7

Numerical Solution of Ordinary Differential Equations

In the field of science and technology, a number of problems can be formulated into differential equations. Such problems in this field are reduced to the problem of solving differential equations satisfying certain given conditions. Hence, the solution of differential equations is a necessity in such studies. There are number of differential equations to get closed form solutions: Even if they possess closed form solutions we do not know the method of getting it. In such situations, we go in for numerical solutions of differential equations. In researches, the numerical solutions of the differential equations have become easy for manipulation. Hence, we present below some of the methods of numerical solutions of the ordinary differential equations. Such numerical solutions are approximate solutions.

The solution of an ordinary differential equation means finding an explicit expression for y, in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the *closed or finite form of solution.*

To describe various numerical methods for the solution of ordinary differential equations, we consider the first-order differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $y(x_0) = y_0$ to study the various numerical methods of solving such equations.

The methods of Picard and Taylor series belong to the former class of solutions. In these methods, y in

$$\frac{dy}{dx} = f(x, y)$$

is approximated by a truncated series, each term of which is a function of x. These are referred to as *single-step method*. The method of Euler, Runge–Kutta, Milne, Adams–Bashforth etc. are belong to the latter class of solutions. In these methods, the point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called *step-by-step methods*.

Euler and Runge–Kutta methods are used for computing y over a limited range of x-values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values.

7.1 TAYLOR SERIES

The Taylor series method provides a solution of the equation

$$\frac{dy}{dx} = f(x, y); \qquad y(x_0) = y_0 \tag{7.1}$$

We assume that $f(x, y)$ is sufficiently differentiable with respect to x and y. If $y(x)$ is the exact solution of Eq. (7.1), we can expand $y(x)$ in a Taylor series about the point $x = x_0$ in powers of $x - x_0$.

$$y(x) = y(x_0) + (x - x_0)\, y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0)$$

$$+ \frac{(x - x_0)^3}{3!} y'''(x_0) + \frac{(x - x_0)^4}{4!} y^{iv}(x_0) + \ldots$$

where $\qquad h = x - x_0 \;\Rightarrow\; x = h + x_0$

$\therefore$ $y(x_0 + h)$ can be expanded in the form of Taylor series as

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{h^2}{2} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \frac{h^4}{4!} y^{iv}(x_0) + \ldots$$

Since the solution is not known, the derivatives in the above expansion are not known explicitly. However, it is assumed to be differentiable and therefore, the derivatives can be obtained directly from the given differential equation itself.

$$\therefore \qquad y' = \frac{dy}{dx} = f(x, y) = f$$

$$y'' = \frac{d}{dx}(y') = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$

$$= f_x + f_y \cdot f$$

$$y'' = f_x + ff_y$$

$$y''' = \frac{d}{dx}(y'') = \frac{\partial}{\partial x}[f_x + ff_y]$$

$$= f_{xx} + 2f_{xy}f + f_xf_y + f_{yy}f^2 + f_y^2 f$$

$$y''' = f_{xx} + 2ff_{xy} + f_xf_y + f^2 f_{yy} + ff_y^2$$

$$y^{iv} = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f_y(f_{xx} + 2ff_{xy} + f^2f_{yy})$$

$$+ 3(f_x + ff_y)(f_{xy} + ff_{yy}) + f_y^2(f_x + ff_y) \text{ etc.}$$

By continuing in this manner, we can express the derivative of y in terms of $f(x, y)$ and its partial derivatives.

Remark

This method is due to determine the value of f_x and f_y and sometimes find f_{xx}, f_{xy} and f_y^2 etc. If we need better approximation then higher derivatives are needed, additional to these functions have to be evaluated at the initial points.

EXAMPLE 7.1 Find $y(1.1)$, given $y' = 2x - y$ and $y(1) = 3$.

Solution

Given

$$y' = 2x - y,\ x_0 = 1,\ y_0 = 3,\ x_1 = 1.1,\ h = 0.1$$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots \qquad \text{(i)}$$

$$y' = 2x - y \Rightarrow y_0' = 2x_0 - y_0$$
$$= 2(1) - 3$$
$$= -1$$

$$y'' = 2 - y' \Rightarrow y_0'' = 2 - y_0'$$
$$= 2 - (-1)$$
$$= 3$$

$$y''' = -y'' \Rightarrow y_0''' = -y_0''$$
$$= -3$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 3 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-3) + \dots$$

$$= 3 - 0.1 + 0.015 - 0.0005 \dots$$

$$= 2.9145$$

EXAMPLE 7.2 Given $\dfrac{dy}{dx} = 3x + \dfrac{y}{2}$ and y(0) = 1. Find the values of $y(0.1)$ and $y(0.2)$, using the Taylor series method.

Solution

Given

$$y' = 3x + y/2 \text{ and } x_0 = 0,\ y_0 = 1$$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots \qquad \text{(i)}$$

To find y(0.1):

$$y' = 3x + \frac{y}{2} \quad \Rightarrow \quad y_0' = 3x_0 + \frac{y_0}{2}$$

$$= 3(0) + \frac{1}{2}$$

$$= 0.5$$

$$y'' = 3 + \frac{y'}{2} \quad \Rightarrow \quad y_0'' = 3 + \frac{y_0'}{2}$$

$$= 3 + \frac{0.5}{2}$$

$$= 3.25$$

$$y''' = \frac{y''}{2} \quad \Rightarrow \quad y_0''' = \frac{y_0''}{2}$$

$$= \frac{3.25}{2}$$

$$= 1.625$$

$$y^{iv} = \frac{y'''}{2} \quad \Rightarrow \quad y_0^{iv} = \frac{y_0'''}{2}$$

$$= \frac{1.625}{2}$$

$$= 0.8125$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.1)(0.5) + \frac{(0.1)^2}{2}(3.25) + \frac{(0.1)^3}{6}(1.625) + \frac{(0.1)^4}{24}(0.8125) + \ldots$$

$$= 1.0665$$

$$y_1 = y(0.1)$$

$$= 1.0665.$$

To find y(0.2):

Taylors series formula for $y(0.2)$ is

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \ldots \tag{ii}$$

$$y' = 3x + \frac{y}{2} \quad \Rightarrow \quad y_1' = 3x_1 + \frac{y_1}{2}$$

$$= 3(0.1) + \frac{1.0665}{2}$$

$$= 0.83325$$

$$y'' = 3 + \frac{y'}{2} \quad \Rightarrow \quad y_1'' = 3 + \frac{y_1'}{2}$$

$$= 3 + \frac{0.83325}{2}$$

$$= 3.416625$$

$$y''' = \frac{y''}{2} \quad \Rightarrow \quad y_1''' = \frac{y_1''}{2}$$

$$= \frac{3.416625}{2}$$

$$= 1.7083125$$

$$y^{iv} = \frac{y'''}{2} \quad \Rightarrow \quad y_1^{iv} = \frac{y_1'''}{2}$$

$$= \frac{1.7083125}{2}$$

$$= 0.85415625$$

Equation (ii) $\Rightarrow$

$$y_2 = 1.0665 + (0.1)(0.83325) + \frac{(0.1)^2}{2}(3.416625)$$

$$+ \frac{(0.1)^3}{6}(1.7083125) + \frac{(0.1)^4}{24}(0.85415625) + \ldots$$

$$= 1.167196.$$

$$y_2 = y(0.2)$$

$$= 1.167196.$$

EXAMPLE 7.3 Obtain $y(4.2)$ and $y(4.4)$, given

$$\frac{dy}{dx} = \frac{1}{x^2 + y}, \; y(4) = 4, \text{ taking } h = 0.2$$

Solution

Given $$y' = \frac{1}{x^2 + y},\ x_0 = 4,\ y_0 = 4,\ x_1 = 4.2,\ x_2 = 4.4,\ h = 0.2.$$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \text{(i)}$$

To find y(4.2):

$$y' = \frac{1}{x^2 + y} \Rightarrow y_0' = \frac{1}{x^2 + y_0}$$

$$= \frac{1}{4^2 + 4}$$

$$= 0.05$$

$$y'' = -1\,(x^2 + y)^{-2}\, y' \Rightarrow y_0'' = \frac{-y_0'}{(x_0^2 + y_0)^2}$$

$$= \frac{-0.05}{(20)^2}$$

$$= -0.000125$$

$$y''' = 2(x^2 + y)^{-3}\,(y')^2 + (-1)\,(x^2 + y)^{-2}\, y''$$

$$y_0''' = \frac{2(y_0')^2}{(x_0^2 + y_0)^3} - \frac{y_0''}{(x_0^2 + y_0)^2}$$

$$= \frac{2(0.05)^2}{(20)^3} + \frac{0.000125}{(20)^2}$$

$$= 0.0000009375.$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 4 + (0.2)\,(0.05) + \frac{(0.2)^2}{2}(-0.000125) + \frac{(0.2)^3}{6}(0.0000009375) + \ldots$$

$$= 4 + 0.01 - 0.0000025 + 0.00000000125$$

$$= 4.0099975$$

$$y_1 = y(4.2)$$

$$= 4.009998$$

To find y(4.4):

Taylor series is given by

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \quad \text{(ii)}$$

$$y' = \frac{1}{x^2 + y} \quad \Rightarrow \quad y_1' = \frac{1}{x_1^2 + y_1}$$

$$= \frac{1}{(4.2)^2 + 4.009998}$$

$$= 0.046189$$

$$y_1'' = \frac{-y_1'}{(x_1^2 + y_1)^2} = \frac{-0.046189}{[(4.2)^2 + 4.009998]^2}$$

$$= -0.000098542$$

$$y_1''' = \frac{2(y_1')^2}{(x_1^2 + y_1)^3} - \frac{y_1''}{(x_1^2 + y_1)^2}$$

$$= \frac{2(0.046189)^2}{[(4.2)^2 + 4.009998]^3} + \frac{0.000098542}{[(4.2)^2 + 4.009998]^2}$$

$$= 0.000000420469 + 0.00000021024$$

$$= 0.000000630704$$

$\therefore$ Equation (ii) $\Rightarrow$

$$y_2 = 4.009998 + (0.2)(0.046189) + \frac{(0.2)^2}{2}(-0.000098542)$$

$$+ \frac{(0.2)^3}{6}(0.000000630704)$$

$$= 4.009998 + 0.0092378 - 0.00000197084 + 0.00000000094094$$

$$= 4.019234$$

EXAMPLE 7.4 Find $y(0.1)$, given $y' = x^2y - 1$, $y(0) = 1$.

Solution

Given $y' = x^2y - 1$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = 0.1$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots \tag{i}$$

$$y' = x^2y - 1 \quad \Rightarrow \quad y_0' = x_0^2\, y_0 - 1$$

$$= (0)(1) - 1$$

$$= -1$$

$$y'' = x^2y' + 2xy \quad \Rightarrow \quad y_0'' = x_0^2\, y_0' + 2x_0y_0$$

$$= 0$$

$$y''' = x^2(y')^2 + 2xy' + 2xy' + 2y$$

$$y_0''' = x_0^2 (y_0')^2 + 2x_0\, y_0' + 2x_0 y_0' + 2y_0$$

$$= 2$$

∴ Equation (i) ⇒

$$y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{6}(2) + \ldots$$

$$= 1 - 0.1 + 0.0003333 + \ldots$$

$$= 0.900333$$

$$y_1 = y(0.1)$$

$$= 0.900333$$

7.1.1 Taylor Series Method for Simultaneous First-order Differential Equations

The simultaneous first-order differential equations of the form

$$\frac{dy}{dx} = f_1(x, y, z)$$

and

$$\frac{dz}{dx} = f_2(x, y, z)$$

with initial values $y(x_0) = y_0$ and $z(x_0) = z_0$

To solve this system of equations at an interval h, the increments in y and z are obtained by using the formulae

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

and

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots$$

EXAMPLE 7.5 Solve the differential equations using the Taylor series

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy, \quad \text{for } x = 0.3$$

given that $x = 0$, $y = 0$, $z = 1$.

Solution

$$\frac{dy}{dx} = 1 + xz$$

$$\frac{dz}{dx} = -xy$$

$$x = 0,\ y = 0,\ z = 1,\ h = 0.3$$

Taylors series for y' is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \text{(i)}$$

$$y' = 1 + xz \Rightarrow y_0' = 1 + x_0 z_0$$
$$= 1 + (0)(1)$$
$$= 1$$

$$y'' = xz' + z \Rightarrow y_0'' = x_0 z_0' + z_0$$
$$= x_0(-x_0 y_0) + z_0$$
$$= 0 + 1$$
$$= 1$$

$$y''' = x(z')^2 + z' + z' \Rightarrow y_0''' = x_0 (z_0')^2 + 2z_0'$$
$$= x_0(-x_0 y_0)^2 + 2(x_0 y_0)$$
$$= 0$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 0 + (0.3)(1) + \frac{(0.3)^2}{2}(1) + \frac{(0.3)^3}{6}(0) + \ldots$$
$$= 0.3 + 0.045$$
$$= 0.345$$

$$y_1 = y(0.3)$$
$$= 0.345$$

Taylor series for z' is given by

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots \quad \text{(ii)}$$

$$z' = -xy \Rightarrow z_0' = -x_0 y_0 = 0$$

$$z'' = -(xy' + y) \Rightarrow z_0'' = -x_0 y_0' + y_0$$
$$= -(0)(1) + 0$$
$$= 0$$

$$z''' = -[x y'' + y' + y'] \Rightarrow z_0''' = -x_0 y_0'' + 2y_0'$$
$$= -(0)(1) + 2(1)$$
$$= 2.$$

$\therefore$ Equation (ii) $\Rightarrow$

$$z_1 = 1 + (0.3)(0) + \frac{(0.3)^2}{2}(0) + \frac{(0.3)^3}{6}(2) + \ldots$$

$$= 1 + 0.009$$
$$= 1.009$$
$$z_1 = z(0.3)$$
$$= 1.009$$

EXAMPLE 7.6 Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$, given

$$\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2, z(0) = 1$$

Solution

Given

$$y' = x + z, z' = x - y^2, x_0 = 0, y_0 = 2, z_0 = 1, h = 0.1$$

To find y(0.1):

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \text{(i)}$$

$$y' = x + z, \qquad z' = x - y^2$$
$$y'' = 1 + z', \qquad z'' = 1 - 2yy'$$
$$y''' = z''$$
$$z''' = -[2yy'' + 2(y')^2]$$

$$\therefore \quad y' = x + z \Rightarrow y_0' = x_0 + z_0$$
$$= 0 + 1$$
$$= 1$$
$$y'' = 1 + z' = 1 + x - y^2 \Rightarrow y_0'' = 1 + x_0 - y_0^2$$
$$= 1 - 2^2$$
$$= -3$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 2 + (0.1)(1) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(-3) + \ldots$$
$$= 2 + 0.1 - 0.015 - 0.0005 \ldots$$
$$= 2.0845$$

To find z(0.1):

Taylor series is given by

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots \quad \text{(ii)}$$

$$z_0' = x_0 - y_0^2$$
$$= 0 - 2^2$$
$$= -9$$

$$z_0'' = 1 - 2y_0\, y_0'$$
$$= 1 - 2(2)\,(1)$$
$$= -3$$

$$z_0''' = -[2y_0 y_0'' + 2(y_0')^2]$$
$$= -[2(2)\,(-3) + 2(1)^2]$$
$$= -[-12 + 2]$$
$$= 10$$

$\therefore$ Equation (ii) $\Rightarrow$

$$z_1 = 1 + (0.1)\,(-4) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(10) + \ldots$$
$$= 1 - 0.4 + (-0.015) + 0.0016667$$
$$= 1 - 0.4 - 0.015 + 0.001667$$
$$= 0.586667$$

To find y(0.2):
Taylor series is given by

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots \qquad \text{(iii)}$$

$$y_1' = x_1 + z_1$$
$$= 0.1 + 0.586667$$
$$= 0.686667$$
$$y_1'' = 1 + z_1'$$
$$= 1 + x_1 - y_1^2$$
$$= 1 + 0.1 - 2.0845^2$$
$$= -3.24514$$
$$y_1''' = z_1''$$
$$= 1 - 2y_1 y_1'$$
$$= 1 - 2(2.0845)\,(0.686667)$$
$$= -1.862415$$

$\therefore$ Equation (iii) $\Rightarrow$

$$y_2 = 2.0845 + (0.1)\,(0.686667) + \frac{(0.1)^2}{2}(-3.24514)$$
$$+ \frac{(0.1)^3}{6}(-1.862715) + \ldots$$

$$= 2.0845 + 0.0686667 - 0.01623 - 0.00031045$$

$$= 2.13663$$

To find z(0.2):

Taylor series is given by

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \ldots$$

$$z_1' = x_1 - y_1^2$$
$$= 0.1 - 2.0845^2$$
$$= -4.24514$$

$$z_1'' = 1 - 2y_1y_1'$$
$$= 1 - 2(2.0845)(0.686667)$$
$$= -1.862715$$

$$z_1''' = -[2y_1 y_1'' + 2(y_1')^2]$$
$$= -[2(2.0845)(-3.24514) + 2(0.686664)^2]$$
$$= -[-13.52899 + 0.943023]$$
$$= 12.585967$$

$\therefore$ Equation $\Rightarrow$

$$z_2 = (0.1) + (0.1)(-4.24514) + \frac{(0.1)^2}{2}(-1.862715)$$
$$+ \frac{(0.1)^3}{6}(12.585967) + \ldots$$
$$= 0.1 - 0.424514 - 0.009313575 + 0.00209766$$
$$= -0.33173$$

7.1.2 Taylor Series Method for Second-order Differential Equations

The differential equation of the second-order can be solved by reducing it to a lower-order differential equation. A second-order differential equation can be reduced to a first-order differential equation by transformation $y' = z$.

