Discrete Structures

Sprin g 2020

Proof: Mathematical Induction

Text book: Kenneth H. Rosen, Discrete Mathematics and Its Applications

Section: 5.1 and 5.2

Mathematical Induction

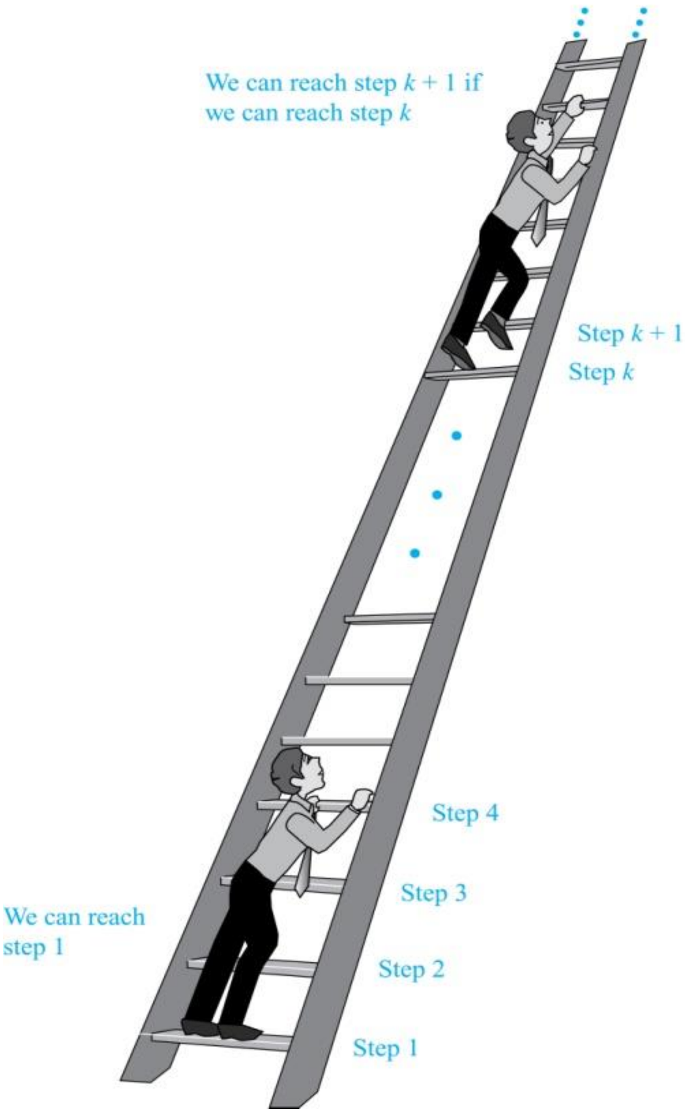
Section 5.1

Section Summary

- Mathematical Induction
- Examples of Proof by Mathematical Induction
- Guidelines for Proofs by Mathematical Induction

Climbing an Infinite Ladder

- Suppose we have an infinite ladder:
  - We can reach the first rung of the ladder.
  - If we can reach a particular rung of the ladder, then we can reach the next rung.
- Can we reach every step on the ladder?



Principle of Mathematical Induction

- Principle of Mathematical Induction: To prove that P(n) is true for all positive integers n, we complete these steps:
  - Basis Step: Show that P(1) is true.
  - Inductive Step: Show that $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.
- To complete the inductive step, assuming the inductive hypothesis that P(k) holds for an arbitrary integer k, show that must P(k + 1) be true.

Principle of Mathematical Induction

- **Climbing an Infinite Ladder Example:**
  - **BASIS STEP: By (1), we can reach rung 1.**
  - **INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung k + 1.**
- **Hence, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k. We can reach every rung on the ladder.**

Important Points

- Mathematical induction can be expressed as the rule of inference

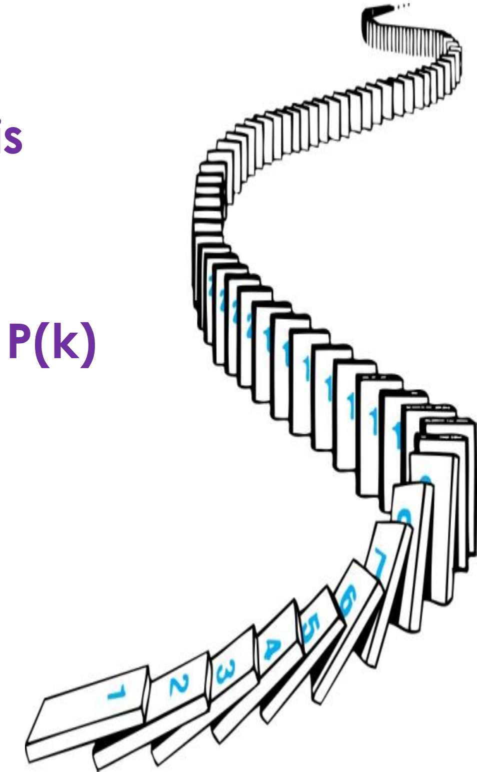
$$\boxed{(P(1) \wedge \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n),}$$

where the domain is the set of positive integers.

