Chapter 5

Numerical Integration

5.1 INTRODUCTION

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to do so? Among the most common examples are finding the velocity of a body from acceleration functions, and displacement of a body from velocity data. Throughout the engineering fields, there are (what sometimes seems like) countless applications for integral calculus. Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods have been developed to find the integral.

The purpose of this chapter is to develop the basic principles of **numerical integration,** which are used to obtain approximate results for some definite integrals. We restrict ourselves to define integrals of the form:

$$I = \int_a^b f(x)dx \qquad \text{.... (5.1)}$$

where a and b are finite and f(x) is a continuous function of x for $a \le x \le b$. Some examples of definite integrals are,

$$\int_2^4 x\,dx\,,\quad \int_{-1}^{\frac{1}{2}} x^{-x^2}\,dx\,,\quad \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin^2 x}\,,\quad \int_0^2 \frac{e^{2x}}{1+x^2}dx\,,\ \text{etc.}$$

The indefinite integrals are included among the solutions of ordinary differential equations discussed in Chapter 6. The value of I is interpreted as an area bounded by the curve $y = f(x)$, the x-axis, and the two ordinates at $x = a$ and $x = b$; I represents a number which is interpreted as an **area**. The numerical integration is often referred to as **quadrature** (also **mechanical quadrature**) which simply means working out an area.

The use of numerical integration becomes necessary when either the function f(x) cannot be integrated analytically or the analytical solution of the integral presents such difficulties that it is faster to find a numerical solution or when the values of functions are available only in a tabular form but no information is available about the function itself.

There are several methods available in the literature for numerical integration, but the most commonly used methods may be classified into the following two groups:

(a) **Newton-Cotes formulas** that employ functional values at equally-spaced data-points, and

(b) The **Gaussian quadrature formulas** that employ functional values at equally spaced data-points determined by certain properties of orthogonal polynomials.

We shall mostly confine ourselves to the Newton-Cotes formulas, which can be derived by integrating one of the interpolation formulas.

We now approach the object of numerical integration: the goal is to approximate the definite integral of f(x) over the interval [a, b] by evaluating f(x) at a definite number of sample points.

Since integration is the inverse of differentiation, we use the following relation for evaluating integrals:

$$\int_{x_0}^{x_n} f(x)\,dx = h\int_0^p f_p\,dp \qquad \ldots (5.2)$$

Integration formulas are used to derive the predictor-corrector formulas for solving differential equations (see Chapter 6).

5.2 DERIVATION OF INTEGRATION FORMULA BASED ON NEWTON'S FORWARD DIFFERENCES

Integrating Newton's forward difference formula (3.2), we get:

$x_p = x_0 + ph$ $\quad p = \frac{x_p - x_0}{h}$

$$\int_{x_0}^{x_n} f(x)\,dx = h\int_0^p f_p\,dp$$

$x_n = x_0 + nh$

$x = x_0 + ph$

$dx = h\,dp$

$$= h\int_0^p \Big[f_0 + p\Delta f_0 + \frac{1}{2}(p^2 - p)\Delta f_0 + \frac{1}{6}(p^3 - 3p^2 + 2p)\Delta^3 f_0 + \frac{1}{24}(p^4 - 6p^3 + 11p^2 - 6p)\Delta^4 f_0 + \cdots \Big]$$

$$= h\left\{ pf_0 + \frac{1}{2}p^2\Delta f_0 + \frac{1}{2}\left[\frac{p^3}{3} - \frac{p^2}{2}\right]\Delta^2 f_0 + \frac{1}{6}\left[\frac{p^4}{4} - p^3 + \frac{p^2}{2}\right]\Delta^3 f_0 + \frac{1}{24}\left[\frac{p^5}{5} - \frac{6p^4}{4} + \frac{11p^3}{3} - 3p^2\right]\Delta^4 f_0 + \cdots \right\} \qquad \ldots (5.3)$$

From (5.3), we can derive several other well-known formulas. For example imposing the limits (0, 1), we get the formula due to **Laplace**:

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx$$

$$= h\int_0^1 f_p\,dp$$

$$= h\left\{ f_0 + \frac{1}{2}\Delta f_0 - \frac{1}{12}\Delta^2 f_0 - \frac{1}{24}\Delta^3 f_0 - \frac{19}{720}\Delta^4 f_0 + \cdots \right\} \qquad \ldots (5.4)$$

5.3 THE NEWTON-COTES FORMULAS

The Newton-Cotes formulas can be derived from the relations (5.3) and (5.4). The following formulas are worth studying:

(a) Trapezoidal rule

(b) Simpson's $\frac{1}{3}$rd rule

(c) Simpson's $\frac{3}{8}$th rule

(d) Boole's rule

(e) Weddle's rule.

The above formulas are simple and some of them are widely used in practice. The use of a particular formula depends upon the nature of the problem to be tackled and also to some extent upon the accuracy desired in the final answers. These rules basically replace f(x) to approximate polynomials, which are then integrated analytically. If the degree of a polynomial is too high, errors due to round-off and local irregularities can cause problems. That is why it is only the lower-degree Newton-Cotes formulas that are often used.