Suppose $$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

i.e., $$y'' = f(x, y, y') \tag{7.2}$$

is the differential equation together with initial conditions.

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_0' \tag{7.3}$$

where y_0, y_0' are known values.

Setting $p = y'$, we get $y'' = p'$

The equation (7.2) becomes

$$p' = f(x, y, p) \text{ with initial conditions}$$

$$y(x_0) = y_0 \tag{7.4}$$

$$y'(x_0) = y_0' \tag{7.5}$$

where y_0, y_0' are known values.

By putting $y' = p$, $y'' = p'$, Eq. (7.2) becomes

$$p' = f(x, y, p)$$

with initial conditions

$$y(x_0) = y_0$$

and $$y'(x_0) = y_0' \Rightarrow p(x_0) = p_0$$

Solving p' by using Eqs. (7.4) and (7.5), we get

$$p_1 = p_0 + hp_0' + \frac{h^2}{2!} p_0'' + \frac{h^3}{3!} p_0''' + \ldots \tag{7.6}$$

where $h = x_1 - x_0$

Since $p = y'$, we get Eq. (7.6) as

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

Similarly, proceeding in similar manner, we get

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

$\therefore$ We calculate $y_1, y_2 \ldots$

EXAMPLE 7.7 Solve $y'' = y + xy'$, given $y(0) = 1$, $y'(0) = 0$ and calculate $y(0.1)$.

Solution

$$x_0 = 0$$

$$y_0 = 1$$

$$y_0' = 0$$

$$y'' = y + xy'$$

$$\Rightarrow \quad y_0'' = y_0 + x_0 y_0'$$

$$= 1 + (0)(0)$$

$$= 1$$

Differentiating with respect to x

$$y''' = y' + y' + xy'' = 2y' + xy''$$

$$\Rightarrow \quad y_0''' = 2y_0' + x_0 y_0''$$

$$= 2(0) + 0(1) = 0$$

$$y^{iv} = 2y'' + y'' + xy'''$$

$$= 3y'' + xy'''$$

$$\Rightarrow \quad y^{iv} = 3y_0'' + x_0 y_0'''$$

$$= 3(1) + (0)(0)$$

$$= 3$$

$$y^{v} = 4y''' + xy^{iv}$$

$$y^{v} = 4y_0''' + x_0 y_0^{iv}$$

$$= 4(0) + 0(3)$$

$$= 0$$

We know that Taylor series is given by

$$y(x) = y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots$$

$$= 1 + (0.1)(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(0) + \frac{(0.1)^4}{24}(3) + \ldots$$

$$= 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24}(3) + \ldots$$

$$= 1.0050125$$

EXAMPLE 7.8 Find $y(0.2)$, given $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution

$$y'' = -y, x_0 = 0, y_0 = 1, y_0' = 0, h = 0.2.$$

To find y(0.2)

We know that Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \tag{i}$$

$$y'' = -y \quad \Rightarrow \quad y_0'' = -y_0 = -1$$

$$y''' = -y' \quad \Rightarrow \quad y_0''' = -y_0' = 0$$

$$y^{iv} = -y'' \quad \Rightarrow \quad y_0^{iv} = -y_0'' = -(-1) = 1$$

$$y^{v} = -y''' \quad \Rightarrow \quad y_0^{v} = -y_0''' = 0$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.2)(0) + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^3}{6}(6) + \frac{(0.2)^4}{24}(1) + \ldots$$

$$= 1 + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^4}{24}(1) + \ldots$$

$= 1 - 0.02 + 0.00006667$

$= 0.9800667$

EXERCISES

7.1 Solve $\frac{dy}{dx} = x + y$, given $y(1) = 0$ and get $y(1.1)$, $y(1.2)$ by Taylor series method.

[*Ans.* $y(1.1) = 0.110342$, $y(1.2) = 0.24281$]

7.2 Using the Taylor series method, find correct to four decimal places, the value of $y(0.1)$, given

$$\frac{dy}{dx} = x^2 + y^2 \text{ and } y(0) = 1.$$

[*Ans.* $y(0.1) = 1.11145$]

7.3 Using the Taylor method, compute $y(0.2)$ and $y(0.4)$ correct to four decimal places, given

$$\frac{dy}{dx} = 1 - 2xy \text{ and } y(0) = 0.$$

[*Ans.* $y(0.2) = 0.19475$, $y(0.4) = 0.359884$]

7.4 Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ and $y(0) = 1$. Find the values of $y(0.1)$ and $y(0.2)$, using the Taylor series method.

[*Ans.* $y(0.1) = 1.0665$, $y(0.2) = 1.167196$]

7.5 Solve by the Taylor series method of third-order the problem

$$\frac{dy}{dx} = (x^3 + xy^2)e^{-x},\ y(0) = 1$$

to find y, for $x = 0.1, 0.2, 0.3$.

[*Ans.* $y(0.1) = 1.0047$, $y(0.2) = 1.01812$, $y(0.3) = 1.03995$]

7.6 Solve by the Taylor series method (of fourth-order)

$$\frac{dy}{dx} = xy^2 + 1, \quad y(0) = 1 \text{ at } x = 0.2, 0.4.$$

[*Ans.* $y(0.2) = 1.226$, $y(0.4) = 1.54205$]

7.7 Using the Taylor series method, solve

$$\frac{dy}{dx} = x^2 - y,\ y(0) = 1 \text{ at } x = 0.1, 0.2, 0.3 \text{ and } 0.4.$$

[*Ans.* $y(0.1) = 0.9052$, $y(0.2) = 0.8213$,
$y(0.3) = 0.7492$, $y(0.4) = 0.6897$]

7.8 Find $y(0.1)$, given $\frac{dy}{dx} = x + y$, $y(0) = 1$.

[*Ans.* $y(0.1) = 1.1103$]

7.9 Find $y(0.1)$, $y(0.2)$, $y(0.3)$, given

$$y' = \frac{x^3 + xy^2}{e^x},\ y(0) = 1.$$

[*Ans.* $y(0.1) = 1.0047$, $y(0.2) = 1.01812$,
$y(0.3) = 1.03995$]

7.10 Solve $\frac{dy}{dx} = y + x^3$, for $x = 1.1, 1.2, 1.3$, given $y(1) = 1$.

[*Ans.* $y(1.1) = 1.225$, $y(1.2) = 1.512$,
$y(1.3) = 1.874$]

7.11 Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$, given

$$\frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2,\ z(0) = 1.$$

[*Ans.* 2.0845, 2.1367, 0.5867, 0.1550]

7.12 Evaluate $x(0.1)$, $y(0.1)$, $x(0.2)$, $y(0.2)$, given

$$\frac{dx}{dt} = t_y + 1,\ \frac{dy}{dt} = -t_x$$

given $x = 0$, $y = 1$ at $t = 0$.

[*Ans.* $x(0.1) = 0.105$, $y(0.1) = 0.9987$,
$x(0.2) = 0.21998$, $y(0.2) = 0.9972$]

7.13 Find $y(0.3)$, $z(0.3)$, given

$$\frac{dz}{dx} = -xy,\ \frac{dy}{dx} = 1 + xz$$

where $y(0) = 0$, $z(0) = 1$.

[*Ans.* $y(0.3) = 0.3448$, $z(0.3) = 0.991$]

7.14 Solve for x and y

$$\frac{dx}{dt} = x + y + t,\quad \frac{dy}{dt} = 2x - t$$

given $x = 0$, $y = 1$ at $t = 1$.

$$\left[\textit{Ans. } x = 2t + t^2 + \frac{5}{6}t^3 + \cdots \right.$$
$$\left. -y = 1 - t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \cdots \right]$$

7.15 Solve numerically, using the Taylor series method find approximate values of y and z corresponding to $x = 0.1, 0.2$ given that

$$y(0) = 2,\ \ z(0) = 1 \text{ and } \frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2$$

[*Ans.* $y(0.1) = 2.0845$, $z(0.1) = 0.5867$
$y(0.2) = 2.1367$, $z(0.2) = 0.15497$]

7.16 Find the value of $y(1.1)$ and $y(1.2)$ from $\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} = x^3$, $y(1) = 1$, $y'(1) = 1$ by using the Taylor series method.

[*Ans.* $y(1.1) = 1.1002$, $y(1.2) = 1.2015$]

7.17 Given $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ with $y(0) = 1, y'(0) = 0$, obtain the values of $y(0.1)$ and $y(0.2)$, correct to 3 decimal places, using the Taylor series method.

[*Ans.* $y(0.1) = 0.995$, $y(0.2) = 0.981$]

7.18 Using the Taylor series method, find $y(0.1)$, $y(0.2)$, given $y'' + xy = 0$ and $y(0) = 1$, $y'(0) = 0.5$.

[*Ans.* $y(0.1) = 1.0498$, $y(0.2) = 1.0986$]

7.2 PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS

Various methods have been formulated for getting to any desired degree of accuracy the numerical solution of the above-mentioned type of differential equation with numerical coefficients and given conditions. In this chapter, we discuss Picard's method for finding an approximate solution of the initial value problem of the form

$$\frac{dy}{dx} = f(x, y),\quad y(x_0) = y_0$$

The condition $y(x_0) = y_0$ is called the *initial condition*. Sometimes $y(x_0) = y_0$ is also expressed by saying that $y = y_0$, when $x = x_0$.

Picard's method is also known as the *method of successive approximation*.

To solve the equation $\frac{dy}{dx} = f(x, y)$ subject to $y(x_0) = y_0$ (7.7)

Integrating Eq. (7.7) w.r.t x between the limits x_0 to x.

$$\int_{x_0}^{x} \frac{dy}{dx}\,dx = \int_{x_0}^{x} f(x, y)\,dx$$

$$[y]_{x_0}^{x} = \int_{x_0}^{x} f(x, y)\, dx$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(x, y)\, dx$$

or

$$y(x) = y(x_0) + \int_{x_0}^{x} f(x, y)\, dx \tag{7.8}$$

The differential equation is transferred to an integral equation. We observe that Eq. (7.7) is an integral equation and can be solved by a process of successive approximations.

For a first approximation, we replace y in the integral of Eq. (7.8) by y_0 so that the integral is a function of x only and hence the integration becomes

$$y_1^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For second approximation to y_1, as

$$y_1^{(2)} = y_0 + \int_{x_0}^{x} f(x_1, y_1^{(1)})\, dx$$

For third approximation, we have

$$y_1^{(3)} = y_0 + \int_{x_0}^{x} f(x_1, y_1^{(2)})\, dx \text{ and so on.}$$

The process is continued till $\left|y^{n+1} - y^{n}\right|$ is less than or equal to the specified degree of accuracy.

In general, we have

$$y_1^{(n)} = y_0 + \int_{x_0}^{x} f(x_1, y_1^{n-1})\, dx$$

The process is continued for next points $y_2 \ldots$.

EXAMPLE 7.9 Solve $\dfrac{dy}{dx} = x + y$, given $y(0) = 1$. Obtain the values of $y(0.1)$, $y(0.2)$, using Picard's method.

Solution

Here $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$

The Picard's algorithm is

$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

$$y = 1 + \int_{0}^{x} f(x, y)\, dx \qquad \text{(i)}$$

Put $y = y_0$, we get

$$y^{(1)} = 1 + \int_{0}^{x} f(x, 1)\, dx$$

$$= 1 + \int_{0}^{x} (x + 1)\, dx$$

$$= 1 + x + \frac{x^2}{2} \qquad \text{(ii)}$$

Again using $y = y^{(1)}$ on R.H.S. of Eq. (i), we get

$$y^{(2)} = 1 + \int_{0}^{x} \left(x + 1 + x + \frac{x^2}{2} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{6} \qquad \text{(iii)}$$

$$y^{(3)} = 1 + \int_{0}^{x} \left(x + 1 + x + x^2 + \frac{x^3}{6} \right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

$$y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + \ldots$$

Setting $x = 0.1$, we get

$$y(0.1) = 1 + 0.1 + 0.01 + \frac{1}{3}(0.001) + \frac{1}{24}(0.0001)$$

$$= 1 + 0.1 + 0.01 + 0.0003333 + 0.0000041$$

$$= 1.1103374$$

$$y(0.2) = 1 + 0.2 + (0.2)^2 + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{24}$$

$$= 1.242733$$

EXAMPLE 7.10 Solve $y' + y = e^x$, $y(0) = 0$, by Picard's method.

Solution

We know that
$$y = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

$$= 0 + \int_0^x (e^x - y)\, dx$$

Here
$$x_0 = 0,\ y_0 = 0$$

$$y^{(1)} = \int_0^x (e^x - 0)\, dx$$

$$= e^x - 1$$

$$y^{(2)} = \int_0^x (e^x - e^x + 1)\, dx$$

$$= x$$

$$y^{(3)} = \int_0^x (e^x - x)\, dx$$

$$= e^x - \frac{x^2}{2} - 1$$

$$y^4 = \int_0^x \left[e^x - \left(e^x - \frac{x^2}{2} - 1 \right) \right] dx$$

$$= \frac{x^3}{6} + x$$

$$y^5 = \int_0^x \left(e^x - x - \frac{x^3}{6} \right) dx$$

$$= e^x - \frac{x^2}{2} - \frac{x^4}{24} - 1$$

EXAMPLE 7.11 Apply Picard's method to solve the following initial value problem upto third approximation.

$$\frac{dy}{dx} = 2y - 2x^2 - 3, \text{ given that } y = 2, \text{ when } x = 0.$$

Solution

Given

$$y' = 2y - 2x^2 - 3 \quad \text{(i)}$$

where

$$y = 2,\ x = 0$$

We know that Picard's method for finding an approximate solution of the initial value problem of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{(ii)}$$

where

$$y = y_0, \text{ when } x = x_0$$

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \quad \text{(iii)}$$

From Eqs. (i) and (ii), we get

$$f(x, y) = 2y - 2x^2 - 3,\ x_0 = 0,\ y_0 = 2$$

Equation (iii) ⇒

$$y_n = 2 + \int_0^x (2y_{n-1} - 2x^2 - 3)\, dx \quad \text{(iv)}$$

When $n = 1$ in Eq. (iv), we have

$$y_1 = 2 + \int_0^x (2y_0 - 2x^2 - 3)\, dx$$

$$= 2 + \int_0^x (4 - 2x^2 - 3)\, dx$$

$$= 2 + \int_0^x (1 - 2x^2)\, dx$$

$$= 2 + \left[x - \frac{2x^3}{3} \right]_0^x$$

$$= 2 + x - \frac{2x^3}{3}$$

When $n = 2$,

$$y_2 = 2 + \int_0^x (2y_1 - 2x^2 - 3)\, dx$$

$$= 2 + \int_0^x \left[2\left(2 + x - \frac{2x^3}{3}\right) - 2x^2 - 3 \right] dx$$

$$= 2 + \int_0^x \left(1 + 2x - 2x^2 - \frac{4x^3}{3}\right) dx$$

$$= 2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3}$$

When $n = 3$,

$$y_3 = 2 + \int_0^x (2y_2 - 2x^2 - 3)\, dx$$

$$= 2 + \int_0^x \left[2\left(2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3}\right) - 2x^2 - 3 \right] dx$$

$$= 2 + \int_0^x \left[1 + 2x - \frac{4x^3}{3} - \frac{2x^4}{3}\right] dx$$

$$= 2 + x + x^2 - \frac{x^4}{3} - \frac{2x^5}{15}$$

EXAMPLE 7.12 Find the third approximation of the solution of the equation

$$\frac{dy}{dx} = 2 - (y/x)$$

by Picard's method, where $y = 2$, when $x = 1$.