  - In a proof by mathematical induction, we don't assume that P(k) is true for all positive integers! We show that if we assume that P(k) is true, then P(k + 1) must also be true.
  - Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a starting point b where b is an integer.
  - Mathematical induction is valid because of the well ordering property

How Mathematical Induction Works

- Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.
  - Let P(n) be the proposition that the n^{th} domino is knocked over. (A domino is a rectangular piece that is one square by two squares).
  - know that the first domino is knocked down, i.e., P(1) is true.
  - We also know that if whenever the k^{th} domino is knocked over, it knocks over the $(k + 1)^{st}$ domino, i.e, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.
  - Hence, all dominos are knocked over.
  - P(n) is true for all positive integers n.



Examples

- **Example: Show that:** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ **for all positive integers.**
- **Solution:**
  - **BASIS STEP**: P(1) is true since 1(1 + 1)/2 = 1.
  - **INDUCTIVE STEP**: Assume true for P(k).
    - The inductive hypothesis is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
    - Under this assumption,

$$1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+1)(k+2)}{2}$$

  - **Hence**, we have shown that P(k + 1) follows from P(k). Therefore the sum of the first n positive integers is $\frac{n(n+1)}{2}$

Examples

- **Example: Use mathematical induction to show that**

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

for all nonnegative integers *n*

- **Solution:**
- **P(n):** $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ **for all nonnegative integers *n***
  - **BASIS STEP: P(0) is true since $2^0 = 1 = 2^1 - 1$. This completes the basis step.**
  - **INDUCTIVE STEP: assume that P(k) is true for an arbitrary nonnegative integer k**
  - $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$
  - **show that assume that P(k) is true, then P(k + 1) is also true.**
  - $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$
  - **Under the assumption of P(k), we see that**
  - $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1.$
  - **Because we have completed the basis step and the inductive step, by mathematical induction we know that P(n) is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.**

Examples

- Example: Conjecture and prove correct a formula for the sum of the first n positive odd integers. Then prove your conjecture.
- Solution:
  - We have:
    - 1= 1,
    - 1 + 3 = 4,
    - 1 + 3 + 5 = 9,
    - 1 + 3 + 5 + 7 = 16,
    - 1 + 3 + 5 + 7 + 9 = 25.
  - We can conjecture that the sum of the first n positive odd integers is n^2,
  - $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Examples

- P(n):1 + 3 + 5 + ⋯+ (2n − 1) =n^2 .
  - **BASIS STEP**: P(1) is true since $1^2 = 1$.
  - **INDUCTIVE STEP**: P(k) → P(k + 1) for every positive integer k.
    - Assume the inductive hypothesis holds and then show that P(k) holds has well.

$$\boxed{\textbf{Inductive Hypothesis: } 1 + 3 + 5 + \cdots + (2k - 1) = k^2}$$

    - So, assuming P(k), it follows that:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

  - **Hence**, we have shown that P(k + 1) follows from P(k). Therefore the sum of the first n positive odd integers is n^2.

Examples

- Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.
- Solution:
- Let P(n) be the proposition that $2^n < n!$.
  - BASIS STEP: P(4) is true since $2^4 = 16 < 4! = 24$.
  - INDUCTIVE STEP: Assume P(k) holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$.
  - To show that $P(k + 1)$ holds:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\ &< (k+1)k! \\ &= (k+1)! \end{aligned}$$

  - Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

Examples

- **Example: Use mathematical induction to show that if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.**
- **Solution:**
- **P(n) be the proposition that a set with n elements has 2^n subsets.**
  - **Basis Step:** P(0) is true, because the empty set has only itself as a subset and $2^0 = 1$.
  - **Inductive Step:** Assume P(k) is true for an arbitrary nonnegative integer k. **Inductive Hypothesis**: For an arbitrary nonnegative integer *k*, every set with *k* elements has 2^k subsets.
    - Let T be a set with $k + 1$ elements. Then $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$.
    - For each subset X of S, there are exactly two subsets of T, i.e., X and $X \cup \{a\}$.
    - By the inductive hypothesis S has 2^k subsets. Since there are two subsets of T for each subset of S, the number of subsets of T is $2 \cdot 2^k = 2^{k+1}$.
  - **Because we have completed the basis step and the inductive step, by mathematical induction** if S is a finite set with n elements, where n is a nonnegative integer, then S has 2^n subsets.

