Discrete Structures

Counting

Text book: Kenneth H. Rosen, Discrete Mathematics and Its Applications

Section: 6.3, 6.4 and 6.5

Summary

- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations

Permutations and Combinations

Section 6.3

Section Summary

- Permutations
- Combinations
- Combinatorial Proofs

Permutations

Definition: A *permutation* of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of r elements of a set is called an *r-permuation*.

Example: Let $S = \{1,2,3\}$.

- The ordered arrangement 3,1,2 is a permutation of S.
- The ordered arrangement 3,2 is a 2-permutation of S.

- The number of r-permuatations of a set with n elements is denoted by $P(n,r)$.
  - The 2-permutations of $S = \{1,2,3\}$ are 1,2; 1,3; 2,1; 2,3; 3,1; and 3,2. Hence, $P(3,2) = 6$.

A Formula for the Number of Permutations

Theorem 1: If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n - 1)(n - 2) \cdots (n - r + 1)$$

r-permutations of a set with n distinct elements.

- Note that $P(n,0) = 1$, since there is only one way to order zero elements.

Corollary 1: If n and r are integers with $1 \leq r \leq n$, then

$$P(n,r) = \frac{n!}{(n-r)!}$$

Examples

Example: How many ways are there to select a first-prize winner, a second prize winner, and a third-prize winner from 100 different people who have entered a contest?

Solution:

$$P(100,3) = 100 \cdot 99 \cdot 98 = 970{,}200$$

Examples

Example: Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

Solution: The first city is chosen, and the rest are ordered arbitrarily. Hence the orders are:

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

If she wants to find the tour with the shortest path that visits all the cities, she must consider 5040 paths!

Examples

Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC* ?

Solution: We solve this problem by counting the permutations of six objects, *ABC*, *D*, *E*, *F*, *G*, and *H*.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

Examples

In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Solution: First, note that the order in which we select the students matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5 \cdot 4 \cdot 3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to arrange all five students in a line for a picture. ◀

Combinations

Definition: An *r-combination* of elements of a set is an unordered selection of *r* elements from the set. Thus, an *r*-combination is simply a subset of the set with *r* elements.

- The number of *r*-combinations of a set with n distinct elements is denoted by $C(n, r)$. The notation $\binom{n}{r}$ is also used and is called a *binomial coefficient*. (*We will see the notation again in the binomial theorem in Section* 6.4.)

Example: Let S be the set $\{a, b, c, d\}$. Then $\{a, c, d\}$ is a 3-combination from S. It is the same as $\{d, c, a\}$ since the order listed does not matter.

- $C(4,2) = 6$ because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, and $\{c, d\}$.

Combinations

Theorem 2: The number of *r*-combinations of a set with *n* elements, where $n \geq r \geq 0$, equals

$$C(n, r) = \frac{n!}{(n-r)!r!}.$$

Combinations

Example: How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a deck of 52 cards?

Solution: Since the order in which the cards are dealt does not matter, the number of five card hands is:

$$C(52,5) = \frac{52!}{5!47!}$$
$$= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2,598,960$$

- The different ways to select 47 cards from 52 is

$$C(52,47) = \frac{52!}{47!5!} = C(52,5) = 2,598,960.$$

This is a special case of a general result. →

Combinations

Corollary 2: Let n and r be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n - r)$.

Proof: From Theorem 2, it follows that

$$C(n,r) = \frac{n!}{(n-r)!r!}$$

and

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} \,.$$

Hence, $C(n, r) = C(n, n - r)$. ◀

This result can be proved without using algebraic manipulation. →

Combinatorial Proofs

- **Definition 1**: A *combinatorial proof* of an identity is a proof that uses one of the following methods.
  - A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
  - A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

Combinatorial Proofs

- Here are two combinatorial proofs that

$$C(n, r) = C(n, n - r)$$

when r and n are nonnegative integers with $r < n$:

  - *Bijective Proof*: Suppose that S is a set with n elements. The function that maps a subset A of S to $\bar{A}$ is a bijection between the subsets of S with r elements and the subsets with $n - r$ elements. Since there is a bijection between the two sets, they must have the same number of elements. ◀
  - *Double Counting Proof*: By definition the number of subsets of S with r elements is $C(n, r)$. Each subset A of S can also be described by specifying which elements are not in A, i.e., those which are in $\bar{A}$. Since the complement of a subset of S with r elements has $n - r$ elements, there are also $C(n, n - r)$ subsets of S with r elements. ◀

Examples

Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school.