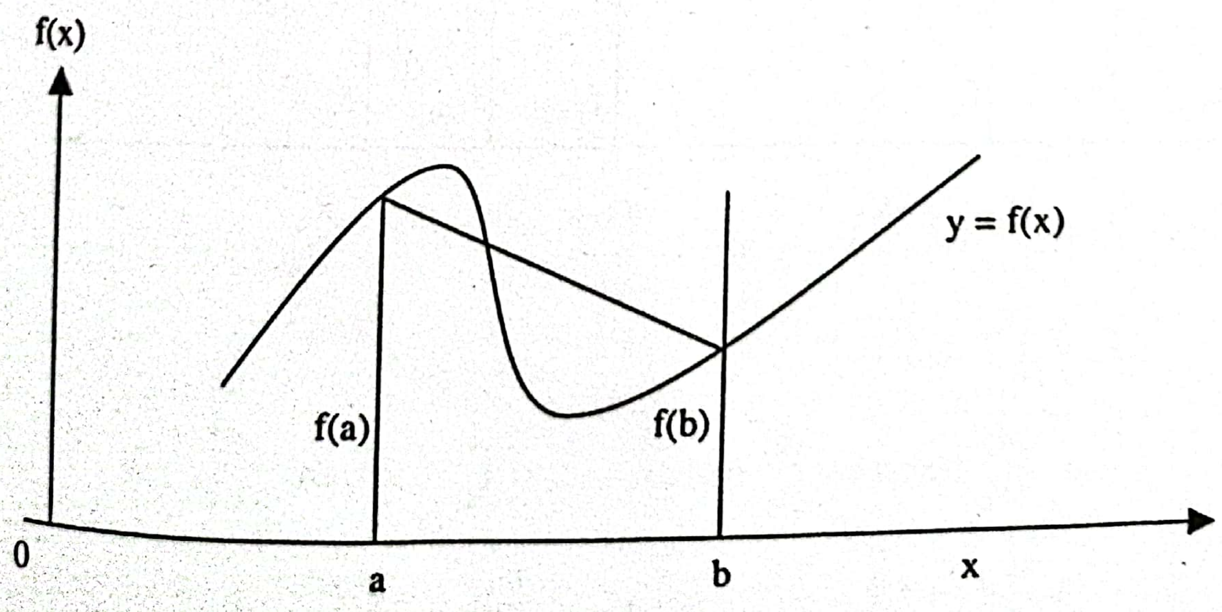
Let us describe the above mentioned formulas one by one.

5.3.1 Trapezoidal Rule

Truncating (5.3) after the first-order differences, we get,

$$I_T = \int_{x_0}^{x_1} f(x)\,dx = h\left[pf_0 + \frac{p^2}{2}\Delta f_0\right]_0^1$$

For fitting a straight line, we use the limits (0, 1), in other words, the integration is over one interval (or two ordinates or two terms):



$$I_T = \int_{x_0}^{x_1} f(x)\,dx$$

$$= h\left[pf_0 + \frac{p^2}{2}\Delta f_0\right]_0^1$$

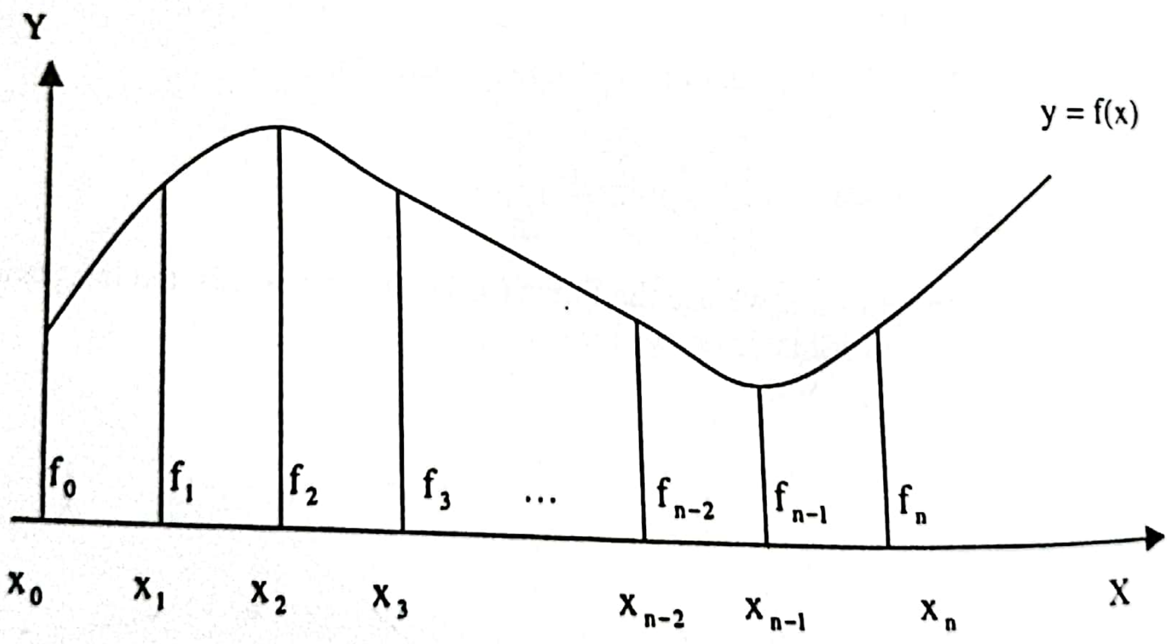
$$= h\left[f_0 + \frac{1}{2}\Delta f_0\right]$$

