11}$$

by first representing f by a half-range sine expansion and then assuming a particular solution of the form

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{p}t. \tag{12}$$

EXAMPLE 4 Particular Solution of a DE

An undamped spring/mass system, in which the mass $m = \frac{1}{16}$ slug and the spring constant $k = 4$ lb/ft, is driven by the 2-periodic external force $f(t)$ shown in Figure 11.3.12. Although the force $f(t)$ acts on the system for $t > 0$, note that if we extend the graph of the function in a 2-periodic manner to the negative t-axis, we obtain an odd function. In practical terms this means that we need only find the half-range sine expansion of $f(t) = \pi t$, $0 < t < 1$. With $p = 1$ it follows from (5) and integration by parts that

$$b_n = 2\int_0^1 \pi t \sin n\pi t\, dt = \frac{2(-1)^{n+1}}{n}.$$

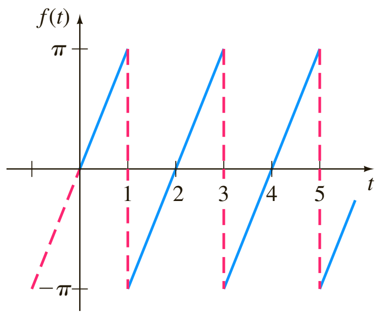


FIGURE 11.3.12 Periodic forcing function for spring/mass system in Example 4

From (11) the differential equation of motion is seen to be

$$\frac{1}{16}\frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin n\pi t. \tag{13}$$

To find a particular solution $x_p(t)$ of (13), we substitute (12) into the differential equation and equate coefficients of sin $n\pi t$. This yields

$$\left(-\frac{1}{16}n^2\pi^2 + 4\right)B_n = \frac{2(-1)^{n+1}}{n} \qquad \text{or} \qquad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)}.$$

Thus
$$x_p(t) = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)} \sin n\pi t. \tag{14}$$

Observe in the solution (14) that there is no integer $n \geq 1$ for which the denominator $64 - n^2\pi^2$ of B_n is zero. In general, if there *is* a value of n, say N, for which $N\pi/p = \omega$, where $\omega = \sqrt{k/m}$, then the system described by (11) is in a state of pure resonance. In other words, we have pure resonance if the Fourier series expansion of the driving force $f(t)$ contains a term $\sin(N\pi/L)t$ (or $\cos(N\pi/L)t$) that has the same frequency as the free vibrations.

Of course, if the $2p$-periodic extension of the driving force f onto the negative t-axis yields an even function, then we expand f in a cosine series.

EXERCISES 11.3

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–10 determine whether the function is even, odd, or neither.

1. $f(x) = \sin 3x$

2. $f(x) = x \cos x$

3. $f(x) = x^2 + x$

4. $f(x) = x^3 - 4x$

5. $f(x) = e^{|x|}$

6. $f(x) = e^x - e^{-x}$

7. $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$

8. $f(x) = \begin{cases} x + 5, & -2 < x < 0 \\ -x + 5, & 0 \leq x < 2 \end{cases}$

9. $f(x) = x^3, \quad 0 \leq x \leq 2$

10. $f(x) = |x^5|$

In Problems 11–24 expand the given function in an appropriate cosine or sine series.

11. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$

12. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$

13. $f(x) = |x|, \quad -\pi < x < \pi$

14. $f(x) = x, \quad -\pi < x < \pi$

15. $f(x) = x^2, \quad -1 < x < 1$

16. $f(x) = x|x|, \quad -1 < x < 1$

17. $f(x) = \pi^2 - x^2, \quad -\pi < x < \pi$

18. $f(x) = x^3, \quad -\pi < x < \pi$

19. $f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 \leq x < \pi \end{cases}$

20. $f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ x - 1, & 0 \leq x < 1 \end{cases}$

21. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$

22. $f(x) = \begin{cases} -\pi, & -2\pi < x < -\pi \\ x, & -\pi \leq x < \pi \\ \pi, & \pi \leq x < 2\pi \end{cases}$

23. $f(x) = |\sin x|, \quad -\pi < x < \pi$

24. $f(x) = \cos x, \quad -\pi/2 < x < \pi/2$

In Problems 25–34 find the half-range cosine and sine expansions of the given function.

25. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} \leq x < 1 \end{cases}$

26. $f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x < 1 \end{cases}$

27. $f(x) = \cos x, \quad 0 < x < \pi/2$

28. $f(x) = \sin x, \quad 0 < x < \pi$

29. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$

30. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ x - \pi, & \pi \leq x < 2\pi \end{cases}$

31. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$

32. $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$

33. $f(x) = x^2 + x, \quad 0 < x < 1$

34. $f(x) = x(2 - x), \quad 0 < x < 2$

In Problems 35–38 expand the given function in a Fourier series.

35. $f(x) = x^2, \quad 0 < x < 2\pi$

36. $f(x) = x, \quad 0 < x < \pi$

37. $f(x) = x + 1, \quad 0 < x < 1$

38. $f(x) = 2 - x, \quad 0 < x < 2$

In Problems 39–42 suppose the function $y = f(x)$, $0 < x < L$, given in the figure is expanded in a cosine series, in a sine series, and in a Fourier series. Sketch the periodic extension to which each series converges.

39.

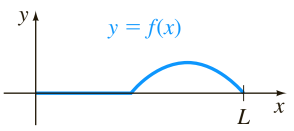


FIGURE 11.3.13 Graph for Problem 39

40.

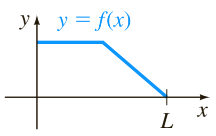


FIGURE 11.3.14 Graph for Problem 40

41.

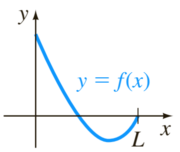


FIGURE 11.3.15 Graph for Problem 41

42.

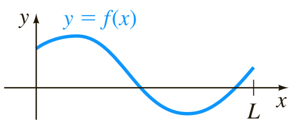


FIGURE 11.3.16 Graph for Problem 42

In Problems 43 and 44 proceed as in Example 4 to find a particular solution $x_p(t)$ of equation (11) when $m = 1$, $k = 10$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is odd.

43. $f(t) = \begin{cases} 5, & 0 < t < \pi \\ -5, & \pi < t < 2\pi \end{cases}; \quad f(t + 2\pi) = f(t)$

44. $f(t) = 1 - t, \quad 0 < t < 2; \quad f(t + 2) = f(t)$

In Problems 45 and 46 proceed as in Example 4 to find a particular solution $x_p(t)$ of equation (11) when $m = \frac{1}{4}$, $k = 12$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is even.

45. $f(t) = 2\pi t - t^2, \quad 0 < t < 2\pi; \quad f(t + 2\pi) = f(t)$

46. $f(t) = \begin{cases} t, & 0 < t < \frac{1}{2} \\ 1 - t, & \frac{1}{2} < t < 1 \end{cases}; \quad f(t + 1) = f(t)$

47. (a) Solve the differential equation in Problem 43, $x'' + 10x = f(t)$, subject to the initial conditions $x(0) = 0$, $x'(0) = 0$.

(b) Use a CAS to plot the graph of the solution $x(t)$ in part (a).

48. (a) Solve the differential equation in Problem 45, $\frac{1}{4}x'' + 12x = f(t)$, subject to the initial conditions $x(0) = 1$, $x'(0) = 0$.

(b) Use a CAS to plot the graph of the solution $x(t)$ in part (a).

49. Suppose a uniform beam of length L is simply supported at $x = 0$ and at $x = L$. If the load per unit length is given by $w(x) = w_0 x/L$, $0 < x < L$, then the differential equation for the deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} = \frac{w_0 x}{L},$$

where E, I, and w_0 are constants. (See (4) in Section 5.2.)

(a) Expand $w(x)$ in a half-range sine series.

(b) Use the method of Example 4 to find a particular solution $y_p(x)$ of the differential equation.

50. Proceed as in Problem 49 to find a particular solution $y_p(x)$ when the load per unit length is as given in Figure 11.3.17.

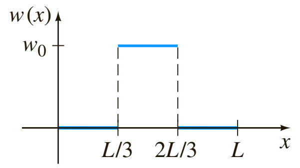


FIGURE 11.3.17 Graph for Problem 50

51. When a uniform beam is supported by an elastic foundation and subject to a load per unit length $w(x)$, the differential equation for its deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} + ky = w(x),$$

where k is the modulus of the foundation. Suppose that the beam and elastic foundation are infinite in length (that is, $-\infty < x < \infty$) and that the load per unit length is the periodic function

$$w(x) = \begin{cases} 0, & -\pi < x < -\pi/2 \\ w_0, & -\pi/2 \le x \le \pi/2; \\ 0 & \pi/2 < x < \pi \end{cases} \quad w(x + 2\pi) = w(x).$$

Use the method of Example 4 to find a particular solution $y_p(x)$ of the differential equation.

Discussion Problems

52. Prove properties (a), (c), (d), (f), and (g) in Theorem 11.3.1.

53. There is only one function that is both even and odd. What is it?

54. As we know from Chapter 4, the general solution of the differential equation in Problem 51 is $y = y_c + y_p$. Discuss why we can argue on physical grounds that the solution of Problem 51 is simply y_p. [*Hint*: Consider $y = y_c + y_p$ as $x \to \pm\infty$.]

Computer Lab Assignments

In Problems 55 and 56 use a CAS to plot graphs of partial sums $\{S_N(x)\}$ of the given trigonometric series. Experiment with different values of N and graphs on different intervals of the x-axis. Use your graphs to conjecture a closed-form expression for a function f defined for $0 < x < L$ that is represented by the series.

55. $f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos nx + \frac{1-2(-1)^n}{n}\sin nx\right]$

56. $f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi}{2}x$

57. Is your answer in Problem 55 or in Problem 56 unique? Give a function f defined on a symmetric interval about the origin $(-a, a)$ that has the same trigonometric series

(a) as in Problem 55,

(b) as in Problem 56.

11.4 Sturm-Liouville Problem

INTRODUCTION In this section we will study some special types of boundary-value problems in which the ordinary differential equation in the problem contains a parameter λ. The values of λ for which the BVP possesses nontrivial solutions are called **eigenvalues,** and the corresponding solutions are called **eigenfunctions.** Boundary-value problems of this type are especially important throughout Chapters 12 and 13. In this section we also see that there is a connection between orthogonal sets and eigenfunctions of a boundary-value problem.

The concept of eigenvalues and eigenfunctions was first introduced in Section 5.2. A review of that section (especially Example 2) is strongly recommended.

REVIEW OF ODEs For convenience we present here a brief review of some of the linear ODEs that will occur frequently in the sections and chapters that follow. The symbol α represents a constant.

Constant-coefficient equations	**General solutions**
$y' + \alpha y = 0$	$y = c_1e^{-\alpha x}$
$y'' + \alpha^2 y = 0, \quad \alpha > 0$	$y = c_1\cos\alpha x + c_2\sin\alpha x$
$y'' - \alpha^2 y = 0, \quad \alpha > 0$	$\begin{cases} y = c_1e^{-\alpha x} + c_2e^{\alpha x}, \quad \text{or} \\ y = c_1\cosh\alpha x + c_2\sinh\alpha x \end{cases}$
Cauchy-Euler equation	**General solutions, $x > 0$**
$x^2y'' + xy' - \alpha^2 y = 0, \quad \alpha \ge 0$	$\begin{cases} y = c_1x^{-\alpha} + c_2x^{\alpha}, & \alpha > 0 \\ y = c_1 + c_2\ln x, & \alpha = 0 \end{cases}$
Parametric Bessel equation ($\nu = 0$)	**General solution, $x > 0$**
$xy'' + y' + \alpha^2 xy = 0,$	$y = c_1J_0(\alpha x) + c_2Y_0(\alpha x)$
Legendre's equation ($n = 0, 1, 2, \ldots$)	**Particular solutions are polynomials**
$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$	$y = P_0(x) = 1,$
	$y = P_1(x) = x,$
	$y = P_2(x) = \frac{1}{2}(3x^2 - 1), \ldots$

Regarding the two forms of the general solution of $y'' - \alpha^2 y = 0$, we will make use of the following informal rule immediately in Example 1 as well as in future discussions:

This rule will be useful in Chapters 12–14.

Use the exponential form $y = c_1e^{-\alpha x} + c_2e^{\alpha x}$ when the domain of x is an infinite or semi-infinite interval; use the hyperbolic form $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$ when the domain of x is a finite interval.

