Satisfiability Checking

The Omega Test

Prof. Dr. Erika Ábrahám

RWTH Aachen University
Informatik 2
LuFG Theory of Hybrid Systems

WS 16/17

The Omega test

- Goal: Decide satisfiability for conjunctions of linear constraints of the form

$$\sum_{0 \leq i \leq n} a_i x_i \geq b$$

  over integers.
- Original application:
  Program optimizations done by a compiler.
- Extension of *Fourier-Motzkin* variable elimination:
  - Pick one variable and eliminate it.
  - Continue until all variables but one are eliminated.

Fourier - Motzkin

$$\exists x.\ \begin{matrix} l_1 \\ \vdots \\ l_n \end{matrix} \le x \le \begin{matrix} u_1 \\ \vdots \\ u_m \end{matrix} \iff \forall 1 \le i \le n.\ \forall 1 \le j \le m.\ l_i \le u_j$$

$$\left.\begin{matrix} p \le bx & \to & p/b \le x \\ cx \le r & \to & x \le r/c \end{matrix}\right\} \to p/b \le r/c$$

$$[p/b, r/c] \ne \emptyset$$

but: is there an integer in it?

$$\left.\begin{matrix} p \le bx & cx \le r \\ c p \le bcx & bcx \le br \end{matrix}\right\} \quad \underline{cp \le br}$$

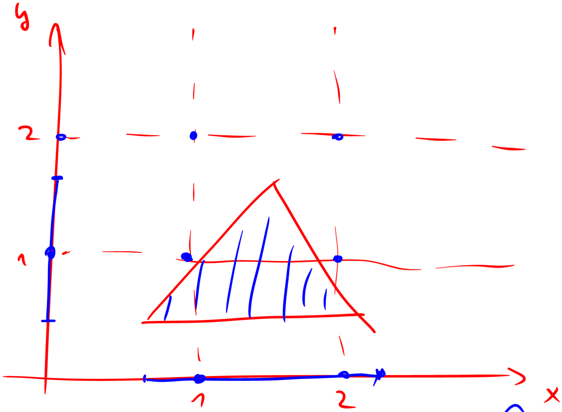
Still does not work for $\mathbb{Z}$!

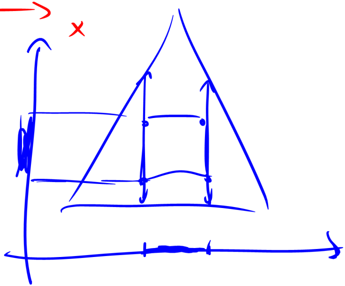
Example:

$$\left.\begin{matrix} x \le 11y \le 2x \\ 1 \le x \le 5 \end{matrix}\right\} \xrightarrow[y]{\text{elim.}} \left\{\begin{matrix} x \le 2x \\ 1 \le x \le 5 \end{matrix}\right\} \xrightarrow[x]{\text{elim.}} \begin{matrix} 0 \le x \\ 1 \le x \le 5 \end{matrix} \rightsquigarrow x \in [1,5]$$

-11-

$$\xrightarrow[x]{\text{elim}} \left.\begin{matrix} x \le 11y \ (1) \\ 11y \le 2x \ (1') \\ 1 \le x \ (2') \\ x \le 5 \ (2) \end{matrix}\right\} \begin{matrix} 11y \le 22y & (1)+(1') \\ 1 \le 11y & (1)+(2') \\ 11y \le 10 & (2)+(1') \\ 1 \le 5 & (2)+(2') \end{matrix} \xrightarrow[y]{\text{elim}} \begin{matrix} 0 \le y \\ 1 \le 11y \\ 11y \le 10 \\ \hline 0 \le 10 \\ 1 \le 10 \end{matrix}$$


y
2
1
x
1
2



Overview of the Omega test

Checks for real solution
with an integer value
in a given dimension.