Solution: By Theorem 2, the number of combinations is

$$C(10,5) = \frac{10!}{5!5!} = 252.$$

Example: A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission?

Solution: By Theorem 2, the number of possible crews is

$$C(30,6) = \frac{30!}{6!24!} = \frac{30\cdot 29\cdot 28\cdot 27\cdot 26\cdot 25}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 593,775\ .$$

Examples

How many bit strings of length n contain exactly r 1s?

Solution: The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s. ◀

Examples

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9,3) \cdot C(11,4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.$$

◀

Exercise Q.13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?

solution There are n men, and all of the $P(n, n) = n!$ arrangements is allowed. Similarly, there are $n!$ arrangements in which the women can appear. Now the men and women must alternate, and there are the same number of men and women; therefore there are exactly two possibilities: either the row starts with a man and ends with a woman ($MWMW \ldots MW$) or else it starts with a woman and ends with a man ($WMWM \ldots WM$). We have three tasks to perform, then: arrange the men among themselves, arrange the women among themselves, and decide which sex starts the row. By the product rule there are $n! \cdot n! \cdot 2 = 2(n!)^2$ ways in which this can be done.

Exercise Q.19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes

a) are there in total?
b) contain exactly two heads?
c) contain at most three tails?
d) contain the same number of heads and tails?

- solution

a) Each flip can be either heads or tails, so there are $2^{10} = 1024$ possible outcomes.

b) To specify an outcome that has exactly two heads, we simply need to choose the two flips that came up heads. There are $C(10,2) = 45$ such outcomes.

c) To contain at most three tails means to contain three tails, two tails, one tail, or no tails. Reasoning as in part (b), we see that there are $C(10,3) + C(10,2) + C(10,1) + C(10,0) = 120 + 45 + 10 + 1 = 176$ such outcomes.

d) To have an equal number of heads and tails in this case means to have five heads. Therefore the answer is $C(10,5) = 252$.

Exercise Q.35. How many bit strings contain exactly eight 0s and 10 1s if every 0 must be immediately followed by a 1?

- solution

To implement the condition that every 0 be immediately followed by a 1, let us think of "gluing" a 1 to the right of each 0. Then the objects we have to work with are eight blocks consisting of the string 01 and two 1's. The question is, then, how many strings are there consisting of these ten objects? This is easy to calculate, for we simply have to choose two of the "positions" in the string to contain the 1's and fill the remaining "positions" with the 01 blocks. Therefore the answer is $C(10,2) = 45$.

01 01 01 01 01 01 01 01 1 1

Binomial Coefficients and Identities

Section 6.4

Section Summary

- The Binomial Theorem
- Pascal's Identity and Triangle
- Other Identities Involving Binomial Coefficients

Powers of Binomial Expressions

Definition: A *binomial* expression is the sum of two terms, such as $x + y$. (More generally, these terms can be products of constants and variables.)

