National University of Computer and Emerging Sciences
Lahore Campus

Discrete Structures (CS1005)

Final Exam *Solution*

Date: June 03, 2024

Course Instructor(s)

Dr. Tahir Ejaz

Total Time (Hrs): 3
Total Marks: 90
Total Questions: 5

____________ ____________ ________________________

Roll No Section Student Signature

Do not write below this line

Use of the **CALCULATOR** *is* ***NOT allowed !!!***

An A4 size <u>only 1 sided</u> HANDWRITTEN (using BLUE ink) help sheet is allowed;

though handwritten but "PHOTOCOPIED" help sheets are NOT allowed.

CLO # 1: Understand the key concepts of Discrete Structures

Q1: Fill in each of the following blanks with the most appropriate answer.

Note: *Answer this question on* ***page 1*** *of the* ***answer sheet***. Do not copy the whole sentences on the answer sheet, just write the answer against each part. **[20*2.5 marks]**

No partial credit would be admissible in this question.

a. My roll number is ___________ and I confirm that I have answered this question on page 1 of the answer sheet according to the given instructions.

b. Suppose that p and q are prime numbers and that $n = pq$, then the number of positive integers not exceeding n that are relatively prime to n is $\boldsymbol{n - \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{n}{q} \right\rfloor + 1}$.

c. If $2k + 1$ is a primitive root modulo 11, then the value of k is **3**.

d. The discrete logarithms of 7 to the base 2 modulo 19 is **6**.

e. There are n people standing in a yard at mutually distinct distances and each person throws a pie at their nearest neighbor. The smallest value of n greater than 20 is **22** when it is possible that everyone is hit by a pie.

f. In the prime factorization of 29! **13** is the exponent of 3.

g. $f(x) = \boldsymbol{-x}$ is a strictly decreasing function from $\mathbf{R}$ (the set of real numbers) to itself.

h. $\boldsymbol{n^2}$ is a formula for the sum of the first n positive odd integers.

i. A computer system considers a string of decimal digits a valid codeword if it does not contain two consecutive 0s. Then $a_n = \boldsymbol{9a_{n-1} + 9a_{n-2}}$ is a recurrence relation for the number of valid n-digit codewords.

j. $a_n = \boldsymbol{a_{n-3} + a_{n-5}}$ is a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take three stairs or five stairs at a time.

k. In **8!** ways a photographer at an event can arrange 6 people in a row from a group of 9 people, where the chief guest is among these 9 people, if the chief guest is in the picture.

l. The Principle of Mathematical Induction can be thought of as a combination of **Modus Ponens** and Hypothetical Syllogism.

m. Let $S = \{-1,0,2,4,7\}$, and $f(x) = \lfloor (x^2+1)/3 \rfloor$, then $f(S) = \boldsymbol{\{0, 1, 5, 16\}}$.

n. There are **zero** one-to-one functions from a set with 5 elements to a set with 4 elements.

o. The name of a variable in the C programming language is a string that can contain uppercase letters or underscores. Further, if the first character in the string is an underscore, then it (i.e. the underscore) must be the last character as well. If the name of a variable is determined by its first m characters, then $\boldsymbol{26 \times 27^{k-1} + 27^{k-2}}$ different variables of length k (where $2 < k < m$) can be named in this version of C?

p. A company stores products in a warehouse. Storage bins in this warehouse are specified by their aisle, location in the aisle, and shelf. There are 30 aisles, 75 horizontal locations in each aisle, and 4 shelves throughout the warehouse. **9001** is the least number of products the company can have so that at least two products must be stored in the same bin?

q. If we know that in a group of n people (where any two people are either friends or enemies), there are not necessarily three mutual friends or three mutual enemies, then the largest value n can have is **five**.

r. $\binom{n+m}{n}$ is the number of paths in the xy plane between the origin (0,0) and point (m,n), where m and n are nonnegative integers, such that each path is made up of a series of steps, where each step is a move one unit to the right or a move one unit upward. (No moves to the left or downward are allowed.)

s. If the characteristic equation of a linear homogeneous recurrence relation is $(r-3)^2(r-7)^2=0$, then the form of the general solution is $\boldsymbol{a_n = (\alpha_{1,0} + \alpha_{1,1}n)3^n + (\alpha_{2,0} + \alpha_{2,1}n)7^n}$.

t. If $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$, then the value of A is **-1**.