1 Check
REAL shadow

Overview of the Omega test

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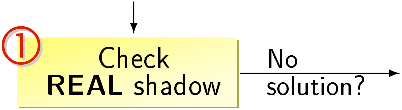


Overview of the Omega test

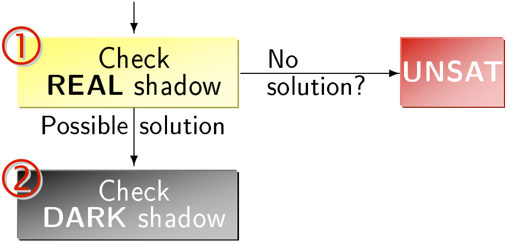
Checks for real solution
with an integer value
in a given dimension.

1 Check **REAL** shadow → No solution? → **UNSAT**

Overview of the Omega test

Checks for real solution with an integer value in a given dimension.

Checks a sufficient condition for integer solution.



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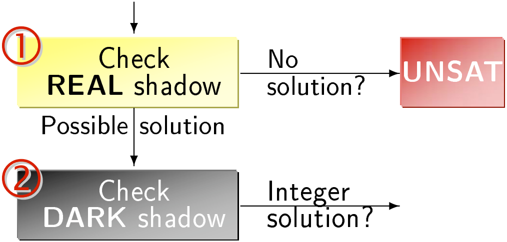

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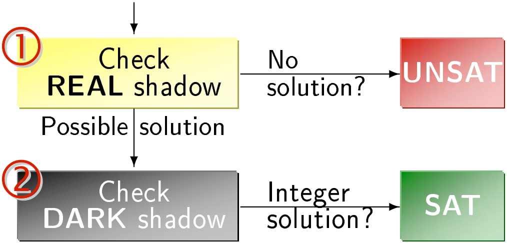
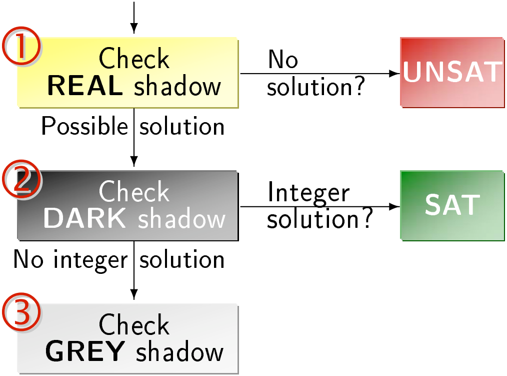


Overview of the Omega test

Checks for real solution with an integer value in a given dimension.

Checks a sufficient condition for integer solution.

Checks for integer solutions not satisfying the sufficient condition.



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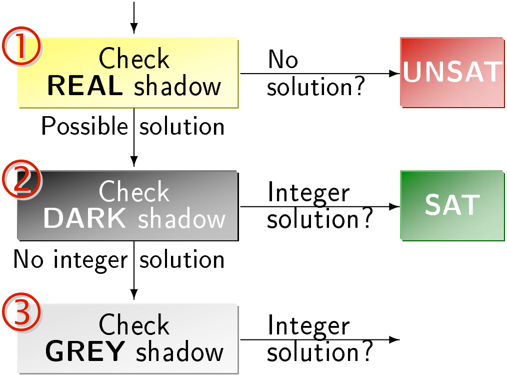

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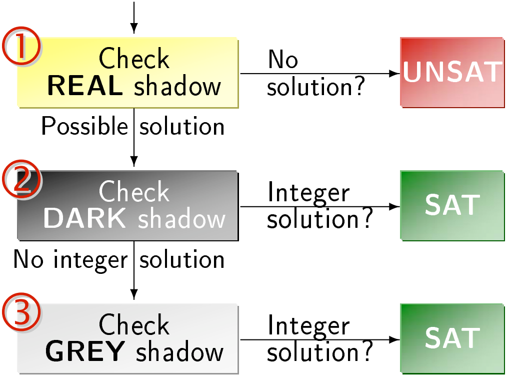
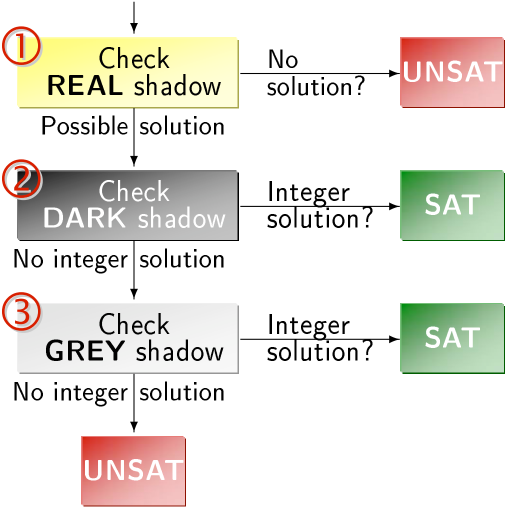


