ASSIGNMENT PROBLEM

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INTRODUCTION TO ASSIGNMENT PROBLEM

- An assignment problem is a particular case of transportation problem.
- The objective is to assign a number of resources to an equal number of activities .
- So as to minimize total cost or maximize total profit of allocation.
- The problem of assignment arises because available resources such as men, machines etc. have varying degrees of efficiency for performing different activities, therefore, cost, profit or loss of performing the different activities is different.

- Suppose that we have n jobs to be performed on m machines (one job to one machine).
- Our objective is to assign the jobs to the machines at the minimum cost (or maximum profit).
- Under the assumption that each machine can perform each job but with varying degree of efficiencies.

MATRIX FORM OF ASSIGNENMT PROBLEM

- The assignment problem can be stated that in the form of mxn matrix c_{ij} called a Cost Matrix (or) Effectiveness Matrix where c_{ij} is the cost of assigning i^{th} machine to j^{th} job.

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}$$

MATHEMATICAL FORMULATION OF AN ASSIGNMENT PROBLEM

- Consider an assignment problem of assigning n jobs to n machines (one job to one machine). Let c_{ij} be the unit cost of assigning ith machine to the jth job and, i^{th} machine to j^{th} job.

- Let $x_{ij} = 1$, if j^{th} job is assigned to i^{th} machine.

  $x_{ij} = 0$, if j^{th} job is not assigned to i^{th} machine.

$$\text{Minimize } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

subject to the constraints

$$\sum_{i=1}^{n} x_{ij} = 1, j = 1, 2, ..., n$$

$$\sum_{j=1}^{n} x_{ij} = 1, i = 1, 2, ..., n$$

$$x_{ij} = 0 \text{ or } 1.$$

DIFFERENCE BETWEEN TRANSPORTATION PROBLEM AND ASSIGMENT PROBLEM

S.No	Transportation Problem	Assignment Problem
1	Supply at any source may be any positive quantity ai.	Supply at any source (machine)will be 1. i.e.,$a_i = 1$.
2	Demand at any destination may be any positive quantity bj.	Demand at any destination (job) will be 1. i.e.,$b_j = 1$.
3	One or more source to any number of destinations.	One source (machine) to only one destination (job).

ASSIGMENT ALGORITHM (OR) HUNGARIAN METHOD

- First check whether the number of rows is equal to number of columns, if it is so, the assignment problem is said to be balanced. Then proceed to step 1. If it is not balanced, then it should be balanced before applying the algorithm.
- **Step 1:** Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.
- **Step 2:** Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1 and make sure each column contains atleast one zero.

- **Step 3: (Assigning the zeros)**

  (a) Examine the rows successively until a row with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it. Cross all other zeros in the column of this encircled zero, as these will not be considered for any future assignment. Continue in this way until all the rows have been examined.

  (b) Examine the columns successively until a column with exactly one unmarked zero is found. Make an assignment to this single unmarked zero by encircling it and cross any other zero in its row. Continue until all the columns have been examined.

- **Step 4: (Apply Optimal Test)**

  (a) If each row and each column contain exactly one encircled zero, then the current assignment is optimal.

  (b) If atleast one row or column is without an assignment (i.e., if there is atleast one row or column is without one encircled zero), then the current assignment is not optimal. Go to step 5. Subtract the smallest cost element of each column from all the elements in the column of the resulting cost matrix obtained by step 1 and make sure each column contains atleast one zero.

- **Step 5: Cover all the zeros by drawing a minimum number of straight lines as follows:**

  (a) Mark the rows that do not have assignment.

  (b) Mark the columns (not already marked) that have zeros in marked rows.

  (c) Mark the rows (not already marked) that have assignments in marked columns.

  (d) Repeat (b) and (c) until no more marking is required.

  (e) Draw lines through all unmarked rows and marked columns. If the number of these lines is equal to the order of the matrix then it is an optimum solution otherwise not.

- **Step 6:** Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

- **Step 7:** Repeat steps (1) to (6), until an optimum assignment is obtained.

EXAMPLE OF ASSIGMENT PROBLEMS

PROBLEM 1: Solve the following assignment problem shown in Table using Hungarian method. The matrix entries are processing time of each man in hours.

$$\begin{bmatrix} & I & II & III & IV & V \\ 1 & 20 & 15 & 18 & 20 & 25 \\ 2 & 18 & 20 & 12 & 14 & 15 \\ 3 & 21 & 23 & 25 & 27 & 25 \\ 4 & 17 & 18 & 21 & 23 & 20 \\ 5 & 18 & 18 & 16 & 19 & 20 \end{bmatrix}$$

Solution: The given problem is balanced with 5 job and 5 men.

$$A = \begin{bmatrix} 20 & 15 & 18 & 20 & 25 \\ 18 & 20 & 12 & 14 & 15 \\ 21 & 23 & 25 & 27 & 25 \\ 17 & 18 & 21 & 23 & 20 \\ 18 & 18 & 16 & 19 & 20 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 5 & 0 & 3 & 5 & 10 \\ 6 & 8 & 0 & 2 & 3 \\ 0 & 2 & 4 & 6 & 4 \\ 0 & 1 & 4 & 6 & 3 \\ 2 & 2 & 0 & 3 & 4 \end{bmatrix}$$

Subtract the smallest cost element of each Column from all the elements in the Column of the given cost matrix. See that each Column contains atleast one zero.

$$A = \begin{bmatrix} 5 & 0 & 3 & 3 & 7 \\ 6 & 8 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 1 \\ 0 & 1 & 4 & 4 & 0 \\ 2 & 2 & 0 & 1 & 1 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 5 & \boxed{0} & 3 & 3 & 7 \\ 6 & 8 & \cancel{0} & \boxed{0} & \cancel{0} \\ \boxed{0} & 2 & 4 & 4 & 1 \\ \cancel{0} & 1 & 4 & 4 & \boxed{0} \\ 2 & 2 & \boxed{0} & 1 & 1 \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to II , 2 to IV , 3 to I , 4 to V and 5 to III.
- The optimal $z = 15 + 14 + 21 + 20 + 16 = 86$ hours.

PROBLEM 2: Solve the following assignment problem shown in Table using Hungarian method. The matrix entries are processing time of each Job to each machine in hours.

$$\begin{bmatrix} J/M & I & II & III & IV & V \\ 1 & 9 & 22 & 58 & 11 & 19 \\ 2 & 43 & 78 & 72 & 50 & 63 \\ 3 & 41 & 28 & 91 & 37 & 45 \\ 4 & 74 & 42 & 27 & 49 & 39 \\ 5 & 36 & 11 & 57 & 22 & 25 \end{bmatrix}$$

Solution: The given problem is balanced with 5 job and 5 machine.

$$A = \begin{bmatrix} 9 & 22 & 58 & 11 & 19 \\ 43 & 78 & 72 & 50 & 63 \\ 41 & 28 & 91 & 37 & 45 \\ 74 & 42 & 27 & 49 & 39 \\ 36 & 11 & 57 & 22 & 25 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

Subtract the smallest cost element of each Column from all the elements in the Column of the given cost matrix. See that each Column contains atleast one zero.

$$A = \begin{bmatrix} 0 & 13 & 49 & 0 & 0 \\ 0 & 35 & 29 & 5 & 10 \\ 13 & 0 & 63 & 7 & 7 \\ 47 & 15 & 0 & 20 & 2 \\ 25 & 0 & 46 & 9 & 4 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$
\begin{bmatrix}
\cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\
\boxed{0} & 35 & 29 & 5 & 10 \\
13 & \boxed{0} & 63 & 7 & 7 \\
47 & 15 & \boxed{0} & 20 & 2 \\
25 & \cancel{0} & 46 & 9 & 4
\end{bmatrix}
$$

Marking the unassigning the zeros row and crosed zero colunm of marked row,we have the following $A =$

$$\begin{array}{cc}
\left[\begin{array}{ccccc}
\cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\
\boxed{0} & 35 & 29 & 5 & 10 \\
13 & \boxed{0} & 63 & 7 & 7 \\
47 & 15 & \boxed{0} & 20 & 2 \\
25 & \cancel{0} & 46 & 9 & 4
\end{array}\right] & \begin{array}{c} \\ \\ \checkmark \\ \\ \checkmark \end{array} \\
\begin{array}{ccccc} & \checkmark & & & \end{array} &
\end{array}$$

Crossing the marked column and unmarked row,we have the following $A =$

$$
\begin{array}{cc}
\begin{bmatrix}
\cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\
\boxed{0} & 35 & 29 & 5 & 10 \\
13 & \boxed{0} & 63 & 7 & 7 \\
47 & 15 & \boxed{0} & 20 & 2 \\
25 & \cancel{0} & 46 & 9 & 4
\end{bmatrix} & \begin{array}{c} \\ \\ \checkmark \\ \\ \checkmark \end{array} \\
\begin{array}{ccccc} & \checkmark & & & \end{array} &
\end{array}
$$

Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

$$A = \begin{bmatrix} 0 & 17 & 49 & 0 & 0 \\ 0 & 39 & 29 & 5 & 10 \\ 9 & 0 & 59 & 3 & 3 \\ 47 & 19 & 0 & 20 & 2 \\ 21 & 0 & 42 & 5 & 0 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} \cancel{0} & 17 & 49 & \boxed{0} & \cancel{0} \\ \boxed{0} & 39 & 29 & 5 & 10 \\ 9 & \boxed{0} & 59 & 3 & 3 \\ 47 & 19 & \boxed{0} & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & \boxed{0} \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to IV , 2 to I , 3 to II , 4 to III and 5 to V.
- The optimal $z = 11 + 43 + 28 + 27 + 25 = 134$ hours.

PROBLEM 3: Solve the assignment problem At the head office of a company there are five registration counters. Five persons are available for service. How should the counters be assigned to persons so as to maximize the profit?

$$\begin{bmatrix} \text{C/P} & A & B & C & D & E \\ 1 & 30 & 37 & 40 & 28 & 40 \\ 2 & 40 & 24 & 27 & 21 & 36 \\ 3 & 40 & 32 & 33 & 30 & 35 \\ 4 & 25 & 38 & 40 & 36 & 36 \\ 5 & 29 & 62 & 41 & 34 & 39 \end{bmatrix}$$

Solution: The given problem is balanced with 5 job and 5 machine.
To convert the problem as minimization we reduce the matrix by subtracting all entry by the largest value , that is 62

$$A = \begin{bmatrix} 32 & 25 & 22 & 34 & 22 \\ 22 & 38 & 35 & 41 & 26 \\ 22 & 30 & 29 & 30 & 27 \\ 37 & 24 & 22 & 26 & 26 \\ 33 & 0 & 21 & 28 & 23 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 10 & 3 & 0 & 12 & 0 \\ 0 & 16 & 13 & 19 & 4 \\ 0 & 8 & 7 & 8 & 5 \\ 15 & 2 & 0 & 4 & 4 \\ 33 & 0 & 21 & 28 & 23 \end{bmatrix}$$

Subtract the smallest cost element of each Column from all the elements in the Column of the given cost matrix. See that each Column contains atleast one zero.

$$A = \begin{bmatrix} 10 & 3 & 0 & 8 & 0 \\ 0 & 16 & 13 & 15 & 4 \\ 0 & 8 & 7 & 4 & 5 \\ 15 & 2 & 0 & 0 & 4 \\ 33 & 0 & 21 & 24 & 23 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 10 & 3 & \cancel{0} & 8 & \boxed{0} \\ \boxed{0} & 16 & 13 & 15 & 4 \\ \cancel{0} & 8 & 7 & 4 & 5 \\ 15 & 2 & \cancel{0} & \boxed{0} & 4 \\ 33 & \boxed{0} & 21 & 24 & 23 \end{bmatrix}$$

Marking the unassigning the zeros row and crosed zero colunm of marked row,we have the following $A =$

$$
\begin{array}{cc}
\begin{bmatrix}
10 & 3 & \cancel{0} & 8 & \boxed{0} \\
\boxed{0} & 16 & 13 & 15 & 4 \\
\cancel{0} & 8 & 7 & 4 & 5 \\
15 & 2 & \cancel{0} & \boxed{0} & 4 \\
33 & \boxed{0} & 21 & 24 & 23
\end{bmatrix}
&
\begin{array}{c}
\\
\checkmark \\
\checkmark \\
\\
\\
\end{array}
\\
\begin{array}{ccccc}
\checkmark & & & &
\end{array}
&
\end{array}
$$

Crossing the marked column and unmarked row,we have the following $A =$

$$\begin{array}{cc}
\begin{bmatrix}
10 & 3 & \cancel{0} & 8 & \boxed{0} \\
\boxed{0} & 16 & 13 & 15 & 4 \\
\cancel{0} & 8 & 7 & 4 & 5 \\
15 & 2 & \cancel{0} & \boxed{0} & 4 \\
33 & \boxed{0} & 21 & 24 & 23
\end{bmatrix} & \begin{array}{c} \\ \checkmark \\ \checkmark \\ \\ \\ \end{array} \\
\begin{array}{ccccc} \checkmark & & & & \end{array} &
\end{array}$$

Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

$$A = \begin{bmatrix} 14 & 3 & 0 & 8 & 0 \\ 0 & 12 & 9 & 11 & 0 \\ 0 & 4 & 3 & 0 & 1 \\ 19 & 2 & 0 & 0 & 4 \\ 37 & 0 & 21 & 24 & 23 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 14 & 3 & \boxed{0} & 8 & \cancel{0} \\ \cancel{0} & 12 & 9 & 11 & \boxed{0} \\ \boxed{0} & 4 & 3 & \cancel{0} & 1 \\ 19 & 2 & \cancel{0} & \boxed{0} & 4 \\ 37 & \boxed{0} & 21 & 24 & 23 \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to C , 2 to E , 3 to A , 4 to D and 5 to B.
- The Maximize profit is $z = 40 + 36 + 40 + 36 + 62 = 214$.