EIGENVALUES AND EIGENFUNCTIONS Orthogonal functions arise in the solution of differential equations. More to the point, an orthogonal set of functions can be generated by solving a certain kind of two-point boundary-value problem involving a linear second-order differential equation containing a parameter λ. In Example 2 of Section 5.2 we saw that the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0, \tag{1}$$

possessed nontrivial solutions only when the parameter λ took on the values $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots,$ called **eigenvalues.** The corresponding nontrivial solutions $y_n = c_2 \sin(n\pi x/L)$, or simply $y_n = \sin(n\pi x/L)$, are called the **eigenfunctions** of the problem. For example, for (1)

$$\textit{BVP}: \qquad y'' \underbrace{-2}_{\text{not an eigenvalue}}y = 0, \quad y(0) = 0, \quad y(L) = 0$$

$$\textit{Trivial solution}: \qquad y = 0 \longleftarrow \text{never an eigenfunction}$$

$$\textit{BVP}: \qquad y'' + \underbrace{\frac{9\pi^2}{L^2}}_{\text{is an eigenvalue } (n = 3)} y = 0, \quad y(0) = 0, \quad y(L) = 0.$$

$$\textit{Nontrivial solution}: \qquad y_3 = \sin(3\pi x/L) \longleftarrow \text{eigenfunction}$$

For our purposes in this chapter it is important to recognize that the set of trigonometric functions generated by this BVP, that is, $\{\sin(n\pi x/L)\}$, $n = 1, 2, 3, \ldots,$ is an orthogonal set of functions on the interval $[0, L]$ and is used as the basis for the Fourier sine series. See Problem 10 in Exercises 11.1.

EXAMPLE 1 Eigenvalues and Eigenfunctions

Consider the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0. \tag{2}$$

As in Example 2 of Section 5.2 there are three possible cases for the parameter λ: zero, negative, or positive; that is, $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$. The solution of the DEs

$$y'' = 0, \quad \lambda = 0, \tag{3}$$

$$y'' - \alpha^2 y = 0, \quad \lambda = -\alpha^2, \tag{4}$$

$$y'' + \alpha^2 y = 0, \quad \lambda = \alpha^2, \tag{5}$$

are, in turn,

$$y = c_1 + c_2x, \tag{6}$$

$$y = c_1 \cosh \alpha x + c_2 \sinh \alpha x, \tag{7}$$

$$y = c_1 \cos \alpha x + c_2 \sin \alpha x. \tag{8}$$

When the boundary conditions $y'(0) = 0$ and $y'(L) = 0$ are applied to each of these solutions, (6) yields $y = c_1$, (7) yields only $y = 0$, and (8) yields $y = c_1 \cos \alpha x$

provided that $\alpha = n\pi/L$, $n = 1, 2, 3, \ldots$. Since $y = c_1$ satisfies the DE in (3) and the boundary conditions for any *nonzero* choice of c_1, we conclude that $\lambda = 0$ is an eigenvalue. Thus the eigenvalues and corresponding eigenfunctions of the problem are $\lambda_0 = 0$, $y_0 = c_1$, $c_1 \neq 0$, and $\lambda_n = \alpha_n^2 = n^2\pi^2/L^2$, $n = 1, 2, \ldots$, $y_n = c_1 \cos(n\pi x/L)$, $c_1 \neq 0$. We can, if desired, take $c_1 = 1$ in each case. Note also that the eigenfunction $y_0 = 1$ corresponding to the eigenvalue $\lambda_0 = 0$ can be incorporated in the family $y_n = \cos(n\pi x/L)$ by permitting $n = 0$. The set $\{\cos(n\pi x/L)\}$, $n = 0, 1, 2, 3, \ldots$, is orthogonal on the interval $[0, L]$. You are asked to fill in the details of this example in Problem 3 in Exercises 11.4. ■

REGULAR STURM-LIOUVILLE PROBLEM The problems in (1) and (2) are special cases of an important general two-point boundary value problem. Let p, q, r, and r' be real-valued functions continuous on an interval $[a, b]$, and let $r(x) > 0$ and $p(x) > 0$ for every x in the interval. Then

$$\textbf{\textit{Solve:}} \qquad \frac{d}{dx}[r(x)y'] + (q(x) + \lambda p(x))y = 0 \tag{9}$$

$$\textbf{\textit{Subject to:}} \qquad A_1 y(a) + B_1 y'(a) = 0 \tag{10}$$

$$A_2 y(b) + B_2 y'(b) = 0 \tag{11}$$

is said to be a **regular Sturm-Liouville problem.** The coefficients in the boundary conditions (10) and (11) are assumed to be real and independent of λ. In addition, A_1 and B_1 are not both zero, and A_2 and B_2 are not both zero. The boundary-value problems in (1) and (2) are regular Sturm-Liouville problems. From (1) we can identify $r(x) = 1$, $q(x) = 0$, and $p(x) = 1$ in the differential equation (9); in boundary condition (10) we identify $a = 0$, $A_1 = 1$, $B_1 = 0$, and in (11), $b = L$, $A_2 = 1$, $B_2 = 0$. From (2) the identifications would be $a = 0$, $A_1 = 0$, $B_1 = 1$ in (10), $b = L$, $A_2 = 0$, $B_2 = 1$ in (11).

The differential equation (9) is linear and homogeneous. The boundary conditions in (10) and (11), both a linear combination of y and y' *equal to zero at a point*, are also **homogeneous.** A boundary condition such as $A_2 y(b) + B_2 y'(b) = C_2$, where C_2 is a nonzero constant, is **nonhomogeneous.** A boundary-value problem that consists of a homogeneous linear differential equation and homogeneous boundary conditions is, of course, said to be a homogeneous BVP; otherwise, it is nonhomogeneous. The boundary conditions (10) and (11) are referred to as **separated** because each condition involves only a single boundary point.

Because a regular Sturm-Liouville problem is a homogeneous BVP, it always possesses the trivial solution $y = 0$. However, this solution is of no interest to us. As in Example 1, in solving such a problem, we seek numbers λ (eigenvalues) and nontrivial solutions y that depend on λ (eigenfunctions).

PROPERTIES Theorem 11.4.1 is a list of the more important of the many properties of the regular Sturm-Liouville problem. We shall prove only the last property.

THEOREM 11.4.1 Properties of the Regular Sturm-Liouville Problem

(a) There exist an infinite number of real eigenvalues that can be arranged in increasing order $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$ such that $\lambda_n \to \infty$ as $n \to \infty$.

(b) For each eigenvalue there is only one eigenfunction (except for nonzero constant multiples).

(c) Eigenfunctions corresponding to different eigenvalues are linearly independent.

(d) The set of eigenfunctions corresponding to the set of eigenvalues is orthogonal with respect to the weight function $p(x)$ on the interval $[a, b]$.

PROOF OF (d) Let y_m and y_n be eigenfunctions corresponding to eigenvalues λ_m and λ_n, respectively. Then

$$\frac{d}{dx}[r(x)y'_m] + (q(x) + \lambda_m p(x))y_m = 0 \tag{12}$$

$$\frac{d}{dx}[r(x)y'_n] + (q(x) + \lambda_n p(x))y_n = 0. \tag{13}$$

Multiplying (12) by y_n and (13) by y_m and subtracting the two equations gives

$$(\lambda_m - \lambda_n)p(x)y_m y_n = y_m \frac{d}{dx}[r(x)y'_n] - y_n \frac{d}{dx}[r(x)y'_m].$$

Integrating this last result by parts from $x = a$ to $x = b$ then yields

$$(\lambda_m - \lambda_n)\int_a^b p(x)y_m y_n\, dx = r(b)[y_m(b)y'_n(b) - y_n(b)y'_m(b)] - r(a)[y_m(a)y'_n(a) - y_n(a)y'_m(a)]. \tag{14}$$

Now the eigenfunctions y_m and y_n must both satisfy the boundary conditions (10) and (11). In particular, from (10) we have

$$A_1 y_m(a) + B_1 y'_m(a) = 0$$
$$A_1 y_n(a) + B_1 y'_n(a) = 0.$$

For this system to be satisfied by A_1 and B_1, not both zero, the determinant of the coefficients must be zero:

$$y_m(a)y'_n(a) - y_n(a)y'_m(a) = 0.$$

A similar argument applied to (11) also gives

$$y_m(b)\, y'_n(b) - y_n(b)\, y'_m(b) = 0.$$

Since both members of the right-hand side of (14) are zero, we have established the orothogonality relation

$$\int_a^b p(x)y_m(x)y_n(x)\, dx = 0, \qquad \lambda_m \neq \lambda_n. \tag{15}$$ ■

EXAMPLE 2 A Regular Sturm-Liouville Problem

Solve the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0. \tag{16}$$

SOLUTION We proceed exactly as in Example 1 by considering three cases in which the parameter λ could be zero, negative, or positive: $\lambda = 0$, $\lambda = -\alpha^2 < 0$, and $\lambda = \alpha^2 > 0$, where $\alpha > 0$. The solutions of the DE for these values are listed in (3)–(5). For the cases $\lambda = 0$ and $\lambda = -\alpha^2 < 0$ we find that the BVP in (16) possesses only the trivial solution $y = 0$. For $\lambda = \alpha^2 > 0$ the general solution of the differential equation is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. Now the condition $y(0) = 0$ implies that $c_1 = 0$ in this solution, so we are left with $y = c_2 \sin \alpha x$. The second boundary condition $y(1) + y'(1) = 0$ is satisfied if

$$c_2 \sin \alpha + c_2 \alpha \cos \alpha = 0.$$

In view of the demand that $c_2 \neq 0$, the last equation can be written

$$\tan \alpha = -\alpha. \tag{17}$$

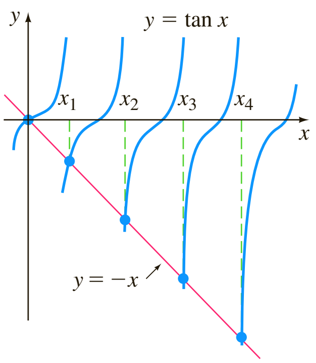


FIGURE 11.4.1 Positive roots $x_1, x_2, x_3, \ldots$ of $\tan x = -x$ in Example 2

If for a moment we think of (17) as $\tan x = -x$, then Figure 11.4.1 shows the plausibility that this equation has an infinite number of roots, namely, the x-coordinates of the points where the graph of $y = -x$ intersects the infinite number of branches of the graph of $y = \tan x$. The eigenvalues of the BVP (16) are then $\lambda_n = \alpha_n^2$, where α_n, $n = 1, 2, 3, \ldots$, are the consecutive *positive* roots $\alpha_1, \alpha_2, \alpha_3, \ldots$ of (17). With the aid of a CAS it is easily shown that, to four rounded decimal places, $\alpha_1 = 2.0288$, $\alpha_2 = 4.9132$, $\alpha_3 = 7.9787$, and $\alpha_4 = 11.0855$, and the corresponding solutions are $y_1 = \sin 2.0288x$, $y_2 = \sin 4.9132x$, $y_3 = \sin 7.9787x$, and $y_4 = \sin 11.0855x$. In general, the eigenfunctions of the problem are $\{\sin \alpha_n x\}$, $n = 1, 2, 3, \ldots$.

With the identification $r(x) = 1$, $q(x) = 0$, $p(x) = 1$, $A_1 = 1$, $B_1 = 0$, $A_2 = 1$, $B_2 = 1$ we see that (16) is a regular Sturm-Liouville problem. We conclude that $\{\sin \alpha_n x\}$, $n = 1, 2, 3, \ldots$, is an orthogonal set with respect to the weight function $p(x) = 1$ on the interval $[0, 1]$. ■

In some circumstances we can prove the orthogonality of solutions of (9) without the necessity of specifying a boundary condition at $x = a$ and at $x = b$.

SINGULAR STURM-LIOUVILLE PROBLEM There are several other important conditions under which we seek nontrivial solutions of the differential equation (9):

- $r(a) = 0$, and a boundary condition of the type given in (11) is specified at $x = b$; (18)
- $r(b) = 0$, and a boundary condition of the type given in (10) is specified at $x = a$; (19)
- $r(a) = r(b) = 0$, and no boundary condition is specified at either $x = a$ or at $x = b$; (20)
- $r(a) = r(b)$, and boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$. (21)

The differential equation (9) along with one of conditions (18)–(20), is said to be a **singular** boundary-value problem. Equation (9) with the conditions specified in (21) is said to be a **periodic** boundary-value problem (the boundary conditions are also said to be periodic). Observe that if, say, $r(a) = 0$, then $x = a$ *may* be a singular point of the differential equation, and consequently, a solution of (9) may become unbounded as $x \to a$. However, we see from (14) that if $r(a) = 0$, then no boundary condition is required at $x = a$ to prove orthogonality of the eigenfunctions provided that these solutions are bounded at that point. This latter requirement guarantees the existence of the integrals involved. By assuming that the solutions of (9) are bounded on the closed interval $[a, b]$, we can see from inspection of (14) that

- if $r(a) = 0$, then the orthogonality relation (15) holds with no boundary condition specified at $x = a$; (22)
- if $r(b) = 0$, then the orthogonality relation (15) holds with no boundary condition specified at $x = b$;* (23)
- if $r(a) = r(b) = 0$, then the orthogonality relation (15) holds with no boundary conditions specified at either $x = a$ or $x = b$; (24)
- if $r(a) = r(b)$, then the orthogonality relation (15) holds with the periodic boundary conditions $y(a) = y(b)$, $y'(a) = y'(b)$. (25)

We note that a Sturm-Liouville problem is also singular when the interval under consideration is infinite. See Problems 9 and 10 in Exercises 11.4.