- We can use counting principles to find the coefficients in the expansion of $(x + y)^n$ where n is a positive integer.
- To illustrate this idea, we first look at the process of expanding $(x + y)^3$.
- $(x + y)\ (x + y)\ (x + y)$ expands into a sum of terms that are the product of a term from each of the three sums.
- Terms of the form x^3, x^2y, $x\,y^2$, y^3 arise. The question is what are the coefficients?
  - To obtain x^3, an x must be chosen from each of the sums. There is only one way to do this. So, the coefficient of x^3 is 1.
  - To obtain x^2y, an x must be chosen from two of the sums and a y from the other. There are $\binom{3}{2}$ ways to do this and so the coefficient of x^2y is 3.
  - To obtain xy^2, an x must be chosen from of the sums and a y from the other two . There are $\binom{3}{1}$ ways to do this and so the coefficient of xy^2 is 3.
  - To obtain y^3, a y must be chosen from each of the sums. There is only one way to do this. So, the coefficient of y^3 is 1.
- We have used a counting argument to show that $(x + y)^3 = x^3 + 3x^2y + 3x\,y^2 + y^3$.
- Next we present the binomial theorem gives the coefficients of the terms in the expansion of $(x + y)^n$.

Binomial Theorem

Binomial Theorem: Let *x* and *y* be variables, and *n* a nonnegative integer. Then:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j}y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \cdots + \binom{n}{n-1} xy^{n-1} + \binom{n}{n} y^n.$$

Using the Binomial Theorem

Example: What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution: We view the expression as $(2x + (-3y))^{25}$.
By the binomial theorem

$$(2x + (-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j.$$

Consequently, the coefficient of $x^{12}y^{13}$ in the expansion is obtained when $j = 13$.

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!}2^{12}3^{13}.$$

A Useful Identity

Corollary 1: With $n \geq 0$,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof (*using binomial theorem*): With $x = 1$ and $y = 1$, from the binomial theorem we see that:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{(n-k)} = \sum_{k=0}^{n} \binom{n}{k}.$$

◀

Proof (*combinatorial*): Consider the subsets of a set with n elements. There are $\binom{n}{0}$ subsets with zero elements, $\binom{n}{1}$ with one element, $\binom{n}{2}$ with two elements, …, and $\binom{n}{n}$ with n elements. Therefore the total is

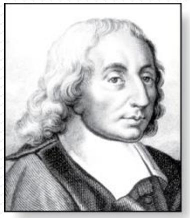
$$\sum_{k=0}^{n} \binom{n}{k}.$$

Since, we know that a set with n elements has 2^n subsets, we conclude:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

◀

Blaise Pascal
(1623-1662)



Pascal's Identity $\binom{3}{2} + \binom{3}{1} = \binom{4}{2}$

Pascal's Identity: If n and k are integers with $n \geq k \geq 0$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Proof (*combinatorial*): Let T be a set where $|T| = n + 1$, $a \in T$, and $S = T - \{a\}$. There are $\binom{n+1}{k}$ subsets of T containing k elements. Each of these subsets either:

- contains a with $k - 1$ other elements, or
- contains k elements of S and not a.

There are

- $\binom{n}{k-1}$ subsets of k elements that contain a, since there are $\binom{n}{k-1}$ subsets of $k - 1$ elements of S,
- $\binom{n}{k}$ subsets of k elements of T that do not contain a, because there are $\binom{n}{k}$ subsets of k elements of S.

Hence,

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$ ◀

See Exercise 19 *for an algebraic proof.*

Pascal's Triangle

The nth row in the triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0,1,\ldots,n$.

$$
\begin{array}{c}
\binom{0}{0} \\
\binom{1}{0}\ \binom{1}{1} \\
\binom{2}{0}\ \binom{2}{1}\ \binom{2}{2} \\
\binom{3}{0}\ \binom{3}{1}\ \binom{3}{2}\ \binom{3}{3} \\
\binom{4}{0}\ \binom{4}{1}\ \binom{4}{2}\ \binom{4}{3}\ \binom{4}{4} \\
\binom{5}{0}\ \binom{5}{1}\ \binom{5}{2}\ \binom{5}{3}\ \binom{5}{4}\ \binom{5}{5} \\
\binom{6}{0}\ \binom{6}{1}\ \binom{6}{2}\ \binom{6}{3}\ \binom{6}{4}\ \binom{6}{5}\ \binom{6}{6} \\
\binom{7}{0}\ \binom{7}{1}\ \binom{7}{2}\ \binom{7}{3}\ \binom{7}{4}\ \binom{7}{5}\ \binom{7}{6}\ \binom{7}{7} \\
\binom{8}{0}\ \binom{8}{1}\ \binom{8}{2}\ \binom{8}{3}\ \binom{8}{4}\ \binom{8}{5}\ \binom{8}{6}\ \binom{8}{7}\ \binom{8}{8} \\
\cdots
\end{array}
$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
\cdots
\end{array}
$$