Solution

Given $$\frac{dy}{dx} = 2 - \left(\frac{y}{x}\right) \qquad \text{(i)}$$

where $y = 2$, when $x = 1$.

wkt $$\frac{dy}{dx} = f(x, y) \qquad \text{(ii)}$$

where $y = y_0$, when $x = x_0$ is givn by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \qquad \text{(iii)}$$

From Eqs. (i) and (ii), we have

$$f(x, y) = 2 - (y/x),\ x_0 = 1,\ y_0 = 2$$

Equation (iii) $\Rightarrow$

$$y_n = 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right) y_{n-1}\right] dx$$

when $n = 1$

$$y_1 = 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right) y_0\right] dx$$

$$= 2 + \int_1^x \left[2 - \frac{2}{x}\right] dx$$

$$= 2 + \left[2x - 2\log x\right]_1^x$$

$$= 2 + 2x - 2\log x - 2$$

$$= 2x - 2\log x$$

when $n = 2$

$$y_2 = 2 + \int_1^x \left(2 - \frac{y_1}{x}\right) dx$$

$$= 2 + \int_1^x \left[2 - \frac{1}{x}(2x - 2\log x)\right] dx$$

$$= 2 + 2\int_1^x \log x \cdot \frac{1}{x}\, dx$$

$$= 2 + \left[(\log x)^2\right]_1^x = 2 + (\log x)^2$$

when $n = 3$

$$y_3 = 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right) y_2\, dx\right]$$

$$= 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right)\{2 + (\log x)^2\}\right] dx$$

$$= 2 + \int_1^x \left[2 - \frac{2}{x} - (\log x)^2\, \frac{1}{x}\right] dx$$

$$= 2 + \left[2x - 2 \log x - \frac{(\log x)^3}{3} \right]_1^x$$

$$= 2 + 2x - 2 \log x - \left(\frac{1}{3}\right)(\log x)^3 - 2$$

$$= 2x - 2 \log x - \left(\frac{1}{3}\right)(\log x)^3$$

EXERCISES

7.19 Apply Picard's method to the following initial value problems and find the first-three successive approximations.

(i) $\frac{dy}{dx} = 2xy,\ y(0) = 1$

$$\left[\textit{Ans. } y_1 = 1 + x^2,\ y_2 = 1 + x^2 + \frac{x^4}{2}, \right.$$
$$\left. y_3 = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} \right]$$

(ii) $\frac{dy}{dx} = 3e^x + 2y, \quad y(0) = 0$

[*Ans.* $y_1 = 3(e^x - 1)$, $y_2 = 9e^x - 6x - 9$,
$y_3 = 21e^x - 6x^2 - 18x - 21$]

(iii) $\frac{dy}{dx} = 1 + xy, \quad y(0) = 1$

$$\left[\textit{Ans. } y_1 = 2 + x + x^2,\ y_2 = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{3}, \right.$$
$$\left. y_3 = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24} \right]$$

(iv) $\frac{dy}{dx} = 2x - y^2$, where $y = 0$ at $x = 0$.

$$\left[\textit{Ans. } y_1 = x^2,\ y_2 = x^2 - \frac{x^5}{5}, \right.$$
$$\left. y_3 = x^2 - \frac{x^5}{5} + \frac{x^8}{20} - \frac{x^{11}}{275} \right]$$

(v) $\dfrac{dy}{dx} = e^x + y^2,\ y(0) = 0$

$$\left[Ans.\ y_1 = e^x - 1,\ y_2 = \frac{e^{2x}}{2} - e^x + x + \frac{1}{2},\ y_3 = \frac{e^{4x}}{16} - \frac{e^{3x}}{3} + \right.$$

$$\left. \frac{xe^{2x}}{2} + \frac{1}{2}e^{2x} - 2xe^x + 2e^x + \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4} - \frac{107}{48}\right]$$

(vi) Obtain $y(0.1)$, given $y' = \dfrac{y - x}{y + x}$ and $y(0) = 1$.

[*Ans.* $y(0.1) = 1.0906$]

(vii) Given $y' = \dfrac{x^2}{1 + y^2}$ and $y(0) = 0$. Find $y(0.25)$, $y(0.5)$.

[*Ans.* $y(0.25) = 0.005$, $y(0.5) = 0.042$]

(viii) Solve $y' = x - y^2$, given $y(0) = 1$.

$$\left[Ans.\ y = 1 - x + \frac{5}{2}x^2 - 2x^3 + x^4 - \frac{x^5}{4}\right]$$

(ix) Solve $y' = x^2 + y^2$, given $y(0) = 0$.

$$\left[Ans.\ y = \frac{x^3}{5} + \frac{x^7}{63} + \frac{2x^{11}}{2079} + \ldots\right]$$

(x) Solve $y' = 2x - y$, with $y(1) = 3$. Find also $y(1.1)$.

$$\left[Ans.\ y = \frac{73}{12} - \frac{35}{6}x + \frac{7}{2}x^2 - \frac{5}{6}x^3 + \frac{x}{12},\right.$$

$$\left. y(1.1) = 2.914508\right]$$

7.3 EULER'S METHOD

In solving a first-order differential equation by numerical methods, we get two types of solutions.

(i) A solution of y in terms of x, which we get the value of y, at a particular value of x, by direct substitution in the series solution.

(ii) Values of y are calculated at specified values of x.

The methods Taylor and Picard's belong to the first category and the another methods like Euler's Runge–Kutta, Adam–Bashforth and Milne's come under the second category. The methods of second category are called *step-by-step methods* or *multistep methods*.

Euler's method is one of the oldest and easiest numerical methods used for integrating the ordinary differential equations. Euler's method, as such, is of every little importance, like the Taylor series solution provides a starting point for the other discussion.

Consider the first-order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{7.9}$$

Let us solve this differential equation under the initial condition $y(x_0) = y_0$.

The solution of Eq. (7.9) gives y as a function of x which may be symbolically written as

$$y = f(x) \tag{7.10}$$

The integral of Eq. (7.10) is a curve in the xy-plane and a smooth curve is practically a straight line for a short distance from any point on it (Figure 7.1), we get the approximate relation as

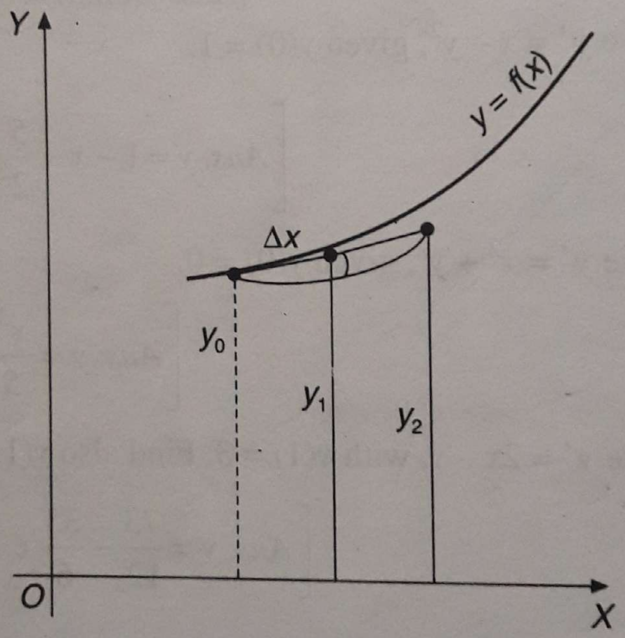

Figure 7.1 Illustration of Euler's method.

$$\tan\theta \approx \frac{\Delta y}{\Delta x}$$

$$\Rightarrow \quad \Delta y \approx \Delta x \tan\theta$$

$$= \Delta x \left(\frac{dy}{dx}\right)_0$$

$$y_1 = y_0 + \Delta y$$

$$= y_0 + \Delta x \left(\frac{dy}{dx}\right)_0 \qquad [\because \Delta x = h]$$

$$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$$

$$= y_0 + h\, f(x_0, y_0) \quad \left[\because \frac{dy}{dx} = f(x, y), \text{ from (7.9)}\right]$$

The next value of y corresponding to $x_2 \cdot x_3 \ldots$ are

$$y_2 = y_1 + h\left(\frac{dy}{dx}\right)_1, \quad x_2 = x_1 + h$$

$$y_3 = y_2 + h\left(\frac{dy}{dx}\right)_2, \quad x_3 = x_2 + h$$

$$\vdots$$

$$y_{n+1} = y_n + h\left(\frac{dy}{dx}\right)_n, \quad x_{n+1} = x_n + h$$

$\therefore$ In general, $y_{n+1} = y_n + hf(x_n, y_n)$

Geometrically, the method has a very simple meaning. By taking h is small enough and proceeding in this manner, we can calculate the expression (7.10) as a set of corresponding values of x and y.

This is the method of Euler.

Note

The Euler's method is used only when the slope at point (x_n, y_n) in computing is y_{n+1}.

7.3.1 Modified Euler's Method

This is a self-starting method of predictor–corrector type having greater accuracy than Euler's method. Herein we refer to this as modified Euler's method. This is illustrated by Figure 7.2.

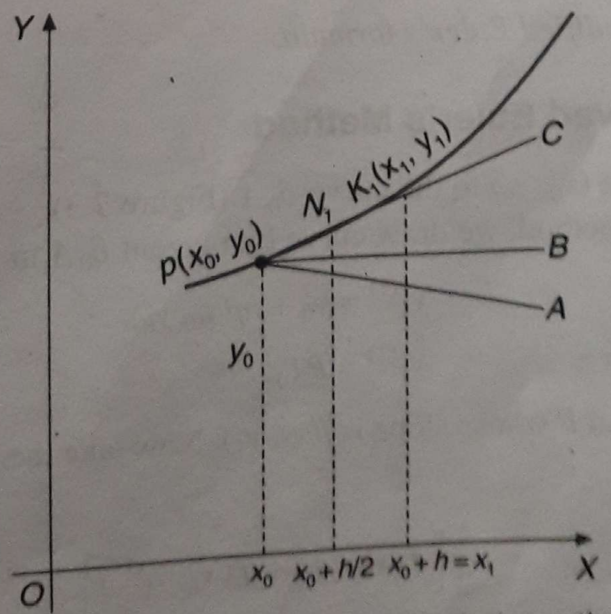

Figure 7.2 Illustration of modified Euler's method.

In this method we approximate the curve in the interval x_0, $x_1 = x_0 + h$, by the line through (x_0, y_0), with slope

$$f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

Geometrically, the line c through (x_0, y_0) which is parallel to

$$C\left\{\text{a line}\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]\right\}$$

with slope

$$\left\{f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]\right\}$$

approximates the curve in the interval (x_0, x_1).

Now draw the line through $p(x_0, y_0)$ with this slope, the line meet $x = x_1$ at k_1 $(x_1, y_1^{(1)})$.

This $y_1^{(1)}$ is taken as approximate value of y at $x = x_1$.

The equation of the line is

$$y - y_0 = (x - x_0)\, f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

Putting $x = x_1$, we get

$$y_1^{(1)} = y_0 + h\left\{f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]\right\}$$

In general,

$$y_{n+1} = y_n + h\left\{f\left[x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right]\right\}$$

This is called *modified Euler's formula.*

7.3.2 Improved Euler's Method

Let the tangent at (x_0, y_0) to the curve $\delta_0 A$ (Figure 7.3).

By Euler's method, we draw curve by tangent $\delta_0 A$ in the interval (x_0, x_1).

$$y_1^{(1)} = y_0 + hf(x_0, y_0) \tag{7.11}$$

where $\quad y_1^{(1)} = PT_1$;

Let P be the line at P whose slope is $f(x_1, y_1)$. Now take the average of the slopes at δ_0 and P.

i.e., $$\frac{1}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

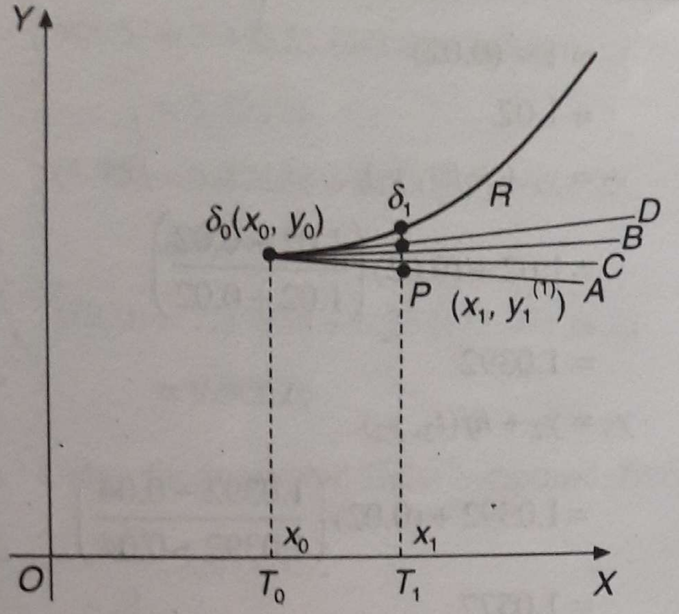

Figure 7.3 Illustration of improved Euler's method.

Now draw a line $\delta_0 D$ through $\delta_0(x_0, y_0)$ with this as the slope.

i.e.,
$$y - y_0 = \frac{1}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right](x - x_0') \tag{7.12}$$

This line intersects $x = x_1$ at

$$\therefore \quad y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1 = y_0 + \frac{1}{2}h\left[f(x_0, y_0) + f\{x_1, y_0 + hf(x_0, y_0)\}\right]$$

In general,

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n) + f\{x_n + h, y_n + hf(x_n, y_n)\}\right]$$

This is called *improved Euler's method.*

EXAMPLE 7.13 Given $\dfrac{dy}{dt} = \dfrac{y - t}{y + t}$ with the initial condition $y = 1$ at $t = 0$.

Find y approximately at $t = 0.1$ in five steps, using Euler's method.

Solution

$$y' = \frac{y - t}{y + t}$$

Taking $h = 0.2$, we compute the value of y at $t = 0.02, 0.04, 0.06, 0.08$ and 0.1.

$$\therefore \quad y_1 = y_0 + h\,f(t_0, y_0)$$

where
$$y_0 = 1,\ t_0 = 0.$$

$$y_1 = 1 + (0.02)\left(\frac{1 - 0}{1 + 0}\right)$$

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$$= 1 + (0.02)$$

$$= 1.02$$

$$y_2 = y_1 + h\,f(t_1\, y_1)$$

$$= 1.02 + (0.02)\left(\frac{1.02 - 0.02}{1.02 + 0.02}\right)$$

$$= 1.0392$$

$$y_3 = y_2 + hf(t_2, y_2)$$

$$= 1.0392 + (0.02)\left(\frac{1.0392 - 0.04}{1.0392 + 0.04}\right)$$

$$= 1.0577$$

$$y_4 = y_3 + hf(t_3, y_3)$$

$$= 1.0577 + (10.02)\left(\frac{1.0577 - 0.06}{1.0577 + 0.06}\right)$$

$$= 1.0738$$

$$y_5 = y_4 + hf(t_4, y_4)$$

$$= 1.0738 + (0.02)\left(\frac{1.0738 - 0.08}{1.0738 + 0.08}\right)$$

$$= 1.0910$$

Hence the value of y corresponding to $t = 0.1$ is 1.0910.

EXAMPLE 7.14 Given the equation $\dfrac{dy}{dx} = 3x^2 + 1$, with $y(1) = 2$. Estimate $y(2)$, by Euler's method, using (i) $h = 0.5$ and (ii) $h = 0.25$.

Solution

(i) When $h = 0.5$

$$y' = 3x^2 + 1, \quad x_0 = 1, y_0 = 2$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y(1.5) = 2 + (0.5)\,[3(1.0)^2 + 1]$$

$$= 4.0$$

$$y(2.0) = 4.0 + (0.5)\,[3(1.5)^2 + 1]$$

$$= 7.875$$

(ii) When $h = 0.25$

$$y(1) = 2$$

$$y(1.25) = 2 + 0.25\,[3(1)^2 + 1]$$

$$= 3.0$$

$$y(1.5) = 3 + 0.25\,[3(1.25)^2 + 1]$$
$$= 5.42188$$
$$y(1.75) = 5.42188 + 0.25\,[3(1.5)^2 + 1]$$
$$= 7.35938$$
$$y(2.0) = 7.35938 + 0.25\,[3(1.75)^2 + 1]$$
$$= 9.90626$$

EXAMPLE 7.15 Using the improved Euler's method, find y at $x = 0.1$ and $x = 0.2$. Given $\dfrac{dy}{dx} = y - \dfrac{2x}{y}$, $y(0) = 1$.