$$= h\left[f_0 + \frac{1}{2}(f_1 - f_0)\right]$$

$$= \frac{h}{2}[f_0 + f_1] \qquad \text{... (5.5)}$$

This is called the **trapezoidal** (or **trapezium**) **rule.** Between x_0 and x_1, the function f(x) is approximated as straight line and the area under the curve representing f(x) is considered to be the area under the straight line.

If n intervals are used, the formula (5.5) is extended as follows to calculate total area between $x_0 = x$ and $x = x_n$.



$$I_T = \int_{x_0}^{x_n} f(x)\,dx$$

$$= \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= \frac{h}{2}[f_0 + f_1] + \frac{h}{2}[f_1 + f_2] + \ldots + \frac{h}{2}[f_{n-1} + f_n]$$

$$= \frac{h}{2}[f_0 + 2(f_1 + f_2 + f_3 + \ldots + f_{n-1}) + f_n]$$

$$= \frac{h}{2}\left[(f_0 + f_n) + 2\sum_{i=1}^{n-1} f_i\right] \qquad \ldots (5.6)$$

The above formula is the **trapezoidal rule** for n intervals. It is also called **multiple-segment** or **composite** trapezoidal rule. Note that all functional values except the first and the last are multiplied by 2. The total area under the curve can, therefore, be approximated by the sum of areas of n trapezia.

Trapezoidal rule is not so accurate, but it is simple and moreover can be used for any number of intervals. Approximations to the integrals can be improved to some extent making the step size h, smaller and smaller (in other words, by increasing the number of intervals). One of the most difficult problems in quadrature is to decide how large n should be taken to achieve the desired accuracy. It is sufficient to say at this point that the error tends to be zero as n tends to infinity.

5.3.2 Simpson's $\frac{1}{3}$rd rule

If we truncate the expression in (5.3) after the second-order differences and impose limits (0, 2), we have,

$$I_S = \int_{x_0}^{x_2} f(x)\,dx$$

$$= h\left\{pf_0 + \frac{p^2}{2}\Delta f_0 + \frac{1}{2}\left[\frac{p^3}{3} - \frac{p^2}{2}\right]\Delta^2 f_0\right\}_0^2$$

$$= h\left[2f_0 + 2\Delta f_0 + \frac{1}{3}\Delta^2 f_0\right]$$

$$= h\left[2f_0 + 2(f_1 - f_0) + \frac{1}{3}(f_2 - 2f_1 + f_0)\right]$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] \qquad \ldots (5.7)$$

The above relation is called **Simpson's $\frac{1}{3}$rd rule** or simply **Simpson's rule.** If n intervals (should be even in numbers) are to be used, we have the following general expression for composite Simpson's rule:

$$I_S = \int_{x_0}^{x_n} f(x)\,dx$$

$$= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx + \ldots + \int_{x_{n-2}}^{x_n} f(x)\,dx$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \frac{h}{3}[f_4 + 4f_5 + f_6] + \dots$$

$$+ \frac{h}{3}[f_{n-2} + 4f_{n-1} + f_n]$$

$$= \frac{h}{3}[(f_0 + f_n) + 4(f_1 + f_3 + f_5 + \dots + f_{n-1}) + 2(f_2 + f_4 + f_6 + \dots + f_{n-2})]$$

$$= \frac{h}{3}\left[(f_0 + f_n) + 4\sum_{\substack{i=1 \\ i=odd}}^{n-1} f_i + 2\sum_{\substack{i=2 \\ i=even}}^{n-2} f_i\right] \quad \dots (5.8)$$

It is obvious from (5.8) that with the exception of the first and the last functional values, all odd functional values are multiplied by 4 and all even functional values are multiplied by 2. The formula is used only when n is even. Simpson's rule gives a more accurate result than the trapezoidal rule and is easier to program and manipulate as well.

5.3.3 Combination of Trapezoidal and Simpson's Rules

Since Simpson's $\frac{1}{3}$rd rule is used when n is even, but if, in some cases, the number of intervals n is odd, we can still find the solution.

For example, we have the following data:

x	f(x)
0	f_0
1	f_1
2	f_2
3	f_3
4	f_4
5	f_5
6	f_6
7	f_7

If we use the values against the points $x = 0$ to $x = 6$ (i.e. $n = 6$) in Simpson's $\frac{1}{3}$rd rule, we get the solution.

$$I_S = \frac{h}{3}[(f_0 + f_6) + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4)]$$

We add to this the result obtained using trapezoidal rule for $x = 6$ to $x = 7$.

$$I_T = \frac{h}{2}[f_6 + f_7]$$

Result = $I_S + I_T$, which is the integral over the entire range.

