Summary

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- Universal Quantifier
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Propositional Logic Not Enough

Propositional logic, studied in Sections 1.1–1.3, cannot adequately express the meaning of all statements in mathematics and in natural language. For example, suppose that we know that

"Every computer connected to the university network is functioning properly."

No rules of propositional logic allow us to conclude the truth of the statement

"MATH3 is functioning properly,"

whereMATH3 is one of the computers connected to the university network. Likewise,

we cannot use the rules of propositional logic to conclude from the statement

"CS2 is under attack by an intruder,"

where CS2 is a computer on the university network, to the truth of

"There is a computer on the university network that is under attack by an intruder."

Propositional Logic Not Enough

If we have:

“All men are mortal.”

“Socrates is a man.”

Does it follow that “Socrates is mortal?”

Can’t be represented in propositional logic. Need a language that talks about objects, their properties, and their relations.

Later we’ll see how to draw inferences.

Introducing Predicate Logic

In this section we will introduce a more powerful type of logic called **predicate logic**.

We will see how predicate logic can be used to express the meaning of a wide range of statements in mathematics and computer science in ways that permit us to reason and explore relationships between objects.

To understand predicate logic, we first need to introduce the concept of a **predicate**.

Afterward, we will introduce the notion of **quantifiers**, which enable us to reason with statements that assert that a certain property **holds for all** objects of a certain type and

With statements that assert the **existence** of an object with a particular property.

Introducing Predicate Logic

Predicate logic uses the following new features:

- Variables: *x*, *y*, *z*
- Predicates: $P(x)$, $M(x)$
- Quantifiers

Propositional functions are a generalization of propositions.

- They contain variables and a predicate, e.g., $P(x)$
- Variables can be replaced by elements from their *domain*.

Predicates

Statements involving variables, such as

"$x > 3$," "$x = y + 3$," "$x + y = z$,"

and

"computer x is under attack by an intruder,"

and

"computer x is functioning properly,"

are often found in mathematical assertions, in computer programs, and in system specifications.

- These statements are neither true nor false when the values of the variables are not specified.
- In this section, we will discuss the ways that propositions can be produced from such statements.

Predicates

Statements involving variables, such as

$$\text{“}x > 3,\text{”} \quad \text{“}x = y + 3,\text{”} \quad \text{“}x + y = z,\text{”}$$

and

“computer x is under attack by an intruder,”

and

“computer x is functioning properly,”

- The statement “*x* is greater than 3” has two parts.
- The first part, the variable *x*, is the subject of the statement.
- The second part—the **predicate**, “is greater than 3”—refers to a property that the subject of the statement can have.
- We can denote the statement “*x* is greater than 3” by *P(x)*,
- where *P* denotes **the predicate** “is greater than 3” and *x* is **the variable**.

Propositional Functions

The statement *P(x)* is said to be the value of the **propositional function** *P* at *x*.

Propositional functions become propositions (and have truth values) when their variables are replaced by a value from the *domain* (or *bound* by a quantifier, as we will see later).

For example, let *P(x)* denote "$x > 0$" and the domain be the integers. Then:

P(-3) is false.

P(0) is false.

P(3) is true.

Propositional Functions

EXAMPLE 2 Let $A(x)$ denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of $A(\text{CS1})$, $A(\text{CS2})$, and $A(\text{MATH1})$?

Propositional Functions

EXAMPLE 3 Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Propositional Functions

In general, a statement involving the n variables $x_1, x_2, \ldots, x_n$ can be denoted by

$$P(x_1, x_2, \ldots, x_n).$$

A statement of the form $P(x_1, x_2, \ldots, x_n)$ is the value of the **propositional function** P at the n-tuple $(x_1, x_2, \ldots, x_n)$, and P is also called an n-**place predicate** or a n-**ary predicate**.