(b)

By Pascal's identity, adding two adjacent bionomial coefficients results is the binomial coefficient in the next row between these two coefficients.

Other Identities Involving Binomial Coefficients

VANDERMONDE'S IDENTITY Let m, n, and r be nonnegative integers with r not exceeding either m or n. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Corollary 4 follows from Vandermonde's identity.

If n is a nonnegative integer, then

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Proof: We use Vandermonde's identity with $m = r = n$ to obtain

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

The last equality was obtained using the identity $\binom{n}{k} = \binom{n}{n-k}$.

Other Identities Involving Binomial Coefficients

- Theorem

Let n and r be nonnegative integers with $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}.$$

Exercise Q.7. What is the coefficient of x^9 in $(2-x)^{19}$?

- Since by Binomial theorem the term involving x^9 in $(2-x)^{19}$ is $\binom{19}{9}2^{10}(-x)^9$.

Thus required coefficient is

$$\binom{19}{9}2^{10}(-1)^9 = -2^{10}\binom{19}{9} = -94{,}595{,}072.$$

$(x+y)^n$

$n = 19, \quad x = 2, \quad y = -x$

Exercise Q.25.

Let n be a positive integer. Show that

$$\binom{2n}{n+1}+\binom{2n}{n}=\binom{2n+2}{n+1}/2.$$

using Pascal's identity twice (Theorem 2 section 6.4) and Corollary 1 of section 6.3, we have

$2 = 1 + 1$

$$\binom{2n}{n+1}+\binom{2n}{n}=\frac{2}{2}\binom{2n+1}{n+1}=\frac{1}{2}\left(\binom{2n+1}{n+1}+\binom{2n+1}{n+1}\right)$$

$$=\frac{1}{2}\left(\binom{2n+1}{n+1}+\binom{2n+1}{n}\right)=\frac{1}{2}\left(\binom{2n+2}{n+1}\right)$$

$$\boxed{\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}.}$$

$C(n, r) = C(n, n - r)$

Generalized Permutations and Combinations

Section 6.5

Section Summary

- Permutations with Repetition
- Combinations with Repetition
- Permutations with Indistinguishable Objects
- Distributing Objects into Boxes

Permutations with Repetition

Theorem 1: The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed. Hence, by the product rule there are n^r r-permutations with repetition. ◀

Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution: The number of such strings is 26^r, which is the number of r-permutations of a set with 26 elements.

Combinations with Repetition

How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution: To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

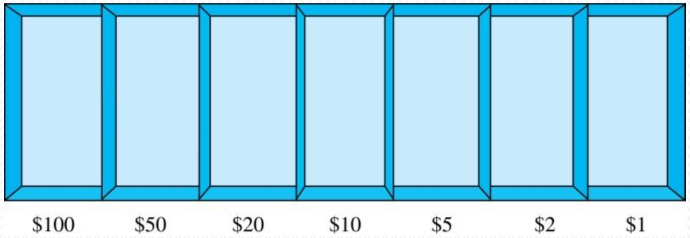
4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

The solution is the number of 4-combinations with repetition allowed from a three-element set, $\{apple, orange, pear\}$.

