Continuous Bivariate Distributions:-

The bivariate probability density function of continuous r.v's X and Y is an integrable function f(x, y) satisfying the following properties:

i) $f(x,y) \geq 0$ for all (x, y)

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

(iii) $P(a \leq X \leq b, c \leq Y \leq d) = \int_{a}^{b}\int_{c}^{d} f(x,y)\,dy\,dx$

The distribution function of the bivariate r.v (X, Y) is defined by

$$F(x,y) \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,dv\,du$$

Also $\frac{\partial^2 F(x,y)}{\partial x\,\partial y} = f(x,y)$ where F is differentiable.

The marginal p.d.f of the continuous r.v X is $g(x) = \int_{-\infty} f(x,y)\,dy$

and marginal p.d.f of the continuous r.v Y is $h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

The conditional p.d.f of r.v X given Y is defined as $f(x/y) = \frac{f(x,y)}{h(y)}$ $h(y)>0$

similarly conditional p.d.f of r.v Y given X is $f(y/x) = \frac{f(x,y)}{g(x)}$ $g(x) > 0$

$\Rightarrow$ Two continuous r.v's X and Y are said to be statistically independent, if and only if their joint density $f(x,y)$ can be factored to form

$f(x,y) = g(x).h(y)$ for all possible values of x and y.

Example 7.8

(Pg 244, Sher M. Chaudhry)

$$f(x,y) = \frac{1}{8}(6-x-y) \quad 0 \le x \le 2, \ 2 \le y \le 4$$
$$= 0 \quad \text{otherwise}$$

a) verify that $f(x,y)$ is a joint density function.

b) calculate (i) $P(X \le \frac{3}{2}, Y \le 5/2)$

(ii) $P(X+Y < 3)$

c) Find the marginal p.d.f $g(x)$ and $h(y)$.

d) Find the conditional p.d.f $f(x/y)$ and $f(y/x)$.

a) The joint density $f(x,y)$ will be a p.d.f if i) $f(x,y) \geq 0$ and

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

As $f(x,y)$ is ≥ 0, so we need to check

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$= \int_0^2\int_2^4 \frac{1}{8}(6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^2\int_2^4 (6-x-y)\,dy\,dx$$

$$= \frac{1}{8}\int_0^2 \left[6y - xy - \frac{y^2}{2}\right]_2^4 dx$$

$$= \frac{1}{8}\int_0^2 \left[(24-4x-8)-(12-2x-2)\right]dx$$

$$= \frac{1}{8}\int_0^2 \left[6-2x\right]dx$$

$$= \frac{1}{8}\left[6x - x^2\right]_0^2 dx$$

$$= \frac{1}{8}\left[(12-4)-0\right]$$

$$= 1$$

Hence proved that $f(x,y)$ has the properties of a joint p.d.f.

b) (i) $P(X \le \frac{3}{2}, Y \le 5/2) = \int_{x=0}^{3/2}\int_{y=2}^{5/2} \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \frac{1}{8}\int_0^{3/2}\left[6y - xy - y^2/2\right]_2^{5/2} dx$$

$$= \frac{1}{8}\int_0^{3/2}\left(\frac{15}{8} - \frac{x}{2}\right)dx$$

$$= \frac{1}{64}\left[15x - 2x^2\right]_0^{3/2}$$

$$= \frac{9}{32}$$

(ii) $P(X+Y<3) = \frac{1}{8}\int_0^1\int_2^{3-x}(6-x-y)\,dy\,dx$

$(\because x+y \le 3$
$\therefore y = 3-x$

$$= \frac{1}{8}\int_0^1\left[6x - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1\left(\frac{x^2}{2} - 4x + \frac{7}{2}\right)dx$$

$$= \frac{1}{8}\left[\frac{x^3}{6} - 2x^2 - \frac{7x}{2}\right]_0^1$$

$$= \frac{1}{8} \times \frac{10}{6}$$

$$= 5/24$$

c) The marginal p.d.f of x is

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \qquad -\infty < x < \infty$$

$$= \frac{1}{8}\int_{2}^{4} (6-x-y)\,dy \qquad 0 \le x \le 2$$

$$= \frac{1}{8}\left[6y - xy - \frac{y^2}{2}\right]_2^4$$

$$= \frac{1}{8}\left[(24-4x-8)-(12-2x-2)\right]$$

$$= \frac{1}{8}\left[16-4x-10+2x\right]$$

$$= \frac{1}{8}\left[6-2x\right]$$

$$= \frac{2}{8}(3-x)$$

$$\boxed{g(x) = \frac{1}{4}(3-x)}$$

Similarly the marginal p.d.f of y is

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \qquad -\infty < y < \infty$$

