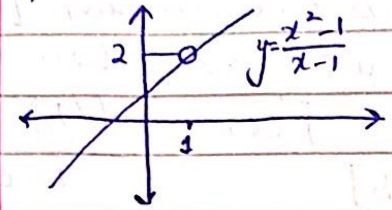
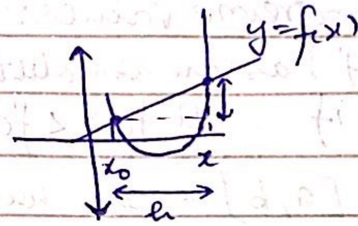
Limits & continuity at $x=c$

- $f(x)$ has a limit L if as x approaches c, f approaches L.



$$\boxed{\lim_{x\to c} f(x) = L}$$ → we find limit when $f(c)$ does not exist.

- gradient of secant line
$$m = \frac{f(x_0+h)-f(x_0)}{h}$$



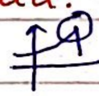
- gradient of tangent at $x=x_0$
$$m = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$ } known as the derivate of f at $x=x_0$.

for a derivative to exist at $x=c$

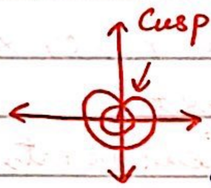
1. Right derivative = Left derivative. ↳ as $x\to c^-$ & $x\to c^+$
2. $f(x)$ is continuous at $x=c$

Differentiability is NOT valid:

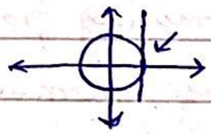
1. At vertex



2. At cusps



3. At vertical tangents



Differentiation of logs (basic log rules are assumed)

- $\frac{d}{dx} \ln[f(x)] = \frac{f'(x)}{f(x)}$
- $\frac{d}{dx} \log_a [f(x)] = \frac{f'(x)}{f(x)\cdot \ln a}$.
- $\log_a 0 = N/A$.

4.5: Applied optimization

Using calculus to better utilise resources around us.

- differentiation of exponents.
- $\frac{d}{dx}(a^x) = a^x \cdot \frac{dx}{dx} \cdot \ln a$
- $\frac{d}{dx} a^{f(x)} = a^{f(x)} \cdot \frac{df(x)}{dx} \cdot \ln a$.
- $\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x) \cdot \underbrace{\boxed{\ln e}}_{=1}$

So $\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x)$

Chain rule at $x=c$

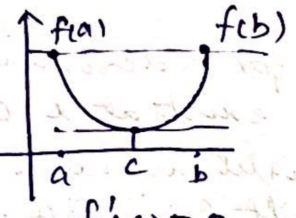
$$\frac{dy}{dt} = \left.\frac{dy}{dx}\right|_{x=c} \times \frac{dx}{dt}.$$

- Local extrema: turning pt on the curve

$f'(x) = 0 \Rightarrow$ solve for x.

- Absolute extremum: the highest/lowest of all local extremas

- Rolle's Theoram: if $f(x)$ is continuous over $[a, b]$ and $f(a) = f(b)$ then there has to be a point $x = c$ where $f'(c) = 0$.



$f'(c) = 0$.

Mean value theoram:

- if f is continous over $[a, b]$ ~~then~~ and $a < c < b$ then

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

- Extreme values: → only in close intervals. $[a, b]$

f has an absolute max at $x = c$ if all $f(x) < f(c)$ in $\forall x \in D$ OR f has an absolute min if all $f(x) > f(c)$ in $\forall x \in D$

$[a, b]$ - can have absolute max and/or min

$[a, b)$ - can only have absolute min

(a, b) - no absolute max or min.

To find Absolute Maxima, Minima

1. Find all critical points
2. Evalute fuction at those points
3. Choose largest & smallest.

- Critical point

where $f'(x)$ is either 0 or ∞.

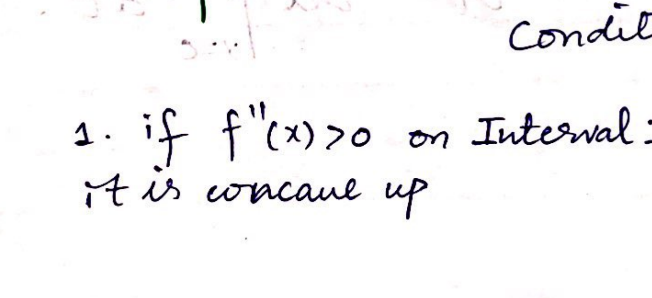
- First derivative test.

say $x = c$ is a critical point, then by moving from left-to right,

if f' chages from -ve to +ve $\Rightarrow$ Min pt

if f' chages from +ve to -ve $\Rightarrow$ Max pt.

Visualisation

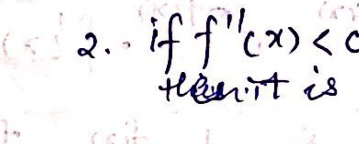


Concavity

refers to the direction of "bulge" of a curve on an interval.

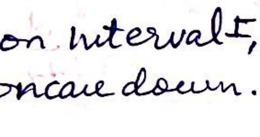
1. Concave Up

$f'(x)$ is inc



2. Concave down

$f'(x)$ is dec



★ Second derivative Test (for concavity)

Condition: let $f(x)$ be twice differentiable.

Then:

1. if $f''(x) > 0$ on Interval I, then it is concave up
2. if $f''(x) < 0$ on interval I, then it is concave down.

Indeterminat Forms.

