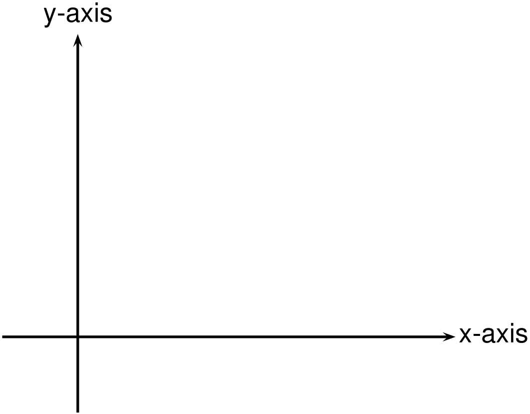
Curvature

Before defining curvature, we first give some basic concepts:



Curvature

1. From the figure, we have

$$|AC| < \widehat{AC} < |AB| + |BC|$$

Dividing by $|AC|$ on both sides, we get

$$\frac{|AC|}{|AC|} < \frac{\widehat{AC}}{|AC|} < \frac{|AB|}{|AC|} + \frac{|BC|}{|AC|}$$

$$1 < \frac{\widehat{AC}}{|AC|} < \frac{|AB|}{|AC|} + \frac{|BC|}{|AC|}$$

$$1 < \frac{\widehat{AC}}{|AC|} < \cos(\alpha) + \sin(\alpha)$$

Curvature

Taking limit, when $C \longrightarrow A$, then $\alpha \longrightarrow 0$. So, we have

$$1 < \frac{\widehat{AC}}{|AC|} < 1 + 0$$

$$\implies \quad 1 < \frac{\widehat{AC}}{|AC|} < 1$$

which is only possible if

$$\frac{\widehat{AC}}{|AC|} = 1.$$

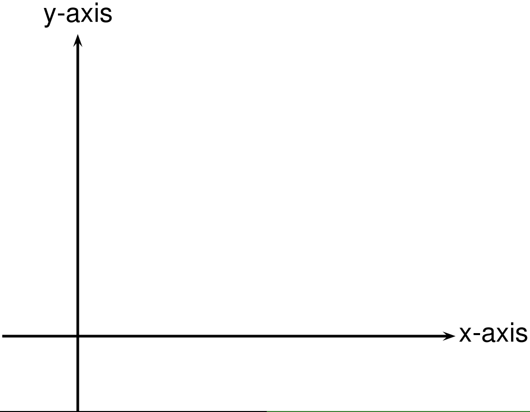
Which shows that the ratio between the arc length and the chord length is unity.

Curvature

2. Prove that

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Proof:



Curvature

From the figure, we have

$$|PQ|^2 = |PK|^2 + |KQ|^2,$$
$$|PQ|^2 = (\delta x)^2 + (\delta y)^2,$$
$$|PQ| = \sqrt{(\delta x)^2 + (\delta y)^2}.$$

Dividing by δx on both sides, we get

$$\frac{|PQ|}{\delta x} = \frac{\sqrt{(\delta x)^2 + (\delta y)^2}}{\delta x},$$
$$\frac{|PQ|}{\delta x} = \frac{\sqrt{(\delta x)^2 + (\delta y)^2}}{\sqrt{(\delta x)^2}},$$
$$\frac{|PQ|}{\delta x} = \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2},$$

Curvature

$$\frac{|PQ|}{\delta s}\frac{\delta s}{\delta x} = \sqrt{1+\left(\frac{\delta y}{\delta x}\right)}^2.$$

Taking limit, when $Q \longrightarrow P$, then $\delta x \longrightarrow 0$. So, we have

$$1.\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)}^2 \qquad \because \frac{|PQ|}{\delta s} = 1, \ \lim_{\delta x \longrightarrow 0}\frac{\delta s}{\delta x} = \frac{ds}{dx}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)}^2$$

Hence proved.

Curvature

3. Prove that

$$\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}.$$

4. Prove that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

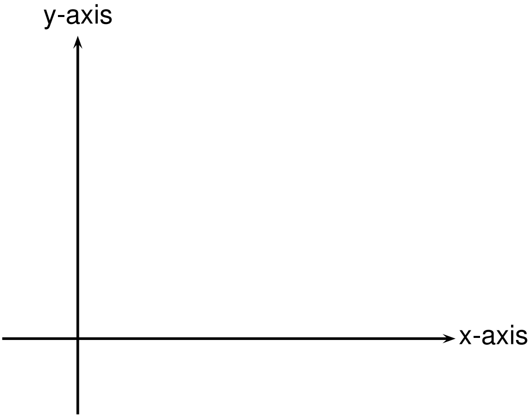
Definition:-

Let y be a function of x defined by the equation $y = f(x)$. Let P be any point on the curve. Draw a tangent to the curve at the point P. Now take another point Q in the neighborhood of the point P (as shown in the figure) such that

$$\widehat{PQ} = \delta s.$$

Let $\delta\psi$ be the angle turned through by the tangent from P to Q, then the ratio $\frac{\delta\psi}{\delta s}$ is called average curvature, i.e.,

