Satisfiability Checking

Propositional Logic

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Informatik 2
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WS 16/17

Propositional logic

The slides are partly taken from:

www.decision-procedures.org/slides/

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

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Syntax of propositional logic

Abstract syntax of well-formed propositional formulae:

$$\varphi \;:=\; a \;\mid\; (\neg\varphi) \;\mid\; (\varphi \wedge \varphi)$$

where AP is a set of (atomic) propositions (Boolean variables) and $a \in AP$.
We write *APForm* for the set of all propositional logic formulae.

Syntactic sugar:

$$\begin{array}{ccccccl}
 & & \bot & & & := & (a \wedge \neg a) \\
 & & \top & & & := & (a \vee \neg a) \\
(& \varphi_1 & \vee & \varphi_2 &) & := & \neg((\neg\varphi_1) \wedge (\neg\varphi_2)) \\
(& \varphi_1 & \rightarrow & \varphi_2 &) & := & ((\neg\varphi_1) \vee \varphi_2) \\
(& \varphi_1 & \leftrightarrow & \varphi_2 &) & := & ((\varphi_1 \rightarrow \varphi_2) \wedge (\varphi_2 \rightarrow \varphi_1)) \\
(& \varphi_1 & \bigoplus & \varphi_2 &) & := & (\varphi_1 \leftrightarrow (\neg\varphi_2))
\end{array}$$

Formulae

- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \land (b \land c))$
  - $(a \rightarrow (b \rightarrow c))$
- We omit parentheses whenever we may restore them through operator precedence:

binds stronger

$$\longleftarrow$$

$$\neg \quad \land \quad \lor \quad \rightarrow \quad \leftrightarrow$$

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Semantics: Assignments

Structures for predicate logic:

- The domain is $\mathbb{B} = \{0, 1\}$.
- The interpretation assigns Boolean values to the variables:

$$\alpha : AP \rightarrow \{0, 1\}$$

We call these special interpretations assignments and use *Assign* to denote the set of all assignments.
Example: $AP = \{a, b\}, \alpha(a) = 0, \alpha(b) = 1$

Equivalently, we can see an assignment α as a set of variables ($\alpha \in 2^{AP}$), defining the variables from the set to be true and the others false.
Example: $AP = \{a, b\}, \alpha = \{b\}$

An assignment can also be seen as being of type $\alpha \in \{0, 1\}^{AP}$, if we have an order on the propositions.
Example: $AP = \{a, b\}, \alpha = 01$

Only the projected assignment matters...

- Let $\alpha_1, \alpha_2 \in$ *Assign* and $\varphi \in$ *APForm*.
- Let $AP(\varphi)$ be the atomic propositions in φ.
- Clearly $AP(\varphi) \subseteq AP$.
- Lemma: if $\alpha_1|_{AP(\varphi)} = \alpha_2|_{AP(\varphi)}$, then

Projection

$$(\alpha_1 \textit{ satisfies } \varphi) \quad \text{iff} \quad (\alpha_2 \textit{ satisfies } \varphi)$$

- We will assume, for simplicity, that $AP = AP(\varphi)$.

Semantics I: Truth tables

- Truth tables define the semantics (=meaning) of the operators. They can be used to define the semantics of formulae inductively over their structure.
- Convention: 0= false, 1= true

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \bigoplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

Each possible assignment is covered by a line of the truth table.
α satisfies φ iff in the line for α and the column for φ the entry is 1.

Q: How many binary operators can we define that have different semantics?
A: 16

Semantics I: Example

- Let φ be defined as $(a \vee (b \rightarrow c))$.
- Let $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ be an assignment with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.
- Q: Does α satisfy φ?
- A1: Compute with truth table:

a	b	c	$b \rightarrow c$	$a \vee (b \rightarrow c)$
0	0	0	1	1
0	0	1	1	1
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	0	1
1	1	1	1	1

Semantics II: Satisfaction relation

Satisfaction relation: $\models \subseteq \mathit{Assign} \times \mathit{APForm}$

Instead of $(\alpha, \varphi) \in \models$ we write $\alpha \models \varphi$ and say that

- α satisfies φ or
- φ holds for α or
- α is a model of φ.