PROBLEM 4: Solve the assignment problem

$$\begin{bmatrix} \text{M/J} & A & B & C & D \\ 1 & 7 & 5 & 8 & 4 \\ 2 & 5 & 6 & 7 & 4 \\ 3 & 8 & 7 & 9 & 8 \end{bmatrix}$$

Solution: The given problem is not balanced with 3 men and 4 job.
To convert the problem as balanced by adding a dummy row with 0 cost we have.

$$A = \begin{bmatrix} 7 & 5 & 8 & 4 \\ 5 & 6 & 7 & 4 \\ 8 & 7 & 9 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Subtract the smallest cost element of each row from all the elements in the row of the given cost matrix. See that each row contains atleast one zero.

$$A = \begin{bmatrix} 3 & 1 & 4 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 3 & 1 & 4 & \boxed{0} \\ 1 & 2 & 3 & \cancel{0} \\ 1 & \boxed{0} & 2 & 1 \\ \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Marking the unassigning the zeros row and crosed zero colunm of marked row,we have the following $A =$

$$
\begin{array}{cc}
\begin{bmatrix}
3 & 1 & 4 & \boxed{0} \\
1 & 2 & 3 & \cancel{0} \\
1 & \boxed{0} & 2 & 1 \\
\boxed{0} & \cancel{0} & \cancel{0} & \cancel{0}
\end{bmatrix}
&
\begin{array}{c}
\checkmark \\
\checkmark \\
\\
\\
\end{array}
\\
\begin{array}{cccc}
 & & & \checkmark
\end{array}
&
\end{array}
$$

Crossing the marked column and unmarked row,we have the following $A =$

$$\begin{array}{c c}
\begin{bmatrix}
3 & 1 & 4 & \boxed{0} \\
1 & 2 & 3 & \cancel{0} \\
1 & \boxed{0} & 2 & 1 \\
\boxed{0} & \cancel{0} & \cancel{0} & \cancel{0}
\end{bmatrix} & \begin{array}{c} \checkmark \\ \checkmark \\ \\ \\ \end{array} \\
\begin{array}{cccc} \quad & \quad & \quad & \qquad\quad\checkmark \end{array} &
\end{array}$$

Determine the smallest cost element not covered by the straight lines. Subtract this smallest cost element from all the uncovered elements and add this to all those elements which are lying in the intersection of these straight lines and do not change the remaining elements which lie on the straight lines.

$$A = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Assigning the zeros,we have the following $A =$

$$\begin{bmatrix} 2 & \cancel{0} & 2 & \boxed{0} \\ \boxed{0} & 1 & 2 & \cancel{0} \\ 1 & \boxed{0} & 2 & 2 \\ \cancel{0} & \cancel{0} & \boxed{0} & 1 \end{bmatrix}$$

- Since each row and each column contain exactly one encircled zero, then the current assignment is optimal.
- Where the optimal assignment is as 1 to D , 2 to A , 3 to B , d to C .
- The optimal solution is $z = 4 + 5 + 7 + 0 = 16$.

QUESTION TO ANSWER

PROBLEM 1: Solve the following assignment problem .

$$\begin{bmatrix} \text{J/M} & A & B & C & D \\ 1 & 1 & 4 & 6 & 3 \\ 2 & 9 & 7 & 10 & 9 \\ 3 & 4 & 5 & 11 & 7 \\ 4 & 8 & 7 & 8 & 5 \end{bmatrix}$$

Answer:1 to A, 2 to C, 3 to B, 4 to D . Min cost =21

PROBLEM 2: Solve the assignment problem

$$
\begin{array}{c c}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{bmatrix}
\end{array}
\quad (\text{M/J})
$$

Answer:1 to A, 2 to B, 3 to C, 4 to D . Min cost =50

PROBLEM 3: Solve the following assignment problem to get maximum profit .

$$\begin{bmatrix} \text{J/M} & A & B & C & D \\ 1 & 35 & 27 & 28 & 37 \\ 2 & 28 & 34 & 29 & 40 \\ 3 & 35 & 24 & 32 & 28 \\ 4 & 24 & 32 & 25 & 28 \end{bmatrix}$$

Answer:1 to A, 2 to D, 3 to C, 4 to B . Max profit $=139$

MCQ QUESTIONS WITH ANSWER

1. If the number of rows and columns in an assignment problem are not equal than it is called problem.
   (a) prohibited
   (b) infeasible
   (c) unbounded
   (d) **unbalanced**
2. The method of solution of assignment problems is called method.
   (a) NWCR
   (b) VAM
   (c) LCM
   (d) **Hungarian**

3. When a maximization assignment problem is converted in minimization problem, the resulting matrix is called
   (a) Cost matrix
   (b) Profit matrix
   (c) **Regret matrix**
   (d) Dummy matrix
4. The extra row or column which is added to balance an assignment problem is called
   (a) regret
   (b) epsilon
   (c) **dummy**
   (d) extra

5. When a particular assignment in the given problem is not possible or restricted as a condition, it is called a problem.
   (a) infeasible
   (b) degenerate
   (c) unbalanced
   (d) **prohibited**
6. If in an assignment problem, number of rows is not equal to number of columns then
   (a) Problem is degenerate
   (b) **Problem is unbalanced**
   (c) It is a maximization problem
   (d) Optimal solution is not possible


THANK YOU

GAME THEORY

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- The aim of the theory of games is to analyse the different situations each player has to face and different situation he has to choose according to those of the opponent.
- The term game represents a conflict between two or more parties. Some of the recreational games such as checkers, chess, bridge, etc., can be analysed as games of strategy.
- The application of game theory is not limited to games in ordinary sense of it but also includes in economics, business, warefare and social behaviour, etc.
- To analyse the theory of games we introduce the terms players, strategy pay off and saddle point.

PROPERTY OF GAME THEORY

A competitive situation is called a game if it has the following properties.

- There are finite number of competitors called players.
- A list of finite or infinite number of possible courses of action is available to each player.
- A play is played when each player chooses one of his courses of action. The courses are assumed to be made simultaneously, so that, no player knows his opponent's choice until he has decided his course of action.

- Every play is associated with an outcome, known as the pay off which determine a set of gains, one to each player. Here a loss considered as a negative gain. The other assumptions are,
    - All players act rationally
    - Each player attempts to maximize his gain or minimize his loss
    - Complete relevant information is known to each player
    - Each player makes individual decisions without direct communications.
- A game involving n players is called n-person game. Two-person games are of more importance.

TWO PERSON ZERO SUM GAME

- If the algebraic sum of gains and losses of all the players is zero in a game then such game is called **Zero sum Game**.
- If the algebraic sum of gains and losses of all the players is not zero in a game then such game is called **Non Zero sum Game**.
- A game with two players wherein one person's gain is the loss of the other is called **two person zero sum game**.
- The pay offs corresponding to various strategies of the players are represented in a matrix called the **game matrix**.
- The gains resulting from a two person zero sum game can be represented in the matrix form, usually called **Pay-off matrix**.

BASIC DEFINATION OF GAME THEORY

- If the number of strategies of the players is finite then the game is said to be a **Finite Game**.
- If at least one of the players has infinite number of strategies, the game is said to be **Infinite Game**.
- If a player known exactly what the other player is going to do, a deterministic situation is obtained and **objective function is to maximize the gain**.
- If the best strategy for each player is to play one particular strategy throughout the game it is called **Pure Strategy Game**.
- If the optimal plan for each player is to choose different strategies at different situations the game is called a **Mixed Strategy Game**.

- If a player known exactly what the other player is going to do, a **deterministic situation** is obtained and objective function is to maximize the gain.
- The maximum value of row minima value of a pay of matrix is called as **Maximin value** and maximin value is denoted as $\underline{v}$.
- The minima value of column maximum value of a pay of matrix is called as **Minimax value** and minimax value is denoted as $\bar{v}$.
- A **saddle point** of a payoff matrix is that position in the payoff matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.

- When the *maximin* $=$ *minimax* value of the game the corresponding **pure strategies** are called optimum strategies.

- The value of the game is denoted as v .

- A game is said to be **fair** if $\underline{v} = \bar{v} = v = 0$

- A game is said to be **Strictly Determinable** if $\underline{v} = \bar{v} = v$

- When *maximin* $\neq$ *minimax*, then pure strategy fails and this condition is **mixed strategy**.

Problem 1 : Solve the following game

$$\begin{array}{cccc} & B1 & B2 & B3 \\ A1 & 20 & 15 & 22 \\ A2 & 35 & 45 & 40 \\ A3 & 18 & 20 & 25 \end{array}$$

Solution:

$$\begin{bmatrix} & B1 & B2 & B3 & ROW\ MIN \\ A1 & 20 & 15 & 22 & 15 \\ A2 & 35 & 45 & 40 & 35 \\ A3 & 18 & 20 & 25 & 18 \\ COLUMN\ MAX & 35 & 45 & 40 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = 35$.
- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = 35$.

- Here $maximin = minimax$, that is $\underline{v} = \bar{v} = 35$.

- The Saddle point is $(A2, B1)$ and Value of the game $v = 35$.

- The game is said to be **Strictly Determinable** , with $\underline{v} = \bar{v} = v = 35.$

- $\therefore$ the value of the game $v = 35$ wich is **Strictly Determinable game** and the corresponding Saddle point is $S = (A2, B1)$

Problem 2 : Solve the following game

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \end{array}\begin{array}{c} \begin{array}{ccc} B1 & B2 & B3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Solution:

$$\begin{bmatrix} & B1 & B2 & B3 & ROW\ MIN \\ A1 & 1 & 3 & 1 & 1 \\ A2 & 0 & -4 & -3 & -4 \\ A3 & 1 & 5 & -1 & -1 \\ COLUMN\ MAX & 1 & 5 & 1 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = 1$.
- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = 1$.

- Here $maximin = minimax$, that is $\underline{v} = \bar{v} = 1$.

- The Saddle point is $(A1, B1)$ *or* $(A1, B3)$ and Value of the game $v = 1$.

- The game is said to be **Strictly Determinable** , with $\underline{v} = \bar{v} = v = 1$.

- $\therefore$ the value of the game $v = 1$ wich is **Strictly Determinable game** and the corresponding Saddle point is $S = (A1, B1)$ *or* $(A1, B3)$

Problem 3 : For what value of λ, the game with the following matrix is strictly determinable.

$$\begin{array}{c c}
 & \begin{array}{ccc} B1 & B2 & B3 \end{array} \\
\begin{array}{c} A1 \\ A2 \\ A3 \end{array} & \begin{bmatrix} \lambda & 6 & 2 \\ -1 & \lambda & -7 \\ -2 & 4 & \lambda \end{bmatrix}
\end{array}$$

Solution: Ignoring the value of λ we have

$$\begin{bmatrix} & B1 & B2 & B3 & ROW\ MIN \\ A1 & \lambda & 6 & 2 & 2 \\ A2 & -1 & \lambda & -7 & -7 \\ A3 & -2 & 4 & \lambda & -2 \\ COLUMN\ MAX & -1 & 6 & 2 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = 2$.

- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = -1$.

- Given that, the game is **Strictly Determinable** , then we have $\underline{v} = \bar{v} = v$.

- $\therefore$ the value of λ lies between $-1 \leq \lambda \leq 2$

Problem 4 : Determine the range of p and q that will make the payoff element a_{22} as the saddle point of the game with the following matrix .

$$\begin{bmatrix} & B1 & B2 & B3 \\ A1 & 2 & 4 & 5 \\ A2 & 10 & 7 & q \\ A3 & 4 & p & 8 \end{bmatrix}$$

Solution: Ignoring the value of p and q we have

$$\begin{bmatrix} & B1 & B2 & B3 & ROW\ MIN \\ A1 & 2 & 4 & 5 & 2 \\ A2 & 10 & 7 & q & 7 \\ A3 & 4 & p & 8 & 4 \\ COLUMN\ MAX & 10 & 7 & 8 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = 7$.
- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = 7$.
- The Saddle point is $(A2, B2)$,that is a_{22}.
- This implies , from column B2 $p \leq 7$ and from row A2 $q \geq 7$.
- $\therefore$ the range of p and q value is as $p \leq 7$ and $q \geq 7$.