SELF-ADJOINT FORM By carrying out the indicated differentiation in (9), we see that the differential equation is the same as

$$r(x)y'' + r'(x)y' + (q(x) + \lambda p(x))y = 0. \tag{26}$$

Examination of (26) might lead one to believe, given the coefficient of y' is the derivative of the coefficient of y'', that few differential equations have form (9).

*Conditions (22) and (23) are equivalent to choosing $A_1 = 0$, $B_1 = 0$, and $A_2 = 0$, $B_2 = 0$, respectively.

On the contrary, if the coefficients are continuous and $a(x) \neq 0$ for all x in some interval, then *any* second-order differential equation

$$a(x)y'' + b(x)y' + (c(x) + \lambda d(x))y = 0 \qquad (27)$$

can be recast into the so-called **self-adjoint form** (9). To this end, we basically proceed as in Section 2.3, where we rewrote a linear first-order equation $a_1(x)y' + a_0(x)y = 0$ in the form $\frac{d}{dx}[\mu y] = 0$ by dividing the equation by $a_1(x)$ and then multiplying by the integrating factor $\mu = e^{\int P(x)\,dx}$, where, assuming no common factors, $P(x) = a_0(x)/a_1(x)$. So first, we divide (27) by $a(x)$. The first two terms are $Y' + \frac{b(x)}{a(x)}Y + \cdots$, where for emphasis we have written $Y = y'$. Second, we multiply this equation by the integrating factor $e^{\int(b(x)/a(x))\,dx}$, where $a(x)$ and $b(x)$ are assumed to have no common factors:

$$\underbrace{e^{\int(b(x)/a(x))\,dx}Y' + \frac{b(x)}{a(x)}e^{\int(b(x)/a(x))\,dx}Y}_{\text{derivative of a product}} + \cdots = \frac{d}{dx}\left[e^{\int(b(x)/a(x))\,dx}Y\right] + \cdots = \frac{d}{dx}\left[e^{\int(b(x)/a(x))\,dx}y'\right] + \cdots.$$

In summary, by dividing (27) by $a(x)$ and then multiplying by $e^{\int(b(x)/a(x))\,dx}$, we get

$$e^{\int(b/a)\,dx}y'' + \frac{b(x)}{a(x)}e^{\int(b/a)\,dx}y' + \left(\frac{c(x)}{a(x)}e^{\int(b/a)\,dx} + \lambda\frac{d(x)}{a(x)}e^{\int(b/a)\,dx}\right)y = 0. \qquad (28)$$

Equation (28) is the desired form given in (26) and is the same as (9):

$$\frac{d}{dx}\left[\underbrace{e^{\int(b/a)\,dx}}_{r(x)}y'\right] + \left(\underbrace{\frac{c(x)}{a(x)}e^{\int(b/a)\,dx}}_{q(x)} + \lambda\underbrace{\frac{d(x)}{a(x)}e^{\int(b/a)\,dx}}_{p(x)}\right)y = 0$$

For example, to express $2y'' + 6y' + \lambda y = 0$ in self-adjoint form, we write $y'' + 3y' + \lambda\frac{1}{2}y = 0$ and then multiply by $e^{\int 3\,dx} = e^{3x}$. The resulting equation is

$$\overset{r(x)}{\overset{\downarrow}{e^{3x}}}y'' + \overset{r'(x)}{\overset{\downarrow}{3e^{3x}}}y' + \lambda\overset{p(x)}{\overset{\downarrow}{\tfrac{1}{2}e^{3x}}}y = 0 \qquad \text{or} \qquad \frac{d}{dx}\left[e^{3x}y'\right] + \lambda\frac{1}{2}e^{3x}y = 0.$$

It is certainly not necessary to put a second-order differential equation (27) into the self-adjoint form (9) to *solve* the DE. For our purposes we use the form given in (9) to determine the weight function $p(x)$ needed in the orthogonality relation (15). The next two examples illustrate orthogonality relations for Bessel functions and for Legendre polynomials.

EXAMPLE 3 Parametric Bessel Equation

In Section 6.4 we saw that the parametric Bessel differential equation of order n is $x^2y'' + xy' + (\alpha^2x^2 - n^2)y = 0$, where n is a fixed nonnegative integer and α is a positive parameter. The general solution of this equation is $y = c_1J_n(\alpha x) + c_2Y_n(\alpha x)$. After dividing the parametric Bessel equation by the lead coefficient x^2 and multiplying the resulting equation by the integrating factor $e^{\int(1/x)\,dx} = e^{\ln x} = x$, $x > 0$, we obtain

$$xy'' + y' + \left(\alpha^2x - \frac{n^2}{x}\right)y = 0 \qquad \text{or} \qquad \frac{d}{dx}[xy'] + \left(\alpha^2x - \frac{n^2}{x}\right)y = 0.$$

By comparing the last result with the self-adjoint form (9), we make the identifications $r(x) = x$, $q(x) = -n^2/x$, $\lambda = \alpha^2$, and $p(x) = x$. Now $r(0) = 0$, and of the two solutions $J_n(\alpha x)$ and $Y_n(\alpha x)$, only $J_n(\alpha x)$ is bounded at $x = 0$. Thus in view of (22) above, the set $\{J_n(\alpha_i x)\}$, $i = 1, 2, 3, \ldots$, is orthogonal with respect to the weight function $p(x) = x$ on the interval $[0, b]$. The orthogonality relation is

$$\int_0^b xJ_n(\alpha_i x)J_n(\alpha_j x)\,dx = 0, \qquad \lambda_i \neq \lambda_j, \tag{29}$$

provided that the α_i, and hence the eigenvalues $\lambda_i = \alpha_i^2$, $i = 1, 2, 3, \ldots$, are defined by means of a boundary condition at $x = b$ of the type given in (11):

The extra factor of α comes from the Chain Rule:

$\frac{d}{dx}J_n(\alpha x) = J'_n(\alpha x)\frac{d}{dx}\alpha x = \alpha J'_n(\alpha x).$

$$A_2 J_n(\alpha b) + B_2\alpha J'_n(\alpha b) = 0. \tag{30}$$

For any choice of A_2 and B_2, not both zero, it is known that (30) has an infinite number of roots $x_i = \alpha_i b$. The eigenvalues are then $\lambda_i = \alpha_i^2 = (x_i/b)^2$. More will be said about eigenvalues in the next chapter.

EXAMPLE 4 Legendre's Equation

Legendre's differential equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ is exactly of the form given in (26) with $r(x) = 1 - x^2$ and $r'(x) = -2x$. Hence the self-adjoint form (9) of the differential equation is immediate,

$$\frac{d}{dx}\left[(1 - x^2)y'\right] + n(n + 1)y = 0. \tag{31}$$

From (31) we can further identify $q(x) = 0$, $\lambda = n(n + 1)$, and $p(x) = 1$. Recall from Section 6.4 that when $n = 0, 1, 2, \ldots$, Legendre's DE possesses polynomial solutions $P_n(x)$. Now we can put the observation that $r(-1) = r(1) = 0$ together with the fact that the Legendre polynomials $P_n(x)$ are the only solutions of (31) that are bounded on the closed interval $[-1, 1]$ to conclude from (24) that the set $\{P_n(x)\}$, $n = 0, 1, 2, \ldots$, is orthogonal with respect to the weight function $p(x) = 1$ on $[-1, 1]$. The orthogonality relation is

$$\int_{-1}^1 P_m(x)P_n(x)\,dx = 0, \qquad m \neq n.$$

EXERCISES 11.4

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1 and 2 find the eigenfunctions and the equation that defines the eigenvalues for the given boundary-value problem. Use a CAS to approximate the first four eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and λ_4. Give the eigenfunctions corresponding to these approximations.

1. $y'' + \lambda y = 0, \quad y'(0) = 0, y(1) + y'(1) = 0$

2. $y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, y(1) = 0$

3. Consider $y'' + \lambda y = 0$ subject to $y'(0) = 0$, $y'(L) = 0$. Show that the eigenfunctions are

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots\right\}.$$

This set, which is orthogonal on $[0, L]$, is the basis for the Fourier cosine series.

4. Consider $y'' + \lambda y = 0$ subject to the periodic boundary conditions $y(-L) = y(L)$, $y'(-L) = y'(L)$. Show that the eigenfunctions are

$$\left\{1, \cos\frac{\pi}{L}x, \cos\frac{2\pi}{L}x, \ldots, \sin\frac{\pi}{L}x, \sin\frac{2\pi}{L}x, \sin\frac{3\pi}{L}x, \ldots\right\}.$$

This set, which is orthogonal on $[-L, L]$, is the basis for the Fourier series.

5. Find the square norm of each eigenfunction in Problem 1.

6. Show that for the eigenfunctions in Example 2,

$$\|\sin\alpha_n x\|^2 = \tfrac{1}{2}[1 + \cos^2\alpha_n].$$

7. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem

$$x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(5) = 0.$$

(b) Put the differential equation in self-adjoint form.

(c) Give an orthogonality relation.

8. (a) Find the eigenvalues and eigenfunctions of the boundary-value problem

$$y'' + y' + \lambda y = 0, \quad y(0) = 0, \quad y(2) = 0.$$

(b) Put the differential equation in self-adjoint form.

(c) Give an orthogonality relation.

9. Laguerre's differential equation

$$xy'' + (1 - x)y' + ny = 0, \quad n = 0, 1, 2, \ldots$$

has polynomial solutions $L_n(x)$. Put the equation in self-adjoint form and give an orthogonality relation.

10. Hermite's differential equation

$$y'' - 2xy' + 2ny = 0, \quad n = 0, 1, 2, \ldots$$

has polynomial solutions $H_n(x)$. Put the equation in self-adjoint form and give an orthogonality relation.

11. Consider the regular Sturm-Liouville problem:

$$\frac{d}{dx}[(1 + x^2)y'] + \frac{\lambda}{1 + x^2}y = 0,$$
$$y(0) = 0, \quad y(1) = 0.$$

(a) Find the eigenvalues and eigenfunctions of the boundary-value problem. [*Hint*: Let $x = \tan\theta$ and then use the Chain Rule.]

(b) Give an orthogonality relation.

12. (a) Find the eigenfunctions and the equation that defines the eigenvalues for the boundary-value problem

$$x^2y'' + xy' + (\lambda x^2 - 1)y = 0, \quad x > 0,$$
$$y \text{ is bounded at } x = 0, \quad y(3) = 0.$$

Let $\lambda = \alpha^2$, $\alpha > 0$.

(b) Use Table 6.4.1 of Section 6.4 to find the approximate values of the first four eigenvalues λ_1, λ_2, λ_3, and λ_4.

Discussion Problems

13. Consider the special case of the regular Sturm-Liouville problem on the interval $[a, b]$:

$$\frac{d}{dx}[r(x)y'] + \lambda p(x)y = 0,$$
$$y'(a) = 0, \quad y'(b) = 0.$$

Is $\lambda = 0$ an eigenvalue of the problem? Defend your answer.

Computer Lab Assignments

14. (a) Give an orthogonality relation for the Sturm-Liouville problem in Problem 1.

(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues λ_1 and λ_2, respectively.

15. (a) Give an orthogonality relation for the Sturm-Liouville problem in Problem 2.

(b) Use a CAS as an aid in verifying the orthogonality relation for the eigenfunctions y_1 and y_2 that correspond to the first two eigenvalues λ_1 and λ_2, respectively.

11.5 Bessel and Legendre Series

INTRODUCTION Fourier series, Fourier cosine series, and Fourier sine series are three ways of expanding a function in terms of an orthogonal set of functions. But such expansions are by no means limited to orthogonal sets of *trigonometric* functions. We saw in Section 11.1 that a function f defined on an interval (a, b) could be expanded, at least in a formal manner, in terms of any set of functions $\{\phi_n(x)\}$ that is orthogonal with respect to a weight function on $[a, b]$. Many of these orthogonal series expansions or generalized Fourier series stem from Sturm-Liouville problems which, in turn, arise from attempts to solve linear partial differential equations that serve as models for physical systems. Fourier series and orthogonal series expansions, as well as the two series considered in this section, will appear in the subsequent consideration of these applications in Chapters 12 and 13.