$\models$ is defined recursively:

$$
\begin{array}{llll}
\alpha & \models p & \textit{iff} & \alpha(p) = \mathit{true} \\
\alpha & \models \neg\varphi & \textit{iff} & \alpha \not\models \varphi \\
\alpha & \models \varphi_1 \wedge \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ and } \alpha \models \varphi_2 \\
\alpha & \models \varphi_1 \vee \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ or } \alpha \models \varphi_2 \\
\alpha & \models \varphi_1 \rightarrow \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ implies } \alpha \models \varphi_2 \\
\alpha & \models \varphi_1 \leftrightarrow \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ iff } \alpha \models \varphi_2
\end{array}
$$

Note: More elegant but semantically equivalent to truth tables.

Semantics II: Example

- Let φ be defined as $(a \vee (b \rightarrow c))$.
- Let $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ be an assignment with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

- Q: Does α satisfy φ?

A2: Compute with the satisfaction relation:

$$
\begin{array}{ll}
 & \alpha \models (a \vee (b \rightarrow c)) \\
\textit{iff} & \alpha \models a \textit{ or } \alpha \models (b \rightarrow c) \\
\textit{iff} & \alpha \models a \textit{ or } (\alpha \models b \textit{ implies } \alpha \models c) \\
\textit{iff} & 0 \textit{ or } (0 \textit{ implies } 1) \\
\textit{iff} & 0 \textit{ or } 1 \\
\textit{iff} & 1
\end{array}
$$

Semantics III: The algorithmic view

- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \rightarrow \{0,1\}$ is a model of a propositional logic formula $\varphi \in APForm$:

```
Eval(α, φ) {
   if φ ≡ a return α(a);
   if φ ≡ (¬φ1) return not Eval(α, φ1);
   if φ ≡ (φ1 op φ2)
         return Eval(α, φ1) ⟦op⟧ Eval(α, φ2);
}
```

- Equivalent to the $\models$ relation, but from the algorithmic view.
- Q: Complexity? A: Polynomial (time and space).

Semantics III: Example

- Recall our example
  - $\varphi = (a \vee (b \to c))$
  - $\alpha : \{a, b, c\} \to \{0, 1\}$ with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

- $$\begin{aligned} \mathit{Eval}(\alpha,\ \varphi) \ = \ & \mathit{Eval}(\alpha,\ a) \ \mathit{or}\ \mathit{Eval}(\alpha,\ b \to c) = \\ & 0 \ \mathit{or}\ (\mathrm{Eval}(\alpha,\ b)\ \mathit{implies}\ \mathrm{Eval}(\alpha,\ c)) = \\ & 0 \ \mathit{or}\ (0\ \mathit{implies}\ 1) = \\ & 0 \ \mathit{or}\ 1 = \\ & 1 \end{aligned}$$

- Hence, $\alpha \models \varphi$.

Satisfying assignments

- Intuition: each formula specifies a set of assignments satisfying it.
- Remember: *Assign* denotes the set of all assignments.
- Function $sat : APForm \to 2^{Assign}$

  (a formula $\to$ set of its satisfying assignments)
- Recursive definition:

$$
\begin{array}{lcl}
sat(a) & = & \{\alpha \mid \alpha(a) = 1\}, \quad a \in AP \\
sat(\neg\varphi_1) & = & Assign \setminus sat(\varphi_1) \\
sat(\varphi_1 \wedge \varphi_2) & = & sat(\varphi_1) \cap sat(\varphi_2) \\
sat(\varphi_1 \vee \varphi_2) & = & sat(\varphi_1) \cup sat(\varphi_2) \\
sat(\varphi_1 \rightarrow \varphi_2) & = & (Assign \setminus sat(\varphi_1)) \cup sat(\varphi_2)
\end{array}
$$