Propositional Functions

Let "$x + y = z$" be denoted by $R(x, y, z)$ and *domain* (for all three variables) be the integers. Find these truth values:

- R$(2,-1,5)$

  Solution: F

- R$(3,4,7)$

  Solution: T

- R$(x, 3, z)$

  Solution: Not a Proposition

Now let "$x - y = z$" be denoted by $Q(x, y, z)$, with domain as the integers. Find these truth values:

- Q$(2,-1,3)$

  Solution: T

- Q$(3,4,7)$

  Solution: F

- Q$(x, 3, z)$

  Solution: Not a Proposition

Compound Expressions

Connectives from propositional logic carry over to predicate logic.

If *P(x)* denotes "$x > 0$," find these truth values:

- $P(3) \lor P(-1)$
- **Solution**: T
- $P(3) \land P(-1)$
- **Solution**: F
- $P(3) \rightarrow P(-1)$
- **Solution**: F
- $P(3) \rightarrow \neg P(-1)$
- **Solution**: T

Expressions with variables are not propositions and therefore do not have truth values. For example,

- $P(3) \land P(y)$
- $P(x) \rightarrow P(y)$

When used with quantifiers (to be introduced next), these expressions (propositional functions) become propositions.

Quantifiers

We need *quantifiers* to express the meaning of English words including *all* and *some*:

- "All men are Mortal."
- "Some cats do not have fur."

The two most important quantifiers are:

- *Universal Quantifier,* **"**For all," symbol: $\forall$
- *Existential Quantifier*, "There exists," symbol: $\exists$

We write as in $\forall x\, P(x)$ and $\exists x\, P(x)$.

$\forall x\, P(x)$ asserts $P(x)$ is true for <u>every</u> x in the *domain*.

$\exists x\, P(x)$ asserts $P(x)$ is true for <u>some</u> x in the *domain*.

The quantifiers are said to bind the variable x in these expressions.

Universal Quantifier

Many mathematical statements assert that a property is true for all values of a variable in a particular domain, called the **domain of discourse** (or the **universe of discourse**), often just referred to as the **domain**.

Such a statement is expressed using universal quantification. The universal quantification of $P(x)$ for a particular domain is the proposition that asserts that $P(x)$ is true for all values of x in this domain.

Note that the domain specifies the possible values of the variable x. The meaning of the universal quantification of $P(x)$ changes when we change the domain.

The domain must always be specified when a universal quantifier is used; without it, the universal quantification of a statement is not defined.

Universal Quantifier

The ***universal quantification*** of $P(x)$ is the statement "$P(x)$ for all values of x in the domain."

- $\forall x\, P(x)$ is read as "For all x, P(x)" or "For every x, P(x)"

Examples:

1) If $P(x)$ denotes "$x > 0$" and *domain* is the integers, then $\forall x\, P(x)$ is false.
2) If $P(x)$ denotes "$x > 0$" and *domain* is the positive integers, then $\forall x\, P(x)$ is true.
3) If $P(x)$ denotes "x is even" and *domain* is the integers, then $\forall x\, P(x)$ is false.

An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.

Universal Quantifier

EXAMPLE 8 Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Universal Quantifier

Remark:

- Generally, an implicit assumption is made that all domains of discourse for quantifiers are nonempty.
- Note that if the domain is empty, then $\forall x P(x)$ is true for any propositional function $P(x)$ because there are no elements x in the domain for which $P(x)$ is false.
- Besides "for all" and "for every," universal quantification can be expressed in many other ways, including "all of," "for each," "given any," "for arbitrary," "for each," and "for any."
- It is better to avoid using "for any x" because it is often ambiguous as to whether "any" means "every" or "some." In some cases, "any" is unambiguous, such as when it is used in negatives, for example, "there is not any reason to avoid studying."

Universal Quantifier

- A statement $\forall x P(x)$ is false, where $P(x)$ is a propositional function, if and only if $P(x)$ is not always true when x is in the domain.
- One way to show that $P(x)$ is not always true when x is in the domain is to find a counterexample to the statement $\forall x P(x)$.
- Note that a single counterexample is all we need to establish that $\forall x P(x)$ is false.