Consequently, we can also select the first interval to integrate by the Trapezoidal rule and the remainder by Simpson's $\frac{1}{3}$rd rule. However, this criterion seems to work for choosing the end for applying the Trapezoidal rule. There may be a little difference in the two results we obtain but the former is slightly better.

5.3.4 Simpson's $\frac{3}{8}$th Rule

If we truncate the expression in (5.3) after the third-order differences and impose the limits (0, 3), we have,

$$I_{SR} = \int_{x_0}^{x_3} f(x)\,dx$$

$$= \int_0^3 f_p\,dp$$

$$= \left[pf_0 + \frac{1}{2}p^2\Delta f_0 + \frac{1}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}\right)\Delta^2 f_0 + \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 f_0\right]_0^3$$

Simplifying and rearranging terms, we get,

$$I_{SR} = \frac{3h}{8}[f_0 + 3(f_1 + f_2) + f_3] \qquad \ldots (5.9)$$

This is called **Simpson's $\frac{3}{8}$th rule.**

Extending the formula (5.9) upto n intervals, we get,

$$I_{SR} = \frac{3h}{8}[f_0 + 3(f_1 + f_2) + 2f_3 + 3(f_4 + f_5) + 2f_6 + 3(f_7 + f_8) + \ldots$$
$$+ 3(f_{n-2} + f_{n-1}) + f_n] \qquad \ldots (5.10)$$

The above formula does not yield more accurate result than the simple Simpson's rule. One useful application is the calculation of a tabulated function with an odd number of panels by doing the first (or the last) three with the $\frac{3}{8}$th rule and the rest with the $\frac{1}{3}$rd rule. There may be a little difference, although the former is slightly better.

5.3.5 Boole's Rule

If we truncate the expression in (5.3) after fourth-order differences and impose the limits (0, 4), we have,

$$I_B = \int_{x_0}^{x_4} f(x)\,dx$$

$$= \int_0^4 f_p\,dp$$

$$= \frac{2h}{45}\{7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4\} \qquad \ldots (5.11)$$

This is called the **Boole's rule.**

5.3.6 Weddle's Rule

If we truncate the expression in (5.3) after sixth differences and impose the limits (0, 6), we have,

$$I_W = \int_{x_0}^{x_6} f(x)\,dx$$

$$= \int_0^6 f_p\,dp$$

$$= \frac{3h}{10}\{f_0 + 5f_1 + f_2 + 6f_3 + f_4 + 5f_5 + f_6\} \qquad \ldots (5.12)$$

This is called the **Weddle's rule.**

In order to illustrate the above methods, we consider the following simple example.

Example 1 The following table represents the values of sine function:

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
f(x)	0.0000	0.0998	0.1987	0.2955	0.3894	0.4794	0.5646

Compute $\int_0^{0.6} f(x)\,dx$ based on,

(a) Trapezoidal rule, (b) Simpson's $\frac{1}{3}$rd rule, (c) Simpson's $\frac{3}{8}$th rule,

(d) Boole's rule, and (e) Weddle's rule.

Solution As the number of functional values is seven, the number of intervals, $n = 6$ and $h = 0.1$.

(a) **Trapezoidal Rule**

$$I_T = \frac{h}{2}[f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5) + f_6]$$

$$= \frac{0.1}{2}\{0.0000 + 2(0.0998 + 0.1987 + 0.2955 + 0.3894 + 0.4794) + 0.5646\}$$

$$= \frac{0.1}{2} \times 3.4902 = 0.1745$$

(b) **Simpson's $\frac{1}{3}$ rd Rule**

$$I_S = \frac{h}{3}[f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6]$$

$$= \frac{0.1}{3}\{0.0000 + 4(0.0998 + 0.2955 + 0.4794) + 2(0.1987 + 0.3894) + 0.5646\}$$

$$= \frac{0.1}{3} \times 5.2396 = 0.1747$$

(c) **Simpson's $\frac{3}{8}$ th Rule**

$$I_{SR} = \frac{3h}{8}\{f_0 + 3(f_1 + f_2) + 2f_3 + 3(f_4 + f_5) + f_6\}$$

$$= 3 \times \frac{0.1}{8}\{0.0000 + 3(0.0998 + 0.1987) + 2 \times 0.2955 + 3(0.3894 + 0.4794)$$
$$+ 0.5646\}$$

$$= 3 \times \frac{0.1}{8} \times 4.6575 = 0.1747$$

(d) **Boole's Rule**

$$I_B = \frac{2h}{45}\{7(f_0 + f_6) + 32(f_1 + f_3 + f_5) + 12(f_2 + f_4)\}$$

$$= 2 \times \frac{0.1}{45}\{7(0.0000 + 0.5646) + 32(0.0998 + 0.2955 + 0.4794)$$
$$+ 12(0.1987 + 0.3894)\}$$

$$= \frac{0.2}{45}[3.9522 + 27.9904 + 7.0572]$$

$$= \frac{0.2}{45} \times 38.9998 = 0.1733$$

(e) Weddle's Rule

$$I_W = 3\times\frac{0.1}{10}[(0.0000+0.5646)+5(0.0998+0.4794)+(0.1987+0.3894)$$
$$+6\times 0.2955]$$

$$=\frac{0.3}{10}[0.5646+2.8960+0.5881+1.7730]$$

$$=\frac{0.3}{10}\times 5.8217 = 0.1747$$

Example 2 Given the following integral:

$$\int_0^2 \frac{e^{2x}}{1+x^2}$$

Use Simpson's $\frac{1}{3}$rd rule to evaluate the integral with $n = 8$.