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \pm \infty$, 1^{∞}, ∞^0, 0^0

Note that

$$\frac{f'(x)}{g'(x)} \neq \left[\frac{f(x)}{g(x)}\right]'$$

Be wary of this

L-Hôpital Rule.

if $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist, or gives any of the indeterminant forms,

try $\lim_{x \to a} \frac{f'(x)}{g'(x)}$, given that $g'(x) \neq 0$ if $x \neq a$.

1. $\frac{0}{0}$ form

Example 1.

$$\lim_{x \to 0} \frac{3x - \sin x}{x}$$

$\Downarrow D^1$ - meaning first derivative

$$\lim_{x \to 0} \frac{3 - \cos x}{1} = 3 - \lim_{x \to 0} \cos x$$

$$= 3 - 1$$

$$= 2.$$

Example 2.

$$\lim_{x \to 0} \frac{x - \sin x}{x^3}$$

$\Downarrow D^1$

$$\lim_{x \to 0} \frac{1 - \cos x}{3x^2}$$

$\Downarrow D^2$

$$\lim_{x \to 0} \frac{1 + \sin x}{6x}$$

$\Downarrow D^3$

$$\lim_{x \to 0} \frac{1 + \cos x}{6}$$

$$\frac{1}{6} + \lim_{x \to 0} \cos x.$$

$$\frac{1}{6} + 1 = \frac{7}{6}.$$

2. $\frac{\infty}{\infty}$ form

Example 1.

$$\lim_{x \to \infty} \frac{e^x}{x^2} \overset{D^1}{\Rightarrow} \frac{e^x}{2x} \overset{D^2}{\Rightarrow} \frac{e^x}{2}.$$

Now $\lim_{x \to \infty} \frac{e^x}{2} = \infty$

Example 2

$$\lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}.$$

D^1: $\lim_{x \to \pi/2} \frac{\sec x \cdot \tan x}{\sec^2 x}.$

$$\lim_{x \to \pi/2} \frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x} = 1$$

3. $\infty \cdot 0$ form

Example 1.

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x}\right)$$

let $h = \frac{1}{x}$.

as $x \to \infty, h \to 0$

So,

$$\lim_{h \to 0.} \frac{1}{h} \sin(h)$$

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

Example 2

$$\lim_{x \to 0^+} \sqrt{x} \ln x$$

$$\lim_{x \to 0^+} \frac{\ln x}{1/\sqrt{x}}$$

$$\lim_{x \to 0^+} \frac{1/x}{-\frac{1}{2}(x)^{-3/2}}$$

$$\frac{\frac{1}{x}}{\frac{-1}{2x^{3/2}}} \Rightarrow \frac{1}{\cancel{x}} \cdot -2x^{3/2 - 1}$$

$$-2\sqrt{x}.$$

$$\lim_{x \to 0^+} -2\sqrt{x} = 0$$

Two tips:

1. either make it a fraction & apply L'Hopital
2. do some algebraic manipulation & see if it becomes simpler

$\infty - \infty$ form

Example.

$$\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$$

$$\lim_{x\to 0} \frac{x - \sin x}{x\sin x}$$

$$D^1: \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} \quad (\text{still } \tfrac{0}{0})$$

$$D^2: \lim_{x\to 0} \frac{\sin x}{\cos x + \cos x - x\sin x}$$

$$\lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2-0} = \frac{0}{2} = 0.$$

1^∞ form

$$\lim_{x\to 0^+} (1+x)^{1/x}$$

let $f(x) = (1+x)^{1/x}$

$\ln f(x) = \ln(1+x)^{1/x}$.

$= \frac{1}{x}\ln(1+x)$

$$\lim_{x\to 0^+} \frac{1}{x}\ln(1+x)$$

$$\lim_{x\to 0^+} \frac{\ln(1+x)}{x}.$$

$$D^1: \lim_{x\to 0^+} \frac{\frac{1}{x+1}}{1.} \Rightarrow \frac{1}{x+1}$$

$$\boxed{\lim_{x\to 0^+} \frac{1}{x+1} = 1}$$

$\therefore \lim_{x\to 0^+} \ln f(x) = 1$

Now, $\lim_{x\to 0^+} (1+x)^{1/x} = e^{\lim_{x\to 0^+} \ln f(x)}$

So in essence,

$= e^{\lim_{x\to 0^+} \ln f(x)}$

$= e^1 = \boxed{e}$

So

$$\lim_{x\to 0^+} (1+x)^{1/x} = e$$

general note: try to form the L.Hopital form & take it from there.

∞^0 form

Life saver tip: if variables appear in exponents, you will most certainly ~~[illegible]~~ need logs to figure shit out.

$$\lim_{x\to\infty} x^{1/x}$$

let $f(x) = x^{1/x}$.

$\ln f(x) = \frac{1}{x}\ln x$.

