Q# 1 [10 marks]

(a) $V = R^2$ → ordered pair of real number with standard vector addition but with scalar multiplication by

$$K(x,y) = (3Kx, 1)$$

Sol:

Let $\vec{u} = (x,y)$, $\vec{v} = (w,z)$

The axioms which are satisfied are as follows

(1) $\vec{u}+\vec{v} \in R^2$ (2) $\vec{0}+\vec{u} = \vec{u}+\vec{0} = \vec{u}$ (3) $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$

(4) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (5) $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ (6) $K\vec{u} \in R^2$

The axioms which do not hold:

(7) $K(\vec{u}+\vec{v}) = K\vec{u}+K\vec{v}$

$K[(x,y)+(w,z)]$	$K(x,y)+K(w,z)$
$K(x+w, y+z)$	$(3Kx,1)+(3Kw,1)$
$(3Kx+3Kw, 1)$	$(3Kx+3Kw, 2)$

⇒ Does not hold.

(8) $(K+m)\vec{u} = K\vec{u}+m\vec{u}$

$(K+m)(x,y)$	$K(x,y)+m(x,y)$
$(3Kx+3mx, 1)$	$(3Kx,1)+(3mx,1)$
	$(3Kx+3mx, 2)$

⇒ Does not hold.

(9) $K(m\vec{u}) = (Km)\vec{u}$

$K[m(x,y)]$	$Km(x,y)$
$K(3mx, 1)$	$(3Kmx, 1)$
$(9Kmx, 1)$	$(3Kmx, 1)$

⇒ Does not hold.

(10) $1\vec{u} = \vec{u}$

$$1.(x,y) = (3x,1) \neq \vec{u}$$

⇒ Does not hold.

(b) Use subspace test ... subspace of M_{nn} [4]

Sol:

$$W = \{A = [a_{ij}]_{n\times n} \,;\, tr(A) = 0\}$$

Let $A = [a_{ij}]_{n\times n} \in W$

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$\Rightarrow Tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii} = 0$

Also

$B = [b_{ij}]_{n\times n} \in W$

$$= \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$\Rightarrow Tr(B) = b_{11} + b_{22} + \cdots + b_{nn} = \sum_{i=1}^{n} b_{ii} = 0$

Subspace Test:

1 $A + B = \cancel{A}\, [a_{ij}]_{n\times n} + [b_{ij}]_{n\times n}$

$= [a_{ij} + b_{ij}]_{n\times n}$

To satisfy the first axiom, its trace, i.e., $Tr(A+B) = 0$.

$Tr(A+B) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii}$

$= 0 + 0$

$\Rightarrow Tr(A+B) = 0$ meaning $A + B \in W$.

2 $K.A = [Ka_{ij}]_{n\times n}$

$$= \begin{bmatrix} Ka_{11} & Ka_{12} & \cdots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \cdots & Ka_{2n} \\ \vdots & \vdots & & \\ Ka_{n1} & Ka_{n2} & \cdots & Ka_{nn} \end{bmatrix}$$

$Tr(KA) = Ka_{11} + Ka_{22} + \cdots + Ka_{nn}$

$= K(a_{11} + a_{22} + \cdots + a_{nn})$

$= K \sum_{i=1}^{n} a_{ii}$

$= K(0) = 0$ $\Rightarrow Tr(KA)0$ meaning $KA \in W$

$\Rightarrow$ This shows that the given matrix is a subspace of M_{nn}.

[10]

(C) Find the basis and dimension

$$2x + y + 3z = 0$$
$$x + 5z = 0$$
$$y + z = 0$$

Sol:

The augmented matrix is

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_1$

$$\begin{bmatrix} 1 & 0 & 5 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -7 \\ 0 & 0 & 8 \end{bmatrix}$$

$x + 5z = 0$ — (i)

$y - 7z = 0$ — (ii)

$8z = 0$ — (iii)

(iii) $\Rightarrow \boxed{z = 0}$

Substitute $z=0$ in (ii) and (iii)

(ii) $\Rightarrow \boxed{y = 0}$, (iii) $\Rightarrow \boxed{x = 0}$

Thus

$$\bar{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

There is no non-trivial solution, so the basis for the solution space is empty, and as the solution space consists only of the zero vector so the dimension is 0.

