EXERCISE 1 — SOLUTION

1. Affine Transformations

(a) Write down a general translation matrix for 3D points. Explain the individual entries.

Solution

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad t_x, t_y, t_z \;:\; \text{translation in } x, y, \text{ and } z, \text{ respectively.}$$

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Write down the general rotation matrices (one for each rotation axis) for 3D points and vectors. Explain the individual entries.

Solution

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_x & -\sin\phi_x & 0 \\ 0 & \sin\phi_x & \cos\phi_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{R}_y = \begin{bmatrix} \cos\phi_y & 0 & \sin\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi_y & 0 & \cos\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

$$\mathbf{R}_z = \begin{bmatrix} \cos\phi_z & -\sin\phi_z & 0 & 0 \\ \sin\phi_z & \cos\phi_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \phi_x, \phi_y, \phi_z \;:\; \text{rotation around } x, y, \text{ and } z, \text{ resp.}$$

$$\mathbf{R}_x^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(-\phi_x) & -\sin(-\phi_x) & 0 \\ 0 & \sin(-\phi_x) & \cos(-\phi_x) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \mathbf{R}_y^{-1} = \begin{bmatrix} \cos(-\phi_y) & 0 & \sin(-\phi_y) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-\phi_y) & 0 & \cos(-\phi_y) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

$$\mathbf{R}_z^{-1} = \begin{bmatrix} \cos(-\phi_z) & -\sin(-\phi_z) & 0 & 0 \\ \sin(-\phi_z) & \cos(-\phi_z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(c) Write down a general scaling matrix for 3D points and vectors. Explain the individual entries.

Solution

$$\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad s_x, s_y, s_z \;:\; \text{scaling in } x, y, \text{ and } z, \text{ resp.}$$

$$\mathbf{S}^{-1} = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 & 0 \\ 0 & 0 & \frac{1}{s_z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(d) Write down a general shearing matrix for 3D points and vectors. Explain the individual entries.

Solution

$$\mathbf{D} = \begin{bmatrix} 1 & d_{x,y} & d_{x,z} & 0 \\ d_{y,x} & 1 & d_{y,z} & 0 \\ d_{z,x} & d_{z,y} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad , \quad \begin{array}{rcl} d_{x,y}, d_{x,z} & : & \text{shearing in } x \text{ depending on } y \text{ and } z \text{ resp.} \\ d_{y,x}, d_{y,z} & : & \text{shearing in } y \text{ depending on } x \text{ and } z \text{ resp.} \\ d_{z,x}, d_{z,y} & : & \text{shearing in } z \text{ depending on } x \text{ and } y \text{ resp.} \end{array}$$

$\mathbf{D}^{-1}$ is only "intuitive" for individual shearing matrices, e.g.:

$$\mathbf{D}_{\cdot,z} = \begin{bmatrix} 1 & 0 & d_{x,z} & 0 \\ 0 & 1 & d_{y,z} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D}_{\cdot,z}^{-1} = \begin{bmatrix} 1 & 0 & -d_{x,z} & 0 \\ 0 & 1 & -d_{y,z} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{D}_{x,\cdot} = \begin{bmatrix} 1 & d_{x,y} & d_{x,z} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{D}_{x,\cdot}^{-1} = \begin{bmatrix} 1 & -d_{x,y} & -d_{x,z} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Animation:
Let $\mathbf{p} = \begin{bmatrix} 1 & 2 & 3 & 1 \end{bmatrix}^\top$ denote a 3D point.
Construct a time-dependent transformation matrix that rotates this point on a circle with

- radius $r = 1$,
- around the z-axis,
- at $z = 0$.

Use t for the elapsed time.

Solution

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

brings it to 1,0,0

$$\mathbf{R}_z(t) = \begin{bmatrix} \cos\phi_z(t) & -\sin\phi_z(t) & 0 & 0 \\ \sin\phi_z(t) & \cos\phi_z(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\phi_z(t) = \omega_z \cdot t \quad , \quad \omega_z \ : \ \text{angular velocity}$$

$$\mathbf{M}(t) = \mathbf{R}_z(t) \cdot \mathbf{T}$$

2. Scene Graph

(a) Construct a scene graph for a model of a car consisting of:

- Chassis,
- Body,
- 4 wheels.

(b) Consider row vectors. Specify the computation of the transformation matrix for the rear wheel on the left side.

Solution

We did not cover this task. Instead, we solved the more complex task of sheet 2.