- For $\varphi \in APForm$ and $\alpha \in Assign$ it holds that

$$\alpha \models \varphi \qquad \textit{iff} \qquad \alpha \in sat(\varphi)$$

Satisfying assignments: Example

$$
\begin{aligned}
& sat(a \vee (b \rightarrow c)) && = \\
& sat(a) \cup sat(b \rightarrow c) && = \\
& sat(a) \cup ((Assign \setminus sat(b)) \cup sat(c)) && = \\
& \{\alpha \in Assign \mid \alpha(a) = 1\} \cup \\
& \{\alpha \in Assign \mid \alpha(b) = 0\} \cup \\
& \{\alpha \in Assign \mid \alpha(c) = 1\} && = \\
& \{\alpha \in Assign \mid \alpha(a) = 1 \text{ or } \alpha(b) = 0 \text{ or } \alpha(c) = 1\}
\end{aligned}
$$

Extensions of $\models$

- We define $\models \subseteq 2^{Assign} \times APForm$ by

$$T \models \varphi \textit{ iff } T \subseteq sat(\varphi)$$

for formulae $\varphi \in APForm$ and assignment sets $T \subseteq 2^{Assign}$.

Examples: $\{\alpha \in Assign \mid \alpha(a) = \alpha(c) = 1\} \models a \lor (b \rightarrow c)$
$\{\alpha \in Assign \mid \alpha(x_1) = 1\} \models x_1 \lor x_2$

- We define $\models \subseteq 2^{APForm} \times 2^{APForm}$ by

$$\varphi_1 \models \varphi_2 \textit{ iff } sat(\varphi_1) \subseteq sat(\varphi_2)$$

for formulae $\varphi_1, \varphi_2 \in APForm$.

Examples: $a \land c \models a \lor (b \rightarrow c)$
$x_1 \models x_1 \lor x_2$

Short summary for propositional logic

- Syntax of propositional formulae $\varphi \in APForm$:

$$\varphi \ := \ AP \mid (\neg\varphi) \mid (\varphi \wedge \varphi)$$

- Semantics:
  - Assignments $\alpha \in Assign$:

$$\begin{array}{l} \alpha : AP \rightarrow \{0,1\} \\ \alpha \in 2^{AP} \\ \alpha \in \{0,1\}^{AP} \end{array}$$

  - Satisfaction relation:

$$\begin{array}{llll} \models \ \subseteq Assign \times APForm & , & (\text{e.g., } \alpha & \models \varphi \ ) \\ \models \ \subseteq 2^{Assign} \times APForm & , & (\text{e.g., } \{\alpha_1, \ldots, \alpha_n\} & \models \varphi \ ) \\ \models \ \subseteq APForm \times APForm & , & (\text{e.g., } \varphi_1 & \models \varphi_2) \\ sat : \ APForm \rightarrow 2^{Assign} & , & (\text{e.g., } \ sat(\varphi) & \ ) \end{array}$$

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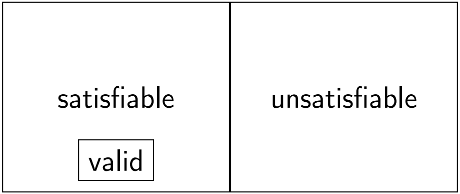
Semantic classification of formulae

- A formula φ is called valid if $sat(\varphi) = Assign$.
  (Also called a tautology).

- A formula φ is called satisfiable if $sat(\varphi) \neq \emptyset$.

- A formula φ is called unsatisfiable if $sat(\varphi) = \emptyset$.
  (Also called a contradiction).



