National University of Computer and Emerging Sciences

Lahore Campus

Operations Research (MT 4031)

Final Exam

Date: May 28, 2024

Course Instructor(s)

Dr. Misbah Irshad, Dr. Uzma Bashir, Dr. Shazia Javed, Dr. Ayesha Razzaq, Miss Aisha Rasheed, Mr. Yasir.

Total Time (Hrs.): 3

Total Marks: 70

Total Questions: 6

Roll No ____________ Section ____________ Student Signature ____________

Do not write below this line.

i) Attempt all the questions neatly on the answer sheet.
ii) Solve all the parts of a question together in order.
iii) Don't use a red pen or lead pencil to solve the paper.

SOLUTION

Question 1: [8+7]

a. For the following LP, find three alternative optimal basic solutions.

$$\text{Max } z = x_1 + 2x_2 + 3x_3$$

subject to

$$x_1 + 2x_2 + 3x_3 \le 10$$
$$x_1 + x_2 \le 5$$
$$x_1 \le 1$$
$$x_1, x_2, x_3 \ge 0.$$

LP in standard form:

$$\text{Max } z = x_1 + 2x_2 + 3x_3$$

subject to

$$x_1 + 2x_2 + 3x_3 + x_4 = 10$$
$$x_1 + x_2 + x_5 = 5$$
$$x_1 + x_6 = 1$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0.$$

(1 mark)

Basic	x_1	x_2	x_3	x_4	x_5	x_6	sol
z	-1	-2	-3	0	0	0	0
x_4	1	2	3	1	0	0	10
x_5	1	1	0	0	1	0	5
x_6	1	0	0	0	0	1	1
Basic	x_1	x_2	x_3	x_4	x_5	x_6	**sol**
z	0	0	0	1	0	0	10
x_3	1/3	2/3	1	1/3	0	0	10/3
x_5	1	1	0	0	1	0	5
x_6	1	0	0	0	0	1	1

x_1 and x_2 are basic variables with 0 z coefficients, indicating that alternative solutions exist.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	sol
z	**0**	**0**	**0**	**1**	**0**	**0**	**10**
x_3	**-1/3**	**0**	**1**	**1/3**	**-2/3**	**0**	**0**
x_2	**1**	**1**	**0**	**0**	**1**	**0**	**5**
x_6	**1**	**0**	**0**	**0**	**0**	**1**	**1**
Basic	x_1	x_2	x_3	x_4	x_5	x_6	**sol**
z	**0**	**0**	**0**	**1**	**0**	**0**	**10**
x_3	**0**	**0**	**1**	**1/3**	**2/3**	**1/3**	**1/3**
x_2	**0**	**1**	**0**	**0**	**1**	**-1**	**4**
x_1	**1**	**0**	**0**	**0**	**0**	**1**	**1**

(1.5 for each iteration)

The three alternative solutions are:

$$x_1 = 0, x_2 = 0, x_3 = \frac{10}{3}$$

$$x_1 = 0, x_2 = 5, x_3 = 0$$

$$x_1 = 1, x_2 = 4, x_3 = \frac{1}{3}$$

(1 mark)

b. Consider the following LP model:

$$\text{Maximize } z = 4x_1 + 14x_2$$

subject to

$$2x_1 + 7x_2 + x_3 \quad = 21$$

$$7x_1 + 2x_2 \quad + x_4 = 21$$

$$x_i \geq 0, \qquad i = 1,2,3,4$$

Construct the entire simplex tableau associated with the following basic variables and check it for optimality and feasibility of the given basic solution,

Basic Variable $= (x_2, x_4)$ and Inverse $= \begin{bmatrix} \frac{1}{7} & 0 \\ \frac{-2}{7} & 1 \end{bmatrix}$

DUAL:

(1 mark)

$$\text{Minimize } w = 21y_1 + 21y_2$$

subject to

$$2y_1 + 7y_2 \geq 4$$

$$7y_1 + 2y_2 \geq 14$$

$$y_i \geq 0, \qquad i = 1,2$$

Solution of dual: $(y_1, y_2) = (14 \quad 0)\begin{pmatrix} \frac{1}{7} & 0 \\ \frac{-2}{7} & 1 \end{pmatrix} = (2 \quad 0)$ (1 mark)

FEASIBILITY:

$\begin{pmatrix} x_2 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{7} & 0 \\ \frac{-2}{7} & 1 \end{pmatrix} \times \begin{pmatrix} 21 \\ 21 \end{pmatrix} = \begin{pmatrix} 3 \\ 15 \end{pmatrix}$. Thus, the solution is feasible. (2 marks)

OPTIMALITY:

