Indeterminate forms and L'Hospital's rule

The following are the so-called indeterminate forms:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^{\infty}, \quad ,(\infty)^0$$

$$(0)^0, \quad 0.\infty, \quad \infty - \infty.$$

One can apply L'Hospital's rule to the forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. It is simple to translate $0.\infty$ in to $\frac{\infty}{1/0}$ or in to $\frac{0}{1/\infty}$.

For example,

$$\lim_{x \longrightarrow \infty} xe^{-x}$$

can be written as

$$\lim_{x \longrightarrow \infty} \frac{x}{1/e^{-x}} \quad \text{or as} \quad \lim_{x \longrightarrow \infty} \frac{e^{-x}}{1/x}.$$

Indeterminate forms and L'Hospital's rule

To see that the exponent forms are indeterminate; see, for example

$$\begin{aligned}
\ln 0^0 &= 0.\ln 0 = 0(-\infty) = 0.\infty, \quad \because \lim_{x \longrightarrow o^+} \ln(x) = -\infty \\
\ln 1^\infty &= \infty.0 = 0.\infty, \\
\ln \infty^0 &= 0.\infty.
\end{aligned}$$

These formula's also suggest ways to compute the limits using L'Hospital's rule. Basically, we use two things; that e^x and $\ln x$ are inverse functions of each other and that they are continuous functions. If $f(x)$ is a continuous function, then we have

$$f(\lim_{x \longrightarrow a} g(x)) = \lim_{x \longrightarrow a} f(g(x)).$$

Indeterminate forms and L'Hospital's rule

Definition:-

Let $f(x)$ and $g(x)$ are two functions defined on an interval I, such that,

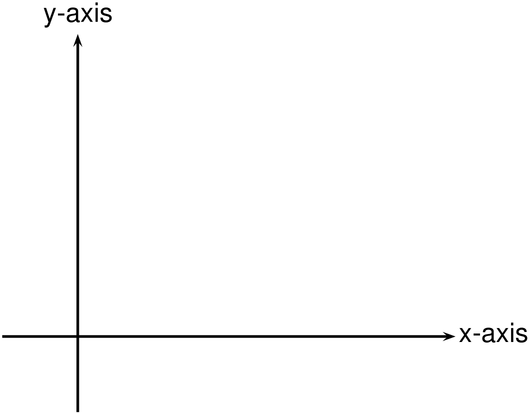
$$f(a) = 0 = g(a), \quad f'(x), g'(x) \text{ exist, and } g'(x) \neq 0.$$

Then according to the L'Hospital's rule, we have

$$\begin{aligned}
\lim_{x \longrightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \longrightarrow a} \frac{f'(x)}{g'(x)} \\
&= \lim_{x \longrightarrow a} \frac{f''(x)}{g''(x)} \\
&\vdots \\
&= \lim_{x \longrightarrow a} \frac{f^{(n)}(x)}{g^{(n)}(x)}.
\end{aligned}$$

Asymptotes

A straight line L is called as asymptote for the curve C if the distance between L and C approaches to zero as the distance moved along the lime L (from some fixed point on the line L) tends to infinity.



Asymptotes

Basically, asymptote is a straight line that continually approaches a given curve but does not meet it at any finite distance.

Types of asymptotes:- There are three types of asymptotes:

(i) Horizontal asymptote

(ii) Vertical asymptote

(iii) Oblique/inclined asymptote

(i) Horizontal asymptote

Let the equation $y = f(x)$ of the curve C is such that y is real and $y \longrightarrow a$ as $x \longrightarrow +\infty$ or $x \longrightarrow -\infty$. Then $y = a$ is called horizontal asymptote.

Asymptotes

Vertical asymptote

Let the equation $y = f(x)$ of the curve C is such that y is real and $|y| \longrightarrow \infty$ as $x \longrightarrow a$, then $x = a$ is called vertical asymptote.

Oblique asymptote

Let the equation $y = f(x)$ of the curve C is such that y is real and $y = mx + c$ as $x \longrightarrow +\infty, x \longrightarrow -\infty$ then $y = mx + c$ is called inclined or oblique asymptote.

Working rules for finding asymptotes parallel to the axes

In an equation of the curve C, the coefficient of the highest power of x (respectively of y) equated to 0 gives (if possible) asymptote parallel to $x - axes$ (respectively $y - axes$).

