All Pairs Shortest Paths

Problem Definition

Input: Directed graph $G = (V, E)$ with edge costs c_e for each edge $e \in E$, [No distinguished source vertex.]

Goal: Either

(A) Compute the length of a shortest $u \to v$ path for all pairs of vertices $u, v \in V$

OR

(B) Correctly report that G contains a negative cycle.

Shortest Path with Negative Cycle

Quiz

Question: How many invocations of a single-source shortest-path subroutine are needed to solve the all-pairs shortest path problem? [n = # of vertices]

a) 1
b) n - 1
c) n
d) n^2

Quiz

Question: How many invocations of a single-source shortest-path subroutine are needed to solve the all-pairs shortest path problem? [n = # of vertices]

a) 1

b) n - 1

c) n

d) n^2

Running time (nonnegative edge costs):

$$n \cdot \text{Dijkstra} = O(nm \log n) = \begin{array}{l} O(n^2 \log n) \text{ if } m = \Theta(n) \\ O(n^3 \log n) \text{ if } m = \Theta(n^2) \end{array}$$

Running time (general edge costs):

$$n \cdot \text{Bellman-Ford} = O(n^2 m) = \begin{array}{l} O(n^3) \text{ if } m = \Theta(n) \\ O(n^4) \text{ if } m = \Theta(n^2) \end{array}$$

Motivation

Floyd-Warshall algorithm: $O(n^3)$ algorithm for APSP.

- Works even with graphs with negative edge lengths.

Thus: (1) At least as good as n Bellman-Fords, better in dense graphs.

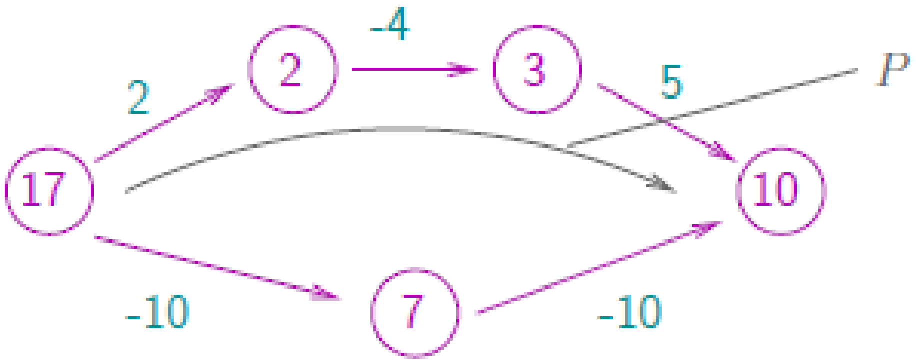
(2) In graphs with nonnegative edge costs, competitive with n Dijkstra's in dense graphs.

Optimal Substructure

Key idea: Order the vertices $V = \{1, 2, \ldots, n\}$ arbitrarily. Let $V^{(k)} = \{1, 2, \ldots, k\}$.

Lemma: Suppose G has no negative cycle. Fix source $i \in V$, destination $j \in V$, and $k \in \{1, 2, \ldots, n\}$. Let P = shortest (cycle-free) i-j path with all internal nodes in $V^{(k)}$.

Example: $[i = 17, j = 10, k = 5]$



Optimal Substructure (con'd)

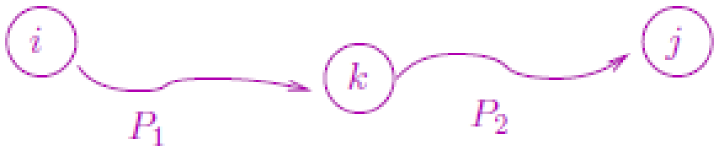
Optimal substructure lemma: Suppose G has no negative cost cycle. Let P be a shortest (cycle-free) i-j path with all internal nodes in $V^{(k)}$. Then:

Case 1: If k not internal to P, then P is a shortest (cycle-free) i-j path with all internal vertices in $V^{(k-1)}$.

Case 2: If k is internal to P, then:

P_1 = shortest (cycle-free) i-k path with all internal nodes in $V^{(k-1)}$ and

P_2 = shortest (cycle-free) k-j path with all internal nodes in $V^{(k-1)}$



Quiz

Setup: Let A = 3-D array (indexed by i , j , k).