Overview of the Omega test

Checks for real solution with an integer value in a given dimension.

Checks a sufficient condition for integer solution.

Checks for integer solutions not satisfying the sufficient condition.



The real shadow

(1) Check
REAL shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\beta \quad \leq \quad bz \qquad\qquad cz \quad \leq \quad \gamma \quad (b, c > 0)$$

The real shadow

① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{rclcrcll} \beta & \leq & bz & \quad & cz & \leq & \gamma & \quad (b, c > 0) \\ c\beta & \leq & cbz & & cbz & \leq & b\gamma & \end{array}$$

Important: All terms are integer-valued. (Instead of cb, we can also use the smallest common multiple of c and b.)

① Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper/lower bound:

$$\begin{array}{ccccccc} \beta & \leq & bz & \quad cz & \leq & \gamma & (b, c > 0) \\ c\beta & \leq & cbz & \quad cbz & \leq & b\gamma & \end{array}$$

Important: All terms are integer-valued. (Instead of cb, we can also use the smallest common multiple of c and b.)

- Constraint for real shadow if z is not the last variable to be eliminated:

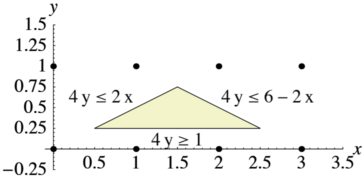
$$c\beta \leq b\gamma$$

- Constraint for real shadow if z is the last variable to be eliminated:

$$\left\lceil \frac{\beta}{b} \right\rceil \leq \left\lfloor \frac{\gamma}{c} \right\rfloor$$

The real shadow: Example I

① Check **REAL** shadow



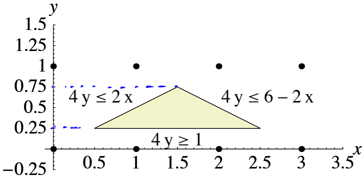
$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

The real shadow: Example I

1 Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x+6$

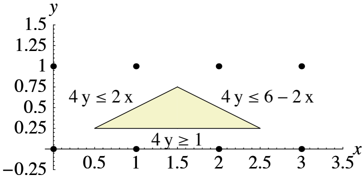
$4y \geq 1$

Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x+6$

$2x \leq -4y+6$

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

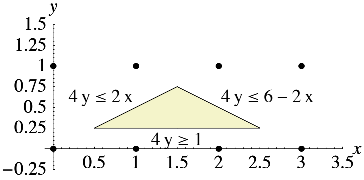
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$

The real shadow: Example I

1 Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

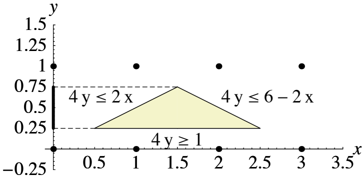
Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$$4y \leq 6 - 4y$$
$$8y \leq 6$$

The real shadow: Example I

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Eliminate x:

$4y \leq 2x$ $\quad$ $4y \leq -2x + 6$

$4y \leq 6 - 4y$

$8y \leq 6$

$\rightarrow \quad y \leq \frac{6}{8}$

$\frac{1}{4} \leq y$

$y \in [\frac{1}{4}, \frac{6}{8}]$

Real Shadow:

