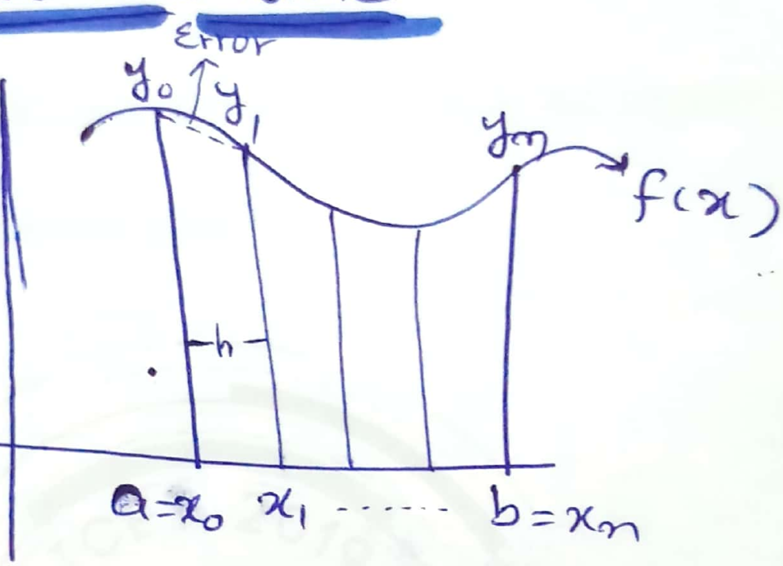
Trapezodial Rule

$$I = \int_a^b f(x)\,dx$$



$$\boxed{h = \frac{b-a}{n}}$$

Formula:

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

Steps:

1. Divide the interval $[a, b]$ into n equal interval with length h (step size) i,e $[a, b] = [a = x_0, x_1, x_2, \ldots, x_n = b]$ where

$$a = x_0$$
$$x_1 = x_0 + h$$
$$x_2 = x_0 + 2h$$
$$\vdots$$
$$x_n = x_0 + nh$$

$$\boxed{n = \frac{b-a}{h}} \quad \text{or} \quad \boxed{h = \frac{b-a}{n}}$$

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- It is applicable on any no. of interval.

Example

Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$, using trapezodial rule.

Sol

$a = 0,\ b = 1,$ let $n = 6$

then $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

$$\boxed{h = \frac{1}{6}}$$

Now,

$a = x_0 = 0$	$y_0 = \frac{1}{1+(0)^2} = 1$
$x_1 = x_0 + h = \frac{1}{6}$	$y_1 = \frac{1}{1+(\frac{1}{6})^2} = \frac{36}{37}$
$x_2 = x_0 + 2h = \frac{2}{6}$	$y_2 = \cancel{18/19}\ \frac{9}{10}$
$x_3 = \frac{3}{6}$	$y_3 = \frac{4}{5}$
$x_4 = \frac{4}{6}$	$y_4 = \frac{9}{13}$
$x_5 = \frac{5}{6}$	$y_5 = \frac{36}{61}$
$x_6 = \frac{6}{6} = 1$	$y_6 = \frac{1}{2}$

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By using trapezoidal rule, we have

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$n=6$

$$= \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{h}{2}\left[1 + 0.5 + 2\left(\frac{36}{37} + \frac{9}{10} + \frac{4}{5} + \frac{9}{13} + \frac{36}{61}\right)\right]$$

$$= \frac{h}{2}\left[\frac{3}{2} + 7.910889\right]$$

$$= \frac{1}{12}\left[9.410889\right]$$

$$\boxed{\int_0^1 \frac{1}{1+x^2}\,dx = 0.7842}$$

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ERROR IN TRAPEZODIAL RULE

Consider,

$$\int_{x_0}^{x_0+h} y(x)\,dx \quad \text{———} \quad (1)$$

where $y(x)$ is the taylor series i.e

$$y(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2!}y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots \quad \text{———} \quad (2)$$

put Eq (2) in Eq (1), we have

$$\int_{x_0}^{x_0+h} y(x)\,dx = \int_{x_0}^{x_0+h} \left[y_0 + (x-x_0)y_0' + \frac{1}{2!}(x-x_0)^2 y_0'' + \frac{(x-x_0)^3}{3!}y_0''' + \cdots\right]dx$$

