Discrete Structures

Advanced Counting Techniques

Text book: Kenneth H. Rosen, Discrete Mathematics and Its Applications

Section: 8.2

Solving Linear Recurrence Relations

Section 8.2

Section Summary

- Linear Homogeneous Recurrence Relations
- Solving Linear Homogeneous Recurrence Relations with Constant Coefficients.
- Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + ... + c_k\, a_{n-k}$, where $c_1, c_2, ...,c_k$ are real numbers, and $c_k \neq 0$

• it is ***linear*** because the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of *n*.
• it is ***homogeneous*** because no terms occur that are not multiples of the a_js.
•The coefficients of the terms of the sequence are all ***constants***, rather than functions that depend on *n*
• the ***degree*** is *k* because a_n is expressed in terms of the previous *k* terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the *k* initial conditions $a_0 = C_0$, $a_0 = C_1$,... , $a_{k-1} = C_{k-1}$.

Examples of Linear Homogeneous Recurrence Relations

- $P_n = (1.11)P_{n-1}$ linear homogeneous recurrence relation of degree one
- $f_n = f_{n-1} + f_{n-2}$ linear homogeneous recurrence relation of degree two
- $a_n = a_{n-1} + a_{n-2}^2$ not linear
- $H_n = 2H_{n-1} + 1$ not homogeneous
- $B_n = nB_{n-1}$ coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

- The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.
- Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$.
- Algebraic manipulation yields the ***characteristic equation***:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

- The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.
- The solutions to the characteristic equation are called the ***characteristic roots*** of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots,$ where α_1 and α_2 are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2} \text{ with } a_0 = 2 \text{ and } a_1 = 7?$$

Solution: The characteristic equation is $r^2 - r - 2 = 0$.
Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.

To find the constants α_1 and α_2, note that

$$a_0 = 2 = \alpha_1 + \alpha_2 \text{ and } a_1 = 7 = \alpha_1 2 + \alpha_2(-1).$$

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3 \cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers

example: We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers (*continued*)

Therefore by Theorem 1

$$f_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1 .$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$.

Hence,

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n$$

The Solution when there is a Repeated Root

Theorem 2:

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Using Theorem 2

Example: What is the solution to the recurrence relation
$a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.
The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.
Hence,

$$a_n = 3^n + n3^n .$$

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3:

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

EXAMPLE 6 Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with the initial conditions $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$.

Solution: The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1$, $r = 2$, and $r = 3$, because $r^3 - 6r^2 + 11r - 6 = (r-1)(r-2)(r-3)$. Hence, the solutions to this recurrence relation are of the form

$$a_n = \alpha_1 \cdot 1^n + \alpha_2 \cdot 2^n + \alpha_3 \cdot 3^n.$$

To find the constants α_1, α_2, and α_3, use the initial conditions. This gives

$$\begin{aligned}
a_0 &= 2 = \alpha_1 + \alpha_2 + \alpha_3,\\
a_1 &= 5 = \alpha_1 + \alpha_2 \cdot 2 + \alpha_3 \cdot 3,\\
a_2 &= 15 = \alpha_1 + \alpha_2 \cdot 4 + \alpha_3 \cdot 9.
\end{aligned}$$

When these three simultaneous equations are solved for α_1, α_2, and α_3, we find that $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = 1 - 2^n + 2 \cdot 3^n.$$

The General Case with Repeated Roots Allowed

Theorem 4:

Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = {} & (\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ & + (\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ & + \cdots + (\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$.

Solution: The characteristic equation of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

Because $r^3 + 3r^2 + 3r + 1 = (r + 1)^3$, there is a single root $r = -1$ of multiplicity three of the characteristic equation. By Theorem 4 the solutions of this recurrence relation are of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$, and $\alpha_{1,2}$, use the initial conditions. This gives

$$\begin{aligned}
a_0 &= 1 = \alpha_{1,0},\\
a_1 &= -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2},\\
a_2 &= -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}.
\end{aligned}$$

The simultaneous solution of these three equations is $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$. Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k\, a_{n-k} + F(n)\,,$$

where c_1, c_2, ….,c_k are real numbers, and $F(n)$ is a function not identically zero depending only on *n*.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots + c_k\, a_{n-k}\,,$$

is called the associated homogeneous recurrence relation.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*cont.*)

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$a_n = a_{n-1} + 2^n$,

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$,

$a_n = 3a_{n-1} + n3^n$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$

where the following are the associated linear homogeneous recurrence relations, respectively:

$a_n = a_{n-1}$,

$a_n = a_{n-1} + a_{n-2}$,

$a_n = 3a_{n-1}$,

$a_n = a_{n-1} + a_{n-2} + a_{n-3}$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients

Theorem 5:

If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients (*continued*)

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Because $F(n)= 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $p_n = cn + d$, where c and d are constants. Suppose that $p_n = cn + d$ is such a solution.

Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d)+ 2n$.

Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if $2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = -1$ and $d = -3/2$. Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution. Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$. Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.