$8y \leq 6$

$4y \geq 1$

$\Rightarrow$

$y \leq 0.75$

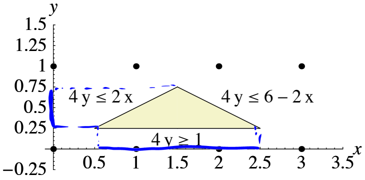
$y \geq 0.25$

No integer solution

$\Longrightarrow$ Original problem has no solution

The real shadow: Example II

① Check **REAL** shadow



$4y \leq 2x$

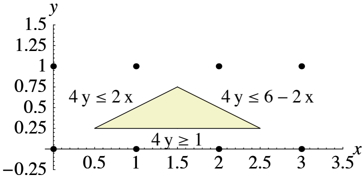
$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ $\qquad$ $4y \leq 2x$

1 Check
REAL shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

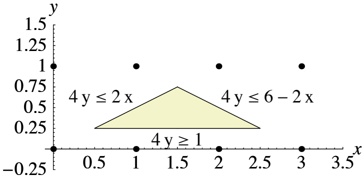
Let's eliminate y instead:

$1 \leq 4y$ $\quad$ $4y \leq 2x$

$$1 \leq 2x$$

The real shadow: Example II

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

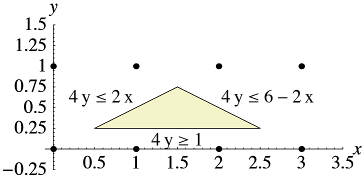
$$1 \leq 4y \qquad 4y \leq 2x$$

$$1 \leq 2x$$

$$1 \leq 4y \qquad 4y \leq -2x + 6$$

The real shadow: Example II

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

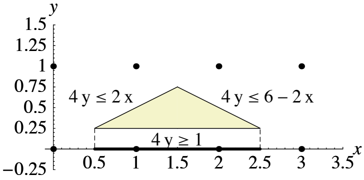
$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$ and $4y \leq 2x$ yield $1 \leq 2x$

$1 \leq 4y$ and $4y \leq -2x + 6$ yield $1 \leq -2x + 6$

① Check **REAL** shadow



$4y \leq 2x$

$4y \leq -2x + 6$

$4y \geq 1$

Let's eliminate y instead:

$1 \leq 4y$, $4y \leq 2x$ → $1 \leq 2x$

$1 \leq 4y$, $4y \leq -2x + 6$ → $1 \leq -2x + 6$

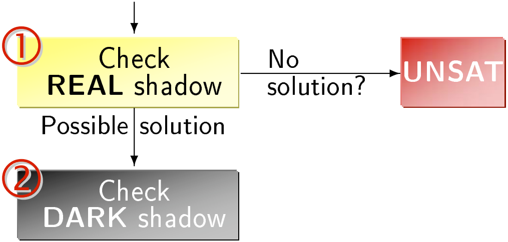
Real Shadow:

$$1 \leq 2x \qquad 1 \leq -2x + 6 \quad \Rightarrow \quad x \geq 0.5 \qquad x \leq 2.5$$

Real solution with integer x-value!

But there exists
no pure integer solution!

From real to dark shadow



- A solution for the REAL shadow does not guarantee that there is an integer solution for the original problem.
- Thus, we check the DARK shadow next.

Define sufficient condition $(*) \rightarrow (\text{sol})$

Idea to derive: assume no solution

$$\neg(\text{Sol}) \rightarrow \neg(*)$$

$$\Updownarrow$$

$$(*) \rightarrow (\text{Sol})$$

Idea of the dark shadow

② Check
DARK shadow

- Idea of the DARK shadow:

$$\beta \quad \leq \quad bz \qquad\qquad\qquad cz \quad \leq \quad \gamma$$

Idea of the dark shadow

② Check **DARK** shadow

- Idea of the DARK shadow:

$$\begin{array}{rclcrclcl} \beta & \leq & bz & \mid : b & cz & \leq & \gamma & \mid : c & \\ \frac{\beta}{b} & \leq & z & & z & \leq & \frac{\gamma}{c} & & z \in \mathbb{N} \end{array}$$

Idea of the dark shadow

② Check **DARK** shadow

- Idea of the DARK shadow:

$$
\begin{array}{rclcrclcl}
\beta & \leq & bz & \mid : b & cz & \leq & \gamma & \mid : c & \\
\frac{\beta}{b} & \leq & z & & z & \leq & \frac{\gamma}{c} & & z \in \mathbb{N}
\end{array}
$$

- Try to prove that there is an integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$.