Solution

The improved Euler's method is

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n\, y_n) + f\{x_n + h,\, y_n + hf(x_n, y_n)\}\right] \tag{i}$$

Put $n = 0$ in Eq. (i), we get

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\{x_0 + h,\, y_0 + hf(x_0, y_0)\}\right]$$

Given

$$x_0 = 0,$$
$$y_0 = 1$$
$$y' = y - \frac{2x}{y} \cdot f(x_0, y_0)$$
$$= 1 - \frac{2(0)}{1} = 1$$

$\therefore$

$$y_1 = y_0 + \frac{h}{2}\{1 + f(x_0 + h,\, y_0 + h \cdot 1)\}$$
$$= 1 + \frac{0.1}{2}\{1 + f(0 + 0.1,\, 1 + 0.1)\}$$
$$= 1 + \frac{0.1}{2}\{1 + f(0.1, 1.1)\} \tag{ii}$$

To find f(0.1, 1.1):

$$f(0.1,\, 1.1) = 1.1 - \frac{2(0.1)}{1.1}$$
$$= 0.9182 \tag{iii}$$

Substituting Eq. (iii) in Eq. (ii), we get

$$y_1 = 1 + \frac{0.1}{2}(1 + 0.9182)$$

$$= 1.09591$$

Putting $n = 1$ in Eq. (i), we get

$$y_2 = y_1 + h/2\,[f(x_1, y_1) + f\{x_1 + h, y_1 + hf(x_1, y_1)\}] \quad \text{(iv)}$$

when $x_1 = 0.1$, $y_1 = 1.0959$

$$f(x_1, y_1) = f(0.1, 1.0959)$$

$$= 1.0959 - \frac{2(0.1)}{1.0959}$$

$$= 0.9135 \quad \text{(v)}$$

Substituting Eq. (v) in Eq. (iv), we get

$$y_2 = y_1 + h/2\,[0.9135 + f\{0.2, y_1 + h(0.9135)\}]$$

$$= 1.0959 + \frac{0.1}{2}[0.9135 + f\{0.2, 1.0959 + (0.1)(0.9135)\}]$$

$$= 1.0959 + 0.05\,[0.9135 + f(0.2, 1.1872)] \quad \text{(vi)}$$

Now, $$f(0.2, 1.1872) = 1.1872 - \frac{2(0.2)}{1.1872}$$

$$= 0.8503 \quad \text{(vii)}$$

Substituting Eq. (vii) in Eq. (vi), we get

$$y_2 = 1.0959 + 0.05\,[0.9135 + 0.8503]$$

$$= 1.1841$$

$$\therefore \quad y(0.2) = 1.1841$$

EXAMPLE 7.16 Using the modified Euler's method, find $y(0.2)$, $y(0.1)$, given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$.

Solution

$x_0 = 0$, $y_0 = 1$, $h = 0.1$, $x_1 = 0.1$, $y' = x^2 + y^2$

By the modified Euler's method

$$y_1 = y_0 + hf[x_0 + h/2, y_0 + h/2\, f(x_0, y_0)]$$

$$= 1 + (0.1)\, f\left[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2} f(0.1)\right]$$

$$= 1 + (0.1) f[0.05, 1.05 f(0.1)] \quad \text{(i)}$$

Now, $$f(0.1) = x^2 + y^2 \quad \text{(ii)}$$

$$= 0^2 + 1^2$$

$$= 1$$

Substituting Eq. (ii) in Eq. (i), we get

$$y_1 = 1 + (0.1) f(0.05, 1.05)$$

$$= 1 + (0.1) [(0.05)^2 + (1.05)^2]$$

$$= 1.1105$$

$$y_1 = y(0.1) = 1.1105$$

$$y_2 = y_1 + hf[x_1 + h/2, y_1 + h/2 f(x_1, y_1)]$$

$$= 1.1105 + (0.1) f\left[0.1 + \frac{0.1}{2}, 1.1105 + \frac{0.1}{2} f(0.1, 1.1105)\right]$$

$$= 1.1105 + (0.1) f[0.15, 1.1605 f(0.1, 1.1105)] \quad \text{(iii)}$$

Now, $$f(0.1, 1.1105) = 0.1^2 + 1.1105^2 \quad \text{(iv)}$$

$$= 1.24321$$

Substituting Eq. (iv) in Eq. (iii), we get

$$y_2 = 1.1105 + (0.1) f[0.15, 1.1605 (1.24321)]$$

$$= 1.1105 + (0.1) f(0.15, 1.44245)$$

$$= 1.1105 + (0.1) (0.15^2 + 1.44245^2)$$

$$= 1.1105 + (0.1) (2.10403)$$

$$= 1.1105 + 0.210403$$

$$= 1.320903$$

$$y_2 = y(0.2) = 1.320903$$

EXAMPLE 7.17 Solve numerically $y' = y + e^x$, $y(0) = 0$, for $x = 0.2, 0.4$, by the improved Euler's method.

Solution

$y' = y + e^x, y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0, x_1 = 0.2, x_2 = 0.4, h = 0.2$

By improved Euler's method

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f\{x_1, y_0 + hf(x_0, y_0)\}]$$

$$= 0 + \frac{0.2}{2}[y_0 + e^{x_0} + y_0 + h(y_0 + e^{x_0}) + e^{x_0 + h}]$$

$$= 0.1[0 + 1 + 0 + 0.2\,(0 + 1) + e^{0.2}]$$

$$= (0.1)\,[1 + 0.2 + 1.2214]$$

$$= 0.24214$$

$$y_2 = y_1 + h/2\,[f(x_1, y_1) + f\{x_1 + h, y_1 + hf(x_1, y_1)\}]$$

$$f(x_1, y_1) = y_1 + e^{x_1}$$

$$= 0.24214 + e^{0.2}$$

$$= 1.46354$$

$$y_1 + hf(x_1, y_1) = 0.24214 + (0.2)\,(1.46354)$$

$$= 0.53485$$

$$= f[x_1 + h, y_1 + hf(x_1, y_1)]$$

$$= f(0.4, 0.53485)$$

$$= 0.53485 + e^{0.4}$$

$$= 2.02667$$

$$y_2 = y(0.4)$$

$$= 0.24214 + (0.1)\,[1.46354 + 2.02667]$$

$$= 0.59116$$

$$y(0.4) = 0.59116$$

EXERCISES

7.20 Given $y' = -y$ and $y(0) = 1$. Determine the values of y at $x(0.01)\,(0.01)\,(0.04)$, by Euler's method.

[*Ans.* $y(0.01) = 0.9900$, $y(0.02) = 0.9801$, $y(0.03) = 0.9703$, $y(0.04) = 0.9606$]

7.21 Using Euler's method, solve numerically the equation,

$$y' = x + y,\ y(0) = 1, \text{ for } x = (0.0)\,(0.2)\,(1.0).$$

[*Ans.* $y(0.2) = 1.2$, $y(0.4) = 1.48$, $y(0.6) = 1.856$ $y(0.8) = 2.3472$, $y(1.0) = 2.94664$]

7.22 Using the modified Euler's method, solve $\dfrac{dy}{dx} = y + x^2$, $y(0) = 1$ to find $y(0.2)$ and $y(0.4)$, correct to 3 decimal places.

[*Ans.* $y(0.2) = 1.224$, $y(0.4) = 1.514$]

7.23 Solve $\dfrac{dy}{dx} = \dfrac{2y}{x} + x^3$ to obtain $y(1.2)$ and $y(1.4)$, given $y = 0.5$ when $x = 1$, by the modified Euler's method.

[*Ans.* $y(1.2) = 1.0228$, $y(1.4) = 1.8847$]

7.24 Given $\frac{dy}{dx} = \frac{x - y}{x + y}$, $y(2) = 1$. Find $y(1)$ by taking $h = -0.2$, using the modified Euler's method.

[*Ans.* $y(1) = 0.73207$]

7.25 Solve $\frac{dy}{dx} = x\sqrt{1 + y^2}$, $y(1) = 0$ at $x = 3$, taking $h = 0.4$, using the modified Euler's method.

[*Ans.* $y(3) = 21.671$]

7.26 Find $y(0.6)$, $y(0.8)$, $y(1)$, given $\frac{dy}{dx} = x + y$, $y(0) = 0$, taking $h = 0.2$, by the improved Euler's method.

[*Ans.* $y(0.6) = 0.2158$, $y(0.8) = 0.4153$, $y(1) = 0.7027$]

7.4 RUNGE–KUTTA METHODS

The most efficient methods in terms of getting exact values were developed by two German mathematicians, Runge and Kutta. These methods are well-known as Runge–Kutta methods. They are distinguished by their orders that they agree with the Taylor's series solution upto the terms of h^r, where r differs from method to method and is known as the *order of the Runge–Kutta method.* The Runge–Kutta methods are designed to give greater accuracy for the given function values at different points.

Since the derivation of fourth-order Runge–Kutta methods are widely used for finding the solutions of linear or non-linear ordinary differential equations. The fourth-order Runge–Kutta method is algebraically complicated to find the solutions so we convey the basic idea of these methods by developing the second-order Runge–Kutta and third-order Runge–Kutta methods which we mention it as R–K method for our convenience.

Now we shall derive the second-order R–K method as follows:

To solve
$$y' = \frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0 \tag{7.13}$$

By Taylor series, we have
$$y(x + h) = y(x) + hy'(x) + \frac{h^2}{2!}y''(x) + O(h^3) \tag{7.14}$$

Differentiating Eq. (7.13) w.r.t. x, we get
$$y'' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx}$$
$$= f_x + y' f_y$$

Rewriting the derivatives of y in terms of f, we get
$$= f_x + ff_y \tag{7.15}$$

Equation (7.15) $\Rightarrow$

$$y(x+h) - y(x) = hy'(x) + \frac{h^2}{2!} y''(x) + O(h^3)$$

$$y(x+h) - y(x) = hf + \frac{1}{2} h^2 (f_x + ff_y) + O(h^3)$$

$$\Delta y = hf + \frac{1}{2} h^2 (f_x + ff_y) + O(h^3) \tag{7.16}$$

$$[\because \Delta y = y(x+h) - y(x)]$$

Let us assume that

$$\Delta y = ak_1 + bk_2 \tag{7.17}$$

where $k_1 = hf(x, y)$, $k_2 = hf(x + mh, y + mk_1)$

and a, b and m are constants to get the greater accuracy of Δy.

Expanding k_2, by Taylor series for two variables, we have

$$k_2 = hf(x + mh, y + mk_1)$$

$$= h\left[f(x, y) + \left(mh \frac{\partial f}{\partial x} + mk_1 \frac{\partial f}{\partial y} \right) + \frac{1}{2!} \left(mh \frac{\partial f}{\partial x} + mk \frac{\partial f^2}{\partial y} \right) + \ldots \right]$$

$$= h\left[f(x, y) + \left(mh \frac{\partial}{\partial x} + mk_1 \frac{\partial}{\partial y} \right) f + \frac{1}{2!} \left(mh \frac{\partial}{\partial x} + mk_1 \frac{\partial}{\partial y} \right)^2 f + \ldots \right]$$

$$= h\left[f + mhf_x + mhff_y + \frac{1}{2!} \left(mh \frac{\partial}{\partial x} + mk_1 \frac{\partial}{\partial y} \right)^2 f + \ldots \right] \quad [\because k_1 = hf]$$

$$= hf + mh^2 f_x + mh^2 ff_y + \ldots \text{ higher powers of } h$$

$$= hf + mh^2 (f_x + ff_y) + O(h^3) \tag{7.18}$$

Substituting k_1, k_2 in Eq. (7.18), we get

$$\Delta y = ahf + b[hf + mh^2 (f_x + ff_y) + O(h^3)]$$

$$= ahbf + bhf + bmh^2 (f_x + ff_y) + O(h^3).$$

$$= (a + b)hf + bmh^2 (f_x + ff_y) + O(h^3) \tag{7.19}$$

Equating Δy from Eqs. (7.16) and (7.19), we get

$$a + b = 1 \text{ and } bm = \frac{1}{2}.$$

To find the value for a, b, m from the above two equations. we get

$$a + b = 1 \quad \Rightarrow \quad a = 1 - b \quad \text{and} \quad m = \frac{1}{2b}$$

Equation (7.17) ⇒

$$\Delta y = (1-b)\,k_1 + bk_2 \tag{7.20}$$

where

$$k_1 = h\,f(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

Equation (7.20) ⇒

$$\Delta y = (1-b)\,hf(x, y) + b\,hf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

$$y(x+h) - y(x) = (1-h)hf + bhf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

$$y(x+h) = y(x) + (1-b)hf + bhf\left(x + \frac{h}{2b}, y + \frac{hf}{2b}\right)$$

In general we can return it as

$$y_{n+1} = y_n + (1-b)\,hf(x_n, y_n) + bhf\left[x_n + \frac{h}{2b}, \frac{h}{2b} f(x_n, y_n)\right] + O(h^3)$$

By setting $a = 0$, $b = 1$, $m = 1/2$ in above equations, we get the second-order R–K algorithm as

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right) \text{ and } \Delta y = k_2$$

Similarly, we can state third and fourth-order R–K algorithm because their derivatives are tedious.

The third-order Runge–Kutta algorithm is stated as

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$k_3 = hf(x + h, y + 2k_2 - k_1)$$

and

$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

The fourth-order R–K algorithm is stated as

$$k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x + \frac{h}{2}, y + \frac{k_2}{2}\right)$$

$$k_4 = hf(x + h, y + k_3)$$

and $$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $$y_1 = y_0 + \Delta y$$

Notes

(i) In all the three methods, i.e. second-order, third-order, and fourth-order R–K algorithms the values of k_1 and k_2 are same. Therefore, we need not repeat k_1 and k_2 while doing all the three methods in same problem.

(ii) The fourth-order Runge–Kutta algorithm can be derived independently.

Let us assume that it has the function x done, we get

$$k_1 = hf(x)$$

$$k_2 = hf\left(x + \frac{1}{2}h\right)$$

$$k_3 = hf\left(x + \frac{1}{2}h\right)$$

$$k_4 = hf(x + h)$$

$$\Delta y = \frac{1}{6}\left[hf(x) + 2hf\left(x + \frac{1}{2}h\right) + 2hf\left(x + \frac{1}{2}h\right) + hf(x + h)\right]$$

$$= \frac{1}{6}h\left[f(x) + 4f\left(x + \frac{h}{2}\right) + f(x + h)\right]$$

$$= \frac{h/2}{3}\left[f(x) + 4f(x + h/2) + f(x + h)\right]$$

= Equal intervals of length $h/2$ by Simpson's one-third rule to the interval $x = x_0$ to $x = x_0 + h$.

(iii) The Runge–Kutta method of second-order is nothing but the modified Euler's method.

EXAMPLE 7.18 Obtain the values of y at $x = 0.1, 0.2$, for the differential equation $y' = -y$, given $y(0) = 1$, using the R–K methods of (i) second-order, (ii) third-order and (iii) fourth-order.