So $\lim_{x\to\infty} \frac{1}{x}\ln x$

$$D^1: \frac{1/x}{1} = \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

So

$$\boxed{\lim_{x\to\infty} \ln f(x) = 0}$$

Now,

$$\lim_{x\to\infty} x^{1/x} = e^{\lim_{x\to\infty} \ln f(x)}$$

$$= e^{\boxed{\lim_{x\to\infty} \ln f(x)}} \quad (= 0)$$

$= e^0$

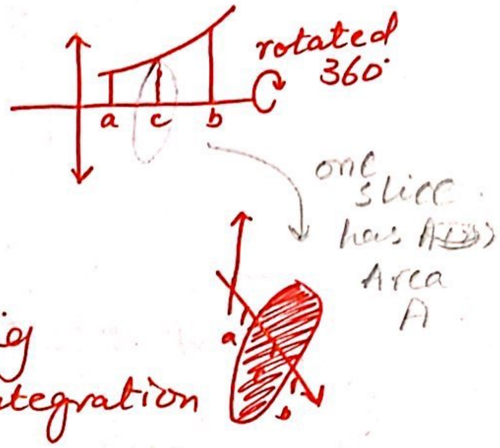
$\boxed{= 1}$

So

$$\lim_{x\to\infty} x^{1/x} = 1$$

6 Applications of definite integrals.

6.1 Volumes using cross-sections



$$V = \pi \int_a^b A(x) \cdot dx.$$

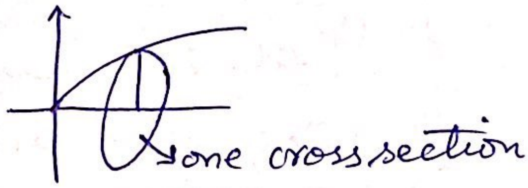
where A(x) is just the area at one point.

So if we know the formula for calculatig A at any given point, we can utilise integration

1. Disk Method (Same variable, same integrad)

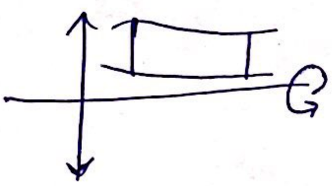
$$V = \pi \int_a^b y^2 \cdot dx$$

y is a fuction of x — These two have to be same.



2. Washer Method

Two fuctions, some gap between fuction and axis.



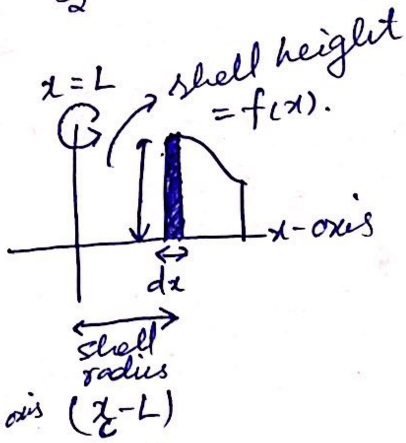
$$V = \int_a^b A(x) \cdot dx = \int_a^b \pi \left[(R(x))^2 - (r(x))^2 \right] \cdot dx.$$

3. Shell Method

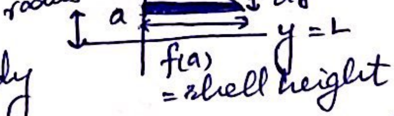
Shell formula about a vertical line, $x = L$ $\quad y = f(x) \geq 0, \; L \leq a \leq x \leq b$

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell height} \end{pmatrix} dx.$$

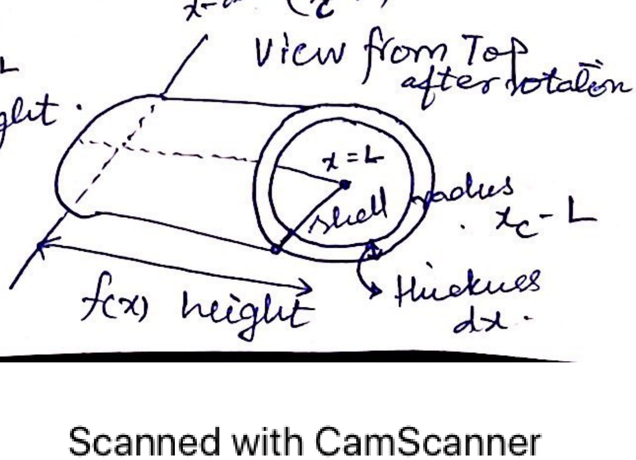
$$V = \int_a^b 2\pi (x) f(x) \, dx.$$



shell formula about a horizontal line $y = P$

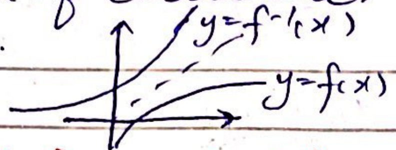


$$V = \int_a^b 2\pi (y) (f(y)) \, dy$$



7.1 Inverse Functions

- f is one-to-one if $f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$ in D
- ~~reflection of each other~~ in $y = x$.



Horizontal line test: if f is one-to-one, horizontal line must cut at most once ~~only~~.

Domain of $f(x)$ = Range of $f^{-1}(x)$
Range of $f^{-1}(x)$ = Domain of $f(x)$

$f^{-1}(b) = a$ if $\Rightarrow f(a) = b$.

Derivative rule for the inverse.

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Note to self: clarify the formula stated above in meeting

Think about domain as horizontal values and range as vertical ones.

7.2 Natural logarithms.

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0.$$

Visually speaking, the area under the curve $y = \frac{1}{t}$ starting from $t = 1$ uptill $t = x$ ~~under~~ is given by $\ln x$.

Definition of e.
e is the special number which gives area under the curve as 1.

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1.$$

Recall fundamental theorem of calculus which establishes the opposite nature of Differentiation & Integration.

Derivatives & Integrals.

- $\int \tan x\,dx = \ln|\sec x| + C$
- $\int \cot x\,dx = \ln|\csc x| + C.$
- $\int \sec x \cdot dx = \ln|\sec x + \tan x|$
- $\int \csc x\,dx = \ln|\csc x + \cot x|$

7.3 Exponential Function

$$\frac{d}{dx} e^{f(x)} = e^{f(x)} \cdot f'(x).$$

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7.4 Exponential change & separable Differential Equation

Tip: take each variable on seperate sides.

$$\frac{dy}{dt} = ky \rightarrow \boxed{y = Ae^{kt}}$$

Apply same rule to other differential equations.