Dark shadow: Proof by contradiction

②
Check
DARK shadow

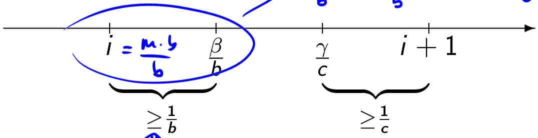
Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$.

② Check
DARK shadow

Assume there is no integer z between $\frac{\beta}{b}$ and $\frac{\gamma}{c}$. Then:

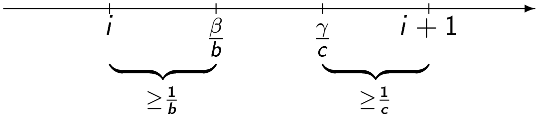
Let $i := \lfloor \frac{\beta}{b} \rfloor$ $\qquad i \in \mathbb{Z}$

$m \in \mathbb{Z}$ $\qquad \Rightarrow \frac{\beta}{b} - \frac{m \cdot b}{b} = \frac{\beta - m \cdot b}{b} \in \mathbb{Z}$

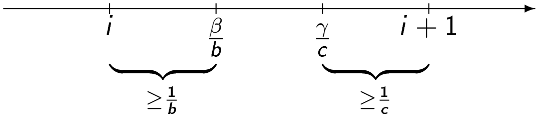


Note: Fourier-Motzkin applied previously only multiplication
$\Rightarrow$ β, γ are integer valued + coefficients $b, c \in \mathbb{Z}$

Dark shadow: Proof by contradiction

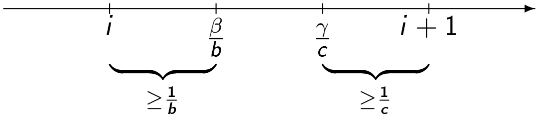


Dark shadow: Proof by contradiction



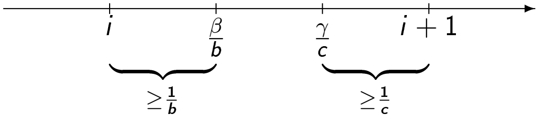
$$\begin{aligned} \frac{\beta}{b} - i &\geq \frac{1}{b} \\ i + 1 - \frac{\gamma}{c} &\geq \frac{1}{c} \end{aligned}$$

Dark shadow: Proof by contradiction



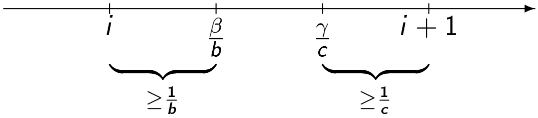
$$\begin{array}{rcl}
\frac{\beta}{b} - i & \geq & \frac{1}{b} \\
i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c}
\end{array}$$

Dark shadow: Proof by contradiction



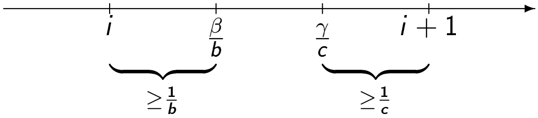
$$
\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} & \\
\hline
\frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} & \quad | \cdot c \cdot b \\
c\beta + cb - b\gamma & \geq & c + b &
\end{array}
$$

Dark shadow: Proof by contradiction



$$\begin{array}{rcll} \frac{\beta}{b}-i & \geq & \frac{1}{b} & \\ i+1-\frac{\gamma}{c} & \geq & \frac{1}{c} & \\ \hline\hline \frac{\beta}{b}+1-\frac{\gamma}{c} & \geq & \frac{1}{b}+\frac{1}{c} & \mid \cdot c \cdot b \\ c\beta+cb-b\gamma & \geq & c+b & \mid -cb \\ c\beta-b\gamma & \geq & -cb+c+b & \end{array}$$

