LECTURE NOTES
CG RECAP – TRANSFORMATION

We have decided to use column vectors here:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

To save space they may be denoted using the transpose operator:

$$\mathbf{v} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top$$

Part I
The Model Matrix

1 Translation I

- Objects are translated by adding a translation vector $\mathbf{t}$ to all of their points $\mathbf{p}$:

$$\mathbf{p}' = \mathbf{p} + \mathbf{t} \quad .$$

- Vectors do not change under translation.

2 Scaling

- Objects are scaled with respect to the origin by multiplying all of their points with a diagonal matrix

$$\mathbf{p}' = \mathbf{S} \cdot \mathbf{p} \quad \text{, with}$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

3 Rotation

- Objects are rotated by multiplying all of their points with a rotation matrix:

$$\mathbf{p}' = \mathbf{R}_i \cdot \mathbf{p} \quad , \quad i \in \{x, y, z\}$$

- Rotation with respect to the origin around the x-axis:

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

- Rotation with respect to the origin around the y-axis:

$$\mathbf{R}_y = \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix}$$

- Rotation with respect to the origin around the z-axis:

$$\mathbf{R}_z = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

How to remember these matrices?

- The axis of rotation is denoted by the position of the 1.
- All the remaining elements in the respecitve row and column are 0.
- The remaining elements form the block
$$\begin{matrix} \cos\varphi & \sin\varphi \\ \sin\varphi & \cos\varphi \end{matrix}$$
- x, z rotation: minus sign at the top right $\sin$. y rotation: minus sign at the bottom left $\sin$.

4 Shear

- Shearing an object pushes its points sideways depending on their distance to the origin.
- Objects are sheared by multiplying all of their points with a shearing matrix:
$$\mathbf{p}' = \mathbf{D} \cdot \mathbf{p} \quad \text{, with}$$
$$\mathbf{D} = \begin{bmatrix} 1 & d_{xy} & d_{xz} \\ d_{yx} & 1 & d_{yz} \\ d_{zx} & d_{zy} & 1 \end{bmatrix}$$
- d_{ij} denotes the shearing into i with respect to the j-coordinate, or
- shearing into direction of *row* with respect to *column*.

5 Concatenation I

- Let an object be
  - translated ($\mathbf{t}_1$)
  - scaled ($\mathbf{S}$)
  - rotated ($\mathbf{R}$)
  - translated ($\mathbf{t}_2$)
  - sheared ($\mathbf{D}$)

  then
$$\mathbf{p}' = \mathbf{D} \cdot (\mathbf{R} \cdot \mathbf{S} \cdot (\mathbf{p} + \mathbf{t}_1) + \mathbf{t}_2)$$
- Thus, translation the way it is now hinders concatenation of transformation into a single matrix.

6 Homogeneous Coordinates

This secion will omit mathematical rigor in favour of an intuitive understanding.

- Recall from linear algebra that
$$\mathbf{p} = \begin{bmatrix} x & y & z \end{bmatrix}^\top = x \cdot \mathbf{e}_x + y \cdot \mathbf{e}_y + z \cdot \mathbf{e}_z = \begin{bmatrix} | & | & | \\ \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ | & | & | \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
- Then intuitively, translation as above yields
$$\mathbf{p} + \mathbf{t} = x \cdot \mathbf{e}_x + y \cdot \mathbf{e}_y + z \cdot \mathbf{e}_z + 1 \cdot \mathbf{t} = \begin{bmatrix} | & | & | & | \\ \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z & \mathbf{t} \\ | & | & | & | \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- However, the above yields vectors/points with 3 coordinates. In order to yield the same number of coordinates as the input has

$$\mathbf{p} + \mathbf{t} = \begin{bmatrix} | & | & | & | \\ \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z & \mathbf{t} \\ | & | & | & | \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- Consequently, for points:

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

  and for vectors

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix} \quad ,$$

  since the latter do not change under transloation.

7 Translation II

- Objects are translated by multiplying all of their points with a translation matrix:

$$\mathbf{p}' = \mathbf{T} \cdot \mathbf{p} \quad \text{, with}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

8 Concatenation II

- Scaling, rotation, and shearing matrices need to be extended to 4×4 in order to make them compatible:

$$\begin{bmatrix} & & & 0 \\ & \mathbf{S}, \mathbf{R}, \mathbf{D} & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Let an object now be
  - translated ($\mathbf{T}_1$)
  - scaled ($\mathbf{S}$)
  - rotated ($\mathbf{R}$)
  - translated ($\mathbf{T}_2$)
  - sheared ($\mathbf{D}$)

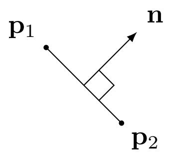
  then

$$\mathbf{p}' = \mathbf{D} \cdot \mathbf{T}_2 \cdot \mathbf{R} \cdot \mathbf{S} \cdot \mathbf{T}_1 \cdot \mathbf{p} = \mathbf{M} \cdot \mathbf{p}$$

- The matrix closest to the point affects the point first – no matter if using column or row vectors.

