Discrete Structures

Graphs

Text book: Kenneth H. Rosen, Discrete Mathematics and Its Applications

Section: 10.2

Graph Terminology and Special Types of Graphs

Section 10.2

Section Summary

- Basic Terminology
- Some Special Types of Graphs
- Bipartite Graphs
- New Graphs from Old

Basic Terminology

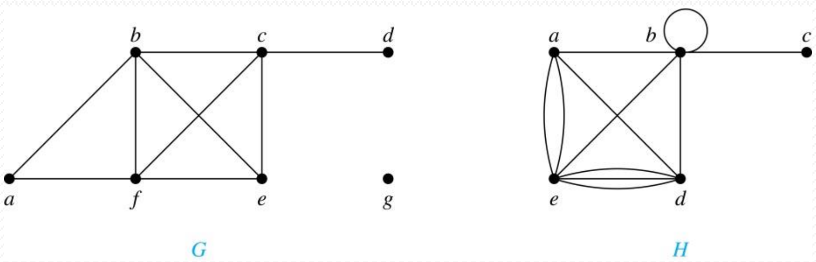
Definition 1. Two vertices u, v in an undirected graph G are called *adjacent* (or *neighbors*) in G if there is an edge e between u and v. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.

Definition 2. The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.

Definition 3. The *degree of a vertex in a undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Degrees and Neighborhoods of Vertices

Example: What are the degrees and neighborhoods of the vertices in the graphs G and H?



Solution:

G: $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, $\deg(g) = 0$.

$N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, $N(g) = \varnothing$.

H: $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, $\deg(d) = 5$.

$N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, $N(e) = \{a, b, d\}$.

Degrees of Vertices

Theorem 1 (***Handshaking Theorem***): If $G = (V,E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Think about the graph where vertices represent the people at a party and an edge connects two people who have shaken hands.

Handshaking Theorem

We now give two examples illustrating the usefulness of the handshaking theorem.

Example: How many edges are there in a graph with 10 vertices of degree six?

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, the handshaking theorem tells us that $2m = 60$. So the number of edges $m = 30$.

Example: If a graph has 5 vertices, can each vertex have degree 3?

Solution: This is not possible by the handshaking theorem, because the sum of the degrees of the vertices $3 \cdot 5 = 15$ is odd.

Degree of Vertices (*continued*)

Theorem 2: An undirected graph has an even number of vertices of odd degree.

Proof: Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in an undirected graph $G = (V, E)$ with m edges. Then

even

$$2m = \sum_{v \in V} \deg(v) = \sum_{v \in V_1} \deg(v) + \sum_{v \in V_2} \deg(v).$$

must be even since deg(v) is even for each $v \in V_1$

This sum must be even because 2m is even and the sum of the degrees of the vertices of even degrees is also even. Because this is the sum of the degrees of all vertices of odd degree in the graph, there must be an even number of such vertices.

Directed Graphs

Recall the definition of a directed graph.

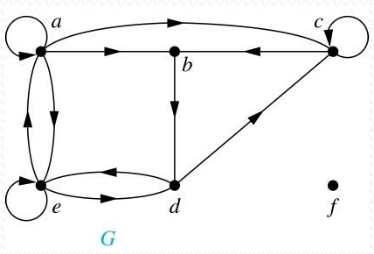
Definition: An *directed graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*), and E, a set of *directed edges* or *arcs*. Each edge is an ordered pair of vertices. The directed edge (u,v) is said to start at u and end at v.

Definition: Let (u,v) be an edge in G. Then u is the *initial vertex* of this edge and is *adjacent to* v and v is the *terminal* (or *end*) *vertex* of this edge and is *adjacent from* u. The initial and terminal vertices of a loop are the same.

Directed Graphs (*continued*)

Definition: The *in-degree of a vertex v*, denoted *deg*$^-(v)$, is the number of edges which terminate at v. The *out-degree of v*, denoted *deg*$^+(v)$, is the number of edges with v as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

Example: In the graph G we have



$\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$,
$\deg^-(e) = 3$, $\deg^-(f) = 0$.