Solution

Here $y' = -y$, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $y_0 = 1$

For second-order:

$$k_1 = hf(x_0, y_0)$$
$$= (0.1)(-y_0)$$
$$= (0.1)(-1)$$
$$= -0.1$$

To find y_1:

$$\Delta y = k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= (0.1)\, f(0.05, 0.95)$$
$$= -0.095$$
$$y_1 = y_0 + \Delta y$$
$$= 1 - 0.095$$
$$= 0.905$$

$\therefore$
$$y_1 = y(0.1)$$
$$= 0.905$$

To find y_2:

$$k_1 = hf(x_1, y_1)$$
$$= (0.1)(-y_1)$$
$$= (0.1)(-0.905)$$
$$= -0.0905$$
$$\Delta y = k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= (0.1)\, f(0.15, 0.85975)$$
$$= (0.1)(-0.85975)$$
$$= -0.085975$$

$\therefore$
$$y_2 = y_1 + \Delta y$$
$$= 0.905 - 0.085975$$
$$= 0.819025$$

For third-order

To find y_1:

$$k_1 = hf(x_0, y_0)$$
$$= (0.1)(-y_0)$$
$$= -0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.1)\, f(0.05, 0.95)$$
$$= -\,0.095$$
$$k_3 = hf(x_0 + h, y_0 + 2k_2 - k_1)$$
$$= (0.1)\, f(0.1, 0.9)$$
$$= (0.1)\,(-\,0.9)$$
$$= -\,0.09$$
$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= \frac{1}{6}[-\,0.1 - 4(0.095) - 0.09]$$
$$= -\,0.095$$

$\therefore$
$$y_1 = y_0 + \Delta y$$
$$= 1 - 0.095$$
$$= 0.905$$

$\therefore$
$$y_1(0.1) = 0.905$$

To find y_2:

$$k_1 = hf(x_1, y_1)$$
$$= (0.1)\,(-\,0.905)$$
$$= -\,0.0905$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= (0.1)\, f(0.15, 0.85975)$$
$$= (0.1)\,(-\,0.85975)$$
$$= -\,0.085975$$
$$k_3 = hf(x_1 + h, y_1 + 2k_2 - k_1)$$
$$= (0.1)\, f(0.2, 0.82355)$$
$$= (0.1)\,(-\,0.82355)$$
$$= -\,0.082355$$

$\therefore$
$$y_2 = y_1 + \Delta y = 0.905 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$
$$= 0.905 + \frac{1}{6}(-\,0.516755)$$
$$= 0.905 - 0.08613$$
$$= 0.81887$$

$$y_2 = y(0.2)$$
$$= 0.81887$$

For fourth-order:

To find y_1: $k_1 = hf(x_0, y_0)$

$$= (0.1)(-1)$$
$$= -0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= (0.1) f(0.05, 0.95)$$
$$= -0.095$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= (0.1) f(0.05, 0.9525)$$
$$= -0.09525$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= (0.1) f(0.1, 0.90475)$$
$$= (0.1)(-0.90475)$$
$$= -0.090475$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[-0.1 - 2(0.095) - 2(0.09525) - 0.090475]$$
$$= \frac{1}{6}(-0.1 - 0.19 - 0.1905 - 0.090475)$$
$$= -0.09516$$

$\therefore$
$$y_1 = y(0.1)$$
$$= 1 - 0.09516$$
$$= 0.9048375$$
$$= 0.90484$$

To find y_2:

$$k_1 = hf(x_1, y_1)$$
$$= (0.1)(-y_1)$$
$$= -0.09048375$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.1) f(0.15, 0.8595956)$$

$$= -0.08595956$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.1) f(0.15, 0.8618577)$$

$$= -0.08618577$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= (0.1)(-0.8186517)$$

$$= -0.08186517$$

$$\Delta y = \frac{1}{6}[-0.09048375 - 2(0.08595956) - 2(0.08618577) - 0.08186517]$$

$$= -0.0861066067$$

$$y_2 = y(0.2)$$

$$= 0.81873089$$

EXAMPLE 7.19 Using Runge–Kutta method of fourth-order, solve for y at $x = 1.2, 1.4$, from $y' = \dfrac{2xy + e^x}{x^2 + xe^x}$, with $x_0 = 1$, $y_0 = 0$.

Solution

$$y' = \frac{2xy + e^x}{x^2 + xe^x}, \text{ with } x_0 = 1, y_0 = 0, x_1 = 1.2, x_2 = 1.4, h = 0.2$$

To find y_1:

$$k_1 = hf(x_0, y_0)$$

$$= (0.2)\left(\frac{2x_0y_0 + e^{x_0}}{x_0^2 + x_0e^{x_0}}\right)$$

$$= (0.2)\left(\frac{2.7182}{1 + 2.7182}\right)$$

$$= 0.1462$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2)\left[\frac{2\left(x_0+\frac{h}{2}\right)\left(y_0+\frac{k_1}{2}\right)+e^{x_0+\frac{h}{2}}}{\left(x_0+\frac{h}{2}\right)^2+\left(x_0+\frac{h}{2}\right)e^{x_0+\frac{h}{2}}}\right]$$

$$= (0.2)\left[\frac{2\left(1+\frac{0.2}{2}\right)\left(0+\frac{0.1462}{2}\right)+e^{1+\frac{0.2}{2}}}{\left(1+\frac{0.2}{2}\right)^2+\left(1+\frac{0.2}{2}\right)e^{1+\frac{0.2}{2}}}\right]$$

$$= (0.2)\left(\frac{3.1649}{4.5146}\right)$$

$$= 0.1402$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right)$$

$$= (0.2)\left[\frac{2\left(1+\frac{0.2}{2}\right)\left(0+\frac{0.1402}{2}\right)+e^{1+\frac{0.2}{2}}}{\left(1+\frac{0.2}{2}\right)^2+\left(1+\frac{0.2}{2}\right)e^{1+\frac{0.2}{2}}}\right]$$

$$= (0.2)\left(\frac{3.1583}{4.5145}\right)$$

$$= 0.1399$$

$$k_4 = h\,f(x_0+h, y_0+k_3)$$

$$= (0.2)\left[\frac{2(1+0.2)\,(0+0.1399)+e^{1+0.2}}{(1+0.2)^2+(1+0.2)\,e^{1+0.2}}\right]$$

$$= (0.2)\left(\frac{3.6559}{5.4241}\right)$$

$$= 0.1348$$

$$\Delta y = \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$= \frac{1}{6}(0.1462+0.2804+0.2798+0.1348)$$

$$= 0.1402$$

$$
\begin{aligned}
y_1 &= y_0 + \Delta y \\
&= 0 + 0.1402 \\
&= 0.1402 \\
y(1.2) &= 0.1402
\end{aligned}
$$

To find y(1.4):

$$
\begin{aligned}
k_1 &= h\,f(x_1, y_1) \\
&= (0.2)\left[\frac{2(1.2)(0.1402) + e^{1.2}}{(1.2)^2 + (1.2)e^{1.2}}\right] \\
&= (0.2)\left(\frac{3.6565}{5.4241}\right) \\
&= 0.1348
\end{aligned}
$$

$$
k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)
$$

$$
= h\left[\frac{2\left(x_1 + \frac{h}{2}\right)\left(y_1 + \frac{k_1}{2}\right) + e^{x_1 + \frac{h}{2}}}{\left(x_1 + \frac{h}{2}\right)^2 + \left(x_1 + \frac{h}{2}\right)e^{x_1 + \frac{h}{2}}}\right]
$$

$$
= (0.2)\left[\frac{2\left(1.2 + \frac{0.2}{2}\right)\left(0.1402 + \frac{0.1348}{2}\right) + e^{1.2 + \frac{0.2}{2}}}{\left(1.2 + \frac{0.2}{2}\right)^2 + \left(1.2 + \frac{0.2}{2}\right)e^{1.2 + \frac{0.2}{2}}}\right]
$$

$$
= 0.1303
$$

$$
k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)
$$

$$
= (0.2)\left[\frac{2\left(1.2 + \frac{0.2}{2}\right)\left(0.1402 + \frac{0.1303}{2}\right)e^{1.2 + \frac{0.2}{2}}}{\left(1.2 + \frac{0.2}{2}\right)^2 + \left(1.2 + \frac{0.2}{2}\right)e^{1.2 + \frac{0.2}{2}}}\right]
$$

$$
= (0.2)\left(\frac{0.5339 + 3.6692}{6.46}\right)
$$

$$
= 0.1301
$$

$$k_4 = h\,f(x_1 + h,\, y_1 + k_3)$$

$$= h\left[\frac{2(x_1 + h)(y_1 + k_3) + e^{x_1+h}}{(x_1 + h)^2 + (x_1 + h)\, e^{x_1+h}}\right]$$

$$= (0.2)\left[\frac{2(1.2 + 0.2)(0.1402 + 0.1301) + e^{1.2+0.2}}{(1.2 + 0.2)^2 + (1.2 + 0.2)\, e^{1.2+0.2}}\right]$$

$$= (0.2)\left(\frac{4.8184}{7.6372}\right)$$

$$= 0.1262$$

$$y_2 = y_1 + \Delta y$$

$$= y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.1402 + \frac{1}{6}(0.1348 + 0.2606 + 0.2602 + 0.1262)$$

$$= 0.1402 + 0.1303$$

$$= 0.2705$$

$$\therefore \quad y_2 = y(1.4)$$

$$= 0.2705$$

EXAMPLE 7.20 Using the Runge–Kutta method of fourth-order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ at $x = 0.2$, 0.4.

Solution

Given $y' = \frac{y^2 - x^2}{y^2 + x^2}$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $x_2 = 0.4$

To find y(0.2):

$$k_1 = h\,f(x_0, y_0)$$

$$= (0.2)\, f(0.\ 1)$$

$$= 0.2\left(\frac{1^2 - 0^2}{1^2 + 0^2}\right)$$

$$= 0.2000$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.2)\, f(0.1,\ 1.1)$$

$$= 0.19672$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

$$= (0.2)\, f(0.1,\ 1.09836)$$

$$= 0.1967$$

$$k_4 = hf(x_0 + h,\ y_0 + k_3)$$

$$= (0.2)\, f(0.2,\ 1.1967)$$

$$= (0.2)\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right]$$

$$= 0.1891$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.2 + 2(0.19672) + 2(0.1967) + 0.1891]$$

$$= 0.19599$$

$$y(0.2) = y_0 + k$$

$$= 1 + 0.19599$$

$$= 1.19599$$

To find $y(0.4)$:

Here, $x_1 = 0.2$, $y_1 = 1.196$, $h = 0.2$

$$k_1 = hf(x_1,\ y_1)$$

$$= (0.2)\left[\frac{1.196^2 - 0.2^2}{1.196^2 + 0.2^2}\right]$$

$$= 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right)$$

$$= (0.2)\, f(0.3,\ 1.2906)$$

$$= (0.2)\left[\frac{1.2906^2 - 0.3^2}{1.2906^2 + 0.3^2}\right]$$

$$= 0.1795$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.2\,f(0.3,\ 1.2858)$$

$$= (0.2)\left[\frac{1.2858^2 - 0.2^2}{1.2858^2 + 0.2^2}\right]$$

$$= 0.1793$$

$$k_4 = hf(x_1 + h,\ y_1 + k_3)$$

$$= (0.2)\,f(0.4,\ 1.3753)$$

$$= (0.2)\left[\frac{1.3753^2 - 0.4^2}{1.3753^2 + 0.4^2}\right]$$

$$= 0.1688$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1891 + 2(0.1795) + 2(0.1793) + 0.1688]$$

$$= 0.1792$$

$$y(0.4) = y_1 + k$$

$$= 1.196 + 0.1792$$

$$y(0.4) = 1.3752$$

7.4.1 Runge–Kutta Method for Simultaneous First-order Differential Equations

It is necessary to solve sets of simultaneous first-order differential equations in analyzing the engineering systems. Such equations occur in obtaining solutions of higher-order differential equations which are transformed to sets of first-order differential equations as part of solution process. Runge–Kutta methods are well-suited for the solution of such equations.

Let us consider the solution of two simultaneous first-order differential equations of the form

$$\frac{dy}{dx} = f(x, y, z), \quad \frac{dz}{dx} = g(x, y, z)$$

where initial values $y(x_0) = y_0$ and $z(x_0) = z_0$.

To solve this system of equations at an interval of h, the increments in y and z for the first increment in x are computed by using the formulae.

$$k_1 = hf(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_1 = y_0 + \Delta y$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4), \quad z_1 = z_0 + \Delta z$$

In a similar manner, we can find the next value by replacing x_0, y_0, z_0 by x_1, y_1, z_1 and so on.

EXAMPLE 7.21 Use the Runge–Kutta method to find the solution of the system

$$\frac{dy_1}{dx} = y_1 - y_2, \qquad y_1(0) = 0$$

$$\frac{dy_2}{dx} = -y_1 + y_2, \qquad y_2(0) = 1$$

at $x = 0.1$, taking $h = 0.1$.

Solution

Here $h = 0.1$, $x_{10} = 0$, $y_{10} = 0.0$, $y_{20} = 1.0$

$$f(x, y_1, z_2) = \frac{dy_1}{dx}$$

$$= y_1 - y_2$$

$$g(x, y_1, y_2) = -y_1 + y_2$$

To find $y_1(0.1)$:

$$k_1 = hf(x_{10}, y_{10}, z_{10})$$

$$= (0.1)(y_{10} - y_{20})$$

$$= (0.1)(0.0 - 1.0)$$

$$= -0.1$$

$$k_2 = hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_1}{2}, y_{20} + \frac{l_1}{2}\right)$$

$$= (0.1)\, f(0.05, -0.05, 1.05)$$

$$= (0.1)\, (-0.05 - 1.05)$$

$$= -0.11$$

$$k_3 = hf\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_2}{2}, y_{20} + \frac{l_2}{2}\right)$$

$$= hf(0.05, -0.055, 1.055)$$

$$= (0.1)\, (-0.055 - 1.055)$$

$$= -0.1110$$

$$k_4 = hf(x_{10} + h, y_{10} + k_3, y_{20} + l_3)$$

$$= (0.1)\, f(0.1, -0.111, 1.111)$$

$$= (0.1)\, (-0.111 - 1.111)$$

$$= -0.1222$$

$$\Delta y_1 = 1/6\, (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= -0.1107$$

$$y_1(0.1) = y_{10} + \Delta y_1$$

$$= 0 - 0.1107$$

$$= -0.1107$$

$$l_1 = hg(x_{10}, y_{10}, y_{20})$$

$$= (0.1)\, (-y_{10} + y_{20})$$

$$= (0.1)\, (-0 + 1)$$

$$= 0.1$$

$$l_2 = hg\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_1}{2}, y_{20} + \frac{l_1}{2}\right)$$

$$= (0.1)\, g(0.05, -0.05, 1.05)$$

$$= (0.1)\, [-(-0.05) + 1.05]$$

$$= 0.11$$

$$l_3 = hg\left(x_{10} + \frac{h}{2}, y_{10} + \frac{k_2}{2}, y_{20} + \frac{l_2}{2}\right)$$

$$= (0.1)\, g(0.05, -0.055, 1.055)$$

$$= (0.1)\, (0.055 + 1.055)$$

$$= 0.1110$$

$$l_4 = hg(x_{10} + h, y_{10} + k_3, y_{20} + l_3)$$
$$= (0.1)\, g(0.1, -0.111, 1.111)$$
$$= (0.1)\,(0.111 + 1.111)$$
$$= 0.1222$$
$$\Delta y_2 = 1/6\,(l_1 + 2l_2 + 2l_3 + l_4)$$
$$= 0.1107$$
$$y_2(0.1) = y_{20} + \Delta y_2$$
$$= 1 + 0.1107$$
$$= 1.1107$$
$$y_1(0.1) = -0.1107$$
$$y_2(0.1) = 1.1107$$

EXAMPLE 7.22 Using the Runge–Kutta method, tabulate the solution of the system $\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y, y = 0, z = 1$ when $x = 0$ at intervals of $h = 0.1$, from $x = 0.0$ to $x = 0.2$.

Solution

To compute y(0.1) and z(0.1):

$$k_1 = hf(x_0, y_0, z_0)$$
$$= h(x_0 + z_0)$$
$$= (0.1)\,(0 + 1)$$
$$= 0.1$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(z_0 + \frac{l_1}{2}\right)\right]$$
$$= (0.1)\left[0 + \frac{0.1}{2}, 1 + \frac{0}{2}\right]$$
$$= 0.105$$
$$k_3 = hf\left[x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right]$$
$$= h\left[\left(x_0 + \frac{h}{2}\right) + \left(z_0 + \frac{l_2}{2}\right)\right]$$

$$= (0.1)\left[\left(0+\frac{0.1}{2}\right)+\left(1+\frac{0}{2}\right)\right]$$

$$= 0.105$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0.1)\,[(0 + 0.1) + (1 - 0.00026)]$$

$$= 0.1099$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$= h(x_0 - y_0)$$

$$= (0.1)\,(0 - 0)$$

$$= 0$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{k_1}{2}\right)\right]$$

$$= (0.1)\left[\left(0+\frac{0.1}{2}\right)-\left(0+\frac{0.1}{2}\right)\right]$$

$$= 0$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= h\left[\left(x_0 + \frac{h}{2}\right) - \left(y_0 + \frac{k_2}{2}\right)\right]$$

$$= (0.1)\left[\left(0+\frac{0.1}{2}\right)-\left(0+\frac{0.105}{2}\right)\right]$$

$$= -0.00026$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= h[(x_0 + h) - (y_0 + k_3)]$$

$$= -0.0005$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1 + 2(0.105) + 2(0.105) + 0.1099]$$

$$= 0.1050$$

$$y_1 = y_0 + \Delta y$$
$$= 0 + 0.1050$$
$$= 0.1050$$
$$y(0.1) = 0.1050$$
$$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 - l_4)$$
$$= \frac{1}{6}[0 + 0 + 2(-0.00026) - 0.0005]$$
$$= -0.00017$$
$$z_1 = z_0 + \Delta z$$
$$= 1 - 0.00017$$
$$z(0.1) = 0.9998$$

To compute y(0.2) and z(0.2):