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + C$$

$$|y| = e^{kt+C}$$

$$|y| = e^{kt} \cdot e^{C}$$

$$y = \pm e^{C} \cdot e^{kt}$$

Radioactivity

$$y = y_0 e^{-kt}$$

half-life

$$\cancel{y_0} e^{-kt} = \frac{1}{2}\cancel{y_0}$$

$$e^{kt} = 2$$

$$kt = \ln 2$$

$$\boxed{t = \frac{\ln 2}{k}} \text{ half life}$$

7.5 Lopital Rule. Already done

7.6 Inverse trignometric (See Below)

7.7 Hyper Bolic functions (See Below)

8.1 Basic Integration

- Make the form that is most suitable
- if $\frac{f(x)}{g(x)}$, then ensure it is proper. And then see the cases below.
- 8.2 Integration By Parts (See below)

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Further Differentiation

Trigonometric

1 $\frac{d}{dx} \sin^n(ax) = n \sin^{n-1}(ax) \cos(ax)(a)$

2. $\frac{d}{dx} \cos^n(ax) = n \cos^{n-1}(ax)(-\sin(ax))(a)$

3. $\frac{d}{dx} \tan^n(ax) = n \tan^{n-1}(ax) \sec^2(ax) \cdot (a)$

4. $\frac{d}{dx} \sec x = \sec x \cdot \tan x$. ☆ $\int \sec x \, dx = \ln(\sec x + \tan x)$

5 $\frac{d}{dx} \operatorname{cosec} x = -\cot x \cdot \operatorname{cosec} x$.

6 $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

Identities

Inverse trigonometric

All are proved by phythagorean

1 $\frac{d}{dx} \sin^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2 - x^2}}$.

2 $\frac{d}{dx} \cos^{-1}\left(\frac{x}{a}\right) = -\frac{1}{\sqrt{a^2 - x^2}}$

3. $\frac{d}{dx} \tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2 + x^2}$.

4. $\frac{d}{dx} \sec^{-1}\left(\frac{x}{a}\right) = \frac{a}{|x|\sqrt{x^2 - a^2}}$

5. $\frac{d}{dx} \operatorname{cosec}^{-1}\left(\frac{x}{a}\right) = -\frac{a}{|x|\sqrt{x^2 - a^2}}$

6 $\frac{d}{dx} \cot^{-1}\left(\frac{x}{a}\right) = \frac{-a}{a^2 + x^2}$

Extra knowledge.

$$\boxed{\cosh^2 x - \sinh^2 x \equiv 1}$$

$\Downarrow \div \cosh^2 x$

Hyperbolic

1) $1 - \tanh^2 x \equiv \operatorname{sech}^2 x$

$\div$ by $\sinh^2 x$

2) $\coth^2 x - 1 \equiv \operatorname{cosech}^2 x$

1) $\frac{d}{dx} \sinh x = \cosh x$.

2) $\frac{d}{dx} \cosh x = \sinh x$

3) $\frac{d}{dx} \tanh x = \operatorname{sech}^2 x$.

4) $\frac{d}{dx} \operatorname{sech} x = -\operatorname{sech} x . \tanh x$.

5 $\frac{d}{dx} \operatorname{csch} x = -\operatorname{csch} x . \coth x$.

6 $\frac{d}{dx} \coth x = -\operatorname{csch}^2 x$.

Inverse hyperbolic

1 $\frac{d}{dx} \sinh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{a^2+x^2}}$

2 $\frac{d}{dx} \cosh^{-1}\left(\frac{x}{a}\right) = \frac{1}{\sqrt{x^2-a^2}}$

3 $\frac{d}{dx} \tanh^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2-x^2}$.

4 $\frac{d}{dx} \operatorname{sech}^{-1}\left(\frac{x}{a}\right) = \frac{-a}{(x)\sqrt{a^2-x^2}}$

5. $\frac{d}{dx} \operatorname{csch}^{-1}\left(\frac{x}{a}\right) = \frac{-a}{x\sqrt{x^2+a^2}}$

6. $\frac{d}{dx} \coth^{-1}\left(\frac{x}{a}\right) = \frac{-a}{x^2-a^2}$.

$\Downarrow$

$\frac{\cancel{-}a}{\cancel{-}(a^2-x^2)}$

$\frac{a}{a^2-x^2}$

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April 12, 19.