$\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$,
$\deg^+(e) = 3$, $\deg^+(f) = 0$.

Directed Graphs (*continued*)

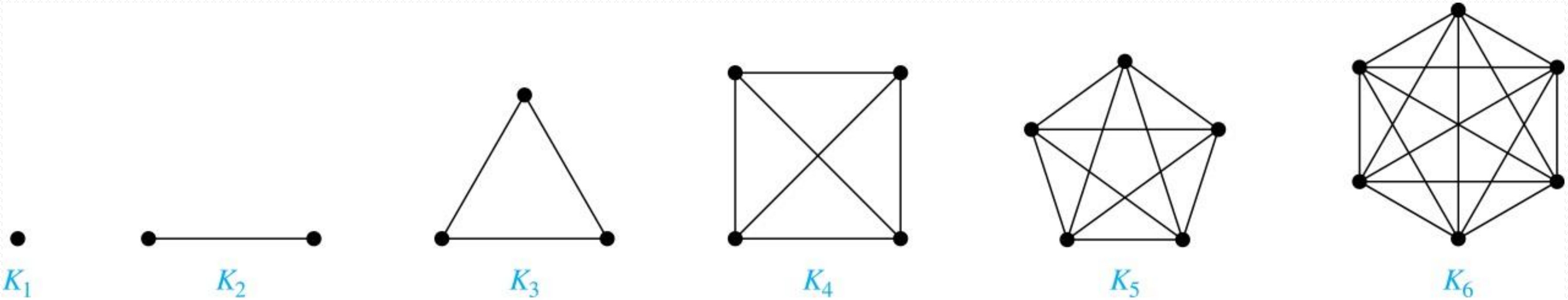
Theorem 3: Let $G = (V, E)$ be a graph with directed edges. Then:

$$|E| = \sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v).$$

Proof: The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. It follows that both sums equal the number of edges in the graph. ◀

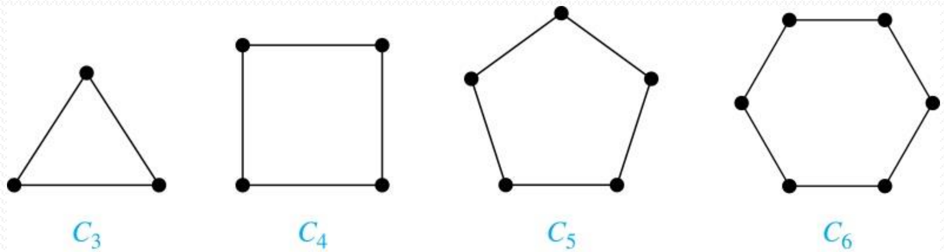
Special Types of Simple Graphs: Complete Graphs

A *complete graph on n vertices*, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

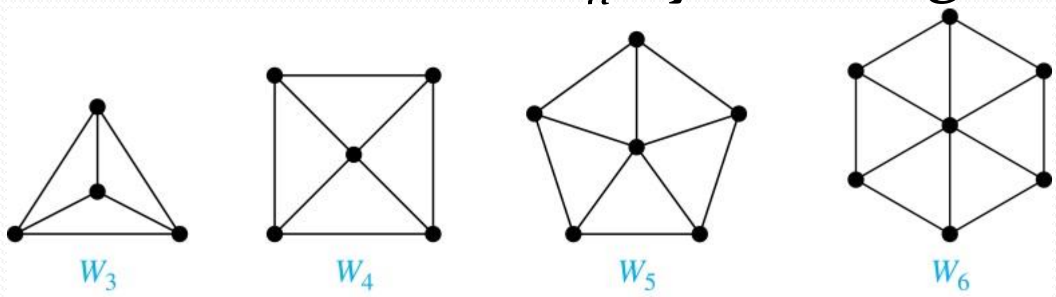


Special Types of Simple Graphs: Cycles and Wheels

A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \cdots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \cdots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