Remark

- Note : When we show that the inductive step is true, we do not show P(k+1) is true.

  Instead, we show the conditional statement

  $$P(k) \rightarrow P(k+1) \text{ is true.}$$

  This allows us to use P(k) as the premise, and gives us an easier way to show P(k+1)

- Once basis step and inductive step are proven, by mathematical induction, $\forall n\, P(n)$ is true

Remark

- Mathematical induction is a very powerful technique, because we show just two statements, but this can imply infinite number of cases to be correct

- However, the technique does not help us find new theorems. In fact, we have to obtain the theorem (by guessing) in the first place, and induction is then used to formally confirm the theorem is correct

Template for Proofs by Mathematical Induction

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer b.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of b is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k+1)$ says.
6. Prove the statement $P(k+1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers k with $k \geq b$, taking care that the proof works for small values of k, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers n with $n \geq b$.

Strong Induction and Well-Ordering

Section 5.2

Section Summary

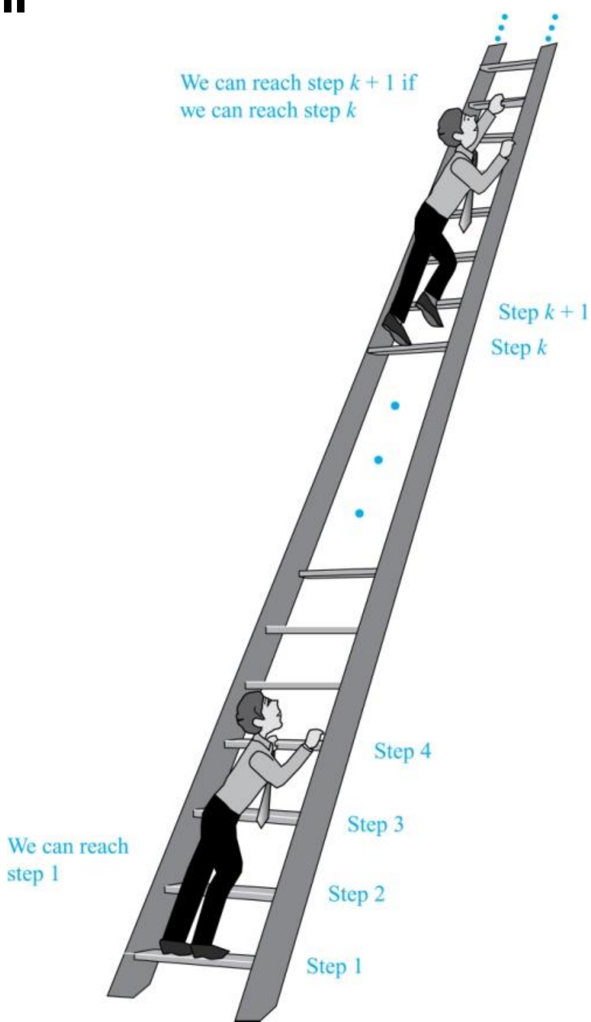
- Strong Induction
- Example Proofs using Strong Induction
- Well-ordering property

Strong Induction

- ***Strong Induction***: To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional function, complete two steps:
  - **Basis Step**: Verify that the proposition P(1) is true.
  - **Inductive Step**: Show the conditional statement $[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ holds for all positive integers k.
- Strong Induction is sometimes called the second principle of mathematical induction or complete induction.

Strong Induction and the Infinite Ladder

- Strong induction tells us that we can reach all rungs if:
  - We can reach the first rung of the ladder.
  - For every integer k, if we can reach the first k rungs, then we can reach the (k + 1)st rung.