Curvature



Curvature

$$\text{Average curvature} = \frac{\delta\psi}{\delta s}.$$

Taking limit, when $Q \longrightarrow P$, then $\delta s \longrightarrow 0$. So, we get

$$\lim_{\delta s \longrightarrow 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds},$$

which is called as curvature and is denoted by K, i.e.,

$$K = \frac{d\psi}{ds}.$$

Note. Basically, curvature shows the sharpness of the bending of the curve and its reciprocal is called as radius of curvature denoted by ρ, i.e.,

$$\rho = \frac{1}{K} = \frac{ds}{d\psi}.$$

Curvature

Formula 1:-

When the equation of the curve is given in rectangular coordinate system, i.e., $y = f(x)$. Then, we have

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}$$

Proof:- Since, we know that

$$\tan(\psi) = \frac{dy}{dx}.$$

Differentiating both sides with respect to s, we get

Curvature

$$\sec^2(\psi)\frac{d\psi}{ds} = \frac{d^2y}{dx^2}\frac{dx}{ds}$$

$$\frac{d\psi}{ds} = \frac{1}{\sec^2(\psi)}\frac{d^2y}{dx^2}\frac{dx}{ds}$$

$$\frac{d\psi}{ds} = \frac{1}{1+\tan^2(\psi)}\frac{d^2y}{dx^2}\frac{dx}{ds}$$

$$\frac{d\psi}{ds} = \frac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]}\frac{1}{\frac{ds}{dx}}, \qquad \because \tan(\psi) = \frac{dy}{dx}$$

$$\frac{d\psi}{ds} = \frac{\frac{d^2y}{dx^2}}{[1+(\frac{dy}{dx})^2]}\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

Curvature

$$\frac{d\psi}{ds} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}, \qquad \because \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$K = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}, \qquad \because K = \frac{d\psi}{ds}$$

and hence

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|}.$$

Example. Show that the radius of curvature of $y = \sqrt{r^2 - x^2}$ is r.

Curvature

Formula 2:-

When the equation of the curve is given in parametric form, i.e., $x = f(t)$, and $y = g(t)$. Then, we have

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{\frac{3}{2}}}{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}} \quad \text{or} \quad \rho = \frac{\left[f'^2 + g'^2\right]^{\frac{3}{2}}}{f'g'' - g'f''}.$$

Formula 3:-

When the equation of the curve is given in the polar form, i.e., $r = f(\theta)$. Then, we have

$$\rho = \frac{\left(r^2 + r'^2\right)^{\frac{3}{2}}}{r^2 - rr'' + 2r'^2}.$$

Curvature

Exercise

1. Show that the radius of curvature of the curve $y = \sqrt{r^2 - x^2}$ is r.

2. Find the radius of curvature for any value of t of each of the following curves:

$$(i) \qquad x = a(t + \sin t), \quad y = a(1 - \cos t)$$

$$(ii) \qquad x = a\cos t, \quad y = b\sin t$$

3. Find the radius of curvature of the following curves at the indicated points:

$$(i) \qquad r = \frac{a}{1+\cos(\theta)}, \quad \text{at } \theta = \frac{\pi}{2}$$

$$(ii) \qquad r = 2\cos(2\theta), \quad \text{at } \theta = \frac{\pi}{2}$$

Curvature

4. Find the radius of curvature to the curve $r = a(1 + \cos(\theta))$ at the point where tangent is parallel to the initial line.

5. Find the radius of curvature of each of the following curves:

$$(i) \qquad r(t) = t\hat{i} + ln(\cos t)\hat{j}, \quad \frac{-\pi}{2} < t < \frac{\pi}{2}$$

$$(ii) \qquad r(t) = (\cos t + t\sin t)\hat{i} + (\sin t - t\cos t)\hat{j}$$

$$(iii) \qquad r(t) = (e^t \cos t)\hat{i} + (e^t \sin t)\hat{j}$$

$$(iv) \qquad r(t) = (\cos^3 t)\hat{i} + (\sin^3 t)\hat{j}, \quad 0 < t < \frac{\pi}{2}$$

6. Show that the parabola $y = ax^2$, $a \neq 0$ has its largest curvature at its vertex and has no minimum curvature. **Note.** Since the curvature of a curve remains the same if the curve is rotated or translated. This result is true for any parabola.