Some notations

- We can write:
  - $\models \varphi$ when φ is valid
  - $\not\models \varphi$ when φ is not valid
  - $\not\models \neg\varphi$ when φ is satisfiable
  - $\models \neg\varphi$ when φ is unsatisfiable

Examples

- $(x_1 \wedge x_2) \to (x_1 \vee x_2)$ is valid
- $(x_1 \vee x_2) \to x_1$ is satisfiable
- $(x_1 \wedge x_2) \wedge \neg x_1$ is unsatisfiable

Examples

- Here are some valid formulae:
  - $\models a \wedge 1 \leftrightarrow a$
  - $\models a \wedge 0 \leftrightarrow 0$
  - $\models \neg\neg a \leftrightarrow a$ (double-negation rule)
  - $\models a \wedge (b \vee c) \leftrightarrow (a \wedge b) \vee (a \wedge c)$
- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
  - $\models \neg(a \vee b) \leftrightarrow (\neg a \wedge \neg b)$

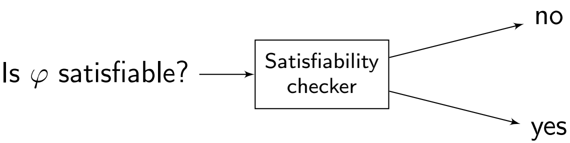
The satisfiability problem for propositional logic

- The satisfiability problem for propositional logic is as follows:

  Given an input propositional formula φ, decide whether φ is satisfiable.

- This problem is decidable but NP-complete.
- An algorithm that always terminates for each propositional logic formula with the correct answer is called a decision procedure for propositional logic.

Goal: Design and implement such a decision procedure:



Note: A formula φ is valid iff $\neg\varphi$ is unsatisfiable.

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Before we solve this problem...

- Suppose we can solve the satisfiability problem... how can this help us?
- There are numerous problems in the industry that are solved via the satisfiability problem of propositional logic
  - Logistics
  - Planning
  - Electronic Design Automation industry
  - Cryptography
  - ...

Example 1: Placement of wedding guests

- Three chairs in a row: $1, 2, 3$
- We need to place Aunt, Sister and Father.
- Constraints:
  - Aunt doesn't want to sit near Father
  - Aunt doesn't want to sit in the left chair
  - Sister doesn't want to sit to the right of Father

- Q: Can we satisfy these constraints?

Example 1 (continued)

- Notation: Aunt = 1, Sister = 2, Father = 3
  Left chair = 1, Middle chair = 2, Right chair = 3

  Introduce a propositional variable for each pair (person, chair):

  $x_{p,c}$ = "person p is sited in chair c" for $1 \leq p, c \leq 3$
- Constraints:

  Aunt doesn't want to sit near Father:

  $$((x_{1,1} \vee x_{1,3}) \rightarrow \neg x_{3,2}) \wedge (x_{1,2} \rightarrow (\neg x_{3,1} \wedge \neg x_{3,3}))$$

  Aunt doesn't want to sit in the left chair:

  $$\neg x_{1,1}$$

  Sister doesn't want to sit to the right of Father:

  $$(x_{3,1} \rightarrow \neg x_{2,2}) \wedge (x_{3,2} \rightarrow \neg x_{2,3})$$

Example 1 (continued)

Each person is placed:

$$(x_{1,1} \vee x_{1,2} \vee x_{1,3}) \wedge (x_{2,1} \vee x_{2,2} \vee x_{2,3}) \wedge (x_{3,1} \vee x_{3,2} \vee x_{3,3})$$

$$\bigwedge_{p=1}^{3} \bigvee_{c=1}^{3} x_{p,c}$$

No person is placed in more than one chair:

$$\bigwedge_{p=1}^{3} \bigwedge_{c1=1}^{3} \bigwedge_{c2=c1+1}^{3} (\neg x_{p,c1} \vee \neg x_{p,c2})$$

At most one person per chair:

$$\bigwedge_{p1=1}^{3} \bigwedge_{p2=p1+1}^{3} \bigwedge_{c=1}^{3} (\neg x_{p1,c} \vee \neg x_{p2,c})$$

Example 2: Assignment of frequencies

- n radio stations
- For each station assign one of k transmission frequencies, $k < n$.
- E – set of pairs of stations, that are too close to have the same frequency.

- Q: Can we assign to each station a frequency, such that no station pairs from E have the same frequency?

