$Z = R + j\omega L$

$Y = G + j\omega C$

$\dfrac{d^2V}{dz^2} = (ZY)V$

$Z_0 \equiv \dfrac{V^+(z)}{I^+(z)}$ $\quad Z_0 = \dfrac{V_0^+}{I_0^+}$

$R=0,\ G=0$ $\quad \alpha = 0$ $\quad \beta = \omega\sqrt{LC}$

$Z_0 = \sqrt{\dfrac{L}{C}}$ $\quad v_p = \dfrac{1}{\sqrt{LC}}$

$LC = \mu\varepsilon$

Quarter-Wave Transformer

$Z_{in} = \dfrac{Z_{0T}^2}{Z_L}$ $\quad \Gamma_{in} = 0$

$Z_{0T} = [Z_0 Z_L]^{1/2}$

$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$

$I(z) = \dfrac{V_0^+}{Z_0} e^{-\gamma z} - \dfrac{V_0^-}{Z_0} e^{+\gamma z}$ $\quad v_p = \dfrac{\omega}{\beta}$

$\gamma = \alpha + j\beta = [(R + j\omega L)(G + j\omega C)]^{1/2}$

$Z_0 = \left(\dfrac{R + j\omega L}{G + j\omega C}\right)^{1/2}$ $\quad \lambda_g = \dfrac{2\pi}{\beta}$

$V(-\ell) = V_0^+ e^{j\beta\ell}\left(1 + \Gamma_L e^{-2j\beta\ell}\right)$

$I(-\ell) = \dfrac{V_0^+}{Z_0} e^{j\beta\ell}\left(1 - \Gamma_L e^{-2j\beta\ell}\right)$

$Z(-\ell) = \dfrac{V(-\ell)}{I(-\ell)} = Z_0\left(\dfrac{1 + \Gamma_L e^{-2j\beta\ell}}{1 - \Gamma_L e^{-2j\beta\ell}}\right) = Z_0\left(\dfrac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)}\right)$

$\Gamma_L = \dfrac{Z_L - Z_0}{Z_L + Z_0}$

$P(-d) = \underbrace{\dfrac{1}{2}\dfrac{|V_0^+|^2}{Z_0}e^{2\alpha\ell}}_{\text{power in forward wave}} - \underbrace{\dfrac{1}{2}\dfrac{|V_0^+|^2}{Z_0}|\Gamma_L|^2 e^{-2\alpha\ell}}_{\text{power in backward wave}}$

Lossless line $(\alpha = 0)$

$P(-d) = \dfrac{1}{2}\dfrac{|V_0^+|^2}{Z_0}\left(1 - |\Gamma_L|^2\right)$

$\nabla^2\underline{E} \equiv \nabla(\nabla\cdot\underline{E}) - \nabla\times\nabla\times\underline{E}$

$\nabla^2\underline{E} = \hat{x}(\nabla^2 E_x) + \hat{y}(\nabla^2 E_y) + \hat{z}(\nabla^2 E_z)$

helmholtz

$\nabla^2 E_z + k^2 E_z = 0$

$\nabla^2 H_z + k^2 H_z = 0$

$(\nabla^2 + k^2)E_z = 0$

$k_z = \beta - j\alpha = k = \omega\sqrt{\mu\varepsilon_c} = k' - jk''$

Coaxial Cable

$C = \dfrac{2\pi\varepsilon_0\varepsilon_r}{\ln\left(\dfrac{b}{a}\right)}$ [F/m] $\quad G = \dfrac{2\pi\sigma}{\ln\left(\dfrac{b}{a}\right)}$ $\quad \varepsilon_c = \varepsilon - j\left(\dfrac{\sigma}{\omega}\right)$

$L = \dfrac{\mu_0\mu_r}{2\pi}\ln\left(\dfrac{b}{a}\right)$ $\quad G = C\left(\dfrac{\sigma}{\varepsilon}\right)$ $\quad \tan\delta \equiv \dfrac{\varepsilon_c''}{\varepsilon_c'} = \dfrac{\varepsilon'' + \left(\dfrac{\sigma}{\omega}\right)}{\varepsilon'}$

$\eta_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} = 376.7303$ $\quad \dfrac{G}{\omega C} = \tan\delta$ $\quad \varepsilon_0 = 8.854 \cdot 10^{-12}$ $\quad \mu_0 = 4\pi\times 10^{-7}$

$R = R_a + R_b$ $\quad R_a = R_{sa}\left(\dfrac{1}{2\pi a}\right)$ $\quad R_b = R_{sb}\left(\dfrac{1}{2\pi b}\right)$

$H_x = \dfrac{j}{k_c^2}\left(\omega\varepsilon_c\dfrac{\partial E_z}{\partial y} \mp k_z\dfrac{\partial H_z}{\partial x}\right)$

$H_y = \dfrac{-j}{k_c^2}\left(\omega\varepsilon_c\dfrac{\partial E_z}{\partial x} \pm k_z\dfrac{\partial H_z}{\partial y}\right)$

$E_x = \dfrac{-j}{k_c^2}\left(\pm k_z\dfrac{\partial E_z}{\partial x} + \omega\mu\dfrac{\partial H_z}{\partial y}\right)$

$E_y = \dfrac{j}{k_c^2}\left(\mp k_z\dfrac{\partial E_z}{\partial y} + \omega\mu\dfrac{\partial H_z}{\partial x}\right)$

$E_{1t} = 0$

$\hat{n}\times H_1 = J_s$

$\hat{n}\cdot D_1 = \rho_s$

$B_{1n} = 0$

$w >> d,\ \lambda \Rightarrow \dfrac{\partial}{\partial x} = 0$

$\left(\cancel{\dfrac{\partial^2}{\partial x^2}} + \dfrac{\partial^2}{\partial y^2} + k_c^2\right)h_z(x,y) = 0$

TE

$E_x = 0 \quad @\ y = 0, d$

$E_x = \dfrac{1}{j\omega\varepsilon_c}\left(\dfrac{\partial H_z}{\partial y} - \cancel{\dfrac{\partial H_y}{\partial z}}\right)$

$H_z = B_n\cos\left(\dfrac{n\pi}{d}y\right)e^{\mp jk_z z}$

PEC: $\underline{H}\cdot\hat{n} = 0$

$E_x = \dfrac{j\omega\mu}{k_c}B_n\sin\left(\dfrac{n\pi}{d}y\right)e^{\mp jk_z z}$

$H_y = \dfrac{\pm jk_z}{k_c}B_n\sin\left(\dfrac{n\pi}{d}y\right)e^{\mp jk_z z}$

$H_x = E_y = E_z = 0$

$h_z = A\sin(k_c y) + B\cos(k_c y)$

$k_c = \dfrac{n\pi}{d};\quad n = 1, 2, \ldots$ $\quad k_z = \sqrt{k^2 - \left(\dfrac{n\pi}{d}\right)^2}$

Parallel plate

$\Phi(x,0) = 0\ ;\ \Phi(x,d) = V_0$ $\quad$ TEM

$\nabla_t^2\Phi = \left(\cancel{\dfrac{\partial^2}{\partial x^2}} + \dfrac{\partial^2}{\partial y^2}\right)\Phi = 0$

$\underline{E}(x,y,z) = -\hat{y}\dfrac{V_0}{d}e^{\mp jkz}$

$\Phi(x,y) = \dfrac{V_0}{d}y$ $\quad I(z) = \pm\overbrace{\dfrac{V_0}{\eta}\dfrac{w}{d}}^{I_0}e^{\mp jkz}$

$\underline{H}(x,y,z) = \pm\hat{x}\dfrac{V_0}{\eta d}e^{\mp jkz}$

$V(z) = \int_d^0 \underline{E}\cdot\hat{y}\,dy = V_0 e^{\mp jkz}$

$Z_0 = \eta\dfrac{d}{w}$

TM

$\left(\cancel{\dfrac{\partial^2}{\partial x^2}} + \dfrac{\partial^2}{\partial y^2} + k_c^2\right)e_z = 0,\quad k_c = [k^2 - k_z^2]^{\frac{1}{2}}$

$e_z(x,y) = A\sin(k_c y) + B\cos(k_c y)$

$f_{cn} = \dfrac{n}{2d}\dfrac{1}{\sqrt{\mu\varepsilon}}$ $\quad$ برای TE و TM

$E_z = A_n\sin\left(\dfrac{n\pi}{d}y\right)e^{\mp jk_z z}$

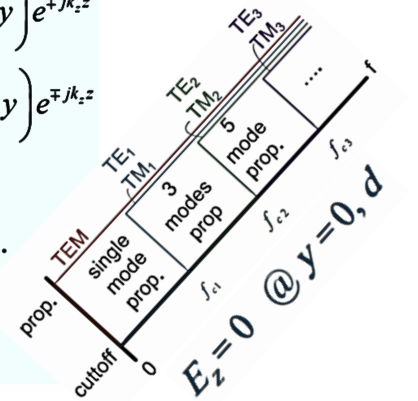
$E_y = \mp\dfrac{jk_z}{k_c}A_n\cos\left(\dfrac{n\pi}{d}y\right)e^{\mp jk_z z}$

$H_x = \dfrac{j\omega\varepsilon_c}{k_c}A_n\cos\left(\dfrac{n\pi}{d}y\right)e^{\mp jk_z z}$