EXAMPLE 9 Let $Q(x)$ be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: $Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus

$$\forall x Q(x)$$

is false. ◀

Universal Quantifier

EXAMPLE 10 Suppose that $P(x)$ is "$x^2 > 0$." To show that the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers, we give a counterexample. We see that $x = 0$ is a counterexample because $x^2 = 0$ when $x = 0$, so that x^2 is not greater than 0 when $x = 0$. ◀

Looking for counterexamples to universally quantified statements is an important activity in the study of mathematics, as we will see in subsequent sections of this book.

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n),$$

because this conjunction is true if and only if $P(x_1), P(x_2), \ldots, P(x_n)$ are all true.

Universal Quantifier

EXAMPLE 11 What is the truth value of $\forall x P(x)$, where $P(x)$ is the statement "$x^2 < 10$" and the domain consists of the positive integers not exceeding 4?

Solution: The statement $\forall x P(x)$ is the same as the conjunction

$$P(1) \wedge P(2) \wedge P(3) \wedge P(4),$$

because the domain consists of the integers 1, 2, 3, and 4. Because $P(4)$, which is the statement "$4^2 < 10$," is false, it follows that $\forall x P(x)$ is false.

EXAMPLE 12 What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

Solution: The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as "Every computer on campus is connected to the network."

Universal Quantifier

EXAMPLE 13 What is the truth value of $\forall x(x^2 \geq x)$ if the domain consists of all real numbers? What is the truth value of this statement if the domain consists of all integers?

Solution: The universal quantification $\forall x(x^2 \geq x)$, where the domain consists of all real numbers, is false. For example, $(\frac{1}{2})^2 \not\geq \frac{1}{2}$. Note that $x^2 \geq x$ if and only if $x^2 - x = x(x-1) \geq 0$. Consequently, $x^2 \geq x$ if and only if $x \leq 0$ or $x \geq 1$. It follows that $\forall x(x^2 \geq x)$ is false if the domain consists of all real numbers (because the inequality is false for all real numbers x with $0 < x < 1$). However, if the domain consists of the integers, $\forall x(x^2 \geq x)$ is true, because there are no integers x with $0 < x < 1$. ◀

Existential Quantifier

The ***existential quantification*** of *P(x)* is the proposition "There exists an element *x* in the domain such that *P(x)*." We use the notation $\exists x P(x)$ for the existential quantification of *P(x)*. Here $\exists$ is called **the *existential quantifier***.

A domain must always be specified when a statement $\exists x P(x)$ is used.

Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes.

Without specifying the domain, the statement $\exists x P(x)$ has no meaning.

Existential Quantifier

Besides the phrase "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is."

The existential quantification $\exists x P(x)$ is read as "There is an x such that $P(x)$," or

"There is at least one x such that $P(x)$," or

"For some $xP(x)$."

Examples:

1. If $P(x)$ denotes "$x > 0$" and *domain* is the integers, then $\exists x\, P(x)$ is true. It is also true if domain is the positive integers.
2. If $P(x)$ denotes "$x < 0$" and *domain* is the positive integers, then $\exists x\, P(x)$ is false.
3. If $P(x)$ denotes "x is even" and *domain* is the integers, then $\exists x\, P(x)$ is true.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Existential Quantifier

EXAMPLE 14 Let $P(x)$ denote the statement "$x > 3$." What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Solution: Because "$x > 3$" is sometimes true—for instance, when $x = 4$—the existential quantification of $P(x)$, which is $\exists x P(x)$, is true. ◀

Uniqueness Quantifier

$\exists! x\, P(x)$ means that $P(x)$ is true for one and only one x in the universe of discourse.

This is commonly expressed in English in the following equivalent ways:

- "There is a unique x such that $P(x)$."
- "There is one and only one x such that $P(x)$"

Examples:

1. If $P(x)$ denotes "$x + 1 = 0$" and domain is the integers, then $\exists! x\, P(x)$ is true.
2. But if $P(x)$ denotes "$x > 0$," then $\exists! x\, P(x)$ is false.