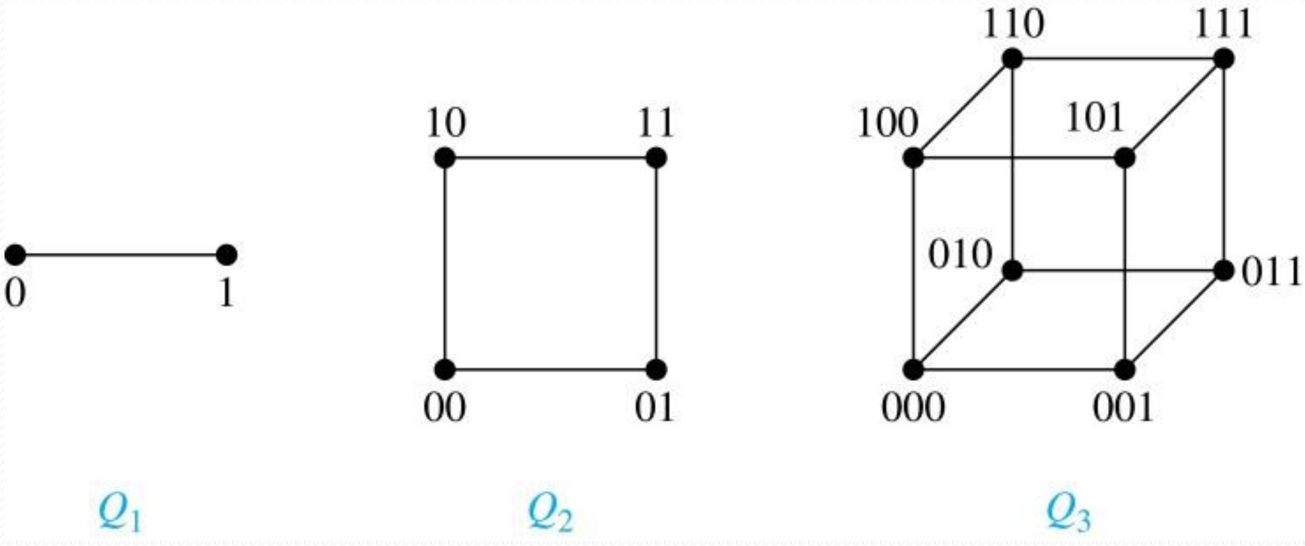


A *wheel* W_n is obtained by adding an additional vertex to a cycle C_n for $n \geq 3$ and connecting this new vertex to each of the n vertices in C_n by new edges.



Special Types of Simple Graphs: *n*-Cubes

An *n-dimensional hypercube*, or *n-cube*, $\mathbf{Q}_n$, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices that differ in exactly one bit position.



Bipartite Graphs

Definition: A simple graph G is bipartite if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2. In other words, there are no edges which connect two vertices in V_1 or in V_2.

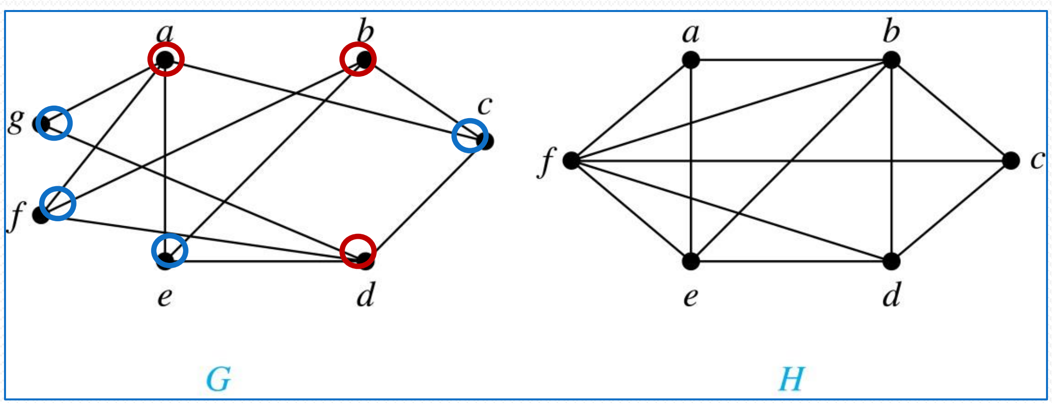
- It is not hard to show that an equivalent definition of a bipartite graph is a graph where it is possible to color the vertices red or blue so that no two adjacent vertices are the same color.