Integration

(I) $f(x) \rightarrow \{\text{Polynomial}\}$

① $f(x) \rightarrow$ Linear

$$y = \int [f(x)]^n \cdot dx$$

$$= \frac{[f(x)]^{n+1}}{(n+1)(f'(x))}$$

② $f(x) \rightarrow$ Non-linear

$$y = \int [f(x)]^n \cdot f'(x) \cdot dx$$

(AA) Allowed Alternative

$$y = \frac{[f(x)]^{n+1}}{n+1}$$

$$y = \frac{1}{n+1}\int n+1 [f(x)]^n \cdot f'(x) \cdot dx$$

$$y = \frac{1}{n+1}[f(x)]^n + c$$

(II) Exponential if $p(x) \rightarrow$ linear

$$y = \int e^{P(x)} \cdot dx$$

AA

$$y = \frac{e^{P(x)}}{P'(x)}$$

OG

$$y = \int P'(x) \cdot e^{P(x)} dx$$

$$y = e^{P(x)}$$

if $p(x) \rightarrow$ non-linear

$$y = \int e^{P(x)} \cdot dx$$

N/A if $p'(x)$ not multiplied

$$y = \int e^{P(x)} \cdot p'(x)\, dx \Rightarrow e^{P(x)}$$

(III) Reciprocal

① $f(x) \rightarrow$ linear $\quad y = \int \frac{1}{[f(x)]^1} \cdot dx.$

$$y = \frac{\ln[f(x)]}{f'(x)} + c$$

② $f(x) \rightarrow$ Non-linear

$$y = \int \frac{f'(x)}{[f(x)]} \cdot dx.$$

$$y = \ln(f(x))$$

(IV) Trigonometric

① $\int \cos\theta \cdot d\theta = \sin\theta + c$

② $\int \sin\theta \cdot d\theta = -\cos\theta + c$

③ $\int \sec^2\theta \cdot d\theta = \tan\theta + c$

④ $\int \sec\theta \cdot \tan\theta \cdot d\theta = \sec\theta$

⑤ $\int \mathrm{cosec}^2\theta \cdot d\theta = -\cot\theta + c$

⑥ $\int \mathrm{cosec}\theta \cdot \cot\theta \cdot d\theta = -\mathrm{cosec}\theta + c$.

⑦ $\int \tan x \cdot dx \cdot \Rightarrow -\int \frac{-\sin x}{\cos x} \cdot dx$

$\Rightarrow -\ln(\cos x)$

$\Rightarrow \ln(\cos x)^{-1}$.

$\int \tan x \cdot dx = \boxed{+\ln(\sec x)}$

⑧ $\int \cot x \cdot dx \Rightarrow \int \frac{\cos x}{\sin x} \cdot dx$.

$\int \cot x \cdot dx = \ln(\sin x) = -\ln(\operatorname{cosec} x)$

⑨: $\int \tan 2x \cdot dx = -\frac{1}{2}\int \frac{-2(\sin 2x)}{\cos 2x}$.

$= -\frac{1}{2}\ln(\cos 2x)$

$= -\frac{1}{2}\ln(\sec 2x)$.

⑨ $\int \cancel{\sec\theta \cdot d\theta}$ $y = \sec x$.

$\sec^{-1} y = x$.

$\cancel{\sec\theta = \frac{1}{\cos\theta}}$.

$\cancel{y = \frac{1}{\cos\theta}}$.

$\cancel{\cos\theta\, y} = 1$.

April 13, 2019.

Integration Applications

Integration By Parts

↳ Applies to a product of two unrelated functions.

Proving the formula of Integration By Parts.

Proved by Product Rule

$y = xe^x$.

$\frac{dy}{dx} = e^x(1) + (x)(e^x)$

$= e^x + xe^x$

$\Downarrow$

$\int (1e^x + xe^x)\,dx = xe^x$.

$\int 1e^x + \int xe^x = xe^x$.

$\int xe^x\,dx = xe^x - \int 1e^x\,dx$.

$$\boxed{\int xe^x\,dx = xe^x - e^x.}$$

$\int_a^b u.v\,dx = \left|u\int_a^b v\,dx\right| - \left|\int_a^b \left[u'\int v\,dx\right]\right|dx.$

$y = uv$

$\frac{dy}{dx} = vu' + uv'$

$\int vu' + uv' = \int uv.$

Case ①

$\int x^n . \text{Trig} . dx$.

$\int x^n . e^{P(x)} . dx$.

where x^n will always be taken as u.

Case 2 ~~No~~ Tested.

$y. \int \ln(x)\, dx.$

cannot be integrated

$y = \int \overset{v}{1} . \overset{u}{\ln(x)}\, dx.$

$y = \ln x \int 1 . dx - \int \left(\frac{1}{x}\right)(x) . dx$

$y = \int x \ln x - x$

$\cancel{\int x \ln x - \int x = \int x}$

$I_n = \int x^n \ln x\, dx.$

$\ln x \int x^n - \int \left(\frac{1}{x}\right)\left(\frac{x^{n+1}}{n+1}\right) dx$

$\frac{1}{n+1} x^{n+1} \ln x - \frac{1}{n+1} \int x^n\, dx.$

$\frac{1}{n+1} x^{n+1} \ln x - \frac{1}{(n+1)^2} x^{n+1}$

Q

$\int x . \ln x\, dx.$

$\ln x \int x . dx \int \left(\frac{1}{x}\right)\left(\frac{1}{2} x^2\right) dx.$

$\frac{1}{2} x^2 \ln x - \int \frac{1}{2} x\, dx.$

$\frac{1}{2} x^2 \ln x - \frac{1}{4} x^2.$

Case 2

1) $\int_a^b \overset{v}{K} . \overset{u}{\ln x} . dx.$

2) $\int_a^b \overset{v}{x^n} . \overset{u}{\ln x} . dx$

$\boxed{\text{Case 3}}$

$$\boxed{y = \int e^x \cdot \cos x \, dx.}$$

If $e^x = u$.

$e^x \int \cos x \cdot dx - \int \left[e^x \int \cos x \right] dx.$

$e^x (\sin x) \cancel{+} \int \left[e^x \sin x \right]$

$e^x (\sin x) - \int e^x \sin x.$

if $u = \cos x.$

$\cos x \int e^x - \int \left[(-\sin x) \, e^x \right] dx$

$e^x \cos x + \underbrace{\int e^x \sin x \, dx}.$

This makes more I. sense as it can simplify easily.

→ consistency policy, because we took u = trig function

$$I = \sin x \cdot e^x - \underbrace{\boxed{\int e^x (\cos x) \, dx}}_{y}$$

$$y = e^x \cos x + e^x \sin x - y.$$

$$2y = e^x (\cos x + \sin x)$$

$$\boxed{y = \frac{1}{2} e^x (\cos x + \sin x) + c}$$

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Flowchart of Integration

otherwise
1) Integration by substitution
2) Integration due to trapezium Rule.