Combinations with Repetition

Example: How many ways are there to select five bills from a cash box containing \$1 bills, \$2 bills, \$5 bills, \$10 bills, \$20 bills, \$50 bills, and \$100 bills? Assume that the order in which the bills are chosen does not matter, that the bills of each denomination are indistinguishable, and that there are at least five bills of each type.

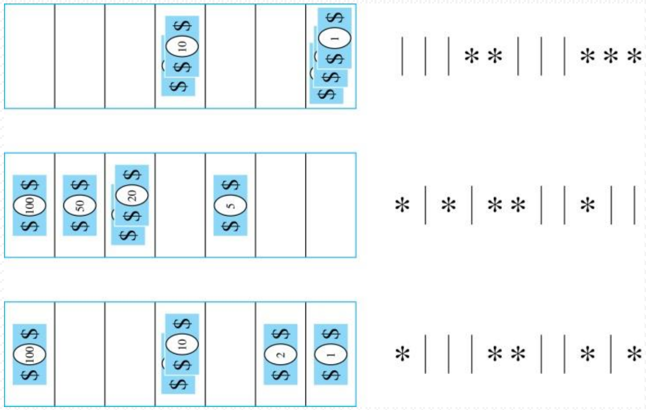
Solution: Place the selected bills in the appropriate position of a cash box illustrated below:



continued →

Combinations with Repetition

- Some possible ways of placing the five bills:



- The number of ways to select five bills corresponds to the number of ways to arrange six bars and five stars in a row.
- This is the number of unordered selections of 5 objects from a set of 11. Hence, there are

$$C(11,5) = \frac{11!}{5!6!} = 462$$

ways to choose five bills with seven types of bills.

Combinations with Repetition

Theorem 2: The number of r-combinations from a set with n elements when repetition of elements is allowed is

$$C(n + r - 1, r) = C(n + r - 1, n - 1).$$

Combinations with Repetition

Example: How many solutions does the equation

$x_1 + x_2 + x_3 = 11$

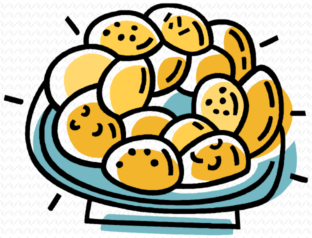
have, where x_1 , x_2 and x_3 are nonnegative integers?

Solution: Each solution corresponds to a way to select 11 items from a set with three elements; x_1 elements of type one, x_2 of type two, and x_3 of type three.

By Theorem 2 it follows that there are

$$C(3+11-1,11) = C(13,11) = C(13,2) = \frac{13 \cdot 12}{1 \cdot 2} = 78$$

solutions.



Combinations with Repetition

Example: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Solution: The number of ways to choose six cookies is the number of 6-combinations of a set with four elements. By Theorem 2

$$C(9,6) = C(9,3) = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} = 84$$

is the number of ways to choose six cookies from the four kinds.

Summarizing the Formulas for Counting Permutations and Combinations with and without Repetition

TABLE 1 **Combinations and Permutations With and Without Repetition.**

Type	*Repetition Allowed?*	*Formula*
r-permutations	No	$\frac{n!}{(n-r)!}$
r-combinations	No	$\frac{n!}{r!\,(n-r)!}$
r-permutations	Yes	n^r
r-combinations	Yes	$\frac{(n+r-1)!}{r!\,(n-1)!}$

Permutations with Indistinguishable Objects

Example: How many different strings can be made by reordering the letters of the word *SUCCESS*.

Solution: There are seven possible positions for the three Ss, two Cs, one U, and one E.

- The three Ss can be placed in $C(7,3)$ different ways, leaving four positions free.
- The two Cs can be placed in $C(4,2)$ different ways, leaving two positions free.
- The U can be placed in $C(2,1)$ different ways, leaving one position free.
- The E can be placed in $C(1,1)$ way.

