Optimization (Extreme values)

Before discussing optimization of a function of one variable, we give some basic concepts:

Increasing function.

Decreasing function.

Constant function.

Example:-

Determine whether $y = f(x)$ is increasing or decreasing in the given interval

$$(i) \qquad f(x) = \frac{-x^2}{4} + 4, \quad]-\infty, 0[$$

$$(ii) \qquad f(x) = \frac{1}{2}(x^2 - 4x + 4), \quad]2, \infty[$$

Optimization (Extreme values)

Stationary and critical points.

Maxima and minima of a function of one variable.

First derivative test.

Special cases:

Local verses absolute extrema.

Second derivative test.

Optimization (Extreme values)

Exercise

1. Determine the values of x for which each of the following function has

i) relative maxima ii) relative minima.

Then find the relative extreme values of the given functions:

$$(i) \qquad f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$$

$$(ii) \qquad f(x) = 12x^5 - 45x^4 + 40x^3 + 6$$

$$(iii) \qquad f(x) = (x-1)(x-2)(x-3)$$

$$(iv) \qquad f(x) = \cos(3x) - 3\cos(x), \qquad [-\pi, \pi]$$

Optimization (Extreme values)

$$(v) \qquad f(x) = \frac{\ln x}{x}, \qquad]0, \infty[$$

$$(vi) \qquad f(x) = x^x, \qquad x \neq 0$$

$$(vii) \qquad f(x) = \left(\frac{1}{x}\right)^x, \qquad x \neq 0$$

2. The height above ground of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

where s in feet and t in seconds. Find
a. the object's velocity when $t = 0$,
b. its maximum height and when it occurs,
c. its velocity when $s = 0$.

Optimization (Extreme values)

3. An open rectangular box is to be made from a sheet of cardboard 8*dm* and 5*dm*, by cutting equal square from each corner and turning up the sides. Find the edge of the square which makes the volume maximum.

4. Find the dimensions of a rectangle of maximum area that can be inscribed in a circle of radius r.

5. Find the dimensions of a rectangle of maximum area that can be inscribed in a semi-circle of radius r.

6. Find two numbers whose sum is 8 and the product of one number with cube of the other shall be maximum possible.

7. Find the number which when added to its reciprocal gives the least possible sum.

8. A rectangle has its base on the x-axis and its upper two vertices on the curve $y = 12 - x^2$. What is the largest area the rectangle can have?

Optimization (Extreme values)

9. A window is in the form of a rectangle surmounted by a semi-circle. The rectangle is of clear glass, while the semi-circle is of tinted glass that transmits only half as much light per unit area as clear glass does. Total perimeter of the window is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.

10. A rectangular plot of farmland is bounded on one side by a river and on the other three sides by a single-stand electric fence with 800m of wire at your disposal. What is the largest area you can enclose?

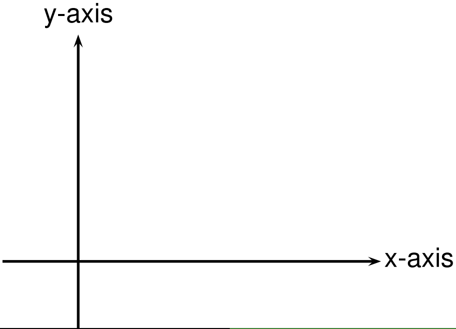
11. One side of an open rectangular field is bounded by a straight river. How would you put a fence around the outer three sides of a rectangular plot in order to enclose greatest area as possible with a given length of fence.

12. Determine the constants a and b, so that the function $f(x) = x^3 + ax^2 + bx + c$ may have

i) relative maxima at $x = -1$ ii) relative minima at $x = 3$.

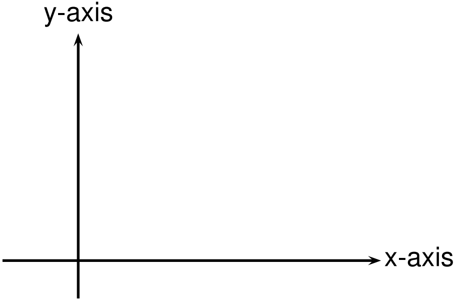
Concavity

Let y be a function of x defined by the equation $y = f(x)$. The graph of the function $f(x)$ is said to be concave-up (faces upward) if it lies above the tangents between the points $(a, f(a))$ and $(b, f(b))$. If we travel from left to right alongside the curve, the tangents rotate anti-clockwise. Since angle increases in the anti-clockwise direction, so that their slopes increase in the anti-clockwise direction, see Figure (a).