Asymptotes

Example:-

Find the asymptotes of the following curve

$$y = \frac{1}{(x-2)^2}$$

Solution:-

Horizontal asymptote.

If $|x| \longrightarrow \infty$, then $y = 0$. Thus, $y = 0$ is the horizontal asymptote.

Vertical asymptote.

when $x \longrightarrow 2$ from either side, then $y \longrightarrow \infty$. Thus, $x = 2$ is a vertical asymptote.

Asymptotes

These asymptotes can readily be obtained by the working rules as follows.

Horizontal asymptote. Equate to zero the coefficient of highest power of x, we have

$$y = 0.$$

Vertical asymptote. Equate to zero the coefficient of highest power of y, we have

$$(x-2)^2 = 0, \quad \implies x = 2.$$

Example:-

Find the asymptotes of the following curve

$$2xy + 2y = (x-2)^2$$

Solution:-

The equation of the given curve can be written as

Asymptotes

$$x^2 - 2xy - 4x - 2y + 4 = 0.$$

Horizontal asymptote. Equate to zero the coefficient of highest power of x. Since, the coefficient of highest power of x is constant. So, there is no horizontal asymptote.

Vertical asymptote. Equate to zero the coefficient of highest power of y, we have

$$-2x - 2 = 0, \quad \Longrightarrow x = -1.$$

Hence, $x = -1$ is the vertical asymptote.

Inclined asymptote. Let $y = mx + c$ is the inclined asymptote of the given curve. Then, we have

$$\begin{aligned}
&2x(mx + c) + 2(mx + c) - x^2 + 4x - 4 = 0\\
&2mx^2 + 2cx + 2mx + 2c - x^2 + 4x - 4 = 0\\
&(2m - 1)x^2 + (2c + 2m + 4)x + (2c - 4) = 0.
\end{aligned}$$

Asymptotes

Dividing the above equation by x^2 and taking limit as $x \longrightarrow \infty$, we get

$$2m - 1 = 0, \quad \Longrightarrow m = \frac{1}{2}.$$

So, we have

$$(2c + 5)x + (2c - 4) = 0$$

Now dividing the above equation by x and taking limit as $x \longrightarrow \infty$, we get

$$(2c + 5) = 0, \quad \Longrightarrow c = -\frac{5}{2},$$

Finally, put $m = \frac{1}{2}$ and $c = -\frac{5}{2}$ in $y = mx + c$, we get

$$y = \frac{1}{2}x - \frac{5}{2}.$$

Asymptotes

Exercise

1. Find the asymptotes of the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$.

2. Find the asymptotes of the rectangular hyperbola $x^2 - y^2 = 25$.

3. Find the asymptotes of the equilateral hyperbola $xy = a$, where a is some constant.

4. Find asymptotes of each of the following curves:

(i) $$9x^2 - 16y^2 + 144 = 0$$

(ii) $$y^2(x-1) = x^2(x+1)$$

(iii) $$x^3 - 2x^2y - 3xy^2 - 4x^2 - 12xy - 12x + 4 = 0$$

Asymptotes

(*iv*) $x^2y - (x+2)(y^2-4) = 0$

(*v*) $x^3 - x^2y - x - 6 = 0$

(*vi*) $x^3 + y^3 = 3axy$

Differentiation

Differentiation is a process of finding the rate at which one variable quantity changes with respect to another. Differentiation is one of the primary technique of calculus and the part of calculus that is associated with differentiation is called as **Differential calculus**. Whereas, the part of calculus that is associated with integration is called as **Integral calculus**. The primary objective of differential calculus is to establish the measure of the changes in a function with mathematical accuracy.

Applications of differentiation are enormous in both academia and industry, confined not only physics and engineering, its far reaching tentacles touch a myriad of other areas, for example, business and other sciences. The key idea being modelled is **change**, and how quickly things are changing. We expect you to have met the derivative before in your past study. Here, we will give the description, definition, and some working rules for computing derivatives in the proceeding few slides.

Differentiation

Increment of a function

Literally, the word increment means an increase in the value of a quantity. However, in mathematics, increment means a small change in the value of a variable. The increment in the value of a variable may be positive or negative. The increment in the value of the variable x is denoted by δx and is read as **delta** x. Some authors use Δx instead of δx. Note it that, δx is not the product of the quantity δ and the variable x, but is merely symbol for the increment of the variable x.

