Satisfiability Checking - WS 2016/2017

Series 11

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Exercise 1: Interval arithmetic

Apply basic interval arithmetic as presented in the lecture.

$$x \in I_x = [-1;3], y \in I_y = [2;6]$$

Calculate:

1. $2 \cdot x + y$
2. $x^2 - 4 \cdot x + 7$
3. $x \cdot x \cdot y$
4. $\frac{2 \cdot x}{y}$
5. $z \in (3;4]$, calculate: $x + z - y$

Solution:

1. $2 \cdot x + y = [2;2] \cdot [-1;3] + y = [-2;6] + [2;6] = [0;12]$
2. $x^2 - 4 \cdot x + 7 = [0;9] - 4 \cdot x + 7 = [0;9] - [4;4] \cdot [-1;3] + [7;7] = [0;9] - [-4;12] + [7;7] = [-12;13] + [7;7] = [-5;20]$
3. $x \cdot x \cdot y = [-1;3] \cdot [-1;3] \cdot [2;6] = [-3;9] \cdot [2;6] = [-18;54]$
4. $\frac{2 \cdot x}{y} = \frac{[2;2] \cdot [-1;3]}{[2;6]} = \frac{[-2;6]}{[2;6]} = [-2;6] \cdot [\frac{1}{6}; \frac{1}{2}] = [-1;3]$
5. $z \in (3;4]$, calculate: $x + z - y = [-1;3] + (3;4] - [2;6] = (2;7] - [2;6] = (-4;5]$

Exercise 2: Propagation

a) Given the following constraints

$$c_1 : 2 \cdot x - 3 \cdot y = 0, c_2 : x^2 - 2 \cdot y = 0,$$

perform two interval propagation steps. In each step choose the most appropriate contraction candidate. The initial intervals of x and y are $x, y \in [1;10]$.

b) Given the constraints

$$a^2 + b^2 < 1 \text{ and } a \cdot b > 1$$

preprocessing yields the following equations and initial bounds:

$$\begin{array}{lll} e_1: & h_1 = a \cdot b & h_1 \in (1, \infty) \\ e_2: & h_2 = a^2 & h_2 \in (-\infty, \infty) \\ e_3: & h_3 = b^2 & h_3 \in (-\infty, \infty) \\ e_4: & h_4 = h_2 + h_3 & h_4 \in (-\infty, 1) \\ & & a \in (\infty, \infty) \\ & & b \in (\infty, \infty) \end{array}$$

Propagate using these equations until unsatisfiability is proven for at least one of the variables.

Solution:

a) Possible contractions in the first step:

- $(c_1, x) : x = \frac{3 \cdot y}{2} = \frac{[3;30]}{[2;2]} = [\frac{3}{2}; 15] \rightarrow x \in [1; 10] \cap [\frac{3}{2}; 15] = [\frac{3}{2}; 10]$
  (relative contraction: ~ 0.056)
- $(c_1, y) : y = \frac{2 \cdot x}{3} = \frac{[2;20]}{[3;3]} = [\frac{2}{3}; \frac{20}{3}] \rightarrow y \in [1; 10] \cap [\frac{2}{3}; \frac{20}{3}] = [1; \frac{20}{3}]$
  (relative contraction: ~ 0.370)
- $(c_2, x) : x = \sqrt{2 \cdot y} = [\sqrt{2}; \sqrt{20}] \rightarrow x \in [1; 10] \cap [\underbrace{\sqrt{2}}_{\sim 1.41}; \underbrace{\sqrt{20}}_{\sim 4.47}] = [\sqrt{2}; \sqrt{20}]$
  (relative contraction: ~ 0.843)
  Note that we skipped the case $-\sqrt{2 \cdot y}$ because $x \in [1; 10]$ and thus we know that negative values would be cut away anyway.
- $(c_2, y) : y = \frac{x^2}{2} = \frac{[1;100]}{2} = [\frac{1}{2}; 50] \rightarrow y \in [1; 10] \cap [\frac{1}{2}; 50] = [1; 10]$
  (relative contraction: 0)

Choose (c_2, x) as the first contraction candidate. Updated intervals: $x \in [\sqrt{2}; \sqrt{20}], y \in [1; 10]$.

