Mean and variance of continuous Uniform distribution

$$f(x) = \frac{1}{b-a} \quad a \leq x \leq b$$

$$= 0 \quad \text{Elsewhere}$$

$$E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b x\, dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b$$

$$= \frac{1}{b-a} \left[\frac{b^2}{2} - \frac{a^2}{2} \right]$$

$$= \frac{1}{b-a} \left[\frac{(b^2-a)^2}{2} \right]$$

$$= \frac{1}{\cancel{b-a}} \frac{(\cancel{b-a})(b+a)}{2}$$

$$= \frac{b+a}{2} \Rightarrow \frac{a+b}{2}$$

So

$$\boxed{E(x) = \frac{a+b}{2}}$$

As $\quad var(x) = E(x^2) - [E(x)]^2$

$$E(x^2) = \int_a^b x^2 f(x)\, dx$$

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$$E(x^2) = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx \Rightarrow$$

$$= \frac{1}{b-a}\int_a^b x^2\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b \Rightarrow \frac{1}{b-a}\left[\frac{b^3-a^3}{3}\right]$$

$$= \frac{1}{3\cancel{(b-a)}}\left[\cancel{(b-a)}(b^2+ab+a^2)\right]$$

$$\boxed{E(x^2) = \frac{a^2+ab+b^2}{3}}$$

Now $\quad var(x) = E(x^2) - [E(x)]^2$

$$= \frac{a^2+ab+b^2}{3} - \left[\left(\frac{a+b}{2}\right)\right]^2$$

$$= \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{1}{12}\left[4(a^2+ab+b^2) - 3(a+b)^2\right]$$

$$= \frac{1}{12}\left[4a^2+4ab+4b^2-3a^2-6ab-3b^2\right]$$

$$= \frac{1}{12}\left[a^2-2ab+b^2\right]$$

$$\boxed{var(x) = \frac{(b-a)^2}{12}}$$

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Exponential distribution:-

$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x > 0$$
$$= 0 \quad \text{else where}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

Integrating by Parts

$$= \lambda \left[x \int_0^{\infty} e^{-\lambda x} dx - \int_0^{\infty} \left[\frac{d}{dx}(x) \int e^{-\lambda x} dx \right] \right]$$

$$\int u\,dv = uv - \int v\,du$$

$$\lambda e^{-\lambda x} dx = dv$$

$$x = u \quad v = -e^{-\lambda x}, \quad du = dx$$

$$= \left[-x e^{-\lambda x} \right]_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx$$

$$= 0 + \left[\frac{-e^{-\lambda x}}{\lambda} \right]_0^{\infty}$$

$$\boxed{E(x) = \frac{1}{\lambda}}$$

$$var(x) = E(x^2) - [E(x)]^2$$

$$E[x^2] = \int_0^{\infty} x^2 \cdot \lambda e^{-\lambda x} dx$$

$$= \lambda \left[x^2 \left(-\frac{1}{\lambda} e^{-\lambda x} \right) \right]_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x} dx$$

$$= 0 + \frac{2}{\lambda} E(x)$$

$$E(x^2) = \frac{2}{\lambda^2}$$

$$\text{var}(x) = \sigma^2 = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}$$

$$S.D = \sqrt{\sigma^2} = \frac{1}{\lambda}$$

Properties of variance of random ~~variance~~ variable :-

1) $\text{var}(a) = 0$
2) If $Y = x + b$ then
   $\text{var}(Y) = \text{var}(x)$
3) If $Y = aX$ then
   $\text{var}(Y) = a^2 \text{var}(x)$.

Expected value of $Y = g(x)$

Suppose If we are interested in finding the expected value of $Y = g(x)$ the $E(Y) = \int_{-\infty}^{\infty} g(x) \cdot f_x(x)\, dx$

Example 3.33

(Gartia Pg 130)

$$E(Y) = E[a \cos(\omega t + \theta)]$$

$$E(Y) = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx$$

Now

$$g(x) = a\cos(\omega t + \theta)$$

$$f_x(x) = \frac{1}{2\pi - 0}$$

Continuous uniform dist $\frac{1}{b-a} = f(x)$

So

$$E(Y) = \int_0^{2\pi} [a\cos(\omega t + \theta)]\left(\frac{1}{2\pi - 0}\right) d\theta$$

$$= \frac{a}{2\pi} \int_0^{2\pi} \cos(\omega t + \theta)\, d\theta$$

$$= \frac{a}{2\pi} \Big| \sin(\omega t + \theta) \Big|_0^{2\pi}$$

$$= \frac{a}{2\pi} \left[\sin(\omega t + 2\pi) - \sin \omega t \right]$$

$$= \frac{a}{2\pi} \left[\sin \omega t - \sin \omega t \right]$$

$$\therefore \boxed{E(Y) = 0}$$

Also $E(Y^2) = E[a^2\cos^2(\omega t+\theta)]$

$$= \int_0^{2\pi} [a^2\cos^2(\omega t+\theta)]\left[\frac{1}{2\pi - 0}\right] d\theta$$

$$= \frac{a^2}{2\pi}\int_0^{2\pi} \cos^2(\omega t+\theta)\, d\theta$$

$$= \frac{a^2}{2\pi}\int_0^{2\pi} \left[\frac{1+\cos(2\omega t+2\theta)}{2}\right] d\theta$$

$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$\cos^2(\omega t+\theta)$

$= \frac{1+\cos(2\omega t+2\theta)}{2}$

$$= \frac{a^2}{4\pi}\int_0^{2\pi} [1+\cos(2\omega t+2\theta)]\, d\theta$$

$$= \frac{a^2}{4\pi}\left[\theta + \sin(2\omega t+2\theta)\right]_0^{2\pi}$$

$$= \frac{a^2}{4\pi}\left[2\pi + \sin(2\omega t+4\pi) - \sin 2\omega t\right]$$

$$= \frac{a^2}{4\pi}\left[2\pi + \cancel{\sin 2\omega t} - \cancel{\sin 2\omega t}\right]$$

$$= \frac{a^2}{\cancel{4\pi}_{\,2}} \cdot \cancel{2\pi} = \frac{a^2}{2}$$

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Example 3-32

(Gartia)
Pg 129

$$E(N) = \sum_{k=1}^{\infty} k p q^{k-1}$$

$$= p \sum_{k=1}^{\infty} k q^{k-1}$$

Since

$$\sum x^k = 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{1-x} \quad \because \text{Infinite geometric series}$$

$$\frac{d}{dx} \sum_{k=0}^{\infty} x^k = \frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \because S_n = \frac{1}{1-r}$$

$$\Rightarrow \sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

Let $x = q$ we get

$$E(N) = p \sum_{k=1}^{\infty} k q^{k-1}$$

$$= p \frac{1}{(1-q)^2}$$

$$= \cancel{p} \cdot \frac{1}{p^2}$$

$$\boxed{E(N) = \frac{1}{p}}$$

If probability of success in one trial is $p = 1/10$ then we expect that on the average $1/p = \frac{1}{1/10} = 10$ trials are required to obtain success