Covariance of two random variables:-

The covariance of two r.v's x and y is a numerical measure of the extent to which their values tend to increase or decrease together. It is denoted by σ_{xy} or $cov(x,y)$ and is defined as the expected value of the product $[X - E(x)][Y - E(Y)]$.

i.e

$$\begin{aligned} cov(x,y) &= E[\{X - E(x)\}\{Y - E(Y)\}] \\ &= E[XY - XE(X) - YE(Y) + E(X)E(Y)] \\ &= E(XY) - E(X)\cancel{E(Y)} - E(Y)E(X) + E(X)\cancel{E(Y)} \end{aligned}$$

$$Cov(x,y) = E(XY) - E(X)E(Y)$$

$\Rightarrow$ When x and y are independent then $cov(x,y) = 0$ but its converse is not generally true.

Correlation co-efficient of random variables:-

Let x and y be two r.v's with non-zero variance σ_x^2 and σ_y^2. Then the correlation co-efficient which is a measure of linear relationship between x and is denoted ρ_{xy} or $corr(x,y)$, is defined as

$$\rho_{xy} = \frac{E[X - E(X)][Y - E(Y)]}{\sigma_x \sigma_y} = \frac{cov(X,Y)}{\sqrt{var(X)\,var(Y)}}$$

$\Rightarrow$ If x and y are independent r.v's then ρ_{xy} will be zero but zero correlation does not necessarily imply independence.

⇒ correlation co-efficient has the following properties.

1. correlation co-efficient is unit less and symmetric in x and y. i.e $\rho_{xy} = \rho_{yx}$.
2. correlation co-efficient remains unchanged if constants are added to the r.v's or if the r.v's are multiplied by constants having same sign.
3. correlation co-efficient lies between -1 and $+1$ inclusive.

$$-1 \leq \rho \leq 1.$$

Example 7.22 (pg 264).

From the joint p.d of x and y find $var(x)$, $var(y)$, $cov(x,y)$ and ρ.

x \ y	0	1	2	3	$g(x)$
0	0.05	0.05	0.10	0	0.20
1	0.05	0.10	0.25	0.10	0.50
2	0	0.15	0.10	0.05	0.30
$h(y)$	0.10	0.30	0.45	0.15	1.00

$E(x) = \sum x_i g(x_i) = 0 \times 0.20 + 1 \times 0.50 + 2 \times 0.30$

$E(x) = 1.10$

$E(y) = \sum y_j h(y_j) = 0 \times 0.10 + 1 \times 0.30 + 2 \times 0.45 + 3 \times 0.15$

$= 0 + 0.30 + 0.90 + 0.45$

$E(y) = 1.65$

$E(x^2) = \sum x_i^2 g(x_i) = 0 \times 0.20 + 1 \times 0.50 + 4 \times 0.30$

$E(x^2) = 1.70$

$E(Y^2) = \sum y_j^2 h(y_j) = 0\times 0.10 + 1\times 0.30 + 4\times 0.45 + 9\times 0.15$

$$\boxed{E(Y^2) = 3.45}$$

$Var(X) = E(X^2) - [E(X)]^2 = 1.70 - (1.10)^2$

$$\boxed{Var(X) = 0.49}$$

$Var(Y) = E(Y^2) - [E(Y)]^2 = 3.45 - (1.65)^2$

$$\boxed{Var(Y) = 0.7275}$$

$E(XY) = \sum_i \sum_j (x_i, y_j) f(x_i, y_j)$

$= 1\times 0.10 + 2\times 0.15 + 2\times 0.25 + 4\times 0.10 + 3\times 0.10 + 6\times 0.05$

$= 0.10 + 0.30 + 0.50 + 0.40 + 0.30 + 0.30$

$$\boxed{E(XY) = 1.90}$$

As $Cov(X, Y) = E(XY) - E(X)E(Y)$

$= 1.90 - [1.10\times 1.65]$

$$\boxed{Cov(X, Y) = 0.085}$$

$$\rho = \frac{cov(X, Y)}{\sqrt{var(X)\,var(Y)}} = \frac{0.085}{\sqrt{(0.49)(0.7275)}}$$

$$= \frac{0.085}{0.595}$$

$$\boxed{\rho = 0.14}$$

Example 7.23 (Pg 265)

If $f(x,y) = x^2 + \frac{xy}{3} \quad 0 \le x \le 1, \; 0 \le y \le 2$

$= 0$ else where.

find var(X), var(Y), corr(X, Y).

Solution:—

$$g(x) = \int_0^2 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= 2x^2 + \frac{2}{3}x$$

$$h(y) = \int_0^1 \left(x^2 + \frac{xy}{3}\right) dy$$

$$= \frac{1}{3} + \frac{y}{6}$$

$$E(X) = \int_{-\infty}^{\infty} x\, g(x)\, dx = \int_0^1 x\left(2x^2 + \frac{2x}{3}\right) dx = \frac{13}{18}$$

$$E(Y) = \int_{-\infty}^{\infty} y\, h(y)\, dy = \int_0^2 y\left(\frac{1}{3} + \frac{y}{6}\right) dy = \frac{10}{9}$$

$$var(X) = E(X^2) - [E(X)]^2$$

OR

$$= E[X - E(X)]^2 =$$

$$var(X) = \int_{-\infty}^{\infty} (x - \mu_x)^2 g(x)\, dx$$

$$= \int_0^1 \left(x - \frac{13}{18}\right)^2 \left(2x^2 + \frac{2x}{3}\right) dx$$

$$var(X) = \frac{73}{1620}$$

$$Var(Y) = E[Y - E(Y)]^2 = \int_{-\infty}^{\infty} (Y - \mu_y)^2 h(y)\, dy$$

$$= \int_0^2 \left(y - \frac{10}{9}\right)^2 \left(\frac{1}{3} + \frac{y}{6}\right) dy$$

$$= \frac{26}{81}$$

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

or

$$= E[\{X - E(X)\}\{Y - E(Y)\}]$$

$$= \int_0^1 \int_0^2 \left(x - \frac{13}{18}\right)\left(Y - \frac{10}{9}\right)\left(x^2 + \frac{xy}{3}\right) dy\, dx$$

$$= \int_0^1 \left(-\frac{2}{9}x^3 + \frac{25}{81}x^2 - \frac{26}{243}x\right) dx$$

$$Cov(X, Y) = -\frac{1}{162}$$

Hence

$$Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) Var(Y)}}$$

$$= \frac{-1/162}{\sqrt{(73/1620)(26/81)}}$$

$$\boxed{Corr(X, Y) = -0.05}$$