PROBLEMS ON MIXED STRATEGY - SOLUTION OF 2×2 MATRIX WITHOUT SADDLE POINT

METHOD 1:
solve the game with 2×2 pay of matrix

$$\begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} & B1 & B2 & ROW\ MIN \\ A1 & 2 & 5 & 2 \\ A2 & 7 & 3 & 3 \\ COLUMN\ MAX & 7 & 5 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = 3$.
- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = 5$.
- Here $maximin \neq minimax$, therefore the game is **mixed strategy** and no Saddle point .

- To find the Strategy of Player B

- Let player B choose 2 and 5 with the problity x and $1 - x$ and 7 and 3 with the problity x and $1 - x$

- $\Rightarrow 2x + 5(1 - x) = 7x + 3(1 - x)$

  $\Rightarrow -7x = -2$

  $\Rightarrow x = \frac{2}{7}$ and $(1 - x) = \frac{5}{7}$

- Strategy of player B is as $S_B = (\frac{2}{7}, \frac{5}{7})$

- Value of the game $v = (2 \times \frac{2}{7}) + (5 \times \frac{5}{7}) = \frac{29}{7}$

  $\Rightarrow v = \frac{29}{7}$

- To find the Strategy of Player A
- Let player A choose 2 and 7 with the problity y and $1-y$ and 5 and 3 with the problity y and $1-y$
- $\Rightarrow 2y + 7(1-y) = 5y + 3(1-y)$

  $\Rightarrow 7y = 4$

  $\Rightarrow y = \frac{4}{7}$ and $(1-y) = \frac{3}{7}$
- Strategy of player A is as $S_A = (\frac{4}{7}, \frac{3}{7})$
- Value of the game $v = (2 \times \frac{4}{7}) + (7 \times \frac{3}{7}) = \frac{29}{7}$

  $\Rightarrow v = \frac{29}{7}$

METHOD 2:
solve the game with 2×2 pay of matrix

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

To find the strategy of the player's A and B as

- The strategy of player A, $S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix}$ and
- The strategy of player B, $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$

To find the Value of the game using the formulas

- $\lambda = [a_{11} + a_{22}] - [a_{12} + a_{21}]$
- $p_1 = \frac{a_{22} - a_{21}}{\lambda}$ and $p_2 = 1 - p_1$
- $q_1 = \frac{a_{22} - a_{12}}{\lambda}$ and $q_2 = 1 - q_1$
- $v = \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda}$

Problem 1:

solve the game with 2×2 pay of matrix with out saddle point

$$\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$$

Solution: To find the value of the game and strategy of the player's A and B as

- $\lambda = [a_{11} + a_{22}] - [a_{12} + a_{21}] = [2+1] - [5+4] = -6$
- $p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{1 - 4}{-6} = \frac{1}{2}$ and

  $p_2 = 1 - p_1 = 1 - \frac{1}{2} = \frac{1}{2}$

- $q_1 = \frac{a_{22} \;-\; a_{12}}{\lambda} = \frac{1 \;-\; 5}{-6} = \frac{2}{3}$ and

  $q_2 = 1 \;-\; q_1 = 1 - \frac{2}{3} = \frac{1}{3}$

- $v = \frac{a_{11}a_{22} \;-\; a_{12}a_{21}}{\lambda} = \frac{2-20}{-6} = \frac{-18}{-6} = 3.$

- The strategy of player A, $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and

- The strategy of player B, $S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

Problem 2:
solve the game with 2×2 pay of matrix with out saddle point

$$\begin{bmatrix} 6 & 9 \\ 8 & 4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} & B1 & B2 & ROW\ MIN \\ A1 & 6 & 9 & 6 \\ A2 & 8 & 4 & 4 \\ COLUMN\ MAX & 8 & 9 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = 6$.
- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = 8$.
- Here *maximin* $\neq$ *minimax*, therefore the game is **mixed strategy** and no Saddle point .
- To find the value of the game and strategy of the player's A and B as
- $\lambda = [a_{11} + a_{22}] - [a_{12} + a_{21}] = [6 + 4] - [9 + 8] = -7$
- $p_1 = \frac{a_{22} - a_{21}}{\lambda} = \frac{4 - 8}{-7} = \frac{4}{7}$ and

  $p_2 = 1 - p_1 = 1 - \frac{4}{7} = \frac{3}{7}$

- $q_1 = \frac{a_{22} \quad - \quad a_{12}}{\lambda} = \frac{4 \quad - \quad 9}{-7} = \frac{5}{7}$ and

  $q_2 = 1 \quad - \quad q_1 = 1 - \frac{5}{7} = \frac{2}{7}$

- $v = \frac{a_{11}a_{22} \quad - \quad a_{12}a_{21}}{\lambda} = \frac{24-72}{-7} = \frac{48}{7}$.

- The strategy of player A, $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix}$ and

- The strategy of player B, $S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{5}{7} & \frac{2}{7} \end{bmatrix}$

Problem 3:
solve the game with 2×2 pay of matrix

$$\begin{bmatrix} 2 & -1 \\ -1 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} & H & T & ROW\ MIN \\ H & 2 & -1 & -1 \\ T & -1 & 0 & -1 \\ COLUMN\ MAX & 2 & 0 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = -1$.
- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = 0$.
- Here *maximin* $\neq$ *minimax*, therefore the game is **mixed strategy** and no Saddle point .
- To find the value of the game and strategy of the player's A and B as
- $\lambda = [a_{11} + a_{22}] - [a_{12} + a_{21}] = [2 + 0] - [-1 + -1] = 4$
- $p_1 = \frac{a_{22} \; - \; a_{21}}{\lambda} = \frac{0 \; - \; -1}{4} = \frac{1}{4}$ and
  $p_2 = 1 \; - \; p_1 = 1 - \frac{1}{4} = \frac{3}{4}$

- $q_1 = \frac{a_{22} - a_{12}}{\lambda} = \frac{0 - -1}{4} = \frac{1}{4}$ and
  $q_2 = 1 - q_1 = 1 - \frac{1}{4} = \frac{3}{4}$
- $v = \frac{a_{11}a_{22} - a_{12}a_{21}}{\lambda} = \frac{0-1}{4} = \frac{-1}{4}$.
- The strategy of player A, $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ and
- The strategy of player B, $S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

Problem 4: In a game of matching coins with two players, suppose A wins one unit value when there are two heads, wins nothing when there are two tails, and losses $\frac{1}{2}$ unit value when there are one head and one tail. Determine the payoff matrix, the best strategy for each player, and the value of the game.

Solution:The pay off matrix is as

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} & H & T & ROW\ MIN \\ H & 1 & -\frac{1}{2} & -\frac{1}{2} \\ T & -\frac{1}{2} & 0 & -\frac{1}{2} \\ COLUMN\ MAX & 1 & 0 & \end{bmatrix}$$

- Where Maximin $\underline{v}$ = Maximum of row minimum , that is $\underline{v} = -\frac{1}{2}$.

- Where Minimax $\bar{v}$ = minimum of column maximum , that is $\bar{v} = 0$.

- Here *maximin* $\neq$ *minimax*, therefore the game is **mixed strategy** and no Saddle point .
- To find the value of the game and strategy of the player's A and B as

- $\lambda = [a_{11} + a_{22}] - [a_{12} + a_{21}] = [1 + 0] - [-\frac{1}{2} + -\frac{1}{2}] = 2$

- $p_1 = \frac{a_{22} \; - \; a_{21}}{\lambda} = \frac{0 \; - \; -\frac{1}{2}}{2} = \frac{1}{4}$ and
  $p_2 = 1 \; - \; p_1 = 1 - \frac{1}{4} = \frac{3}{4}$

- $q_1 = \frac{a_{22} \quad - \quad a_{12}}{\lambda} = \frac{0 \quad - \quad -\frac{1}{2}}{2} = \frac{1}{4}$ and
  $q_2 = 1 \quad - \quad q_1 = 1 - \frac{1}{4} = \frac{3}{4}$

- $v = \frac{a_{11}a_{22} \quad - \quad a_{12}a_{21}}{\lambda} = \frac{0-\frac{1}{4}}{2} = \frac{-1}{8}$.

- The strategy of player A, $S_A = \begin{bmatrix} A_1 & A_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$ and

- The strategy of player B, $S_B = \begin{bmatrix} B_1 & B_2 \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}$

MATRIX ODDMENT METHOD FOR $n \times n$ GAMES OR ARITHEMETIC METHOD

- **Step 1:** Let $A = (a_{ij})$ be $n \times n$ pay off matrix. Obtain a new matrix C, whose first column is obtained form A by subtracting 2nd column from 1st ; second column is obtained by subtracting 3rd column from 2nd and so on till the last column of A is taken care of.Thus C is a $n \times (n-1)$ matrix.

- **Step 2:** Obtain a new matrix R, from A, by subtracting its successive rows from the preceding ones, in exactly the same manner as was done for columns is step 1. Thus R is a $(n-1) \times n$ matrix.

- **Step 3:** Determine the magnitude of oddments corresponding to each row and each column of A. The oddment corresponding to ith row of A is defined as the determinant C_i where C_i is obtained from C by deleting ith row. Similarly oddment corresponding jth column of A is R_j, defined as determinant where R_j is obtained from R by deleting its jth column.
- **Step 4:** Write the magnitude of oddments (after ignoring negative signs, if any) against their respective rows and columns.
- **Step 5:** Check whether the sum of row oddments is equal to the sum of column oddments. If so, the oddments expressed as fractions of the grand total yields the optimum strategies. If not , the method fails.
- **Step 6:** Calculate the expected value of the game corresponding to the optimum mixed strategy determined above for the row player (against any move of the column player).

PROBLEM 1:
Solve the following 3×3 game

$$\begin{bmatrix} 3 & -1 & -3 \\ -3 & 3 & -1 \\ -4 & -3 & 3 \end{bmatrix}$$

Solution : we obtain the matrix C and R.

$$C = \begin{bmatrix} 4 & 2 \\ -6 & 4 \\ -1 & -6 \end{bmatrix} \text{ and } R = \begin{bmatrix} 6 & -4 & -2 \\ -2 & 6 & -4 \end{bmatrix}$$

Now to get the row and column oddments from matrix C and R.

$$|C1| = \begin{vmatrix} -6 & -4 \\ -1 & -6 \end{vmatrix} = 40 \ , \qquad |R1| = \begin{vmatrix} -4 & -2 \\ 6 & -4 \end{vmatrix} = 28$$

$$|C2| = \begin{vmatrix} 4 & 2 \\ -1 & -6 \end{vmatrix} = -22, \qquad |R2| = \begin{vmatrix} 6 & -2 \\ -2 & -4 \end{vmatrix} = -22$$

$$|C3| = \begin{vmatrix} 4 & 2 \\ -6 & 4 \end{vmatrix} = 28 \ , \qquad |R3| = \begin{vmatrix} 6 & -4 \\ -2 & 6 \end{vmatrix} = 40$$

The augmented payoff matrix is as followes

				Row oddments
	3	−1	−3	40
	−3	3	−1	22
	−4	−3	3	28
Column oddments	28	22	40	90

- Strategy of Row Player $= (\frac{40}{90}, \frac{22}{90}, \frac{28}{90}) = (\frac{4}{9}, \frac{11}{45}, \frac{14}{45})$
- Strategy of Column Player $= (\frac{28}{90}, \frac{22}{90}, \frac{40}{90}) = (\frac{14}{45}, \frac{11}{45}, \frac{4}{9})$
- Value of the Game
$= [(\frac{4}{90} \times (3) + \frac{11}{45} \times (-3) + \frac{14}{45} \times (-4)] = \frac{-29}{45}$
- $\therefore V = \frac{-29}{45}$

PROBLEM 2:

Solve the following 3×3 game

$$\begin{bmatrix} -1 & 2 & 1 \\ 1 & -2 & 2 \\ 3 & 4 & -3 \end{bmatrix}$$

Solution : we obtain the matrix C and R.

$$C = \begin{bmatrix} -3 & 1 \\ 3 & -4 \\ -1 & 7 \end{bmatrix} \text{ and } R = \begin{bmatrix} -2 & 4 & -1 \\ -2 & -6 & 5 \end{bmatrix}$$

Now to get the row and column oddments from matrix C and R.

$$|C1| = \begin{vmatrix} 3 & -4 \\ -1 & 7 \end{vmatrix} = 17\ , \qquad |R1| = \begin{vmatrix} 4 & -1 \\ -6 & 5 \end{vmatrix} = 14$$

$$|C2| = \begin{vmatrix} -3 & 1 \\ -1 & 7 \end{vmatrix} = -20, \qquad |R2| = \begin{vmatrix} -2 & -1 \\ -2 & 5 \end{vmatrix} = -12$$

$$|C3| = \begin{vmatrix} -3 & 1 \\ 3 & -4 \end{vmatrix} = 9\ , \qquad |R3| = \begin{vmatrix} -2 & 4 \\ 4 & -6 \end{vmatrix} = 20$$

The augmented payoff matrix is as followes

$$\begin{array}{lccccc} & & & & & \text{Row oddments} \\ & -1 & 2 & 1 & & 17 \\ & 1 & -2 & 2 & & 20 \\ & 3 & 4 & -3 & & 9 \\ \text{Column oddments} & 14 & 12 & 20 & & 46 \end{array}$$

- Strategy of Row Player $= (\frac{17}{46}, \frac{20}{46}, \frac{9}{46})$
- Strategy of Column Player $= (\frac{14}{46}, \frac{12}{46}, \frac{20}{46})$
- Value of the Game $= [(\frac{17}{46} \times (-1) + \frac{20}{46} \times (1) + \frac{9}{46} \times (3)] = \frac{15}{23}$
- $\therefore V = \frac{15}{23}$

DOMINANCE PROPERTY

- It is observed that one of the pure strategies of either player is inferior to atleast one of the remaining ones.
- The superior strategies are said to dominate the inferior ones.
- In such cases of dominance, we can reduce the size of the payoff matrix by deleting those strategies which are dominated by other.
- The following rules are applied:
  - If all the elements of a row, say kth, are less than or equal to the corresponding elements of any other row, say rth , the kth row is dominated by rthrow. Omit dominated rows.
  - If all the elements of a column, say kth, are greater than or equal to the corresponding elements of any other column, say rth , the kth column is dominated by rth column. Omit dominated columns.
  - If some linear combination of some rows dominates ith row, the ith row will be deleted. Similar rules follow for columns.