9 Transforming Normals

- Normals $\mathbf{n}$ are perpendicular to a surface:

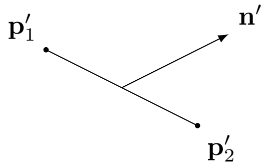


Thus, $\mathbf{n}^\top \cdot (\mathbf{p}_2 - \mathbf{p}_1) = 0$

- Let all be non-uniformly scaled:

$$\mathbf{p}' = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \mathbf{p} \quad , \quad \mathbf{n}' = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \mathbf{n}$$

- Then,



and $\mathbf{n}'^\top \cdot (\mathbf{p}_2' - \mathbf{p}_1') \neq 0$

- Thus, in general, i.e., if non-uniform scaling and shearing are allowed: if points are transformed by $\mathbf{M}$, normals require a separate matrix $\mathbf{N}$, so that

$$(\mathbf{N} \cdot \mathbf{n})^\top \cdot (\mathbf{M} \cdot (\mathbf{p}_2 - \mathbf{p}_1)) = 0$$

- Let $\mathbf{v} = \mathbf{p}_2 - \mathbf{p}_1$, then

$$\begin{aligned} 0 = \mathbf{n}^\top \cdot \mathbf{v} &= \mathbf{n}^\top \cdot \mathbf{I} \cdot \mathbf{v} \\ &= \mathbf{n}^\top \cdot \mathbf{M}^{-1} \cdot \mathbf{M} \cdot \mathbf{v} \\ &= \mathbf{n}^\top \cdot \mathbf{M}^{-1} \cdot \mathbf{v}' \\ &= \left((\mathbf{M}^{-1})^\top \cdot \mathbf{n}\right)^\top \cdot \mathbf{v}' = 0 \end{aligned}$$

- By comparing these equations

$$\mathbf{N} = (\mathbf{M}^{-1})^\top$$

10 Turning Column Vectors Into Row Vectors

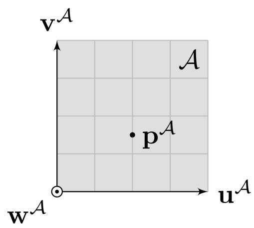
- $\begin{bmatrix} x & y & z & 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}^\top$

- $\begin{bmatrix} x' & y' & z' & 1 \end{bmatrix} = \left(\mathbf{A} \cdot \mathbf{B} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \right)^\top = \begin{bmatrix} x & y & z & 1 \end{bmatrix} \cdot \mathbf{B}^\top \cdot \mathbf{A}^\top$

Part II
The View Matrix

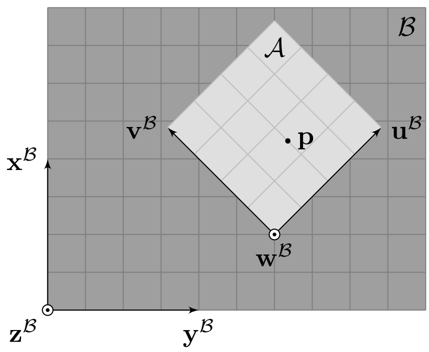
11 Coordinate Transformation

- Let a point $\mathbf{p}^{\mathcal{A}} = \begin{bmatrix} p_x^{\mathcal{A}} & p_y^{\mathcal{A}} & p_z^{\mathcal{A}} & 1 \end{bmatrix}^\top$ be specified with respect to some local coordinate frame $\mathcal{A}$



$$\mathbf{u}^{\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top \quad , \quad \mathbf{v}^{\mathcal{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top \quad , \quad \mathbf{w}^{\mathcal{A}} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top$$

- Let the coordinate frame $\mathcal{A}$ be embedded in some other coordinate frame $\mathcal{B}$ by first rotating ($\mathbf{R}$) and then translating ($\mathbf{T}$) it:



$$\mathbf{x}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top \quad , \quad \mathbf{y}^{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^\top \quad , \quad \mathbf{z}^{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top$$

- Consequently, $\mathbf{p}^{\mathcal{B}} = \begin{bmatrix} p_x^{\mathcal{B}} & p_y^{\mathcal{B}} & p_z^{\mathcal{B}} & 1 \end{bmatrix}^\top$ can be specified with respect to the second coordinate frame $\mathcal{B}$ by transforming $\mathbf{p}^{\mathcal{A}}$
$$\mathbf{p}^{\mathcal{B}} = \mathbf{T} \cdot \mathbf{R} \cdot \mathbf{p}^{\mathcal{A}}$$

- The same applies to the basis vectors of the coordinate frame $\mathcal{A}$
$$\mathbf{u}^{\mathcal{B}} = \mathbf{R} \cdot \mathbf{u}^{\mathcal{A}} \quad , \quad \mathbf{v}^{\mathcal{B}} = \mathbf{R} \cdot \mathbf{v}^{\mathcal{A}} \quad , \quad \mathbf{w}^{\mathcal{B}} = \mathbf{R} \cdot \mathbf{w}^{\mathcal{A}}$$

  Note that the vectors are not affected by translation.