11.5.1 FOURIER-BESSEL SERIES

We saw in Example 3 of Section 11.4 that for a fixed value of n the set of Bessel functions $\{J_n(\alpha_i x)\}$, $i = 1, 2, 3, \ldots$, is orthogonal with respect to the weight function $p(x) = x$ on an interval $[0, b]$ whenever the α_i are defined by means of a boundary condition of the form

$$A_2 J_n(\alpha b) + B_2 \alpha J_n'(\alpha b) = 0. \tag{1}$$

The eigenvalues of the corresponding Sturm-Liouville problem are $\lambda_i = \alpha_i^2$. From (7) and (8) of Section 11.1 the orthogonal series, or generalized Fourier series, expansion of a function f defined on the interval $(0, b)$ in terms of this orthogonal set is

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x), \tag{2}$$

where
$$c_i = \frac{\int_0^b x J_n(\alpha_i x) f(x)\, dx}{\|J_n(\alpha_i x)\|^2}. \tag{3}$$

The square norm of the function $J_n(\alpha_i x)$ is defined by (11) of Section 11.1.

$$\|J_n(\alpha_i x)\|^2 = \int_0^b x J_n^2(\alpha_i x)\, dx. \tag{4}$$

The series (2) with coefficients (3) is called a **Fourier-Bessel series,** or simply, a **Bessel series.**

DIFFERENTIAL RECURRENCE RELATIONS The differential recurrence relations that were given in (23) and (22) of Section 6.4 are often useful in the evaluation of the coefficients (3). For convenience we reproduce those relations here:

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \tag{5}$$

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x). \tag{6}$$

SQUARE NORM The value of the square norm (4) depends on how the eigenvalues $\lambda_i = \alpha_i^2$ are defined. If $y = J_n(\alpha x)$, then we know from Example 3 of Section 11.4 that

$$\frac{d}{dx}[xy'] + \left(\alpha^2 x - \frac{n^2}{x}\right) y = 0.$$

After we multiply by $2xy'$, this equation can be written as

$$\frac{d}{dx}[xy']^2 + (\alpha^2 x^2 - n^2)\frac{d}{dx}[y]^2 = 0.$$

Integrating the last result by parts on $[0, b]$ then gives

$$2\alpha^2 \int_0^b xy^2\, dx = \left([xy']^2 + (\alpha^2 x^2 - n^2) y^2\right)\Big|_0^b.$$

Since $y = J_n(\alpha x)$, the lower limit is zero because $J_n(0) = 0$ for $n > 0$. Furthermore, for $n = 0$ the quantity $[xy']^2 + \alpha^2 x^2 y^2$ is zero at $x = 0$. Thus

$$2\alpha^2 \int_0^b x J_n^2(\alpha x)\, dx = \alpha^2 b^2 [J_n'(\alpha b)]^2 + (\alpha^2 b^2 - n^2)[J_n(\alpha b)]^2, \tag{7}$$

where we have used the Chain Rule to write $y' = \alpha J_n'(\alpha x)$.

We now consider three cases of (1).

CASE I: If we choose $A_2 = 1$ and $B_2 = 0$, then (1) is

$$J_n(\alpha b) = 0. \tag{8}$$

There are an infinite number of positive roots $x_i = \alpha_i b$ of (8) (see Figure 6.4.1), which define the α_i as $\alpha_i = x_i/b$. The eigenvalues are positive and are then $\lambda_i = \alpha_i^2 = x_i^2/b^2$. No new eigenvalues result from the negative roots of (8), since $J_n(-x) = (-1)^n J_n(x)$. (See page 267.) The number 0 is not an eigenvalue for any n because $J_n(0) = 0$ for $n = 1, 2, 3, \ldots$ and $J_0(0) = 1$. In other words, if $\lambda = 0$, we get the trivial function (which is never an eigenfunction) for $n = 1, 2, 3, \ldots$, and for $n = 0$, $\lambda = 0$ (or, equivalently, $\alpha = 0$) does not satisfy the equation in (8). When (6) is written in the form $xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x)$, it follows from (7) and (8) that the square norm of $J_n(\alpha_i x)$ is

$$\|J_n(\alpha_i x)\|^2 = \frac{b^2}{2} J_{n+1}^2(\alpha_i b). \tag{9}$$

CASE II: If we choose $A_2 = h \geq 0$, and $B_2 = b$, then (1) is

$$hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0. \tag{10}$$

Equation (10) has an infinite number of positive roots $x_i = \alpha_i b$ for each positive integer $n = 1, 2, 3, \ldots$. As before, the eigenvalues are obtained from $\lambda_i = \alpha_i^2 = x_i^2/b^2$. $\lambda = 0$ is not an eigenvalue for $n = 1, 2, 3, \ldots$. Substituting $\alpha_i b J_n'(\alpha_i b) = -hJ_n(\alpha_i b)$ into (7), we find that the square norm of $J_n(\alpha_i x)$ is now

$$\|J_n(\alpha_i x)\|^2 = \frac{\alpha_i^2 b^2 - n^2 + h^2}{2\alpha_i^2} J_n^2(\alpha_i b). \tag{11}$$

CASE III: If $h = 0$ and $n = 0$ in (10), the α_i are defined from the roots of

$$J_0'(\alpha b) = 0. \tag{12}$$

Even though (12) is just a special case of (10), it is the only situation for which $\lambda = 0$ is an eigenvalue. To see this, observe that for $n = 0$ the result in (6) implies that $J_0'(\alpha b) = 0$ is equivalent to $J_1(\alpha b) = 0$. Since $x_1 = \alpha_1 b = 0$ is root of the last equation, $\alpha_1 = 0$, and because $J_0(0) = 1$ is nontrivial, we conclude from $\lambda_1 = \alpha_1^2 = x_1^2/b^2$ that $\lambda_1 = 0$ is an eigenvalue. But obviously, we cannot use (11) when $\alpha_1 = 0$, $h = 0$, and $n = 0$. However, from the square norm (4),

$$\|1\|^2 = \int_0^b x\,dx = \frac{b^2}{2}. \tag{13}$$

For $a_i > 0$ we can use (11) with $h = 0$ and $n = 0$:

$$\|J_0(\alpha_i x)\|^2 = \frac{b^2}{2} J_0^2(\alpha_i b). \tag{14}$$

The following definition summarizes three forms of the series (2) corresponding to the square norms in the three cases.

DEFINITION 11.5.1 Fourier-Bessel Series

The **Fourier-Bessel series** of a function f defined on the interval $(0, b)$ is given by

(i)
$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{15}$$

$$c_i = \frac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\,dx, \tag{16}$$

where the α_i are defined by $J_n(\alpha b) = 0$.

(continued)

(ii)

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\alpha_i x) \tag{17}$$

$$c_i = \frac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2)J_n^2(\alpha_i b)} \int_0^b x J_n(\alpha_i x) f(x)\, dx, \tag{18}$$

where the α_i are defined by $hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0$.

(iii)

$$f(x) = c_1 + \sum_{i=2}^{\infty} c_i J_0(\alpha_i x) \tag{19}$$

$$c_1 = \frac{2}{b^2}\int_0^b x f(x)\, dx, \quad c_i = \frac{2}{b^2 J_0^2(\alpha_i b)} \int_0^b x J_0(\alpha_i x) f(x)\, dx, \tag{20}$$

where the α_i are defined by $J_0'(\alpha b) = 0$.

CONVERGENCE OF A FOURIER-BESSEL SERIES Sufficient conditions for the convergence of a Fourier-Bessel series are not particularly restrictive.

THEOREM 11.5.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[0, b]$. Then for all x in the interval $(0, b)$, the Fourier-Bessel series of f converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier-Bessel series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

EXAMPLE 1 Expansion in a Fourier-Bessel Series

Expand $f(x) = x$, $0 < x < 3$, in a Fourier-Bessel series, using Bessel functions of order one that satisfy the boundary condition $J_1(3\alpha) = 0$.

SOLUTION We use (15) where the coefficients c_i are given by (16) with $b = 3$:

$$c_i = \frac{2}{3^2 J_2^2(3\alpha_i)} \int_0^3 x^2 J_1(\alpha_i x)\, dx.$$

To evaluate this integral, we let $t = \alpha_i x$, $dx = dt/\alpha_i$, $x^2 = t^2/\alpha_i^2$, and use (5) in the form $\dfrac{d}{dt}[t^2 J_2(t)] = t^2 J_1(t)$:

$$c_i = \frac{2}{9\alpha_i^3 J_2^2(3\alpha_i)} \int_0^3 \alpha_i \frac{d}{dt}[t^2 J_2(t)]\, dt = \frac{2}{\alpha_i J_2(3\alpha_i)}.$$

Therefore the desired expansion is

$$f(x) = 2\sum_{i=1}^{\infty} \frac{1}{\alpha_i J_2(3\alpha_i)} J_1(\alpha_i x).$$

You are asked to find the first four values of the α_i for the foregoing Fourier-Bessel series in Problem 1 in Exercises 11.5.

EXAMPLE 2 Expansion in a Fourier-Bessel Series

If the α_i in Example 1 are defined by $J_1(3\alpha) + \alpha J_1'(3\alpha) = 0$, then the only thing that changes in the expansion is the value of the square norm. Multiplying the boundary condition by 3 gives $3J_1(3\alpha) + 3\alpha J_1'(3\alpha) = 0$, which now matches (10) when $h = 3$, $b = 3$, and $n = 1$. Thus (18) and (17) yield, in turn,

$$c_i = \frac{18\alpha_i J_2(3\alpha_i)}{\left(9\alpha_i^2 + 8\right)J_1^2(3\alpha_i)}$$

and

$$f(x) = 18\sum_{i=1}^{\infty} \frac{\alpha_i J_2(3\alpha_i)}{\left(9\alpha_i^2 + 8\right)J_1^2(3\alpha_i)} J_1(\alpha_i x).$$

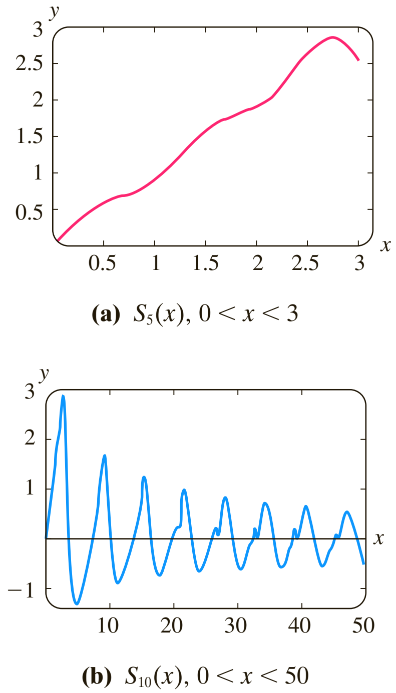


FIGURE 11.5.1 Graphs of two partial sums of the Fourier-Bessel series in Example 2

USE OF COMPUTERS Since Bessel functions are "built-in functions" in a CAS, it is a straightforward task to find the approximate values of the α_i and the coefficients c_i in a Fourier-Bessel series. For example, in (10) we can think of $x_i = \alpha_i b$ as a positive root of the equation $hJ_n(x) + xJ_n'(x) = 0$. Thus in Example 2 we have used a CAS to find the first five positive roots x_i of $3J_1(x) + xJ_1'(x) = 0$, and from these roots we obtain the first five values of α_i: $\alpha_1 = x_1/3 = 0.98320$, $\alpha_2 = x_2/3 = 1.94704$, $\alpha_3 = x_3/3 = 2.95758$, $\alpha_4 = x_4/3 = 3.98538$, and $\alpha_5 = x_5/3 = 5.02078$. Knowing the roots $x_i = 3\alpha_i$ and the α_i, we again use a CAS to calculate the numerical values of $J_2(3\alpha_i)$, $J_1^2(3\alpha_i)$, and finally the coefficients c_i. In this manner we find that the fifth partial sum $S_5(x)$ for the Fourier-Bessel series representation of $f(x) = x$, $0 < x < 3$ in Example 2 is

$$\begin{aligned} S_5(x) &= 4.01844\, J_1(0.98320x) - 1.86937\, J_1(1.94704x) \\ &\quad + 1.07106\, J_1(2.95758x) - 0.70306\, J_1(3.98538x) + 0.50343\, J_1(5.02078x). \end{aligned}$$

The graph of $S_5(x)$ on the interval (0, 3) is shown in Figure 11.5.1(a). In Figure 11.5.1(b) we have graphed $S_{10}(x)$ on the interval (0, 50). Notice that outside the interval of definition (0, 3) the series does not converge to a periodic extension of f because Bessel functions are not periodic functions. See Problems 11 and 12 in Exercises 11.5.