Quantifiers with Restricted Domains

An abbreviated notation is often used to restrict the domain of a quantifier. In this notation, a condition a variable must satisfy is included after the quantifier.

EXAMPLE 17 What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$, and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Solution: The statement $\forall x < 0\,(x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x < 0 \rightarrow x^2 > 0)$.

The statement $\forall y \neq 0\,(y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." Note that this statement is equivalent to $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.

Finally, the statement $\exists z > 0\,(z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$. ◀

Note that the restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0\,(x^2 > 0)$ is another way of expressing $\forall x(x < 0 \rightarrow x^2 > 0)$. On the other hand, the restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists z > 0\,(z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$.

More about Quantifiers

When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.

To evaluate $\forall x\, P(x)$ loop through all x in the domain.

- If at every step $P(x)$ is true, then $\forall x\, P(x)$ is true.
- If at a step $P(x)$ is false, then $\forall x\, P(x)$ is false and the loop terminates.

To evaluate $\exists x\, P(x)$ loop through all x in the domain.

- If at some step, $P(x)$ is true, then $\exists x\, P(x)$ is true and the loop terminates.
- If the loop ends without finding an x for which $P(x)$ is true, then $\exists x\, P(x)$ is false.

Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

Properties of Quantifiers

The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain.

Examples:

1. If *domain* is the positive integers and *P(x)* is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
2. If *domain* is the negative integers and *P(x)* is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
3. If *domain* consists of 3, 4, and 5, and *P(x)* is the statement "$x > 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.

   But if *P(x)* is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are false.

Binding Variables

When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**.

An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.

All the variables that occur in a propositional function must be bound or set equal to a particular value to turn it into a proposition. This can be done using a combination of universal quantifiers, existential quantifiers, and value assignments.

The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.

Consequently, a variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.

Binding Variables

EXAMPLE 18 In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable. This illustrates that in the statement $\exists x(x + y = 1)$, x is bound, but y is free.

In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, all variables are bound. The scope of the first quantifier, $\exists x$, is the expression $P(x) \wedge Q(x)$ because $\exists x$ is applied only to $P(x) \wedge Q(x)$, and not to the rest of the statement. Similarly, the scope of the second quantifier, $\forall x$, is the expression $R(x)$. That is, the existential quantifier binds the variable x in $P(x) \wedge Q(x)$ and the universal quantifier $\forall x$ binds the variable x in $R(x)$. Observe that we could have written our statement using two different variables x and y, as $\exists x(P(x) \wedge Q(x)) \vee \forall y R(y)$, because the scopes of the two quantifiers do not overlap. The reader should be aware that in common usage, the same letter is often used to represent variables bound by different quantifiers with scopes that do not overlap. ◀

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all the logical operators.

For example, $\forall x\, P(x) \vee Q(x)$ means $(\forall x\, P(x)) \vee Q(x)$

$\forall x\, (P(x) \vee Q(x))$ means something different.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain .

Solution 1: If *domain* is all students in this class, define a propositional function J(*x*) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if *domain* is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

because this statement says that all people are students in this class and have taken a course in java.

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain .

Solution 1: If *domain* is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if *domain* is all people, then translate as $\exists x\, (S(x) \wedge J(x))$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Equivalences in Predicate Logic

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value

- for every predicate substituted into these statements and
- for every domain of discourse used for the variables in the expressions.

The notation $S \equiv T$ indicates that S and T are logically equivalent.

Example: $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Equivalences in Predicate Logic

EXAMPLE 19 Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent (where the same domain is used throughout). This logical equivalence shows that we can distribute a universal

Solution: To show that these statements are logically equivalent, we must show that they always take the same truth value, no matter what the predicates P and Q are, and no matter which domain of discourse is used.

Suppose we have particular predicates P and Q, with a common domain. We can show that $\forall x(P(x) \wedge Q(x))$ and $\forall xP(x) \wedge \forall xQ(x)$ are logically equivalent by doing two things.