EXAMPLE 11 Are the graphs G and H displayed in Figure 8 bipartite?

Solution: Graph G is bipartite because its vertex set is the union of two disjoint sets, $\{a, b, d\}$ and $\{c, e, f, g\}$, and each edge connects a vertex in one of these subsets to a vertex in the other subset. (Note that for G to be bipartite it is not necessary that every vertex in $\{a, b, d\}$ be adjacent to every vertex in $\{c, e, f, g\}$. For instance, b and g are not adjacent.)

Graph H is not bipartite because its vertex set cannot be partitioned into two subsets so that edges do not connect two vertices from the same subset. (The reader should verify this by considering the vertices a, b, and f.) ◀

G is bipartite



H is not bipartite since if we color a red, then the adjacent vertices f and b must both be blue.

EXAMPLE 9 C_6 is bipartite, as shown in Figure 7, because its vertex set can be partitioned into the two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$, and every edge of C_6 connects a vertex in V_1 and a vertex in V_2. ◀

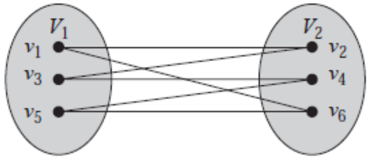
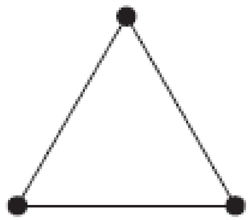


FIGURE 7 **Showing That C_6 Is Bipartite.**

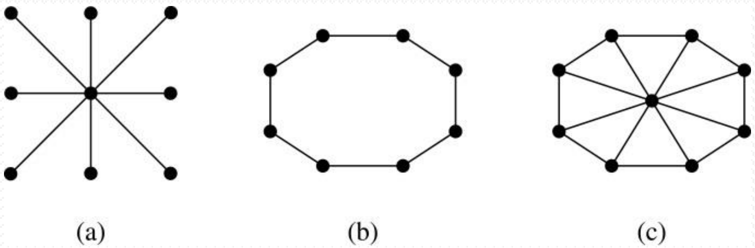
EXAMPLE 10 K_3 is not bipartite. To verify this, note that if we divide the vertex set of K_3 into two disjoint sets, one of the two sets must contain two vertices. If the graph were bipartite, these two vertices could not be connected by an edge, but in K_3 each vertex is connected to every other vertex by an edge. ◀



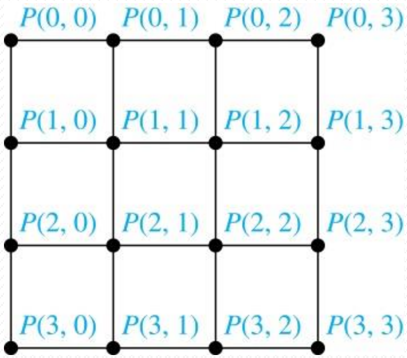
K_3

Special Types of Graphs and Computer Network Architecture

Various special graphs play an important role in the design of computer networks.



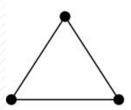
- Some local area networks use a *star topology*, which is a complete bipartite graph $K_{1,n}$, as shown in (a). All devices are connected to a central control device.
- Other local networks are based on a *ring topology*, where each device is connected to exactly two others using C_n, as illustrated in (b). Messages may be sent around the ring.
- Others, as illustrated in (c), use a W_n – based topology, combining the features of a star topology and a ring topology.
- Various special graphs also play a role in parallel processing where processors need to be interconnected as one processor may need the output generated by another.
  - The *n-dimensional hypercube*, or *n-cube*, $\boldsymbol{Q_n}$, is a common way to connect processors in parallel, e.g., Intel Hypercube.
  - Another common method is the *mesh* network, illustrated here for 16 processors.