11.5.2 FOURIER-LEGENDRE SERIES

From Example 4 of Section 11.4 we know that the set of Legendre polynomials $\{P_n(x)\}$, $n = 0, 1, 2, \ldots$, is orthogonal with respect to the weight function $p(x) = 1$ on the interval $[-1, 1]$. Furthermore, it can be proved that the square norm of a polynomial $P_n(x)$ depends on n in the following manner:

$$\|P_n(x)\|^2 = \int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n + 1}.$$

The orthogonal series expansion of a function in terms of the Legendre polynomials is summarized in the next definition.

DEFINITION 11.5.2 Fourier-Legendre Series

The **Fourier-Legendre series** of a function f defined on the interval $(-1, 1)$ is given by

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \tag{21}$$

where
$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx. \tag{22}$$

CONVERGENCE OF A FOURIER-LEGENDRE SERIES Sufficient conditions for convergence of a Fourier-Legendre series are given in the next theorem.

THEOREM 11.5.2 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $[-1, 1]$. Then for all x in the interval $(-1, 1)$, the Fourier-Legendre series of f converges to $f(x)$ at a point of continuity. At a point of discontinuity the Fourier-Legendre series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

EXAMPLE 3 Expansion in a Fourier-Legendre Series

Write out the first four nonzero terms in the Fourier-Legendre expansion of

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \le x < 1. \end{cases}$$

SOLUTION The first several Legendre polynomials are listed on page 271. From these and (22) we find

$$c_0 = \frac{1}{2}\int_{-1}^{1} f(x)P_0(x)\,dx = \frac{1}{2}\int_0^1 1 \cdot 1\,dx = \frac{1}{2}$$

$$c_1 = \frac{3}{2}\int_{-1}^{1} f(x)P_1(x)\,dx = \frac{3}{2}\int_0^1 1 \cdot x\,dx = \frac{3}{4}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)\,dx = \frac{5}{2}\int_0^1 1 \cdot \frac{1}{2}(3x^2 - 1)\,dx = 0$$

$$c_3 = \frac{7}{2}\int_{-1}^{1} f(x)P_3(x)\,dx = \frac{7}{2}\int_0^1 1 \cdot \frac{1}{2}(5x^3 - 3x)\,dx = -\frac{7}{16}$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} f(x)P_4(x)\,dx = \frac{9}{2}\int_0^1 1 \cdot \frac{1}{8}(35x^4 - 30x^2 + 3)\,dx = 0$$

$$c_5 = \frac{11}{2}\int_{-1}^{1} f(x)P_5(x)\,dx = \frac{11}{2}\int_0^1 1 \cdot \frac{1}{8}(63x^5 - 70x^3 + 15x)\,dx = \frac{11}{32}.$$

Hence
$$f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) + \cdots.$$

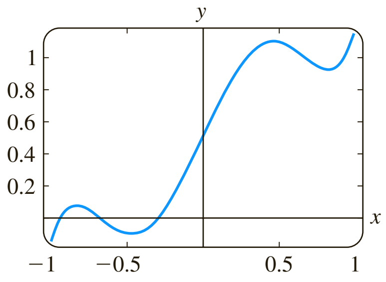


FIGURE 11.5.2 Partial sum $S_5(x)$ of the Fourier-Legendre series in Example 3

Like the Bessel functions, Legendre polynomials are built-in functions in computer algebra systems such as *Mathematica* and *Maple,* so each of the coefficients just listed can be found by using the integration application of such a program. Indeed, using a CAS, we further find that $c_6 = 0$ and $c_7 = -\frac{65}{256}$. The fifth partial sum of the Fourier-Legendre series representation of the function f defined in Example 3 is then

$$S_5(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \frac{11}{32}P_5(x) - \frac{65}{256}P_7(x).$$

The graph of $S_5(x)$ on the interval $(-1, 1)$ is given in Figure 11.5.2.

ALTERNATIVE FORM OF SERIES In applications the Fourier-Legendre series appears in an alternative form. If we let $x = \cos\theta$, then $x = 1$ implies that $\theta = 0$ whereas $x = -1$ implies that $\theta = \pi$. Since $dx = -\sin\theta\, d\theta$, (21) and (22) become, respectively,

$$F(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta) \tag{23}$$

$$c_n = \frac{2n+1}{2}\int_0^{\pi} F(\theta)P_n(\cos\theta)\sin\theta\, d\theta, \tag{24}$$

where $f(\cos\theta)$ has been replaced by $F(\theta)$.

EXERCISES 11.5

Answers to selected odd-numbered problems begin on page ANS-19.

11.5.1 Fourier-Bessel Series

In Problems 1 and 2 use Table 6.4.1 in Section 6.4.

1. Find the first four $\alpha_i > 0$ defined by $J_1(3\alpha) = 0$.

2. Find the first four $\alpha_i \geq 0$ defined by $J_0'(2\alpha) = 0$.

In Problems 3–6 expand $f(x) = 1$, $0 < x < 2$, in a Fourier-Bessel series, using Bessel functions of order zero that satisfy the given boundary condition.

3. $J_0(2\alpha) = 0$

4. $J_0'(2\alpha) = 0$

5. $J_0(2\alpha) + 2\alpha J_0'(2\alpha) = 0$

6. $J_0(2\alpha) + \alpha J_0'(2\alpha) = 0$

In Problems 7–10 expand the given function in a Fourier-Bessel series, using Bessel functions of the same order as in the indicated boundary condition.

7. $f(x) = 5x$, $0 < x < 4$, $\quad 3J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0$

8. $f(x) = x^2$, $0 < x < 1$, $\quad J_2(\alpha) = 0$

9. $f(x) = x^2$, $0 < x < 3$, $\quad J_0'(3\alpha) = 0 \quad$ [*Hint*: $t^3 = t^2 \cdot t$.]

10. $f(x) = 1 - x^2$, $0 < x < 1$, $\quad J_0(\alpha) = 0$

Computer Lab Assignments

11. (a) Use a CAS to plot the graph of $y = 3J_1(x) + xJ_1'(x)$ on an interval so that the first five positive x-intercepts of the graph are shown.

(b) Use the root-finding capability of your CAS to approximate the first five roots x_i of the equation $3J_1(x) + xJ_1'(x) = 0$.

(c) Use the data obtained in part (b) to find the first five positive values of α_i that satisfy $3J_1(4\alpha) + 4\alpha J_1'(4\alpha) = 0$. (See Problem 7.)

(d) If instructed, find the first ten positive values of α_i.

12. (a) Use the values of α_i in part (c) of Problem 11 and a CAS to approximate the values of the first five coefficients c_i of the Fourier-Bessel series obtained in Problem 7.

(b) Use a CAS to plot the graphs of the partial sums $S_N(x)$, $N = 1, 2, 3, 4, 5$ of the Fourier-Bessel series in Problem 7.

(c) If instructed, plot the graph of the partial sum $S_{10}(x)$ on the interval $(0, 4)$ and on $(0, 50)$.

Discussion Problems

13. If the partial sums in Problem 12 are plotted on a symmetric interval such as $(-30, 30)$ would the graphs possess any symmetry? Explain.

14. (a) Sketch, by hand, a graph of what you think the Fourier-Bessel series in Problem 3 converges to on the interval $(-2, 2)$.

(b) Sketch, by hand, a graph of what you think the Fourier-Bessel series would converge to on the interval $(-4, 4)$ if the values α_i in Problem 7 were defined by $3J_2(4\alpha) + 4\alpha J_2'(4\alpha) = 0$.

11.5.2 Fourier-Legendre Series

In Problems 15 and 16 write out the first five nonzero terms in the Fourier-Legendre expansion of the given function. If instructed, use a CAS as an aid in evaluating the coefficients. Use a CAS to plot the graph of the partial sum $S_5(x)$.

15. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$

16. $f(x) = e^x, \quad -1 < x < 1$

17. The first three Legendre polynomials are $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$. If $x = \cos\theta$, then $P_0(\cos\theta) = 1$ and $P_1(\cos\theta) = \cos\theta$. Show that $P_2(\cos\theta) = \frac{1}{4}(3\cos 2\theta + 1)$.

18. Use the results of Problem 17 to find a Fourier-Legendre expansion (23) of $F(\theta) = 1 - \cos 2\theta$.

19. A Legendre polynomial $P_n(x)$ is an even or odd function, depending on whether n is even or odd. Show that if f is an even function on the interval $(-1, 1)$, then (21) and (22) become, respectively,

$$f(x) = \sum_{n=0}^{\infty} c_{2n} P_{2n}(x) \tag{25}$$

$$c_{2n} = (4n + 1) \int_0^1 f(x) P_{2n}(x)\, dx. \tag{26}$$

The series (25) can also be used when f is defined only on the interval $(0, 1)$. The series then represents f on $(0, 1)$ and an even extension of f on the interval $(-1, 0)$.

20. Show that if f is an odd function on the interval $(-1, 1)$, then (21) and (22) become, respectively,

$$f(x) = \sum_{n=0}^{\infty} c_{2n+1} P_{2n+1}(x) \tag{27}$$

$$c_{2n+1} = (4n + 3) \int_0^1 f(x) P_{2n+1}(x)\, dx. \tag{28}$$

The series (27) can also be used when f is defined only on the interval $(0, 1)$. The series then represents f on $(0, 1)$ and an odd extension of f on the interval $(-1, 0)$.

In Problems 21 and 22 write out the first four nonzero terms in the indicated expansion of the given function. What function does the series represent on the interval $(-1, 1)$? Use a CAS to plot the graph of the partial sum $S_4(x)$.

21. $f(x) = x, \quad 0 < x < 1; \quad$ use (25)

22. $f(x) = 1, \quad 0 < x < 1; \quad$ use (27)

Discussion Problems

23. Discuss: Why is a Fourier-Legendre expansion of a polynomial function that is defined on the interval $(-1, 1)$ necessarily a finite series?

24. Using only your conclusions from Problem 23—that is, do not use (22)—find the finite Fourier-Legendre series of $f(x) = x^2$. The series of $f(x) = x^3$.

Chapter 11 In Review

Answers to selected odd-numbered problems begin on page ANS-19.

In Problems 1–6 fill in the blank or answer true or false without referring back to the text.

1. The functions $f(x) = x^2 - 1$ and $g(x) = x^5$ are orthogonal on the interval $[-\pi, \pi]$. ________

2. The product of an odd function f with an odd function g is ________.

3. To expand $f(x) = |x| + 1$, $-\pi < x < \pi$, in an appropriate trigonometric series, we would use a ________ series.

4. $y = 0$ is never an eigenfunction of a Sturm-Liouville problem. ________

5. $\lambda = 0$ is never an eigenvalue of a Sturm-Liouville problem. ________

6. If the function $f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ -x, & 0 < x < 1 \end{cases}$ is expanded in a Fourier series, the series will converge to ________ at $x = -1$, to ________ at $x = 0$, and to ________ at $x = 1$.

7. Suppose the function $f(x) = x^2 + 1$, $0 < x < 3$, is expanded in a Fourier series, a cosine series, and a sine series. Give the value to which each series will converge at $x = 0$.

8. What is the corresponding eigenfunction for the boundary-value problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(\pi/2) = 0$$

for $\lambda = 25$?

9. Chebyshev's differential equation

$$(1 - x^2)y'' - xy' + n^2 y = 0$$

has a polynomial solution $y = T_n(x)$ for $n = 0, 1, 2, \ldots$. Specify the weight function $w(x)$ and the interval over which the set of Chebyshev polynomials $\{T_n(x)\}$ is orthogonal. Give an orthogonality relation.

10. The set of Legendre polynomials $\{P_n(x)\}$, where $P_0(x) = 1$, $P_1(x) = x, \ldots$, is orthogonal with respect to the weight function $w(x) = 1$ on the interval $[-1, 1]$. Explain why $\int_{-1}^{1} P_n(x)\, dx = 0$ for $n > 0$.

11. Without doing any work, explain why the cosine series of $f(x) = \cos^2 x$, $0 < x < \pi$ is the finite series $f(x) = \frac{1}{2} + \frac{1}{2}\cos 2x$.

12. (a) Show that the set

$$\left\{\sin\frac{\pi}{2L}x, \sin\frac{3\pi}{2L}x, \sin\frac{5\pi}{2L}x, \ldots\right\}$$

is orthogonal on the interval $[0, L]$.