Product

$y = \int u \cdot v \cdot dx$

Related (i.e $u = v'$)

Either of the two functions is the derivative of the other

① $y = \int [f(x)]^n \cdot f'(x) \cdot dx$

$\Downarrow$

$y = \dfrac{[f(x)]^{n+1}}{n+1}$

② $y = \int e^{P(x)} \cdot p'(x) \cdot dx$

$y = e^{P(x)}$

UnRelated (Integration by Parts)

① $\int x^n e^x \cdot dx$.

$\int x^n \cdot \text{trig}$

x^n is always taken as u.

② $\int \underbrace{K}_{v} \cdot \underbrace{\ln x}_{u} \cdot dx$.

$\int x^7 \cdot \ln x \cdot dx$.

③ $\int e^{P(x)} \cdot \text{trig} \cdot dx$.

take any as u and stick with it.

Quotient

$y = \int \frac{u}{v} \cdot dx$

$u = v'$

Related

① $y = \int \dfrac{1}{[f(x)]^1} \cdot dx$

$\Downarrow$

$y = \dfrac{\ln(f(x))}{f'(x)} + c$

where $f(x)$ is linear.

② $y = \int \dfrac{f'(x)}{[f(x)]^1}$ → mandatory.

where $f(x)$ is non-linear

$= \ln(f(x)) + c$

Unrelated

$\Downarrow$

By Partial Fractions

April 20, 2019

Differential Equations

- Every derivative is a differential Equations.

Q1: $\frac{dy}{dx}$ = $(x^2-4)\frac{dy}{dx} = xy$

$$\frac{dy}{y} = \frac{x}{(x^2-4)}dx$$

$$\ln y = \tfrac{1}{2}\ln(x^2-4) + c.$$

$$\ln(1) = \tfrac{1}{2}\ln(5) + c.$$

$$c = -\tfrac{1}{2}\ln(5).$$

$$\ln y = \tfrac{1}{2}\ln(x^2-4) - \tfrac{1}{2}\ln(5)$$

$$\ln y = \ln(x^2-4)^{1/2} - \ln(5)^{1/2}.$$

$$\cancel{\ln} y = \cancel{\ln}\left(\frac{x^2-4}{5}\right)^{1/2}$$

$$y = \frac{(x^2-4)^{1/2}}{\sqrt{5}}$$

Partial Fractions (P3)

1) Case #1:- Linear Denominator

$$\frac{4x}{(x-4)(x-1)} \equiv \frac{A}{(x-4)} + \frac{B}{(x-1)}.$$

so,

$$4x \equiv A(x-1) + B(x-4).$$

2) Case #2: Repeated Denominator

$$\frac{x}{(x-1)(x-2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}.$$

$(x-2)$ is repeated as $(x-2)^1$ hides in the $(x-2)^2$ when we take LCM of the denominator.

To generalize,

$$\frac{x}{(x-1)(x-3)^n} = \frac{A}{(x-1)} + \frac{B}{(x-3)^1} + \frac{C}{(x-3)^2} + \dots \frac{D}{(x-3)^{n-1}} + \frac{E}{(x-3)^n}$$

But a third degree Binomial Term will not come in the denominator.

x^2

3) Case # 3 : Quadratic Denominator

$$\frac{2}{(x-1)(x^2-4)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2-4)}.$$

The goal is that we need to make the denominator linear. And that is only possible if the denominator is of degree n then numerator is of degree $n-1$.

For instance,

$$\frac{x}{(x-2)(x^3-3)} = \frac{A}{(x-2)} + \frac{Bx^2+Cx+D}{x^3-3}.$$

But third degree fractions are not in syllabus.

⇒ Condition # 1 : Only Proper fractions can get partialized.

In algebra, improper fraction is the one having a numerator with a degree higher than the denominator

i.e Numerator > Denominator

or = Num = Deno.

$\frac{x^3-2}{x^2+2x-3}$ ⇒ cannot be expressed in partial fractions as yet.

By long division,

$$\begin{array}{r|l} & x \\ \hline x^2+2x-3 & x^3+0x^2+0x+2 \\ & -(x^3+2x^2-3x) \\ \hline & 0 \quad -2x^2+3x+2 \end{array}$$

Now,

$$\text{quotient} + \frac{-2x^2+3x+2 \;\to \text{Remainder}}{x^2+2x-3 \;\to \text{Divisor}}.$$

$$\underbrace{x + \frac{-2x^2+3x+2}{x^2+2x-3}}$$

Now it can be expressed in partial fractions.

$$-\frac{(2x^2-3x-2)}{x^2+2x-3} \Rightarrow \frac{2x^2-4x+x-2}{x^2+3x-x-3}.$$

$$\frac{2x(x-2)+1(x-2)}{x(x+3)-1(x+3)} \Rightarrow \frac{(2x+1)(x-2)}{(x-1)(x+3)}$$

~~Some points~~: Recall Trig identities Below! $\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

$\tan^2 x + 1 = \sec^2 x$.

8.3 Trigonometric Integrals

- We have to use basic trig identity & work our way through

$$\int \sin^m x \cos^n x \cdot dx$$

② if n odd then write:

$$\cos^{2k+1} x = (\cos^2 x)^k \cos x.$$
$$= (1-\sin^2 x)^k \cos x \, dx$$

① if m is odd,
then write $\sin^m x = \sin^{2k+1} x$

$$\sin^{2k} x \cdot \sin x = (\sin^2 x)^k \sin x$$
$$= (1-\cos^2 x)^k \sin x \, dx$$

Goal is to bring to one trig ~~identity~~ func only.