Dark shadow: Proof by contradiction



$$
\begin{array}{rcll}
\frac{\beta}{b} - i & \geq & \frac{1}{b} & \\
i + 1 - \frac{\gamma}{c} & \geq & \frac{1}{c} & \\
\hline\hline
\frac{\beta}{b} + 1 - \frac{\gamma}{c} & \geq & \frac{1}{b} + \frac{1}{c} & \mid \cdot c \cdot b \\
c\beta + cb - b\gamma & \geq & c + b & \mid -cb \\
c\beta - b\gamma & \geq & -cb + c + b & \mid \cdot (-1) \\
{\color{red} b\gamma - c\beta} & {\color{red}\leq} & {\color{red} cb - c - b} &
\end{array}
$$

Dark shadow: Proof by contradiction

- From previous slide:

$$b\gamma - c\beta \quad \leq \quad cb - c - b$$

Dark shadow: Proof by contradiction

- From previous slide:

$$\begin{array}{rrcl} & b\gamma - c\beta & \leq & cb - c - b \\ \leftrightarrow & \neg(b\gamma - c\beta & > & cb - c - b) \end{array}$$

Dark shadow: Proof by contradiction

- From previous slide:

$$
\begin{array}{rrcl}
 & b\gamma - c\beta & \leq & cb - c - b \\
\leftrightarrow & \neg(b\gamma - c\beta & > & cb - c - b) \\
\leftrightarrow & \neg(b\gamma - c\beta & \geq & cb - c - b + 1)
\end{array}
$$

Dark shadow: Proof by contradiction

- From previous slide:

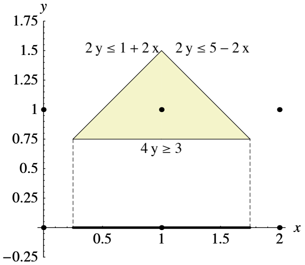
$$
\begin{array}{lrcl}
 & b\gamma - c\beta & \leq & cb - c - b \\
\leftrightarrow & \neg(b\gamma - c\beta & > & cb - c - b) \\
\leftrightarrow & \neg(b\gamma - c\beta & \geq & cb - c - b + 1) \\
\leftrightarrow & \neg(\underbrace{b\gamma - c\beta \;\; \geq \;\; (c-1)(b-1)}_{*}) & &
\end{array}
$$

$\neg(\text{sol}) \rightarrow \neg(*) \rightsquigarrow (*) \rightarrow (\text{sol})$

- Thus, if * holds, we know that there must be an integer solution.
- If $c = 1$ or $b = 1$, then this is the same as the real shadow. This case is called an exact projection.

Example for the dark shadow

② Check **DARK** shadow



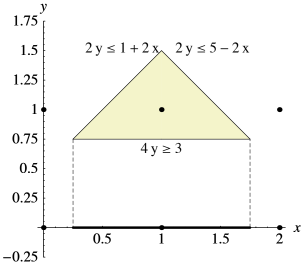
$2y \leq 2x + 1$

$2y \leq -2x + 5$

$4y \geq 3$

Reminder: $(\beta \leq bz \wedge cz \leq \gamma) \rightarrow b\gamma - c\beta \geq (c-1)(b-1)$

② Check **DARK** shadow



$2y \leq 2x+1$

$2y \leq -2x+5$

$4y \geq 3$

;

Eliminate y with the dark shadow:

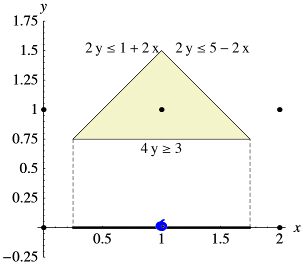
$\overset{c}{2}y \leq \overset{\gamma}{2x+1}$ $\quad$ $\overset{b}{4}y \geq \overset{\beta}{3}$

$$\underset{b}{4}(\underbrace{2x+1}_{\gamma}) - \underset{c}{2} \cdot \underset{\beta}{3} \geq (\underset{c}{2}-1)(\underset{b}{4}-1)$$