Let y be a function of x defined by the equation

$$y = f(x). \tag{1}$$

If y is a continuous function of x then corresponding to a small change δx in the value of the variable x, there will be a small change δy in the value of the variable y. Then from equation (1), we have

$$y + \delta y = f(x + \delta x). \tag{2}$$

Differentiation

It is obvious that if

$$\delta x \longrightarrow 0, \quad \text{then}$$
$$\delta y \longrightarrow 0.$$

Subtracting equation (1) from equation (2), we obtain the increment of y or $f(x)$ as

$$\delta y = f(x + \delta x) - f(x). \tag{3}$$

Average rate of change

If δx is the distance covered by a body during the time interval δt, then the average speed of the body is given by

$$\text{Average speed} = \frac{\delta x}{\delta t}.$$

Thus the average speed may be described as average rate of change of x with respect to t during the interval δt.

Differentiation

Similarly, if δv is the change in the velocity of a body during the time interval δt, then the average acceleration of the body is given by

$$\text{Average acceleration} = \frac{\delta v}{\delta t},$$

which shows that average acceleration may be described as the average rate of change of v with respect to t during the interval δt.

By analogy, the equation

$$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x} \tag{4}$$

is obtained by dividing equation (3) by δx, gives the average rate of change of y with respect to x during the interval δx.

Differentiation

Rate of change

If δt is of extremely short duration, the average speed over this interval is nearly equal to the instantaneous speed of the body at the beginning of the interval. Thus if $\lim\limits_{\delta t \longrightarrow 0} \frac{\delta x}{\delta t}$ exists, it is called as instantaneous speed or the rate of change of x with respect to the time t for the initial value of t.

Similarly, if $\lim\limits_{\delta t \longrightarrow 0} \frac{\delta v}{\delta t}$ exists, it is called as instantaneous acceleration or the rate of change of v with respect to t for the initial value of t.

By analogy, from equation (4), we have

$$\lim_{\delta x \longrightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \longrightarrow 0} \frac{f(x+\delta x) - f(x)}{\delta x}. \tag{5}$$

If limit on the right side of equation (4) exists, it is called as the derivative or rate of change of y or $f(x)$ with respect to x.

Differentiation

The following symbols are commonly used to denote the derivative of y or $f(x)$ with respect to x.

$\frac{d}{dx}(y), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}(f(x)), \quad \frac{df}{dx}, \quad f'(x).$

Definition:-

Given a function f and a point x_0 inside an open interval (a, b) contained in the domain of f, the limit

$$f^{'}(x_0) = \lim_{h \longrightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

is called the **derivative of** f at x_0 **with respect to** x. If this limit exists then the function f is called **differentiable** at the point x_0. Otherwise, it is not differentiable at that point, and hence the derivative at that point does not exist.

Differentiation

Sometimes people prefer the following equivalent form of the derivative

$$f^{'}(x_0) = \lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c}.$$

Definition:-

Given a function f, the derivative function $f^{'}$ is defined as

$$f^{'}(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

We say that $f^{'}$ is the **derivative of** f **with respect to** x. The domain of $f^{'}$ is assumed to be the natural domain of $f^{'}$, i.e., set of all points where the derivative exists.

Differentiation

Remarks:-

1. The process of finding the derivative of a function is called as differentiation.
2. A function $f(x)$ is said to be differentiable at $x = a$, if
$$\lim_{x \longrightarrow a} \frac{f(x) - f(a)}{x - a} \quad \textit{exists}.$$
3. If $f(x)$ is differentiable at $x = a$, then it is denoted by $\frac{dy}{dx}\Big|_{x=a}$ or $f'(a)$.
4. If a function is differentiable at every point of its domain, then it is called as differentiable function.

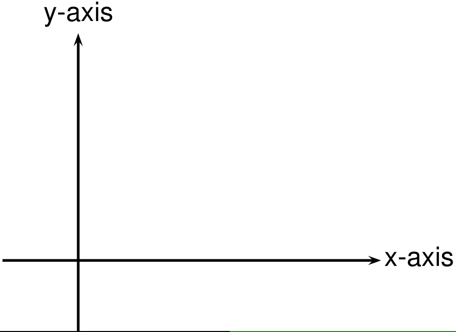
Differentiation

Inclination of a line

Let a line L makes an angle α with x-axis in the positive direction. Then α is called inclination of that line.