As for these two, consider:

③ if m & n are even

Recall that

$$2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$= \cos 2x$$

Replace dx using this substitution.

① $\frac{d}{dx} \sin x = \cos x$

$d \sin x = \boxed{\cos x \, dx}$

② $\frac{d}{dx} \cos x = -\sin x$

$d \cos x = -\sin x \, dx$.

$\boxed{\sin x \, dx = -d \cos x}$

Replace all expressions with $\cos 2x$!

- Recall More Trig identities

$\cos 4x = 2\cos^2 2x - 1$

Say $I = \int_0^{\pi/4} \sqrt{1+\cos 4x} \, dx$.

$$I = \int_0^{\pi/4} \sqrt{\cancel{1} + 2\cos^2 2x - \cancel{1}} \cdot dx$$

By eliminating radicals, Integrations becomes easier.

$$I = \sqrt{2} \int_0^{\pi/4} \cos 2x \cdot dx$$

Recall trigonometric Compound Angle formulas.

① $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B.$

② $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.$

③ $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}.$

Double/ Half angle formulas.

① $\boxed{\sin 2x \equiv 2\sin x \cos x}$

$\sin(x+x)$

$\sin x \cos x + \cos x \sin x$

$\boxed{= 2\sin x \cos x.}$

② $\boxed{\cos 2x \equiv \cos^2 x - \sin^2 x}$ ①

$\cos^2 x - (1-\cos^2 x)$

$= 2\cos^2 x - 1$ ②

$(1-\sin^2 x) - \sin^2 x.$

$= 1 - 2\sin^2 x$ ③

$\cos 2x = (\cos x - \sin x)(\cos x + \sin x).$

$\cos(x+x)$

$= \cos x \cos x - \sin x \sin x$

$\boxed{= \cos^2 x - \sin^2 x.}$

③ $\tan(x+x) = \dfrac{\tan x + \tan x}{1 - \tan x \tan x}$

$\boxed{= \dfrac{2\tan x}{1-\tan^2 x}.}$

★ $\operatorname{cosec}(A+B) = \frac{1}{\sin(A+B)}$

★ $\sec(A+B) = \frac{1}{\cos(A+B)}$

★ $\cot(A+B) = \frac{1}{\cot(A+B)} = \frac{1-\tan^2 x}{2\tan x}$.

$\operatorname{cosec} 2x = \frac{1}{2\sin x\cos x}$.

$$\boxed{\cos n\theta = \cos^2 \frac{n}{2}\theta - \sin^2 \frac{n}{2}\theta}$$

$\sec 2x = \frac{1}{\cos 2x} = \frac{1}{\cos^2 x - \sin^2 x}$.

$\sin 2x = 2\sin x\cos x$.

$$\boxed{\sin x\cos x = \frac{\sin 2x}{2}} \text{ — } 1$$

FORMULA.

NO NEED FOR REDUCTION

$\cos 2x = 2\cos^2 x - 1$ $\Rightarrow \int \cos^2 x\,dx = \frac{1}{2}\int (\cos 2x + 1)\,dx$.

$$\boxed{\cos^2 x = \frac{\cos 2x + 1}{2}} \text{ — } 2$$

$\cos 2x = 1 - 2\sin^2 x$.

$$\boxed{\sin^2 x = \frac{1-\cos 2x}{2}} \text{ — } 3$$

$$\boxed{\sin nx = 2\sin \frac{n}{2}x \cos \frac{n}{2}x}$$

for our convenience.

$\sin 4x = 2\sin 2x\cos 2x$

$\sin 2x\cos 2x = \frac{\sin 4x}{2}$ — 1.

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8.5 Integration by partial fractions (See Below)

8.6 Reduction Formulas (See Below)

8.8 Improper integrals. → i.e $\int_{\infty}^{\infty} f(x).dx$.

Integrals with either ① infinite ~~range~~ domain or ② the area of the given integrad is infinite → i.e the integrad is problematic.

Type ① improper integrals.

An improper integral either

Converges	Diverges.
Meaning the limit approaches a finite value	The limit does ~~not app~~ not exists.

1. $\int_a^{\infty} f(x).dx = \lim_{b \to \infty} \int_a^b f(x).dx$.

2. $\int_{-\infty}^{b} f(x) = \lim_{a \to -\infty} \int_a^b f(x).dx$.

3. $\int_{-\infty}^{\infty} f(x).dx = \int_{-\infty}^{0} f(x).dx + \int_0^{\infty} f(x).dx$.

Special improper integral $\int_1^{\infty} \frac{dx}{x^p}$ (We can do some specific examples if you like)

→ when $p \leq 1$, it diverges

→ when $p > 1$, it converges.

Type ② improper integrals.

Here we have an infinite discontinuity at either of the limits, meaning there is a vertical asymptote on either side.

1. $\int_a^b f(x).dx = \lim_{c \to a^+} \int_c^b f(x).dx$ (discontinuity at left end)

2. $\int_a^b f(x).dx = \lim_{c \to b^-} \int_a^c f(x).dx$ (discontinuity at right end)

3. $\int_a^b f(x).dx = \int_a^c f(x).dx + \int_c^b f(x).dx$, where $a < c < b$ and f is discontinuous at c.

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April 16, 19.

Integration

↳ By Partial fraction

Quotient of unrelated Functions

$$\int \frac{2x+1}{x^2-5x+6}\,dx = \int \frac{2x+1}{(x-2)(x-3)}$$

when Integrated, C has to be zero

$$\int \frac{4x}{(x+4)(x^2+3)}\,dx = \frac{A}{(x+4)} + \frac{Bx+C}{x^2+3}.$$

since $(x^2+3)' = 2x$ and no constant.