$$= \frac{1}{8}\int_{0}^{2} (6-x-y)\,dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - xy\right]_0^2$$

$$= \frac{1}{8}\left[(12-2-2y)-(0)\right]$$

$$= \frac{1}{8}(10-2y)$$

$$= \frac{2}{8}(5-y)$$

$$\boxed{h(y) = \frac{1}{4}(5-y)}$$

d) $f(x/y) = \dfrac{f(x,y)}{h(y)}$ $h(y) > 0$

$$= \frac{\frac{1}{8}(6-x-y)}{(1/4)(5-y)}$$

$$= \frac{6-x-y}{2(5-y)}$$

$f(y/x) = \dfrac{f(x,y)}{g(x)}$ $g(x) > 0$

$$= \frac{1/8(6-x-y)}{1/4(3-x)}$$

$$= \frac{6-x-y}{2(3-x)}$$

Ans

Example 7.9 pg 246

Example 3.15 & 3.17, 3.19 (walpole)

Show that

$$E(x+y) = E(x) + E(y)$$

Proof :-

Let x and y be two random variables then

$$E(x+y) = E(x) + E(y)$$

$$E(x+y) = \sum_i \sum_j (x_i + y_j) f(x_i, y_j)$$

$$= \sum_i \sum_j x_i f(x_i, y_j) + \sum_i \sum_j y_j f(x_i, y_j)$$

As

$$\sum_i \sum_j x_i f(x_i, y_j) = \sum_i x_i \sum_j f(x_i, y_j)$$

$$= \sum_i x_i [f(x_i, y_1) + f(x_i, y_2) + \cdots + f(x_i, y_n)]$$

$$= \sum_i x_i g(x_i)$$

$(\because$ All possible values of y are included in $\sum_j)$

$$= \sum_i x_i g(x) = E(x)$$

Similarly $\sum_i \sum_j y_j f(x_i, y_j) = \sum_j y_j \sum_i f(x_i, y_j)$

$$= \sum_j y_j [f(x_1, y_j) + f(x_2, y_j) + \cdots + f(x_m, y_j)]$$

$$= \sum_j y_j h(y_j)$$

$$= E(y)$$

Hence

$$E(x+y) = E(x) + E(y)$$

$E(XY) = E(X).E(Y)$ if X and Y are independent.

As $f(x_i, y_j) = g(x_i)h(y_j)$ [when x and y are independent]

$$E(XY) = \sum_i \sum_j x_i y_j f(x_i, y_j)$$

$$= \sum_i \sum_j x_i y_j g(x_i) h(y_j) \quad \because f(x_i, y_j) = g(x)h(y)$$

$$= \sum_i x_i g(x_i) \sum_j y_j h(y_j)$$

$$= E(X).\ E(Y)$$

Example 7.19:- X & Y are two independent r.v's such that (Pg 259)

$g(x) = \frac{1}{3}$ $\quad x = 1, 2, 3$

$h(y) = \frac{1}{2}$ $\quad y = 0, 1$

if $Z = 2X - Y$, then verify $E(Z) = 2E(X) - E(Y)$

Solution:-

The joint distribution of the two independent r-v's X and Y is

Y	X			h(y)
	1	2	3	
0	1/6	1/6	1/6	1/2
1	1/6	1/6	1/6	1/2
g(x)	1/3	1/3	1/3	1

$E(X) = \sum x g(x) = \left[(1 \times \frac{1}{3}) + (2 \times \frac{1}{3}) + (3 \times \frac{1}{3})\right] = 2$

$E(Y) = \sum y h(y) = \left[(0 \times \frac{1}{2}) + (1 \times \frac{1}{2})\right] = \frac{1}{2}$

$$E(Z) = E[2X - Y] = 2E(X) - E(Y)$$
$$= 2 \times 2 - \frac{1}{2}$$
$$= 7/2$$

For verification, we find $E(2X-Y)$ directly as below:

$$E(2X - Y) = \sum_i \sum_j (2x_i - y_j) f(x_i, y_j)$$
$$= (2\times1 - 0)\tfrac{1}{6} + (2\times1 - 1)\tfrac{1}{6} + (2\times2 - 0)\tfrac{1}{6} + (2\times2 - 1)\tfrac{1}{6} + (2\times3 - 0)\tfrac{1}{6} + (2\times3 - 1)\tfrac{1}{6}$$
$$= \frac{2}{6} + \frac{1}{6} + \frac{4}{6} + \frac{3}{6} + \frac{6}{6} + \frac{5}{6}$$
$$= \frac{21}{6} = \frac{7}{2}$$