$E_x = H_y = H_z = 0$

$k_c = \dfrac{n\pi}{d};\quad n = 0, 1, 2, \ldots$

$k_z = \sqrt{k^2 - \left(\dfrac{n\pi}{d}\right)^2}$



$E_z = 0 \quad @\ y = 0, d$

$\alpha = \alpha_c + \alpha_d$ $\quad$ Lossy dielectric $\Rightarrow$ complex permittivity $\Rightarrow$ complex wavenumber k

$k^2 = \omega^2\mu\varepsilon_c = \omega^2\mu\varepsilon_c'(1 - j\tan\delta)$

$k = k' - jk''$

$\varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega} = \varepsilon_c' - j\varepsilon_c'' = \varepsilon_c'\left(1 - j\dfrac{\varepsilon_c''}{\varepsilon_c'}\right) = \varepsilon_c'(1 - j\tan\delta)$

Perturbation Method for α_c

The value k_c is always real

$k_z = \beta - j\alpha_d = \sqrt{k^2 - k_c^2}$

$k = \omega\sqrt{\mu\varepsilon_c}$

Power flow along the guide:

$P(z) = P_0 e^{-2\alpha z}$

$P_l = -\dfrac{dP}{dz}$

$\alpha = \dfrac{P_l(z)}{2P(z)} = \dfrac{P_l(0)}{2P_0}$

Small dielectric loss in medium:

$\Rightarrow \tan\delta << 1 \Rightarrow k = \omega\sqrt{\mu\varepsilon_c'}\left(1 - j(\tan\delta)/2\right)$

$k' \approx \omega\sqrt{\mu\varepsilon_c'}$

$k'' \approx k'(\tan\delta)/2$

$\beta \approx \sqrt{k'^2 - k_c^2}$

$\alpha_d \approx \dfrac{k'^2\tan\delta}{2\beta}$

For TEM mode

$\beta = k'$

$\alpha_d = k''$

$P_0 = \mathrm{Re}\left\{\iint_S\left(\dfrac{1}{2}\underline{E}\times\underline{H}^*\right)\cdot\hat{z}\,dS\Big|_{z=0}\right\}$

$P_l(0) = \dfrac{R_s}{2}\int_C |\underline{J}_s|^2\Big|_{z=0} d\ell$

Attenuation

For these calculations, we neglect dielectric loss.

$\underline{E}_t$ = tangential electric field at surface

$\underline{H}_t$ = tangential magnetic field at surface

$\hat{n}$ = outward unit normal to conductor surface

Summary for a Good Conductor $\quad \underline{J}_s^{eff} = (\hat{n}\times\underline{H}_t)$ $\quad \underline{E}_t = Z_s\underline{J}_s^{eff}$

$\underline{E}_t = Z_s(\hat{n}\times\underline{H}_t)$ $\quad \left|\dfrac{\sigma}{\omega\varepsilon}\right| \gg 1$ $\quad k' \approx k'' \approx \sqrt{\dfrac{\omega\mu\sigma}{2}}$ $\quad \delta \equiv d_p = \dfrac{1}{k''}$ "skin depth"

$Z_s = \eta$

$Z_s = (1 + j)R_s$ $\quad R_s = \sqrt{\dfrac{\omega\mu}{2\sigma}} = \dfrac{1}{\sigma\delta}$ $\quad k' \approx k'' \approx \dfrac{1}{\delta}$ $\quad \delta = d_p = \sqrt{\dfrac{2}{\omega\mu\sigma}}$

$\langle P_d\rangle = \dfrac{1}{2}R_s\left|\underline{J}_s^{eff}\right|^2$

$P_0 = \dfrac{1}{2}\mathrm{Re}\left\{\int_0^w\int_0^d\left(\dfrac{|V_0|^2}{\eta^* d^2}\hat{z}\right)\cdot\hat{z}\,dydx\right\} = \dfrac{1}{2}|V_0|^2\left(\dfrac{w}{d}\right)\mathrm{Re}\left(\dfrac{1}{\eta^*}\right) \approx \dfrac{1}{2}|V_0|^2\left(\dfrac{w}{\eta^{lossless}d}\right)$

$\eta^{lossless} = \sqrt{\dfrac{\mu}{\varepsilon'}}$

$\underline{J}_s^{top} = -\hat{y}\times\underline{H} = \hat{z}\dfrac{V_0}{\eta d}e^{-jkz}$

$\underline{J}_s^{bot} = \hat{y}\times\underline{H} = -\hat{z}\dfrac{V_0}{\eta d}e^{-jkz}$

$P_l(0) \approx R_s\dfrac{|V_0|^2}{(\eta^{lossless}d)^2}w$

$\alpha_c = \dfrac{R_s}{\eta^{lossless}d}$

$\nabla\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ $\quad \oint\vec{E}\cdot d\vec{a} = \dfrac{Q_{enc}}{\varepsilon_0}$

$\nabla\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ $\quad \oint\vec{E}\cdot d\vec{l} = -\int\dfrac{\partial\vec{B}}{\partial t}\cdot d\vec{a}$

$\nabla\cdot\vec{B} = 0$ $\quad \oint\vec{B}\cdot d\vec{a} = 0$

$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t}$ $\quad \oint\vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0\int\dfrac{\partial\vec{E}}{\partial t}$

$e^{\mp\gamma z} = e^{\mp jk_z z}$ $\quad \nabla\times\underline{E} = -j\omega\mu\underline{H}$

$\gamma = jk_z$ $\quad \nabla\times\underline{H} = j\omega\varepsilon\underline{E} + \underline{J}$

$k_z = \beta - j\alpha$ $\quad \nabla\cdot\underline{E} = \dfrac{\rho_v}{\varepsilon}$

$k^2 = \omega^2\mu\varepsilon_c$ $\quad \nabla\cdot\underline{H} = 0$ $\quad k_c = \left(k^2 - k_z^2\right)^{1/2}$

$\underline{h}_t(x,y) = \pm\dfrac{1}{Z_{TE}}\left(\hat{z}\times\underline{e}_t(x,y)\right)$

$Z_{TE} = \dfrac{\omega\mu}{k_z}$ $\quad Z_{TM} = \dfrac{k_z}{\omega\varepsilon_c}$

For TEM waves $\quad k_c^2 = k^2 - k_z^2 = 0$

$k_z = k = \omega\sqrt{\mu\varepsilon_c}$

$\nabla_t\cdot\underline{E} = 0$

$\Rightarrow \nabla_t\cdot\left(\underline{e}_t(x,y)e^{\mp jk_z z}\right) = 0$

$\Rightarrow e^{\mp jk_z z}\nabla_t\cdot\left(\underline{e}_t(x,y)\right) = 0$

$\nabla_t\cdot\left(\underline{e}_t(x,y)\right) = 0$

$\nabla_t\times\left(\underline{e}_t(x,y)\right) = 0$

$\underline{e}_t(x,y) = -\nabla_t\Phi(x,y)$

$\nabla_t^2\Phi(x,y) = 0$

Rectangular

$a \geq b$. No TEM mode

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)h_z(x,y)=-k_c^2 h_z(x,y)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)e_z(x,y)=-k_c^2 e_z(x,y)$$

$$E_z=0 \ @\ x=0,\ a \quad @\ y=0,\ b$$

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-k_c^2$$

$$E_z=B_{mn}\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

$m=1,2,3,\ldots$

$n=1,2,3,\ldots$

$$k_x^2+k_y^2=k_c^2$$

$$k_z=\sqrt{k^2-k_c^2}=\sqrt{k^2-\left(\frac{m\pi}{a}\right)^2-\left(\frac{n\pi}{b}\right)^2}$$

$$h_z(x,y)=\underbrace{(A\cos k_x x+B\sin k_x x)}_{X(x)}\underbrace{(C\cos k_y y+D\sin k_y y)}_{Y(y)}$$

$$\frac{\partial h_z}{\partial y}=0 @ y=0,\ b \qquad k_y=\frac{n\pi}{b} \qquad n=0,1,2,\ldots$$

$$\frac{\partial h_z}{\partial x}=0 @ x=0,\ a \qquad k_x=\frac{m\pi}{a} \qquad m=0,1,2,\ldots$$

$$f_c^{mn}=\frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

Lossless Case TM$_{11}$ mode

$$H_z=A_{mn}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)e^{\mp jk_z z}$$

But $m=n=0$ is not allowed!