By the product rule, the number of different strings is:

$$C(7,3)C(4,2)C(2,1)C(1,1) = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420.$$

The reasoning can be generalized to the following theorem. →

Permutations with Indistinguishable Objects

Theorem 3: The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, …., and n_k indistinguishable objects of type k, is:

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof: By the product rule the total number of permutations is:

$C(n, n_1)\, C(n - n_1, n_2) \cdots C(n - n_1 - n_2 - \cdots - n_k, n_k)$ since:

- The n_1 objects of type one can be placed in the n positions in $C(n, n_1)$ ways, leaving $n - n_1$ positions.
- Then the n_2 objects of type two can be placed in the $n - n_1$ positions in $C(n - n_1, n_2)$ ways, leaving $n - n_1 - n_2$ positions.
- Continue in this fashion, until n_k objects of type k are placed in $C(n - n_1 - n_2 - \cdots - n_k, n_k)$ ways.

The product can be manipulated into the desired result as follows:

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2!)}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!} = \frac{n!}{n_1!n_2!\cdots n_k!}.$$

◀

Distributing Objects into Boxes

- Many counting problems can be solved by counting the ways objects can be placed in boxes.
  - The objects may be either different from each other (*distinguishable*) or identical (*indistinguishable*).
  - The boxes may be labeled (*distinguishable*) or unlabeled (*indistinguishable*).

Distributing Objects into Boxes

- *Distinguishable objects* and *distinguishable boxes*.
  - There are $n!/(n_1!n_2!\cdots n_k!)$ ways to distribute n distinguishable objects into k distinguishable boxes.
  - (*See Exercises* 47 *and* 48 *for two different proofs.*)
  - Example: There are 52!/(5!5!5!5!32!) ways to distribute hands of 5 cards each to four players.
- *Indistinguishable objects* and *distinguishable boxes*.
  - There are $C(n + r - 1, n - 1)$ ways to place r indistinguishable objects into n distinguishable boxes.
  - Proof based on one-to-one correspondence between n-combinations from a set with k-elements when repetition is allowed and the ways to place n indistinguishable objects into k distinguishable boxes.
  - Example: There are $C(8 + 10 - 1, 10) = C(17,10) = 19{,}448$ ways to place 10 indistinguishable objects into 8 distinguishable boxes.

Distributing Objects into Boxes

- *Distinguishable objects* and *indistinguishable boxes*.
  - Example: There are 14 ways to put four employees into three indistinguishable offices (*see Example* 10).
  - There is no simple closed formula for the number of ways to distribute n distinguishable objects into j indistinguishable boxes.
  - See the text for a formula involving *Stirling numbers of the second kind*.
- *Indistinguishable objects* and *indistinguishable boxes*.
  - Example: There are 9 ways to pack six copies of the same book into four identical boxes (*see Example* 11).
  - The number of ways of distributing n indistinguishable objects into k indistinguishable boxes equals $p_k(n)$, the number of ways to write n as the sum of at most k positive integers in increasing order.
  - No simple closed formula exists for this number.

Exercise Q.15.

How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21,$$

where x_i, $i = 1, 2, 3, 4, 5$, is a nonnegative integer such that

a) $x_1 \geq 1$?
b) $x_i \geq 2$ for $i = 1, 2, 3, 4, 5$?
c) $0 \leq x_1 \leq 10$?
d) $0 \leq x_1 \leq 3$, $1 \leq x_2 < 4$, and $x_3 \geq 15$?

Solution: (a) $x_1 = x_1' + 1$; thus x_1' is the value that x_1 has in excess of its required 1. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2 + x_3 + x_4 + x_5 = 20$. By Theorem 2 there are $C(5+20-1, 20) = C(24, 20) = C(24, 4) = 10{,}626$ of them.

b) Substitute $x_i = x_i' + 2$ into the equation for each i; thus x_i' is the value that x_i has in excess of its required 2. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2' + x_3' + x_4' + x_5' = 11$. By Theorem 2 there are $C(5+11-1, 11) = C(15, 11) = C(15, 4) = 1365$ of them.