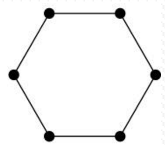
Bipartite Graphs (*continued*)

Example: Show that C_6 is bipartite.

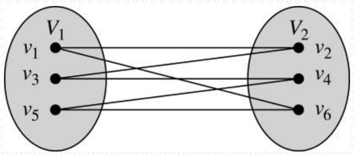
Solution: We can partition the vertex set into $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ so that every edge of C_6 connects a vertex in V_1 and V_2 .



C_3



C_6



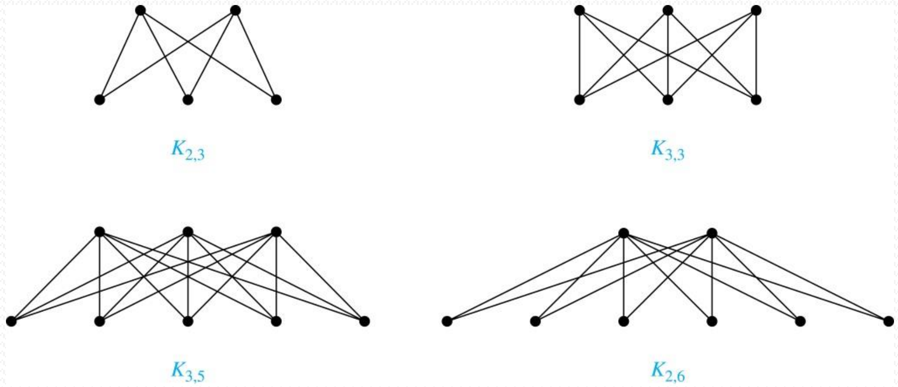
Example: Show that C_3 is not bipartite.

Solution: If we divide the vertex set of C_3 into two nonempty sets, one of the two must contain two vertices. But in C_3 every vertex is connected to every other vertex. Therefore, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

Complete Bipartite Graphs

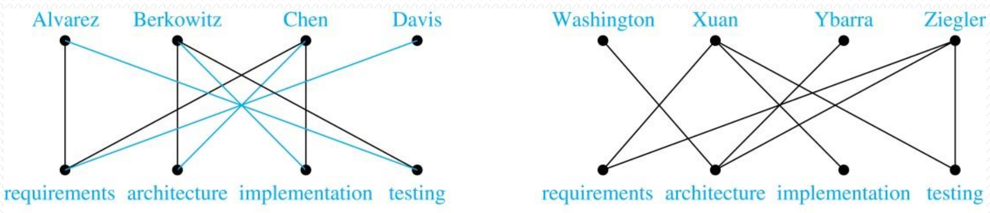
Definition: A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

Example: We display four complete bipartite graphs here.



Bipartite Graphs and Matchings

- Bipartite graphs are used to model applications that involve matching the elements of one set to elements in another, for example:
- *Job assignments* - vertices represent the jobs and the employees, edges link employees with those jobs they have been trained to do. A common goal is to match jobs to employees so that the most jobs are done.



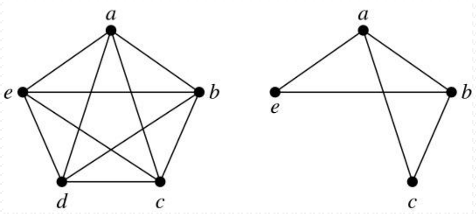
(a) (b)

- *Marriages on an island* - vertices represent the men and the women and edges link a a man and a woman if they are an acceptable spouse. We may wish to find the largest number of possible marriages.