First, we show that if $\forall x(P(x) \wedge Q(x))$ is true, then $\forall xP(x) \wedge \forall xQ(x)$ is true.

Second, we show that if $\forall xP(x) \wedge \forall xQ(x)$ is true, then $\forall x(P(x) \wedge Q(x))$ is true.

So, suppose that $\forall x(P(x) \wedge Q(x))$ is true. This means that if a is in the domain, then $P(a) \wedge Q(a)$ is true. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true for every element in the domain, we can conclude that $\forall xP(x)$ and $\forall xQ(x)$ are both true.

This means that $\forall xP(x) \wedge \forall xQ(x)$ is true.

Next, suppose that $\forall xP(x) \wedge \forall xQ(x)$ is true. It follows that $\forall xP(x)$ is true and $\forall xQ(x)$ is true.

Hence, if a is in the domain, then $P(a)$ is true and $Q(a)$ is true [because $P(x)$ and $Q(x)$ are both true for all elements in the domain, there is no conflict using the same value of a here].

It follows that for all a, $P(a) \wedge Q(a)$ is true. It follows that $\forall x(P(x) \wedge Q(x))$ is true. We can now conclude that $\forall x(P(x) \wedge Q(x)) \equiv \forall xP(x) \wedge \forall xQ(x)$.

Equivalences in Predicate Logic

EXAMPLE 19 Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent (where the same domain is used throughout). This logical equivalence shows that we can distribute a universal

This logical equivalence shows that we can distribute a universal quantifier over a conjunction.

Furthermore, we can also distribute an existential quantifier over a disjunction.

However, we cannot distribute a universal quantifier over a disjunction, nor can we distribute an existential quantifier over a conjunction.

Quantifiers as Conjunctions and Disjunctions

If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

If *U* consists of the integers 1,2, and 3:

$$\forall x P(x) \equiv P(1) \wedge P(2) \wedge P(3)$$

$$\exists x P(x) \equiv P(1) \vee P(2) \vee P(3)$$

Even if the domains are infinite, you can still think of the quantifiers in this fashion, but the equivalent expressions without quantifiers will be infinitely long.

Negating Quantified Expressions

Consider $\forall x\, J(x)$

"Every student in your class has taken a course in Java."

Here $J(x)$ is "x has taken a course in Java" and

the domain is students in your class.

Negating the original statement gives "It is not the case that every student in your class has taken Java." This implies that "There is a student in your class who has not taken Java."

Symbolically $\neg\forall x\, J(x)$ and $\exists x\, \neg J(x)$ are equivalent

Negating Quantified Expressions

Now Consider $\exists x\, J(x)$

"There is a student in this class who has taken a course in Java."

Where $J(x)$ is "x has taken a course in Java."

Negating the original statement gives "It is not the case that there is a student in this class who has taken Java." This implies that "Every student in this class has not taken Java"

Symbolically $\neg\exists x\, J(x)$ and $\forall x\, \neg J(x)$ are equivalent

De Morgan's Laws for Quantifiers

The rules for negating quantifiers are:

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

The reasoning in the table shows that:

$$\neg\forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg\exists x P(x) \equiv \forall x \neg P(x)$$

These are important. You will use these.

De Morgan’s Laws for Quantifiers

- ***Remark:*** When the domain of a predicate $P(x)$ consists of n elements, where n is a positive integer greater than one, the rules for negating quantified statements are exactly the same as De Morgan’s laws
- This is why these rules are called De Morgan’s laws for quantifiers. When the domain has n elements $x1, x2, \ldots, xn$, it follows that $\neg\forall xP(x)$ is the same as $\neg(P(x1) \wedge P(x2) \wedge \cdots \wedge P(xn))$, which is equivalent to $\neg P(x1) \vee \neg P(x2) \vee \cdots \vee \neg P(xn)$ by De Morgan’s laws, and this is the same as $\exists x \neg P(x)$.
- Similarly, $\neg\exists xP(x)$ is the same as $\neg(P(x1) \vee P(x2) \vee \cdots \vee P(xn))$, which by De Morgan’s laws is equivalent to $\neg P(x1) \wedge \neg P(x2) \wedge \cdots \wedge \neg P(xn)$, and this is the same as $\forall x \neg P(x)$.