Example 2 (continued)

- Notation:
  $x_{s,f}$ = "station s is assigned frequency f" for $1 \leq s \leq n$, $1 \leq f \leq k$
- Constraints:
  Every station is assigned at least one frequency:

$$\bigwedge_{s=1}^{n} \left(\bigvee_{f=1}^{k} x_{s,f} \right)$$

  Every station is assigned at most one frequency:

$$\bigwedge_{s=1}^{n} \bigwedge_{f1=1}^{k-1} \bigwedge_{f2=f1+1}^{k} (\neg x_{s,f1} \vee \neg x_{s,f2})$$

  Close stations are not assigned the same frequency:
  For each $(s1, s2) \in E$,

$$\bigwedge_{f=1}^{k} (\neg x_{s1,f} \vee \neg x_{s2,f})$$

Example 3: Seminar topic assignment

- n participants
- n topics
- Set of preferences $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$
  $(p, t) \in E$ means: participant p would take topic t

- Q: Can we assign to each participant a topic which he/she is willing to take?

Example 3 (continued)

- Notation: $x_{p,t}$ = "participant p is assigned topic t"
- Constraints:

Each participant is assigned at least one topic:

$$\bigwedge_{p=1}^{n}\left(\bigvee_{t=1}^{n} x_{p,t}\right)$$

Each participant is assigned at most one topic:

$$\bigwedge_{p=1}^{n}\bigwedge_{t1=1}^{n-1}\bigwedge_{t2=t1+1}^{n}(\neg x_{p,t1} \vee \neg x_{p,t2})$$

Each participant is willing to take his/her assigned topic:

$$\bigwedge_{p=1}^{n}\bigwedge_{(p,t)\notin E}\neg x_{p,t}$$

Example 3 (continued)

Each topic is assigned to at most one participant:

$$\bigwedge_{t=1}^{n} \bigwedge_{p1=1}^{n} \bigwedge_{p2=p1+1}^{n} (\neg x_{p1,t} \vee \neg x_{p2,t})$$

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Definitions

- Definition: A literal is either a variable or a negation of a variable.
- Example: $\varphi = \neg(a \vee \neg b)$
  Variables: $AP(\varphi) = \{a, b\}$
  Literals: $lit(\varphi) = \{a, \neg b\}$
- Note: Equivalent formulae can have different literals.
  Example: $\varphi' = \neg a \wedge b$
  Literals: $lit(\varphi') = \{\neg a, b\}$

Definitions

- Definition: a term is a conjunction of literals
  - Example: $(a \land \neg b \land c)$

- Definition: a clause is a disjunction of literals
  - Example: $(a \lor \neg b \lor c)$

Negation Normal Form (NNF)

- Definition: A formula is in Negation Normal Form (NNF) iff
  (1) it contains only $\neg$, $\wedge$ and $\vee$ as connectives and
  (2) only variables are negated.

- Examples:
- $\varphi_1 = \neg(a \vee \neg b)$ is not in NNF
- $\varphi_2 = \neg a \wedge b$ is in NNF

Converting to NNF

- Every formula can be converted to NNF in linear time:
  - Eliminate all connectives other than $\wedge$, $\vee$, $\neg$
  - Use De Morgan and double-negation rules to push negations to operands

- Example: $\varphi = \neg(a \rightarrow \neg b)$
  - Eliminate $'\rightarrow'$: $\varphi = \neg(\neg a \vee \neg b)$
  - Push negation using De Morgan: $\varphi = (\neg\neg a \wedge \neg\neg b)$
  - Use double-negation rule: $\varphi = (a \wedge b)$

Disjunctive Normal Form (DNF)

- Definition: A formula is said to be in Disjunctive Normal Form (DNF) iff it is a disjunction of terms.
- In other words, it is a formula of the form

$$\bigvee_i \left(\bigwedge_j l_{i,j} \right)$$

where $l_{i,j}$ is the j-th literal in the i-th term.

- Example:

$$\varphi = (a \wedge \neg b \wedge c) \vee (\neg a \wedge d) \vee (b) \quad \textit{is in DNF}$$

- DNF is a special case of NNF.