Coefficients of non basic variables:

$$x_1 - - - - 2y_1 + 7y_2 - 4 = 4 + 0 - 4 = 0$$

$$x_3 - - - - y_1 - 0 = 2 - 0 = 2$$

Thus, the solution is optimal. **(2 marks)**

Constraint coefficients in the optimal tableau: **(1 mark)**

$$\begin{pmatrix} \frac{1}{7} & 0 \\ \frac{-2}{7} & 1 \end{pmatrix} \times \begin{pmatrix} 2 & 7 & 1 & 0 \\ 7 & 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2/7 & 1 & 1/7 & 0 \\ 45/7 & 0 & -2/7 & 1 \end{pmatrix}$$

The entire simplex tableau is:

Basic	x_1	x_2	x_3	x_4	**sol**
z	**0**	**0**	**2**	**0**	**42**
x_2	**2/7**	**1**	**1/7**	**0**	**3**
x_4	**45/7**	**0**	**-2/7**	**1**	**15**

Question 2: [7+8]

a. There are five registration counters in a university. Five persons are available for service. The details of the expected number of registered students is given below. How should the counters be assigned to persons to register the maximum number of students.

Counters	Persons				
	A	B	C	D	E
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

Transform the problem into minimization type first. The maximum entry is 62. (2 marks)

Counters	Persons				
	A	B	C	D	E
1	32	25	22	34	22
2	22	38	35	41	26
3	22	30	29	32	27
4	37	24	22	26	26
5	33	0	21	28	23

$p_1 = 22, p_2 = 22,\ p_3 = 22,\ p_4 = 22,$ (1 mark)

Counters	Persons				
	A	B	C	D	E
1	10	3	0	12	0
2	0	16	13	19	4
3	0	8	7	10	5
4	15	2	0	4	4
5	33	0	21	28	23

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$q_4 = 4$

Draw minimum number of lines to cross all zeros. (2 marks)

Counters	Persons				
	A	B	C	D	E
1	10	3	0	8	0
2	0	16	13	15	4
3	0	8	7	6	5
4	15	2	0	0	4
5	33	0	21	24	23

The smallest uncrossed entry=4 (1 mark)

Counters	Persons				
	A	B	C	D	E
1	14	3	0	8	0
2	0	12	9	11	0
3	0	4	3	2	1
4	19	2	0	0	4
5	37	0	21	24	23

Optimal assignment is:

A------3, B------5, C-------1, D------4 and E------2.

Maximum no. of registered students:214 (1 mark)

b. In the following transportation problem, the total demand exceeds the total supply. Suppose that the penalty costs per unit of unsatisfied demand are \$2, \$5, and \$3 for destinations 1, 2, and 3, respectively. Find the optimal minimum cost of the problem by using VAM for starting solution.

\$5	\$1	\$7	**10**
\$6	\$4	\$6	**80**
\$3	\$2	\$5	**15**
75	**20**	**50**	

Balancing the model:

5	1	7	10
6	4	6	80
3	2	5	10
2	5	3	40
75	20	50	

(1 mark)

	1	2	3	
1	5	$x_{12}=10$ 1	7	~~10~~
2	6	4	6	80
3	15 3	2	5	~~15~~
4	2	5	40 3	~~40~~
	~~75~~ 60	~~20~~ 10	~~50~~ 10	

Row penalties

4	—	—
2	2	2
1	1	1
1	1	—

	1	2	3	
2	6 60	4 10	6 10	~~80~~ ~~70~~ 60
	~~60~~	~~10~~	~~10~~	

(2marks)

Col Penalities 1 1 3
1 2 2

Assign multiplier u_i to each row i & v_j to each column j using $u_i + v_j = c_{ij}$ (for basic variables)

	$v_1 = 3$	$v_2 = 1$	$v_3 = 3$
$u_1 = 0$	[-1] 5	10 1	[-4] 7
$u_2 = 3$	60 (−) 6	10 4	10 (+) 6
$u_3 = 0$	15 3	[-1] 2	[-2] 5
$u_4 = 0$	[1] x_{14} (+) 2	[-4] 5	40 (−) 3

(2 marks)

For each non basic var. compute $\bar{c}_{ij} = u_i + v_j - c_{ij}$

x_{41} enters & x_{43} leaves

	$v_1 = 3$	$v_2 = 1$	$v_3 = 3$	
$u_1 = 0$	[-2] 5	10 1	[-4] 7	10
$u_2 = 3$	20 6	10 4	50 6	80
$u_3 = 0$	15 3	[-1] 2	[-2] 5	15
$v_4 = -1$	40 2	[-5] 5	[-1] 3	40
	75	20	50	