Here $x_1 = 0.1$, $y_1 = 0.1050$, $z_1 = 0.9998$

$$k_1 = hf(x_1, y_1, z_1)$$
$$= (0.1)\,(0.1 + 0.9998)$$
$$= 0.1099$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$
$$= h\left[\left(x_1 + \frac{h}{2}\right) + \left(z_1 + \frac{l_1}{2}\right)\right]$$
$$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right) + \left(0.9998 - \frac{0.0005}{2}\right)\right]$$
$$= 0.1149$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$
$$= h\left[\left(x_1 + \frac{h}{2}\right) + \left(z_1 + \frac{l_2}{2}\right)\right]$$
$$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right) - \left(0.9998 + \frac{0.00099}{2}\right)\right]$$
$$= 0.1149$$

$$k_4 = hf\left(x_1 + h,\, y_1 + k_3,\, z_1 + l_3\right)$$

$$= h[(x_1 + h) + (z_1 + l_3)]$$

$$= (0.1)\,[(0.1 + 0.1) + (0.9998 - 0.00125)]$$

$$= 0.1198$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.1099 + 2(0.1149) + 2(0.1149) + 0.1198]$$

$$= 0.1149$$

$$y_2 = y_1 + \Delta y$$

$$= 0.1050 + 0.1149$$

$$= 0.2199$$

$$l_1 = hg(x_1, y_1, z_1)$$

$$= (0.1)\,(0.1 - 0.1050)$$

$$= -\,0.0005$$

$$l_2 = hg\left(x_1 + \frac{h}{2},\, y_1 + \frac{k_1}{2},\, z_1 + \frac{l_1}{2}\right)$$

$$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right) - \left(0.1050 + \frac{0.1099}{2}\right)\right]$$

$$= -\,0.00099$$

$$l_3 = hg\left(x_1 + \frac{h}{2},\, y_1 + \frac{k_2}{2},\, z_1 + \frac{l_2}{2}\right)$$

$$= h\left[\left(x_1 + \frac{h}{2}\right) - \left(y_1 + \frac{k_2}{2}\right)\right]$$

$$= (0.1)\left[\left(0.1 + \frac{0.1}{2}\right) - \left(0.1050 + \frac{0.1149}{2}\right)\right]$$

$$= -\,0.00125$$

$$l_4 = hg(x_1 + h,\, y_1 + k_3,\, z_1 + l_3)$$

$$= h[(x_1 + h) - (y_1 + k_3)]$$

$$= (0.1)\,[(0.1 + 0.1) - (0.1050 + 0.1149)]$$

$$= -\,0.00199$$

$$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$$= \frac{1}{6}[-0.0005 + 2(-0.00049) + 2(-0.00125) - 0.00199]$$

$$= \frac{1}{6}[-0.0005 - 0.00198 - 0.0025 - 0.00199]$$

$$= -0.00116$$

$$z_2 = z_1 + \Delta z$$

$$= 0.9998 - 0.00116$$

$$= 0.9986$$

ANSWERS:

$$y(0.1) = 0.1050$$

$$z(0.1) = 0.9998$$

$$y(0.2) = 0.2199$$

$$z(0.2) = 0.9986$$

7.4.2 Runge–Kutta Method for Second-order Differential Equations

Let us assume the second-order differential equation be

$$\frac{d^2y}{dx^2} = g\left(x, y, \frac{dy}{dx}\right) \tag{7.21}$$

By setting

$$\frac{dy}{dx} = y' = z \tag{7.22}$$

Differentiating Eq. (7.22) on bothsides, we get

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} \tag{7.23}$$

Substituting Eq. (7.23) in Eq. (7.21), we get

$$\frac{dz}{dx} = g\left(x, y, \frac{dy}{dx}\right)$$

∴ Substituting Eq. (7.23) in above, we get

$$\frac{dz}{dx} = g(x, y, z) \tag{7.24}$$

$$\frac{dy}{dx} = f(x, y, z) \tag{7.25}$$

Equations (7.24) and (7.25) give a simultaneous equations which can be solved as in previous articles.

EXAMPLE 7.23 Somlve $\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0$

Using the Runge–Kutta method for $x = 0.2$, with initial conditions $x = 0, y = 1, y' = 0$.

Solution

$$\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^2 + y^2 = 0 \tag{i}$$

We know that

$$\frac{dy}{dx} = z$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

Equation (i) ⇒

$$\frac{dz}{dx} - xz^2 + y^2 = 0$$

$$\frac{dz}{dx} = xz^2 - y^2$$

∴

$$\frac{dy}{dx} = z$$

$$\frac{dz}{dx} = xz^2 - y^2$$

with $x_0 = 0,\ y_0 = 1,\ y_0' = 0,\ h = 0.2$

i.e., $y' = z \Rightarrow y_0' = z_0 = 0$

To find y(0.2):

$$k_1 = hf(x_0, y_0, z_0)$$
$$= hz_0 = (0.2)\,(0) = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$
$$= (0.2)\left(z_0 + \frac{(-0.2)}{2}\right)$$
$$= (0.2)\,(0 - 0.1)$$
$$= -0.02$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= h\left(z_0 + \frac{l_2}{2}\right)$$

$$= (0.2)\left(0 - \frac{0.1998}{2}\right)$$

$$= -0.01998$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= h(z_0 + l_3)$$

$$= (0.2)(0 - 0.1958)$$

$$= -0.0392$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$= h(x_0 z_0^2 - y_0^2)$$

$$= (0.2)(0 - 1)$$

$$= -0.2$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= (0.2)\left[\left(x_0 + \frac{h}{2}\right)\left(z_0 + \frac{l_1}{2}\right)^2 - \left(y_0 + \frac{k_1}{2}\right)\right]$$

$$= (0.2)\left[\left(0 + \frac{0.2}{2}\right)\left(0 - \frac{0.2}{2}\right)^2 - \left(1 + \frac{0}{2}\right)^2\right]$$

$$= -0.1998$$

$$l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= (0.2)\left[\left(0 + \frac{0.2}{2}\right)\left(0 - \frac{0.1998}{2}\right)^2 - \left(1 - \frac{0.02}{2}\right)^2\right]$$

$$= -0.1958$$

$$l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0.2)\,[(0.2)(0 - 0.1958)^2 - (1 - 0.01998)^2]$$

$$= -0.1906$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0 + 2(-0.02) + 2(-0.01998) - 0.0392]$$

$$= -0.0199$$

$$y(0.2) = y_1 = y_0 + \Delta y$$

$$= 1 - 0.0199$$

$$= 0.9801$$

$$y(0.2) = 0.9801$$

EXAMPLE 7.24 Find $y(0.1)$, from

$$\frac{d^2y}{dx^2} - y^3 = 0,\ y(0) = 10,\ y'(0) = 50$$

using the Runge–Kutta method.

Solution

Let us assume
$$\frac{dy}{dx} = z$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$\therefore$
$$\frac{dz}{dx} = y^3$$

$$\frac{dy}{dx} = z$$

with $x_0 = 0$, $y_0 = 10$, $y_0' = z_0 = 50$, $h = 0.1$

To find y(0.1):

$$k_1 = hf(x_0, y_0, z_0)$$

$$= hz_0$$

$$= (0.1)(50)$$

$$= 5$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= h\left(z_0 + \frac{l_1}{2}\right)$$

$$= (0.1)\left(50 + \frac{100}{2}\right)$$

$$= 10$$

$$l_1 = hg(x_0, y_0, z_0)$$

$$= (0.1)\,(y_0^{\,3})$$

$$= (0.1)\,(10^3)$$

$$= 100$$

$$l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= (0.1)\,g\left(y_0 + \frac{k_1}{2}\right)^3$$

$$= (0.1)\left(10 + \frac{5}{2}\right)^3$$

$$= 195.313$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= h\left(z_0 + \frac{l_2}{2}\right)$$

$$= (0.1)\left(50 + \frac{195.313}{2}\right)$$

$$= 14.7657$$

$$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= h(z_0 + l_3)$$

$$= (0.1)\,(50 + 675)$$

$$= 72.5$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(5 + 2(10) + 2(14.7657) + 72.5)$$

$$= \frac{1}{6}(5 + 20 + 29.5314 + 72.5)$$

$$= \frac{1}{6}(127.0314)$$

$$= 21.1719$$

$$y(0.1) = y_1 = y_0 + \Delta y$$

$$= 10 + 21.1719$$

$$= 31.1719$$

$$y(0.1) = 31.1719$$

EXERCISES

7.27 Using the Runge–Kutta method of fourth-order, solve for y at $x = 1.2$, 1.4, from

$$\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}, \text{ with } x_0 = 1, y_0 = 0.$$

[*Ans.* $y(1.2) = 0.1402$, $y(1.4) = 0.2705$]

7.28 Use the Runge–Kutta method to approximate y, when $x = 0.1, 0.2, 0.3$, $h = 0.1$, given $x = 0$, when $y = 1$ and $\frac{dy}{dx} = x + y$.

[*Ans.* $y(0.1) = 1.1103$, $y(0.2) = 1.2428$, $y(0.3) = 1.3997$]

7.29 Given

$$\frac{dy}{dx} = x^3 + \frac{y}{2}, \; y(1) = 2$$

Find $y(1.1)$ and $y(1.2)$, using the Runge–Kutta method of fourth-order.

[*Ans.* $y(1.1) = 2.2213$, $y(1.2) = 2.4914$]

7.30 Solve the initial value problem

$$\frac{dy}{dx} = 3x - 4y, \; y(0) = 2$$

at $x = 0.4$, taking $h = 0.2$, by the Runge–Kutta method of fourth-order.

[*Ans.* $y(0.4) = 0.5543$]

7.31 Solve $y' = 1 + xy$, at $x = 0.2, 0.4, 0.6$, given that $y(0) = 2$, by taking $h = 0.2$.

[*Ans.* $y(0.2) = 2.243$, $y(0.4) = 2.589$,
$y(0.6) = 3.072$]

7.32 Using the Runge–Kutta method of fourth-order, find y, for $x = 0.1$, $x = 0.2$, $x = 0.3$, given that

$$\frac{dy}{dx} = xy + y^2, \; y(0) = 1.$$

[*Ans.* $y(0.1) = 1.1168$, $y(0.2) = 1.2740$,
$y(0.3) = 1.488$]

$$y(0.1) = y_1 = y_0 + \Delta y$$
$$= 10 + 21.1719$$
$$= 31.1719$$
$$y(0.1) = 31.1719$$

EXERCISES

7.27 Using the Runge–Kutta method of fourth-order, solve for y at $x = 1.2$, 1.4, from

$$\frac{dy}{dx} = \frac{2xy + e^x}{x^2 + xe^x}, \text{ with } x_0 = 1,\ y_0 = 0.$$

[*Ans.* $y(1.2) = 0.1402$, $y(1.4) = 0.2705$]

7.28 Use the Runge–Kutta method to approximate y, when $x = 0.1, 0.2, 0.3$, $h = 0.1$, given $x = 0$, when $y = 1$ and $\frac{dy}{dx} = x + y$.

[*Ans.* $y(0.1) = 1.1103$, $y(0.2) = 1.2428$, $y(0.3) = 1.3997$]

7.29 Given

$$\frac{dy}{dx} = x^3 + \frac{y}{2},\ y(1) = 2$$

Find $y(1.1)$ and $y(1.2)$, using the Runge–Kutta method of fourth-order.

[*Ans.* $y(1.1) = 2.2213$, $y(1.2) = 2.4914$]

7.30 Solve the initial value problem

$$\frac{dy}{dx} = 3x - 4y,\ y(0) = 2$$

at $x = 0.4$, taking $h = 0.2$, by the Runge–Kutta method of fourth-order.

[*Ans.* $y(0.4) = 0.5543$]

7.31 Solve $y' = 1 + xy$, at $x = 0.2, 0.4, 0.6$, given that $y(0) = 2$, by taking $h = 0.2$.

[*Ans.* $y(0.2) = 2.243$, $y(0.4) = 2.589$, $y(0.6) = 3.072$]

7.32 Using the Runge–Kutta method of fourth-order, find y, for $x = 0.1$, $x = 0.2$, $x = 0.3$, given that

$$\frac{dy}{dx} = xy + y^2,\ y(0) = 1.$$

[*Ans.* $y(0.1) = 1.1168$, $y(0.2) = 1.2740$, $y(0.3) = 1.488$]

7.33 Using the Runge–Kutta method of fourth-order, determine and correct to three decimal places the value of y, at $x = 0.1, 0.2$, if y satisfies the equation

$$\frac{dy}{dx} = x^2 y + x,\ y(0) = 1.$$

[*Ans.* $y(0.1) = 1.005$, $y(0.2) = 1.0224$]

7.34 Using the Runge–Kutta method of fourth-order, find $y(0.2)$, $y(0.4)$ and $y(0.6)$, correct to four decimal places, when

$$\frac{dy}{dx} = y - x^2;\ y(0) = 1.$$

[*Ans.* $y(0.2) = 1.2185$, $y(0.4) = 1.46882$,
$y(0.6) = 1.738$]

7.35 Solve $\dfrac{dy}{dx} = -\dfrac{y}{x} + \dfrac{1}{x^2},\ y(1) = 1.0$

using the fourth-order Runge–Kutta method. Evaluate the value of y, when $x = 1.1$.

[*Ans.* $y(1.1) = 0.9958$]

7.36 Given $\dfrac{dy}{dx} = 1 + y^2$, where $y = 0$, when $x = 0$, find $y(0.2)$, $y(0.4)$ and $y(0.6)$.

[*Ans.* $y(0.2) = 0.2027$, $y(0.4) = 0.4228$,
$y(0.6) = 0.6841$]

7.37 Solve the system of differential equations $\dfrac{dy}{dx} = xz + 1$, $\dfrac{dz}{dx} = -xy$, for $x = 0.3\ (0.3)\ 0.9$ using the fourth-order Runge–Kutta method. Initial values are $x = 0$, $y = 0$, $z = 1$.

[*Ans.* $y(0.3) = 0.3448$, $z(0.3) = 0.99$, $y(0.6) = 0.7738$,
$z(0.6) = 0.9121$, $y(0.9) = 1.255$, $z(0.9) = 0.6806$]

7.38 Given $y' = xyz,\ z' = \dfrac{xy}{z},\ y(1) = \dfrac{1}{3},\ z(1) = 1.$

Evaluate $y(1.1)$, $z(1.1)$, by the second-order Runge–Kutta method.

[*Ans.* $y(1.1) = 0.3704$, $z(1.1) = 1.03615$]

7.39 Solve the system of differential equations at $x = 0.1$, $\dfrac{dx}{dt} = y - t$, $\dfrac{dy}{dt} = x + t$

with $x = 1$, $y = 1$, when $t = 0$, taking $h = 0.1$, by using the Runge–Kutta method.

[*Ans.* $x(0.1) = 1.1003$, $y(0.1) = 1.1102$]

7.40 Solve $\frac{dy}{dx} = -xz, \frac{dz}{dx} = y^2$

at $x = 0.2$ and $x = 0.4$, given that $y(0) = 1$, $z(0) = 1$, by using the Runge–Kutta method.

[*Ans.* $y(0.2) = 0.978$, $z(0.2) = 1.2$
$y(1.4) = 0.9003$, $z(0.4) = 1.382$]

7.41 Solve by the Runge–Kutta method the differential equation $y'' = xy' - 4y$, $y(0) = 3$, $y'(0) = 0$ to approximate $y(0.1)$.

[*Ans.* $y(0.1) = 2.9399$]

7.42 Given $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Find $y(0.1)$ and $y(0.2)$, using the Runge–Kutta method.

[*Ans.* $y(0.1) = 0.995$, $y(0.2) = 0.980$]

7.43 Given $\frac{d^2y}{dx^2} = y^3$, $y(0) = 10$, $y'(0) = 5$

Evaluate $y(0.1)$, using the Runge–Kutta method.

[*Ans.* $y(0.1) = 14.42$]

7.5 PREDICTOR–CORRECTOR METHOD

The methods which we have presented so far are called *Single-step methods*. All of them use information only from the last computed point (x_i, y_i) to compute the next point (x_{i+1}, y_{i+1}). It is possible to improve the efficiency of estimation by using the information at several previous points. Methods that use information from more than one previous points to compute the next point are called *multistep method*. Sometimes, a pair of multistep methods are used, such as one for predicting the value of y_{i+1} and the other for correcting the predicted value of y_{i+1}. Such methods are termed as *predictor–corrector methods.*

The multistep methods can be derived by using Milne's predictor–corrector and Adams–Bashforth predictor–corrector.

In solving the equation $\frac{dy}{dx} = f(x, y)$ with initial values $y(x_0) = y_0$

By using Euler's formula

$$y_{i+1} = y_i + hf'(x_i, y_i),\ i = 0, 1, 2 \ldots \tag{7.26}$$

and this value can be improved by using the improved Euler's method.

$$y_{i+1} = y_i + \frac{h}{2}[f(x_i, y_i) + f(x_{i+1}, y_{i+1})] \tag{7.27}$$

In Eq. (7.27), we calculate the value of y_{i+1}, using Euler's formula (7.26) and use it on the R.H.S. of (7.27) to get the value of y_{i+1}. This y_{i+1} can be used further to predict a value of y_{i+1}, from formula (7.26), and use in (7.27) to correct a value. Hence Eq. (7.26) is a predictor and Eq. (7.27) is a corrector. A predictor formula is used to predict the value of y and a corrector formula is used to correct the error.