Lossless Case $$f_c^{mn}=\frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

$$k_z=\beta=\sqrt{k^2-\left(\frac{\pi}{a}\right)^2} \qquad \lambda_g=\frac{2\pi}{\beta}$$

TE$_{10}$ mode $(a>b)$

$$a\big|_{f=f_c}=\lambda_d/2$$

$$P_{10}^+=\frac{1}{2}\mathrm{Re}\left\{\int_0^a\int_0^b\left(\underline{E}\times\underline{H}^*\right)\cdot\hat{\underline{z}}\,dy\,dx\right\}$$

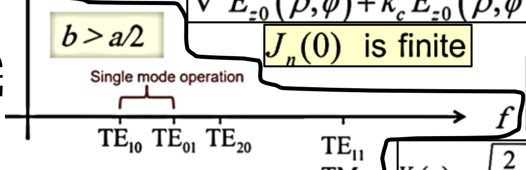
$$P_{10}^+=\left(\frac{ab}{4\omega\mu}\right)\mathrm{Re}\{k_z\}|E_{10}|^2 \qquad A_{10}=\frac{-\pi}{j\omega\mu a}E_{10}$$

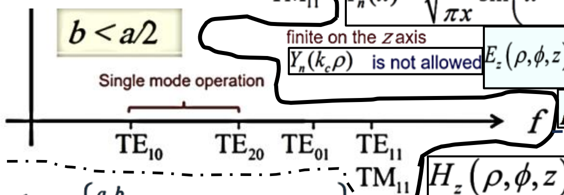
$$\alpha_c=\frac{P_l(0)}{2P_0} \qquad P_0=P_{10}^+ \qquad \alpha_c=\frac{R_s}{a^3 b\beta(k\eta)}\left(2b\pi^2+a^3k^2\right)$$

$$P_l(0)=R_s|A_{10}|^2\left(b+\left(\frac{\beta^2a^3}{2\pi^2}+\frac{a}{2}\right)\right)$$

$$P_l(0)=\frac{R_s}{2}\int_C\left|\underline{J}_s\right|^2d\ell \qquad \beta^2=k^2-k_c^2 \qquad k_z\approx\beta \qquad k_c^{10}=\frac{\pi}{a}$$

Rectangular





Circular

$$\nabla^2E_{z0}(\rho,\phi)+k_c^2E_{z0}(\rho,\phi)=0$$

$J_n(0)$ is finite

$$E_{z0}(\rho,\phi)=\begin{Bmatrix}J_n(k_c\rho)\\Y_n(k_c\rho)\end{Bmatrix}\begin{Bmatrix}\sin(n\phi)\\\cos(n\phi)\end{Bmatrix}$$

$$E_z(\rho,\phi,z)=\cos(n\phi)J_n(k_c\rho)e^{\mp jk_z z}$$

$$E_z(a,\phi,z)=0 \qquad J_n(k_c a)=0$$

$$J_n(x)\sim\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right) \quad J_n(x)$$

$$Y_n(x)\sim\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)$$

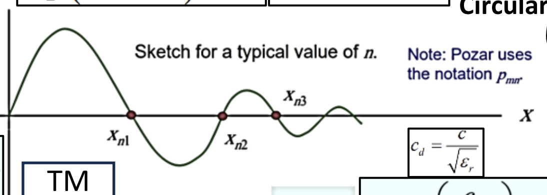
$$k_z=\sqrt{k^2-\left(\frac{x_{np}}{a}\right)^2}$$

finite on the z axis

$Y_n(k_c\rho)$ is not allowed

$$E_z(\rho,\phi,z)=\cos(n\phi)J_n\left(x_{np}\frac{\rho}{a}\right)e^{\mp jk_z z} \qquad n=0,1,2\ldots$$

$p=1,2,3,\ldots\ldots\ldots$



Note: Pozar uses the notation p_{mn}

TM

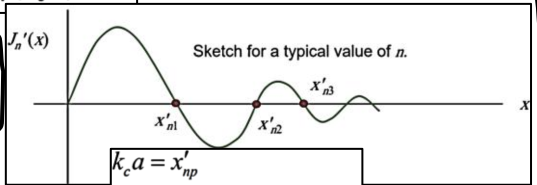
$$c_d=\frac{c}{\sqrt{\varepsilon_r}}$$

$$k_c a=x_{np} \Rightarrow k_c=\frac{x_{np}}{a} \qquad f_c^{TM}=\left(\frac{c_d}{2\pi a}\right)x_{np}$$

$$H_z(\rho,\phi,z)=\cos(n\phi)J_n(k_c\rho)e^{\mp jk_z z}$$

$$E_\phi(a,\phi,z)=0 \qquad E_\phi=\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial\rho}\Rightarrow\frac{\partial H_z}{\partial\rho}=0\Big|_{\rho=a}$$

$$J_n'(k_c a)=0$$



$$c_d=\frac{c}{\sqrt{\varepsilon_r}}$$

$$f_c^{TE}=\left(\frac{c_d}{2\pi a}\right)x'_{np}$$

$$k_c a=x'_{np} \qquad k_c=\frac{x'_{np}}{a} \qquad p=1,2,3,\ldots\ldots$$

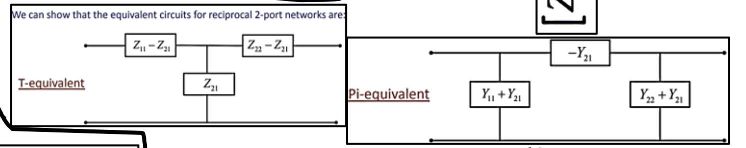
$$H_z(\rho,\phi,z)=\cos(n\phi)J_n\left(x'_{np}\frac{\rho}{a}\right)e^{\mp jk_z z} \qquad p=1,2,\ldots$$

Reciprocal Networks

$$[Z][Y]=[U]=\text{Identity Matrix}$$

$$[Y]=[Z]^{-1}$$

Reciprocal Networks $[Z]$ and $[Y]$ are symmetric



$$\begin{bmatrix}V_1\\I_1\end{bmatrix}=\begin{bmatrix}A&B\\C&D\end{bmatrix}\begin{bmatrix}V_2\\I_2'\end{bmatrix}$$

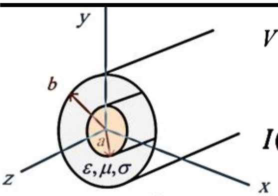
$$\Gamma_L=\frac{V_1^-(0)/\sqrt{Z_{01}}}{V_1^+(0)/\sqrt{Z_{01}}}=\frac{b_1(0)}{a_1(0)} \qquad b_1(0)=\Gamma_L a_1(0)=S_{11}a_1(0)$$

$$S_{ij}=\frac{b_i(0)}{a_j(0)}\bigg|_{a_k=0\ k\neq j}$$

reciprocal

$$S_{ij}=S_{ji} \qquad i\neq j$$

Coaxial Cable



$$V(z)=\int_A^B\underline{E}\cdot d\underline{r}=\int_a^b\frac{V_0}{\rho\ln\left(\frac{b}{a}\right)}e^{\mp jk_z z}d\rho=V_0e^{\mp jk_z z}$$

TEM

$$I(z)=\int_0^{2\pi}J_{sz}\,\rho d\phi=\int_0^{2\pi}\left(\underline{H}\cdot\hat{\underline{\phi}}\right)\rho d\phi=\pm\frac{2\pi V_0}{\eta\ln\left(\frac{b}{a}\right)}e^{\mp jk_z z}$$

$$\frac{\partial}{\partial\phi}=0$$

$$\nabla_t^2\Phi=0; \qquad \Phi(a)=V_0 \qquad \Phi(b)=0$$

$$\Rightarrow\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right)=0$$

$$\Rightarrow\Phi(\rho)=C\ln\rho+D$$

or $\Phi(\rho)=C\ln\left(\frac{\rho}{\rho_0}\right)$

$$\Phi(\rho)=\frac{V_0}{\ln\left(\frac{b}{a}\right)}\ln\left(\frac{b}{\rho}\right) \qquad \underline{E}(x,y,z)=\hat{\underline{\rho}}\frac{V_0}{\rho\ln\left(\frac{b}{a}\right)}e^{\mp jk_z z}$$

$$\underline{H}=\pm\hat{\underline{\phi}}\frac{V_0}{\eta\rho\ln\left(\frac{b}{a}\right)}e^{\mp jk_z z} \qquad \eta=\sqrt{\frac{\mu}{\varepsilon_c}} \qquad Z_0^{lossless}=\frac{\eta^{lossless}}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$Z_0=\frac{\eta}{2\pi}\ln\left(\frac{b}{a}\right) \qquad P_l(0)=|I|^2\left(\frac{1}{4\pi}\right)\left(\frac{R_{sa}}{a}+\frac{R_{sb}}{b}\right)$$

$$\alpha=\alpha_d+\alpha_c \qquad \alpha_c=\left(\frac{1}{Z_0^{lossless}}\right)\left(\frac{1}{4\pi}\right)\left(\frac{R_{sa}}{a}+\frac{R_{sb}}{b}\right)$$

TEM: $\alpha_d=k''$

$$P_0\approx\frac{1}{2}Z_0^{lossless}|I|^2$$

$$L=Z_0^{lossless}\sqrt{\mu\varepsilon'} \qquad C=\sqrt{\mu\varepsilon'}/Z_0^{lossless}$$

$$G=(\omega C)\tan\delta \qquad R=\alpha_c\left(2Z_0^{lossless}\right)$$

lowest mode is the TE$_{11}$ mode

$$\nabla^2H_{z0}(\rho,\phi)+k_c^2H_{z0}(\rho,\phi)=0$$

$$H_z(\rho,\phi,z)=H_{z0}(\rho,\phi)e^{\mp jk_z z}$$

$$E_\phi(a,\phi)=0 \qquad E_\phi(b,\phi)=0$$

$$E_\phi=\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial\rho} \qquad \frac{\partial H_z}{\partial\rho}=0\Big|_{\rho=a,b}$$

$$k_c\left(AJ_n'(k_c a)+BY_n'(k_c a)\right)=0$$

$$k_c\left(AJ_n'(k_c b)+BY_n'(k_c b)\right)=0$$

$$J_n'(k_c a)Y_n'(k_c b)-J_n'(k_c b)Y_n'(k_c a)=0$$

$$F(x;n,b/a)=J_n'(x)Y_n'(x(b/a))-J_n'(x(b/a))Y_n'(x)=0$$

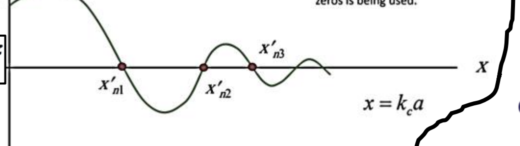
For a given choice of n and a given value of b/a, we can solve the above equation for x to find the zeros.

$$k_c a=\frac{2}{1+b/a} \quad n=1$$

TE$_{11}$ mode: $$f_c\approx\frac{c}{a\sqrt{\varepsilon_r}}\left(\frac{1}{\pi}\right)\left(\frac{1}{1+b/a}\right)$$

$x'_{np}=p^{th}$ zero

Note: These values are not the same as those of the circular waveguide, although the same notation for the zeros is being used.