Note:- (i) The inclination of the line x-axis is 0.

(ii) The inclination of the line y-axis is 90^0.



Differentiation

Slope or gradient of a line

Let a line L makes an angle α with x-axis in the positive direction. Then $\tan(\alpha)$ is called as slope of that line denoted by

$$m = \tan(\alpha).$$

Note:- (i) The slope of the line x-axis is

$$m = \tan(0) = 0.$$

(ii) The slope of the line y-axis is

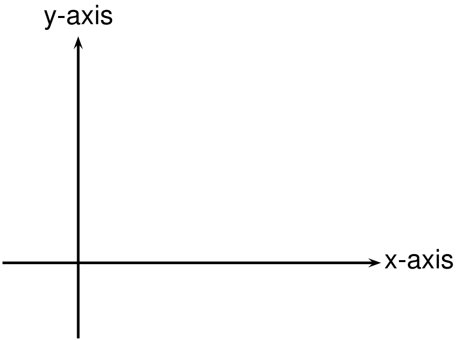
$$m = \tan(90^0) = \text{undefined}.$$

Geometrical interpretation of derivative

If we wish to obtain the tangent line to a point on a smooth looking function, how would you do it? The slope of the tangent line is approximately the slope of the secant line between (x, y) and $(x + \delta x, y + \delta y)$.

Differentiation

As we make arbitrarily δx and δx smaller, our estimate for the slope of the tangent should get arbitrarily accurate. Note it that in each approximation, we are essentially estimating the rate of change over a smaller and smaller interval. In this way, the derivative is often thought of as the instantaneous rate of change at a given point.



Differentiation

Remarks:- The following are all different interpretations for the limit of the difference quotient

$$\lim_{h \longrightarrow 0} \frac{f(x_0+h)-f(x_0)}{h}.$$

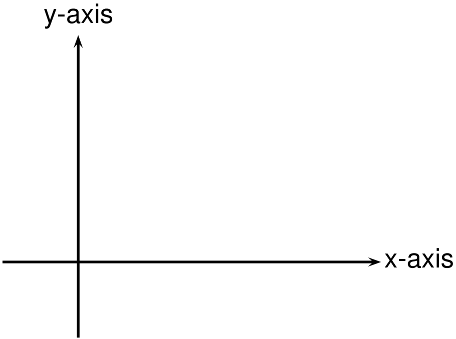
1. The slope of the graph of $y = f(x)$ at $x = x_0$.
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$.
3. The rate of change of $f(x)$ with respect to x at $x = x_0$.
4. The derivative $f^{'}(x_0)$ at a point.

Tangents and normals

Equation of tangent to an arbitrary curve $y = f(x)$ at any point $P(x_1, y_1)$.

Derivation:

$$y - y_1 = f^{'}(x_1)(x - x_1)$$



Tangents and normals

Example:-

Find an equation for the tangent to the curve at the given point:

$$\begin{array}{lrcl} (i) & y & = & 4-x^2, \quad (-1,3) \\ (ii) & y & = & (x-1)^2+1, \quad (1,1) \\ (iii) & y & = & x^3, \quad (-2,-8) \\ (iv) & y & = & \dfrac{1}{x^3}, \quad (-2,-\dfrac{1}{8}) \end{array}$$

Example:-

Find the points on the given curves where the tangent line is parallel to x-axis and where it is parallel to y-axis:

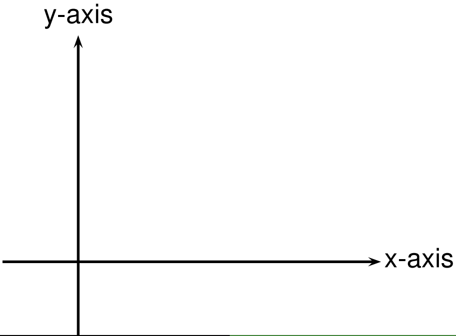
$$\begin{array}{lrcl} (i) & x^3+y^3 & = & a^3 \\ (ii) & x^3+y^3 & = & 3axy \end{array}$$

Tangents and normals

Equation of normal to an arbitrary curve $y = f(x)$ at any point $P(x_1, y_1)$.

