National University of Computer and Emerging Sciences
Lahore Campus

0224

Linear Algebra (MT1004)

Final Exam

Date: Ist January, 2024

Course Instructor(s)

Dr. Akhlaq Ahmad
Dr. Tayyaba Naz
Dr. Nazish Iftikhar
Dr. Nasir Ali
Dr. Sonia Hanif
Dr. Komal Hassan
Dr. Muhammad Rizwan
Ms. Maria Shabir

Total Time (Hrs): 3
Total Marks: 90
Total Questions: 5

Section

Student Signature

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Instruction/Notes:

1. **Programmable calculators are not allowed.**
2. **Wrong calculation work found (if any) at a step will not be further marked. Marks will be awarded till the correct calculations.**
3. **Attempt all question parts together. Question attempted in separate parts will not be marked.**

CLO #1: Use concept of elementary row operations to find the inverse of square matrices, determinant of a matrix and solving the system of linear equations.
CLO #5: Express a linear transformation graphically using matrices and to solve problems.

Application in Computer Graphics:

Question#1: [3+2 +3+5+2+5 marks]

a) Use **Inversion Algorithm** to find the Inverse of matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$. Write down A^{-1} as a product of elementary matrices $A^{-1} = E_k E_{k-1} \ldots\ldots E_3 E_2 E_1$.

$\begin{bmatrix} 9 & -4 \\ -2 & 1 \end{bmatrix}$

b) Verify that $A = E_1^{-1}\, E_2^{-1}\, E_3^{-1} \ldots\, E_{k-1}^{-1}\, E_k^{-1}$ for some k.

c) Sketch the image of the triangle with vertices $(0,0), (1.5,2), (3,1)$ under multiplication by the invertible matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}$.

d) Discuss the **Geometric Effect** on the triangle of multiplication by the given matrix A, using the following steps:

i) Show the effect of $E_1^{-1}\, E_2^{-1}\, E_3^{-1} \ldots\, E_{k-1}^{-1}\, E_k^{-1}$ on the triangle with vertices $(0,0), (1.5,2), (3,1)$ step by step.

ii) Show mathematically action of each elementary matrix on the end points of the edges and graphically show the output images at each step.

e) Show that succession of shears, compressions, expansions, and reflections that obtained in part **(d)** produces the same image as obtained in part **(c)**.

f) Find an equation for the image of the line $y = -4x + 3$ under multiplication by the matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}.$$

CLO #2: Properties of vectors in 2-space, 3-space and n-space and recognize vector spaces and/or subspaces to compute their bases and its dimension.

Question#2: [5 +5+10 marks]

a) Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$:

$$u + v = (u_1 + v_1 - 3,\ u_2 + v_2 - 3),\quad ku = (ku_1,\ 0).$$

i) Show that Axiom **4** holds by producing a zero vector such that $\boldsymbol{u} + (\mathbf{0}) = \boldsymbol{u}$ for $u = (u_1, u_2)$.

ii) Show that Axiom **5** holds by producing a negative vector $(-\boldsymbol{u})$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$ for $\boldsymbol{u} = (u_1, u_2)$.

iii) Show that Axiom **8**, *i.e.* $[(k+m)u = ku + mu]$ and Axiom **10** $[1.u = u]$, fail and hence that V is not a vector space under the given operations.

b) Consider the bases $B = \{u_1, u_2\}$ and $B' = \{u_1', u_2'\}$ for R^2, where $u_1 = \begin{bmatrix}1\\0\end{bmatrix}, u_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$$u_1' = \begin{bmatrix}3\\1\end{bmatrix} \ \& \ u_2' = \begin{bmatrix}2\\-1\end{bmatrix}$$

Find the transition matrix $P_{B\to B'}$.

$$\begin{bmatrix} 1/5 & 2/5 \\ 1/5 & -3/5 \end{bmatrix}$$

c) For the given matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \\ 4 & 8 & -4 & 4 \end{bmatrix}$$

i) Find the basis for the row space of A.

ii) Find the basis for the column space of A.

iii) Find the basis for the null space of A and explain geometrically the solution space/sub-space/null space spanned by the basis for the null space of A. **Hint:** Write the solution space say "X" in Matrix column notation and then check cardinality of X to determine basis for null space.

iv) Find rank and nullity for the given matrix A.