Example

Find all solutions of the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution: This is a linear nonhomogeneous recurrence relation. The solutions of its associated homogeneous recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

are $a_n^{(h)} = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n$, where α_1 and α_2 are constants. Because $F(n) = 7^n$, a reasonable trial solution is $a_n^{(p)} = C \cdot 7^n$, where C is a constant. Substituting the terms of this sequence into the recurrence relation implies that $C \cdot 7^n = 5C \cdot 7^{n-1} - 6C \cdot 7^{n-2} + 7^n$. Factoring out 7^{n-2}, this equation becomes $49C = 35C - 6C + 49$, which implies that $20C = 49$, or that $C = 49/20$. Hence, $a_n^{(p)} = (49/20)7^n$ is a particular solution. By Theorem 5, all solutions are of the form

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot 2^n + (49/20)7^n.$$

Exercise

solve c) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

The characteristic equation is $r^2 - 5r + 6 = 0$, which factors as $(r-2)(r-3) = 0$, so the roots are $r = 2$ and $r = 3$. Therefore by Theorem 1 the general solution to the recurrence relation is $a_n = \alpha_1 2^n + \alpha_2 3^n$ for some constants α_1 and α_2. We plug in the initial condition to solve for the α's. Since $a_0 = 1$ we have $1 = \alpha_1 + \alpha_2$, and since $a_1 = 0$ we have $0 = 2\alpha_1 + 3\alpha_2$. These linear equations are easily solved to yield $\alpha_1 = 3$ and $\alpha_2 = -2$. Therefore the solution is $a_n = 3 \cdot 2^n - 2 \cdot 3^n$.

Exercise

13. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

Characteristic equation: $r^3 - 7r - 6 = 0$.
This is a 3rd degree equation, so we have to use Theorem 3 or 4.
Need to find roots. Use rational root test.
We know our possible roots are $\pm 1, \pm 2, \pm 3, \pm 6$.
Test to see which of these number will be our root for the equation.
Test -1.
$(-1)^3 - 7(-1) - 6 = -1 + 7 - 6 = 0$ Good! We know -1 is a root.
Factor out $(r - (-1))$ in $r^3 - 7r - 6 = 0$ to get $(r+1)(r^2 - r - 6) = 0$. Continue factoring and we get $(r+1)(r-3)(r+2) = 0$.
We know our characteristic roots: $r_1 = -1, r_2 = 3, r_3 = -2$.
Our general solution using Theorem 3 is: $a_n = \alpha_1(-1)^n + \alpha_2 3^n + \alpha_3(-2)^n$.
Find $\alpha_1, \alpha_2, \alpha_3$ by using the initial conditions.
For $a_0 = 9$
$9 = \alpha_1(-1)^0 + \alpha_2 3^0 + \alpha_3(-2)^0$
$9 = \alpha_1 + \alpha_2 + \alpha_3$
For $a_1 = 10$
$10 = \alpha_1(-1)^1 + \alpha_2 3^1 + \alpha_3(-2)^1$
$10 = -\alpha_1 + 3\alpha_2 - 2\alpha_3$
For $a_2 = 32$
$32 = \alpha_1(-1)^2 + \alpha_2 3^2 + \alpha_3(-2)^2$

Exercise

13. Find the solution to $a_n = 7a_{n-2} + 6a_{n-3}$ with $a_0 = 9$, $a_1 = 10$, and $a_2 = 32$.

$32 = \alpha_1 + 9\alpha_2 + 4\alpha_3$

Solve the system of equations:
$9 = \alpha_1 + \alpha_2 + \alpha_3$
$10 = -\alpha_1 + 3\alpha_2 - 2\alpha_3$
$32 = \alpha_1 + 9\alpha_2 + 4\alpha_3$

Add the first equation and second equation together to get:
$19 = 4\alpha_2 - \alpha_3$
Add the second equation and third equation together to get:
$42 = 12\alpha_2 + 2\alpha_3$
Multiply $19 = 4\alpha_2 - \alpha_3$ by 2 and add with $41 = 12\alpha_2 + 2\alpha_3$ to get:
$80 = 20\alpha_2$
$4 = \alpha_2$
Substitute α_2 back in.
$19 = 4(4) - \alpha_3$
$19 = 16 - \alpha_3$
$-3 = \alpha_3$
With α_2 and α_3, we can find α_1
$9 = \alpha_1 + 4 - 3$
$9 = \alpha_1 + 1$
$8 = \alpha_1$

Our solution is $a_n = 8(-1)^n + 4(3)^n + (-3)(-2)^n$.

Exercise

21. What is the general form of the solutions of a linear homogeneous recurrence relation if its characteristic equation has roots 1, 1, 1, 1, -2, -2, -2, 3, 3, -4?

- Solution

Use Theorem 4 with 4 roots that have multiple multiplicities.
$r_1 = 1$ with multiplicity 4, $r_2 = -2$ with multiplicity 3, $r_3 = 3$ with multiplicity 2, and $r_4 = -4$ with multiplicity 1.
So our general solution is of the form:
$a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2 + \alpha_{1,3}n^3)(1)^n + (\alpha_{2,0} + \alpha_{2,1}n + \alpha_{2,2}n^2)(-2)^n + (\alpha_{3,0} + \alpha_{3,1}n)(3)^n + (\alpha_{4,0})(-4)^n$

Exercise

29. a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 3^n$.

b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 5$.

- Solution

The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$.
The characteristic equation is $r - 2 = 0$.
Since our characteristic root is $r = 2$, we know by Theorem 3 that $a_n^{(h)} = \alpha 2^n$.
Note that $F(n) = 3^n$, so we know by Theorem 6 that $s = 3$ and 3 is not a root, the particular solution is of the form $a_n^{(p)} = c \cdot 3^n$. Plug $a_n^{(p)} = c \cdot 3^n$ into the recurrence relation and you'll get $c \cdot 3^n = 2c \cdot 3^{n-1} + 3^n$.
Simplify $c \cdot 3^n = 2c \cdot 3^{n-1} + 3^n$:

$c \cdot 3 = 2c + 3$

$3c = 2c + 3$

$c = 3$

Therefore, the particular solution that we seek is $a_n^{(p)} = 3 \cdot 3^n = 3^{n+1}$.
So the general solution is the sum of the homogeneous solution and the particular solution: $a_n = \alpha 2^n + 3^{n+1}$.