DS 501: STATISTICAL AND MATHEMATICAL METHODS FOR DATA SCIENCE

Quiz 4

PROBLEM

maximize: $-x_1^2 - x_2^2$

Subject to: $x_1 \geq 3$

$x_1 \geq 5$

SOLUTION

NOTE: The easier method would be to combine the two constraints into one, i.e., $x_1 \geq 5$. We can see by observation that $x_1 \geq 3$ is an inactive constraint.

As most of the students have used two Lagrange multipliers to solve this the solution below is with 2 constraints:

The Lagrange function is given by:

$$L(x,\lambda_1, \lambda_2) = -x_1^2 - x_2^2 + \lambda_1 (x_1-3) + \lambda_2 (x_1-5) \quad (1)$$

The stationary points of the function are when

$\nabla_x L = 0$ with the following KKT conditions:

$\lambda_1 (x_1-3) = 0$

$\lambda_1 \geq 0$

$x_1-3 \geq 0$

$\lambda_2 (x_1-5) = 0$

$\lambda_2 \geq 0$

$x_1-5 \geq 0$

Solving the above via differentiating the Lagrange function:

$$\frac{\partial L}{\partial x_1} = -2x_1 + \lambda_1 + \lambda_2 = 0 \quad (2)$$

$$\frac{\partial L}{\partial x_1} = -2x_2 = 0 \quad (3)$$

$$x_2 = 0 \quad (4)$$

Lets now look at 4 cases:

CASE 1: $\lambda_1 = 0$ and $\lambda_2 = 0$. Substituting in (2) we get $x_1 = 0$. (0,0) being the optimal point. However, this violates two of our KKT conditions and hence we reject this case.

CASE 2: $\lambda_1 = 0$ and $\lambda_2 > 0$. This would mean an active constraint given by $x_1 = 5$. Giving us the optimal point (5,0), which satisfies all our KKT conditions.

CASE 3: $\lambda_1 > 0$ and $\lambda_2 = 0$. This would mean $x_1 = 3$. However, this violates our KKT condition $x_1-5 \geq 0$ and hence we reject this case.

CASE 4: $\lambda_1 > 0$ and $\lambda_2 > 0$. This would mean two active constraints $x_1 = 3$ and $x_1 = 5$. As x_1 cannot have both values at the same time we reject this case.

Hence only case 2 is possible giving us the optimal point (5,0).

The feasible region along with the optimal point and the contours of the objective function are given below:

The shaded region is the feasible region: