Discrete Mathematics

Chapter 9

Relations

Outline

- 9.1 Relations and their properties
- 9.3 Representing Relations
- 9.4 Closures of Relations (not included)
- 9.5 Equivalence Relations

9.1 Relations and their properties.

※The most direct way to express a relationship between elements of two sets is to use ordered pairs.

For this reason, sets of ordered pairs are called **binary relations**.

Def 1

Let A and B be sets. A binary relation from A to B is a subset R of $A \times B = \{ (a, b) : a \in A, b \in B \}$.

Example 1.

A : the set of students in your school.

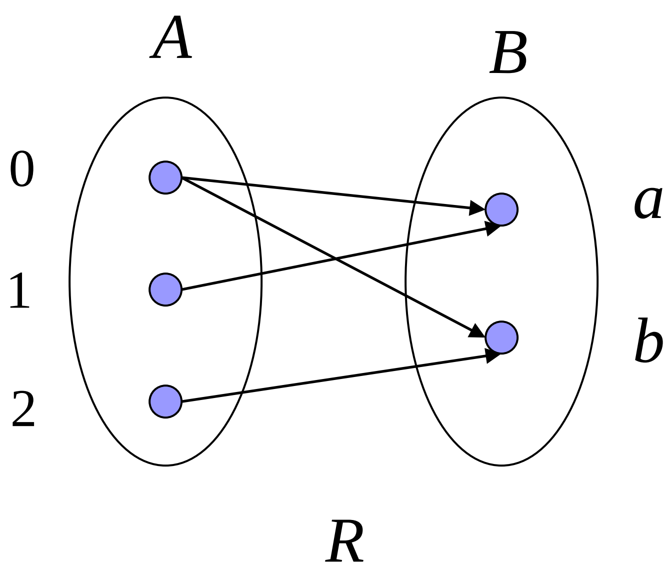
B : the set of courses.

$R = \{ (a, b) : a \in A, b \in B, a \text{ is enrolled in course } b \}$

Def 1'. We use the notation $\boldsymbol{aRb}$ to denote that $(a, b) \in R$, and $a \not{R} b$ to denote that $(a,b) \notin R$.

Moreover, a is said to be related to b by R if aRb.

Example 3. Let $A=\{0, 1, 2\}$ and $B=\{a, b\}$, then $\{(0,a),(0,b),(1,a),(2,b)\}$ is a relation R from A to B. This means, for instance, that $0Ra$, but that $1 \not{R} b$.



$$R \subseteq A \times B = \{ (0,a), (0,b), (1,a), \underbrace{(1,b)}_{\notin R}, \underbrace{(2,a)}_{\notin R}, (2,b)\}$$

Note. **Relations vs. Functions**

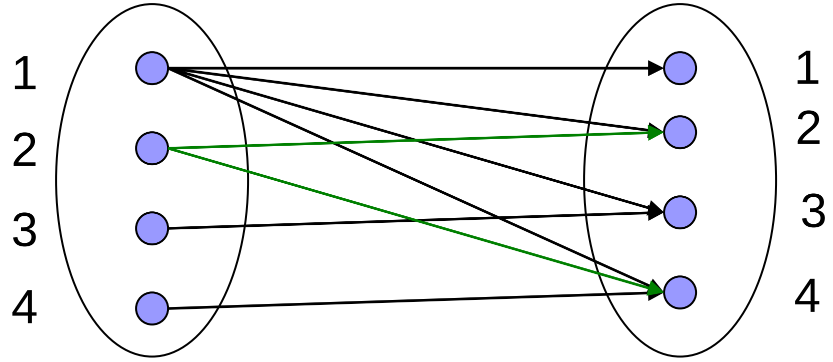
A relation can be used to express a 1-to-many relationship between the elements of the sets $\boldsymbol{A}$ and $\boldsymbol{B}$.

Def 2. A relation on the set $\boldsymbol{A}$ is a subset of $\boldsymbol{A} \times \boldsymbol{A}$ (i.e., a relation from $\boldsymbol{A}$ to $\boldsymbol{A}$).

Example 4.

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{ (a, b) | \ a \text{ divides } b \}$?

Sol :



$R = \{ (1,1), (1,2), (1,3), (1,4),$
$(2,2), (2,4),$
$(3,3),$
$(4,4) \}$

Example 5. Consider the following relations on **Z**.

$R_1 = \{ (a, b) \mid a \leq b \}$

$R_2 = \{ (a, b) \mid a > b \}$

$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$

$R_4 = \{ (a, b) \mid a = b \}$

$R_5 = \{ (a, b) \mid a = b+1 \}$

$R_6 = \{ (a, b) \mid a + b \leq 3 \}$

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1,–1), and (2,2)?

Sol :

	(1,1)	(1,2)	(2,1)	(1,–1)	(2,2)
R_1	●	●			●
R_2			●	●	
R_3	●			●	●
R_4	●				●
R_5			●		
R_6	●	●	●	●	

Example 6. How many relations are there on a set with n elements?

Sol :

A relation on a set A is a subset of $A \times A$.

$\Rightarrow A \times A$ has n^2 elements.

$\Rightarrow A \times A$ has 2^{n^2} subsets.

$\Rightarrow$ There are 2^{n^2} relations.

Properties of Relations

Def 3. A relation R on a set A is called reflexive if $(a,a) \in R$ for every $a \in A$.

Example 7. Consider the following relations on $\{1, 2, 3, 4\}$:

$R_2 = \{ (1,1), (1,2), (2,1) \}$

$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

which of them are reflexive ?

Sol :

R_3

Example 8. Which of the relations from Example 5 are reflexive?

$$R_1 = \{ (a, b) \mid a \le b \}$$
$$R_2 = \{ (a, b) \mid a > b \}$$
$$R_3 = \{ (a, b) \mid a = b \text{ or } a = -b \}$$
$$R_4 = \{ (a, b) \mid a = b \}$$
$$R_5 = \{ (a, b) \mid a = b+1 \}$$
$$R_6 = \{ (a, b) \mid a + b \le 3 \}$$

Sol : R_1, R_3 and R_4

Example 9. Is the “divides” relation on the set of positive integers reflexive?

Sol : Yes.

Def 4.

(1) A relation R on a set A is called symmetric

if for $a, b \in A$,

$$(a, b) \in R \Rightarrow (b, a) \in R.$$

(2) A relation R on a set A is called antisymmetric if for $a, b \in A$,

$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b.$$

i.e., $a \neq b$ and $(a,b) \in R \Rightarrow (b, a) \notin R$

$a=b$ ， $(a,a) \in R$ *or* $(a, a) \notin R$

Example 10. Which of the relations from Example 7 are symmetric or antisymmetric ?

$\boldsymbol{R_2} = \{ (1,1), (1,2), (2,1) \}$

$\boldsymbol{R_3} = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$\boldsymbol{R_4} = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

$\boldsymbol{R_2}$, $\boldsymbol{R_3}$ are symmetric

$\boldsymbol{R_4}$ are antisymmetric.

Example 11. Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?

Sol : It is not symmetric since $1|2$ but $2 \nmid 1$.

It is antisymmetric since $a|b$ and $b|a$ implies $a=b$.

antisymmetric symmetric

$$\forall (a,\, b)\in R,\ a\neq b \quad \begin{cases} \text{sym.} \Rightarrow (b,\, a)\in R \\ \\ \text{antisym.} \Rightarrow (b,a)\notin R \end{cases}$$

eg. Let A = {1,2,3}, give a relation R on A s.t. R is both symmetric and antisymmetric, but not reflexive.

Sol :

$$R = \{\ (1,1),(2,2)\ \}$$

Def 5. A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is called transitive if for $\boldsymbol{a, b, c \in A}$,

$(\boldsymbol{a, b}) \in \boldsymbol{R}$ and $(\boldsymbol{b, c}) \in \boldsymbol{R} \Rightarrow (\boldsymbol{a, c}) \in \boldsymbol{R}$.

Example 15. Is the "divides" relation on the set of positive integers transitive?

Sol : Suppose $a|b$ and $b|c$

$\Rightarrow a|c$

$\Rightarrow$ transitive

Example 13. Which of the relations in Example 7 are transitive ?

$R_2 = \{ (1,1), (1,2), (2,1) \}$

$R_3 = \{ (1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4) \}$

$R_4 = \{ (2,1), (3,1), (3,2), (4,1), (4,2), (4,3) \}$

Sol :

R_2 is not transitive since

$(2,1) \in R_2$ and $(1,2) \in R_2$ but $(2,2) \notin R_2$.

R_3 is not transitive since

$(2,1) \in R_3$ and $(1,4) \in R_3$ but $(2,4) \notin R_3$.

R_4 is transitive.

Example 16. How many reflexive relation are there on a set with $\boldsymbol{n}$ elements?

Sol : A relation $\boldsymbol{R}$ on a set $\boldsymbol{A}$ is a subset of $\boldsymbol{A}\times\boldsymbol{A}$.

$\Rightarrow$ $\boldsymbol{A}\times\boldsymbol{A}$ has $\boldsymbol{n}^2$ elements

$\Rightarrow$ $\boldsymbol{R}$ contains $(\boldsymbol{a},\boldsymbol{a})$ $\forall \boldsymbol{a}\in\boldsymbol{A}$ since $\boldsymbol{R}$ is reflexive

$\Rightarrow$ There are $2^{\boldsymbol{n}^2-\boldsymbol{n}}$ reflexive relations.

Exercise: 7, 43

Combining Relations

Example 17. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$.

The relation $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined to obtain

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2), (3,3)\}$

$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$

$R_1 \oplus R_2 = \{(2,2), (3,3), (1,2), (1,3), (1,4)\}$

symmetric difference, $(R_1 \cup R_2) - (R_1 \cap R_2)$

Def 6. Let R be a relation from a set A to a set B and S a relation from B to a set C. The composite of R and S is the relation consisting of ordered pairs (a,c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$. We denote the composite of R and S by $S \circ R$.

Example 20. What is the composite of relations R and S, where R is the relation from $\{1, 2, 3\}$ to $\{1, 2, 3, 4\}$ with $R = \{(1, 1), (1, 4), (2, 3), (3, 1), (3, 4)\}$ and S is the relation from $\{1, 2, 3, 4\}$ to $\{0, 1, 2\}$ with $S = \{(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)\}$?

Sol. $S \circ R$ is the relation from $\{1, 2, 3\}$ to $\{0, 1, 2\}$ with $S \circ R = \{(1, 0), (1,1), (2, 1), (2, 2), (3, 0), (3, 1)\}$.

Def 7. Let R be a relation on the set A.
The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$.

Example 22. Let $R = \{(1, 1), (2, 1), (3, 2), (4, 3)\}$.
Find the powers R^n, $n=2, 3, 4, \ldots$.

Sol. $R^2 = R \circ R = \{(1, 1), (2, 1), (3, 1), (4, 2)\}$.

$R^3 = R^2 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\}$.

$R^4 = R^3 \circ R = \{(1, 1), (2, 1), (3, 1), (4, 1)\} = R^3$.

Therefore $R^n = R^3$ for $n=4, 5, \ldots$.

Exercise: 54

Thm 1. The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

9.3 Representing Relations

Representing Relations using Matrices

Suppose that R is a relation from $A=\{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.

The relation R can be represented by the matrix $M_R = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R \\ 0, & \text{if } (a_i, b_j) \notin \mathrm{R} \end{cases}$$

Example 1. Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$
Let $\boldsymbol{R = \{(a, b) \mid a > b, a \in A, b \in B\}}$.
What is the matrix M_R representing R?

Sol :

$$\boldsymbol{R = \{(2, 1), (3, 1), (3, 2)\}}$$

$$\begin{array}{cc} & B \\ & \begin{array}{cc} 1 & 2 \end{array} \\ A & \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \end{array} \qquad \therefore M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Exercise: 1

※ Let $A=\{a_1, a_2, \ldots, a_n\}$.
A relation R on A is reflexive iff $(a_i,a_i)\in R, \forall i$.

i.e.,

$$M_R = \begin{array}{c} \\ a_1 \\ a_2 \\ \vdots \\ \vdots \\ a_n \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & \cdots & \cdots & a_n \end{array} \\ \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \end{array}$$

對角線上全為1

※ The relation R is symmetric iff $(a_i,a_j)\in R \Rightarrow (a_j,a_i)\in R$.
This means $m_{ij} = m_{ji}$.

$$M_R = \begin{bmatrix} & & 1 & & \\ & & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & & 0 \\ & & & \cdot^{\cdot^{\cdot}} & \\ & & 0 & & \end{bmatrix} = (M_R)^t$$

※ The relation R is antisymmetric iff

$$(a_i, a_j) \in R \text{ and } i \neq j \Rightarrow (a_j, a_i) \notin R.$$

This means that if $m_{ij}=1$ with $i \neq j$, then $m_{ji}=0$.

i.e.,

$$M_R = \begin{bmatrix} & 1 & & \\ 0 & & 0 & \\ & 1 & & 0 \\ & & 0 & \end{bmatrix}$$

Example 3. Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric ?

Sol :

reflexive

symmetric

not antisymmetric

eg. Suppose that $S=\{0, 1, 2, 3\}$. Let R be a relation containing (a, b) if $a \le b$, where $a \in S$ and $b \in S$. Is R reflexive, symmetric, antisymmetric ?

Sol :

$$M_R = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\therefore$ R is reflexive and antisymmetric, not symmetric.

Exercise: 7

Example 4. Suppose the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Sol :

$$M_{R_1 \cup R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R_1 \cap R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 5. Find the matrix representing the relation $S \circ R$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Sol :

$$M_{S \circ R} = M_R \odot M_S = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 6. Find the matrix representing the relation R^2, where the matrix representing R is

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Sol :

$$M_{R^2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Exercise: 14

Representing Relations using Digraphs

Def 1. A directed graph (digraph) consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or arcs).

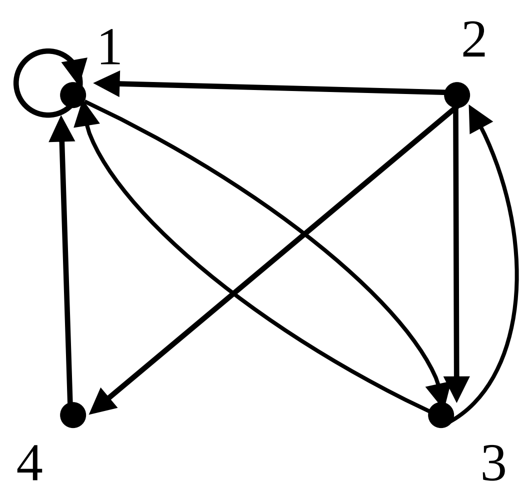
Example 8. Show the digraph of the relation $R=\{(1,1),(1,3),(2,1),(2,3),(2,4), (3,1),(3,2),(4,1)\}$ on the set $\{1,2,3,4\}$.

Exercise: 26,27

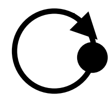
Sol :

vertex : 1, 2, 3, 4

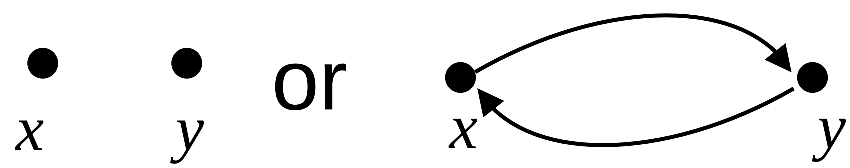
edge : (1,1), (1,3),
(2,1), (2,3), (2,4),
(3,1), (3,2),
(4,1)



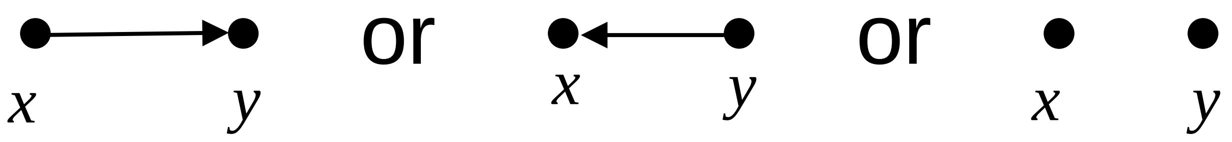
※ The relation R is reflexive iff for every vertex,



※ The relation R is symmetric iff for any vertices $x \neq y$, either



※ The relation R is antisymmetric iff for any $x \neq y$,

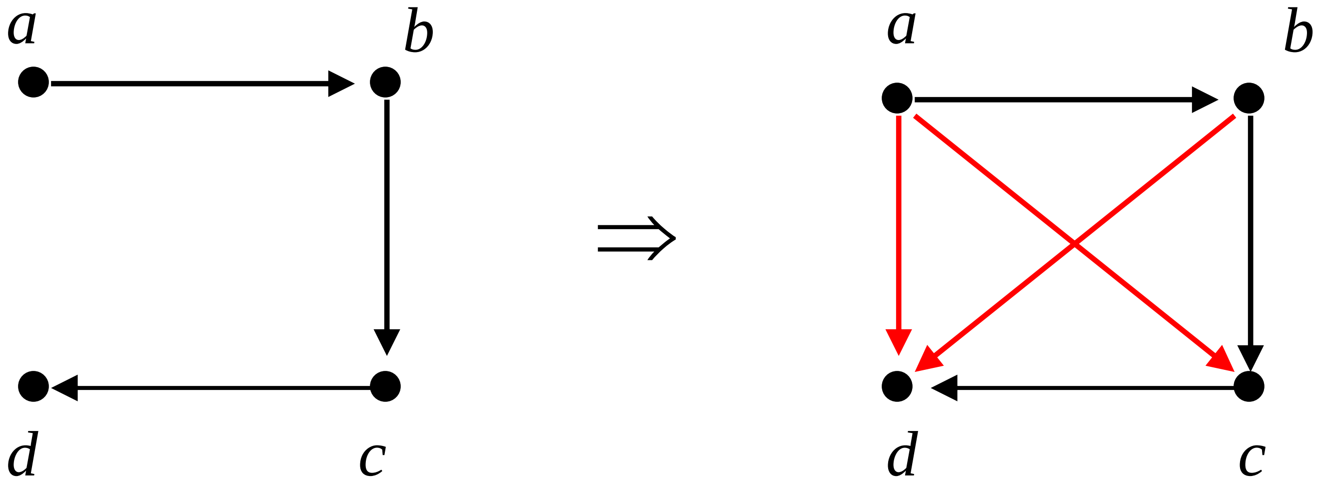


※ The relation R is transitive iff

for $\boldsymbol{a, b, c} \in \boldsymbol{A}$,

$$(\boldsymbol{a, b}) \in \boldsymbol{R} \text{ and } (\boldsymbol{b, c}) \in \boldsymbol{R} \Rightarrow (\boldsymbol{a, c}) \in \boldsymbol{R}.$$

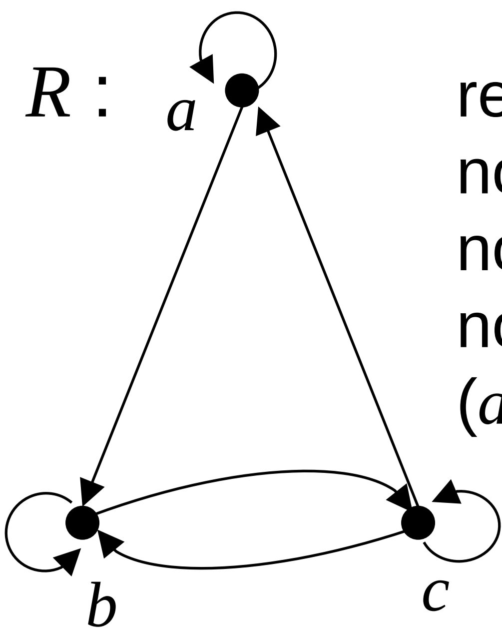
This means:



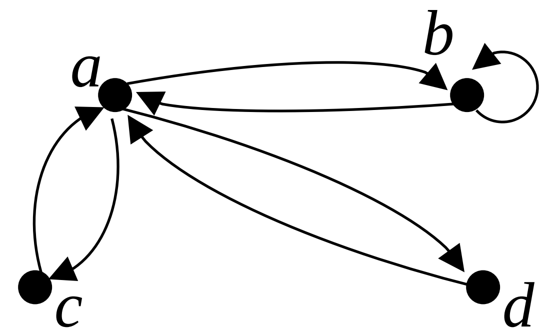
Example 10. Determine whether the relations R and S are reflexive, symmetric, antisymmetric, and/or transitive

Sol :

R :



reflexive,
not symmetric,
not antisymmetric,
not transitive
($a \to b$, $b \to c$, $a \not\to c$)



S

not reflexive,
symmetric
not antisymmetric
not transitive
($b \to a$, $a \to c$, $b \not\to c$)

Exercise: 31

9.4 Closures of Relations

※ Closures

The relation $R=\{(1,1), (1,2), (2,1), (3,2)\}$ on the set $A=\{1, 2, 3\}$ is not reflexive.

Q: How to construct a smallest reflexive relation R_r such that $R \subseteq R_r$?

Sol: Let $R_r = R \cup \{(2,2), (3,3)\}$.

i. e., $R_r = R \cup \Delta$, where $\Delta=\{(a, a) \mid a \in A\}$.

R_r is a reflexive relation containing R that is as small as possible. It is called the reflexive closure of R.

Example 1. What is the reflexive closure of the relation $R=\{(a,b) \mid a < b\}$ on the set of integers ?

Sol : $R_r = R \cup \Delta = \{(a,b) \mid a < b\} \cup \{ (a, a) \mid a \in \mathbf{Z} \}$
$= \{ (a, b) \mid a \le b, \quad a, b \in \mathbf{Z} \}$

Example :

The relation $R=\{ (1,1),(1,2),(2,2),(2,3),(3,1),(3,2) \}$ on the set $A=\{1,2,3\}$ is not symmetric. Find a smallest symmetric relation R_s containing R.

Sol : Let $R^{-1}=\{ (b, a) \mid (a, b) \in R \}$

Let $R_s = R \cup R^{-1} = \{ (1,1),(1,2),(2,1),(2,2),(2,3),$
$(3,1),(1,3),(3,2) \}$

R_s is called the symmetric closure of R.

Example 2. What is the symmetric closure of the relation $R=\{(a, b) \mid a > b\}$ on the set of positive integers ?

Sol :

$$R \cup \{ (b, a) \mid a > b \} = \{ (c, d) \mid c \neq d \}$$

Exercise: 1,9

Def :

1.(reflexive closure of R on A)

R_r=the smallest reflexive relation containing R.

$R_r = R \cup \{ (a, a) \mid a \in A , (a, a) \notin R\}$

2.(symmetric closure of R on A)

R_s=the smallest symmetric relation containing R.

$R_s = R \cup \{ (b, a) \mid (a, b) \in R \text{ and } (b, a) \notin R\}$

3.(transitive closure of R on A)

R_t=the smallest transitive relation containing R.

$R_t = R \cup \{(a, c) \mid (a, b) \in R \text{ and } (b, c) \in R, \text{ but } (a, c) \notin R\}$(repeat)

Note. There is no antisymmetric closure，

Definition . *Let A be a set and R a relation on A. The transitive closure of R is the relation R^t on A that satisfies the following three properties:*

1. *R^t is transitive.*
2. *$R \subseteq R^t$.*
3. *If S is any other transitive relation that contains R, then $R^t \subseteq S$.*

Example . *Let $A = \{0, 1, 2, 3\}$ and consider the relation R on A as follows:*

$$R = \{(0, 1), (1, 2), (2, 3)\}.$$

Find the transitive closure of R.

Solution. Every pair in R is in R^t, so

$$\{(0, 1), (1, 2), (2, 3)\} \subseteq R^t.$$

Thus the directed graph of R contains the arrows shown below.

$$\begin{tikzcd} 0 \arrow[r] & 1 \arrow[d] \\ 3 & 2 \arrow[l] \end{tikzcd}$$

Since there are arrows going from 0 to 1 and from 1 to 2, R^t must have an arrow going from 0 to 2. Hence $(0, 2) \in R^t$. Also, since $(1, 2) \in R^t$ and $(2, 3) \in R^t$, then $(1, 3) \in R^t$.

Adding these pairs does not produce a transitive relation, because the resulting relation contains $(0, 2)$ and $(2, 3)$ but does not contain $(0, 3)$. This shows that constructing the transitive closure of a relation is more complicated than constructing either the reflexive or symmetric closure. The transitive closure of a relation can be found by adding new ordered pairs that must be present and then repeating this process until no new ordered pairs are needed.

Then $(0, 2) \in R^t$ and $(2, 3) \in R^t$, so since R^t is transitive, $(0, 3) \in R^t$. Thus R^t contains at least the following ordered pairs:

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

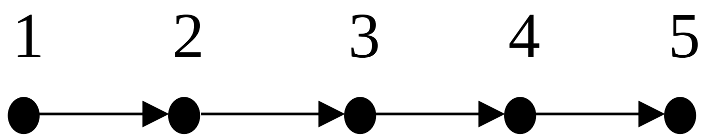
But this relation is transitive; hence it equals R^t. Note that the directed graph of R^t is as shown below. □

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \arrow[dl] \\ 3 & 2 \arrow[l] \end{tikzcd}$$

Paths in Directed Graphs

Def 1. A path from a to b in the digraph G is a sequence of edges $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)$ in G, where $n \in \mathbf{Z}^+$, and $x_0 = a$, $x_n = b$. This path is denoted by $x_0, x_1, x_2, \ldots, x_n$ and has length n.

Ex.



A path from 1 to 5 of length 4

Theorem 1 Let R be a relation on a set A. There is a path of length n, where $n \in \mathbf{Z}^+$, from a to b if and only if $(a, b) \in R^n$.

Transitive Closures

Def 2. Let R be a relation on a set A. The connectivity relation R^* consists of pairs (a, b) such that there is a path of length at least one from a to b in R.

i.e., $$R^* = \bigcup_{i=1}^{\infty} R^i$$

Theorem 2 The transitive closure of a relation R equals the connectivity relation R^*.

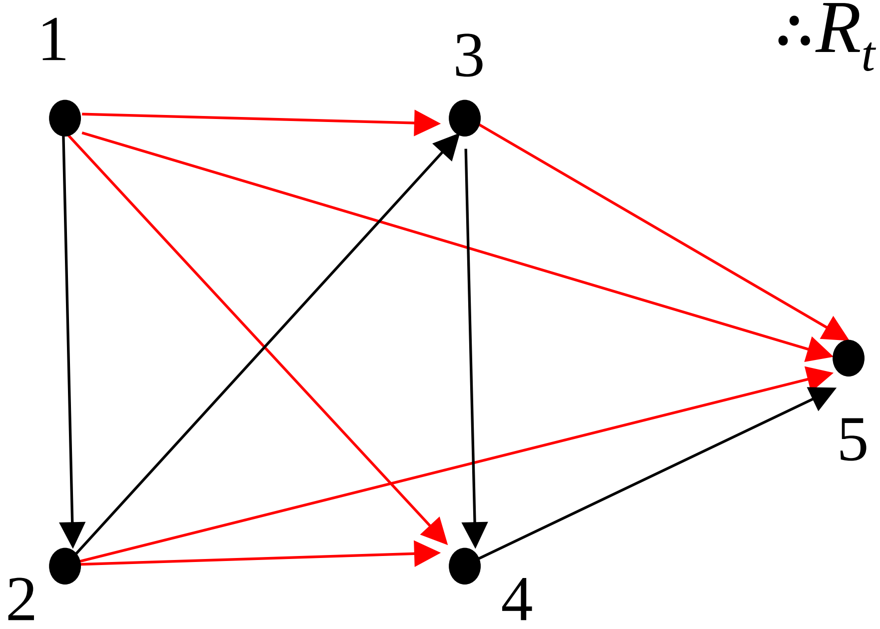
Lemma 1 Let R be a relation on a set A with $|A|=n$. then
$$R^* = \bigcup_{i=1}^{n} R^i$$

Example. Let R be a relation on a set A, where

$A=\{1,2,3,4,5\}$, $R=\{(1,2),(2,3),(3,4),(4,5)\}$.

What is the transitive closure R_t of R ?

Sol :



$$\therefore R_t = R \cup R^2 \cup R^3 \cup R^4 \cup R^5$$
$$= \{(1,2),(2,3),(3,4),(4,5), (1,3), (2,4), (3,5), (1,4), (2,5), (1,5)\}$$

Theorem 3 Let M_R be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R^* is

$$M_{R^*} = M_R \vee M_R^{[2]} \vee \cdots \vee M_R^{[n]}.$$

Example 7. Find the zero-one matrix of the transitive closure of the relation R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Sol :

$$M_{R^*} = M_R \vee M_R^{[2]} \vee M_R^{[3]} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Exercise: 25

9.5 Equivalence Relations

Def 1. A relation R on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Example 1.

Let R be the relation on the set of integers such that aRb if and only if $a=b$ or $a=-b$. Then R is an equivalence relation.

Example 2.

Let R be the relation on the set of real numbers such that aRb if and only if $a-b$ is an integer. Then R is an equivalence relation.

Example 3. (Congruence Modulo m)

Let $m \in \mathbf{Z}$ and $m > 1$. Show that the relation $R=\{ (a,b) \mid a\equiv b \pmod{m} \}$ is an equivalence relation on the set of integers.

(a is congruent to b modulo m,

Sol : Note that $a\equiv b \pmod{m}$ iff $m \mid (a-b)$.

$\because$ ① $a\equiv a \pmod{m} \Rightarrow (a, a)\in R \Rightarrow$ **reflexive**

② If $a\equiv b \pmod{m}$, then $a-b=km$, $k\in \mathbf{Z}$

$\Rightarrow b-a= (-k)m \Rightarrow b\equiv a \pmod{m} \Rightarrow$ **symmetric**

③ If $a\equiv b \pmod{m}$, $b\equiv c \pmod{m}$

then $a-b=km$, $b-c=lm$

$\Rightarrow a-c=(k+l)m \Rightarrow a\equiv c \pmod{m} \Rightarrow$ **transitive**

$\therefore R$ is an equivalence relation.

Example 4.

Let $l(x)$ denote the length of the string x.

Suppose that the relation

$R=\{(a,b) \mid l(a)=l(b)$, a,b are strings of English letters $\}$

Is R an equivalence relation?

Sol :

$$\left.\begin{array}{lll} ① & (a,a)\in R \quad \forall \text{string } a & \Rightarrow \text{reflexive} \\ ② & (a,b)\in R \Rightarrow (b,a)\in R & \Rightarrow \text{symmetric} \\ ③ & (a,b)\in R,(b,c)\in R \Rightarrow (a,c)\in R & \Rightarrow \text{transitive} \end{array}\right\} \text{Yes.}$$

Example 7.

Let R be the relation on the set of real numbers such that xRy if and only if x and y differ by less than 1, that is $|x-y| < 1$. Show that R is not an equivalence relation.

Sol :

① $xRx \ \forall x$ since $x-x=0$ $\Rightarrow$ reflexive

② $xRy \Rightarrow |x-y| < 1 \Rightarrow |y-x| < 1 \Rightarrow yRx$

$\Rightarrow$ symmetric

③ $xRy,\ yRz \Rightarrow |x-y| < 1,\ |y-z| < 1 \ \not\Rightarrow |x-z| < 1$

$\Rightarrow$ Not transitive

Exercise: 3, 23

Equivalence Classes

Def 3.

Let R be an equivalence relation on a set A.

The equivalence class of the element $a \in A$ is

$$[a]_R = \{ s \mid (a, s) \in R \}$$

For any $b \in [a]_R$, b is called a representative of this equivalence class.

Note:

If $(a, b) \in R$, then $[a]_R = [b]_R$.

Example 9.

What are the equivalence class of 0 and 1 for congruence modulo 4 ?

Sol :

Let $R=\{\ (a,b) \mid a\equiv b \pmod 4)\ \}$

Then $[0]_R = \{\ s \mid (0,s)\in R\ \}$

$= \{\ \ldots, -8, -4, 0, 4, 8, \ldots\ \}$

$[1]_\mathrm{R} = \{\ t \mid (1,t)\in R\ \} = \{\ \ldots,-7, -3, 1, 5, 9,\ldots\}$

Exercise: 25, 29

Equivalence Classes and Partitions

Def.

A partition of a set S is a collection of disjoint nonempty subsets A_i of S that have S as their union.

In other words, we have $A_i \neq \emptyset, \forall i$,

$A_i \cap A_j = \emptyset$, for any $i \neq j$, and $\cup A_i = S$.

Example 12.

Suppose that $S=\{1, 2, 3, 4, 5, 6\}$. The collection of sets $A_1=\{1, 2, 3\}$, $A_2=\{4, 5\}$, and $A_3=\{6\}$ form a partition of S.

Thm 2.

Let R be an equivalence relation on a set A.
Then the equivalence classes of R form a partition of A.

Example 13.

List the ordered pairs in the equivalence relation R produced by the partition $A_1=\{1, 2, 3\}$, $A_2=\{4, 5\}$, and $A_3=\{6\}$ of $S=\{1, 2, 3, 4, 5, 6\}$.

Sol :

$$R=\{ (a, b) \mid a, b\in A_1\}\cup \{ (a, b) \mid a, b\in A_2\}$$
$$\cup \{ (a, b) \mid a, b\in A_3\}$$
$$=\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3), (4,4), (4, 5), (5,4), (5, 5), (6, 6)\}$$

Exercise: 47

Example 14.

The equivalence classes of the congruence modulo 4 relation form a partition of the integers.

Sol :

$[0]_4 = \{ \ldots, -8, -4, 0, 4, 8, \ldots \}$

$[1]_4 = \{ \ldots, -7, -3, 1, 5, 9, \ldots \}$

$[2]_4 = \{ \ldots, -6, -2, 2, 6, 10, \ldots \}$

$[3]_4 = \{ \ldots, -5, -1, 3, 7, 11, \ldots \}$