$x=k_c a$

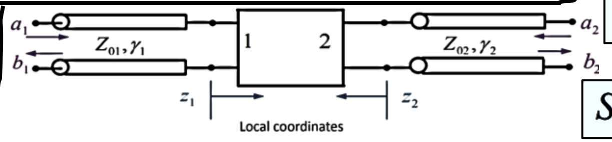
$$x=x'_{np} \Rightarrow k_c a=x'_{np}$$

At the cutoff frequency, the wavelength (in the dielectric) is

$$2(a+b)\approx\frac{\lambda_d}{\pi/2}$$

Coaxial Cable

S-parameters



On each transmission line:

$$V_i(z_i)=V_{i0}^+e^{-\gamma_i z_i}+V_{i0}^-e^{+\gamma_i z_i}=V_i^+(z_i)+V_i^-(z_i)$$

$$I_i(z_i)=\frac{V_i^+(z_i)}{Z_{0i}}-\frac{V_i^-(z_i)}{Z_{0i}} \qquad i=1,2$$

Incoming wave function $\equiv a_i(z_i)\equiv V_i^+(z_i)/\sqrt{Z_{0i}}$

Outgoing wave function $\equiv b_i(z_i)\equiv V_i^-(z_i)/\sqrt{Z_{0i}}$

$$P_i^+(0)=\frac{1}{2}\mathrm{Re}\left[V_i^+(0)I_i^{+*}(0)\right]=\frac{1}{2}\frac{\left|V_i^+(0)\right|^2}{Z_{0i}} \qquad P_i^-(0)=\frac{1}{2}\frac{\left|V_i^-(0)\right|^2}{Z_{0i}}=\frac{1}{2}\left|b_i(0)\right|^2$$

$$a_i(0)=V_i^+(0)/\sqrt{Z_{0i}} \qquad \Rightarrow P_i^+(0)=\frac{1}{2}\left|a_i(0)\right|^2$$

$$V_i^+(-l_i)=V_i^+(0)\,e^{+\gamma_i l_i} \qquad V_i^-(-l_i)=V_i^-(0)\,e^{-\gamma_i l_i}$$

$$V_1^+(0)=a_1(0)\sqrt{Z_0}$$

$$S_{21}=\frac{b_2(0)}{a_1(0)}\bigg|_{a_2=0}=\frac{V_2^-(0)}{V_1^+(0)}\bigg|_{a_2=0}$$

$a_2=0\Rightarrow V_2^-(0)=V_2(0)$

$$V_2(0)=V_1(0)\left(\frac{Z_0}{Z+Z_0}\right)$$

$$V_1(0)=a_1\sqrt{Z_0}\left(1+S_{11}\right)$$

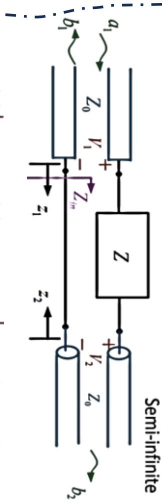
$$\Rightarrow V_2^-(0)=V_2(0)=a_1\sqrt{Z_0}\left(1+S_{11}\right)\left(\frac{Z_0}{Z+Z_0}\right)$$

$$S_{21}=\frac{2Z_0}{Z+2Z_0} \qquad S_{12}=S_{21}$$

$$S_{11}=\frac{Z}{Z+2Z_0}$$

$$S_{11}=\frac{b_1(0)}{a_1(0)}\bigg|_{a_2=0}=\frac{V_1^-(0)}{V_1^+(0)}=\frac{Z_{in}-Z_0}{Z_{in}+Z_0}\bigg|_{a_2=0}=\frac{(Z+Z_0)-Z_0}{(Z+Z_0)+Z_0}$$

By symmetry: $S_{22}=S_{11}$



$[S]=[S]^T$ reciprocal

$[S]^T[S]^*=[S]^*[S]^T=[U]$

$[S]^{T*}=[S]^{-1}$ lossless

$$[S]=\begin{bmatrix}S_{11}&S_{12}&S_{13}\\S_{21}&S_{22}&S_{23}\\S_{31}&S_{32}&S_{33}\end{bmatrix}$$

$$\begin{bmatrix}b_1(0)\\b_2(0)\end{bmatrix}=\begin{bmatrix}S_{11}&S_{12}\\S_{21}&S_{22}\end{bmatrix}\begin{bmatrix}a_1(0)\\a_2(0)\end{bmatrix}$$

Unitary

$S_{11}S_{11}^*+S_{21}S_{21}^*+S_{31}S_{31}^*=1$

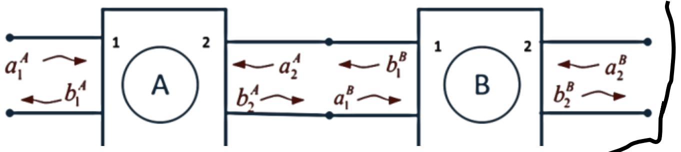
$S_{12}S_{12}^*+S_{22}S_{22}^*+S_{32}S_{32}^*=1$

$S_{13}S_{13}^*+S_{23}S_{23}^*+S_{33}S_{33}^*=1$

$S_{11}S_{12}^*+S_{21}S_{22}^*+S_{31}S_{32}^*=0$

$S_{11}S_{13}^*+S_{21}S_{23}^*+S_{31}S_{33}^*=0$

$S_{12}S_{13}^*+S_{22}S_{23}^*+S_{32}S_{33}^*=0$



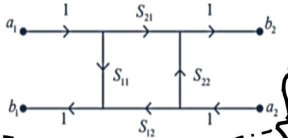
$$\begin{bmatrix}a_1\\b_1\end{bmatrix}=\begin{bmatrix}T_{11}&T_{12}\\T_{21}&T_{22}\end{bmatrix}\begin{bmatrix}b_2\\a_2\end{bmatrix}=[T]\begin{bmatrix}b_2\\a_2\end{bmatrix}$$

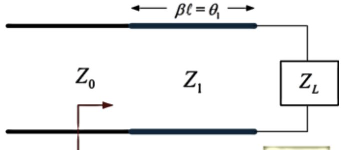
$$[T]=\begin{bmatrix}\frac{1}{S_{21}}&\frac{-S_{22}}{S_{21}}\\\frac{S_{11}}{S_{21}}&S_{12}-\frac{S_{11}S_{22}}{S_{21}}\end{bmatrix}$$

$$[S]=\begin{bmatrix}\frac{-T_{21}}{T_{22}}&\frac{1}{T_{22}}\\T_{11}-\frac{T_{12}^2}{T_{22}}&\frac{T_{12}}{T_{22}}\end{bmatrix}$$

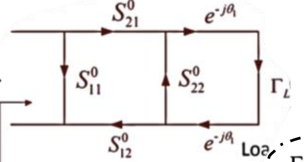
$b_1=S_{11}a_1+S_{12}a_2$

$b_2=S_{21}a_1+S_{22}a_2$





Z_L is real



$S_{21}^0S_{12}^0=\left(1-S_{11}^{0\,2}\right)$

Step Impedance change

$S_{11}^0=-S_{22}^0=\dfrac{Z_1-Z_0}{Z_1+Z_0}$

$S_{21}^0=S_{12}^0=\sqrt{\dfrac{Z_0}{Z_1}}\left(1+S_{11}^0\right)=\dfrac{2\sqrt{Z_0Z_1}}{Z_1+Z_0}$ $\Rightarrow$ $\Gamma=S_{11}^0+\dfrac{S_{21}^0S_{12}^0\,\Gamma_L\,e^{-j2\theta_1}}{1-S_{22}^0\,\Gamma_L\,e^{-j2\theta_1}}$

$\Gamma=\dfrac{Z_{in}-Z_0}{Z_{in}+Z_0}$ $\qquad$ $\Gamma_L=\dfrac{Z_L-Z_1}{Z_L+Z_1}$

$\Gamma=\dfrac{S_{11}^0+\Gamma_L\,e^{-j2\theta_1}}{1+S_{11}^0\,\Gamma_L\,e^{-j2\theta_1}}$ $\quad$ $\left|S_{11}^0\,\Gamma_L\right|<<1$ $\Rightarrow$ $\Gamma\approx S_{11}^0+\Gamma_L\,e^{-j2\theta_1}$

$\Gamma_0=\dfrac{Z_1-Z_0}{Z_1+Z_0}$ $\quad$ $\Gamma_1=\dfrac{Z_L-Z_1}{Z_L+Z_1}$

assume symmetric reflections

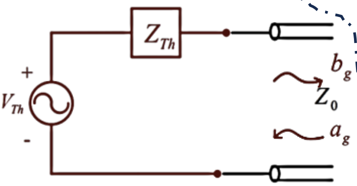
$\Gamma_0=\Gamma_N,\ \Gamma_1=\Gamma_{N-1},\ \Gamma_2=\Gamma_{N-2},\ \ldots$

$\Rightarrow\Gamma(\theta)\approx e^{-jN\theta}\left[\Gamma_0\left(e^{jN\theta}+e^{-jN\theta}\right)+\Gamma_1\left(e^{j(N-2)\theta}+e^{-j(N-2)\theta}\right)+\ldots\right]$