- This can be rewritten using matrix notation
$$\begin{bmatrix} | & | & | \\ \mathbf{u}^{\mathcal{B}} & \mathbf{v}^{\mathcal{B}} & \mathbf{w}^{\mathcal{B}} \\ | & | & | \end{bmatrix} = \mathbf{R} \cdot \begin{bmatrix} | & | & | \\ \mathbf{u}^{\mathcal{A}} & \mathbf{v}^{\mathcal{A}} & \mathbf{w}^{\mathcal{A}} \\ | & | & | \end{bmatrix} = \mathbf{R} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{R_{4\times 3}} \quad ,$$

  with $\mathbf{R_{4\times 3}}$ denoting the matrix that contains the 3 leftmost columns of $\mathbf{R}$:
$$\mathbf{R} = \left[\begin{array}{c|c} \boxed{\mathbf{R_{4\times 3}}} & \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} \end{array} \right]$$

- Thus, the basis vectors $\mathbf{u}^{\mathcal{B}}, \mathbf{v}^{\mathcal{B}}, \mathbf{w}^{\mathcal{B}}$ of frame $\mathcal{A}$ measured in frame $\mathcal{B}$ create the first three columns of $\mathbf{R}$.
- Let $\mathbf{e}^{\mathcal{B}} = \begin{bmatrix} e_x^{\mathcal{B}} & e_y^{\mathcal{B}} & e_z^{\mathcal{B}} & 1 \end{bmatrix}^\top$ denote the position of frame $\mathcal{A}$'s origin measured in frame $\mathcal{B}$.
- Consequently, the matrix

$$\mathbf{M}_{\mathcal{A}\to\mathcal{B}} = \mathbf{T} \cdot \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & | \\ 0 & 1 & 0 & \mathbf{e} \\ 0 & 0 & 1 & | \\ 0 & 0 & 0 & \end{bmatrix} \cdot \begin{bmatrix} | & | & | & 0 \\ \mathbf{u}^{\mathcal{B}} & \mathbf{v}^{\mathcal{B}} & \mathbf{w}^{\mathcal{B}} & 0 \\ | & | & | & 0 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} | & | & | & | \\ \mathbf{u}^{\mathcal{B}} & \mathbf{v}^{\mathcal{B}} & \mathbf{w}^{\mathcal{B}} & \mathbf{e}^{\mathcal{B}} \\ | & | & | & | \end{bmatrix}$$

  transforms (points and vetors from) coordinate frame $\mathcal{A}$ into frame $\mathcal{B}$.

12 Camera Transform

- By convention, for rendering, all objects, points, vectors are transformed from the canonical world space into the coordinate frame of the camera.
- The user defines a camera by specifying the
  - eye position $\mathbf{e}$,
  - gaze vector $\mathbf{g}$,
  - view-up vector $\mathbf{t}$.
- From these, the basis vectors of th camera coordinate frame are computed
  - $\mathbf{w} = -\dfrac{\mathbf{g}}{\|\mathbf{g}\|}$
  - $\mathbf{u} = \dfrac{\mathbf{t} \times \mathbf{w}}{\|\mathbf{t} \times \mathbf{w}\|}$
  - $\mathbf{v} = \mathbf{w} \times \mathbf{u}$

  Note: By convention, the camera looks along $-z$.
- In terms of the notation of Sec. 11, the canonical world space resembles frame $\mathcal{B}$, the camera's coordinate frame resembles frame $\mathcal{A}$. The above $\mathbf{e}, \mathbf{u}, \mathbf{v}, \mathbf{w}$ are measured in frame $\mathcal{B}$. They thus resemble $\mathbf{e}^{\mathcal{B}}, \mathbf{u}^{\mathcal{B}}, \mathbf{v}^{\mathcal{B}}, \mathbf{w}^{\mathcal{B}}$.
- Consequently, the following matrix transforms all points from world space into camera space

$$\mathbf{M}_{\mathcal{B}\to\mathcal{A}} = (\mathbf{M}_{\mathcal{A}\to\mathcal{B}})^{-1} = (\mathbf{T} \cdot \mathbf{R})^{-1} = \mathbf{R}^{-1} \cdot \mathbf{T}^{-1} = \begin{bmatrix} & — \mathbf{u} — & & \\ & — \mathbf{v} — & & \\ & — \mathbf{u} — & & \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -e_x \\ 0 & 1 & 0 & -e_y \\ 0 & 0 & 1 & -e_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Part III
Further Reading

Peter Shirley, Steve Marschner:
Fundamentals of Computer Graphics

- Chapter “Transformation Matrices” (Ch. 6 in 3rd Edition)
- Subsection “The Camera Transformation” (Subsec. 7.1.3 in 3rd Edition)