Example for the dark shadow

Reminder: $(\beta \leq bz \wedge cz \leq \gamma) \rightarrow b\gamma - c\beta \geq (c-1)(b-1)$

2 Check **DARK** shadow



$2y \leq 2x + 1$

$2y \leq -2x + 5$

$4y \geq 3$

;

Eliminate y with the dark shadow:

$2y \leq 2x + 1$ $\quad$ $4y \geq 3$

$4(2x+1) - 2 \cdot 3 \geq (\underset{1}{2-1})(\underset{3}{4-1})$

$8x \geq 5$

$4y \geq 3$ $\quad$ $2y \leq -2x + 5$

$4(-2x+5) - 2 \cdot 3 \geq (2-1)(4-1)$

$-8x + 20 - 6 \geq 3$

$8x \leq 11 \quad \rightarrow \quad 8x \in [5, 11]$

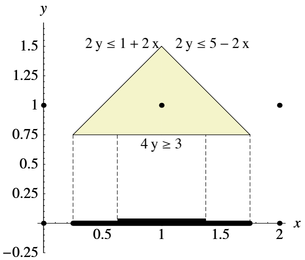
$x \in [5/8, 11/8]$

$x = 1$

Example for the dark shadow

Reminder: $(\beta \leq bz \wedge cz \leq \gamma) \rightarrow b\gamma - c\beta \geq (c-1)(b-1)$

2 Check
DARK shadow



$2y \leq 2x+1$

$2y \leq -2x+5$

$4y \geq 3$

;

Eliminate y with the dark shadow:

$2y \leq 2x+1$ $\quad$ $4y \geq 3$

$$4(2x+1) - 2\cdot 3 \geq (2-1)(4-1)$$

$4y \geq 3$ $\quad$ $2y \leq -2x+5$

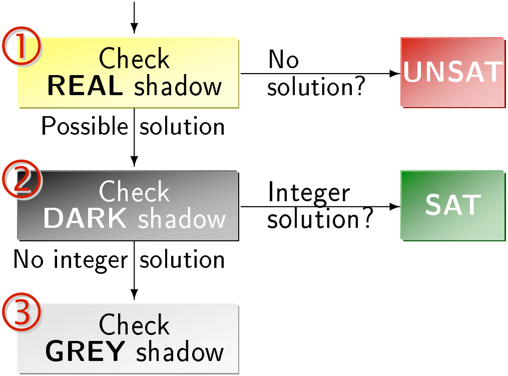
$$4(-2x+5) - 2\cdot 3 \geq (2-1)(4-1)$$

Dark Shadow:

$$\begin{array}{lcl} x & \geq & 5/8 \\ x & \leq & 11/8 \end{array}$$

$\Longrightarrow$ Integer solution!

From dark to grey shadow



- No integer solution in the DARK shadow does not guarantee that there is no integer solution for the original problem.
- Thus, we check the GREY shadow next.

The grey shadow

③ Check
GREY shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

The grey shadow

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$
\begin{array}{rl}
\text{In R:} & b\gamma \geq cbz \geq c\beta \\
\textbf{Not} \text{ in D:} & cb - c - b \geq b\gamma - c\beta \\
\leftrightarrow & cb - c - b + c\beta \geq b\gamma \\
\Leftrightarrow & cb - c - b + c\beta \geq cbz \geq c\beta
\end{array}
$$

In R and not in D

The grey shadow

③ Check **GREY** shadow

Idea of the Grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$
\begin{array}{rll}
\text{In R:} & b\gamma \geq cbz \geq c\beta & \\
\textbf{Not in D:} & cb - c - b \geq b\gamma - c\beta & \\
\leftrightarrow & cb - c - b + c\beta \geq b\gamma & \\
\Rightarrow & cb - c - b + c\beta \geq cbz \geq c\beta & \mid : c \\
& (cb - c - b)/c + \beta \geq bz \geq \beta &
\end{array}
$$

The grey shadow

③ Check
GREY shadow

- Try all values of z such that

$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

③ Check **GREY** shadow

- Try all values of z such that

$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

- Optimization: find the largest coefficient c in any upper bound and try the following for each lower bound $bz \geq \beta$:

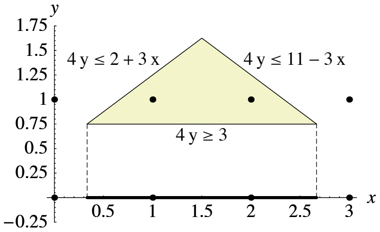
$$bz = \beta + i \quad \text{for } 0 \leq i \leq (cb - c - b)/c$$

- As before, combine this with the original problem, and solve recursively.