Hence
$$E(2X - Y) = 2E(X) - E(Y)$$
Ans

Example 7.18 (Same as 7.19)

Example 7·21 Let X and Y be independent r.v's with joint p.d.f.

$$f(x,y) = \frac{x(1+3y^2)}{4} \quad 0<x<2,\ 0<y<1$$

$$= 0 \quad \text{Else where.}$$

Find $E(X), E(Y), E(X+Y)$ and $E(XY)$.

Solution :—

For $E(X)$ and $E(Y)$, marginal functions of x and y required. i.e $g(x)$ and $h(y)$

$$g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_0^1 \frac{x(1+3y^2)}{4}\,dy$$

$$= \frac{1}{4}\left[xy + xy^3\right]_0^1$$

$$\boxed{g(x) = \frac{x}{2}} \quad \text{for} \quad 0<x<2$$

$$h(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \frac{1}{4}\int_0^2 x(1+3y^2)\,dx = \frac{1}{4}\int_0^2 (x+3xy^2)\,dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} + \frac{3x^2y^2}{2}\right]_0^2 = \frac{1}{4}\left[\frac{x^2}{2}(1+3y^2)\right]_0^2$$

$$\boxed{h(y) = \frac{1}{2}(1+3y^2)} \quad \text{for} \quad 0<y<1$$

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Now $E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx$

$$= \int_0^2 x \cdot \frac{x}{2}\, dx \Rightarrow \int_0^2 \frac{x^2}{2}\, dx$$

$$= \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{6}\left[8 - 0\right] = \frac{8}{6}$$

$$\boxed{E(X) = \frac{4}{3}}$$

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy = \frac{1}{2}\int_0^1 y \cdot (1+3y^2)\, dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} + \frac{3y^4}{4}\right]_0^1 = \frac{1}{2}\left[\frac{1}{2} + \frac{3}{4}\right]$$

$$\boxed{E(Y) = \frac{5}{8}}$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f(x,y)\, dx\, dy$$

$$= \int_0^2\int_0^1 (x+y)\frac{x(1+3y^2)}{4}\, dy\, dx$$

$$= \int_0^2\int_0^1 \frac{x^2 + 3x^2y^2}{4}\, dy\, dx + \int_0^2\int_0^1 \frac{xy + 3xy^3}{4}\, dy\, dx$$

$$= \int_0^2 \frac{1}{4}\left[x^2y + x^2y^3\right]_0^1 dx + \int_0^2 \frac{1}{4}\left[\frac{xy^2}{2} + \frac{3xy^4}{4}\right]_0^1 dx$$

$$= \int_0^2 \frac{1}{4}(2x^2)\, dx + \int_0^2 \frac{1}{4}\left(\frac{x}{2} + \frac{3x}{4}\right) dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{4}\left[\frac{x^2}{4} + \frac{3x^2}{8}\right]_0^2$$

$$= \frac{4}{3} + \frac{5}{8}$$

$$E(X+Y) = \frac{47}{24}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$$

$$= \int_0^2\int_0^1 (xy)\frac{x(1+3y^2)}{4}\, dy\, dx$$

$$= \int_0^2\int_0^1 \frac{x^2y + 3x^2y^3}{4}\, dy\, dx$$

$$= \int_0^2 \frac{1}{4}\left[\frac{x^2y^2}{2} + \frac{3x^2y^4}{4}\right]_0^1 dx$$

$$= \int_0^2 \frac{1}{4}\left(\frac{5x^2}{4}\right) dx = \frac{1}{4}\left[\frac{5x^3}{12}\right]_0^2$$

$$\boxed{E(XY) = \frac{5}{6}}$$

i) $E(X+Y) = E(X) + E(Y)$

$$\frac{47}{24} = \frac{4}{3} + \frac{5}{8} = \frac{47}{24}$$ proved.

ii) $E(XY) = E(X) \cdot E(Y)$

$$\frac{5}{6} = \left(\frac{\cancel{4}}{3}\right)\left(\frac{5}{\cancel{8}\,2}\right)$$

$$\frac{5}{6} = \frac{5}{6}$$ proved.