(2marks)

Optimal transportation cost is=$595. 1 mark

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Question 3: [8+4]

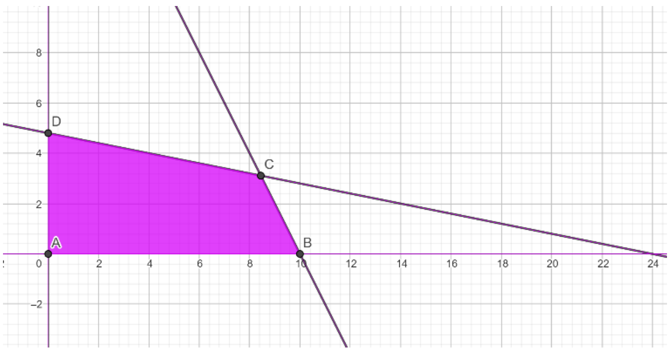
a. Solve the following integer linear programming problem using branch and bound algorithm by demonstrating the portioning graphically. Develop B & B tree as well.

$$\text{Maximize } z = 2x_1 + 9x_2$$

subject to

$$2x_1 + x_2 \leq 20$$
$$x_1 + 5x_2 \leq 24$$
$$x_1, x_2 \geq 0 \text{ and integers.}$$

Eliminating integer restriction and solving the problem using graphical method, the optimal solution is:



A(0,0), B(10,0), C$\left(8\frac{4}{9}, 3\frac{1}{9}\right)$, D$\left(0, 4\frac{4}{5}\right)$

$z_{max} = 44\frac{8}{9}$ at point C.

The variable x_1 got the maximum fractional part, select it for branching. The two constraints $x_1 \leq 8$ and $x_1 \geq 9$ are added to original problem. [1mark]

Subproblem 1

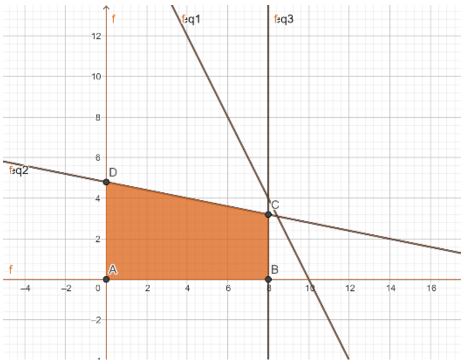
$$\text{Maximize } z = 2x_1 + 9x_2$$

subject to

$$2x_1 + x_2 \le 20$$
$$x_1 + 5x_2 \le 24$$
$$x_1 \le 8$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

A(0,0)
B(8,0)
$C\left(8, 3\frac{1}{5}\right)$
$D\left(0, 4\frac{4}{5}\right)$

$$z_{max} = 44\frac{4}{5}$$



Subproblem 2

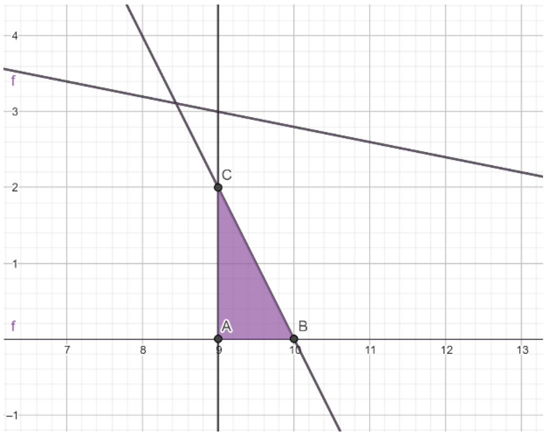
$$\text{Maximize } z = 2x_1 + 9x_2$$

subject to

$$2x_1 + x_2 \le 20$$
$$x_1 + 5x_2 \le 24$$
$$x_1 \ge 9$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

A(9,0)
B(10,0)
C(9,2)

$z_{max} = 38$ exists at point C.



Next select x_2 for branching. Subproblem 3 and 4 are formed using subproblem 1.