PROBLEM 1:Using DOMINANCE PROPERTY solve the following game

$$
\begin{array}{c}
\begin{array}{cccccc}
 & B1 & B2 & B3 & B4 & B5 \\
A1 & 2 & 4 & 3 & 8 & 4 \\
A2 & 5 & 6 & 3 & 7 & 8 \\
A3 & 6 & 7 & 9 & 8 & 7 \\
A4 & 4 & 2 & 8 & 4 & 2
\end{array}
\end{array}
$$

Solution:

- $B2$ is dominated by $B4 \implies$ omite the column $B4$.
- Also $B2$ is dominated by $B5 \implies$ omite the column $B5$
- Omiting $B4$ and $B5$, we have the reduced matrix as

$$\begin{array}{c} \\ A1 \\ A2 \\ A3 \\ A4 \end{array} \begin{bmatrix} B1 & B2 & B3 \\ 2 & 4 & 3 \\ 5 & 6 & 3 \\ 6 & 7 & 9 \\ 4 & 2 & 8 \end{bmatrix}$$

- $A1$ is dominated by $A2 \implies$ omite the row $A1$.
- Also $A1$ is dominated by $A3 \implies$ omite the row $A1$
- Omiting $A1$, we have the reduced matrix as

$$\begin{bmatrix} & B1 & B2 & B3 \\ A2 & 5 & 6 & 3 \\ A3 & 6 & 7 & 9 \\ A4 & 4 & 2 & 8 \end{bmatrix}$$

- $A2$ is dominated by $A3 \implies$ omite the row $A2$.
- Also $A4$ is dominated by $A3 \implies$ omite the row $A4$
- Omiting $A2$ and $A4$, we have the reduced matrix as

$$\begin{array}{cccc} & B1 & B2 & B3 \\ A3 & 6 & 7 & 9 \end{array}$$

- $B1$ is dominated by $B2 \implies$ omite the column $B2$.

- Also $B1$ is dominated by $B3 \implies$ omite the column $B3$

- Omiting $B2$ and $B3$, we have the reduced matrix as

$$\begin{array}{cc} & B1 \\ A3 & \begin{bmatrix} 6 \end{bmatrix} \end{array}$$

$\therefore$ value of the game $V = 6$ with the saddle point $S = (A3, B1)$.

PROBLEM 2:Using DOMINANCE PROPERTY solve the following game

$$\begin{bmatrix} & B1 & B2 & B3 & B4 \\ A1 & -5 & 3 & 1 & 20 \\ A2 & 5 & 5 & 4 & 6 \\ A3 & -4 & -2 & 0 & -5 \end{bmatrix}$$

Solution:

- $B1$ is dominated by $B2 \implies$ omite the column $B2$.

- Omiting $B2$, we have the reduced matrix as

$$
\begin{array}{c c}
 & \begin{array}{ccc} B1 & B3 & B4 \end{array} \\
\begin{array}{c} A1 \\ A2 \\ A3 \end{array} & \begin{bmatrix} -5 & 1 & 20 \\ 5 & 4 & 6 \\ -4 & 0 & -5 \end{bmatrix}
\end{array}
$$

- $A3$ is dominated by $A2 \implies$ omite the row $A3$.
- Omiting $A3$, we have the reduced matrix as

$$\begin{array}{cccc} & B1 & B3 & B4 \\ A1 & -5 & 1 & 20 \\ A2 & 5 & 4 & 6 \end{array}$$

- $B1$ is dominated by $B4 \implies$ omite the column $B4$.
- Also $B3$ is dominated by $B4 \implies$ omite the column $B4$
- Omiting $B4$, we have the reduced matrix as

$$\begin{array}{ccc} & B1 & B3 \\ A1 & -5 & 1 \\ A2 & 5 & 4 \end{array}$$

- $A1$ is dominated by $A2 \implies$ omite the row $A1$.
- Omiting $A1$, we have the reduced matrix as

$$\begin{array}{ccc} & B1 & B3 \\ A2 & 5 & 4 \end{array}$$

- $B3$ is dominated by $B1 \implies$ omite the column $B1$.
- Omiting $B1$, we have the reduced matrix as

$$\begin{array}{cc} & B3 \\ A2 & 4 \end{array}$$

$\therefore$ value of the game $V = 4$ with the saddle point $S = (A2, B3)$.

- If dominance property fails to solve a game problem we can try to solve it by graphical method.
- But this method can be applied only to game problem of the type $2 \times n$ or $m \times 2$.
- That is either the player A or B should have only 2 strategies.
- **Method of solving** $2 \times n$ **game** Plot the pairs of pay off of the n strategies of the players A and B on two vertical axes (axis 1 and axis 2) and connect the pairs of point by straight line.
- Locate the highest point on the line segments that form the lower boundary of the graph.

- The line that intersect at this point identify the strategies player B should adopt in his optimum strategy.

- **Method of solving** $m \times 2$ **game** Plot the pairs of pay off of the m strategies of players A and B on the two vertical axes and connect the pairs of point by straight line.

- Locate the lowest point on the line of segment that form the upper boundary.

- The lines that intersect at this point identify the strategies of the player A.

- Thereby we reduce the game into a 2 x 2 game.

PROBLEM 1: Solve the following game

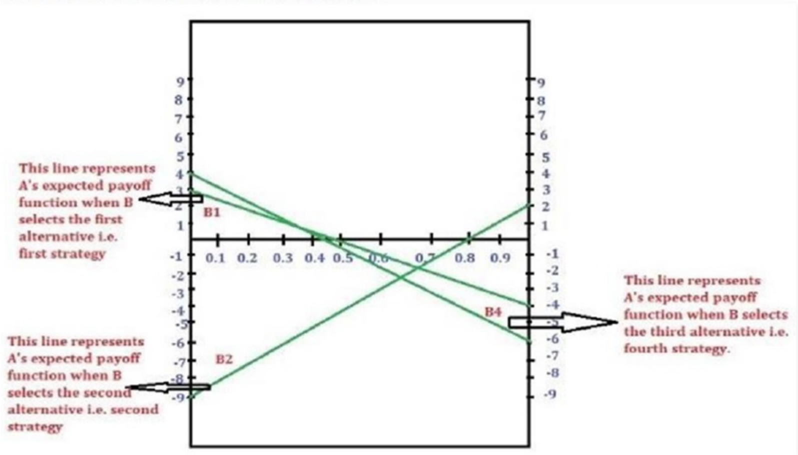
$$\begin{array}{c|ccccc} & B1 & B2 & B3 & B4 & B5 \\ \hline A1 & -4 & 2 & 5 & -6 & 6 \\ A2 & 3 & -9 & 7 & 4 & 8 \end{array}$$

SOLUTION:

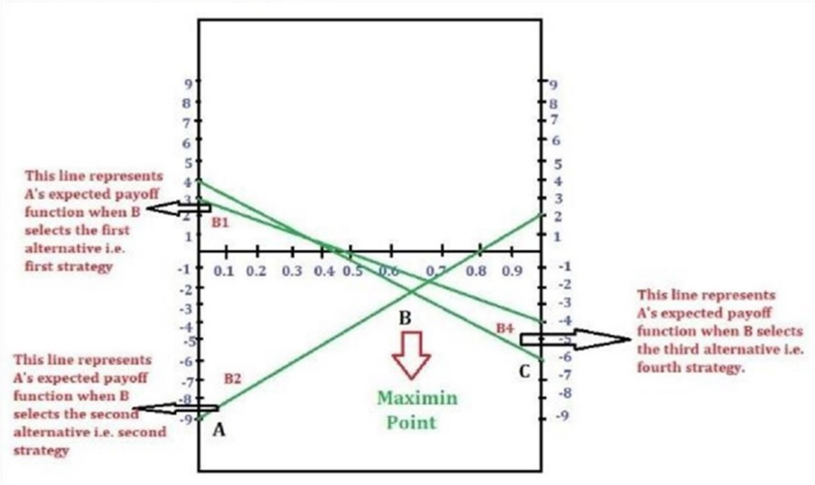
- The game has no saddle point.
- That is $\underline{v} = -6 \neq \bar{v} = 2$.
- So using dominance property , $B1$ is dominated by $B3$ and $B5$. Omitting the column $B3$ and $B5$ we have

$$\begin{bmatrix} & B1 & B2 & B4 \\ A1 & -4 & 2 & -6 \\ A2 & 3 & -9 & 4 \end{bmatrix}$$

Now Using graphical method we have,



To get Maximini point we have,



- From the maximini point in the graph we have the payoff matrix has interection of $B2$ and $B4$.

$$\begin{bmatrix} & B2 & B4 \\ A1 & 2 & -6 \\ A2 & -9 & 4 \end{bmatrix}$$

- Value of the game is $V = \frac{-46}{21}$.
- The Strategy of row player $S_A = (\frac{13}{21}, \frac{8}{21})$.
- The Strategy of column player $S_B = (0, \frac{10}{21}, 0, \frac{11}{21})$.

PROBLEM 1: In a game of matching coins with two players, suppose A wins one unit value when there are two heads, wins nothing when there are two tails, and wins $\frac{1}{2}$ unit value when there are one head and one tail. Determine the payoff matrix, the best strategy for each player, and the value of the game.

Answer: Value of the game $V = \frac{1}{2}$, Saddle point $S = (H, T)$.

PROBLEM 2: solve the game with 2×2 pay of matrix

$$\begin{bmatrix} 6 & -3 \\ -3 & 0 \end{bmatrix}$$

Answer: Value of the game $V = -\frac{3}{4}$.

PROBLEM 3: Solve the following 3×3 game

$$\begin{bmatrix} 1 & -1 & -1 \\ -1 & -1 & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Answer: Value of the game $V = -\frac{2}{26}$.

PROBLEM 4:Using DOMINANCE PROPERTY solve the following game

$$\begin{bmatrix} & B1 & B2 & B3 & B4 \\ A1 & 3 & 2 & 4 & 0 \\ A2 & 3 & 4 & 2 & 4 \\ A3 & 4 & 2 & 4 & 0 \\ A4 & 0 & 4 & 0 & 8 \end{bmatrix}$$

Answer: Value of the game $V = \frac{8}{3}$.

MCQ QUESTIONS WITH ANSWER

1. A competitive situation is called a
   a. **game**
   b. sequencing
   c. probability
   d. transportation

2. In game theory, there are finite number of competitors called
   a. gainers
   b. **players**
   c. travelling salesman
   d. machines

3. In game theory, a situation in which one firm can gain only what another firm loses is called
   a. nonzero-sum game
   b. prisoner's dilemma
   c. **Zero-sum game**.
   d. Predation game.

4. The payoff value at the saddle point is called the
   a. **value of the game**
   b. strategy of the game
   c. minimax value
   d. maximin value

5. When maximin $\neq$ minimax, then this type of strategy is called
   a. pure strategy
   b. **mixed strategy**
   c. zero sum game
   d. fair game

6. If a player known exactly what the other player is going to do is known as
   a. **pure strategy**
   b. mixed strategy
   c. value of the game
   d. saddle point

THANK YOU

LPP FORMULATION - GRAPHICAL SOLUTION AND SIMPLEX METHOD

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INTRODUCTION TO LINEAR PROGRAMMING

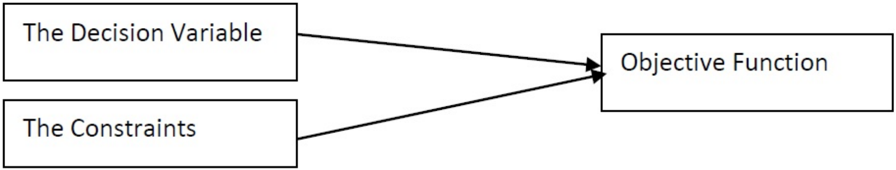
- Linear programming is an optimization technique for a system of linear constraints and a linear objective function.
- An objective function defines the quantity to be optimized, and the goal of linear programming is to find the values of the variables that maximize or minimize the objective function.
- Linear programming is useful for many problems that require an optimization of resources.
- It could be applied to manufacturing, to calculate how to assign labor and machinery to minimize cost of operations.
- It could be applied in high-level business operations, to decide which products to sell and in what quantity in order to maximize profit.
- It could also be applied in logistics, to decide how to apply resources to get a job done in the minimum amount of time.

IMPORTANT TERMS OF LINEAR PROGRAMMING

- **Decision variables:** Mathematical symbols representing levels of activity of a firm.
- **Objective function:** Linear mathematical relationship describing an objective of the firm, in terms of decision variables, that is maximized or minimized.
- **Constraints:** Restrictions placed on the firm by the operating environment stated in linear relationships of the decision variables.
- **Parameters:** Numerical coefficients and constants used in the objective function and constraint equations.

COMPONENTS OF LINEAR PROGRAMMING

- **The decision variable**
- **The environment (uncontrollable) parameters**
- **The result (dependent) variable**
- Linear Programming Model is composed of the same components



MATHEMATICAL FORMULATION OF LINEAR PROGRAMMING

- If $x_j, (j = 1, 2, 3, \ldots, n)$ are n decision variables and the system is subject to m constraint equations , then the general mathematical formulation of LPP is as
- **OBJECTIVE FUNCTION**

  Optimize (Maximize or Miminize) $Z = f(x_1, x_2, \ldots, x_n)$
- **CONSTRAINT EQUATIONS**

  subject to $g_i(x_1, x_2, \ldots, x_n) \leq, =, \geq b_i, (i = 1, 2, 3, \ldots, m)$
- **NON-NEGATIVE RESTRICTION OR CONSTRAINT**

  and $x_1, x_2, x_3, \ldots, x_n \geq 0$

- The general form of mathematical formulation of LPP is as

$$\text{Optimize (Maximize or Miminize) } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\text{subject to } a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq, =, \geq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq, =, \geq b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

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$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq, =, \geq b_n$$

$$\text{and } x_1, x_2, x_3, \ldots, x_n \geq 0$$

FORMULATION OF A LINEAR PROGRAMMING PROBLEM

The formulation of the LPP as mathematical model involves the following key steps:

- **Step 1.** Identify the decision variables to be determined and express them in terms of algebraic symbols as $x_1, x_2, \ldots, x_n$.
- **Step 2.** Identify the objective which is to be optimized (maximized or minimized) and express it as a linear function of the above defined decision variables.
- **Step 3.** Identify all the constraints in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.
- **Step 4.** Non-negativity restrictions on decision variables.

PROBLEMS ON FORMULATION OF LPP

PROBLEM 1 : **Production allocation problem** Four different type of metals, namely, iron, copper, zinc and manganese are required to produce commodities A, B and C. To produce one unit of A, 40kg iron, 30kg copper, 7kg zinc and 4kg manganese are needed. Similarly, to produce one unit of B, 70kg iron, 14kg copper and 9kg manganese are needed and for producing one unit of C, 50kg iron, 18kg copper and 8kg zinc are required. The total available quantities of metals are 1 metric ton iron, 5 quintals copper, 2 quintals of zinc and manganese each. The profits are Rs 300, Rs 200 and Rs 100 by selling one unit of A, B and C respectively. Formulate the problem mathematically.

SOLUTION : The above said problem can be tabulated as follows:

	Iron in Kg	Copper in Kg	Zinc in Kg	Manganese in Kg	Profit per unit in Rs
A	40	30	7	4	300
B	70	14	0	9	200
C	60	18	8	0	100
Available quantities	100	500	200	200	

- Let us consider the decision variables for the three products as:
    - x_1 unit of product A
    - x_2 unit of product B
    - and x_3 unit of product C

- The objective is to maximize profit , $\implies$ let z be the total profit and the problem is to maximize z
- $\therefore$ The objective function is Maximize $z = 300x_1 + 200x_2 + 100x_3$.
- For the above tabulation we have the following constraint as follows:
  - $40x_1 + 70x_2 + 60x_3 \leq 1000$
  - $30x_1 + 14x_2 + 18x_3 \leq 500$
  - $7x_1 + 0x_2 + 8x_3 \leq 200$
  - $4x_1 + 9x_2 + 0x_3 \leq 200$

- $\therefore$ The LPP formulation is as :

  Maximize $z = 300x_1 + 200x_2 + 100x_3$.

  subject to

$$40x_1 + 70x_2 + 60x_3 \leq 1000$$

$$30x_1 + 14x_2 + 18x_3 \leq 500$$

$$7x_1 + 0x_2 + 8x_3 \leq 200$$

$$4x_1 + 9x_2 + 0x_3 \leq 200$$

GRAPHICAL SOLUTION OF LPP MODELS

If the objective function z is a function of two variables then the problem can be solved by graphical method. The procedure is as follows.

- **Step 1.** First of all we consider the constraints as equalities or equations.
- **Step 2.** Then we draw the lines in the plane corresponding to each equation obtained in step 1 and non-negative restrictions.
- **Step 3.** Then we find the permissible region(region which is common to all the equations) for the values of the variables which is the region bounded by the lines drawn in step 2.
- **Step 4.** Finally we find a point in the permissible region which gives the optimum value of the objective function.

Note:

- If there is no permissible region in a problem then we say that the problem has no solution using graphical method.

- If the maximum value of z appears only at ∞ then the problem has unbounded solution.

- If the maximum value of z appears at 2 points there exists infinite number of solutions to the LPP.

PROBLEM 1: Solve graphically the given linear programming problem.

Maximize $Z = 40x_1 + 50x_2$

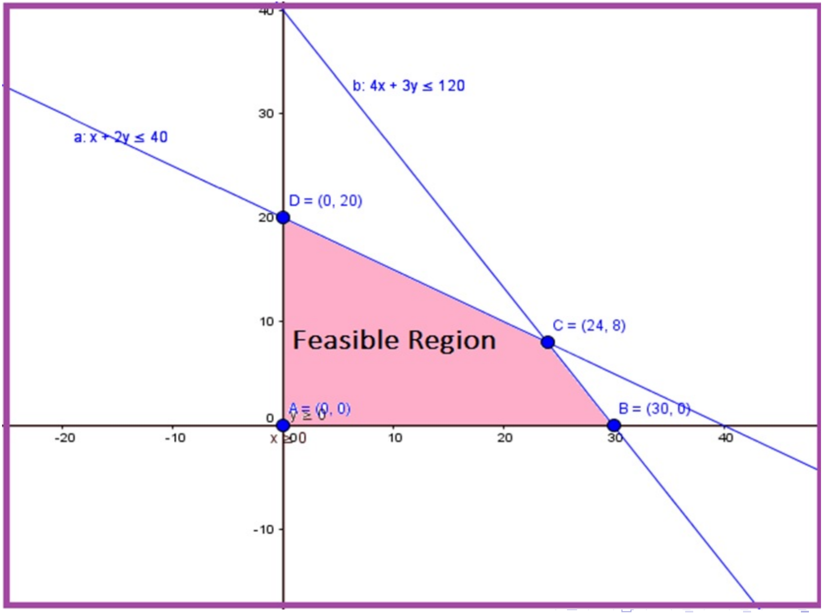
subject to

$$x_1 + 2x_2 \leq 40$$
$$4x_1 + 3x_2 \leq 120$$
$$\text{and } x_1, x_2 \geq 0.$$

Solution: Considering the constraint inequality as equality we have,

- Let constraint 1: $x_1 + 2x_2 \leq 40 \Rightarrow x_1 + 2x_2 = 40$ gives $(0, 20)$ and $(40, 0)$.
- Let constraint 2: $4x_1 + 3x_2 \leq 120 \Rightarrow 4x_1 + 3x_2 = 120$ gives $(0, 40)$ and $(30, 0)$
- Let constraint 3 : $x_1, x_2 \geq 0 \Rightarrow x_1 = 0, x_2 = 0$.

Ploting the points in the graph we have,



From the graph the Feasible region is as ABCD :

S.No	Point	$Z = 40x_1 + 50x_2$
1	$A(0,0)$	0
2	$B(30,0)$	1200
3	**C(24,8)**	**1360 = Max z**
4	$D(0,20)$	1000

- $\therefore$ the $Max \quad Z = 1360, x_1 = 24$ and $x_2 = 8$

PROBLEM 2: Solve graphically the given linear programming problem.

Minimize $Z = 3x_1 + 5x_2$

subject to

$$-3x_1 + 4x_2 \leq 12$$

$$x_1 \leq 4$$

$$2x_1 - x_2 \geq -2$$

$$x_2 \geq 2$$

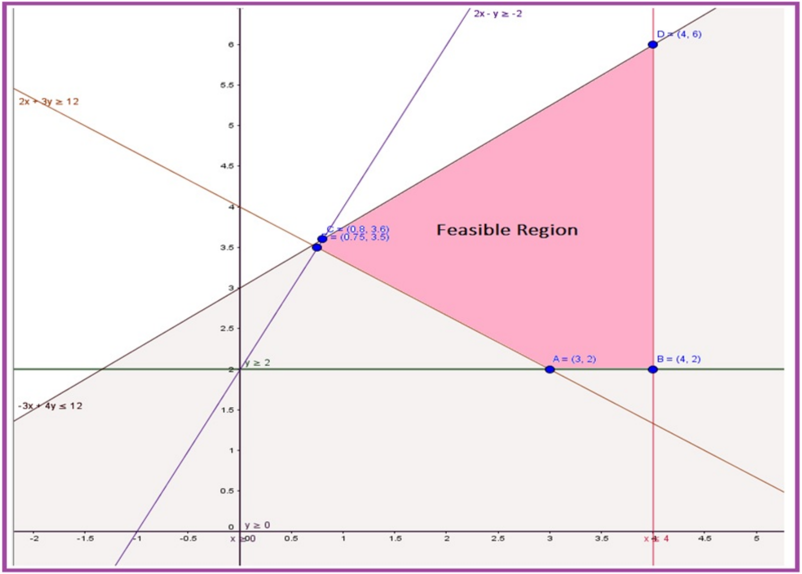
$$2x_1 + 3x_2 \geq 12$$

and $x_1, x_2 \geq 0$.

Solution: Considering the constraint inequality as equality we have,

- Let constraint 1: $-3x_1 + 4x_2 \leq 12 \Rightarrow -3x_1 + 4x_2 = 12$ gives $(0, 3)$ and $(-4, 0)$.
- Let constraint 2: $x_1 \leq 4 \Rightarrow x_1 = 4$ is a vertical line passing through point $(4, 0)$
- Let constraint 3 : $2x_1 - x_2 \geq -2 \Rightarrow 2x_1 - x_2 = -2$ gives $(0, 2)$ and $(-1, 0)$.
- Let constraint 4: $x_2 \geq 2 \Rightarrow x_2 = 2$ is a horizontal line passing through point $(0, 2)$
- Let constraint 5 : $2x_1 + 3x_2 \geq 12 \Rightarrow 2x_1 + 3x_2 = 12$ gives $(0, 4)$ and $(6, 0)$.
- Let constraint 6 : $x_1, x_2 \geq 0 \Rightarrow x_1 = 0, x_2 = 0$.

Ploting the points in the graph we have,



From the graph the Feasible region is as ABDCE :

S.No	Point	$Z = 3x_1 + 5x_2$
1	**A(3,2)**	**19 = Min Z**
2	$B(4,2)$	22
3	$C(\frac{4}{5}, \frac{18}{5})$	$\frac{102}{5}$
4	$D(4,6)$	42
5	$E(\frac{3}{4}, \frac{7}{2})$	$\frac{79}{4}$

- $\therefore$ the *Min* $Z = 19, x_1 = 3$ and $x_2 = 2$

PROBLEM 3: Solve graphically the given linear programming problem.

Maximize $Z = x_1 - 2x_2$

subject to

$$-x_1 + x_2 \leq 1$$

$$6x_1 + 4x_2 \geq 24$$

$$0 \leq x_1 \leq 5$$

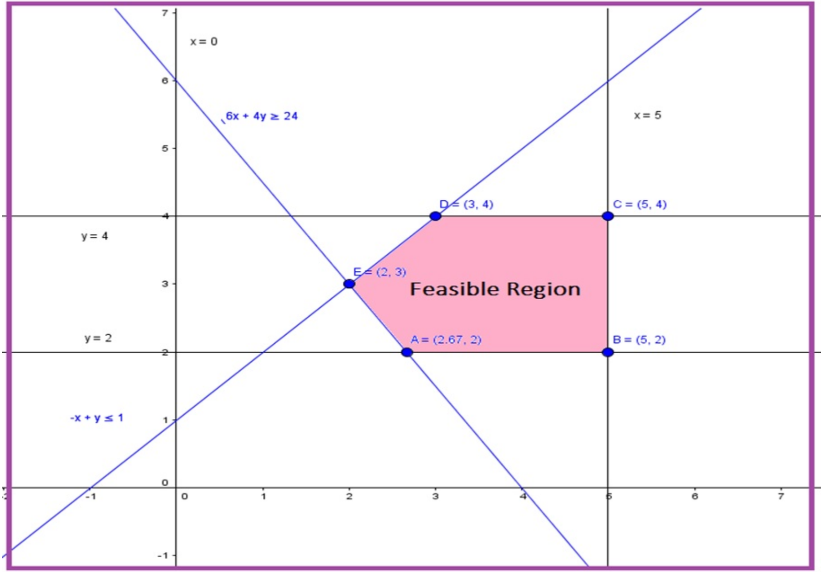
$$2 \leq x_2 \leq 4$$

and $x_1, x_2 \geq 0$.

Solution: Considering the constraint inequality as equality we have,

- Let constraint 1: $-x_1 + x_2 \leq 1 \Rightarrow -x_1 + x_2 = 1$ gives $(0, 1)$ and $(-1, 0)$.
- Let constraint 2: $6x_1 + 4x_2 \geq 24 \Rightarrow 6x_1 + 4x_2 = 24$ gives $(0, 6)$ and $(4, 0)$.
- Let constraint 3 : $0 \leq x_1 \leq 5 \Rightarrow x_1 = 0, x_1 = 5$ is vertical line passing through point $(0, 0)$ and $(0, 5)$.
- Let constraint 4: $2 \leq x_2 \leq 4 \Rightarrow x_2 = 2, x_2 = 4$ is a horizontal line passing through point $(0, 2)$ and $(0, 4)$
- Let constraint 5 : $x_1, x_2 \geq 0 \Rightarrow x_1 = 0, x_2 = 0$.

Ploting the points in the graph we have,



From the graph the Feasible region is as ABCDE :

S.No	Point	$Z = x_1 - 2x_2$
1	$A(\frac{8}{3}, 2)$	$-\frac{4}{3}$
2	**B(5,2)**	$\mathbf{1 = Max\ z}$
3	$C(5, 4)$	-3
4	$D(3, 4)$	-5
5	$E(2, 3)$	-4

- $\therefore$ the $Max \quad z = 1, x_1 = 5$ and $x_2 = 2$

PROBLEM 4: Solve graphically the given linear programming problem.

Maximize $Z = 2x_1 + 3x_2$

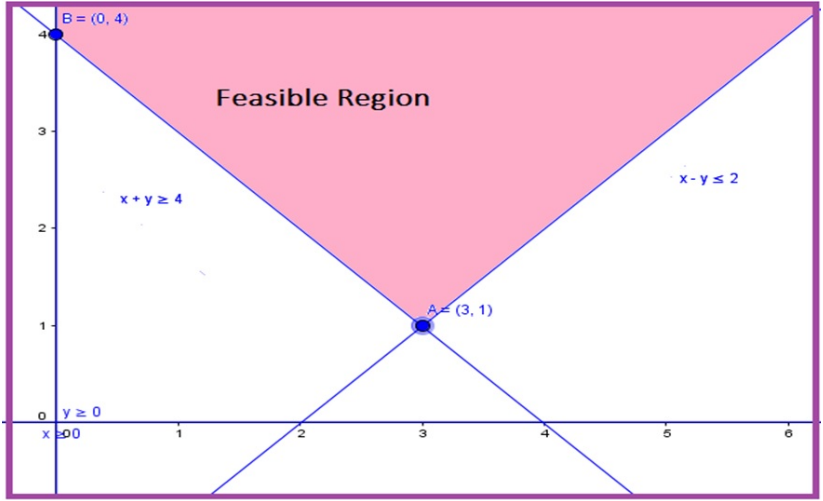
subject to

$$x_1 - x_2 \leq 2$$
$$x_1 + x_2 \geq 4$$
$$\text{and } x_1, x_2 \geq 0.$$

Solution: Considering the constraint inequality as equality we have,

- Let constraint 1: $x_1 - x_2 \leq 2 \Rightarrow x_1 - x_2 = 2$ gives $(0, -2)$ and $(2, 0)$.
- Let constraint 2: $x_1 + x_2 \geq 4 \Rightarrow x_1 + x_2 = 4$ gives $(0, 4)$ and $(4, 0)$.
- Let constraint 3 : $x_1, x_2 \geq 0 \Rightarrow x_1 = 0, x_2 = 0$.

Ploting the points in the graph we have,



- From the graph the Feasible region here the solution space is unbounded.

- So considering the point on the feasible region Points $A(3, 1)$ and $B(0, 4)$

- $\therefore$ The $Max \quad z = 12$ at $x_1 = 0$ and $x_2 = 4$.

PROBLEM 5: Solve graphically the given linear programming problem.

Maximize $Z = 4x_1 + 3x_2$

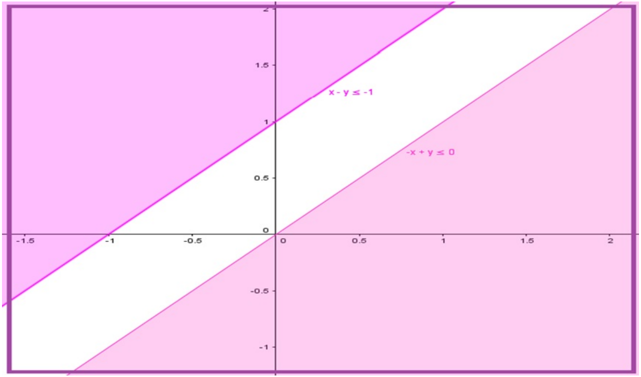
subject to

$$x_1 - x_2 \leq -1$$
$$-x_1 + x_2 \leq 0$$
$$\text{and } x_1, x_2 \geq 0.$$

Solution: Considering the constraint inequality as equality we have,

- Let constraint 1: $x_1 - x_2 \leq -1 \Rightarrow x_1 - x_2 = -1$ gives $(0, 1)$ and $(-1, 0)$.
- Let constraint 2: $-x_1 + x_2 \leq 0 \Rightarrow x_1 = x_2$ is the slop line passing through $(0, 0)$.
- Let constraint 3 : $x_1, x_2 \geq 0 \Rightarrow x_1 = 0, x_2 = 0$.

Ploting the points in the graph we have,



- Here there no feasible region exist. That is no solution space.

BASIC DEFINITION OF LPP

- **The Simplex method** is an approach to solving linear programming models by hand using slack variables, tableaus, and pivot variables as a means to finding the optimal solution of an optimization problem. When decision variables are more than 2 , we always use Simplex Method in solving LPP

- **Slack Variable:** Variable added to a $\leq$ constraint to convert it to an equation, that represents unused resources and contributes nothing to the objective function value.

- **Surplus Variable:** Variable subtracted from a $\geq$ constraint to convert it to an equation, that represents an excess above a constraint requirement level and contribute nothing to the calculated value of the objective function.

- **Basic Solution(BS):** This solution is obtained by setting any n variables (among $m + n$ variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these variables is non zero. Such m variables are called basic variables and remaining n zero valued variables are called non basic variables.

- **Basic Feasible Solution(BFS):** It is a basic solution which satisfies the non negativity restrictions.Basic Feasible Solution are of two types,
  - Degenerate BFS: If one or more basic variables are zero.
  - Non Degenerate BFS : All basic variables are non zero.

- **Optimal Basic Feasible Solution :** Basic Feasible Solution which optimizes the objective function.

STANDARD FORM OF LPP

- The properties of the standard LP form is :
- All the constraints (with the exception of the non negativity restrictions on the variables are equations with non negative right hand side.
- All the variables are non negative.
- The objective function may be of the maximization or minimization type.

- A linear programming problem is in standard form if it seeks to maximize the objective function

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

$$\text{subject to } a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n + s_1 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n + s_2 = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n + s_m = b_n$$

$$\text{where } x_1, x_2, x_3, \ldots, x_n, s_1, s_2, \ldots, s_m, \geq 0$$

$$\text{and } b_1, b_2, b_3, \ldots, b_n$$

SIMPLEX PROCEDURE

- **Step 1:** The objective function of the given LPP is to be maximized if it is minimized convert into a maximisation problem by , Mimimum $z. = -$ Maximimum $(-z)$.

- **Step 2:** All the b_j should be non-negative. If some b_j is negative multiply the corresponding equation by -1.

- **Step 3:** The inequations of the constraint must be converted into equations by introducing slack or surplus variables in the constraints. The cost of these variables are taken to be zero.

- **Step 4:** Find an initial basic feasible solution and formulate the simplex table as follows

		C_j	C_1	C_2	$\dots$	0	0	$\dots$
C_B	Y_B	X_B	x_1	x_2	$\dots$	s_1	s_2	$\dots$
C_{B_1}	s_1	b_1	a_{11}	a_{12}	$\dots$	1	0	$\dots$
C_{B_2}	s_2	b_2	a_{21}	a_{22}	$\dots$	0	1	$\dots$
C_{B_3}	s_3	b_3	a_{31}	a_{32}	$\dots$	0	0	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\dots$
$(Z_j - C_j)$		Z_0	$(Z_1 - C_1)$	$(Z_2 - C_2)$	$\dots$	$\dots$	$\dots$	$\dots$

- C_j - Coefficient of objective function.
- C_B - Coefficient of basic variables.
- Y_B - Basic variable.
- X_B - Value of the basic variable.
- $Z_j - C_j$ - Net evaluation.

- **Step 5:** Examine the sign of $Z_j - C_j$
  - If all $Z_j - C_j \geq 0$ the current feasible solution is optimal.
  - If all $Z_j - C_j < 0$ the current feasible solution is not optimal go to step.
- **Step 6:Finding the entering variable:** The entering variable is the non basic varible corresponding to the most negative value of $Z_j - C_j$. The entering column is know as key column or pivoting column.
- **Step 7:Finding the leaving variable:** The leaving variable is computed as $\theta = Min(\frac{X_{B_i}}{a_{ir}}, a_{ir} > 0)$. The leaving row is know as key row or pivoting row. The intersection element of entering column and leaving row is called as key element or pivoting element or leading element.

- **Step 8:** Drop the leaving variable and introduce the entering variable under C_B column.

  - New pivot equation $=$ old pivotequation $\div$ pivot element.
  - New equation(all the other row including $Z_j - C_j$ row) $=$ old equation - (corresponding column coefficent) $\times$ (New pivot equation).

- **Step 9:** go to step 5 and repeat the procedure untill optimality is reached.

PROBLEM ON SIMPLEX METHOD

PROBLEM 1: Solve by Simplex method and find the non-negative roots.

$$\text{Maximize } Z = 4x_1 + 10x_2$$

$$\text{subject to } 2x_1 + x_2 \leq 50$$

$$2x_1 + 5x_2 \leq 100$$

$$2x_1 + 3x_2 \leq 90$$

$$\text{and } x_1, x_2 \geq 0$$

SOLUTION : The given problem is to maximize and all $b_j \geq 0$, then the standard form of LPP by introducing slack variables is as

$$\text{Maximize } Z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } 2x_1 + x_2 + s_1 = 50$$

$$2x_1 + 5x_2 + s_2 = 100$$

$$2x_1 + 3x_2 + s_3 = 90$$

$$\text{and } x_1, x_2, s_1, s_2, s_3 \geq 0$$

$\therefore$ the initial basic feasible solution is as $s_1 = 50, s_2 = 100, s_3 = 90$.

The initial simplex table is as

		C_j	4	10	0	0	0	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	$\theta = Min$ $(\frac{X_{B_i}}{a_{ir}}, a_{ir} > 0)$
0	s_1	50	2	1	1	0	0	50
0	s_2	100	2	[5]	0	1	0	$20 \rightarrow$
0	s_3	90	2	3	0	0	1	30
$(Z_j - C_j)$		0	-4	$-10 \uparrow$	0	0	0	

- x_2 - Entering variable.
- s_2 - Leaving variable Coefficient of basic variables.
- $a_{22} = 5$ - Pivoting element.