Solution $n = 8,\quad a = 0, b = 2$

$$h = \frac{b-a}{n} = \frac{2-0}{8} = 0.25$$

Table of Values:

x	$f=\frac{e^{2x}}{1+x^2}$
$x_0=0$	$f_0= 1.0000$
$x_0=0.25$	$f_1= 1.5500$
$x_1=0.50$	$f_2 = 2.1746$
$x_2=0.75$	$f_3 = 2.8683$
$x_3= 1.00$	$f_4 = 3 .6945$
$x_4= 1.25$	$f_5 = 4.7542$
$x_5= 1.50$	$f_6 = 6.1802$
$x_6= 1.75$	$f_7 = 12.8132$
$x_7= 2.00$	$f_8 = 10.9197$

Using Simpson's $\frac{1}{3}$ rd Rule:

$$I_S = \frac{h}{3}[(f_0 + f_8) + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6)]$$

$$= \frac{0.25}{3}[(0.0000 + 10.9197) + 4(1.5500 + 2.8683 + 4.7542 + 11.8132)$$
$$+ 2(2.1746 + 3.6945 + 6.02)]$$

$$= \frac{0.25}{3}[11.9197 + 83.9428 + 24.0986] = 9.9968$$

5.4 ESTIMATION OF ERRORS IN SOME NEWTON-COTES FORMULAS

In this section, we explain ways of analysing errors in the Trapezoidal and Simpson rules:

Let $F(x) = \int f(x)\,dx$ .... (5.13)

Then $I = \int_{x_0}^{x_0+h} f(x)\,dx$

$= F(x_0 + h) - F(x_0)$ .... (5.14)

5.4.1 Error in Trapezoidal Rule

From (5.5), we have,

$$I_T = \frac{h}{2}[f(x_0) + f(x_0 + h)]$$

The error E_T in the Trapezoidal rule can be defined by the following relation:

$$E_T = I - I_T$$

$$= [F(x_0 + h) - F(x_0)] - \frac{h}{2}[f(x_0) + f(x_0 + h)] \quad .... (5.15)$$

Expanding terms $F(x_0 + h)$ and $f(x_0 + h)$ in (5.15) by Taylor series and setting,

$F'(x_0) = f(x_0)$

$F''(x_0) = f'(x_0)$, etc., we get,

$$E_T = \left[F(x_0) + hF'(x_0) + \frac{h^2}{2!}F''(x_0) + \frac{h^3}{3!}F'''(x_0) + \cdots - F(x_0)\right]$$

$$-\frac{h}{2}\left[f(x_0) + f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \cdots\right]$$

$$= h\left[f(x_0) + \frac{h}{2}f'(x_0) + \frac{h^2}{6}f''(x_0) + \cdots\right]$$

$$-\frac{h}{2}\left[2f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \cdots\right]$$

$$= \frac{-h^3}{12}f''(x_0) \qquad \ldots (5.16)$$

The above error is the error in a single step and is called the **local error.** When using Trapezoidal rule over n intervals, the error is as follows:

$$E_T = \frac{-nh^3}{12}f''(Z) \qquad \ldots (5.17(a))$$

$$= \frac{-(b-a)h^2}{12}f''(Z) \qquad \ldots (5.17(b))$$

where $a \le Z \le b$, and $h = \frac{(b-a)}{n}$.

The above error is called the **global error,** which is the total error.

In order to obtain the upper bound, choose Z in (a, b) such that $f''(Z)$ is the largest in magnitude; similarly lower bound can be obtained choosing Z in (a, b) such that $f''(Z)$ is the smallest in magnitude. It follows from (5.17) that the error in the Trapezoidal rule is of the order h^2 and is conventionally written as "error $O(h^2)$". Its significance lies in the fact that as h→0, the error falls quadratically with h.