c) There are $C(5+21-1, 21) = C(25, 21) = C(25, 4) = 12650$ solutions with no restriction on x_1. The restriction on x_1 will be violated if $x_1 \geq 11$. Following the procedure in part **(a)**, we find that there are $C(5+10-1, 10) = C(14, 10) = C(14, 4) = 1001$ solutions in which the restriction is violated. Therefore there are $12650 - 1001 = 11{,}649$ solutions of the equation with its restriction.

d) First let us impose the restrictions that $x_3 \geq 15$ and $x_2 \geq 1$. Then the problem is equivalent to counting the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subject to the constraints that $x_1 \leq 3$ and $x_2' \leq 2$ (the latter coming from the original restriction that $x_2 < 4$). Note that these two restrictions cannot be violated simultaneously. Thus if we count the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subtract the number of its solutions in which $x_1 \geq 4$, and subtract the numbers of its solutions in which $x_2' \geq 3$, then we will have the answer. By Theorem 2 there are $C(5+5-1, 5) = C(9, 5) = 126$ solutions of the unrestricted equation. Applying the first restriction reduces the equation to $x_1' + x_2' + x_3' + x_4 + x_5 = 1$, which has $C(5+1-1, 1) = C(5, 1) = 5$ solutions. Applying the second restriction reduces the equation to $x_1 + x_2'' + x_3' + x_4 + x_5 = 2$, which has $C(5+2-1, 2) = C(6, 2) = 15$ solutions. Therefore the answer is $126 - 5 - 15 = 106$.

Exercise Q.23.

How many ways are there to distribute 12 distinguishable objects into six distinguishable boxes so that two objects are placed in each box?

Solution:-

first choose the two objects to go into box #1 ($C(12,2)$ ways), then choose the two objects to go into box #2 ($C(10,2)$ ways, since only 10 objects remain), then choose the two objects to go into box #3 ($C(8,2)$ ways), and so on. So the answer is $C(12,2) \cdot C(10,2) \cdot C(8,2) \cdot C(6,2) \cdot C(4,2) \cdot C(2,2) = (12 \cdot 11/2)(10 \cdot 9/2)(8 \cdot 7/2)(6 \cdot 5/2)(4 \cdot 3/2)(2 \cdot 1/2) = 12!/2^6 = 7{,}484{,}400$.

Exercise 6.5

7. How many ways are there to select three unordered elements from a set with five elements when repetition is allowed?

$$x_1 + x_2 + x_3 + x_4 + x_5 = 3$$

$$C(5 + 3 - 1, 3) = C(7, 3) = 35$$

Exercise 6.5

21. How many ways are there to distribute six indistinguishable balls into nine distinguishable bins?

Using Theorem 2

$$C(9+6-1, 6) = C(14, 6)$$

$$= 3003$$

Exercise 6.5

20. How many solutions are there to the inequality

$$x_1 + x_2 + x_3 \leq 11,$$

where x_1, x_2, and x_3 are nonnegative integers? [*Hint:* Introduce an auxiliary variable x_4 such that $x_1 + x_2 + x_3 + x_4 = 11$.]

$$x_1 + x_2 + x_3 \leq 11$$

$$\Rightarrow \quad x_1 + x_2 + x_3 + x_4 = 11 \quad , \quad x_4 \geq 0$$

$$C(11+4-1, 11) = C(14, 11)$$

$$\frac{14 \times 13 \times 12}{3!} = 28 \times 13$$

$$= 364$$

Exercise 6.5

31. How many different strings can be made from the letters in *ABRACADABRA*, using all the letters?

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

Using Theorem 3

Here $n = 11$

$n_1 = 5$ [5 As]

$n_2 = 2$ [2 Bs]

$n_3 = 1$ [1 C]

$n_4 = 1$ [1 D]

$n_5 = 2$ [2 Rs]

Answer:

$$\frac{11!}{5!\, 2!\, 1!\, 1!\, 2!}$$

$= 83160$