Converting to DNF

- Every formula can be converted to DNF in exponential time and space:
  1. Convert to NNF
  2. Distribute disjunctions following the rule: $\models \varphi_1 \land (\varphi_2 \lor \varphi_3) \leftrightarrow (\varphi_1 \land \varphi_2) \lor (\varphi_1 \land \varphi_3)$
- Example:

$$
\begin{aligned}
\varphi \quad &= (a \lor b) \land (\neg c \lor d) \\
&= ((a \lor b) \land (\neg c)) \lor ((a \lor b) \land d) \\
&= (a \land \neg c) \lor (b \land \neg c) \lor (a \land d) \lor (b \land d)
\end{aligned}
$$

- Now consider $\varphi_n = (a_1 \lor b_1) \land (a_2 \lor b_2) \land \ldots \land (a_n \lor b_n)$.
- Q: How many clauses will the DNF have?

  A: 2^n

Satisfiability of DNF

- Q: Is the following DNF formula satisfiable?

  $$(a_1 \land a_2 \land \neg a_1) \lor (a_2 \land a_1) \lor (a_2 \land \neg a_3 \land a_3)$$

  A: Yes, because the term $a_2 \land a_1$ is satisfiable.
- Q: What is the complexity of the satisfiability check of DNF formulae?

  A: Linear (time and space).
- Q: Can there be any polynomial transformation into DNF?
- A: No, it would violate the NP-completeness of the problem.

Conjunctive Normal Form (CNF)

- Definition: A formula is said to be in Conjunctive Normal Form (CNF) iff it is a conjunction of clauses.
- In other words, it is a formula of the form

$$\bigwedge_i \left(\bigvee_j l_{i,j} \right)$$

where $l_{i,j}$ is the j-th literal in the i-th clause.

- Example:

$$\varphi = (a \vee \neg b \vee c) \wedge (\neg a \vee d) \wedge (b) \;\; \textit{is in CNF}$$

- Also CNF is a special case of NNF.

Converting to CNF

- Every formula can be converted to CNF in exponential time and space:
  1. Convert to NNF
  2. Distribute disjunctions following the rule:
     $\models \varphi_1 \vee (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3)$
- Consider the formula $\varphi = (a_1 \wedge b_1) \vee (a_2 \wedge b_2)$.

  Transformation: $(a_1 \vee a_2) \wedge (a_1 \vee b_2) \wedge (b_1 \vee a_2) \wedge (b_1 \vee b_2)$
- Now consider $\varphi_n = (a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee \ldots \vee (a_n \wedge b_n)$.

  Q: How many clauses does the resulting CNF have?

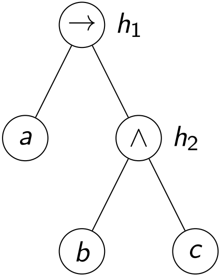
  A: 2^n

Converting to CNF: Tseitin's encoding

- Every formula can be converted to CNF in linear time and space if new variables are added.
- The original and the converted formulae are not equivalent but equi-satisfiable.

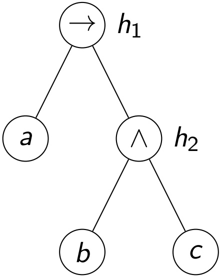
- Consider the formula
  $\varphi = (a \to (b \wedge c))$
- Associate a new auxiliary variable with each gate.
- Add constraints that define these new variables.
- Finally, enforce the root node.

Parse tree:



Converting to CNF: Tseitin's encoding

- Need to satisfy:
  $(h_1 \leftrightarrow (a \ \rightarrow \ h_2))\wedge$
  $(h_2 \leftrightarrow (b \wedge c))\wedge$
  (h_1)



- Each gate encoding has a CNF representation with 3 or 4 clauses.