5.4.2 Error in Simpson's $\frac{1}{3}$rd Rule

The error in Simpson's rule is derived in the following manner:

Let $I = \int_a^{a+2h} f(x)\,dx$

$$= F(a+2h) - F(a) \qquad \ldots (5.18)$$

From (5.7), we have

$$I_S = \frac{h}{3}[f(a)+4f(a+h)+f(a+2h)]$$

The Error in Simpson's rule can be defined by,

$$E_S = I - I_S$$

$$E_S = [F(a+2h) - F(a)] - \frac{h}{3}[f(a)+4f(a+h)+f(a+2h)] \qquad \ldots (5.19)$$

Expanding terms f(a + h), f(a + 2h) and F(a + 2h) in (5.19), we get,

$$\begin{aligned} E_S = &\Big[F(a) + 2hF'(a) + \frac{(2h)^2}{2!}F''(a) + \frac{(2h)^3}{3!}F'''(a) + \frac{(2h)^4}{4!}F^{(iv)}(a) \\ &+ \frac{(2h)^5}{5!}F^{(v)}(a) + \cdots - F(a)\Big] - \frac{h^3}{3!}\Big[f(a) + 4\{f(a) + hf'(a) \\ &+ \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}F'''(a) + \frac{h^4}{4!}F^{(iv)}(a) + \cdots\} + \{f(a) + 2hf'(a) \\ &+ \frac{(2h)^2}{2!}f''(a) + \frac{(2h)^3}{3!}f'''(a) + \frac{(2h)^4}{4!}f^{(iv)}(a) + \cdots\}\Big] \end{aligned}$$

$$\begin{aligned} = &\Big[2hF'(a) + (2h)^2F''(a) + \frac{4h^3}{3}F'''(a) + \frac{2h^4}{3}F^{(iv)}(a) + \frac{4}{15}h^5F^{(v)}(a) + \cdots\Big] \\ &- \frac{h}{3}f(a) - \frac{4h}{3}f(a) - \frac{4h^2}{3}f'(a) - \frac{2h^3}{3}f''(a) - \frac{2h^4}{9}f'''(a) - \frac{h^5}{18}f^{(iv)}(a)] \\ &- \cdots - \frac{h}{3}f(a) - \frac{2h^2}{3}f'(a) - \frac{2h^3}{3}f''(a) - \frac{4h^4}{9}f'''(a) - \frac{2h^5}{9}f^{(iv)}(a) - \cdots \end{aligned} \qquad \ldots (5.20)$$

Let $F'(a) = f(a)$

$F''(a) = f'(a)$

$F'''(a) = f''(a)$, and so on.

Simplifying (5.20), we get,

$$\begin{aligned} E_S = &\ 2hf(a) + 2h^2f'(a) + \frac{4h^3}{3}f''(a) + \frac{2h^4}{3}f'''(a) + \frac{4h^5}{15}f^{(iv)}(a) - 2hf(a) \\ &- 2h^2f'(a) - \frac{4h^3}{3}f''(a) - \frac{2h^4}{3}f'''(a) - \frac{4h^5}{18}f^{(iv)}(a) \\ = &-\frac{h^5}{90}f^{(iv)}(a)\cdots \end{aligned} \qquad \ldots (5.21)$$

The error in (5.21) is called the **local error**. If we integrate over n intervals, we get the **global error** and is as follows,

$$E_S = \frac{-nh^5}{90} f^{(iv)}(Z) \quad \text{... (5.22(a))}$$

$$E_S = \frac{-(b-a)h^4}{90} f^{(iv)}(Z) \quad \text{... (5.22(b))}$$

where $a \le Z \le b$.

The error in Simpson's $\frac{1}{3}$rd rule of the order of h^4, i.e., "$0(h^4)$". This is equivalent to saying that for h (small enough), the error is proportional to h^4.

Example 3 Evaluate $\int_1^2 \sqrt{x}\, dx$, using

(a) Trapezoidal and Simpson's $\frac{1}{3}$rd rules, taking h = 0.25 in each case. Write computer programs in each case also.

(b) Calculate exact value to 4 dp. Compare the results obtained in (a) above with the exact value.

(c) Compute the error bounds in each case.

Solution Tabular Values:

x	$f(x) = \sqrt{x}$
1.00	1.0000
1.25	1.1180
1.50	1.2247
1.75	1.3229
2.00	1.4142

(a) Trapezoidal Rule

$$I_T = \int_1^2 \sqrt{x}\, dx$$

$$= \frac{h}{2}[(f_0 + f_4) + 2(f_1 + f_2 + f_3)]$$

$$= \frac{0.25}{2}[(1.0000 + 1.4142) + 2(1.1180 + 1.2247 + 1.3229)]$$

$$= \frac{0.25}{2} \times 9.7454 = 1.2182$$

(b) Simpson's $\frac{1}{3}$ rd Rule

$$I_S = \frac{h}{3}[(f_0 + f_4) + 4(f_1 + f_3) + 2f_2]$$

$$= \frac{0.25}{3}[(1.0000 + 1.4142) + 4(1.1180 + 1.3229) + 2 \times 1.2247]$$

$$= \frac{0.25}{3}[2.4142 + 9.7636 + 2.4494]$$

$$= \frac{0.25}{3} \times 14.6272 = 1.2189$$

5.5.2 Romberg Integration

Although the Trapezoidal rule is the easiest Newton-Cotes formula to apply, it lacks the degree of accuracy generally required. **Romberg Integration** is a method that has wide application because it improves the approximation fairly rapidly. Romberg integration is mostly designed for cases where the function to be integrated is known. This is because knowledge of the function permits the evaluation required for the initial implementations of the Trapezoidal rule.

Let f(x) be known either explicitly or as a tabulation of equispaced data:

x	x_0	x_1	x_2	...	x_n
f(x)	f_0	f_1	f_2	...	f_n

The first step in Romberg's method is to define a series of sums: I_{11}, I_{12}, I_{13}, ..., where

$$I_{11} = \frac{1}{2}(f_0 + f_n);\ h' = \frac{(b-a)}{n}, \quad \text{where } n = 1.$$

$$I_{12} = \left[I_{11} + f\left(a + \frac{h'}{2}\right)\right]$$

$$I_{13} = \left[I_{12} + f\left(a + \frac{h'}{4}\right) + f\left(a + \frac{3h'}{4}\right)\right]$$

$$I_{14} = \left[I_{13} + f\left(a + \frac{h'}{8}\right) + f\left(a + \frac{3h'}{8}\right) + f\left(a + \frac{5h'}{8}\right) + f\left(a + \frac{7h'}{8}\right)\right]$$

From these sums, various other values T_{11}, T_{12}, T_{13}, ..., are computed using the following relations:

$$T_{11} = h' I_{11}$$

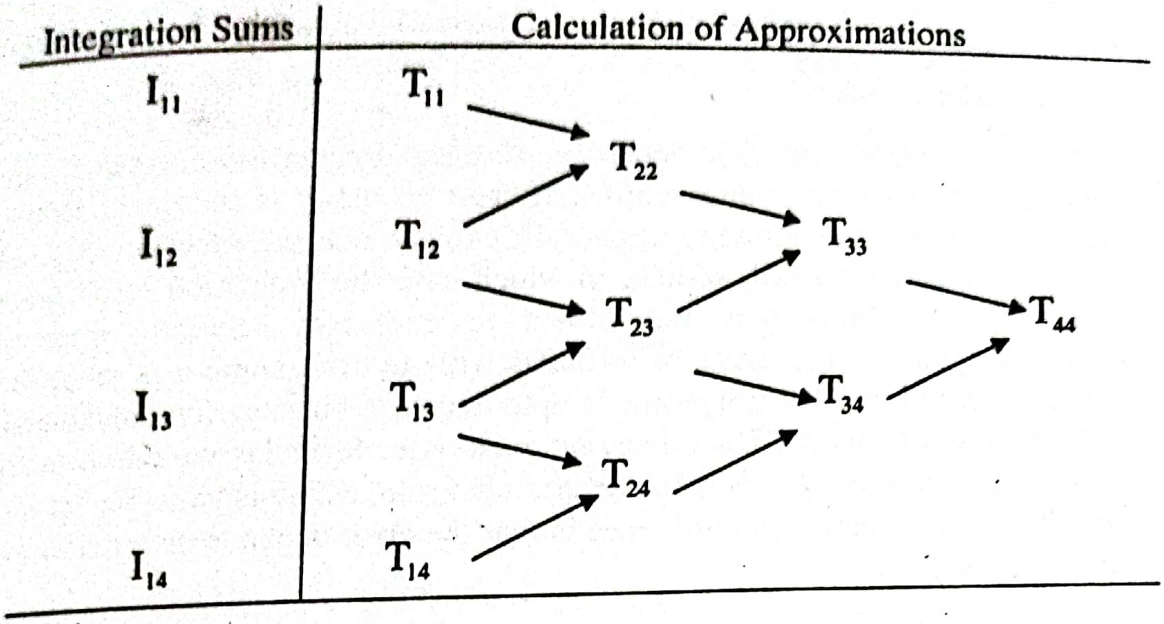
$$T_{12} = \frac{h'}{2} I_{12}$$

$$T_{13} = \frac{h'}{4} I_{13}$$

$$T_{14} = \frac{h'}{8} I_{14}, \text{ and so on.}$$

Note: h is the difference between consecutive values of x, but h′ is the difference between the upper and lower limits of the integral.

Romerg's table is as follows:



With the values of T_{11}, T_{12}, ..., we compute the first-order Romberg integration as follows:

$$T_{22} = T_{12} + \frac{1}{3}\left(T_{12} - T_{11}\right)$$

$$T_{23} = T_{13} + \frac{1}{3}\left(T_{13} - T_{12}\right)$$

$$T_{24} = T_{14} + \frac{1}{3}\left(T_{14} - T_{13}\right)$$

We now compute the second-order Romberg integration:

$$T_{33} = T_{23} + \frac{1}{15}\left(T_{23} - T_{22}\right)$$

$$T_{34} = T_{24} + \frac{1}{15}\left(T_{24} - T_{23}\right)$$

Calculation of third-order Romberg integration:

$$T_{44} = T_{34} + \frac{1}{63}\left(T_{34} - T_{33}\right), \text{ etc.}$$

General formula to calculate various values in the table is,

$$T_{j+1,K+1} = T_{j,K+1} + \frac{1}{4^j - 1}\left[T_{j,K+1} + T_{jK}\right] \qquad \ldots (5.37)$$

The procedure continues until the difference between two successive values on the diagonal agree to the desired accuracy. In each column, the bottom number is hopefully the most accurate number. Trapezoidal and Simpson's rules are sometimes inadequate for problem contexts where high efficiency and low errors are needed.