Intent: A[i , j , k] = length of a shortest i -j path with all internal nodes in {1, 2, … k} (or +∞ if no such paths)

Question: What is A[i , j , 0]

if (1) $i = j$ (2) $(i , j) \in E$ (3) $i \neq j$ and $(i , j) \notin E$

a) 0, 0, and $+\infty$
b) 0, c_{ij} , and c_{ij}
c) 0, c_{ij} , and $+\infty$
d) $+\infty$, c_{ij} , and $+\infty$

Quiz

Setup: Let A = 3-D array (indexed by i , j , k).

Intent: $A[i , j , k]$ = length of a shortest i -j path with all internal nodes in $\{1, 2, \ldots k\}$ (or $+\infty$ if no such paths)

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a) 0, 0, and $+\infty$

b) 0, c_{ij} , and c_{ij}

c) 0, c_{ij} , and $+\infty$

d) $+\infty$, c_{ij} , and $+\infty$

The Floyd-Warshall Algorithm

Let A = 3-D array (indexed by i, j, k)

Base cases: For all $i, j \in V$:

$$A[i,j,0] = \left\{ \begin{array}{l} 0 \text{ if } i = j \\ c_{ij} \text{ if } (i,j) \in E \\ +\infty \text{ if } i \neq j \text{ and } (i,j) \notin E \end{array} \right\}$$

For $k = 1$ to n

For $i = 1$ to n

For $j = 1$ to n

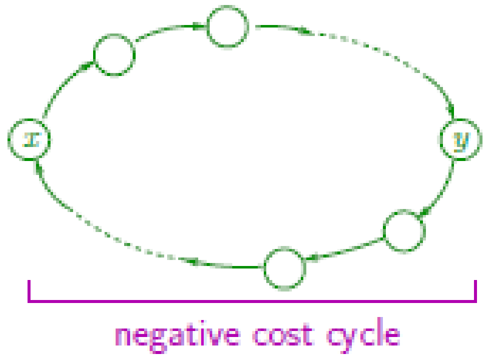
$$A[i,j,k] = \min \left\{ \begin{array}{ll} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{array} \right\}$$

Correctness: From optimal substructure + induction, as usual.

Running time: $O(1)$ per subproblem, $O(n^3)$ overall.

Odds and Ends

Question #1: What if input graph G has a negative cycle?



Answer: Will have $A[i, i, n] < 0$ for at least one $i \in V$ at end of algorithm.

Odds and Ends

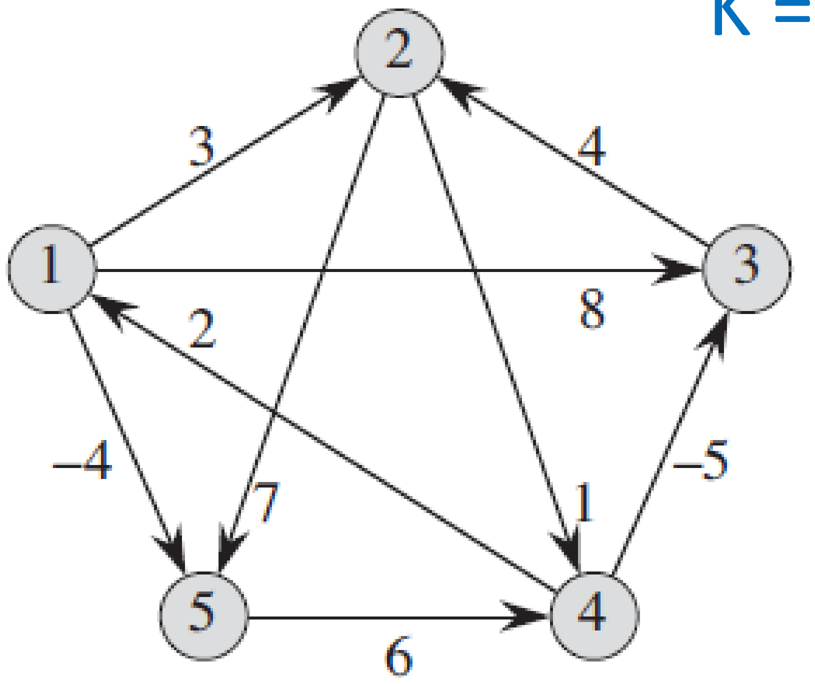
Question #2: How to reconstruct a shortest i-j path?