(b) Find the norm of each function in part (a). Construct an orthonormal set.

13. Expand $f(x) = |x| - x$, $-1 < x < 1$ in a Fourier series.

14. Expand $f(x) = 2x^2 - 1$, $-1 < x < 1$ in a Fourier series.

15. Expand $f(x) = e^{-x}$, $0 < x < 1$

(a) in a cosine series **(b)** in a sine series.

16. In Problems 13, 14, and 15, sketch the periodic extension of f to which each series converges.

17. Discuss: Which of the two Fourier series of f in Problem 15 converges to

$$F(x) = \begin{cases} f(x), & 0 < x < 1 \\ f(-x), & -1 < x < 0 \end{cases}$$

on the interval $(-1, 1)$?

18. Consider the portion of the periodic function f shown in Figure 11.R.1. Expand f in an appropriate Fourier series.

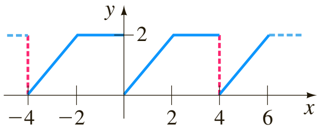


FIGURE 11.R.1 Graph for Problem 18

19. Find the eigenvalues and eigenfunctions of the boundary-value problem

$$x^2y'' + xy' + 9\lambda y = 0, \quad y'(1) = 0, \quad y(e) = 0.$$

20. Give an orthogonality relation for the eigenfunctions in Problem 19.

21. Expand $f(x) = \begin{cases} 1, & 0 < x < 2 \\ 0, & 2 < x < 4 \end{cases}$ in a Fourier-Bessel series, using Bessel functions of order zero that satisfy the boundary-condition $J_0(4\alpha) = 0$.

22. Expand $f(x) = x^4$, $-1 < x < 1$, in a Fourier-Legendre series.

23. Suppose the function $y = f(x)$ is defined on the interval $(-\infty, \infty)$.

(a) Verify the identity $f(x) = f_e(x) + f_o(x)$, where

$$f_e(x) = \frac{f(x) + f(-x)}{2} \quad \text{and} \quad f_o(x) = \frac{f(x) - f(-x)}{2}.$$

(b) Show that f_e is an even function and f_o an odd function.

24. The function $f(x) = e^x$ is neither even or odd. Use Problem 23 to write f as the sum of an even function and an odd function. Identify f_e and f_o.

25. Suppose that f is an integrable $2p$-periodic function. Prove that for any number a,

$$\int_0^{2p} f(x)\,dx = \int_a^{a+2p} f(x)\,dx.$$

this theorem is one of the most monumental results of modern mathematical analysis and has widespread physical and engineering applications.

We express the exponential factor $\exp[ik(x-\xi)]$ in (2.2.4) in terms of trigonometric functions and use the even and odd nature of the cosine and the sine functions respectively as functions of k so that (2.2.4) can be written as

$$f(x) = \frac{1}{\pi}\int_0^\infty dk \int_{-\infty}^\infty f(\xi)\cos k(x-\xi)d\xi. \tag{2.2.7}$$

This is another version of the *Fourier integral formula.* In many physical problems, the function $f(x)$ vanishes very rapidly as $|x| \to \infty$, which ensures the existence of the repeated integrals as expressed.

We now assume that $f(x)$ is an even function and expand the cosine function in (2.2.7) to obtain

$$f(x) = f(-x) = \frac{2}{\pi}\int_0^\infty \cos kx\,dk \int_0^\infty f(\xi)\cos k\xi\,d\xi. \tag{2.2.8}$$

This is called the *Fourier cosine integral formula.*

Similarly, for an odd function $f(x)$, we obtain the *Fourier sine integral formula*

$$f(x) = -f(-x) = \frac{2}{\pi}\int_0^\infty \sin\, kx\,dk \int_0^\infty f(\xi)\sin k\xi\,d\xi. \tag{2.2.9}$$

These integral formulas were discovered independently by Cauchy in his work on the propagation of waves on the surface of water.

2.3 Definition of the Fourier Transform and Examples

We use the Fourier integral formula (2.2.4) to give a formal definition of the Fourier transform.

DEFINITION 2.3.1 *The Fourier transform of $f(x)$ is denoted by $\mathscr{F}\{f(x)\} = F(k)$, $k \in \mathbb{R}$, and defined by the integral*

$$\mathscr{F}\{f(x)\} = F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty e^{-ikx}f(x)dx, \tag{2.3.1}$$

where $\mathscr{F}$ is called the Fourier transform operator or the Fourier transformation and the factor $\frac{1}{\sqrt{2\pi}}$ is obtained by splitting the factor $\frac{1}{2\pi}$ involved in

(2.2.4). This is often called the complex Fourier transform. A sufficient condition for $f(x)$ to have a Fourier transform is that $f(x)$ is absolutely integrable on $(-\infty, \infty)$. The convergence of the integral (2.3.1) follows at once from the fact that $f(x)$ is absolutely integrable. In fact, the integral converges uniformly with respect to k.

Thus, the definition of the Fourier transform is restricted to absolutely integrable functions. This restriction is too strong for many physical applications. Many simple and common functions, such as constant function, trigonometric functions $\sin ax$, $\cos ax$, exponential functions, and $x^n H(x)$ do not have Fourier transforms, even though they occur frequently in applications. The integral in (2.3.1) fails to converge when $f(x)$ is one of the above elementary functions. This is a very unsatisfactory feature of the theory of Fourier transforms. However, this unsatisfactory feature can be resolved by means of a natural extension of the definition of the Fourier transform of a generalized function, $f(x)$ in (2.3.1). We follow Lighthill (1958) and Jones (1982) to discuss briefly the theory of the Fourier transforms of good functions.

The inverse Fourier transform, denoted by $\mathscr{F}^{-1}\{F(k)\} = f(x)$, is defined by

$$\mathscr{F}^{-1}\{F(k)\} = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\, F(k)\, dk, \tag{2.3.2}$$

where $\mathscr{F}^{-1}$ is called the inverse Fourier transform operator.

Clearly, both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are linear integral operators. In applied mathematics, x usually represents a space variable and $k(=\frac{2\pi}{\lambda})$ is a wavenumber variable where λ is the wavelength. However, in electrical engineering, x is replaced by the time variable t and k is replaced by the frequency variable $\omega(=2\pi\nu)$ where ν is the frequency in cycles per second. The function $F(\omega) = \mathscr{F}\{f(t)\}$ is called the *spectrum* of the *time signal function* $f(t)$. In electrical engineering literature, the Fourier transform pairs are defined slightly differently by

$$\mathscr{F}\{f(t)\} = F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi\nu i t} dt, \tag{2.3.3}$$

and

$$\mathscr{F}^{-1}\{F(\nu)\} = f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi i\nu t} d\nu = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega, \tag{2.3.4}$$

where $\omega = 2\pi\nu$ is called the *angular frequency*. The Fourier integral formula implies that any function of time $f(t)$ that has a Fourier transform can be equally specified by its spectrum. Physically, the signal $f(t)$ is represented as an integral superposition of an infinite number of sinusoidal oscillations with different frequencies ω and complex amplitudes $\frac{1}{2\pi}F(\omega)$. Equation (2.3.4) is called the *spectral resolution* of the signal $f(t)$, and $\frac{F(\omega)}{2\pi}$ is called the *spectral density.* In summary, the Fourier transform maps a function (or signal) of time t to a function of frequency ω. In the same way as the Fourier series expansion of a periodic function decomposes the function into harmonic components, the Fourier transform generates a function (or signal) of a continuous variable whose value represents the frequency content of the original signal. This led to the successful use of the Fourier transform to analyze the form of time-varying signals in electrical engineering and seismology.

Next we give examples of Fourier transforms.

Example 2.3.1

Find the Fourier transform of $\exp(-ax^2)$. In fact, we prove

$$F(k) = \mathscr{F}\{\exp(-ax^2)\} = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \quad a > 0. \tag{2.3.5}$$

Here we have, by definition,

$$\begin{aligned} F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx-ax^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-a\left(x+\frac{ik}{2a}\right)^2 - \frac{k^2}{4a}\right] dx \\ &= \frac{1}{\sqrt{2\pi}} \exp(-k^2/4a) \int_{-\infty}^{\infty} e^{-ay^2} dy = \frac{1}{\sqrt{2a}} \exp\left(-\frac{k^2}{4a}\right), \end{aligned}$$

in which the change of variable $y = x + \frac{ik}{2a}$ is used. The above result is correct, but the change of variable can be justified by the method of complex analysis because $(ik/2a)$ is complex. If $a = \frac{1}{2}$

$$\mathscr{F}\{e^{-x^2/2}\} = e^{-k^2/2}. \tag{2.3.6}$$

This shows $\mathscr{F}\{f(x)\} = f(k)$. Such a function is said to be *self-reciprocal* under the Fourier transformation. Graphs of $f(x) = \exp(-ax^2)$ and its Fourier transform is shown in Figure 2.1 for $a = 1$. $\square$

Example 2.3.2

Find the Fourier transform of $\exp(-a|x|)$, i.e.,

$$\mathscr{F}\{\exp(-a|x|)\} = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{(a^2+k^2)}, \quad a > 0. \tag{2.3.7}$$

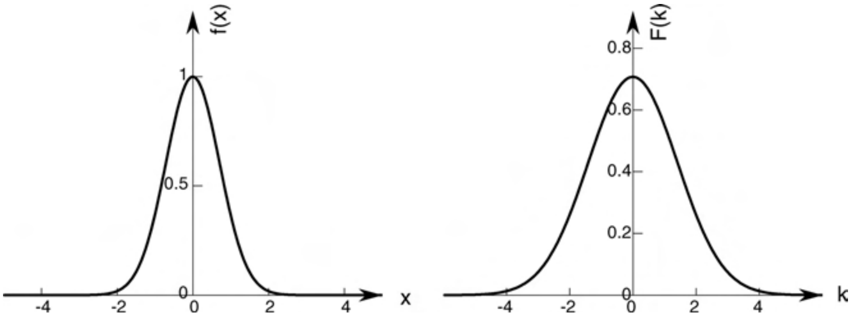


Figure 2.1 Graphs of $f(x) = \exp(-ax^2)$ and $F(k)$ with $a = 1$.

Here we can write

$$\begin{aligned}\mathscr{F}\left\{e^{-a|x|}\right\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a|x|-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-(a+ik)x}dx + \int_{-\infty}^{0} e^{(a-ik)x}dx\right] \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+ik} + \frac{1}{a-ik}\right] = \sqrt{\frac{2}{\pi}}\,\frac{a}{(a^2+k^2)}.\end{aligned}$$

We note that $f(x) = \exp(-a|x|)$ decreases rapidly at infinity, it is not differentiable at $x = 0$. Graphs of $f(x) = \exp(-a|x|)$ and its Fourier transform is displayed in Figure 2.2 for $a = 1$. $\square$

Example 2.3.3

Find the Fourier transform of

$$f(x) = \left(1 - \frac{|x|}{a}\right) H\left(1 - \frac{|x|}{a}\right),$$

where $H(x)$ is the *Heaviside unit step function* defined by

$$H(x) = \begin{Bmatrix} 1, & x > 0 \\ 0, & x < 0 \end{Bmatrix}. \tag{2.3.8}$$

Or, more generally,

$$H(x-a) = \begin{Bmatrix} 1, & x > a \\ 0, & x < a \end{Bmatrix}, \tag{2.3.9}$$

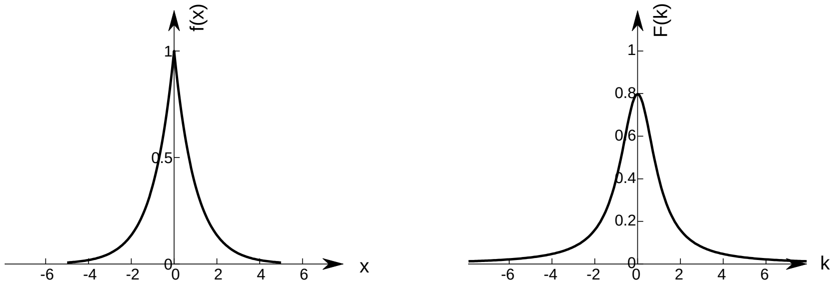


Figure 2.2 Graphs of $f(x) = \exp(-a|x|)$ and $F(k)$ with $a = 1$.

where a is a fixed real number. So the Heaviside function $H(x-a)$ has a finite discontinuity at $x = a$.

$$\begin{aligned}
\mathscr{F}\{f(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx} \left(1 - \frac{|x|}{a}\right) dx = \frac{2}{\sqrt{2\pi}} \int_{0}^{a} \left(1 - \frac{x}{a}\right) \cos kx \, dx \\
&= \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \cos(akx) dx = \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} (1-x) \frac{d}{dx} \left(\frac{\sin akx}{ak}\right) dx \\
&= \frac{2a}{\sqrt{2\pi}} \int_{0}^{1} \frac{\sin(akx)}{ak} dx = \frac{a}{\sqrt{2\pi}} \int_{0}^{1} \frac{d}{dx} \left[\frac{\sin^2\left(\frac{akx}{2}\right)}{\left(\frac{ak}{2}\right)^2} \right] dx \\
&= \frac{a}{\sqrt{2\pi}} \frac{\sin^2\left(\frac{ak}{2}\right)}{\left(\frac{ak}{2}\right)^2}.
\end{aligned} \tag{2.3.10}$$

▯

Example 2.3.4

Find the Fourier transform of the characteristic function $\chi_{[-a,a]}(x)$, where

$$\chi_{[-a,a]}(x) = H(a - |x|) = \left\{ \begin{matrix} 1, & |x| < a \\ 0, & |x| > a \end{matrix} \right\}. \tag{2.3.11}$$

We have

$$F_a(k) = \mathscr{F}\{\chi_{[-a,a]}(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \chi_{[-a,a]}(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-ikx}\, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\sin ak}{k}\right). \quad (2.3.12)$$

Graphs of $f(x) = \chi_{[-a,a]}(x)$ and its Fourier transform are shown in Figure 2.3 for $a = 1$.