De Morgan's Laws for Quantifiers

EXAMPLE 20 What are the negations of the statements "There is an honest politician" and "All Americans eat cheeseburgers"?

Solution: Let $H(x)$ denote "x is honest." Then the statement "There is an honest politician" is represented by $\exists x H(x)$, where the domain consists of all politicians. The negation of this statement is $\neg\exists x H(x)$, which is equivalent to $\forall x \neg H(x)$. This negation can be expressed as "Every politician is dishonest." (*Note:* In English, the statement "All politicians are not honest" is ambiguous. In common usage, this statement often means "Not all politicians are honest." Consequently, we do not use this statement to express this negation.)

Let $C(x)$ denote "x eats cheeseburgers." Then the statement "All Americans eat cheeseburgers" is represented by $\forall x C(x)$, where the domain consists of all Americans. The negation of this statement is $\neg\forall x C(x)$, which is equivalent to $\exists x \neg C(x)$. This negation can be expressed in several different ways, including "Some American does not eat cheeseburgers" and "There is an American who does not eat cheeseburgers." ◀

De Morgan’s Laws for Quantifiers

EXAMPLE 21 What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain. ◀

De Morgan’s Laws for Quantifiers

EXAMPLE 22 Show that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent.

Solution: By De Morgan’s law for universal quantifiers, we know that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(\neg(P(x) \to Q(x)))$ are logically equivalent. By the fifth logical equivalence in Table 7 in Section 1.3, we know that $\neg(P(x) \to Q(x))$ and $P(x) \land \neg Q(x)$ are logically equivalent for every x. Because we can substitute one logically equivalent expression for another in a logical equivalence, it follows that $\neg\forall x(P(x) \to Q(x))$ and $\exists x(P(x) \land \neg Q(x))$ are logically equivalent. ◀

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and *domain* be all people.

   $$\exists x \ (S(x) \land M(x))$$

[***Caution!*** *above* statement cannot be expressed as $\exists x(S(x) \rightarrow M(x))$, which is true when there is someone not in the class because, in that case, for such a person x, $S(x) \rightarrow M(x)$ becomes either **F→T** or **F→F**, both of which are true.]

2. If the domain for x consists of the students in this class, we can translate this first statement as $\exists x M(x)$.

3. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada." domain being all people

   $$\forall x \ (S(x) \rightarrow (M(x) \lor C(x)))$$

4. if the domain for x consists of the students in this class, above statement can be expressed as $\forall x(C(x) \ \lor \ M(x))$.

Translation from English to Logic

EXAMPLE 23 Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

Solution: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain:

"For every student in this class, that student has studied calculus."

Next, we introduce a variable x so that our statement becomes

"For every student x in this class, x has studied calculus."

Continuing, we introduce $C(x)$, which is the statement "x has studied calculus." Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

However, there are other correct approaches; different domains of discourse and other predicates can be used. The approach we select depends on the subsequent reasoning we want to carry out. For example, we may be interested in a wider group of people than only those in this class. If we change the domain to consist of all people, we will need to express our statement as

"For every person x, if person x is a student in this class then x has studied calculus."

Translation from English to Logic

EXAMPLE 23 Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

If $S(x)$ represents the statement that person x is in this class, we see that our statement can be expressed as $\forall x(S(x) \rightarrow C(x))$. [*Caution!* Our statement *cannot* be expressed as $\forall x(S(x) \wedge C(x))$ because this statement says that all people are students in this class and have studied calculus!]