Answer: In addition to A, have Floyd-Warshall compute $B[i,j] =$ max label of an internal node on a shortest i-j path for all $i,j \in V$.

[Reset $B[i,j] = k$ if 2nd case of recurrence used to compute $A[i,j,k]$]

$\Rightarrow$ Can use the $B[i,j]$'s to recursively reconstruct shortest paths!

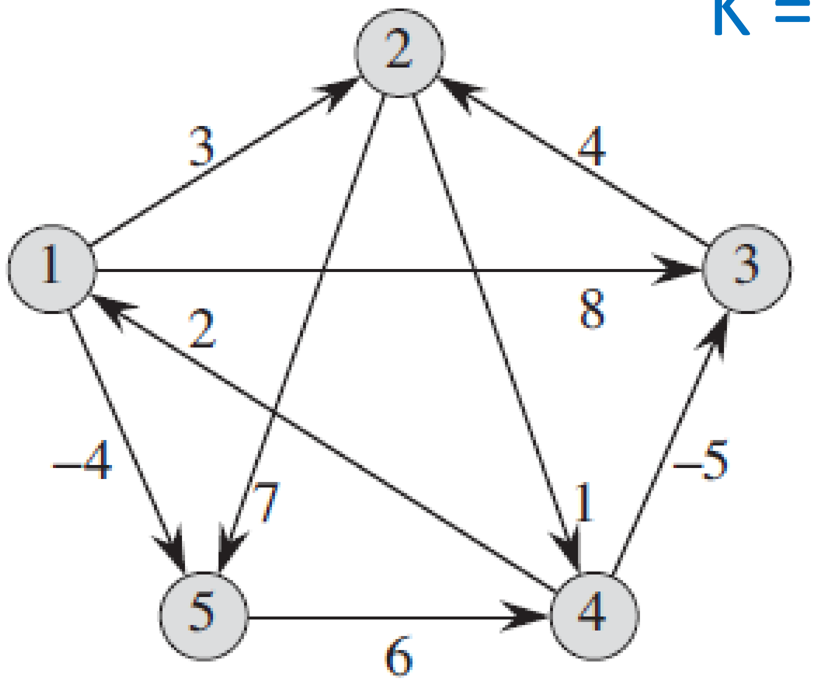
K = 0



$$A^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A[i,j,k] = \min \left\{ \begin{array}{ll} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{array} \right\}$$

K = 1

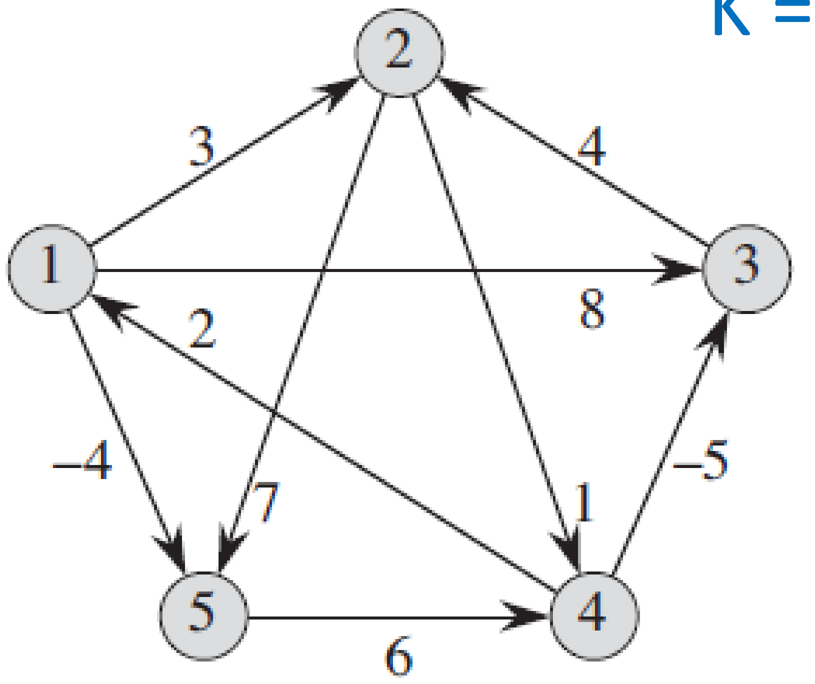


$$A^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A[i,j,k] = \min \begin{cases} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{cases}$$