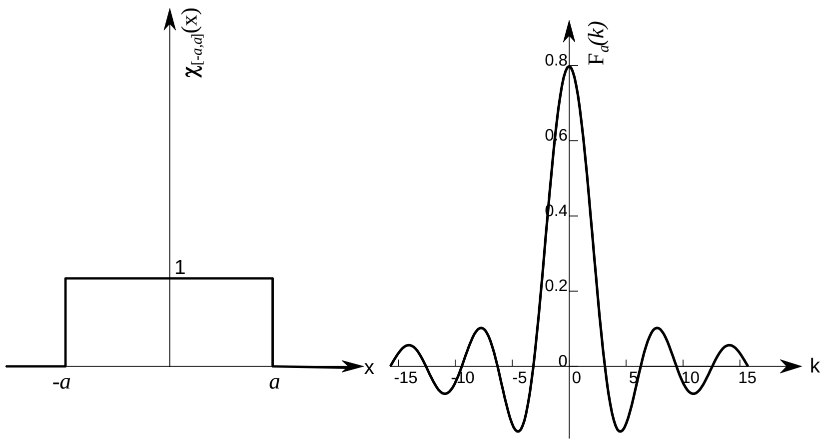


Figure 2.3 Graphs of $\chi_{[-a,a]}(x)$ and $F_a(k)$ with $a = 1$.

□

2.4 Fourier Transforms of Generalized Functions

The natural way to define the Fourier transform of a generalized function, is to treat $f(x)$ in (2.3.1) as a generalized function. The advantage of this is that every generalized function has a Fourier transform and an inverse Fourier transform, and that the ordinary functions whose Fourier transforms are of interest form a subset of the generalized functions. We would not go into great detail, but refer to the famous books of Lighthill (1958) and Jones (1982) for

Result (12.2.9) can be considered as the amplitude modulation of unit impulses by the waveform $f(t)$. Evidently, this result is very useful for analyzing the systems where signals are sampled at a time interval T. Thus, the above discussion enables us to introduce the Z transform in the next section.

12.3 Definition of the Z Transform and Examples

We take the Laplace transform of the sampled function given by (12.2.9) so that

$$\mathscr{L}\{f^*(t)\} = \bar{f}^*(s) = \sum_{n=0}^{\infty} f(nT)\exp(-nsT). \tag{12.3.1}$$

It is convenient to make a change of variable $z = \exp(sT)$ so that (12.3.1) becomes

$$\mathscr{L}\{f^*(t)\} = F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}. \tag{12.3.2}$$

Thus, $F(z)$ is called the Z *transform* of $f(nT)$. Since the interval T between the samples has no effect on the properties and the use of the Z transform, it is convenient to set $T = 1$. We now define the Z *transform* of a sequence $\{f(n)\}$ as the function $F(z)$ of a complex variable z defined by

$$Z\{f(n)\} = F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}. \tag{12.3.3}$$

Thus, Z is a linear transformation and can be considered as an operator mapping sequences of scalars into functions of the complex variable z. It is assumed in this chapter that there exists an R such that (12.3.3) converges for $|z| > R$. Since $|z| = |\exp(sT)| = |\exp(\sigma + i\mu)T| = |\exp(\sigma T)|$, it follows that, when $\sigma < 0$ (that is, in the left half of the complex s plane), $|z| < 1$, and thus, the left half of the s plane corresponds to the interior of the unit circle in the complex z plane. Similarly, the right half of the s plane corresponds to the exterior ($|z| > 1$) of the unit circle in the z plane. And $\sigma = 0$ in the s plane corresponds to the unit circle in the z plane.

The inverse Z transform is given by the complex integral

$$Z^{-1}\{F(z)\} = f(n) = \frac{1}{2\pi i}\oint_C F(z)\, z^{n-1}\, dz, \tag{12.3.4}$$

where C is a simple closed contour enclosing the origin and lying outside the circle $|z| = R$. The existence of the inverse imposes restrictions on $f(n)$ for uniqueness. We require that $f(n) = 0$ for $n < 0$.

To obtain the inversion integral, we consider

$$\begin{aligned}F(z) &= \sum_{n=0}^{\infty} f(n)\, z^{-n} \\ &= f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \cdots + f(n)\, z^{-n} + f(n+1)\, z^{-(n+1)} + \cdots .\end{aligned}$$

Multiplying both sides by $(2\pi i)^{-1} z^{n-1}$ and integrating along the closed contour C, which usually encloses all singularities of $F(z)$, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\oint_C F(z) z^{n-1}\, dz = \frac{1}{2\pi i}\Bigg[&\oint_C f(0) z^{n-1}\, dz + \oint_C f(1) z^{n-2}\, dz \\ &+ \cdots + \oint_C f(n) z^{-1}\, dz + \oint_C f(n+1) z^{-2}\, dz + \cdots\Bigg].\end{aligned}$$

By Cauchy's Fundamental Theorem all integrals on the right vanish except

$$\frac{1}{2\pi i}\oint_C f(n)\, \frac{dz}{z} = f(n).$$

This leads to the inversion integral for the Z transform in the form

$$Z^{-1}\{F(z)\} = f(n) = \frac{1}{2\pi i}\oint_C F(z)\, z^{n-1}\, dz.$$

Similarly, we can define the so called *bilateral Z transform* by

$$Z\{f(n)\} = F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}, \tag{12.3.5}$$

for all complex numbers z for which the series converges. This reduces to the unilateral Z transform (12.3.3) if $f(n) = 0$ for $n < 0$. The inverse Z transform is given by a complex integral similar to (12.3.4). Substituting $z = re^{i\theta}$ in (12.3.5), we obtain the Z transform evaluated at $r = 1$

$$\mathscr{F}\{f(n)\} = F(\theta) = \sum_{n=-\infty}^{\infty} f(n) e^{-in\theta}.$$

This is known as the *Fourier transform of the sequence* $\{f(n)\}_{-\infty}^{\infty}$.

Example 12.3.1

If $f(n) = a^n$, $n \geq 0$, then

$$Z\{a^n\} = \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}, \quad |z| > a. \tag{12.3.6}$$

When $a=1$, we obtain

$$Z\{1\}=\sum_{n=0}^{\infty} z^{-n}=\frac{z}{z-1}, \quad |z|>1. \tag{12.3.7}$$

If $f(n)=n\,a^n$ for $n\geq 0$, then

$$Z\{n\,a^n\}=\sum_{n=0}^{\infty} n\,a^n\,z^{-n}=\frac{az}{(z-a)^2}, \quad |z|>|a|. \tag{12.3.8}$$

□

Example 12.3.2

If $f(n)=\exp(inx)$, then

$$Z\{\exp(inx)\}=\frac{z}{z-\exp(ix)}. \tag{12.3.9}$$

This follows immediately from (12.3.6).

Furthermore,

$$Z\{\cos nx\}=\frac{z(z-\cos x)}{z^2-2z\cos x+1}, \quad Z\{\sin nx\}=\frac{z\sin x}{z^2-2z\cos x+1}. \tag{12.3.10}$$

These follow readily from (12.3.9) by writing $\exp(inx)=\cos nx+i\sin nx$. □

Example 12.3.3

If $f(n)=n$, then

$$\begin{aligned} Z\{n\} &= \sum_{n=0}^{\infty} n\,z^{-n}=z\sum_{n=0}^{\infty} n\,z^{-(n+1)} \\ &= -z\frac{d}{dz}\left(\sum_{n=0}^{\infty} z^{-n}\right)=\frac{z}{(z-1)^2}, \quad |z|>1. \end{aligned} \tag{12.3.11}$$

□

Example 12.3.4

If $f(n)=\frac{1}{n!}$, then

$$Z\left\{\frac{1}{n!}\right\}=\sum_{n=0}^{\infty}\frac{1}{n!}\,z^{-n}=\exp\left(\frac{1}{z}\right) \quad \text{for all } z. \tag{12.3.12}$$

□

Example 12.3.5

If $f(n)=\cosh nx$, then

$$Z\{\cosh nx\}=\frac{z(z-\cosh x)}{z^2-2z\cosh x+1}. \tag{12.3.13}$$

We have

$$\begin{aligned}Z\{\cosh nx\} &= \frac{1}{2}Z\{e^{nx}+e^{-nx}\}\\ &= \frac{1}{2}\left[\frac{z}{z-e^x}+\frac{z}{z-e^{-x}}\right]\\ &= \frac{z(z-\cosh x)}{z^2-2z\cosh x+1}.\end{aligned}$$

□

Example 12.3.6

Show that

$$Z\{n^2\}=\frac{z(z+1)}{(z-1)^3}. \tag{12.3.14}$$

We have, from (12.4.13) in section 12.4,

$$Z\{n\cdot n\}=-z\,\frac{d}{dz}Z\{n\}=-z\,\frac{d}{dz}\,\frac{z}{(z-1)^2}=\frac{z(z+1)}{(z-1)^3}.$$

□

Example 12.3.7

If $f(n)$ is a periodic sequence of integral period N, then

$$F(z)=Z\{f(n)\}=\frac{z^N}{z^N-1}\,F_1(z),$$

where

$$F_1(z)=\sum_{k=0}^{N-1} f(k)\,z^{-k}. \tag{12.3.15}$$

We have, by definition,

$$\begin{aligned}F(z)=Z\{f(n)\} &= \sum_{n=0}^{\infty} f(n)\,z^{-n}=z^N\sum_{n=0}^{\infty} f(n+N)\,z^{-(n+N)}\\ &= z^N\sum_{k=N}^{\infty} f(k)\,z^{-k}, \quad (n+N=k)\\ &= z^N\left[\sum_{k=0}^{\infty} f(k)\,z^{-k}-\sum_{k=0}^{N-1} f(k)\,z^{-k}\right]\\ &= \left\{z^N F(z)-z^N F_1(z)\right\}.\end{aligned}$$

Thus,

$$F(z) = \frac{z^N}{(z^N - 1)} F_1(z).$$

□

12.4 Basic Operational Properties of *Z* Transforms

THEOREM 12.4.1
(*Translation*). If $Z\{f(n)\} = F(z)$ and $m \geq 0$, then

$$Z\{f(n-m)\} = z^{-m} \left[F(z) + \sum_{r=-m}^{-1} f(r)\, z^{-r} \right], \qquad (12.4.1)$$

$$Z\{f(n+m)\} = z^{m} \left[F(z) - \sum_{r=0}^{m-1} f(r)\, z^{-r} \right]. \qquad (12.4.2)$$

In particular, if $m = 1, 2, 3, \ldots$, then

$$Z\{f(n-1)\} = z^{-1} F(z) - f(-1)\, z. \qquad (12.4.3)$$

$$Z\{f(n-2)\} = z^{-2} \left[F(z) + \sum_{r=-2}^{-1} f(r) z^{-r} \right]. \qquad (12.4.4)$$

and so on.

Similarly, it follows from (12.4.2) that

$$Z\{f(n+1)\} = z\{F(z) - f(0)\}, \qquad (12.4.5)$$

$$Z\{f(n+2)\} = z^2\{F(z) - f(0)\} - z\, f(1), \qquad (12.4.6)$$

$$Z\{f(n+3)\} = z^3\{F(z) - f(0)\} - z^2\, f(1) - z\, f(2). \qquad (12.4.7)$$

More generally, for $m > 0$,

$$Z\{f(n+m)\} = z^m\{F(z) - f(0)\} - z^{m-1} f(1) - \cdots - z\, f(m-1). \qquad (12.4.8)$$

All these results are widely used for the solution of initial value problems involving difference equations. Result (12.4.8) is somewhat similar to (3.4.12) for this Laplace transform, and has been used to solve initial value problems involving differential equations.