New Graphs from Old

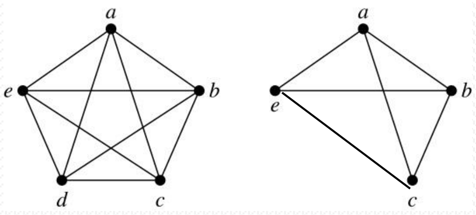
Definition: A *subgraph of a graph* $G = (V,E)$ is a graph (W,F), where $W \subset V$ and $F \subset E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

Example: Here we show K_5 and one of its subgraphs.



Definition: Let $G = (V, E)$ be a simple graph. The *subgraph induced* by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E if and only if both endpoints are in W.

Example: Here we show K_5 and the subgraph induced by $W = \{a,b,c,e\}$.



New Graphs from Old

REMOVING OR ADDING EDGES OF A GRAPH Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a subgraph of G by removing the edge e. The resulting subgraph, denoted by $G - e$, has the same vertex set V as G. Its edge set is $E - e$. Hence,

$$G - e = (V, E - \{e\}).$$

Similarly, if E' is a subset of E, we can produce a subgraph of G by removing the edges in E' from the graph. The resulting subgraph has the same vertex set V as G. Its edge set is $E - E'$.

We can also add an edge e to a graph to produce a new larger graph when this edge connects two vertices already in G. We denote by $G + e$ the new graph produced by adding a new edge e, connecting two previously nonincident vertices, to the graph G Hence,

$$G + e = (V, E \cup \{e\}).$$

The vertex set of $G + e$ is the same as the vertex set of G and the edge set is the union of the edge set of G and the set $\{e\}$.

New Graphs from Old (*continued*)

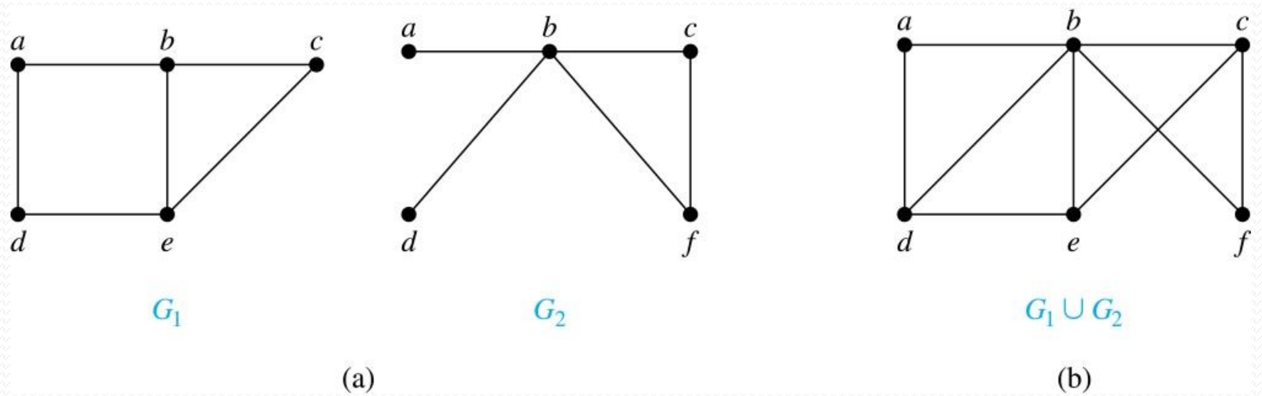
EDGE CONTRACTIONS Sometimes when we remove an edge from a graph, we do not want to retain the endpoints of this edge as separate vertices in the resulting subgraph. In such a case we perform an **edge contraction** which removes an edge e with endpoints u and v and merges u and w into a new single vertex w, and for each edge with u or v as an endpoint replaces the edge with one with w as endpoint in place of u or v and with the same second endpoint. Hence, the contraction of the edge e with endpoints u and v in the graph $G = (V, E)$ produces a new graph $G' = (V', E')$ (which is not a subgraph of G), where $V' = V - \{u, v\} \cup \{w\}$ and E' contains the edges in E which do not have either u or v as endpoints and an edge connecting w to every neighbor of either u or v in V. For example, the contraction of the edge connecting the vertices e and c in the graph G_1 in Figure 16 produces a new graph G_1' with vertices a, b, d, and w. As in G_1, there is an edge in G_1' connecting a and b and an edge connecting a and d. There also is an edge in G_1' that connects b and w that replaces the edges connecting b and c and connecting b and e in G_1 and an edge in G_1' that connects d and w replacing the edge connecting d and e in G_1.