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

③ Check **GREY** shadow



$$4y \leq 3x + 2$$

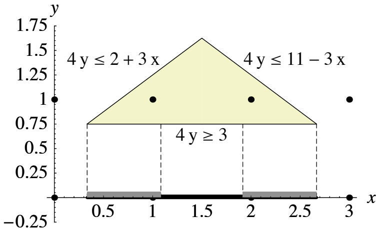
$$4y \leq -3x + 11$$

$$4y \geq 3$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

③ Check **GREY** shadow



$$4y \leq 3x + 2$$

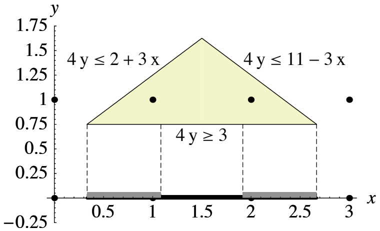
$$4y \leq -3x + 11$$

$$4y \geq 3$$

Example of the grey shadow

Reminder: $bz = \beta + i$ for $0 \leq i \leq (cb - c - b)/c$

3 Check **GREY** shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

$$4y \geq 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$ $\boxed{3 \leq 4y} \leq \dots$

  max. $c = 4$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:

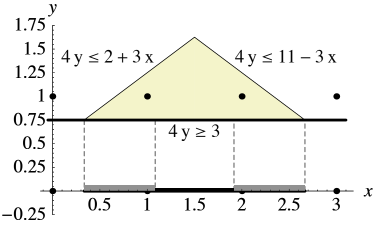
$$\frac{c \cdot b - c - b}{c} = \frac{4 \cdot 4 - 4 - 4}{4} = 2$$

$$\beta \leq y \leq \beta + i \Rightarrow y \in [3,5]$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

③ Check **GREY** shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

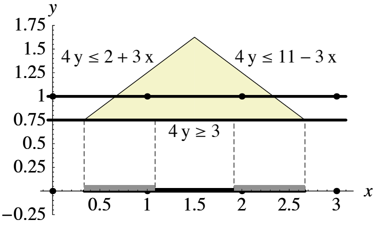
$$4y \geq 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i \quad$ for
  $2 \geq i \geq 0$:
  $$4y = 3$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad \text{for } 0 \leq i \leq (cb - c - b)/c$

③ Check **GREY** shadow



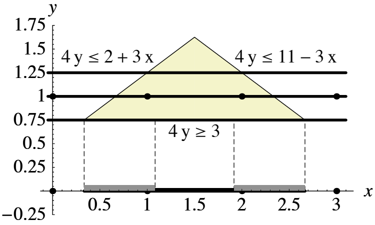
$$4y \leq 3x + 2$$
$$4y \leq -3x + 11$$
$$4y \geq 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i \quad \text{for}$
  $2 \geq i \geq 0$:
  $$4y = 3$$
  $$4y = 4$$

Example of the grey shadow

Reminder: $bz = \beta + i \quad$ for $0 \leq i \leq (cb - c - b)/c$

3 Check **GREY** shadow



$$4y \leq 3x + 2$$

$$4y \leq -3x + 11$$

$$4y \geq 3$$

- Eliminate y:
  $c = 4,\ b = 4,\ \beta = 3$
- New constraint:
  $4y = 3 + i$ for
  $2 \geq i \geq 0$:

  $4y = 3$

  $4y = 4$

  $4y = 5$

$\Longrightarrow$ Integer solution with $4y = 4$