Examples

- Example: Show that if n is an integer greater than 1, then n can be written as the product of primes.
- Solution:
  - P(n) be the proposition that n can be written as the product of primes.
  - BASIS STEP: P(2) is true. 2 can be written as the product of itself.
  - INDUCTIVE STEP:
    - inductive hypothesis :P(j) is true for all integers j with $2 \leq j \leq k$, that is, j can be written as the product of primes whenever j is a positive integer at least 2 and not exceeding k.
    - To complete the inductive step, prove P(k + 1) is true under the assumption.
    - There are two cases to consider, namely.
      - If k + 1 is prime, we immediately see that P(k + 1) is true.
      - Otherwise, k + 1 is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < k + 1$. Because both a and b are integers at least 2 and not exceeding k, we can use the inductive hypothesis to write both a and b as the product of primes. Thus, if k + 1 is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.
  - Conclusion..

Examples

- Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- Prove the result using Mathematical induction.
  - BASIS STEP: Postage of 12 cents can be formed using three 4-cent stamps.
  - INDUCTIVE STEP: assume $P(k)$ is true. That is, postage of k cents can be formed using 4-cent and 5-cent stamps.
  - To complete the inductive step, assume $P(k)$ is true, then $P(k + 1)$ is also true where $k \geq 12$. That is, if we can form postage of k cents, then we can form postage of $k + 1$ cents. So, assume the inductive hypothesis is true;
    - two cases, when at least one 4-cent stamp has been used and when no 4-cent stamps have been used.
    - First, suppose that at least one 4-cent stamp was used to form postage of k cents. Then we can replace this stamp with a 5-cent stamp to form postage of $k + 1$ cents.
    - But if no 4-cent stamps were used, we can form postage of k cents using only 5-cent stamps. Moreover, because $k \geq 12$, we needed at least three 5-cent stamps to form postage of k cents. So, we can replace three 5-cent stamps with four 4-cent stamps to form postage of $k + 1$ cents. This completes the inductive step.
  - Conclusion…

Examples

- Example: Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.
- Prove the result using Strong induction.
  - BASIS STEP: Show that $P(12)$, $P(13)$, $P(14)$, and $P(15)$ are true. This completes the basis step.
  - INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for $12 \leq j \leq k$, where k is an integer with $k \geq 15$.
  - To complete the inductive step, assume that we can form postage of j cents, where $12 \leq j \leq k$. Then show that under the assumption that $P(k + 1)$ is true, we can also form postage of $k + 1$ cents.
  - Using the inductive hypothesis, we can assume that $P(k - 3)$ is true because $k - 3 \geq 12$, that is, we can form postage of $k - 3$ cents using just 4-cent and 5-cent stamps. To form postage of $k + 1$ cents, we need only add another 4-cent stamp to the stamps we used to form postage of $k - 3$ cents. That is, we have shown that if the inductive hypothesis is true, then $P(k + 1)$ is also true. This completes the inductive step.
  - Conclusion…

Examples

Consider a game in which two players take turns removing any positive number of matches they want from one of two piles of matches. The player who removes the last match wins the game. Show that if the two piles contain the same number of matches initially, the second player can always guarantee a win.

Solution: Let n be the number of matches in each pile. We will use strong induction to prove $P(n)$, the statement that the second player can win when there are initially n matches in each pile.

BASIS STEP: When $n = 1$, the first player has only one choice, removing one match from one of the piles, leaving a single pile with a single match, which the second player can remove to win the game.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(j)$ is true for all j with $1 \leq j \leq k$, that is, the assumption that the second player can always win whenever there are j matches, where $1 \leq j \leq k$ in each of the two piles at the start of the game. We need to show that $P(k+1)$ is true, that is, that the second player can win when there are initially $k+1$ matches in each pile, under the assumption that $P(j)$ is true for $j = 1, 2, \ldots, k$. So suppose that there are $k+1$ matches in each of the two piles at the start of the game and suppose that the first player removes r matches $(1 \leq r \leq k)$ from one of the piles, leaving $k+1-r$ matches in this pile. By removing the same number of matches from the other pile, the second player creates the situation where there are two piles each with $k+1-r$ matches. Because $1 \leq k+1-r \leq k$, we can now use the inductive hypothesis to conclude that the second player can always win. We complete the proof by noting that if the first player removes all $k+1$ matches from one of the piles, the second player can win by removing all the remaining matches. ◀

Examples

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Show that whenever $n \geq 3$, $f_n > \alpha^{n-2}$, where $\alpha = (1+\sqrt{5})/2$.

Solution: We can use strong induction to prove this inequality. Let $P(n)$ be the statement $f_n > \alpha^{n-2}$. We want to show that $P(n)$ is true whenever n is an integer greater than or equal to 3.