Subproblem3:

$$\text{Maximize } z = 2x_1 + 9x_2$$

subject to

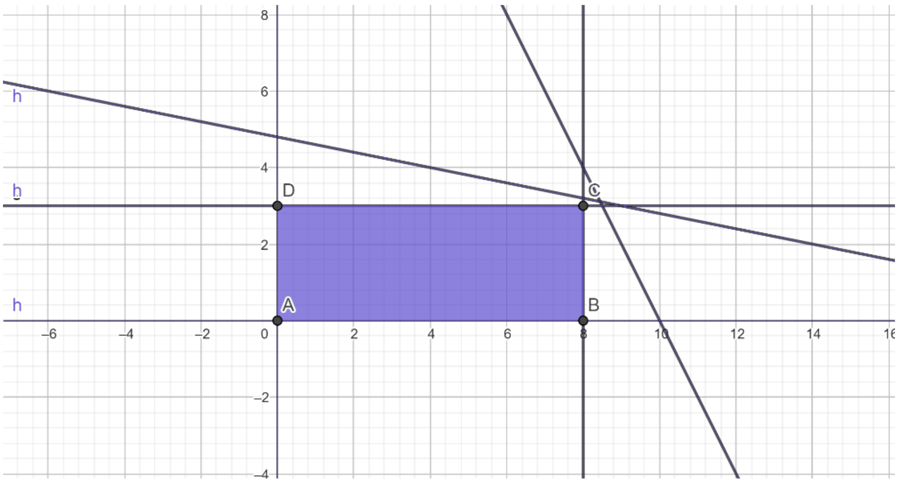
$$2x_1 + x_2 \le 20$$
$$x_1 + 5x_2 \le 24$$
$$x_1 \le 8$$
$$x_2 \le 3$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$

Subproblem 4:

$$\text{Maximize } z = 2x_1 + 9x_2$$

subject to

$$2x_1 + x_2 \le 20$$
$$x_1 + 5x_2 \le 24$$
$$x_1 \le 8$$
$$x_2 \ge 4$$
$$x_1, x_2 \ge 0 \text{ and integers.}$$



Subproblem 3

A(0,0), B(8,0) C(8, 3) and D(0,3). $z_{max} = 38$

A(0,4), B(4,4) and C(0,4.8) $z_{max} = 44$ at B. Which is desired integer solution.

[1.5 marks for each subproblem]

The branch and bound tree is:

$\mathbf{x} = (8\frac{4}{9},\ 3\frac{1}{9})$
$z = 44\frac{8}{9}$

$x_1 \le 8$
$\mathbf{x} = (8,\ 3\frac{1}{5})$
$z = 44\frac{4}{5}$

$x_1 \ge 9$
$\mathbf{x} = (9,\ 2)$
$z = \cancel{38}$

Z=36

$x_2 \le 3$
$\mathbf{x} = (8,\ 3)$
$z = 43$

$x_2 \ge 4$
$\mathbf{x} = (4,\ 4)$
$z = 44$

[1 mark]

b. A continuous optimal solution for an integer programming problem is given. Find a legitimate cut that will force the basic variable x_2 to take the integer value. Write the resulting solution as well.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	sol
z	550/3	0	0	250/3	200/3	0	23000/3
x_2	1/3	1	0	1/3	-1/3	0	20/3
x_3	5/6	0	1	-1/6	2/3	0	50/3
x_6	-5/3	0	0	-2/3	-1/3	1	80/3

Since the basic variable x_2 is restricted to take integer value, we select x_2 row as the source row to generate a cut.

$$x_2 + \frac{1}{3}x_1 + \frac{1}{3}x_4 - \frac{1}{3}x_5 = 6\frac{2}{3}$$

Fractional cut is: $-\frac{1}{3}x_1 - \frac{1}{3}x_4 - \frac{2}{3}x_5 \le -\frac{2}{3}$ **[1 mark]**

Adding this cut to the optimal tableau.

Basic	x_1	x_2	x_3	x_4	x_5	x_6	s_1	sol
z	550/3	0	0	250/3	200/3	0	0	23000/3
x_2	1/3	1	0	1/3	-1/3	0	0	20/3
x_3	5/6	0	1	-1/6	2/3	0	0	50/3
x_6	-5/3	0	0	-2/3	-1/3	1	0	80/3
s_1	-1/3	0	0	-1/3	-2/3	0	1	-2/3
Basic	x_1	x_2	x_3	x_4	x_5	x_6	s_1	sol
z	150	0	0	50	0	0		7600
x_2	1/2	1	0	1/2	0	0	-1/2	7
x_3	1/2	0	1	-1/2	0	0	-1	48
x_6	-3/2	0	0	-1/2	0	1	-1/2	27
x_5	1/2	0	0	1/2	1	0	-3/2	1

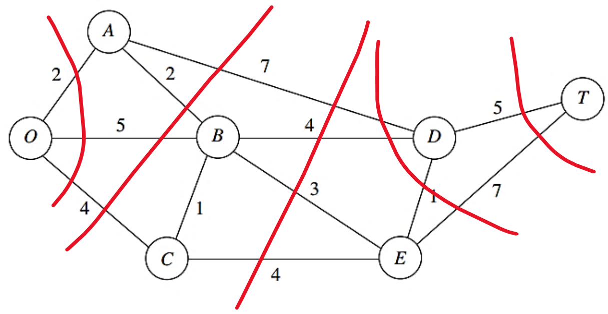
[1 mark for each iteration+ 1 mark for final solution]

Question 4: [8+4]

a.

i. For the given network, enumerate all cuts that will terminate the flow between source node O and sink node T, hence identify the minimum cut.

ii. Find the maximal flow between source and sink nodes and verify that the maximal flow is equal to the minimum cut capacity.