THE BEST ESTIMATE IS : 0.955517

PROBLEMS

1. (a) The values of a certain function f(x) are given in the following table:

x	0	.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
f(x)	1	1.649	2.718	4.482	7.389	12.18	20.09	33.12	45.60

By using Trapezoidal and Simpson's $\frac{1}{3}$rd rules, compute the integral: $\int_0^4 f(x)\,dx$.

(b) Given the following table:

x	−1	0	1	2	3	4	5	6	7
f(x)	0.9848	1	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420

Evaluate $\int_{-1}^{7} f(x)\,dx$ using Trapezoidal and Simpson's $\frac{1}{3}$rd rules.

(c) Use Trapezoidal and Simpson's rules to estimate the numbers of square feet of land in given lots when x and y are measured in feet:

(i)

x	0	10	20	30	40	50	60	70	80	90	100	110	120
y	75	81	84	76	67	68	69	72	68	56	42	44	0

(ii)

x	0	100	200	300	400	500	600	700	800	900	1000
y	125	125	120	112	90	90	95	88	75	35	0

2. (a) Evaluate the integral $e^{\sqrt{x}}$ correct to 3 dp, using (i) Trapezoidal rule and (ii) Simpson's rule from the values given below:

x	0	1	2	3	4
$e^{\sqrt{x}}$	1	2.7185	4.1132	5.6522	7.3891

Using a suitable substitution to evaluate the integral, determine which of these numerical answers is nearer to the exact value.

(b) Compute $\int_0^1 \frac{dx}{2+x^2}$ by Simpson's $\frac{1}{3}$rd rule with n = 8. Evaluate the function analytically and comment on the outcomes in each case.

(c) Evaluate the integral $\int_{-1}^{1} x^2 e^{-x} dx$ using Simpson's $\frac{1}{3}$rd rule with n = 8.

3. Evaluate $\int_0^1 \frac{dx}{1+x}$ using Trapezoidal and Simpson rules. Take n = 8. Compare your results with the exact answer. Compute the error bounds in both cases.

4. (i) The function f(x) is well-defined by the following table and is well-behaved in the given domain:

x	2.03	2.04	2.05	2.06	2.07	2.08	2.09
f(x)	10.13916	10.26167	10.34737	10.45643	10.56905	10.68531	10.80547

a) The value given for f(2.07) is in error by 3×10^{-5}. Find the correct value and show why this is likely to be correct.

b) Compute the integral $\int_{2.03}^{2.09} f(x)\,dx$ from the values given above in 4(i) using Trapezoidal and Simpson rules.

(ii) Evaluate $\int_0^{\frac{\pi}{3}} \sqrt{\sin x}\,dx$ by Simpson's $\frac{1}{3}$rd rule, using 6 intervals.

(iii) Evaluate $\int_0^{\frac{\pi}{2}} \sin dx$ using the Trapezoidal and Simpson's $\frac{1}{3}$rd rules. Find the exact solution and the error involved. Take n = 6.

(iv) A pin moves along a straight guide so that its velocity v(cm/s) when it is a distance x(cm) from the beginning of the guide at time t(s) is given in the table below:

t(s)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
v(cm/s)	0	4.00	7.94	11.68	14.97	17.39	18.25	16.08	0.00

(b) Evaluate $\int_0^1 (1+x^2)^{-1}\,dx$ to 6 dp, using Romberg's method. Take n = 8.

(c) Calculate $\int_0^8 f(x)\,dx$, for the following table using Romberg's method:

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
f(x)	1.000000	0.990050	0.960789	0.913831	0.852144	0.778801	0.697676	0.612626	0.527292

9. (a) Calculate $\int_1^2 \frac{dx}{\sqrt{x}}$ using Romberg's method. Taking n = 8.

(b) Calculate $\int_1^{1.5} e^{-x^2}\,dx$, for the following table using Romberg's method:

x	1	1.125	1.250	1.375	1.5
f(x)	0.3678794	0.2820629	0.2096113	0.1509774	0.1053992

10. Solve the following integrals using Romberg's integration method correct to 4 dp:

a) $\int_1^2 \frac{\sqrt{1-e^{-x}}}{x}\,dx$; Take n = 8

b) $\int_1^2 \frac{dx}{\sqrt{e^x + x - 1}}$; Take n = 8

c) $\int_0^3 xe^{2x}\,dx$; Take n = 8

d) $\int_0^{0.8} e^{-x^2}\,dx$; Take n = 8

e) $\int_1^2 \ln x \cos x\,dx$; Take n = 8

11. Evaluate the following integrals, correct to 6 dp, using Romberg's integration method. Compare these results with their exact answers. What can you say about this comparison?

a) $\int_0^{\frac{\pi}{4}} \sec dx$; Take n = 8