Finally, when we are interested in the background of people in subjects besides calculus, we may prefer to use the two-variable quantifier $Q(x, y)$ for the statement "student x has studied subject y." Then we would replace $C(x)$ by $Q(x, \text{calculus})$ in both approaches to obtain $\forall x Q(x, \text{calculus})$ or $\forall x(S(x) \rightarrow Q(x, \text{calculus}))$. ◀

Using quantifier in System Specification

Predicate logic is used for specifying properties that systems must satisfy.

EXAMPLE 25 Use predicates and quantifiers to express the system specifications "Every mail message larger than one megabyte will be compressed" and "If a user is active, at least one network link will be available."

Solution: Let $S(m, y)$ be "Mail message m is larger than y megabytes," where the variable x has the domain of all mail messages and the variable y is a positive real number, and let $C(m)$ denote "Mail message m will be compressed." Then the specification "Every mail message larger than one megabyte will be compressed" can be represented as $\forall m(S(m, 1) \rightarrow C(m))$.

Let $A(u)$ represent "User u is active," where the variable u has the domain of all users, let $S(n, x)$ denote "Network link n is in state x," where n has the domain of all network links and x has the domain of all possible states for a network link. Then the specification "If a user is active, at least one network link will be available" can be represented by $\exists u A(u) \rightarrow \exists n S(n, \text{available})$. ◀

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)
(1832-1898)

The first two are called *premises* and the third is called the *conclusion*.

1. "All lions are fierce."
2. "Some lions do not drink coffee."
3. "Some fierce creatures do not drink coffee."

Here is one way to translate these statements to predicate logic. Let P(x), Q(x), and R(x) be the propositional functions "x is a lion," "x is fierce," and "x drinks coffee," respectively.Assuming that the domain consists of all creatures

1. $\forall x\,(P(x) \rightarrow Q(x))$
2. $\exists x\,(P(x) \wedge \neg R(x))$
3. $\exists x\,(Q(x) \wedge \neg R(x))$

Later we will see how to prove that the conclusion follows from the premises.

Notice that the second statement cannot be written as $\exists x(P(x) \rightarrow \neg R(x))$. The reason is that $P(x) \rightarrow \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \rightarrow \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as

$$\exists x(Q(x) \rightarrow \neg R(x)).$$

Lewis Carroll Example

Charles Lutwidge Dodgson
(AKA Lewis Caroll)

EXAMPLE 27 Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly colored."
"No large birds live on honey."
"Birds that do not live on honey are dull in color."
"Hummingbirds are small."

Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly colored," respectively. Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.

Solution: We can express the statements in the argument as

$$\forall x(P(x) \rightarrow S(x)).$$
$$\neg\exists x(Q(x) \land R(x)).$$
$$\forall x(\neg R(x) \rightarrow \neg S(x)).$$
$$\forall x(P(x) \rightarrow \neg Q(x)).$$

Section Summary

1. Nested Quantifiers
2. Order of Quantifiers
3. Translating from Nested Quantifiers into English
4. Translating Mathematical Statements into Statements involving Nested Quantifiers.
5. Translated English Sentences into Logical Expressions.
6. Negating Nested Quantifiers.

Nested Quantifiers

Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

Example: “Every real number has an inverse” is

$$\forall x \exists y (x + y = 0)$$

where the domains of x and y are the real numbers.

We can also think of nested propositional functions:

$\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Nested Quantification as Nested Loops

Nested Loops

- To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

  $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
- To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
  - At each step, loop through the values for y.
  - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
  - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

  $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

If the domains of the variables are infinite, then this process can not actually be carried out.

Order of Quantifiers

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that *domain* is the real numbers. Then $\forall x\ \forall y P(x,y)$ and $\forall y\ \forall x P(x,y)$ have the same truth value.
1. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that *domain* is the real numbers. Then $\forall x \exists y Q(x,y)$ is true, but $\exists y\ \forall x Q(x,y)$ is false.