Possible contractions in the second step:

- $(c_1, x) : x = \frac{3 \cdot y}{2} = \frac{[3;30]}{[2;2]} = [\frac{3}{2}; 15] \rightarrow x \in [\sqrt{2}; \sqrt{20}] \cap [\frac{3}{2}; 15] = [\frac{3}{2}; \sqrt{20}]$
  (relative contraction: ~ 0.028)
- $(c_1, y) : y = \frac{2 \cdot x}{3} = \frac{2 \cdot [\sqrt{2}; \sqrt{20}]}{[3;3]} = [\frac{2 \cdot \sqrt{2}}{3}; \frac{2 \cdot \sqrt{20}}{3}] \rightarrow y \in [1; 10] \cap \left[\underbrace{\frac{2 \cdot \sqrt{2}}{3}}_{\sim 0.94}; \underbrace{\frac{2 \cdot \sqrt{20}}{3}}_{\sim 2.98}\right] = [1; \frac{2 \cdot \sqrt{20}}{3}]$
  (relative contraction: ~ 0.669)
- $(c_2, x) : x = \sqrt{2 \cdot y} = [\sqrt{2}; \sqrt{20}] \rightarrow x \in [\sqrt{2}; \sqrt{20}] \cap [\sqrt{2}; \sqrt{20}] = [\sqrt{2}; \sqrt{20}]$
  (relative contraction: 0)
- $(c_2, y) : y = \frac{x^2}{2} = \frac{[2;20]}{2} = [1; 10] \rightarrow y \in [1; 10] \cap [1; 10] = [1; 10]$
  (relative contraction: 0)

Intervals after two optimal contractions: $x \in [\sqrt{2}; \sqrt{20}], y \in [1; \frac{2 \cdot \sqrt{20}}{3}]$.

b) In the following we mean by $\leadsto_{i,e_j}$ that we used the equation e_j in the i-th step for contraction.

$$
\begin{array}{llllll}
h_1 \in (1, \infty) & & & \leadsto_{8,e_1} & (1, \infty) \cap \overbrace{((-1, 1) \cdot (-1, 1))}^{(-1,1)} = \emptyset \\
h_2 \in (-\infty, \infty) & \leadsto_{1,e_2} & [0, \infty) & \leadsto_{4,e_4} & [0, \infty) \cap \overbrace{([0, 1) - [0, \infty))}^{(-\infty,1)} = [0, 1) \\
h_3 \in (-\infty, \infty) & \leadsto_{2,e_3} & [0, \infty) & \leadsto_{5,e_4} & [0, \infty) \cap \overbrace{([0, 1) - [0, 1))}^{(-1,1)} = [0, 1) \\
h_4 \in (-\infty, 1) & \leadsto_{3,e_4} & [0, 1) & & \\
a \in (-\infty, \infty) & \leadsto_{6,e_2} & \overbrace{(-1, 1)}^{\pm\sqrt{[0,1)}} & & \\
b \in (-\infty, \infty) & \leadsto_{7,e_3} & \overbrace{(-1, 1)}^{\pm\sqrt{[0,1)}} & &
\end{array}
$$

Exercise 3: Questions

Give a short answer to the following questions:

1. The ICP algorithm from the lecture maintains two threshold values as parameters. Describe the purpose of these values.
2. Which are the two events causing a split in the ICP algorithm presented in the lecture?
3. ICP is not a complete method. Why does it still make sense to use it as a preprocessing to a complete method, such as CAD or VS?

Solution:

1. ICP maintains a threshold for the weight of the contraction candidates as well as a threshold for the target diameter of the box. The first one is required to stop contraction: If the weight of a contraction candidate is below the threshold, the candidate is not longer considered for contraction, which means that the contraction gain of this candidate has been too low during the last runs and the candidate does not seem to be promising for contraction. The second parameter is used to stop the overall loop. The loop is stopped as soon as the box is smaller than this threshold in every dimension. This parameter specifies the precision of the algorithm.
2. On the one hand a split is performed whenever there is no promising contraction candidate available (autonomous split) - the contraction gain has been too low and the algorithm does not make enough progress and the target diameter is not yet reached. On the other hand a split can occur, when during contraction (e.g. via the componentwise Newton operator) a division by an interval containing 0 occurs (heteronomous split).
3. Complete methods such as CAD or VS profit from a reduced search space as they are able to drop samples/test candidates not contained in the search space.