CLO # 2: Apply formal logic proofs and/or informal, but rigorous, logical reasoning

Q2: Show that the premises “An animal in the National Park has not been fed today,” and “Every animal in the National Park is healthy” imply the conclusion “An animal that is healthy has not been fed today.”

The domain of discourse consists of all animals in the world.

Note: First clearly define your predicates. **[10 marks]**

Let $C(x)$ be “x lives in the National Park,” $B(x)$ be “x has been fed today,” and $P(x)$ be “x is healthy.”

- The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$.
- The conclusion is $\exists x(P(x) \wedge \neg B(x))$.

These steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

CLO # 4: Differentiate various discrete structures

Q3: Show that 1105 is a Carmichael number. **[10 marks]**

Note: If $m_1, m_2, \cdots, m_n$ are pairwise relatively prime integers greater than or equal to 2, then if $a \equiv b \ (mod\ m_i)$ for $i = 1, 2, \ldots, n$, then $a \equiv b \ (mod\ m)$, where $m = m_1 m_2 \cdots m_n$

A composite integer n that satisfies the congruence $b^{n-1} \equiv 1 \ (mod\ n)$ for all positive integers b with $gcd(b, n) = 1$ is called a *Carmichael number*.

As 1105 = 5 · 13 · 17, it is a composite integer. Moreover, if $gcd(b, 1105) = 1$, then $gcd(b, 5) = gcd(b, 13) = gcd(b, 17) = 1$.

Using **Fermat's little theorem** we find that $b^4 \equiv 1(mod\ 5), b^{12} \equiv 1(mod\ 13)$, and $b^{16} \equiv 1(mod\ 17)$.

It follows that

$$b^{1104} = (b^4)^{276} \equiv 1(mod\ 5)$$
$$b^{1104} = (b^{12})^{92} \equiv 1(mod\ 13)$$
$$b^{1104} = (b^{16})^{69} \equiv 1(mod\ 17)$$

Therefore, using the statement give in the **note** for the question, we conclude that

$$b^{1104} \equiv 1(mod\ 1105)$$

for all positive integers b with $gcd(b, 1105) = 1$. Hence, 1105 is a **Carmichael number**.

CLO # 3: Apply discrete structures into computing problems

Q4: The Fibonacci sequence $F_0, F_1, F_2, \ldots$ satisfies the recurrence relation

$$F_k = F_{k-1} + F_{k-2} \qquad \text{for all integers} \qquad k \geq 2$$

with initial conditions

$$F_0 = F_1 = 1.$$

Find an explicit formula for this sequence. **[10 marks]**

Important: The version defined above is different from the one given in the book, so pay necessary attention while solving for the formula.

The roots of the characteristic equation

$$r^2 - r - 1 = 0$$

are $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$.

Therefore, the Fibonacci numbers are given by

$$F_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2. The initial conditions $F_0 = 1$ and $F_1 = 1$ can be used to find these constants.We have

$$F_0 = \alpha_1 + \alpha_2 = 1,$$
$$F_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right).$$

The solution to these simultaneous equations for α_1 and α_2 is

$$\alpha_1 = \frac{1+\sqrt{5}}{2\sqrt{5}}, \qquad \alpha_2 = \frac{-1+\sqrt{5}}{2\sqrt{5}} = \frac{-(1-\sqrt{5})}{2\sqrt{5}}$$

Consequently, the Fibonacci numbers are given by

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

CLO # 4: Differentiate various discrete structures

Q5: **[2+2+2+4 marks]**

Note: Show adequate working where needed.

I. Let R be the relation represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the matrix representing R^2.

The matrix for R^2 is

$$M_{R^2} = M_R^{[2]} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

II. How can the directed graph of a relation R on a finite set A be used to determine whether the relation is symmetric?

The relation is symmetric, if and only if, for every edge, there is an edge in the opposite direction as well.

III. Find the transitive closures of the following relation on $\{1, 2, 3, 4\}$.

$$\{(1, 2), (2,1), (2,3), (3,4), (4,1)\}$$

The transitive closure is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

IV. Show that the relation R on the set of all bit strings such that sRt if and only if s and t contain the same number of 1s is an equivalence relation.

- R is **reflexive** because a bit string s has the same number of 1s as *itself*. Hence, **every** bit string relates to **itself** under R.
- R is **symmetric** because s and t having the same number of 1s implies that t and s do.
- R is **transitive** because s and t having the same number of 1s, and t and u having the same number of 1s implies that s and u have the same number of 1s, hence; sRu whenever sRt and tRu.
- Therefore, R is an **equivalence** relation.