Converting to CNF: Tseitin's encoding

- Need to satisfy:
  $(h_1 \leftrightarrow (a \rightarrow h_2)) \wedge (h_2 \leftrightarrow (b \wedge c)) \wedge (h_1)$

- First: $(h_1 \vee a) \wedge (h_1 \vee \neg h_2) \wedge (\neg h_1 \vee \neg a \vee h_2)$
- Second: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Converting to CNF: Tseitin's encoding

- Let's go back to

  $\varphi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$

- With Tseitin's encoding we need:
  - n auxiliary variables $h_1, \ldots, h_n$.
  - Each adds 3 constraints.
  - Top clause: $(h_1 \vee \cdots \vee h_n)$

- Hence, we have
  - $3n + 1$ clauses, instead of 2^n.
  - $3n$ variables rather than $2n$.

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Two classes of algorithms for validity

- Q: Is φ satisfiable? (Is $\neg\varphi$ valid?)
- Complexity: NP-Complete (Cook's theorem)
- Two classes of algorithms for finding out:
  - Enumeration of possible solutions (Truth tables etc.)
  - Deduction

- More generally (beyond propositional logic):
  - Enumeration is possible only in some logics.
  - Deduction cannot necessarily be fully automated.

The satisfiability problem

- Given a formula φ, is φ satisfiable?

Enumeration the first:

```
Boolean SAT(φ){
    for all α ∈ Assign
        if Eval(α, φ) return true;
    return false;
}
```

Enumeration the second:

Use substitution to eliminate all variables one by one:

$$\varphi \quad \text{iff} \quad \varphi[0/a] \vee \varphi[1/a]$$

- Q: What is the difference?

  A: Branching on complete vs. partial assignments.

Deduction requires axioms and inference rules

- Inference rules:

$$\frac{\textit{Antecedents}}{\textit{Consequents}} \qquad (\textit{rule name})$$

  Meaning: If all antecedents hold then at least one of the consequents can be derived.

- Examples:

$$\frac{a \rightarrow b \qquad b \rightarrow c}{a \rightarrow c} \qquad (\textit{Trans})$$

$$\frac{a \rightarrow b \qquad a}{b} \qquad (M.P.)$$

Axioms

- Axioms are inference rules with no antecedents, e.g.,

$$\frac{}{a \rightarrow (b \rightarrow a)} \quad (H1)$$

- A proof system consists of a set of axioms and inference rules.

Proofs

- Let $\mathcal{H}$ be a proof system.
- $\Gamma \vdash_{\mathcal{H}} \varphi$ means: There is a proof of φ in system $\mathcal{H}$ whose premises are included in Γ
- $\vdash_{\mathcal{H}}$ is called the provability (derivability) relation.

Example

- Let $\mathcal{H}$ be the proof system comprised of the rules Trans and M.P. that we saw earlier:

$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \qquad (\mathit{Trans})$$

$$\frac{a \rightarrow b \quad a}{b} \qquad (M.P.)$$

- Does the following relation hold?

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

Deductive proof: Example

$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \quad (Trans) \qquad \frac{a \rightarrow b \quad a}{b} \quad (M.P.)$$

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1.	$a \rightarrow b$	*premise*
2.	$b \rightarrow c$	*premise*
3.	$a \rightarrow c$	1, 2, *Trans*
4.	$c \rightarrow d$	*premise*
5.	$d \rightarrow e$	*premise*
6.	$c \rightarrow e$	4, 5, *Trans*
7.	$a \rightarrow e$	3, 6, *Trans*
8.	a	*premise*
9.	e	7, 8, *M.P.*

Proof graph

$$
\begin{tikzcd}[column sep=small]
a \rightarrow b \arrow[dr] & & b \rightarrow c \arrow[dl, "\text{trans}"'] & & c \rightarrow d \arrow[dr] & & d \rightarrow e \arrow[dl, "\text{trans}"'] & \\
& a \rightarrow c \arrow[drr] & & & & c \rightarrow e \arrow[dll, "\text{trans}"'] & & \\
& & & a \rightarrow e \arrow[drr] & & & & a \arrow[dll, "\text{M.P.}"'] \\
& & & & & \mathrm{e} & &
\end{tikzcd}
$$

Soundness and completeness

- For a given proof system $\mathcal{H}$,
  - Soundness: Does $\vdash$ conclude "correct" conclusions from premises?
  - Completeness: Can we conclude all true statements with $\mathcal{H}$?
- Correct with respect to what?