K = 1



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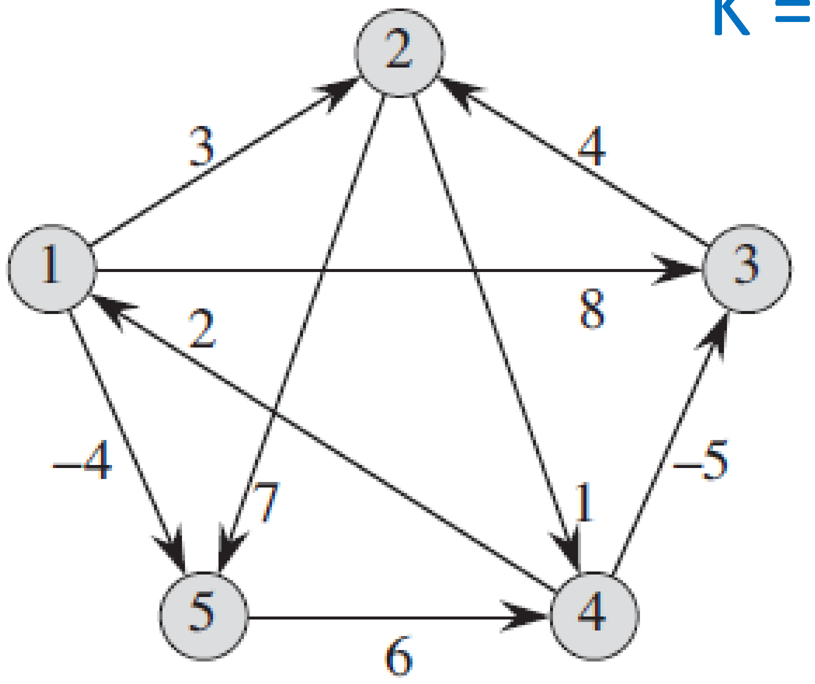


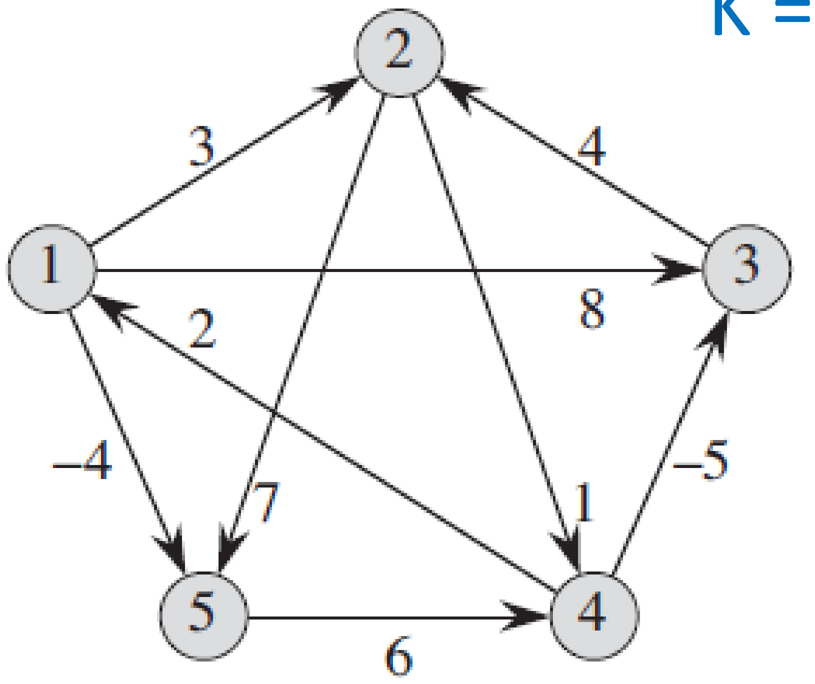
K = 2

$$A^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A[i,j,k] = \min \begin{Bmatrix} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{Bmatrix}$$