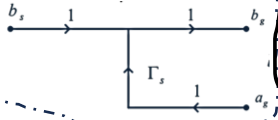
N odd $\rightarrow$ last term $=\Gamma_{\frac{N-1}{2}}\left(e^{j\theta}+e^{-j\theta}\right)$

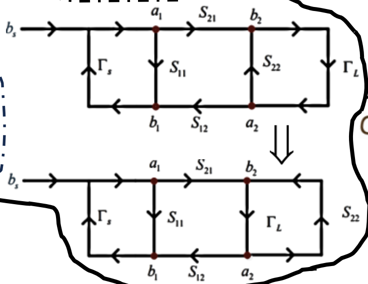
N even $\rightarrow$ last term $=\Gamma_{\frac{N}{2}}$

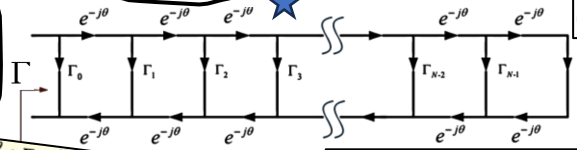
$$\Gamma(\theta)\approx\begin{cases}2e^{-jN\theta}\left[\Gamma_0\cos(N\theta)+\Gamma_1\cos((N-2)\theta)+\ldots+\Gamma_n\cos((N-2n)\theta)+\ldots+\frac{1}{2}\Gamma_{\frac{N}{2}}\right]; & N\text{ even}\\2e^{-jN\theta}\left[\Gamma_0\cos(N\theta)+\Gamma_1\cos((N-2)\theta)+\ldots+\Gamma_n\cos((N-2n)\theta)+\ldots+\Gamma_{\frac{N-1}{2}}\cos\theta\right]; & N\text{ odd}\end{cases}$$



$b_g=a_g\Gamma_s+V_{Th}\left(\dfrac{Z_0}{Z_{Th}+Z_0}\right)\dfrac{1}{\sqrt{Z_0}}$







$\theta_i=\beta_i\ell_i$

Binomial Multistage Transformer

$\left|\Gamma(\theta)\right|=|A|\ 2^N\left|\cos\theta\right|^N$

$\Gamma(\theta)=A\left(1+e^{-j2\theta}\right)^N=Ae^{-jN\theta}\left(e^{j\theta}+e^{-j\theta}\right)^N=A\,e^{-jN\theta}\,2^N\cos^N\theta$

$\theta=\beta\ell=\left(\dfrac{f}{f_0}\right)\dfrac{\pi}{2}$ $\quad$ $C_n^N=\dfrac{N!}{(N-n)!\ n!}$

$\Gamma(\theta)\approx\Gamma_0+\Gamma_1e^{-j2\theta}+\Gamma_2e^{-j4\theta}+\Gamma_3e^{-j6\theta}+\ldots+\Gamma_Ne^{-j2N\theta}$

$\Gamma_n=\dfrac{Z_{n+1}-Z_n}{Z_{n+1}+Z_n}$

Assume $\theta_1=\theta_2=\theta_3\cdots=\theta_N=\theta$

$\Gamma_n=\Gamma_{N-n}$

$f=f_0\ \Rightarrow\ \theta_i=\theta=\beta\ell=\dfrac{\pi}{2}\ \Rightarrow\ \left|\Gamma\left(\dfrac{\pi}{2}\right)\right|=0$

$\Gamma(\theta)=A\left(1+e^{-j2\theta}\right)^N=A\sum_{n=0}^{N}C_n^N\,e^{-j2n\theta}$

$A=2^{-N}\left(\dfrac{Z_L-Z_0}{Z_L+Z_0}\right)$ $\quad$ $\ln(Z_{n+1})\approx2^{-N}C_n^N\ \ln\left(\dfrac{Z_L}{Z_0}\right)+\ln(Z_n)$ $\quad$ $\dfrac{\Delta f}{f_0}=2-\dfrac{4}{\pi}\cos^{-1}\left(\dfrac{1}{2}\left(\dfrac{\Gamma_m}{|A|}\right)^{\frac{1}{N}}\right)$

Chebyshev Multistage Transformer

$T_n(x)\equiv\begin{cases}\cos\left(n\cos^{-1}x\right), & |x|\le1\\\cosh\left(n\cosh^{-1}x\right), & |x|\ge1\end{cases}$

$T_1(x)=x$

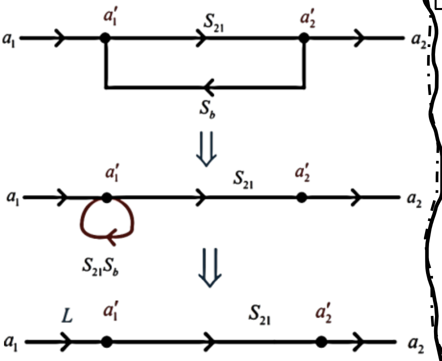
$T_2(x)=2x^2-1$

$T_3(x)=4x^3-3x$

$\Gamma(\theta)=Ae^{-jN\theta}T_N\left(\sec\theta_m\cos\theta\right)$

$=2e^{-jN\theta}\left[\Gamma_0\cos N\theta+\Gamma_1\cos(N-2)\theta+\ldots+\Gamma_n\cos(N-2n)\theta+\ldots\right]$

3) Self-loop



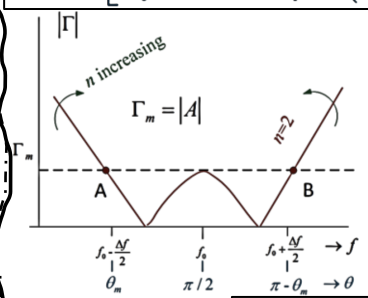
$L=\dfrac{1}{1-S_{21}S_b}$

$a_1'=a_1+a_2'S_b$

$a_2'=a_1'S_{21}$

$a_1'=\left(\dfrac{1}{1-S_{21}S_b}\right)a_1$

$a_1'=a_1+a_1'S_{21}S_b$



As $f\rightarrow0$ $(\beta\ell=\theta\rightarrow0)$

$\Gamma(0)=AT_N(\sec\theta_m)=\dfrac{Z_L-Z_0}{Z_L+Z_0}$

$A=\left(\dfrac{Z_L-Z_0}{Z_L+Z_0}\right)\dfrac{1}{T_N(\sec\theta_m)}$

$A=\mathrm{sgn}(Z_L-Z_0)\Gamma_m$

$BW=\dfrac{\Delta f}{f_0}=2-\dfrac{4\theta_m}{\pi}$ $\quad$ $\sec\theta_m\approx\cosh\left[\dfrac{1}{N}\cosh^{-1}\left(\dfrac{1}{2\Gamma_m}\left|\ln\left(\dfrac{Z_L}{Z_0}\right)\right|\right)\right]$

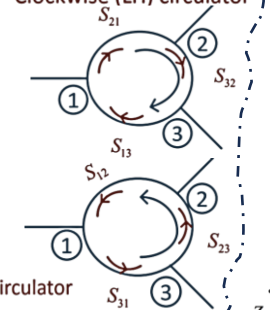
$Z_{n+1}=Z_n\dfrac{(1+\Gamma_n)}{(1-\Gamma_n)}$

"Circulator"

Clockwise (LH) circulator

$$[S]=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

$$[S]=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$$



Counter clockwise (RH) circulator

Power Dividers and Directional Couplers

lossless matched non-reciprocal

$$[S]=\begin{pmatrix}0&S_{12}&S_{13}\\S_{21}&0&S_{23}\\S_{31}&S_{32}&0\end{pmatrix}$$

(There are six distinct values.)

$|S_{21}|^2+|S_{31}|^2=1$

$|S_{12}|^2+|S_{32}|^2=1$

$|S_{13}|^2+|S_{23}|^2=1$

1. $S_{12}=S_{23}=S_{31}=0$, $|S_{21}|=|S_{32}|=|S_{13}|=1$

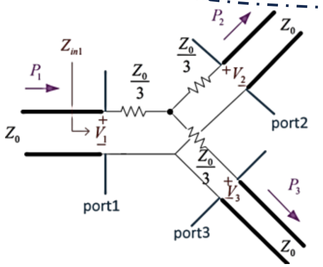
or $\qquad$ Note

2. $S_{21}=S_{32}=S_{13}=0$, $|S_{12}|=|S_{23}|=|S_{31}|=1$

$S_{31}^*S_{32}=0$

$S_{21}^*S_{23}=0$

$S_{12}^*S_{13}=0$



$S_{21}=\left.\dfrac{V_2^-/\sqrt{Z_0}}{V_1^+/\sqrt{Z_0}}\right|_{a_2=a_3=0}$

$V_1=V_1^+(1+S_{11})=V_1^+$

$V_2^-=V_2=V_1\left(\dfrac{\frac{2}{3}Z_0}{\frac{Z_0}{3}+\frac{2}{3}Z_0}\right)\left(\dfrac{Z_0}{\frac{Z_0}{3}+Z_0}\right)$

$V_1^+\left(\dfrac{2}{3}\right)\left(\dfrac{3}{4}\right)=\dfrac{1}{2}V_1^+$ $\quad$ $S_{21}=\dfrac{1}{2}=S_{12}=S_{31}=S_{13}=S_{32}=S_{23}$

$$[S]=\frac{1}{2}\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}$$

$P_1=P_{in}=\dfrac{1}{2}\dfrac{|V_1|^2}{Z_0}=\dfrac{1}{2}|a_1|^2$

$P_2=P_3=\dfrac{1}{2}|b_2|^2=\dfrac{1}{2}|a_1S_{21}|^2=P_1|S_{21}|^2=P_1\left|\dfrac{1}{2}\right|^2=\dfrac{P_{in}}{4}$

1/2 P_{in} is dissipated by resistors

$S_{11}=0$

- The input port is matched, but not the output ports.
- The output ports are not isolated.