7.6 MILNE'S PREDICTOR–CORRECTOR FORMULAE

To solve the equation

$$\frac{dy}{dt} = f(t, y), \text{ with } y(t_0) = y_0 \tag{7.28}$$

By Newton's forward interpolation formula, we have

$$y = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots$$

where $\quad n = \dfrac{t - t_0}{h}, \quad \text{i.e., } t - t_0 = nh$

Changing y to y', we get

$$y' = y_0' + n\Delta y_0' + \frac{n(n-1)}{2!}\Delta^2 y_0' + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0' + \ldots \tag{7.29}$$

On integration between the limits t_0 and t_4, we get

$$\int_{t_0}^{t_0+4h} y' dt = \int_{t_0}^{t_0+4h}\left(y_0' + n\Delta y_0' + \frac{n(n-1)}{2!}\Delta^2 y_0 + \ldots\right)dt$$

$$[y]_{t_0}^{t_0+4h} = \int_{t_0}^{t_0+4h}\left(y_0' + n\Delta y_0' + \frac{n(n-1)}{2!}\Delta^2 y_0' + \ldots\right)dt$$

Put $t = t_0 + nh$

$dt = hdn$

$$y_4 - y_0 = h\int_0^4\left(y_0' + n\Delta y_0' + \frac{n(n-1)}{2}\Delta^2 y_0' + \ldots\right)dn$$

Put $\quad t = t_0$

$t_0 - t_0 = nh$

$n = 0$

$$y_4 - y_0 = h\left[y_0' n + \frac{n^2}{2}\Delta y_0' + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0'}{2}\right.$$

$$+\frac{1}{6}\Delta^3 y_0'\left(\frac{n^4}{4}-n^3+n^2\right)+\ldots\Bigg]_0^4$$

Put $t = t_0 + 4h$

$$t_0 + 4h - t_0 = nh$$

$$n = 4$$

$$= h\left[4y_0' + 8\Delta y_0' + \frac{1}{2}\left(\frac{64}{3}-8\right)\Delta^2 y_0' + \frac{1}{6}\Delta^3 y_0'\,(64-64+16)+\ldots\right]$$

$$= h\left[4y_0' + 8\Delta y_0' + \frac{20}{3}\Delta^2 y_0' + \frac{8}{3}\Delta^3 y_0' + \frac{14}{45}\Delta^4 y_0' + \ldots\right]$$

Since $\Delta = E - 1$, we get

$$y_4 - y_0 = h\left[4y_0' + 8(E-1)\,y_0' + \frac{20}{3}(E-1)^2\,y_0'\right.$$
$$\left.+\frac{8}{3}(E-1)^3\,y_0' + \frac{14}{45}\Delta^4 y_0' + \ldots\right]$$

$$= h\left[4y_0' + 8(y_1' - y_0') + \frac{20}{3}(y_2' - 2y_1' + y_0')\right.$$
$$\left.+\frac{8}{3}(y_3' - 3y_2' + 3y_1' - y_0') + \frac{14}{45}\Delta^4 y_0' + \ldots\right]$$

$$= h\left[4y_0' + 8y_1' - 8y_0' + \frac{20}{3}y_2' - \frac{40}{3}y_1' + \frac{20}{3}y_0' + \frac{8}{3}y_3'\right.$$
$$\left.-\frac{24}{3}y_2' + \frac{24}{3}y_1' - \frac{8}{3}y_0' + \frac{14}{45}\Delta^4 y_0' + \ldots\right]$$

$$= h\left[\left(4-8+\frac{20}{3}-\frac{8}{3}\right)y_0' + \left(8-\frac{40}{3}+8\right)y_1'\right.$$
$$\left.+\left(\frac{20}{3}-8\right)y_2' + \frac{8}{3}y_3' + \frac{14}{45}\Delta^4 y_0 + \ldots\right]$$

$$= h\left[\frac{8}{3}y_1' - \frac{4}{3}y_2' + \frac{8}{3}y_3'\right] + \frac{14h}{45}\Delta^4 y_0' + \ldots$$

$$= \frac{4h}{3}(2y_1' - y_2' + 2y_3') + \frac{14h}{45}\Delta^4 y_0' + \ldots \tag{7.30}$$

In Eq. (7.30), taking into account upto the third-order equation and neglecting $\Delta^4 y_0'$ etc. we get,

$$y_4 - y_0 = \frac{4h}{3}(2y_1' - y_2' + 2y_3')$$

$$y_4 = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad (7.31)$$

This is known as *Milne's predictor formula.*

To get Milne's corrector formula, integrate Eq. (7.27) between the limits t_0 to $t_0 + 2h$.

$$\int_{t_0}^{t_0+2h} y'\,dt = \int_{t_0}^{t_0+2h}\left(y_0' + n\Delta y_0' + \frac{n(n-1)}{2}\Delta^2 y_0' + \ldots\right)dt$$

$$y_2 - y_0 = h\int_0^2\left(y_0' + n\Delta y_0' + \frac{n^2-n}{2}\Delta^2 y_0' + \ldots\right)dn$$

$$= h\left[y_0' n + \frac{n^2}{2}\Delta y_0' + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0' + \ldots\right]_0^2$$

$$= h\left[2y_0' + 2\Delta y_0' + \frac{1}{2}\left(\frac{8}{3} - 2\right)\Delta^2 y_0' - \frac{4}{15}\cdot\frac{1}{24}\Delta^4 y_0' + \ldots\right]$$

$$= h\left[2y_0' + 2(E-1)\,y_0' + \frac{1}{2}\left(\frac{8}{3} - 2\right)(E-1)^2 y_0' - \frac{1}{90}\Delta^4 y_0' + \ldots\right]$$

$$= h\left[2y_0' + 2(y_1' - y_0') + \frac{1}{3}(y_2' - 2y_1' + y_0') - \frac{1}{90}\Delta^4\, y_0' + \ldots\right]$$

$$= h\left[\,2y_0 + 2y_1' - 2y_0' + \frac{y_2'}{3} - \frac{2}{3}y_1' + \frac{y_0'}{3} - \frac{1}{90}\Delta^4\, y_0' + \ldots\right]$$

$$y_2 - y_0 = h\left[\frac{y_0'}{3} + \frac{4}{3}y_1' + \frac{y_2'}{3}\right] - \frac{h}{9}\Delta^4 y_0' + \ldots \qquad (7.32)$$

In Eq. (7.32), we neglect $\Delta^4 y_0'$, then we get

$$y_2 = y_0 + \frac{h}{3}(y_0' + 4y_1' + y_2')$$

This is known as *Milne's corrector formula.*

In general, Milne's predictor–corrector pair can be written as

$$\left.\begin{aligned} p: y_{n+1,\,p} &= y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n') \\ C: y_{n+1,\,c} &= y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}') \end{aligned}\right\} \qquad (7.33)$$

The magnitude of the truncation error in corrector formula is $\frac{1}{90}h\Delta^4 y_0'$, while the truncation error in predictor formula is $\frac{45}{45}h\Delta^4 y_0'$. From this, we notice that the truncation error in corrector formula is less than that in the truncation error in predictor formula.

Using the predicted value y_{n+1}, we calculate the derivative y_{n+1}' from the given differential equation and then we use the corrector formula for the pair (7.33) to have the corrected value of y_{n+1}. This cycle is repeated until we achieve the required accuracy.

Remark: In P–C method, the first-four values of y are given. In case, if it is not mentioned, we can determine these by using the Runge–Kutta method.

EXAMPLE 7.25 Given $y' = \frac{1}{x+y}$, $y(0) = 2$, $y(0.2) = 2.0933$, $y(0.4) = 2.1755$, $y(0.6) = 2.2493$. Find $y(0.8)$, by Milne's predictor–corrector method.

Solution

Here $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$ $h = 0.2$, $y_0 = 2$, $y_1 = 2.0933$, $y_2 = 2.1755$, $y_3 = 2.2493$.

$$\therefore \qquad f(x, y) = y' = \frac{1}{x+y}$$

By Milne's predictor formula, we have

$$y_{n+1,\,p} = y_{n-3} + \frac{4h}{3}\left(2y_{n-2}' - y_{n-1}' + 2y_n'\right)$$

when $n = 3$,

$$y_{4,\,p} = y_0 + \frac{4h}{3}(2y_1' - y_2' + 2y_3') \qquad \text{(i)}$$

Given

$$y' = \frac{1}{x+y} \quad \Rightarrow \quad y_1' = \frac{1}{x_1 + y_1}$$

$$= \frac{1}{0.2 + 2.0933}$$

$$= 0.43605$$

$$y_2' = \frac{1}{x_2 + y_2}$$

$$= \frac{1}{0.4 + 2.1755}$$

$$= 0.38827$$

$$y_3' = \frac{1}{x_3 + y_3}$$

$$= \frac{1}{0.6 + 2.2493}$$

$$= 0.35096$$

Equation (i) $\Rightarrow$

$$y_{4,p} = y_0 + \frac{4}{3} h\,(2y_1' - y_2' + 2y_3')$$

$$= 2 + \frac{4}{3}(0.2)\,[2(0.43605) - 0.38827 + 2(0.35096)]$$

$$= 2 + 0.26657\,[0.8721 - 0.38827 + 0.70192]$$

$$= 2 + 0.26657\,[1.18575]$$

$$= 2.316085$$

$$= 2.3161$$

By Milne's corrector formula, we have

$$= y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \quad \text{(ii)}$$

$$y_4' = \frac{1}{x_4 + y_4}$$

$$= \frac{1}{0.8 + 2.3161}$$

$$= 0.320914$$

Equation (ii) $\Rightarrow$

$$y_{4,c} = 2.1755 + \frac{0.2}{3}[0.38827 + 4(0.35096) + 0.320914]$$

$$= 2.1755 + 0.06657\,[0.38827 + 1.40384 + 0.320914]$$

$$= 2.1755 + 0.06657\,[2.113024]$$

$$= 2.316164$$

$$y(0.8) = 2.3162$$

EXAMPLE 7.26 Solve $y' = \frac{1}{2}(1 + x)\,y^2$, $y(0) = 1$, by the Taylor series method, at $x = 0.2, 0.4, 0.6$, and hence find $y(0.8)$, by Milne's method.

Solution

Here $x_0 = 0$, $y_0 = 1$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$

To find y_1:

$$y' = \frac{1}{2}(1 + x)\, y^2$$

$$= \frac{1}{2}(y^2 + xy^2)$$

$$y_0' = \frac{1}{2}(y_0^2 + x_0 y_0^2)$$

$$= \frac{1}{2}[1^2 + (0)\,(1^2)]$$

$$= 0.5$$

$$y'' = \frac{1}{2}[2yy' + y^2]$$

$$y_0'' = \frac{1}{2}(2y_0\, y_0' + y_0^2)$$

$$= \frac{1}{2}[2(1)\,(0.5) + 1^2]$$

$$= 1$$

$$y''' = \frac{1}{2}[2yy'' + 2(y')^2 + 2yy']$$

$$= \frac{1}{2}[2yy'' + 2(y')^2 + 2yy']$$

$$y_0''' = \frac{1}{2}[2y_0 y_0'' + 2(y_0')^2 + 2y_0 y_0']$$

$$= \frac{1}{2}[2(1)\,(1) + 2(0.5)^2 + 2(1)\,(0.5)]$$

$$= \frac{1}{2}[2 + 0.5 + 1]$$

$$= 1.75$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2}\, y_0'' + \frac{h^3}{3!}\, y_0''' + \ldots$$

$$= 1 + (0.2)\,(0.5) + \frac{(0.2)^2}{2}\,(1) + \frac{(0.2)^3}{6}\,(1.75) + \ldots$$

$$= 1 + 0.1 + 0.02 + 0.002333$$

$$= 1.12233$$

$$y_1 = y(0.2)$$

$$= 1.12233$$

To find y_2:

$$y' = \frac{1}{2}(y^2 + xy^2)$$

$$y_1' = \frac{1}{2}(y_1^2 + x_1 y_1^2)$$

$$= \frac{1}{2}[(1.12233)^2 + (0.2)(1.12233)^2]$$

$$= 0.755774$$

$$y'' = \frac{1}{2}(2yy' + y^2)$$

$$y_1'' = \frac{1}{2}[2y_1 y_1' + y_1^2]$$

$$= \frac{1}{2}[1(1.12233)(0.75577) + (1.12233)^2]$$

$$= 1.47804$$

$$y''' = \frac{1}{2}[2yy'' + 2(y')^2 + 2yy']$$

$$y_1''' = \frac{1}{2}[2y_1' y_1'' + 2(y_1')^2 + 2y_1 y_1']$$

$$= \frac{1}{2}[2(0.755774)(1.47804) + 2(0.75577)^2 + 2(1.12233)(0.755774)]$$

$$= \frac{1}{2}[2.23413 + 1.142377 + 1.69646]$$

$$= 2.536484$$

$$y_2 = y_1 + hy_1' + \frac{h^2}{2}y_1'' + \frac{h^3}{3!}y_1''' + \ldots$$

$$= 1.12233 + (0.2)(0.75577) + \frac{(0.2)^2}{2}(1.47804) + \frac{(0.2)^3}{6}(2.536484) + \ldots$$

$$= 1.12233 + 0.151154 + 0.029561 + 0.00338198$$

$$= 1.30643$$

$$y_2 = y(0.4)$$

$$= 1.30643$$

To find y_3:

$$y' = \frac{1}{2}(y^2 + xy^2)$$

$$y_2' = \frac{1}{2}(y_2^2 + x_2\, y_2^2)$$

$$= \frac{1}{2}[(1.30643)^2 + (0.4)(1.30643)^2]$$

$$= 1.19473$$

$$y'' = \frac{1}{2}(2yy' + y^2)$$

$$y_2'' = \frac{1}{2}(2y_2\, y_2' + y_2^2)$$

$$= \frac{1}{2}[2(1.30643)(1.19473) + (1.30643)^2]$$

$$= 2.414211$$

$$y''' = \frac{1}{2}[2yy'' + 2(y')^2 + 2yy']$$

$$y_2''' = \frac{1}{2}[2y_2y_2'' + 2(y_2')^2 + 2y_2y_2']$$

$$= \frac{1}{2}[2(1.30643)(2.414211) + 2(1.19473)^2 + 2(1.30643)(1.19473)]$$

$$= 5.14221$$

$$y_3 = y_2 + hy_2' + \frac{h^2}{2}y_2'' + \frac{h^3}{3!}y_2''' + \ldots$$

$$= (1.30643) + (0.2)(1.19473) + \frac{(0.2)^2}{2}(2.414211)$$

$$+ \frac{(0.2)^3}{6}(5.14221) + \ldots$$

$$= 1.30643 + 0.23895 + 0.049284 + 0.00585628$$

$$= 1.60052$$

Milne's method:

$x_0 = 0,\ y_0 = 1,\ x_1 = 0.2,\ x_2 = 0.4,\ x_3 = 0.6$

$y_1 = 1.2233,\ y_2 = 1.30643,\ y_3 = 1.60052$

Milne's predictor formula:

$$y_{n+1,\,p} = y_{n-3} + \frac{4}{3}h[2y_{n-2}' - y_{n-1}' + 2y_n']$$

$$y_{4,\,p} = y_0 + \frac{4}{3}h[2y_1' - y_2' + 2y_3']$$

$$y_1' = \frac{1}{2}(y_1^2 + x_1 y_1^2)$$

$$= \frac{1}{2}[(1.2233)^2 + (0.2)(1.2233)^2]$$

$$= 0.755774$$

$$y_2' = \frac{1}{2}(y_2^2 + x_2 y_2^2)$$

$$= \frac{1}{2}[(1.30643)^2 + (0.4)(1.30643)^2]$$

$$= 1.19473$$

$$y_3' = \frac{1}{2}(y_3^2 + x_3 y_3^2)$$

$$= \frac{1}{2}[(1.60052)^2 + (0.6)(1.60052)^2]$$

$$= 2.04933$$

$$y_{4,p} = y_0 + \frac{4}{3}h[2y_1' - y_2' + 2y_3']$$

$$= 1 + \frac{4}{3}(0.2)[2(0.755774) - (1.19473) + 2(2.04933)]$$

$$y_4' = 1 + 1.177461$$

$$= 2.177461$$

Milne's corrector formula:

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y_{n-1}' + 4y_n' + y_{n+1}')$$