$$= \int_{x_0}^{x_0+h} \left[y_0 + (x-x_0)y_0' + \frac{1}{2!}(x^2 - 2xx_0 + x_0^2)y_0'' + \cdots\right]dx$$

$$= \left[xy_0 + \left(\frac{x^2}{2} - xx_0\right)y_0' + \frac{1}{2!}\left(\frac{x^3}{3} - \frac{2x^2x_0}{2} + xx_0^2\right)\right]_{x_0}^{x_0+h}$$

$$= (x_0+h-x_0)y_0 + \left(\frac{(x_0+h)^2 - x_0^2}{2} - \{(x_0+h) - x_0\}x_0\right)y_0' + \cdots$$

$$= hy_0 + \left(\frac{x_0^2 + 2hx_0 + h^2 - x_0^2}{2} - (x_0 + h - x_0)x_0 \right) y_0' + \text{----}$$

$$= hy_0 + \left(\frac{2x_0h + h^2}{2} - hx_0 \right) y_0' + \text{----}$$

$$= hy_0 + \left(\frac{2\cancel{hx_0} + h^2 - 2\cancel{hx_0}}{2} \right) y_0' + \text{---}$$

$$\boxed{\int_{x_0}^{x_0+h} y(x)\,dx \doteq hy_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \frac{h^4}{4!} y_0''' + \ldots\ldots \quad \text{—— (4)}}$$

Trapezodial rule x_0 to x_1 is

$$\int_{x_0}^{x_0+h} f(x)\,dx = \frac{h}{2}\left[y_0 + y_1 \right] \quad \text{—— (5)}$$

As we know that taylor series is:

$$y = y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots\ldots$$

put $\boxed{y = y_1}$, $\boxed{x = x_0 + h}$

in the above equation, we get

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$$y_1 = y_0 + (x_0+h-x_0)y_0' + \frac{(x_0+h-x_0)^2}{2!}y_0'' + \cdots$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots \quad \text{——— (6)}$$

put Eq (6) in (5), we get

$$A_1 = \frac{h}{2}\left[y_0 + y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots\right]$$

$$= \frac{h}{2}\left[2y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots\right]$$

$$\boxed{A_1 = hy_0 + \frac{h^2 y_0'}{2} + \frac{h^3}{2\cdot 2!}y_0'' + \frac{h^4}{3!\times 2}y_0''' + \cdots}$$

$$\boxed{E_1 = \int_{x_0}^{x_0+h} f(x)\,dx - A}$$

$$\Rightarrow E_1 = \cancel{h}y_0 + \frac{h^2}{2!}y_0' + \frac{h^3}{3!}y_0'' + \cdots - \cancel{h}y_0 - \frac{h^2}{2!}y_0' - \frac{h^3}{2\cdot 2!}y_0'' - \cdots$$

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$$E_1 = \left[\frac{h^3}{3!}y_0'' - \frac{h^3}{2\cdot 2!}y_0''\right]$$

$$= \left(\frac{1}{6} - \frac{1}{4}\right)h^3 y_0''$$

$$\boxed{E_1 = \frac{-1}{12}h^3 y_0''}$$

So,

Error between $[x_0, x_1]$ is $\frac{-1}{12}h^3 y_0''$

Error b/w $[x_1, x_2] = \frac{-1}{12}h^3 y_1''$

Error b/w $[x_2, x_3] = \frac{-1}{12}h^3 y_2''$

$\vdots$

Error b/w $[x_{n-1}, x_n] = \frac{-1}{12}h^3 y_{n-1}''$

So

$$T.\ Error = \frac{-1}{12}h^3 y_0'' - \frac{1}{12}h^3 y_1'' - \frac{1}{12}h^3 y_2'' - \cdots\cdots - \frac{1}{12}h^3 y_{n-1}''$$

$$E = \frac{-1}{12} h^3 \left(y_0'' + y_1'' + y_2'' + \cdots + y_{n-1}'' \right)$$

$$= \frac{-1}{12} n h^3 (\eta_r), \quad x_r < r < x_{r+1}$$

$$r = 0, 1, 2, \ldots n-1$$

$$= \frac{-1}{12} n h \cdot h^2 (\eta_r)$$

$$= \frac{-1}{12} \left(\frac{b-a}{\cancel{h}} \right) \cdot h^{\cancel{3}\,2} (\eta_r)$$

$$E = \frac{-1}{12} (b-a) h^2 (\eta_r)$$

$$\boxed{\text{Order} : O(h^2)}$$