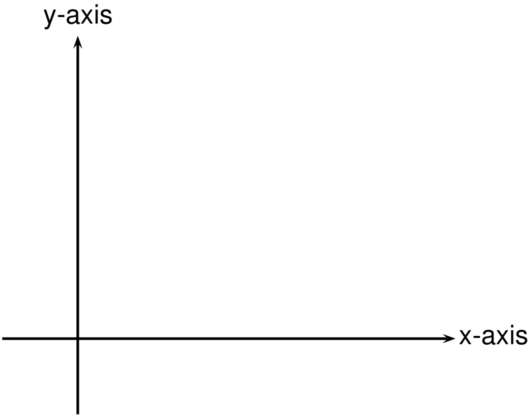
Curvature

7. Show that the ellipse $x = a\cos t$, $y = b\sin t$ for $a > b > 0$, has its largest curvature on its major axis and smallest curvature on its minor axis.

8. Find the point(s) on the curve $y = ln(x)$ where K(curvature) is maximum.

9. At what point(s) does $4x^2 + 9y^2 = 36$ have minimum radius of curvature?

Osculating circle (Circle of curvature)



Osculating circle (Circle of curvature)

Let y be a function of x, defined by the equation $y = f(x)$. Take a point $P(x_1, y_1)$ and draw a tangent to the curve at the point P which makes an angle ψ with $x - axis$.

Now cut the normal line equal to ρ and draw a circle having radius ρ, called as osculating circle (circle of curvature). Let $C(\alpha, \beta)$ be the co-ordinates of the centre of circle then from the figure, we have

$$\begin{aligned} \alpha &= |OM| - |NM| \\ &= x_1 - |KP| \\ \therefore \sin(\psi) &= \frac{|KP|}{\rho} \implies |KP| = \rho \sin(\psi) \\ \implies \alpha &= x_1 - \rho \sin(\psi) \end{aligned}$$

Osculating circle (Circle of curvature)

Now

$$\begin{aligned}\beta &= |NK| + |KC| \\ \beta &= y_1 + |KC|\end{aligned}$$

$$\therefore \cos(\psi) = \frac{|KC|}{\rho} \implies |KC| = \rho\cos(\psi)$$

$$\implies \beta = y_1 + \rho\cos(\psi)$$

Here we know that

$$\tan(\psi) = \frac{dy}{dx}, \quad \sin(\psi) = \frac{\frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}},$$

$$\cos(\psi) = \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}, \text{ and } \rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

Osculating circle (Circle of curvature)

$$\therefore \alpha = x_1 - \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{\frac{dy}{dx}}{\sqrt{1+(\frac{dy}{dx})^2}}$$

$$\alpha = x_1 - \frac{dy}{dx} \frac{[1+(\frac{dy}{dx})^2]}{\frac{d^2y}{dx^2}}$$

and

$$\beta = y_1 + \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \cdot \frac{1}{\sqrt{1+(\frac{dy}{dx})^2}}$$

$$\beta = y_1 + \frac{[1+(\frac{dy}{dx})^2]}{\frac{d^2y}{dx^2}}$$

Hence, the equation of the osculating circle is

$$(x-\alpha)^2 + (y-\beta)^2 = \rho^2.$$

Osculating circle (Circle of curvature)

Exercise

1. Find the equations of osculating circles to the gives curves at the indicated points.

$$(i) \qquad y = ln(x), \quad \text{at } (1,\ 0)$$

$$(ii) \qquad y = x^3, \quad \text{at } (1,\ 1)$$

2. Calculate the radius of curvature for the following curves at the indicated points and sketch the osculating circles:

$$(i) \qquad y = ln(x), \quad \text{at } x = 1$$

$$(ii) \qquad x = t - \sin t,\ y = 1 - \cos t, \quad \text{at } t = \pi$$

Osculating circle (Circle of curvature)

3. Find the radius of curvature of the ellipse $x = 2\cos t$, $y = \sin t$; $0 \leq t \leq 2\pi$ at $t = 0$ and $t = \frac{\pi}{2}$. Sketch the osculating circles at those points.

4. Consider the curve $y = x^4 - 2x^2$

(i) Find the radius of curvature at each of relative extremum.

(ii) Sketch the curve and show the osculating circles at the relative extrema.