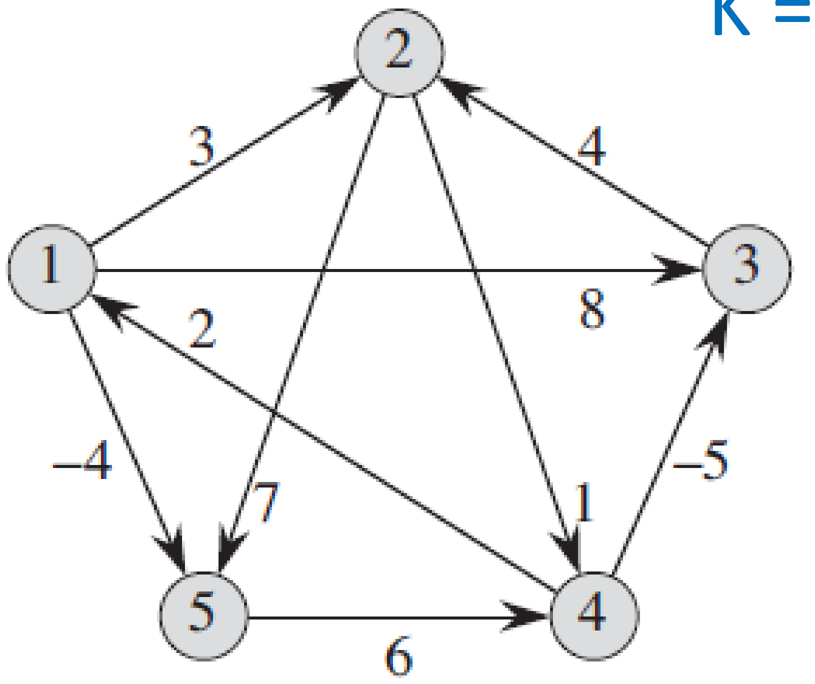
K = 3



$$A^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A[i,j,k] = \min \left\{ \begin{array}{ll} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{array} \right\}$$

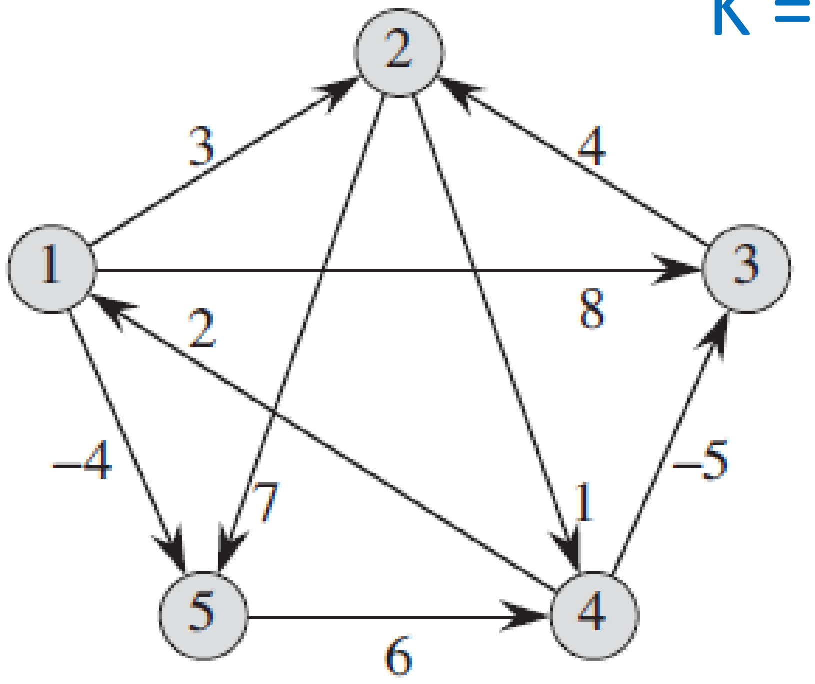


K = 4

$$A^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$A^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$A[i,j,k] = \min \left\{ \begin{array}{ll} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{array} \right\}$$