REMOVING VERTICES FROM A GRAPH When we remove a vertex v and all edges incident to it from $G = (V, E)$, we produce a subgraph, denoted by $G - v$. Observe that $G - v = (V - v, E')$, where E' is the set of edges of G not incident to v. Similarly, if V' is a subset of V, then the graph $G - V'$ is the subgraph $(V - V', E')$, where E' is the set of edges of G not incident to a vertex in V'.

New Graphs from Old (*continued*)

Definition: The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example:



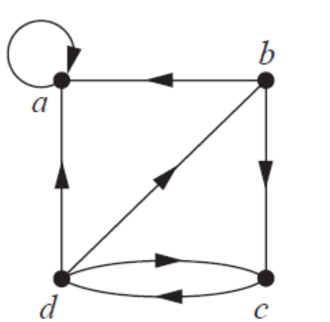
13. What does the degree of a vertex represent in an academic collaboration graph? What does the neighborhood of a vertex represent? What do isolated and pendant vertices represent?

- Solution

Since a person is joined by an edge to each of his or her collaborators, the degree of v is the number of collaborators v has. Similarly, the neighborhood of a vertex is the set of coauthors of the person represented by that vertex. An isolated vertex represents a person who has no coauthors (he or she has published only single-authored papers), and a pendant vertex represents a person who has published with just one other person.

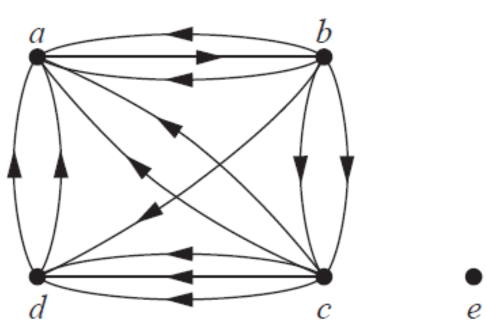
Q. Determine the number of vertices and edges and find the in-degree and out-degree of each vertex for the given directed multigraph.

7.



This directed graph has 4 vertices and 7 edges. The in-degree of vertex *a* is deg- (a) = 3 since there are 3 edges with *a* as their terminal vertex; its out-degree is deg+ (a) = 1 since only the loop has *a* as its initial vertex. Similarly we have deg-(b) = 1, deg+(b) = 2, deg-(c) = 2, deg+(c) = 1, deg-(d) = 1, and deg+ (d) = 3 . As a check we see that the sum of the in-degrees and the sum of the out-degrees are equal (both are equal to 7).

9.



This directed multigraph has 5 vertices and 13 edges. The in-degree of vertex *a* is deg- (a) = 6 since there are 6 edges with *a* as their terminal vertex; its out-degree is deg+(a) = 1. Similarly we have deg-(b) = 1, deg+(b) = 5, deg-(c) = 2, deg+(c) = 5, deg-(d) = 4, deg+(d) = 2, deg-(e) = 0, and deg+(e) = 0 (vertex *e* is isolated). As a check we see that the sum of the in-degrees and the sum of the out-degrees are both equal to the number of edges (13).

Q. How many vertices and how many edges do the graph Q_n have?

Since the vertices of Q_n are the bit strings of length n, there are 2^n vertices. Each vertex has degree n, since there are n strings that differ from any given string in exactly one bit (any one of the n different bits can be changed). Thus the sum of the degrees is $n2^n$. Since this must equal twice the number of edges (by the handshaking theorem), we know that there are $n2^n/2 = n2^{n-1}$ edges.