$\Rightarrow$ Basis = { } (empty set) , Dimension = 0.

(d) Verify that the Cauchy-Schwarz inequality holds. [3]

$\vec{u}=(5,0,-3,7)$, $\vec{v}=(1,2,2,1)$

Sol:

$$|\vec{u}\cdot\vec{v}| \leq \|u\|\,\|v\|$$

$$|\vec{u}\cdot\vec{v}| = |(5,0,-3,7)\cdot(1,2,2,1)|$$

$$= |5+0-6+7| = 6$$

$$\|u\|\,\|v\| = \sqrt{25+0+9+49}\,\sqrt{1+4+4+1} = (9.1)(3.1)$$

$$= 28.2$$

This implies that

$6 \leq 28.2$ so the inequality holds.

(e) Show that the set $S=\{P_1, P_2, P_3\}$ is a basis for P_2 [8]

$P_1 = 1+x+x^2$, $P_2 = x+x^2$, $P_3 = x^2$.

Also find coordinate vector of $p = 7-x+2x^2$ relative to the basis set S.

Solution:

We must show that the given polynomials are linearly independent and span P_2.

For the spanning, we must show that every polynomial $P = ax^2+bx+c$ in P_2 can be expressed as

$$P = k_1P_1 + k_2P_2 + k_3P_3$$

$$ax^2+bx+c = k_1(1+x+x^2) + k_2(x+x^2) + k_3(x^2)$$

$$ax^2+bx+c = (k_1+k_2+k_3)x^2 + (k_1+k_2)x + k_1$$

$$a = k_1+k_2+k_3$$

$$b = k_1+k_2$$

$$c = k_1$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

Expand by R_3

$$|A| = -1$$

So the set span P_2.

For linear independence, we must show

$$k_1P_1 + k_2P_2 + k_3P_3 = 0$$

For the given system, the coefficient matrix is same, i.e., $|A| = -1$

For spanning and linearly independence $|A| \neq 0$.

This shows that the set $S = \{P_1, P_2, P_3\}$ is basis for P_2.

- $P = 7 - x + 2x^2$

$$P = k_1P_1 + k_2P_2 + k_3P_3$$

$$7 - x + 2x^2 = k_1(1 + x + x^2) + k_2(x + x^2) + k_3(x^2)$$

$$7 - x + 2x^2 = (k_1 + k_2 + k_3)x^2 + (k_1 + k_2)x + k_1$$

$7 = k_1$ —— (i)

$-1 = k_1 + k_2$ —— (ii)

$2 = k_1 + k_2 + k_3$ —— (iii)

(i) $\Rightarrow$ $\boxed{k_1 = 7}$.

(ii) $\Rightarrow$ $-1 = 7 + k_2$ $\Rightarrow$ $\boxed{k_2 = -8}$

(iii) $\Rightarrow$ $2 = 7 - 8 + k_3$ $\Rightarrow$ $\boxed{k_3 = 3}$

So the coordinate vector is $(7, -8, 3)$.

(f) Find the volume of parallelpiped and determine whether $\vec{u}$, $\vec{v}$ and $\vec{w}$ lie in the same plane? [5]

$\vec{u} = (2,6,-1)$, $\vec{v} = (1,1,1)$, $\vec{w} = (4,6,2)$.

Volume of parallelpiped $= |\vec{u} \cdot \vec{v} \times \vec{w}| = \begin{vmatrix} 2 & 6 & -1 \\ 1 & 1 & 1 \\ 4 & 6 & 2 \end{vmatrix}$

Expand by R_2

$= -1 \begin{vmatrix} 6 & -1 \\ 6 & 2 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} - 1 \begin{vmatrix} 2 & 6 \\ 4 & 6 \end{vmatrix}$

$= -1(12+6) + 1(4+4) - 1(12-24)$

$= -18 + 8 + 12 = 20 - 18$

$= 2$

The vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ lie in the same plane when

$|\vec{u} \cdot \vec{v} \times \vec{w}| = 0$

we can clearly see that

$|\vec{u} \cdot \vec{v} \times \vec{w}| = 2$

So the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ do not lie in the same plane.