$S_{22}=\dfrac{Z_{01}\|Z_{03}-Z_{02}}{Z_{01}\|Z_{03}+Z_{02}}$

$S_{33}=\dfrac{Z_{01}\|Z_{02}-Z_{03}}{Z_{01}\|Z_{02}+Z_{03}}$

$S_{21}=S_{12}=\sqrt{\dfrac{Z_{03}}{Z_{02}+Z_{03}}}$

$S_{13}=S_{31}=\sqrt{\dfrac{Z_{02}}{Z_{02}+Z_{03}}}$

$S_{32}=S_{23}=(1+S_{22})\sqrt{\dfrac{Z_{02}}{Z_{03}}}$

$Z_{in1}=\dfrac{Z_0}{3}+\dfrac{4Z_0}{3}\Big\|\dfrac{4Z_0}{3}=\dfrac{Z_0}{3}+\dfrac{2Z_0}{3}=Z_0$

Same for Z_{in1} and Z_{in2}

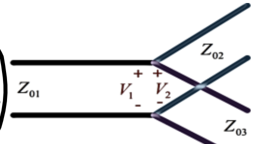
$S_{11}=S_{22}=S_{33}=0$

$S_{21}=\left.\dfrac{V_2^-/\sqrt{Z_{02}}}{V_1^+/\sqrt{Z_{01}}}\right|_{a_2=a_3=0}$

$V_1=V_1^+(1+S_{11})$; $V_2^-=V_2=V_1$ $\Rightarrow$ $V_2^-/V_1^+=V_1/V_1^+=1+S_{11}$

$\Rightarrow S_{21}=(1+S_{11})\sqrt{\dfrac{Z_{01}}{Z_{02}}}=S_{12}$

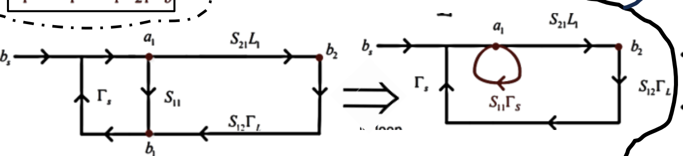
T-Junction: lossless divider



$P_{in_1}=\dfrac{1}{2}\dfrac{|V_1|^2}{Z_{01}}$

$Z_{01}=\dfrac{Z_{02}Z_{03}}{Z_{02}+Z_{03}}$

$\dfrac{P_3}{P_2}=\dfrac{Z_{02}}{Z_{03}}$

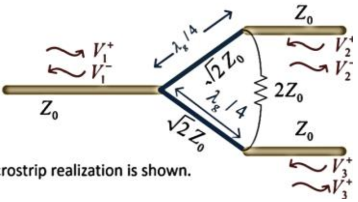


Wilkenson Power Divider

Equal-split (3 dB) power divider

$\boxed{S_{11}=S_{22}=S_{33}=0}$

$\boxed{S_{23}=S_{32}=0}$

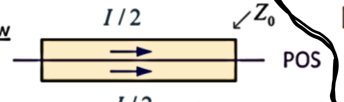


Microstrip realization is shown.

$$|S_{21}|^2=|S_{31}|^2=\frac{1}{2}$$

$$[S]=\frac{-j}{\sqrt{2}}\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix}$$

top view



Voltage is the same for each half of line (V)

Current is halved for each half of line ($I/2$)

$$\Rightarrow Z_0^h=\frac{V}{I/2}=2Z_0$$

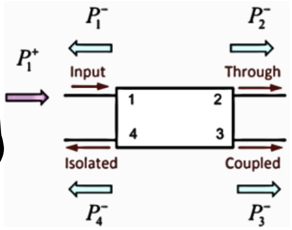
- Even and odd analysis is required to analyze structure when port 2 is excited.

  $\Rightarrow$ To determine S_{22}, S_{32}

- Only even analysis is needed to analyze structure when port 1 is excited.

  $\Rightarrow$ To determine S_{11}, S_{21}

Directional Couplers



$$[S]=\begin{bmatrix}0&S_{12}&S_{13}&0\\S_{12}&0&0&S_{24}\\S_{13}&0&0&S_{34}\\0&S_{24}&S_{34}&0\end{bmatrix}$$

$\boxed{|S_{12}|^2+|S_{13}|^2=1}$

$\boxed{S_{12}S_{13}^*+S_{24}S_{34}^*=0}$

$\boxed{|S_{13}|=|S_{24}|}$

$\boxed{|S_{12}|=|S_{34}|}$

Note: $I^{dB}=D^{dB}+C^{dB}$

Coupling $C^{dB}=10\log\left(\frac{P_1^+}{P_3^-}\right)=20\log\left|\frac{1}{S_{31}}\right|=-20\log|\beta|$

Isolation $I^{dB}=10\log\left(\frac{P_1^+}{P_4^-}\right)=20\log\left|\frac{1}{S_{41}}\right|$

Ideally $+\infty$ dB

Return loss $RL^{dB}=-10\log\left(\frac{P_1^-}{P_1^+}\right)=-20\log|S_{11}|$

Ideally $+\infty$ dB

Directivity $D^{dB}=10\log\left(\frac{P_3^-}{P_4^-}\right)=20\log\left|\frac{S_{31}}{S_{41}}\right|=20\log\left|\frac{\beta}{S_{41}}\right|$

Ideally $+\infty$ dB

$$S_{11}=\frac{V_1^{-e}+V_1^{-o}}{V^++V^+}=\frac{V_1^{-e}+V_1^{-o}}{2V^+}=\frac{1}{2}\left(\frac{V_1^{-e}}{V^+}+\frac{V_1^{-o}}{V^+}\right)=\frac{1}{2}(S_{11}^e+S_{11}^o)=0+0$$

By symmetry: $S_{11}=S_{22}=S_{33}=S_{44}=0$

$$S_{21}=\frac{V_2^{-e}+V_2^{-o}}{V^++V^+}=\frac{V_2^{-e}+V_2^{-o}}{2V^+}=\frac{1}{2}(S_{21}^e+S_{21}^o)=\frac{1}{2}\left[\left(\frac{-1-j}{\sqrt{2}}\right)+\left(\frac{1-j}{\sqrt{2}}\right)\right]=\frac{-j}{\sqrt{2}}$$

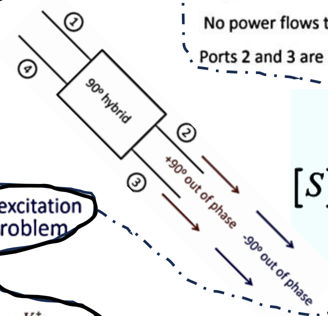
$$S_{21}=S_{12}=S_{43}=S_{34}=\frac{-j}{\sqrt{2}}$$

$$[S]_e=\begin{bmatrix}0&\frac{\mp1-j}{\sqrt{2}}\\\frac{\mp1-j}{\sqrt{2}}&0\end{bmatrix}$$

All ports are matched.

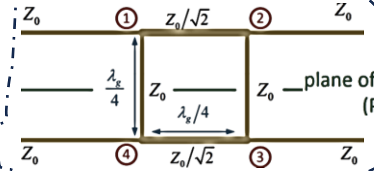
No power flows to port 4 from in the input port 1 (i.e., port 4 is "isolated").

Ports 2 and 3 are isolated (power incident on port 2 does not get to port 3 and vice versa).

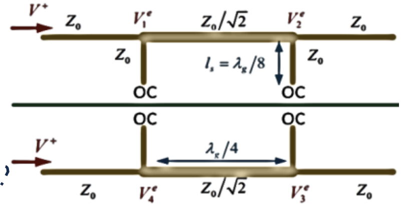


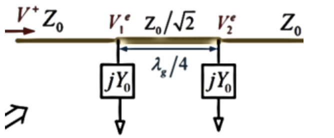
$$[S]=\frac{-1}{\sqrt{2}}\begin{bmatrix}0&j&1&0\\j&0&0&1\\1&0&0&j\\0&1&j&0\end{bmatrix}$$

Quadrature Hybrid Coupler (Branch-line coupler)



Even Analysis



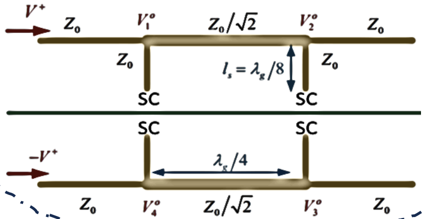


$V_3^e=V_2^e$ $\quad Y_s=jY_0\tan(\beta_l l_s)$

$V_4^e=V_1^e$ $\quad =jY_0\tan(\pi/4)$

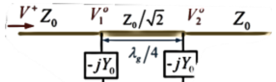
$=jY_0$

Odd Analysis



$V_3^o=-V_2^o$

$V_4^o=-V_1^o$

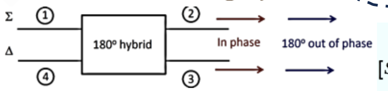


$$Y_s=-jY_0\cot(\beta_l l_s)$$

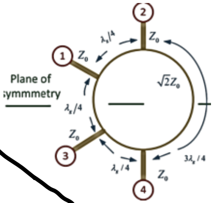
$$=-jY_0\cot(\pi/4)$$

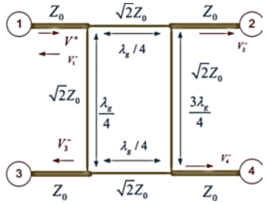
$$=-jY_0$$

Rat-Race Ring Hybrid

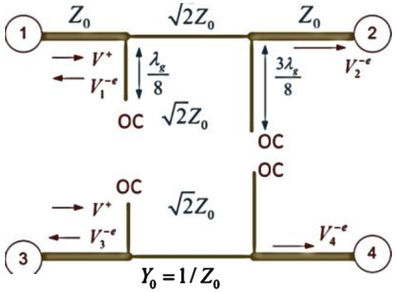


$$[S]=\frac{-j}{\sqrt{2}}\begin{bmatrix}0&1&1&0\\1&0&0&-1\\1&0&0&1\\0&-1&1&0\end{bmatrix}$$