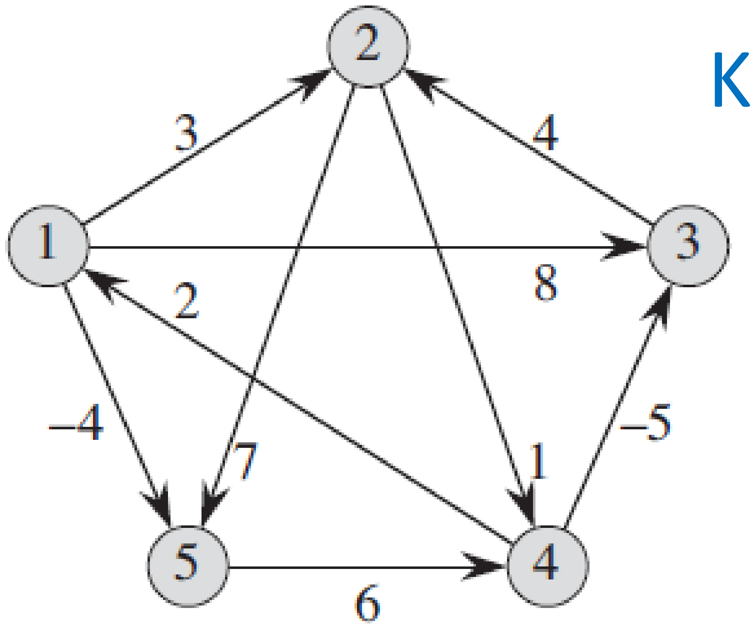


K = 5

$$A^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$A^{(5)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$A[i,j,k] = \min \left\{ \begin{array}{ll} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{array} \right\}$$

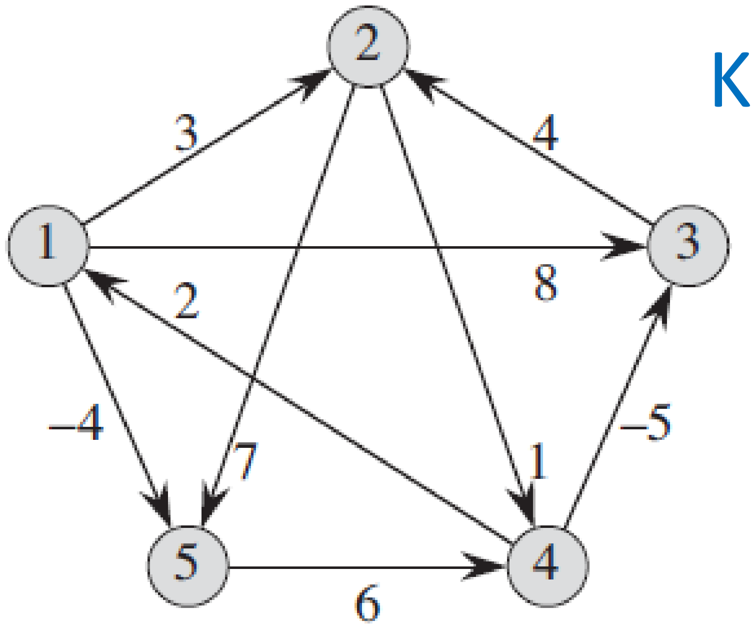


K = 0

$$A^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} \mathrm{N}il & 1 & 1 & \mathrm{N}il & 1 \\ \mathrm{N}il & \mathrm{N}il & \mathrm{N}il & 2 & 2 \\ \mathrm{N}il & 3 & \mathrm{N}il & \mathrm{N}il & \mathrm{N}il \\ 4 & \mathrm{N}il & 4 & \mathrm{N}il & \mathrm{N}il \\ \mathrm{N}il & \mathrm{N}il & \mathrm{N}il & 5 & \mathrm{N}il \end{pmatrix}$$

$$A[i,j,k] = \min \left\{ \begin{array}{ll} A[i,j,k-1] & \text{Case 1} \\ A[i,k,k-1] + A[k,j,k-1] & \text{Case 2} \end{array} \right\}$$