CLO #3: Perform Eigen Value analysis and use it to Diagonalize a matrix and/or find its powers.

Question#3: [5+2+8+5 marks]

a) Find the geometric and algebraic multiplicity of each eigenvalue of the matrix A,

b) Determine whether A is diagonalizable. If A is diagonalizable, then find a matrix P that diagonalizes A.

c) Prove that $P^{-1}A\,P = D$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & -1 \\ -2 & -4 & 4 \end{bmatrix}$$

d) Check that matrix A and $P^{-1}A\,P$ have same trace by using the definition of similarity invariants.

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CLO #4: Identify inner product spaces and/or perform Gram Schmidt process/QR decomposition using inner products.

Question#4: [5+5 marks]

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{matrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1/2 \end{matrix}$$

Suppose $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ define the column vectors u_1, u_2 and u_3 as

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \ \& \ u_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

a) Use the **Gram – Schmidt** process to find an orthogonal set of vectors $\{v_1, v_2, v_3\}$ and then find orthonormal set of vectors $\{q_1, q_2, q_3\}$. Discuss the geometry of Eigen Spaces corresponding to each Eigen value.

$u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2}$

b) Find the **QR**- decomposition of the given matrix. Also verify that $A = QR$ where, $Q = [\ q_1 \mid q_2 \mid q_3\ ]$ consists of the column vectors obtained in part (a) and R is given below

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}.$$

***Note:* Consider standard dot product as standard inner product between vectors.**

CLO #5: Express a linear transformation graphically using matrices and to solve problems.

General Linear Transformations:
Question # 5: [5+10+5 marks]

a) Consider basis $S = \{v_1, v_2\}$ for R^2, where $v_1 = (-2, 1)$ and $v_2 = (1, 3)$, and let $T: R^2 \to R^3$ be the linear transformation such that $T(v_1) = (-1, 2, 0)$ and $T(v_2) = (0, -3, 5)$. Find a formula for $T(x_1, x_2)$, and use that formula to find $T(2, -3)$.

b) Let $T: R^2 \to R^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -4x_1 \\ -x_1 + 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}.$$

i) Find the matrix for the transformation T i.e. $[T]_{B',B} = [\ [T(u_1)]_{B'} \mid [T(u_2)]_{B'}\ ]$ relative to the basis $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2, v_3\}$, where

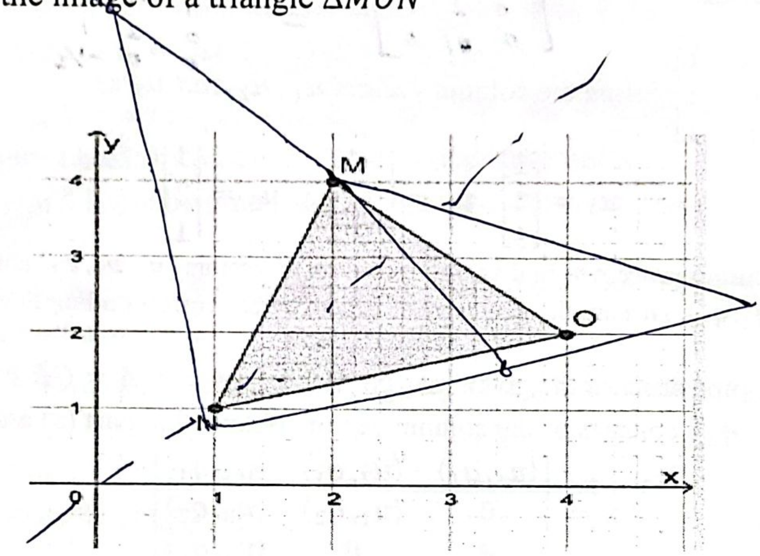
$$u_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, v_1 = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Denote $T = [T]_{B',B}$ and find the following:

ii) Find the Kernal of T i.e $Ker(T)$.

iii) Find the Range of T i.e $R(T)$.

c) The following is the image of a triangle ΔMON



Find standard matrices for the mentioned below parts and plot the final transformed image of a given triangle.

i) Expand by a factor of 2 in the x –direction.

ii) Reflect the given triangle about line $y = x$

Good Luck!