  With respect to the semantic definition of the logic. In the case of propositional logic truth tables give us this.

Soundness and completeness

- Let $\mathcal{H}$ be a proof system

$$\begin{array}{llllll} \textit{Soundness of } \mathcal{H}: & \textit{if} & \vdash_{\mathcal{H}} & \varphi & \textit{then} & \models \quad \varphi \\ \textit{Completeness of } \mathcal{H}: & \textit{if} & \models & \varphi & \textit{then} & \vdash_{\mathcal{H}} \quad \varphi \end{array}$$

- How to prove soundness and completeness?

Example: Hilbert axiom system (H)

- Let H be (M.P.) together with the following axiom schemes:

$$\frac{}{a \rightarrow (b \rightarrow a)} \quad (H1)$$

$$\frac{}{((a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)))} \quad (H2)$$

$$\frac{}{(\neg b \rightarrow \neg a) \rightarrow (a \rightarrow b)} \quad (H3)$$

- H is sound and complete for propositional logic.

Soundness and completeness

- To prove soundness of H, prove the soundness of its axioms and inference rules (easy with truth-tables).
  For example:

a	b	$a \rightarrow (b \rightarrow a)$
0	0	1
0	1	1
1	0	1
1	1	1

- Completeness: harder, but possible.

The resolution proof system

- The resolution inference rule for CNF:

$$\frac{(l \vee l_1 \vee l_2 \vee \ldots \vee l_n) \quad (\neg l \vee l'_1 \vee \ldots \vee l'_m)}{(l_1 \vee \ldots \vee l_n \vee l'_1 \vee \ldots \vee l'_m)} \; \textit{Resolution}$$

- Example:

$$\frac{(a \vee b) \quad (\neg a \vee c)}{(b \vee c)}$$

- We first see some example proofs, before proving soundness and completeness.

Proof by resolution

- Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2 \vee a_5) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4)$
- We want to prove $\varphi \;\rightarrow\; (a_3)$

$$\begin{tikzcd}
 & (\neg a_1 \vee a_4) \arrow[dr] & & (\neg a_1 \vee \neg a_4) \arrow[dl] \\
(a_1 \vee a_3) \arrow[dr] & & (\neg a_1) \arrow[dl] & \\
 & (a_3) & &
\end{tikzcd}$$

Resolution

- Resolution is a sound and complete proof system for CNF.
- If the input formula is unsatisfiable, there exists a proof of the empty clause.

Example

Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4) \wedge (\neg a_3)$.

$$\begin{tikzcd}
(\neg a_1 \vee a_4) \arrow[dr] & & (\neg a_1 \vee \neg a_4) \arrow[dl] & & \\
& (\neg a_1) \arrow[dr] & & (a_1 \vee a_3) \arrow[dl] & \\
& & a_3 \arrow[dr] & & \neg a_3 \arrow[dl] \\
& & & () &
\end{tikzcd}$$

Soundness and completeness of resolution

- Soundness is straightforward. Just prove by truth table that

$$\models ((\varphi_1 \vee a) \wedge (\varphi_2 \vee \neg a)) \rightarrow (\varphi_1 \vee \varphi_2).$$

- Completeness is a bit more involved.
  Basic idea: Use resolution for variable elimination.

$$\begin{array}{c}
(a \vee \varphi_1) \wedge \ldots \wedge (a \vee \varphi_n) \wedge \\
(\neg a \vee \psi_1) \wedge \ldots (\neg a \vee \psi_m) \wedge \\
R \\
\Leftrightarrow \\
(\varphi_1 \vee \psi_1) \wedge \ldots \wedge (\varphi_1 \vee \psi_m) \wedge \\
\ldots \\
(\varphi_n \vee \psi_1) \wedge \ldots (\varphi_n \vee \psi_m) \wedge \\
R
\end{array}$$

  where φ_i ($i = 1, \ldots, n$), ψ_j ($j = 1, \ldots, m$), and R contains neither a nor $\neg a$.