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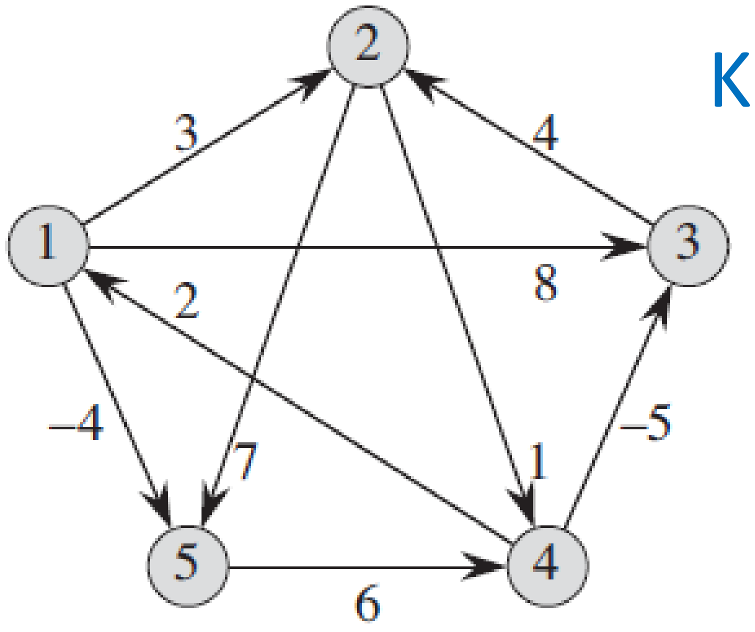
$$A^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} \mathrm{N}il & 1 & 1 & \mathrm{N}il & 1 \\ \mathrm{N}il & \mathrm{N}il & \mathrm{N}il & 2 & 2 \\ \mathrm{N}il & 3 & \mathrm{N}il & \mathrm{N}il & \mathrm{N}il \\ 4 & 1 & 4 & \mathrm{N}il & 1 \\ \mathrm{N}il & \mathrm{N}il & \mathrm{N}il & 5 & \mathrm{N}il \end{pmatrix}$$

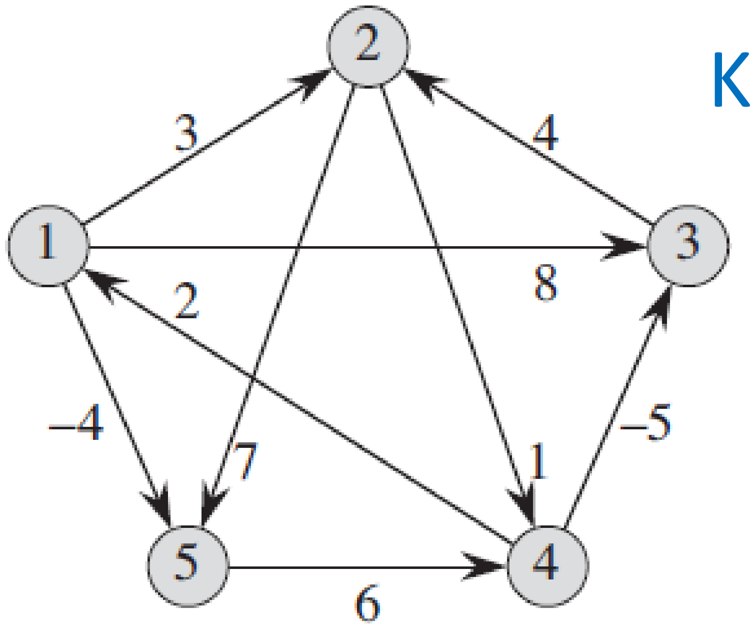
$$A^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$



K = 2

$$A^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$B^{(2)} = \begin{pmatrix} Nil & 1 & 1 & 2 & 1 \\ Nil & Nil & Nil & 2 & 2 \\ Nil & 3 & Nil & 2 & 2 \\ 4 & 1 & 4 & Nil & 1 \\ Nil & Nil & Nil & 5 & Nil \end{pmatrix} \quad A^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