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying reduction formulas like

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \tag{1}$$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \tag{2}$$

$$\int \sin^n x \cos^m x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\,dx \qquad (n \neq -m). \tag{3}$$

By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly. The next example illustrates this procedure.

EXAMPLE 4 Find

$$\int \tan^5 x \, dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x \, dx = \frac{1}{2} \tan^2 x - \int \tan x \, dx = \frac{1}{2} \tan^2 x + \ln |\cos x| + C.$$

The combined result is

$$\int \tan^5 x \, dx = \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \ln |\cos x| + C'.$$ ■

As their form suggests, reduction formulas are derived using integration by parts. (See Example 5 in Section 8.2.)

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Sometimes it's harder to find Convergence/Divergence.
Therefore we have some tests.

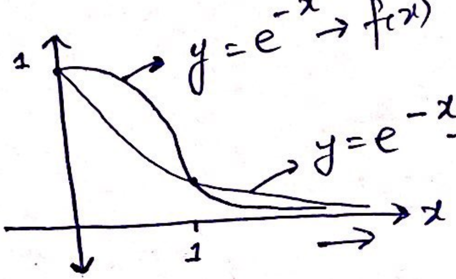
Direct Comparison test

if f & g are continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ $\forall x \ge a$.

Then

① if $\int_a^\infty g(x) \cdot dx$ ~~diverges~~ converges, then $\int_a^\infty f(x) \cdot dx$ also converges.

② if $\int_a^\infty f(x) \cdot dx$ diverges, then $\int_a^\infty g(x) \cdot dx$ also diverges.



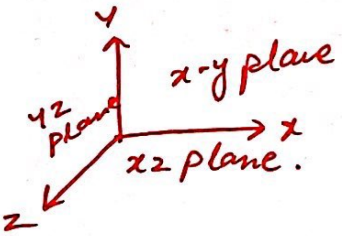
~~Since~~ we don't need to know whether $f(x)$ converges or not, as long as $g(x)$ converges.
(Direct comparison Test.

~~12.1 3D Coordinate System.~~

Limit Comparison Test.

if $\lim_{x \to \infty} \frac{f(x)}{g(x)} = L$ then $\int_a^\infty f(x) \cdot dx$ & $\int_a^\infty g(x) \cdot dx$
→ 1 either both converge
→ 2. or diverge.

12.1 3-D coordinate System.



sphere equation:

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$

12.2 Vectors.

Checklist of learning and understanding

The vector product:

- The vector product is $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\hat{\mathbf{n}} \sin \theta$.
- The result $\mathbf{a} \times \mathbf{b}$ produces a vector perpendicular to the plane that is parallel to both **a** and **b**.
- The area of the triangle OAB is given as $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.
- E The volume of a tetrahedron is given as $\frac{1}{6}|(\mathbf{a} \times \mathbf{b})\cdot\mathbf{c}|$.

Vector equation of a line:

- The vector equation of a line is $\mathbf{r} = \mathbf{a} + \mathbf{b}t$, where **a** is a position vector of a point on the line and **b** is the direction vector of the line.
- The shortest distance between a point and a line is derived from $\overrightarrow{PQ}\cdot\mathbf{b} = 0$, where P is the point and Q is a point on the line.
- The shortest distance between the skew lines $\mathbf{r}_1 = \mathbf{a}_1 + \mathbf{b}_1 s$ and $\mathbf{r}_2 = \mathbf{a}_2 + \mathbf{b}_2 t$ is given by the formula $\left|\dfrac{(\mathbf{a}_2 - \mathbf{a}_1)\cdot(\mathbf{b}_1 \times \mathbf{b}_2)}{|\mathbf{b}_1 \times \mathbf{b}_2|}\right|$.

Planes:

- The scalar equation of a plane is $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$.
- The Cartesian equation of a plane is $ax + by + cz = d$.
- The vector equation of a plane is $\mathbf{r} = \mathbf{a} + \mathbf{b}s + \mathbf{c}t$.
- For the angle between the planes $\mathbf{r}\cdot\mathbf{n}_1 = d_1$ and $\mathbf{r}\cdot\mathbf{n}_2 = d_2$, use $\cos\alpha = \dfrac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}$. The two possible angles are α and $180° - \alpha$.
- For the line of intersection of two planes, set one variable as a free variable (e.g. z) and then obtain a set of equations such as:

  $x = a + bt$
  $y = c + dt$
  $z = e + ft$

  The equation of the line of intersection is then $\mathbf{r} = a\mathbf{i} + c\mathbf{j} + e\mathbf{k} + (b\mathbf{i} + d\mathbf{j} + f\mathbf{k})t$.
- For a line meeting a plane, substitute the parametric form of the equation of the line into the equation of the plane to determine the position vector of their point of intersection.
- For the angle θ between the line $\mathbf{r} = \mathbf{a} + \mathbf{b}t$ and the plane with normal **n**, use $\cos\alpha = \dfrac{\mathbf{n}\cdot\mathbf{b}}{|\mathbf{n}||\mathbf{b}|}$.
- Depending on the size of α, use either $\theta = 90° - \alpha$ or $\theta = \alpha - 90°$.
- For the shortest distance between the point P and the plane $\mathbf{r}\cdot\mathbf{n} = d$ use $\left|\dfrac{\overrightarrow{PQ}\cdot\mathbf{n}}{|\mathbf{n}|}\right|$, where Q can be any point in the plane.