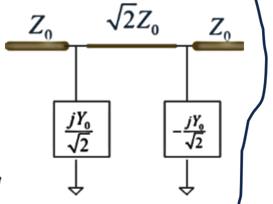


Even Analysis

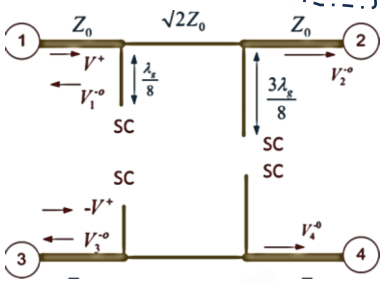


$Y_0=1/Z_0$

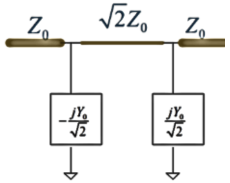
$Y_{0s}=Y_0/\sqrt{2}$



$$Y_{s1}=jY_{0s1}\tan(\beta_s l_{s1})$$
$$=j(Y_0/\sqrt{2})\tan(\pi/4)$$
$$=jY_0/\sqrt{2}$$
$$Y_{s2}=jY_{0s2}\tan(\beta_s l_{s2})$$
$$=j(Y_0/\sqrt{2})\tan(3\pi/4)$$
$$=-jY_0/\sqrt{2}$$

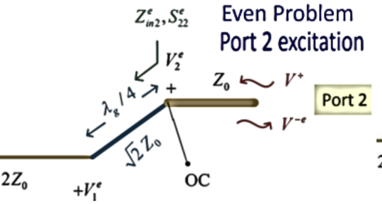


$$Y_{s1}=-jY_{0s1}\cot(\beta_s l_s)$$
$$=-j(Y_0/\sqrt{2})\cot(\pi/4)$$
$$=-jY_0/\sqrt{2}$$
$$Y_{s1}=-jY_{0s2}\cot(\beta_s l_s)$$
$$=-j(Y_0/\sqrt{2})\cot(3\pi/4)$$
$$=jY_0/\sqrt{2}$$



Even Problem

Port 2 excitation



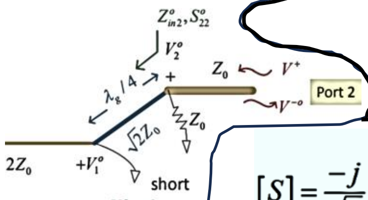
$$Z_{in2}^e=\frac{(\sqrt{2}Z_0)^2}{2Z_0}=Z_0$$

$$\Rightarrow S_{22}^e=\frac{Z_{in2}^e-Z_0}{Z_{in2}^e+Z_0}=0$$

$\boxed{\text{by symmetry, } S_{33}^e=0}$

Port 2 excitation

Odd Problem

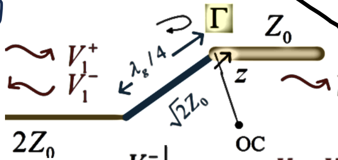


$$Z_{in2}^o=\infty\|Z_0=Z_0$$

$$\Rightarrow S_{22}^o=\frac{Z_{in2}^o-Z_0}{Z_{in2}^o+Z_0}=0$$

$\boxed{\text{by symmetry, } S_{33}^o=0}$

$$[S]=\frac{-j}{\sqrt{2}}\begin{pmatrix}0&1&1\\1&0&0\\1&0&0\end{pmatrix}$$



$$S_{21}=\left.\frac{V_2^-}{V_1^+}\right|_{a_2=a_3=0}$$

$$V_1=V_1^+(1+\cancel{S_{11}})=V_1^+$$

$$V_2=V_2^-+\cancel{V_2^+}$$

$$S_{22}=\left.\frac{V_2^-}{V_2^+}\right|_{a_1=a_3=0}\quad S_{22}=\frac{V^{-e}+V^{-o}}{V^++V^+}=\frac{V^{-e}+V^{-o}}{2V^+}=\frac{1}{2}(S_{22}^e+S_{22}^o)=\frac{1}{2}(0+0)=0$$

$$S_{32}=\left.\frac{V_3^-}{V_2^+}\right|_{a_1=a_3=0}\quad S_{32}=\frac{V^{-e}-V^{-o}}{V^++V^+}=\frac{V^{-e}-V^{-o}}{2V^+}=\frac{1}{2}(S_{22}^e-S_{22}^o)=\frac{1}{2}(0-0)=0$$

$$S_{21}=\frac{V_2^-}{V_1^+}=\frac{V_2}{V_1}=-j\frac{(1+\Gamma)}{(1-\Gamma)}=-j\frac{2}{2\sqrt{2}}$$

$\boxed{S_{23}=0\ \text{(by reciprocity)}}$

Note: Since all ports have the same Z_0, we ignore the normalizing factor $\sqrt{Z_0}$ in the S parameter definition.

$$S_{21}=\frac{-j}{\sqrt{2}}=S_{12}$$

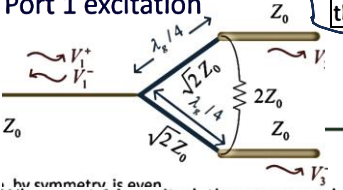
Along $\lambda_g/4$ wave transformer:

$$V(z)=V_0^+e^{-j\beta z}\left(1+\Gamma e^{+j2\beta z}\right)$$

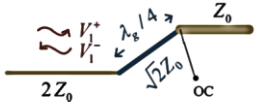
z = distance from port 2

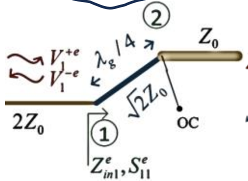
$$\begin{cases}V_2=V(0)=V_0^+(1+\Gamma)\\V_1=V(-\lambda_g/4)=V_0^+j(1-\Gamma)\end{cases}\qquad \Gamma=\frac{Z_0-\sqrt{2}Z_0}{Z_0+\sqrt{2}Z_0}=\frac{1-\sqrt{2}}{1+\sqrt{2}}$$

Port 1 excitation



O.C. symmetry plane





$$Z_{in1}^e=\frac{(\sqrt{2}Z_0)^2}{Z_0}=2Z_0$$

$$S_{11}^e=\frac{Z_{in1}^e-2Z_0}{Z_{in1}^e+2Z_0}=0$$

$$S_{11}=\left.\frac{V_1^-}{V_1^+}\right|_{a_2=a_3=0}=\left.\frac{V_1^{-e}}{V_1^{+e}}\right|_{a_2=0}=S_{11}^e=0$$

When port 1 is excited, the response, by symmetry, is even.

(Hence, the total fields are the same as the even fields.)

Rat-Race Ring Hybrid (cont.)

$$[ABCD]_{\substack{e\\0}} = \begin{bmatrix} 1 & 0 \\ \frac{\pm jY_0}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} 0 & j\sqrt{2}Z_0 \\ j\frac{1}{\sqrt{2}Z_0} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mp\frac{jY_0}{\sqrt{2}} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ \frac{\pm jY_0}{\sqrt{2}} & 1 \end{bmatrix} \begin{bmatrix} \pm 1 & j\sqrt{2}Z_0 \\ j\frac{1}{\sqrt{2}Z_0} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \pm 1 & j\sqrt{2}Z_0 \\ j\frac{\sqrt{2}}{Z_0} & \mp 1 \end{bmatrix}$$

$$[ABCD]_Y = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}$$

Shunt load on line

$$[ABCD]_{\frac{\lambda}{4}} = \begin{bmatrix} 0 & \frac{jZ_0}{\sqrt{2}} \\ \frac{j\sqrt{2}}{Z_0} & 0 \end{bmatrix}$$

Quarter-wave line

In general:

$$[ABCD^{line}] = \begin{bmatrix} \cos(\beta\ell) & jZ_0^{line}\sin(\beta\ell) \\ (j/Z_0^{line})\sin(\beta\ell) & \cos(\beta\ell) \end{bmatrix}$$

Here: $Z_0^{line} = Z_0/\sqrt{2}$

$\beta\ell = \pi/2$

$$[ABCD]_{\substack{e\\0}} = \begin{bmatrix} \pm 1 & j\sqrt{2}Z_0 \\ j\frac{\sqrt{2}}{Z_0} & \mp 1 \end{bmatrix} \qquad [S]_{\substack{e\\0}} = \frac{-j}{\sqrt{2}} \begin{bmatrix} \pm 1 & 1 \\ 1 & \mp 1 \end{bmatrix}$$