Concavity

The graph of the function $f(x)$ is said to be concave-down (faces downward) if it lies below the tangents between the points $(a, f(a))$ and $(b, f(b))$. If we travel from left to right alongside the curve, the tangents rotate clockwise direction. Since angle decreases in the clockwise direction, so that their slopes decrease in the clockwise direction, see Figure (b).



Since $f'(x)$ is the slope of the tangent line, so we are led to the following definition.

Concavity

Definition:- Let $f(x)$ be a differentiable function. Then

(i) The function $f(x)$ is said to be concave-up on an interval I, if $f'(x)$ is increasing on that interval.

(ii) The function $f(x)$ is said to be concave-down on an interval I, if $f'(x)$ is decreasing on that interval.

Since $f''(x)$ is derivative to $f'(x)$, so $f'(x)$ is increasing on an open interval (a, b) if $f''(x) > 0$ for all x in (a, b), and $f'(x)$ is decreasing on an open interval (a, b) if $f''(x) < 0$ for all x in (a, b). Thus we have the following result.

Theorem:- Let y be a function of x, which is twice differentiable on an open interval (a, b). Then

(i) $f(x)$ is said to be concave-up on an open interval (a, b) if $f''(x) > 0 \ \forall \ x \in (a, b)$.
(ii) $f(x)$ is said to be concave-down on an open interval (a, b) if $f''(x) < 0 \ \forall \ x \in (a, b)$.

Concavity

Definition:- A point where the graph of a function has a tangent line and where the concavity changes is called as **point of inflection**.

Working rules:

For point of inflection, put $\frac{d^2y}{dx^2} = 0$, we get

$x = a, b \quad$ (say)

Take $x = a$ and put in $\frac{d^3y}{dx^3}$.

if $\frac{d^3y}{dx^3} \neq 0$, then point of inflection exists and $x = a$ is called x-coordinate of the point of inflection.

Examples:-

$y = x^3$.

$y = x^2$.

$y = 2 + \cos x$.

Concavity

Exercise

1. Find the intervals in which the following curves:

$$(i) \qquad y = 3x^5 - 40x^3 + 3x - 20$$

$$(ii) \qquad y = e^{-x}(x^2 + 4x + 5)$$

(a) Faces upward, (b) faces downward.

2. Find the point(s) of inflection of each of the following curves:

$$(i) \qquad y = \frac{2}{x} - \frac{4}{x^2}$$

$$(ii) \qquad y = \frac{x^3 - x}{3x^2 + 1}$$

Concavity

3. Find the largest intervals on which f is

(a) Increasing, (b) decreasing.

Find the largest open intervals on which f is

(c) concave up, (d) concave down, and

(e) find the x-coordinates of all inflection points for each of the following cases:

$$(i) \qquad f(x) = x^2 - 5x + 6.$$

Answer: $(a)\ [\frac{5}{2}, \infty), (b)\ (-\infty, \frac{5}{2}],\ (c)\ (-\infty, \infty),$
(d) None, (e) None.

$$\begin{aligned} (ii) \qquad f(x) &= 4 - 3x - x^2 \\ (iii) \qquad f(x) &= (x+2)^3 \end{aligned}$$

Concavity

$$(iv) \quad f(x) = 5 + 12x - x^3$$

$$(v) \quad f(x) = 3x^3 - 4x + 3$$

$$(vi) \quad f(x) = x^4 - 8x^2 + 16$$

$$(vii) \quad f(x) = 3x^4 - 4x^3 \; ?$$

$$(viii) \quad f(x) = \frac{x}{x^2 + 2}$$

$$(ix) \quad f(x) = x^{\frac{2}{3}}$$

Concavity

$$(x) \qquad f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$$

$$(xi) \qquad f(x) = x^{\frac{1}{3}}(x+4)$$

$$(xii) \qquad f(x) = \cos x, \quad 0 < x < 2\pi$$

Answer: $(a)\ [\pi, 2\pi), (b)\ (0, \pi],\ (c)\ (\frac{\pi}{2}, \frac{3\pi}{2}),$
$(d)\ (0, \frac{\pi}{2}), (\frac{3\pi}{2}, 2\pi),\ (e)\ \frac{\pi}{2}, \frac{3\pi}{2}.$

$$(xiii) \qquad f(x) = \sin^2 2x, \quad 0 < x < \pi$$

$$(xiv) \qquad f(x) = \tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$$