The first iteration simplex table is as

		C_j	4	10	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	30	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	0
10	x_2	20	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0
0	s_3	30	$\frac{4}{5}$	0	0	$-\frac{3}{5}$	1
$(Z_j - C_j)$		200	0	0	0	2	0

- Since all $Z_j - C_j \geq 0, \implies$ the current basic feasible solution is optimal .
- $\therefore$ The optimal solution is as
- Maximize $Z = 200, x_1 = 0, x_2 = 20.$

PROBLEM 2: Solve by Simplex method and find the non-negative roots.

$$\text{Maximize } Z = 13x_1 + 11x_2$$

$$\text{subject to } 4x_1 + 5x_2 \leq 1500$$

$$5x_1 + 3x_2 \leq 1575$$

$$x_1 + 2x_2 \leq 420$$

$$\text{and } x_1, x_2 \geq 0$$

SOLUTION : The given problem is to maximize and all $b_j \geq 0$, then the standard form of LPP by introducing slack variables is as

$$\text{Maximize } Z = 13x_1 + 11x_2 + 0s_1 + 0s_2 + 0s_3$$

$$\text{subject to } 4x_1 + 5x_2 + s_1 = 1500$$

$$5x_1 + 3x_2 + s_2 = 1575$$

$$x_1 + 2x_2 + s_3 = 420$$

$$\text{and } x_1, x_2, s_1, s_2, s_3 \geq 0$$

$\therefore$ the initial basic feasible solution is as $s_1 = 1500, s_2 = 1575, s_3 = 420.$

The initial simplex table is as

		C_j	13	11	0	0	0	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	$\theta = Min$ $(\frac{X_{B_i}}{a_{ir}}, a_{ir} > 0)$
0	s_1	1500	4	5	1	0	0	375
0	s_2	1575	[5]	3	0	1	0	$315 \rightarrow$
0	s_3	420	1	2	0	0	1	420
$(Z_j - C_j)$		0	$-13 \uparrow$	-11	0	0	0	

- x_1 - Entering variable.
- s_2 - Leaving variable Coefficient of basic variables.
- $a_{12} = 5$ - Pivoting element.

The first iteration simplex table is as

		C_j	13	11	0	0	0	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	$\theta = Min \left(\frac{X_{B_i}}{a_{ir}}, a_{ir} > 0\right)$
0	s_1	240	0	$\frac{13}{5}$	1	$-\frac{4}{5}$	0	92.3
13	x_1	315	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	525
0	s_3	105	0	$[\frac{7}{5}]$	0	$-\frac{1}{5}$	1	$75 \rightarrow$
$(Z_j - C_j)$		4095	0	$-\frac{16}{5} \uparrow$	0	$\frac{13}{5}$	0	

- x_2 - Entering variable.
- s_3 - Leaving variable Coefficient of basic variables.
- $a_{32} = \frac{7}{5}$ - Pivoting element.

The second iteration simplex table is as

		C_j	13	11	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	45	0	0	1	$-\frac{3}{7}$	$-\frac{13}{7}$
13	x_1	270	1	0	0	$\frac{2}{7}$	$-\frac{3}{7}$
11	x_2	75	0	1	0	$-\frac{1}{7}$	$\frac{5}{7}$
$(Z_j - C_j)$		4335	0	0	0	$\frac{15}{7}$	$\frac{16}{7}$

- Since all $Z_j - C_j \geq 0, \implies$ the current basic feasible solution is optimal .
- $\therefore$ The optimal solution is as
- Maximize $Z = 4335, x_1 = 270, x_2 = 75.$

PROBLEM 3: Solve by Simplex method and find the non-negative roots.

$$\text{Minimize } Z = 8x_1 - 2x_2$$

$$\text{subject to } -4x_1 + 2x_2 \leq 1$$

$$5x_1 - 4x_2 \leq 3$$

$$\text{and } x_1, x_2 \geq 0$$

SOLUTION : The given problem is to minimize and all $b_j \geq 0$, then the standard form of LPP by introducing slack variables and
Mimimum $z. = -$ Maximimum $(-z)$, that is
Maximimum $(z^*) = -8x_1 + 2x_2$ is as

$$\text{Maximize } Z^* = -8x_1 + 2x_2 + 0s_1 + 0s_2$$

$$\text{subject to } -4x_1 + 2x_2 + s_1 = 1$$

$$5x_1 - 4x_2 + s_2 = 3$$

$$\text{and } x_1, x_2, s_1, s_2 \geq 0$$

$\therefore$ the initial basic feasible solution is as $s_1 = 1, s_2 = 3$.
The initial simplex table is as

		C_j	4	10	0	0	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	$\theta = Min\ (\frac{X_{B_i}}{a_{ir}}, a_{ir} > 0)$
0	s_1	1	-4	[2]	1	0	$\frac{1}{2} \rightarrow$
0	s_2	3	5	-4	0	1	$--$
$(Z_j - C_j)$		0	8	$-2 \uparrow$	0	0	

- x_2 - Entering variable.
- s_1 - Leaving variable Coefficient of basic variables.
- $a_{12} = 2$ - Pivoting element.

The first simplex table is as

		C_j	4	10	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
0	x_2	$\frac{1}{2}$	-2	1	$\frac{1}{2}$	0
0	s_2	5	-3	0	2	1
$(Z_j - C_j)$		1	4	0	1	0

Since all $Z_j - C_j \geq 0$, $\implies$ the current basic feasible solution is optimal .
$\therefore$ The optimal solution is as Minimize $Z = -1, x_1 = 0, x_2 = \frac{1}{2}$.

QUESTION TO ANSWER

PROBLEM 1: (Diet problem) A patient needs daily 5mg, 20mg and 15mg of vitamins A, B and C respectively. The vitamins available from a mango are 0.5mg of A, 1mg of B, 1mg of C, that from an orange is 2mg of B, 3mg of C and that from an apple is 0.5mg of A, 3mg of B, 1mg of C. If the cost of a mango, an orange and an apple be Rs 0.50, Rs 0.25 and Rs 0.40 respectively, find the minimum cost of buying the fruits so that the daily requirement of the patient be met. Formulate the problem mathematically.

Answer: Minimize $z = 0.50x_1 + 0.25x_2 + 0.40x_3$.

$$\text{Subject to } 0.5x_1 + 0x_2 + 0.5x_3 \geq 5$$

$$x_1 + 2x_2 + 3x_3 \geq 20$$

$$x_1 + 3x_2 + x_3 \geq 15$$

$$\text{and } x_1, x_2, x_3 \geq 0$$

PROBLEM 2: (Transportation problem) Three different types of vehicles A, B and C have been used to transport 60 tons of solid and 35 tons of liquid substance. Type A vehicle can carry 7 tons solid and 3 tons liquid whereas B and C can carry 6 tons solid and 2 tons liquid and 3 tons solid and 4 tons liquid respectively. The cost of transporting is Rs 500, Rs 400 and Rs 450 respectively per vehicle of type A, B and C respectively. Find the minimum cost of transportation. Formulate the problem mathematically.

Answer: Minimize $z = 500x_1 + 400x_2 + 450x_3$

Subject to $7x_1 + 6x_2 + 3x_3 \geq 60$

$3x_1 + 2x_2 + 4x_3 \geq 35$

and $x1, x2, x3 \geq 0$

PROBLEM 3: Solve graphically the given linear programming problem.

Maximize $Z = 150x_1 + 100x_2$

subject to

$$-8x_1 + 5x_2 \leq 60$$

$$4x_1 + 5x_2 \leq 40$$

$$\text{and } x_1, x_2 \geq 0.$$

Answer: Max $z = 1150, x_1 = 5$ and $x_2 = 4$.

PROBLEM 4: Solve using simplex method.

Maximize $Z = x_1 + 4x_2 + 5x_3$

subject to

$$3x_1 + 6x_2 + 3x_3 \leq 22$$
$$x_1 + 2x_2 + 3x_3 \leq 14$$
$$3x_1 + 2x_2 \leq 14$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

Answer: Max $z = \frac{74}{3}, x_1 = 0, x_2 = 2, x_3 = \frac{10}{3}$.

PROBLEM 5: Solve using simplex method.

Minimize $Z = x_1 - x_2 + 2x_3$

subject to

$$3x_1 - x_2 + 2x_3 \leq 7$$

$$-2x_2 + 4x_3 \leq 12$$

$$-4x_1 + 3x_2 + 8x_3 \leq 10$$

and $x_1, x_2, x_3 \geq 0.$

Answer: Min $z = -11, x_1 = 4, x_2 = 5, x_3 = 0.$

MCQ QUESTIONS WITH ANSWER

1. How do we determined the solution of the LPP by?
   A. Objective function
   B. **Decision Variables**
   C. Constraints
   D. Opportunity costs

2. In solving LPP by graphical method, the region of intersection of constraint functions is called as
   A. Infeasible region
   B. Unbounded region
   C. Infinite region
   D. **Feasible region**

3. In the solution of LPP by graphical method, the optimality of Z is obtained from
   A. **Corner points of feasible region**
   B. Both a and c
   C. corner points of the solution region
   D. none of the above

4. Identify the type of the feasible region given by the set of inequalities $x - y \leq 1, x - y \geq 2$ where both x and y are positive.
   A. A triangle
   B. A rectangle
   C. An unbounded region
   D. **An empty region**

5. The feasible region of a linear programming problem has four extreme points: A(0,0), B(30,0), C(24,8), and D(0,20). Identify an optimal solution for maximization problem with the objective function $z = 40x + 50y$
   A. 0
   B. 1400
   C. **1360**
   D. 1700

6. The feasible region of a linear programming problem has five extreme points: A(3,2), B(4,2), C(4/5,18/5), D(4,6) and E(3/4,7/2). Identify an optimal solution for minimization problem with the objective function $z = 3x + 5y$
   A. 0
   B. **19**
   C. 22
   D. 42

7. How the entering variable column is denoted in the simplex algorithm?
   A. **pivoting column**
   B. incoming column
   C. variable column
   D. decision column

8. How the leaving variable row is denoted in the simplex algorithm?
   A. outgoing row
   B. **pivoting row**
   C. interchanging row
   D. basic row

9. What is the intersection element of pivoting column and pivoting row?
   A. vital element
   B. important element
   C. **pivoting element**
   D. basic element

10. The non?negative variables introduced to the to convert the inequality less than or equal to constraint into an equation?
    A. **Slack variables**
    B. Surplus variables
    C. pivot element
    D. basic element

THANK YOU

SEQUENCING PROBLEM

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INTRODUCTION TO SEQUENCING PROBLEM

- A sequence is the order in which the jobs are processed. Sequence problems arise when we are concerned with situations where there is a choice in which a number of tasks can be performed. A sequencing problem could involve:

  - Jobs in a manufacturing plant.
  - Aircraft waiting for landing and clearance.
  - Maintenance scheduling in a factory.
  - Programmes to be run on a computer.
  - Customers in a bank and so on.

TERMS USED IN SEQUENCING PROBLEM

- **JOB :** The jobs or items or customers or orders are the primary stimulus for sequencing. There should be a certain number of jobs say n to be processed or sequenced.

- **NUMBER OF MACHINES :** A machine is characterized by a certain processing capability or facilities through which a job must pass before it is completed in the shop. It may not be necessarily a mechanical device. Even human being assigned jobs may be taken as machines. There must be certain number of machines say k to be used for processing the jobs.

- **Processing Time :** Every operation requires certain time at each of machine. If the time is certain then the determination of schedule is easy. When the processing times are uncertain then the schedule is complex.

- **Total Elapsed Time :** It is the time between starting the first job and completing the last one.

- **Idle time :** it is the time the machine remains idle during the total elapsed time.

- **Technological order :** different jobs may have different technological order. It refers to the order in which various machines are required for completing the jobs e.g.

TYPES OF SEQUENCING PROBLEM

There can be many types of sequencing problems as follows:

- Problem with n jobs through one machine.
- Problem with n jobs through two machines.
- Problem with n jobs through three machines.
- Problem with n jobs through m machines.
- Problem with 2 jobs through m machines.
- Here the objective is to find out the optimum sequence of the jobs to be processed and starting and finishing time of various jobs through all the machines.

BASIC ASSUMPTION OF SEQUENCING PROBLEM

Following are the basic assumptions of a sequencing problem:

- No machine can process more than one job at a time.
- The processing times on different machines are independent of the order in which they are processed.
- The time involved in moving a job from one machine to another is negligibly small.
- Each job once started on a machine is to be performed up to completion on that machine.
- All machines are of different types.
- All jobs are completely known and are ready for processing.
- A job is processed as soon as possible but only in the order specified.

JOHNSON'S METHOD TO SOLVE *n* JOBS ON 2 MACHINES SEQUENCING PROBLEM

- **Step 1:** All jobs are to be listed, and the processing time of each machine is to be listed.
- **Step 2:** Select the job with the shortest processing time.
  - If the shortest time lies on the first machine/work centre, the job is scheduled first.
  - If the shortest time lies on the second machine/work centre, the job is scheduled at the end.
  - Ties in activity times can be broken arbitrarily.
- **Step 3:** Once the job is scheduled, go to step 4.
- **Step 4:** Repeat steps2 and step3 to the remaining jobs, working towards the centre of the sequence.

PROBLEM 1: Suppose that there are five jobs, each of which has to be processed on two machines A and B in the order AB. Processing times are given in the following table:

JOBS	1	2	3	4	5
MACHINE A	6	2	10	4	11
MACHINE B	3	7	8	9	5

Determine a sequence in which these jobs should be processed so as to minimize the total processing time.