BASIS STEP: First, note that

$$\alpha < 2 = f_3, \qquad \alpha^2 = (3+\sqrt{5})/2 < 3 = f_4,$$

so $P(3)$ and $P(4)$ are true.

INDUCTIVE STEP: Assume that $P(j)$ is true, namely, that $f_j > \alpha^{j-2}$, for all integers j with $3 \leq j \leq k$, where $k \geq 4$. We must show that $P(k+1)$ is true, that is, that $f_{k+1} > \alpha^{k-1}$. Because α is a solution of $x^2 - x - 1 = 0$ (as the quadratic formula shows), it follows that $\alpha^2 = \alpha + 1$. Therefore,

$$\alpha^{k-1} = \alpha^2 \cdot \alpha^{k-3} = (\alpha+1)\alpha^{k-3} = \alpha \cdot \alpha^{k-3} + 1 \cdot \alpha^{k-3} = \alpha^{k-2} + \alpha^{k-3}.$$

By the inductive hypothesis, because $k \geq 4$, we have

$$f_{k-1} > \alpha^{k-3}, \qquad f_k > \alpha^{k-2}.$$

Therefore, it follows that

$$f_{k+1} = f_k + f_{k-1} > \alpha^{k-2} + \alpha^{k-3} = \alpha^{k-1}.$$

Hence, $P(k+1)$ is true. This completes the proof. ◀

Well-ordering property

- The well-property property: every nonempty set of nonnegative integers has a least element.
  - Used to prove the validity of mathematical induction and strong induction

Which Form of Induction Should Be Used?

- **We can always use strong induction instead of mathematical induction. But there is no reason to use it if it is simpler to use mathematical induction. (*See page* 335 *of text*.)**
- **In fact, the principles of mathematical induction, strong induction, and the well-ordering property are all equivalent. (*Exercises* 41-43)**

Exercises 5.2

Q.1Use strong induction to show that if you can run one mile or two miles, and if you can always run two more miles once you have run a specified number of miles, then you can run any number of miles.

- **Let P(n) be the statement that you can run n miles. We want to prove that P(n) is true for all positive integers n.**
- **basis step given conditions tell us that P(l) and P(2) are true.**
- **Inductive step, fix $k \geq 2$ and assume that P(j) is true for all $j \leq k$.**
- **We want to show that P(k + 1) is true.**

Since $k \geq 2$, k - 1 is a positive integer less than or equal to k, so by the inductive hypothesis, we know that P(k - 1) is true.

That is, we know that you can run k - 1 miles. We were told that "you can always run two more miles once you have run a specified number of miles," so we know that you can run

(k - 1) + 2 = k + 1 miles. This implies P(k + 1) is true.

Thus P(n) is true for all positive integers n.

Exercises 5.2

Q.4. Let P(n) be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for $n \geq 18$.

a) Show statements P(18), P(19), P(20), and P(21) are true, completing the basis step of the proof.

b) What is the inductive hypothesis of the proof?

c) What do you need to prove in the inductive step?

d) Complete the inductive step for $k \geq 21$.

e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Exercises 5.2

a) To see P(18), P(19), P(20), P(21), we just check:

$18 = 2 \times 7 + 4$

$19 = 3 \times 4 + 7$

$20 = 5 \times 4$

$21 = 3 \times 7$.

b) The inductive hypothesis is that P(k) is true for all k satisfying

$18 \leq k \leq n$ - i.e. k cent postage can be paid for with only 4 and 7 cent stamps for all k between 18 and n (inclusive).

c) In the inductive step, we need to show that if n is some number with $k \geq 18$ and k cent postage can be paid for with only 4 and 7 cent stamps for all k between 18 and n (inclusive), then $k + 1$ cent postage can be paid for using only 4 and 7 cent postage – formally, this says $P(18) \wedge P(19) \wedge . . . \wedge P(k) \rightarrow P(k + 1)$.

d) Suppose $k \geq 21$ and P(j) holds for all k satisfying $18 \leq j \leq k$. We want to show

$P(k + 1)$ is true. As $k \geq 21$ we know $k + 1 \geq 22$. It follows that $(k + 1) - 4 \geq 18$ and therefore by the inductive hypothesis, $(k + 1) - 4$ cent postage can be paid for using only 4 and 7 cent stamps. Then adding one more 4 cent stamp gives postage for an $k + 1$ cent delivery.

e) We established that P(18) is true and for all k, if P(j) is true for all k satisfying $18 \leq j \leq k$ then $P(k + 1)$ is true. It follows by strong induction that P(n) is true for all integers $n \geq 18$.