Order of Quantifiers

Example 1: Let *domain* be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False

2. $\forall x \exists y P(x,y)$

   Answer: True

3. $\exists x \forall y\ P(x,y)$

   Answer: True

4. $\exists x \exists\ y\ P(x,y)$

   Answer: True

Order of Quantifiers

Example 2: Let *domain* be the real numbers,

Define *P(x,y)* : $\frac{x}{y} = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

   Answer: False
2. $\forall x \exists y P(x,y)$

   Answer: False
3. $\exists x \forall y\, P(x,y)$

   Answer: False
4. $\exists x \exists\, y\, P(x,y)$

   Answer: True

Quantifications of Two Variables

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Translating Nested Quantifiers into English

Example 1: Translate the statement

$$\forall x \ (C(x) \vee \exists y \ (C(y) \wedge F(x, y)))$$

where C(x) is "*x* has a computer," and *F*(*x*,*y*) is "*x* and *y* are friends," and the domain for both *x* and *y* consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \, \forall y \, \forall z \ ((F(x, y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There is a student none of whose friends are also friends with each other.

Translating Mathematical Statements into Predicate Logic

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables *x* and *y*, and specify the domain, to obtain:

   "For all positive integers *x* and *y*, *x* + *y* is positive."

3. The result is:

$$\forall x \forall y((x > 0) \land (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

Translating English into Logical Expressions Example

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w\ \forall a\ \exists f\ (P(w,f) \land Q(f,a))$$

Translating English into Logical Expressions Example

Choose the obvious predicates and express in predicate logic.

Example 1: "Brothers are siblings."

Solution: $\forall x\, \forall y\, (B(x,y) \rightarrow S(x,y))$

Example 2: "Siblinghood is symmetric."

Solution: $\forall x\, \forall y\, (S(x,y) \rightarrow S(y,x))$

Example 3: "Everybody loves somebody."

Solution: $\forall x\, \exists y\, L(x,y)$

Example 4: "There is someone who is loved by everyone."

Solution: $\exists y\, \forall x\, L(x,y)$

Example 5: "There is someone who loves someone."

Solution: $\exists x\, \exists y\, L(x,y)$

Example 6: "Everyone loves himself"

Solution: $\forall x\, L(x,x)$

Negating Nested Quantifiers

Example 1: Recall the logical expression developed three slides back:

$$\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

$\neg\exists w \forall a \exists f\ (P(w,f) \land Q(f,a))$

$\forall w \neg \forall a \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$

$\forall w \exists a \neg \exists f\ (P(w,f) \land Q(f,a))$ by De Morgan's for $\forall$

$\forall w \exists a \forall f \neg (P(w,f) \land Q(f,a))$ by De Morgan's for $\exists$

$\forall w \exists a \forall f (\neg P(w,f) \lor \neg Q(f,a))$ by De Morgan's for $\land$.

Part 3: Can you translate the result back into English?

Solution: "For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

More about Quantifiers

Can you switch the order of quantifiers?

- Is this a valid equivalence? $$\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$$

  Solution: Yes! The left and the right side will always have the same truth value. The order in which *x* and *y* are picked does not matter.

- Is this a valid equivalence? $$\forall x \exists y P(x,y) \equiv \exists y \forall x P(x,y)$$

  Solution: No! The left and the right side may have different truth values for some propositional functions for *P*. Try "x + y = 0" for *P(x,y)* with *domain* being the integers. The order in which the values of *x* and *y* are picked does matter.

Can you distribute quantifiers over logical connectives?

$$\forall x (P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

- Is this a valid equivalence?

  Solution: Yes! The left and the right side will always have the same truth value no matter what propositional functions are denoted by *P(x)* and *Q(x)*.

- Is this a valid equivalence? $$\forall x (P(x) \rightarrow Q(x)) \equiv \forall x P(x) \rightarrow \forall x Q(x)$$

  Solution: No! The left and the right side may have different truth values. Pick "x is a fish" for P(x) and "x has scales" for Q(x) with the domain of discourse being all animals. Then the left side is false, because there are some fish that do not have scales. But the right side is true since not all animals are fish.