SOLUTION: First to find the optimal sequence of jobes using Johnson's method.

JOBS	1	2	3	4	5
MACHINE A	6	2	10	4	11
MACHINE B	3	7	8	9	5
Order of cancellation	(5)	(1)	(3)	(2)	(4)

$\therefore$ the optimal sequence of jobs is as :

Optimal Job Sequence	2	4	3	5	1

Now to find Total elapesed tim and Idel times of machines.

JOBS	MACHINE A IN	OUT	MACHINE B IN	OUT
2	0	2	2	9
4	2	6	9	18
3	6	16	18	26
5	16	27	27	32
1	27	33	33	36

$\therefore$ the Total ellapsed time = 36 hours

The Idel time of Machine A = 36 -33 = 3 hours

The Idel time of Machine B = 2+1+1 = 4 hours

PROBLEM 2: A book binder company has one printing machine and one binding machine.There are manuscripts of a number of different books. Processing times for printing and binding are given in the following table:

JOBS	A	B	C	D	E
Printing	5	1	9	3	10
Binding	2	6	7	8	4

Determine the sequence in which books should be processed on the machines so that the total time required is minimized.Determine a sequence in which these jobs should be processed so as to minimize the total processing time.

SOLUTION: First to find the optimal sequence of jobes using Johnson's method.

JOBS	A	B	C	D	E
Printing	5	1	9	3	10
Binding	2	6	7	8	4
Order of cancellation	(5)	(1)	(3)	(2)	(4)

$\therefore$ the optimal sequence of jobs is as :

Optimal Job Sequence	B	D	C	E	A

Now to find Total elapesed tim and Idel times of machines.

JOBS	Printing IN	OUT	Binding IN	OUT
B	0	1	1	7
D	1	4	7	15
C	46	13	15	22
E	13	23	23	27
A	23	28	28	30

$\therefore$ the Total ellapsed time = 30 hours

The Idel time of Printing Machine = 30 -28 = 2 hours

The Idel time of Binding Machine = 1+1+1 = 3 hours

JOHNSON'S METHOD TO SOLVE n JOBS ON 3 MACHINES SEQUENCING PROBLEM

- Let $t_{11}, t_{12}, t_{13}, \ldots, t_{1n}$ be the processing time of machine 1, $t_{21}, t_{22}, t_{23}, \ldots, t_{2n}$ be the processing time of machine 2 and $t_{31}, t_{32}, t_{33}, \ldots, t_{3n}$ be the processing time of machine 3.

- **CONDITION 1:** $min t_{1j} \geq max t_{2j}$, for $j = 1, 2, \ldots, n$.

- **CONDITION 2:** $min t_{3j} \geq max t_{2j}$, for $j = 1, 2, \ldots, n$.

- **Step 1:** Examine the processing times of the given jobs on all three machines and if either one or both the above conditions hold, then go to Step 2; otherwise, the algorithm fails.

- **Step 2:** Introduce two fictitious machines, say G and H, with corresponding processing times given by

  (i) $t_{Gj} = t_{1j} + t_{2j}; j = 1, 2, \ldots, n$ i.e., the processing time on machine G is the sum of the processing times on machines 1 and 2.

  (ii) $t_{Hj} = t_{2j} + t_{3j}; j = 1, 2, \ldots, n$ i.e., the processing time on machine H is the sum of the processing times on machines 2 and 3.

- **Step 3:** the optimal sequence for n jobs and two machine equivalent sequencing problem with the prescribed ordering GH.

SOLUTION OF n JOBS ON 3 MACHINES SEQUENCING PROBLEM

PROBLEM 1: Find the sequence that minimizes the total time required in performing the following jobs on three machines in order ABC. Processing times (in hours) are given in the following table :

JOBS	1	2	3	4	5
MACHINE A	8	10	6	7	11
MACHINE B	5	6	2	3	4
MACHINE C	4	9	8	6	5

Determine a sequence in which these jobs should be processed so as to minimize the total processing time.

SOLUTION: First to find the optimal sequence of jobes using Johnson's method.

$mint_{Aj} = 6, mint_{cj} = 4, maxt_{Bj} = 6 \implies mint_{Aj} = maxt_{Bj}$

$t_{Gj} = t_{Aj} + t_{Bj}; j = 1, 2, \ldots, 5$ and $t_{Hj} = t_{Bj} + t_{Cj}; j = 1, 2, \ldots, 5.$

JOBS	1	2	3	4	5
MACHINE G	13	16	8	10	15
MACHINE H	9	15	10	9	9
Order of cancellation	(4)	(2)	(1)	(5)	(3)

$\therefore$ the optimal sequence of jobs is as :

Optimal Job Sequence	3	2	5	1	4

Now to find Total elapesed tim and Idel times of machines.

JOBS	M - A IN	OUT	M - B IN	OUT	M - C IN	OUT
3	0	6	6	8	8	16
2	6	16	16	22	22	31
5	16	27	27	31	31	36
1	27	35	35	40	40	44
4	35	42	42	45	45	51

$\therefore$ the Total ellapsed time = 51 hours

The Idel time of Machine A = 51 - 42 = 9 hours

The Idel time of Machine B = 6+8+5+4+2+6 = 31 hours

The Idel time of Machine C = 8+6+4+1 = 19 hours

JOHNSON'S METHOD TO SOLVE n JOBS ON m MACHINES SEQUENCING PROBLEM

- Let $t'_{ij}s$ be the processing times of j^th job on i^th of machine , $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$.
- **CONDITION 1:** $mint_{1j} \geq maxt_{ij}$, for $j = 1, 2, \ldots, n$ and $i = 2, 3, \ldots, m-1$.
- **CONDITION 2:** $mint_{mj} \geq maxt_{ij}$, for $j = 1, 2, \ldots, n.$ and $i = 2, 3, \ldots, m-1$.
- **Step 1:** Examine the processing times of the given jobs on all the machines and if either one or both the above conditions hold, then go to Step 2; otherwise, the algorithm fails.

- **Step 2:** Introduce two fictitious machines, say G and H, with corresponding processing times given by

  (i) $t_{Gj} = t_{1j} + t_{2j} + t_{3j} + \ldots + t_{(m-1)j}; j = 1, 2, \ldots, n$ i.e., the processing time on machine G is the sum of the processing times on machines 1 and 2.

  (ii) $t_{Hj} = t_{2j} + t_{3j} + t_{4j} + \ldots + t_{mj}; j = 1, 2, \ldots, n$ i.e., the processing time on machine H is the sum of the processing times on machines 2 and 3.

- **Step 3:** the optimal sequence for n jobs and two machine equivalent sequencing problem with the prescribed ordering GH.

PROBLEM 1: Find the sequence that minimizes the total time required in performing the following jobs on four machines in order ABCD. Processing times (in hours) are given in the following table :

JOBS	1	2	3	4
MACHINE A	13	12	9	8
MACHINE B	8	6	7	5
MACHINE C	7	8	8	6
MACHINE D	14	19	15	15

Determine a sequence in which these jobs should be processed so as to minimize the total processing time.

SOLUTION: First to find the optimal sequence of jobs.
$min t_{Aj} = 8, min t_{Dj} = 14, max t_{ij} = 8 \implies min t_{Aj} = max t_{ij}$
$t_{Gj} = t_{Aj} + t_{Bj} + t_{Cj}; j = 1, 2, \ldots, 4$ and $t_{Hj} = t_{Bj} + t_{Cj} + t_{Dj}; j = 1, 2, \ldots, 4$.

JOBS	1	2	3	4
MACHINE G	28	26	24	19
MACHINE H	29	33	30	26
Order of cancellation	(4)	(3)	(2)	(1)

$\therefore$ the optimal sequence of jobs is as :

Optimal Job Sequence	4	3	2	1

Now to find Total elapesed time and Idel times of machines.

JOBS	M - A IN OUT	M - B IN OUT	M - C IN OUT	M - C IN OUT
4	0 8	8 13	13 19	19 34
3	8 17	17 24	24 32	34 49
2	17 29	29 35	35 43	49 68
1	29 42	42 50	50 57	68 82

$\therefore$ the Total ellapsed time = 82 hours

The Idel time of Machine A = 82 - 42 = 40 hours

The Idel time of Machine B = 8+4+5+7+32 = 56 hours

The Idel time of Machine C = 13+5+3+7+25 = 53 hours

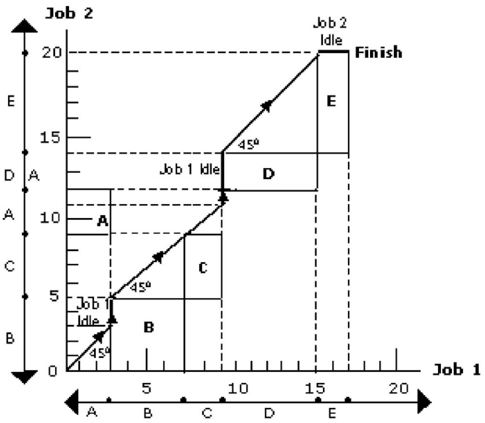
The Idel time of Machine D = 19 = 19 hours

SOLUTION OF 2 JOBS ON m MACHINES - GRAPHICAL METHOD

PROBLEM 1: Using graphical method solve the following sequence that minimizes the total time.

JOB 1	Seq	:	A	B	C	D	E
	Time	:	3	4	2	6	2
JOB 2	Seq	:	B	C	A	D	E
	Time	:	5	4	3	2	6

- Mark the processing time of Job 1 and Job 2 on X-axis and Y-axis respectively.
- By pairing the machines draw the rectangular box.



- Starting from origin O, move through the 45^0 line until a point marked finish is obtained.
- The elapsed time can be calculated by adding the idle time for either job to the processing time for that job. In this illustration, idle time for job 1 is 5 (3+2) hours.

  Elapsed time = Processing time of job 1 + Idle time of job 1

  = (3 + 4 + 2 + 6 + 2) + 5 = 17 + 5 = 22 hours.

  Likewise, idle time for job 2 is 2 hours.

  Elapsed time = Processing time of job 2 + Idle time of job 2

  = (5 + 4 + 3 + 2 + 6) + (2) = 20 + 2 = 22 hours.

1. Find the sequence that minimizes the total elapsed time required to complete the following tasks on machines M_1 and M_2 in the order M_1, M_2. Also find the minimum total elapsed time.

Task	A	B	C	D	E	F	G	H	I
M_1	2	5	4	9	6	8	7	5	4
M_2	6	8	7	4	3	9	3	8	11

Answer:

Optimal Sequence

A	C	I	B	H	F	D	G	E

Minimum total elapsed time = 61 time units
Idle time machine M_1 = 11 time units
Idle time machine M_2 = 2 time units.

2. Five jobs A,B,C,D and E are to be made on three groups of machines F_1, F_2 and F_3 in that order. The time required for each job (in min.) is given below.

	Job				
	A	B	C	D	E
F_1	20	27	31	15	19
F_2	7	9	6	12	14
F_3	27	31	16	11	12

Answer: A-B-E-D-C; 134 mts; 22 mts; 76 mts; 37 mts.

3. Determine the optimal sequence of the following sequencing problem. Find the minimum total elapsed time and idle times of all the machines.

	Machine			
Job	A	B	C	D
J_1	8	3	4	7
J_2	9	2	5	5
J_3	6	4	5	8
J_4	12	5	1	9
J_5	7	1	2	3

Answer:

Optimal Sequence

J_1	J_2	J_3	J_4	J_5

Minimum total elapsed time: 50 hrs.

Idle time of machine A=8 hrs; B=35 hrs; C=33 hrs; D=18 hrs.

4. Determine the minimum time needed to process the two jobs on four machines M_1, M_2, M_3 and M_4. The technological order for these jobs is as given below:

Job1	M_1	M_2	M_3	M_4
Job2	M_4	M_2	M_1	M_3

Processing time (in hours) are as given below:

	M_1	M_2	M_3	M_4
Job1	5	7	8	4
Job2	5	8	6	9

Answer: 31 hrs

MCQ QUESTIONS WITH ANSWER

1. The time required by each job on each machine is called.

a. Elapsed time

b. Idle time

c. Processing time

d. Average time

2. The order in which machines are required for completing the jobs is called.

a. machines order

b. working order

c. processing order

d. job order

3. The time between the starting of the first job and completion of the last job in sequencing problems is called.

a. total time

b. assignment time

c. elapsed time

d. idle time

4. The time during which a machine remains waiting or vacant in sequencing problem is called.

a. Processing time

b. Waiting time

c. Idle time

d. Free time

5. In sequencing problem, the order of completion of jobs is called.

a. completion sequence

b. job sequence

c. processing order

d. job order

6. Find the optimal job sequence of the following data.

				Jobs		
		A	B	C	D	E
Machines	1	5	4	8	7	6
	2	3	9	2	4	10

A. A-B-C-D-E **B. B-E-D-A-C** C. A-D-E B-C D. B-A-E-D-C