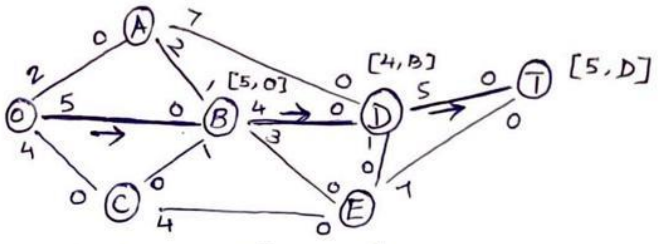


Few cuts

2 marks

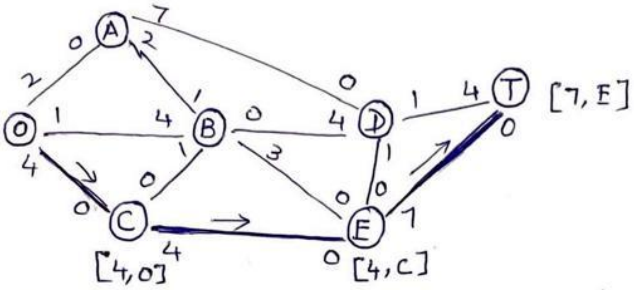
Q4: a(ii)

1)



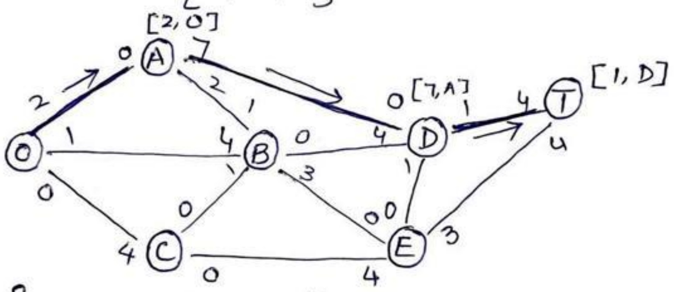
$f_1 = \min\{5, 4, 5\} = 4$

2)



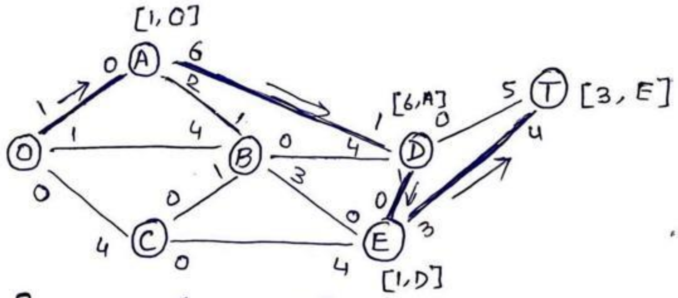
$f_2 = \min\{4, 4, 7\} = 4.$

3)