Derivation:

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$



Tangents and normals

Exercise

1. Find the equations of tangent and normal to the given curves at the indicated points:

(i)	$y = x^2,$	at	$(2, 4)$
(ii)	$y = 3x^2 - 2x + 5,$	at	$(1, 6)$
(iii)	$y^2 - 2x - 4y - 1 = 0,$	at	$(-2, 1)$
(iv)	$9x^2 - 4y^2 = 108,$	at	$(4, 3)$
(v)	$xy + 2x - 5y - 2 = 0,$	at	$(3, 2)$
(vi)	$6x^2 + 3xy + 2y^2 + 17y - 6 = 0,$	at	$(-1, 0)$
(vii)	$2xy + \pi \sin(y) = 2\pi,$	at	$(1, \frac{\pi}{2})$
$(viii)$	$y = 2\sin(\pi x - y),$	at	$(1, 0)$
(ix)	$x^2 \cos^2(y) - \sin(y) = 0,$	at	$(0, \pi)$

Tangents and normals

2. Find the equations of tangent and normal to the curve $y = 2x^3 - 3x^2 - 2x + 5$ at a point whose ordinate is 2.

3. Find the equations of tangent and normal to the given curves at the point (x_1, y_1)

$$
\begin{array}{ll}
(i) & \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \\
(ii) & y^2 = 4ax \\
(iii) & ax^2 + by^2 + 2gx + 2fy + c = 0
\end{array}
$$

4. Find the equations of tangent and normal for any value of t to the given curves:

$$
\begin{array}{ll}
i) & x = a(t + \sin(t)), \quad y = a(1 - \cos(t)) \\
ii) & x = a\sin^3(t), \quad y = b\cos^3(t)
\end{array}
$$

Tangents and normals

5. Find the equations of tangent and normal to the given curve at $\theta = \frac{\pi}{2}$

$$x = a(\theta - \sin(\theta)), \quad y = a(1 - \cos(\theta))$$

6. Find the equation of tangent to the curve $y = x^4 - 4x^2 + 6$ that is parallel to x-axis.

7. Find the points at which tangent to each of the following curve is perpendicular to x-axis.

$$i) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$ii) \qquad y^2 = 4ax$$

8. Find the tangents to the curve $y = x^3 - 6x + 2$ that are parallel to the line $y = 6x - 2$.

9. Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

Tangents and normals

10. Find all lines that can be drawn normal to the curve $x^2 - y^2 = 5$ and are parallel to the line $2x + 3y = 10$.

11. At what points of the curve $y = 2x^3 - 3x^2 - 2x + 4$ are the tangents parallel to the line $10x - y + 7 = 0$. Find the equation of normal at each of these points.

12. At what point of the curve $y = 2x^2 - 3x - 2$ is the normal parallel to the line $x + 9y - 11 = 0$. Find the equation of tangent at this point.

13. Find the coordinates of the point on the curve $y = x^2 + 3x + 4$, the tangent at which passes through origin?

14. For what value of b is the line $y = 12x + b$ tangent to the curve $y = x^3$.

15. The equation of the tangent at (2, 3) on the curve $y^2 = ax^3 + b$ is $y = 4x - 5$. Find the values of a and b.

Tangents and normals

16. Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at a point where it cuts the y-axis.

17. Given that $f(3) = -1$ and $f^{'}(3) = 5$. Find the equation of tangent line to the graph of $y = f(x)$ at $x = 3$.

18. At what point of the curve $y = 2x - x^2$ is the slope equal to 4? Find the equations of tangent and normal at this point.

19. Does the line that is tangent to the curve $y = x^3$ at the point (1, 1) intersect the curve at any other point? If so, find the point.

20. Does the line that is normal to the curve $y = x^2 + 2x - 3$ at the point (1, 0) intersect the curve at any other point? If so, find the point.

Optimization (Extreme values)

Before discussing optimization of a function of one variable, we give some basic concepts:

Increasing function.

Decreasing function.

Constant function.

Example:-

Determine whether $y = f(x)$ is increasing or decreasing in the given interval

$$(i) \qquad f(x) = \frac{-x^2}{4} + 4, \quad]-\infty, 0[$$

$$(ii) \qquad f(x) = \frac{1}{2}(x^2 - 4x + 4), \quad]2, \infty[$$