PROOF We have, by definition,

$$\begin{aligned} Z\{f(n-m)\} &= \sum_{n=0}^{\infty} f(n-m) z^{-n}, \qquad (n-m=r), \\ &= z^{-m} \sum_{r=-m}^{\infty} f(r)\, z^{-r} = z^{-m} \sum_{r=0}^{\infty} f(r) z^{-r} + z^{-m} \sum_{r=-m}^{-1} f(r) z^{-r}. \end{aligned}$$

When $m=1$, we get (12.4.3).

If $f(r)=0$ for all $r<0$, then

$$Z\{f(n-m)\} = z^{-m} \sum_{r=0}^{\infty} f(r)\, z^{-r}. \tag{12.4.9}$$

When $m=1$, this result gives

$$Z\{f(n-1)\} = z^{-1} F(z). \tag{12.4.10}$$

Similarly, we prove (12.4.2) by writing

$$\begin{aligned} Z\{f(n+m)\} &= \sum_{n=0}^{\infty} f(n+m) z^{-n}, \qquad (n+m=r), \\ &= z^{m} \sum_{r=m}^{\infty} f(r)\, z^{-r} = z^{m} \sum_{r=0}^{\infty} f(r)\, z^{-r} - z^{m} \sum_{r=0}^{m-1} f(r)\, z^{-r} \\ &= z^{m} \left[F(z) - \sum_{r=0}^{m-1} f(r) z^{-r} \right]. \end{aligned}$$

When $m=1,2,3,\ldots$, results (12.4.5)–(12.4.7) follow immediately. ∎

THEOREM 12.4.2
(*Multiplication*). If $Z\{f(n)\} = F(z)$, then

$$Z\{a^n f(n)\} = F\left(\frac{z}{a}\right), \qquad |z| > |a|. \tag{12.4.11}$$

$$Z\{e^{-nb} f(n)\} = F(ze^{b}), \qquad |z| > |e^{-b}|. \tag{12.4.12}$$

$$Z\{n\, f(n)\} = -z \frac{d}{dz} F(z). \tag{12.4.13}$$

More generally,

$$Z\left[n^k f(n)\right] = (-1)^k \left(z \frac{d}{dz}\right)^k F(z), \quad k = 0, 1, 2, \ldots, \tag{12.4.14}$$

where

$$\left(z\frac{d}{dz}\right)^k F(z) = \left(z\frac{d}{dz}\right)^{(k-1)} \left(z\,\frac{d}{dz}\right) F.$$

PROOF Result (12.4.11) follows immediately from the definition (12.3.3), and (12.4.12) follows from (12.4.11) by writing $a = e^{-b}$.

If $f(n) = 1$ so that $Z\{f(n)\} = \frac{z}{z-1}$, and if $a = e^b$, then (12.4.1) gives

$$Z\{(e^b)^n\} = \frac{ze^{-b}}{ze^{-b}-1} = \frac{z}{z-e^b}, \quad |z| > |e^b|. \tag{12.4.15}$$

Putting $b = ix$ also gives (12.3.9)

To prove (12.4.13), we use the definition (12.3.3) to obtain

$$\begin{aligned} Z\{n\,f(n)\} &= \sum_{n=0}^{\infty} n\,f(n)\,z^{-n} = z\sum_{n=0}^{\infty} n\,f(n)\,z^{-(n+1)} \\ &= z\sum_{n=0}^{\infty} f(n)\left\{-\frac{d}{dz}z^{-n}\right\} = -z\frac{d}{dz}\left\{\sum_{n=0}^{\infty} f(n)\,z^{-n}\right\} = -z\frac{d}{dz}F(z). \end{aligned}$$

∎

THEOREM 12.4.3
(*Division*).

$$Z\left\{\frac{f(n)}{n+m}\right\} = -z^m \int_0^z \frac{F(\xi)d\xi}{\xi^{m+1}}. \tag{12.4.16}$$

PROOF We have

$$\begin{aligned} Z\left\{\frac{f(n)}{n+m}\right\} &= \sum_{n=0}^{\infty} \frac{f(n)}{n+m}\,z^{-n}, \quad (m \geq 0), \\ &= -z^m \sum_{n=0}^{\infty} f(n)\left[-\int_0^z \xi^{-(n+m+1)}\,d\xi\right] \\ &= -z^m \int_0^z \xi^{-(m+1)}\left[\sum_{n=0}^{\infty} f(n)\,\xi^{-n}\right] d\xi \\ &= -z^m \int_0^z \xi^{-(m+1)} F(\xi)\,d\xi. \end{aligned}$$

When $m = 0, 1, 2, \ldots$, several particular results follow from (12.4.16). ∎

THEOREM 12.4.4
(*Convolution*). If $Z\{f(n)\}=F(z)$ and $Z\{g(n)\}=G(z)$, then the Z transform of the convolution $f(n)*g(n)$ is given by

$$Z\{f(n)*g(n)\}=Z\{f(n)\}Z\{g(n)\}, \tag{12.4.17}$$

where the *convolution* is defined by

$$f(n)*g(n)=\sum_{m=0}^{\infty} f(n-m)g(m). \tag{12.4.18}$$

Or, equivalently,

$$Z^{-1}\{F(z)G(z)\}=\sum_{m=0}^{\infty} f(n-m)g(m). \tag{12.4.19}$$

PROOF We proceed formally to obtain

$$Z\{f(n)*g(n)\}=\sum_{n=0}^{\infty} z^{-n}\sum_{m=0}^{\infty} f(n-m)g(m),$$

which is, interchanging the order of summation,

$$=\sum_{m=0}^{\infty} g(m)\sum_{n=0}^{\infty} f(n-m)z^{-n}.$$

Substituting $n-m=r$, we obtain

$$Z\{f(n)*g(n)\}=\sum_{m=0}^{\infty} g(m)z^{-m}\sum_{r=-m}^{\infty} f(r)z^{-r},$$

which is, in view of $f(r)=0$ for $r<0$,

$$\begin{aligned}&=\sum_{m=0}^{\infty} g(m)\,z^{-m}\sum_{r=0}^{\infty} f(r)\,z^{-r}\\ &=Z\{f(n)\}Z\{g(n)\}.\end{aligned}$$

This proves the theorem.

More generally, the convolution $f(n)*g(n)$ is defined by

$$f(n)*g(n)=\sum_{m=-\infty}^{\infty} f(n-m)\,g(m). \tag{12.4.20}$$

If we assume $f(n)=0=g(n)$ for $n<0$, then (12.4.20) becomes (12.4.18).

However, the Z transform of (12.4.20) gives

$$Z\{f(n)*g(n)\} = \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} f(n-m)\,g(m),$$

which is, interchanging the order of summation,

$$\begin{aligned}
&= \sum_{m=-\infty}^{\infty} g(m) \sum_{n=-\infty}^{\infty} f(n-m)\,z^{-n} \\
&= \sum_{m=-\infty}^{\infty} z^{-m}\,g(m) \sum_{n=-\infty}^{\infty} f(n-m)\,z^{-(n-m)} \\
&= \sum_{m=-\infty}^{\infty} z^{-m} g(m) \sum_{r=-\infty}^{\infty} f(r) z^{-r}, \quad (r = n-m) \\
&= Z\{f(n)\} Z\{g(n)\}. \qquad (12.4.21)
\end{aligned}$$

This is the convolution theorem for the bilateral Z transform. ▮

The Z transform of the product $f(n)\,g(n)$ is given by

$$Z\{f(n)\,g(n)\} = \frac{1}{2\pi i} \oint_C F(w)\,G\left(\frac{z}{w}\right) \frac{dw}{w}, \qquad (12.4.22)$$

where C is a closed contour enclosing the origin in the domain of convergence of $F(w)$ and $G\left(\frac{z}{w}\right)$.

THEOREM 12.4.5
(*Parseval's Formula*). If $F(z) = Z\{f(n)\}$ and $G(z) = Z\{g(n)\}$, then

$$\sum_{n=-\infty}^{\infty} f(n)\,\overline{g}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta})\ \overline{G}(e^{i\theta})\,d\theta. \qquad (12.4.23)$$

In particular,

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{i\theta})|^2\,d\theta. \qquad (12.4.24)$$

THEOREM 12.4.6
(*Initial Value Theorem*). If $Z\{f(n)\} = F(z)$, then

$$f(0) = \lim_{z\to\infty} F(z). \qquad (12.4.25)$$

Also, if $f(0) = 0$, then

$$f(1) = \lim_{z\to\infty} z\,F(z). \qquad (12.4.26)$$

PROOF We have, by definition,

$$F(z)=\sum_{n=0}^{\infty} f(n)\,z^{-n}=f(0)+\frac{f(1)}{z}+\frac{f(2)}{z^2}+\cdots. \tag{12.4.27}$$

The initial value of $f(n)$ at $n=0$ is obtained from (12.4.27) by letting $z\to\infty$, and hence

$$f(0)=\lim_{z\to\infty} F(z).$$

If $f(0)=0$, then (12.4.27) gives

$$f(1)=\lim_{z\to\infty} zF(z).$$

This proves the theorem. ▮

THEOREM 12.4.7
(*Final Value Theorem*). If $Z\{f(n)\}=F(z)$, then

$$\lim_{n\to\infty} f(n)=\lim_{z\to 1}\{(z-1)F(z)\}. \tag{12.4.28}$$

provided the limits exist.

PROOF We have, from (12.3.3) and (12.4.5),

$$Z\{f(n+1)-f(n)\}=z\{F(z)-f(0)\}-F(z).$$

Or, equivalently,

$$\sum_{n=0}^{\infty}[f(n+1)-f(n)]\,z^{-n}=(z-1)\,F(z)-zf(0).$$

In the limit as $z\to 1$, we obtain

$$\lim_{z\to 1}\sum_{n=0}^{\infty}[f(n+1)-f(n)]\,z^{-n}=\lim_{z\to 1}(z-1)\,F(z)-f(0).$$

Or,

$$\lim_{n\to\infty}\,[f(n+1)-f(0)]=f(\infty)-f(0)=\lim_{z\to 1}(z-1)F(z)-f(0)$$

Thus,

$$\lim_{n\to\infty} f(n)=\lim_{z\to 1}(z-1)F(z),$$

provided the limits exist.

This proves the theorem. The reader is referred to Zadeh and Desoer (1963) for a rigorous proof. ▮

Example 12.4.1
Verify the initial value theorem for the function

$$F(z)=\frac{z}{(z-a)(z-b)}.$$

We have

$$f(0)=\lim_{z\to\infty}\frac{z}{(z-a)(z-b)}=0, \qquad f(1)=\lim_{z\to\infty} z\,F(z)=1.$$

□

THEOREM 12.4.8
(*The Z Transform of Partial Derivatives*).

$$Z\left\{\frac{\partial}{\partial a}f(n,a)\right\}=\frac{\partial}{\partial a}[Z\{f(n,a)\}]. \tag{12.4.29}$$

PROOF

$$\begin{aligned}Z\left\{\frac{\partial}{\partial a}f(n,a)\right\}&=\sum_{n=0}^{\infty}\left[\frac{\partial}{\partial a}f(n,a)\right]z^{-n}\\&=\frac{\partial}{\partial a}\left[\sum_{n=0}^{\infty}f(n,a)z^{-n}\right]=\frac{\partial}{\partial a}[Z\{f(n,a)\}].\end{aligned}$$

As an example of this result, we show

$$Z\{n\,e^{an}\}=Z\left\{\frac{\partial}{\partial a}\,e^{na}\right\}=\frac{\partial}{\partial a}Z\{e^{na}\}=\frac{\partial}{\partial a}\left(\frac{z}{z-e^{a}}\right)=\frac{z\,e^{a}}{(z-e^{a})^{2}}.$$

▮

12.5 The Inverse *Z* Transform and Examples

The inverse Z transform is given by the complex integral (12.3.4), which can be evaluated by using the Cauchy residue theorem. However, we discuss other simple ways of finding the inverse transform of a given $F(z)$. These include a method from the definition (12.3.3), which leads to the expansion of $F(z)$ as a series of inverse powers of z in the form

$$F(z)=f(0)+f(1)\,z^{-1}+f(2)\,z^{-2}+\cdots+f(n)z^{-n}+\cdots. \tag{12.5.1}$$