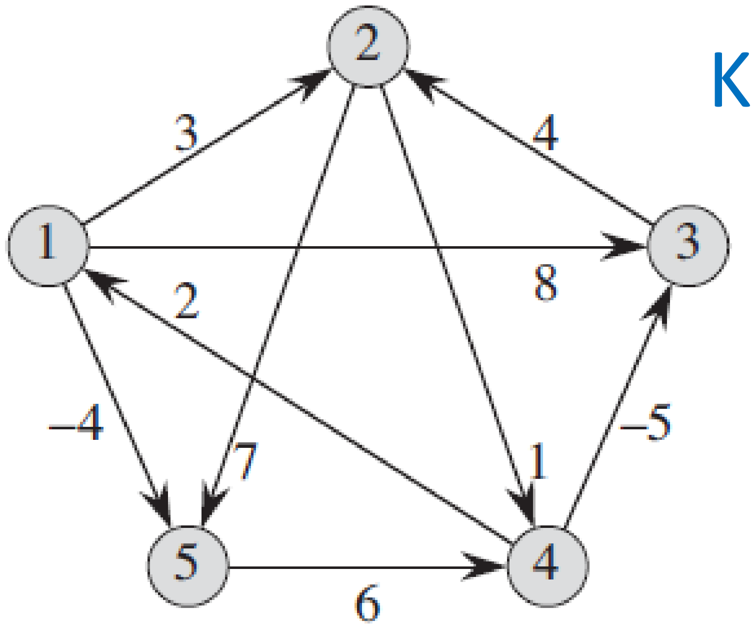


K = 3

$$A^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$B^{(3)} = \begin{pmatrix} \text{Nil} & 1 & 1 & 2 & 1 \\ \text{Nil} & \text{Nil} & \text{Nil} & 2 & 2 \\ \text{Nil} & 3 & \text{Nil} & 2 & 2 \\ 4 & 3 & 4 & \text{Nil} & 1 \\ \text{Nil} & \text{Nil} & \text{Nil} & 5 & \text{Nil} \end{pmatrix}$$

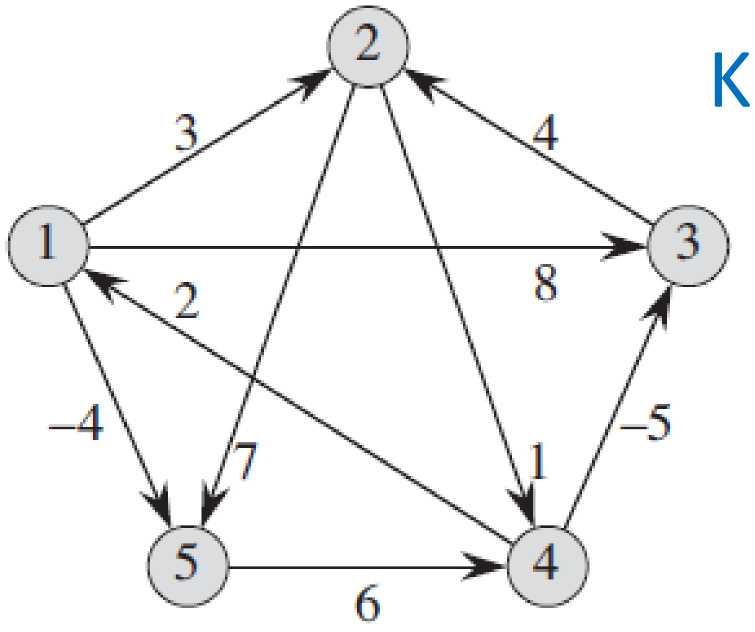
$$A^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$



K = 4

$$A^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$B^{(4)} = \begin{pmatrix} Nil & 1 & 4 & 2 & 1 \\ 4 & Nil & 4 & 2 & 4 \\ 4 & 3 & Nil & 2 & 4 \\ 4 & 3 & 4 & Nil & 1 \\ 4 & 4 & 4 & 5 & Nil \end{pmatrix} \qquad A^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$



K = 5

$$A^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$B^{(4)} = \begin{pmatrix} \mathrm{N}il & 1 & 5 & 5 & 1 \\ 4 & \mathrm{N}il & 4 & 2 & 4 \\ 4 & 3 & \mathrm{N}il & 2 & 4 \\ 4 & 3 & 4 & \mathrm{N}il & 1 \\ 4 & 4 & 4 & 5 & \mathrm{N}il \end{pmatrix}$$

$$A^{(5)} = \begin{pmatrix} 0 & 3 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$