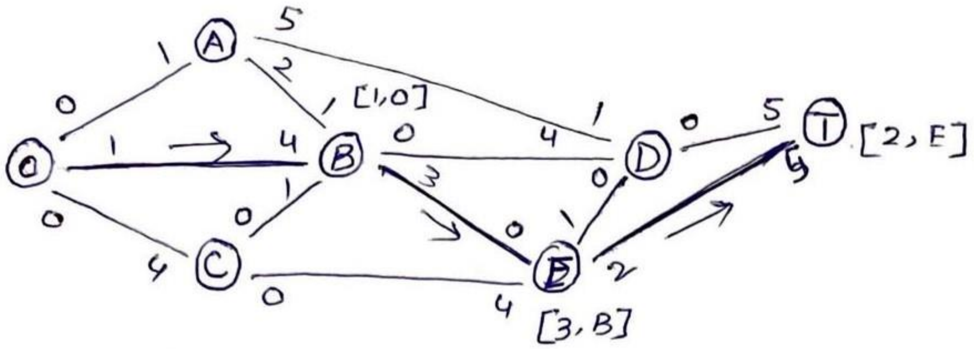


$f_3 = \min\{2, 7, 1\} = 1$

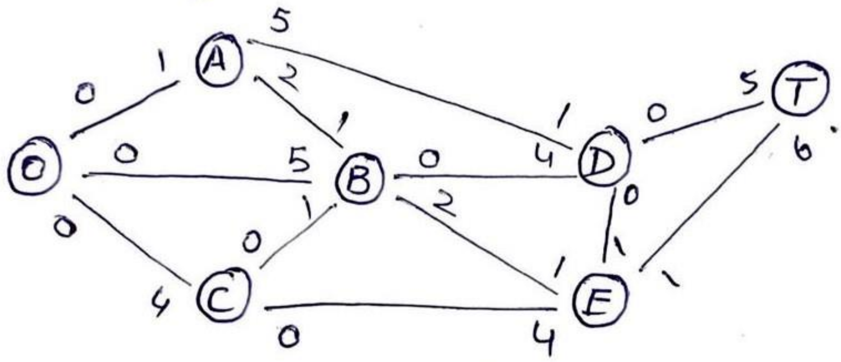
4)



$f_4 = \min\{1, 6, 1, 3\} = 1$



$f_5 = \min\{1, 3, 2\} = 1$



No further break through

$F = f_1 + f_2 + f_3 + f_4 + f_5 = 11$ units (max. flow)

1 mark for each step

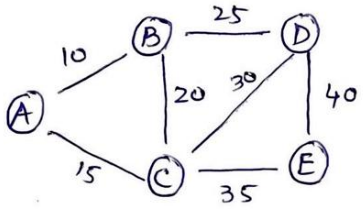
b. A delivery company operates in a city with multiple distribution centers. The city has five distribution centers labeled A, B, C, D, and E. The distances between the distribution centers are as follows:

- Distance between A and B: 10 miles.
- Distance between A and C: 15 miles.
- Distance between B and C: 20 miles.
- Distance between B and D: 25 miles.
- Distance between C and D: 30 miles.
- Distance between C and E: 35 miles.
- Distance between D and E: 40 miles.

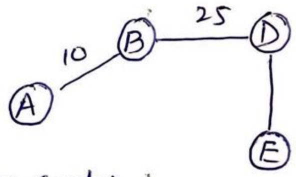
Construct the network and find a tree, a spanning tree and a minimal spanning tree by clearly differentiating the three.

Q4(b)

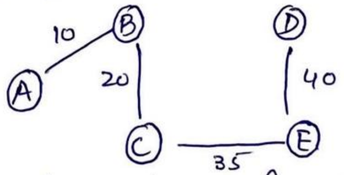
Network:



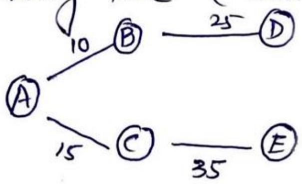
Tree: Any cycle free subgraph.



Spanning tree:- (contains all nodes, no cycle)



Minimal spanning tree (min length)



1 mark for each

Question 5: [4+4]

Suppose that the equation of a circle is $(x - 5)^2 + (y + 7)^2 = 49$.

a) Define the corresponding distributions f (x) and f (y), and then show how a sample point (x, y) is determined using the (0, 1) random pair (R1, R2).

b) Estimate area of the circle for the following random pairs

R1: .0589 .3529 .5869 .7455 .7900 .6307

R2: .6733 .3646 .1281 .4871 .3698 .2346

QNo 5

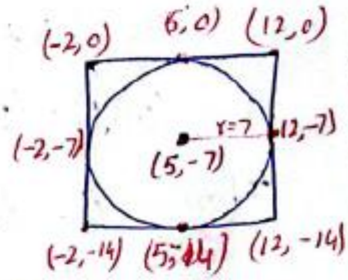
The given Equation of Circle is.

1 mark

$$(x-5)^2 + (y+7)^2 = 49$$

Compare with $(x-h)^2 + (y-k)^2 = r^2$

Center $(h,k) = (5,-7)$, $r = 7$



So, square width = 14 (Circle Diameter).