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + y_4') \qquad \text{(i)}$$

$$y_4' = \frac{1}{2}(y_4^2 + x_4 y_4^2)$$

$$= \frac{1}{2}[(2.177461)^2 + (0.8)(2.177461)^2]$$

$$= 4.26720$$

Equation (i) $\Rightarrow$

$$y_{4,c} = 1.30643 + \frac{0.2}{3}[(1.19473) + 4(2.04933) + (4.26720)]$$

$$y(0.8) = 1.30643 + 0.910517$$

$$= 2.216947$$

7.7 ADAM–BASHFORTH PREDICTOR–CORRECTOR METHOD

Consider the equation $\frac{dy}{dx} = f(x, y)$, with initial values

$$y(x_0) = y_0 \tag{i}$$

Let us integrate (i), with the interval $[x_0, x_1]$, we get

$$\int_{x_0}^{x_1} \frac{dy}{dx}\, dx = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$\int_{x_0}^{x_1} y'\, dx = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$[y]_{x_0}^{x_1} = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$y_1 - y_0 = \int_{x_0}^{x_1} f(x, y)\, dx$$

By Newton's backward difference interpolation formula,

$$f(x, y) = S_0 + n\nabla S_0 + \frac{n(n+1)}{2}\nabla^2 S_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 S_0 + \ldots$$

where

$$n = \frac{x - x_0}{h} \tag{iii}$$

Substituting Eq. (iii) in Eq. (ii), we get

$$y_1 = y_0 + \int_{x_0}^{x_1}\left[S_0 + n\nabla S_0 + \frac{n(n+1)}{2}\nabla^2 S_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 S_0 + \ldots\right] dx \tag{iv}$$

Equation (iii) $\Rightarrow$

$$x = nh + x_0 \quad \Rightarrow \quad dx = h\,dn$$

when

$$x = x_0 \Rightarrow x_0 - x_0 = nh \Rightarrow n = 0$$

$$x = x_1 \Rightarrow x_1 - x_0 = nh \Rightarrow n = 1$$

Equation (iv) $\Rightarrow$

$$y_1 = y_0 + h\int_0^1\left[S_0 + n\nabla S_0 + \frac{n(n+1)}{2}\nabla^2 S_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 S_0 + \ldots\right] dn$$

$$= y_0 + h\left[nS_0 + \frac{n^2}{2}\nabla S_0 + \frac{1}{2}\left(\frac{n^3}{3} + \frac{n^2}{2}\right)\nabla^2 S_0\right.$$

$$\left.+ \frac{\nabla^3 S_0}{6}\left(\frac{n^4}{4} + \frac{3n^3}{3} + \frac{2n^2}{2}\right) + \ldots\right]_0^1 dn$$

$$y_1 = y_0 + h\left[S_0 + \frac{1}{2}\nabla S_0 + \frac{5}{12}\nabla^2 S_0 + \frac{3}{8}\nabla^3 S_0 + \ldots\right] \qquad \text{(v)}$$

By neglecting the fourth and higher-order differences, Eq. (v) can be written as

$$y_1 = y_0 + h\left[S_0 + \frac{1}{2}(S_0 - S_{-1}) + \frac{5}{12}(S_0 - 2S_{-1} + S_{-2})\right.$$

$$\left.+ \frac{3}{8}(\nabla S_0 - 2\nabla S_{-1} + \nabla S_{-2}) + \ldots\right]$$

$$= y_0 + \frac{h}{24}[55S_0 - 59S_{-1} + 37S_{-2} - 9S_{-3}]$$

This method is known as *Adams–Bashforth predictor formula*. In general, we denote it as

$$y_{n+1,\,p} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3} \ldots] \quad [\because S = y']$$

The corrector formula can be derived in a similar manner by Newton's backward difference formula.

$$f(x, y) = S_1 + n\nabla S_1 + \frac{n(n+1)}{2}\nabla^2 S_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 S_1 + \ldots$$

Equation (ii) $\Rightarrow$

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y)\, dx$$

$$= y_0 + \int_{x_0}^{x_1}\left[S_1 + n\nabla S_1 + \frac{n(n+1)}{2}\nabla^2 S_1 + \frac{n(n+1)(n+2)}{6}\nabla^3 S_1 + \ldots\right] dx$$

Put $$\frac{x - x_1}{h} = n.$$

$$x = nh + x_1$$

$$x = x_0$$

$$x_0 - x_1 = nh$$

$$n = \frac{x_0 - x_1}{h}$$

$$= \frac{-h}{n}$$

$$= -1$$

$$x = x_1$$

$$x_1 - x_1 = nh$$

$$n = 0$$

$$= y_0 + h\left[nS_1 + \frac{n^2}{2}\nabla S_1 + \frac{1}{2}\left(\frac{n^3}{3} + \frac{n^2}{2}\right)\nabla^2 S_1 \right.$$

$$\left. + \frac{1}{6}\left(\frac{n^4}{4} + \frac{3n^3}{3} + \frac{2n^2}{2}\right)\nabla^3 S_1 + \dots \right]_{-1}^{0} dn$$

$$y_1 = y_0 + h\left[S_1 - \frac{\nabla S_1}{2} + \frac{1}{12}\nabla^2 S_1 - \frac{1}{24}\nabla^3 S_1 - \dots \right] \quad \text{(vi)}$$

By neglecting fourth and higher-order differences, Eq. (vi) can be written as

Let us assume $\nabla S_0 = S_0 - S_{-1}$

$$\nabla S_1 = S_1 - S_0$$

$$\nabla^2 S_1 = \nabla S_1 - \nabla S_0$$

$$= S_1 - S_0 - S_0 + S_{-1}$$

$$= S_1 - 2S_0 + S_{-1} \text{ etc.}$$

Equation (iv) $\Rightarrow$

$$y_1 = y_0 + h\left[S_1 - \frac{1}{2}(S_1 - S_0) + \frac{1}{12}(S_1 - 2S_0 + S_{-1}) \right.$$

$$\left. - \frac{1}{24}(S_1 - 3S_0 + 3S_{-1} - S_{-2}) \dots \right]$$

$$= y_0 + \frac{h}{24}[9S_1 + 19S_0 - 5S_{-1} + S_{-2}]$$

In general, we denote it as

$$y_{n+1,\,c} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}] \qquad [\because S = y']$$

This is called *Adams–Bashforth corrector formula.*

Note

(i) To use Adam's method we need atleast four values of y to get the required value of y.

(ii) If the prior values of y are not mentioned, we can find the values by using either Taylor series method or Runge–Kutta method.

EXAMPLE 7.27 Given $y' = y - x^2$, $y(0) = 1$, $y(0.2) = 1.1218$, $y(0.4) = 1.4682$, $y(0.6) = 1.7379$. Estimate $y(0.8)$, by the Adam–Bashforth method.

Solution

Hence $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, $h = 0.2$, $y_0 = 1$, $y_1 = 1.1218$, $y_2 = 1.4682$, $y_3 = 1.7379$.

By Adam–Bashforth predictor formula:

$$y_{n+1,p} = y_n + \frac{h}{24}[55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

when $n = 3$,

$$y_{4,p} = y_3 + \frac{h}{24}[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0] \quad \text{(i)}$$

Given

$$y' = y - x^2$$

$$y_0' = y_0 - x_0^2$$

$$= 1 - 0^2$$

$$= 1$$

$$y_1' = y_1 - x_1^2$$

$$= (1.1218) - (0.2)^2$$

$$= 1.0818$$

$$y_2' = y_2 - x_2^2$$

$$= (1.4682) - (0.4)^2$$

$$= 1.3082$$

$$y_3' = y_3 - x_3^2$$

$$= (1.7379) - (0.6)^2$$

$$= 1.3779$$

Equation (i) $\Rightarrow$

$$y_{4,p} = 1.7379 + \frac{0.2}{24}[55(1.3779) - 59(1.3082) + 37(1.0818) - 9(1)]$$

$$= 1.7379 + 0.009333\,[75.7845 - 77.1838 + 40.0266 - 9]$$

$$= 2.01441$$

By Adam–Bashforth corrector formula:

$$y_{n+1,c} = y_n + \frac{h}{24}[9y'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2}]$$

when $n = 3$

$$y_{4,c} = y_3 + \frac{4}{24}[9y_4' + 19y_3' - 5y_2' + y_1'] \quad \text{(ii)}$$

$$y_4' = y_4 - x_4^2$$
$$= 2.01441 - (0.8)^2$$
$$= 1.37441$$

Equation (ii) $\Rightarrow$

$$y_{4,c} = 1.7379 + \frac{0.2}{24}[9(1.37441) + 19(1.3779) - 5(1.3082) + 1.0818]$$
$$= 1.7379 + \frac{0.2}{24}[12.36969 + 26.1801 - 6.541 + 1.0818]$$
$$= 1.7379 + 0.27575$$
$$= 2.01365$$

EXAMPLE 7.28 Find $y(0.4)$, given $y' = y - \frac{2x}{y}$, $y(0) = 1$, $y(0.1) = 1.0959$, $y(0.2) = 1.1841$, $y(0.3) = 1.2662$, using Milne's method and Adam's method.

Solution

$$y' = y - \frac{2x}{y}$$

Here, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, $y_0 = 1$, $y_1 = 1.0959$, $y_2 = 1.1841$, $y_3 = 1.2662$

By Milne's predictor formula:

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n')$$

when $n = 3$

$$y_{4,p} = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3'] \quad \text{(i)}$$

$$y' = y - \frac{2x}{y}$$

$$y_1' = y_1 - \frac{2x_1}{y_1}$$
$$= 1.0959 - \frac{2(0.1)}{1.0959}$$
$$= 0.913402$$

$$y_2' = y_2 - \frac{2x_2}{y_2}$$

$$= 1.1841 - \frac{2(0.2)}{1.1841}$$

$$= 0.846291$$

$$y_3' = y_3 - \frac{2x_3}{y_3}$$

$$= 1.2662 - \frac{2(0.3)}{1.2662}$$

$$= 0.79234$$

Equation (i) $\Rightarrow$

$$y_{4,p} = 1 + \frac{4(0.1)}{3}[2(0.913402) - 0.846291 + 2(0.79234)]$$

$$= 1 + 0.13333\,[1.926804 - 0.846291 + 1.58468]$$

$$= 1 + 0.35535$$

$$= 1.35535$$

By Milne's corrector formula:

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

$$y_{4,c} = y_2 + \frac{h}{3}(y_2' + 4y_3' + 4y_4') \qquad \text{(ii)}$$

$$y_4' = y_4 - \frac{2x_4}{y_4}$$

$$= 1.35535 - \frac{2(0.4)}{1.35535}$$

$$= 0.765097$$

Equation (ii) $\Rightarrow$

$$y_{4,c} = 1.1841 + \frac{0.1}{3}[0.846291 + 4(0.79234) + 0.765097]$$

$$= 1.1841 + 0.03333\,[0.846291 + 3.16936 + 0.765097]$$

$$y(0.8) = 1.1841 + 0.15934$$

$$= 1.34344$$

Adam's predictor method:

$$y_{n+1,p} = y_n + \frac{h}{24}[55y_n' - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3}]$$

$$y_{4,p} = y_3 + \frac{h}{24}[55y_3' - 59y_2' + 37y_1' - 9y_0']$$

$$= 1.2662 + \frac{0.1}{24}[55(0.79234) - 59(0.846291) + 37(0.913402) - 9]$$

$$= 1.2662 + 0.00416657\,[43.5787 - 49.931169 + 33.795874 - 9]$$

$$= 1.2662 + 0.07695$$

$$= 1.34315$$

Adam's corrector method:

$$y_{n+1,c} = y_n + \frac{h}{24}[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}']$$

$$y_{4,c} = y_3 + \frac{h}{24}[9y_4' + 19y_3' - 5y_2' + y_1']$$

$$= 1.2662 + \frac{0.1}{24}[9(0.765097) + 19(0.79234) - 5(0.846291) + 0.913402]$$

$$= 1.2662 + 0.00416657\,[6.985873 + 15.05446 - 4.231455 + 0.913402]$$

$$y(0.8) = 1.2662 + 0.07901$$

$$= 1.3442$$

EXERCISES

7.44 Given $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)\,y^2$ and $y(0) = 1$, $y(0.1) = 1.06$, $y(0.2) = 1.12$, $y(0.3) = 1.21$. Evaluate $y(0.4)$, by Milne's predictor–corrector method.

[*Ans.* $y_p(0.4) = 1.2772$, $y_c(0.4) = 1.2797$]

7.45 Using Milne's predictor and corrector formulae, find $y(4.4)$, given $5xy' + y^2 - 2 = 0$, $y(4) = 1$, $y(4.1) = 1.0049$, $y(4.2) = 1.0097$ and $y(4.3) = 1.0143$.

[*Ans.* $y_c(4.4) = 1.01874$, $y_p(4.4) = 1.01897$]

7.46 Given the differential equation $y' = 1 + y^2$, with $y(0) = 0$. Compute $y(0.8)$ and $y(1.0)$, by Milne's method.

[*Ans.* $y(0.8) = 1.0294$, $y(1.0) = 1.5550$]

7.47 Using the Taylor series method, solve $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$ at $x = 0.1$, 0.2 and 0.3, continue the solution at $x = 0.4$, by Milne's predictor–corrector method.

[*Ans.* $y_c(0.4) = 1.8369$, $y_p(0.4) = 1.8329$]

7.48 If $\frac{dy}{dx} = 2e^x - y$, $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$ and $y(0.3) = 2.090$. Find $y(0.4)$ and $y(0.5)$, correct to three decimal, applying Milne's predictor–corrector method.

[*Ans.* $y_p(0.4) = 2.1623$, $y_c(0.4) = 2.1621$, $y_p(0.5) = 2.2551$, $y_c(0.5) = 2.2546$]

7.49 The differential equation $\frac{dy}{dx} = x^2 + \frac{y}{2}$, satisfied by $y(1) = 2$, $y(1.1) = 2.2156$, $y(1.2) = 2.4649$, $y(1.3) = 2.7514$, using Milne's method, find $y(1.4)$, correct to four decimal places.

[*Ans.* $y_p(1.4) = 3.079$, $y_c(1.4) = 3.0794$]

7.50 Compute $y(0.6)$, by Milne's method, given $y' = x + y$, $y(0) = 1$, with $h = 0.2$. Obtain the required data by the Taylor series method.

[*Ans.* $y(0.6) = 2.0442$]

7.51 Given $y' = x(x^2 + y^2)e^{-x}$, $y(0) = 1$. Find y, at $x = 0.1$, 0.2 and 0.3, by the Taylor method. Compute $y(0.4)$, by Milne's method.

[*Ans.* $y(0.1) = 1.0047$, $y(0.2) = 1.01813$, $y(0.3) = 1.03975$, $y(0.4) = 1.0709$]

7.52 Solve and get $y(2)$, given $\frac{dy}{dx} = \frac{1}{2}(x + y)$, $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$, by Adam's method.

[*Ans.* $y_p(2) = 6.8708$, $y_c(2) = 6.8731$]

7.53 Using Adam's method, find $y(0.4)$, given $\frac{dy}{dx} = \frac{1}{2}xy$, $y(0) = 1$, $y(0.1) = 1.01$, $y(0.2) = 1.022$, $y(0.3) = 1.023$.

[*Ans.* $y_p(0.4) = 1.0408$, $y_c(0.4) = 1.0410$]

7.54 Find $y(0.1)$, $y(0.2)$, $y(0.3)$, from $\frac{dy}{dx} = xy + y^2$, $y(0) = 1$, by using the Runge–Kutta method and hence obtain $y(0.4)$, using Adam's method.

[*Ans.* $y(0.1) = 1.1169$, $y(0.2) = 1.2774$, $y(0.3) = 1.5041$, $y_p(0.4) = 1.8341$, $y_c(0.4) = 1.8389$]

7.55 Obtain $y(0.6)$, given $\frac{dy}{dx} = x + y$, $y(0) = 1$, with $h = 0.2$, by Adam's method.

[*Ans.* $y_p(0.6) = 2.0437$, $y_c(0.6) = 2.0439$]

7.56 Using Adam's predictor–corrector method, find $y(1.4)$, if y satisfies $\frac{dy}{dx} = \frac{1 - xy}{x^2}$ and $y(1) = 1$, $y(1.1) = 0.996$, $y(1.2) = 0.986$, $y(1.3) = 0.972$

[*Ans.* $y(1.4) = 0.9493$]

7.57 Find $y(0.4)$ and $y(0.5)$, from $\frac{dy}{dx} = 3e^x + 2y$ with $x_0 = 0$, $y_0 = 0$, using Adam's–Bashforth formula.

[*Ans.* $y(0.4) = 2.20$, $y(0.5) = 3.20$]

7.58 Obtain the solution of the initial value problem, by Adam's method, at $y(1.4)$, given $\frac{dy}{dx} - x^2 y = x^2$, $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$.

[*Ans.* $y(1.4) = 2.572$]