$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{a_2=a_3=a_4=0} \qquad S_{11} = \frac{V_1^{-e}+V_1^{-o}}{2V^+} = \frac{1}{2}\left(S_{11}^e + S_{11}^o\right) = \frac{1}{2}\left(\frac{-j}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) = 0$$

$S_{11} = S_{33} = 0$
(symmetry)

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{a_2=a_3=a_4=0} \qquad S_{21} = \frac{V_2^{-e}+V_2^{-o}}{2V^+} = \frac{1}{2}\left(S_{21}^e + S_{21}^o\right) = \frac{1}{2}\left(\frac{-j}{\sqrt{2}} + \frac{-j}{\sqrt{2}}\right) = \frac{-j}{\sqrt{2}}$$

$S_{21} = S_{12} = S_{34} = S_{43} = {}^{-j}/_{\sqrt{2}}$
(symmetry and reciprocity)

$$S_{31} = \left.\frac{V_3^-}{V_1^+}\right|_{a_2=a_3=a_4=0} \qquad S_{31} = \frac{V_3^{-e}+V_3^{-o}}{2V^+} = \frac{1}{2}\left(S_{11}^e - S_{11}^o\right) = \frac{1}{2}\left(\frac{-j}{\sqrt{2}} - \frac{j}{\sqrt{2}}\right) = \frac{-j}{\sqrt{2}}$$

$S_{31} = S_{13} = -\frac{j}{\sqrt{2}}$
(symmetry)

Similarly, exciting port 2, and using symmetry and reciprocity, we have the following results:

$$S_{22} = S_{44} = 0$$
$$S_{23} = S_{32} = S_{14} = S_{41} = 0$$
$$S_{24} = S_{42} = {}^{j}/_{\sqrt{2}}$$

TABLE 5.1 Binomial Transformer Design

	$N=2$		$N=3$			$N=4$			
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1067	1.3554	1.0520	1.2247	1.4259	1.0257	1.1351	1.3215	1.4624
2.0	1.1892	1.6818	1.0907	1.4142	1.8337	1.0444	1.2421	1.6102	1.9150
3.0	1.3161	2.2795	1.1479	1.7321	2.6135	1.0718	1.4105	2.1269	2.7990
4.0	1.4142	2.8285	1.1907	2.0000	3.3594	1.0919	1.5442	2.5903	3.6633
6.0	1.5651	3.8336	1.2544	2.4495	4.7832	1.1215	1.7553	3.4182	5.3500
8.0	1.6818	4.7568	1.3022	2.8284	6.1434	1.1436	1.9232	4.1597	6.9955
10.0	1.7783	5.6233	1.3409	3.1623	7.4577	1.1613	2.0651	4.8424	8.6110

	$N=5$					$N=6$					
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_5/Z_0	Z_6/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0128	1.0790	1.2247	1.3902	1.4810	1.0064	1.0454	1.1496	1.3048	1.4349	1.4905
2.0	1.0220	1.1391	1.4142	1.7558	1.9569	1.0110	1.0790	1.2693	1.5757	1.8536	1.9782
3.0	1.0354	1.2300	1.7321	2.4390	2.8974	1.0176	1.1288	1.4599	2.0549	2.6577	2.9481
4.0	1.0452	1.2995	2.0000	3.0781	3.8270	1.0225	1.1661	1.6129	2.4800	3.4302	3.9120
6.0	1.0596	1.4055	2.4495	4.2689	5.6625	1.0296	1.2219	1.8573	3.2305	4.9104	5.8275
8.0	1.0703	1.4870	2.8284	5.3800	7.4745	1.0349	1.2640	2.0539	3.8950	6.3291	7.7302
10.0	1.0789	1.5541	3.1623	6.4346	9.2687	1.0392	1.2982	2.2215	4.5015	7.7030	9.6228

TABLE 4.2 Conversions Between Two-Port Network Parameters

	S	Z	Y	$ABCD$
S_{11}	S_{11}	$\frac{(Z_{11}-Z_0)(Z_{22}+Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0-Y_{11})(Y_0+Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{A+B/Z_0-CZ_0-D}{A+B/Z_0+CZ_0+D}$
S_{12}	S_{12}	$\frac{2Z_{12}Z_0}{\Delta Z}$	$\frac{-2Y_{12}Y_0}{\Delta Y}$	$\frac{2(AD-BC)}{A+B/Z_0+CZ_0+D}$
S_{21}	S_{21}	$\frac{2Z_{21}Z_0}{\Delta Z}$	$\frac{-2Y_{21}Y_0}{\Delta Y}$	$\frac{2}{A+B/Z_0+CZ_0+D}$
S_{22}	S_{22}	$\frac{(Z_{11}+Z_0)(Z_{22}-Z_0)-Z_{12}Z_{21}}{\Delta Z}$	$\frac{(Y_0+Y_{11})(Y_0-Y_{22})+Y_{12}Y_{21}}{\Delta Y}$	$\frac{-A+B/Z_0-CZ_0+D}{A+B/Z_0+CZ_0+D}$
Z_{11}	$Z_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{11}	$\frac{Y_{22}}{\|Y\|}$	$\frac{A}{C}$
Z_{12}	$Z_0\frac{2S_{12}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{12}	$\frac{-Y_{12}}{\|Y\|}$	$\frac{AD-BC}{C}$
Z_{21}	$Z_0\frac{2S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{21}	$\frac{-Y_{21}}{\|Y\|}$	$\frac{1}{C}$
Z_{22}	$Z_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1-S_{11})(1-S_{22})-S_{12}S_{21}}$	Z_{22}	$\frac{Y_{11}}{\|Y\|}$	$\frac{D}{C}$
Y_{11}	$Y_0\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{22}}{\|Z\|}$	Y_{11}	$\frac{D}{B}$
Y_{12}	$Y_0\frac{-2S_{12}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{12}}{\|Z\|}$	Y_{12}	$\frac{BC-AD}{B}$
Y_{21}	$Y_0\frac{-2S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{-Z_{21}}{\|Z\|}$	Y_{21}	$\frac{-1}{B}$
Y_{22}	$Y_0\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{(1+S_{11})(1+S_{22})-S_{12}S_{21}}$	$\frac{Z_{11}}{\|Z\|}$	Y_{22}	$\frac{A}{B}$
A	$\frac{(1+S_{11})(1-S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{11}}{Z_{21}}$	$\frac{-Y_{22}}{Y_{21}}$	A
B	$Z_0\frac{(1+S_{11})(1+S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{\|Z\|}{Z_{21}}$	$\frac{-1}{Y_{21}}$	B
C	$\frac{1}{Z_0}\frac{(1-S_{11})(1-S_{22})-S_{12}S_{21}}{2S_{21}}$	$\frac{1}{Z_{21}}$	$\frac{-\|Y\|}{Y_{21}}$	C
D	$\frac{(1-S_{11})(1+S_{22})+S_{12}S_{21}}{2S_{21}}$	$\frac{Z_{22}}{Z_{21}}$	$\frac{-Y_{11}}{Y_{21}}$	D

$|Z| = Z_{11}Z_{22} - Z_{12}Z_{21}$; $|Y| = Y_{11}Y_{22} - Y_{12}Y_{21}$; $\Delta Y = (Y_{11}+Y_0)(Y_{22}+Y_0) - Y_{12}Y_{21}$; $\Delta Z = (Z_{11}+Z_0)(Z_{22}+Z_0) - Z_{12}Z_{21}$; $Y_0 = 1/Z_0$

TABLE 5.2 Chebyshev Transformer Design

	$N=2$				$N=3$					
	$\Gamma_m=0.05$		$\Gamma_m=0.20$		$\Gamma_m=0.05$			$\Gamma_m=0.20$		
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.1347	1.3219	1.2247	1.2247	1.1029	1.2247	1.3601	1.2247	1.2247	1.2247
2.0	1.2193	1.6402	1.3161	1.5197	1.1475	1.4142	1.7429	1.2855	1.4142	1.5558
3.0	1.3494	2.2232	1.4565	2.0598	1.2171	1.7321	2.4649	1.3743	1.7321	2.1829
4.0	1.4500	2.7585	1.5651	2.5558	1.2662	2.0000	3.1591	1.4333	2.0000	2.7908
6.0	1.6047	3.7389	1.7321	3.4641	1.3383	2.4495	4.4833	1.5193	2.4495	3.9492
8.0	1.7244	4.6393	1.8612	4.2983	1.3944	2.8284	5.7372	1.5766	2.8284	5.0742
10.0	1.8233	5.4845	1.9680	5.0813	1.4385	3.1623	6.9517	1.6415	3.1623	6.0920

	$N=4$							
	$\Gamma_m=0.05$				$\Gamma_m=0.20$			
Z_L/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0	Z_1/Z_0	Z_2/Z_0	Z_3/Z_0	Z_4/Z_0
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5	1.0892	1.1742	1.2775	1.3772	1.2247	1.2247	1.2247	1.2247
2.0	1.1201	1.2979	1.5409	1.7855	1.2727	1.3634	1.4669	1.5715
3.0	1.1586	1.4876	2.0167	2.5893	1.4879	1.5819	1.8965	2.0163
4.0	1.1906	1.6414	2.4369	3.3597	1.3692	1.7490	2.2870	2.9214
6.0	1.2290	1.8773	3.1961	4.8820	1.4415	2.0231	2.9657	4.1623
8.0	1.2583	2.0657	3.8728	6.3578	1.4914	2.2428	3.5670	5.3641
10.0	1.2832	2.2268	4.4907	7.7930	1.5163	2.4210	4.1305	6.5950