To Ensure all points ~~to~~ are Equally Divided in the square, Let we take.

$$f_x(x) = \frac{1}{14} \quad , \quad \overset{x_1}{-2} \le x \le \overset{x_2}{12}.$$

$$f(y) = \frac{1}{14} \quad \quad \overset{y_1}{-14} \le x \le \overset{y_2}{0}$$

1.5 marks

A pair of 0-1 Random variables R_1 and R_2 Can be used to generate random points (x,y) in the square as

$$x = x_1 + [x_2 - x_1]R_1 \quad , \; y = y_1 + [y_2 - y_1]R_2$$

$$x = -2 + 14R_1 \quad , \; y = -14 + 14R_2$$

1.5 marks

Let us select a Random point from (0, 1)
Let $R_1 = 0.32$, $R_2 = 0.45$, (change w.r.t student)
Then

$$x = -2 + 14R_1 = -2 + 14(0.32) = 2.48$$
$$y = -14 + 14R_2 = -14 + 14(0.45) = -7.7$$
$$(x, y) = (2.48, -7.7)$$

Then $(2.48 - 5)^2 + (-7.7 + 7)^2 = 6.84 < 49$

So, point lie inside the circle.

b)

$$\text{Approximate Area of circle} = \frac{m}{n}\left(\text{Area of square}\right)$$

where m means points lie inside the circle and n means total sample points.

(i) $R_1 = 0.0589$, $R_2 = 0.6733$

$x = -2 + 14(0.0589) = -1.1754$

$y = -14 + 14(0.6733) = -4.5738$

$(-1.1754 - 5)^2 + (-4.5738 + 7)^2 = 44.02 < 49$ ✓

Inside.

(ii) $R_1 = 0.3529$, $R_2 = 0.3646$

$x = -2 + 14(0.3529) = 2.9406$

$y = -14 + 14(0.3646) = -8.8956$

$(2.9406 - 5)^2 + (-8.8956 + 7)^2 = 7.83$ (inside)

(iii) $R_1 = 0.5869$ $R_2 = 0.1281$

$x = -2 + 14(0.5869) = 6.2166$

$y = -14 + 14(0.1281) = -12.2066$

$(6.2166 - 5)^2 + (-12.2066 + 7)^2 = 28.589$ (Inside)

iv) $R_1 = 0.7455$ $R_2 = 0.4871$

$x = -2 + 14(0.7455) = 8.4370$

$y = -14 + 14(0.4871) = -7.1806$

$(8.4370 - 5)^2 + (-7.1806 + 7)^2 = 11.85$

v) $R_1 = 0.7900$ $R_2 = 0.3698$

$x = -2 + 14(0.7900) = 9.06$

$y = -14 + 14(0.3698) = -8.8228$

$$(9.06-5)^2+(-8.8228+7)^2 \leq 49 \quad (\text{Inside})$$

vi) $R_1 = 0.6307, \quad R_2 = 0.2346$

$$x = -2+14(0.6307) = 6.8298$$

$$y = -14+14(0.2346) = -10.7156$$

0.5 for each

$$(6.8298-5)^2+(-10.7156+7)^2 \leq 49 \quad (\text{Inside})$$

All points lie within circle.

So $\quad m=6 \quad n=6$

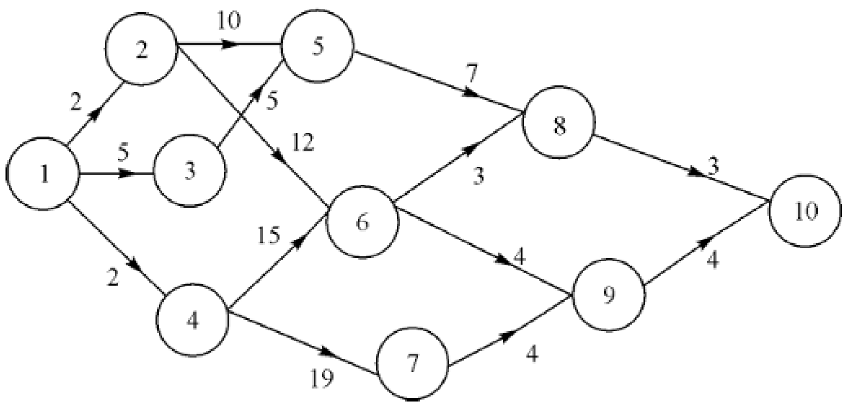
1 mark

Hence,

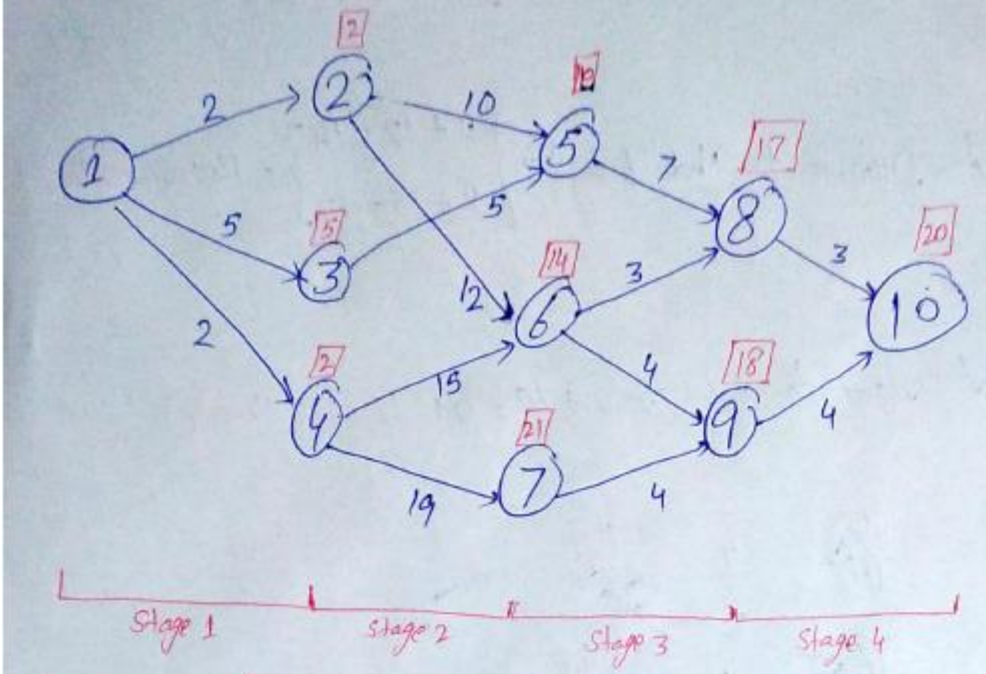
$$\text{App. Area of Circle} = \frac{6}{6} \times (14 \times 14) = 196$$

Question 6: [8]

Mr. Ali, a sales manager, has decided to travel from city 1 to city 10. He wants to plan for a minimum distance program and visit maximum number of branch offices as possible on the route but no restrictions to visit all the offices. The route map of the various ways of reaching city 10 from city 1 is shown below. The number on the arrow indicates the distance in km. (× 100). **Using dynamic programing suggest a feasible minimum distance path plan to Mr. Ali.**



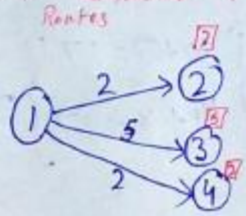
Q No ~~8~~ 6 [8]



Formula: $f_i(x_i) = \min_{\text{all feasible Routes}} \{ d(x_{i-1}, x_i) + f_{i-1}(x_{i-1}) \}$, $i = 1, 2, \ldots$

$f_0(x_0 = 1) = 0$.

Stage 1:



Shortest distance from Node 1 to 2 = 2

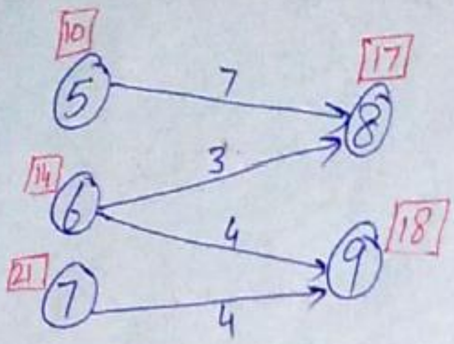
" " " Node 1 to 3 = 5

" " " Node 1 to 4 = 2.

Shortest Distance to Node 6 $= \text{Min}\begin{Bmatrix} 2+12=14 \\ 2+15=17 \end{Bmatrix} = 14$ (From Node 2)

Shortest Distance to Node 7 $= 2+19 = 21$ (From Node 4)

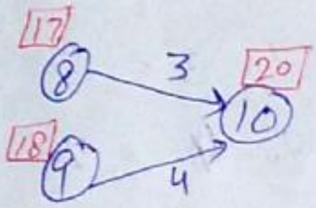
Stage 3



Shortest Distance to Node 8 $= \text{Min}\begin{Bmatrix} 10+7=17 \\ 14+3=17 \end{Bmatrix} = 17$ (From Node 5 and 6)

Shortest Distance to Node 9 $= \text{Min}\begin{Bmatrix} 14+4=18 \\ 21+4=25 \end{Bmatrix} = 18$ (From Node 6)

Stage 4



Shortest Distance to Node 10 $= \text{Min}\begin{Bmatrix} 17+3=20 \\ 18+4=22 \end{Bmatrix} = 20$ (From Node 8)

1.5 for each stage

Routes

(i) $1 \longrightarrow 2 \longrightarrow 6 \longrightarrow 8 \longrightarrow 10$

(ii) $1 \longrightarrow 3 \longrightarrow 5 \longrightarrow 8 \longrightarrow 10$

2 marks

Minimum Distance = 20×100 Km = 2000 Km.