Satisfiability Checking - WS 2016/2017

Series 4

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Exercise 1

In this exercise, we give some more details on the concept of *logical theory* and how it is related to axioms.

We fix an arbitrary signature Σ and an arbitrary structure $\mathcal{S}$ over Σ. In the following, all sentences are over Σ and Φ[1] is a set of sentences. We use the following notation:

- $\mathcal{S} \models \varphi$: $\mathcal{S}$ is a model of a sentence φ.
- $\mathcal{S} \models \Phi$: $\mathcal{S}$ is a model of all sentences φ from the set Φ.

Definitions:

- A sentence φ **is a consequence of** Φ ($\Phi \models \varphi$) iff $\mathcal{S} \models \varphi$ for each model $\mathcal{S} \models \Phi$.
- $\Phi^{\models} := \{\varphi \mid \Phi \models \varphi\}$ denotes the **set of consequences of** Φ.
- Φ is called **consistent** if there is no sentence φ with $\Phi \models \varphi$ and $\Phi \models \neg\varphi$.
- A satisfiable set of sentences T is called a **theory** if for all sentences φ

$$T \models \varphi \quad \iff \quad \varphi \in T.$$

- A theory T is **complete** iff for all sentences φ

$$\text{either} \quad \varphi \in T \quad \text{or} \quad \neg\varphi \in T.$$

Prove the following three statements.

1. Each theory T is consistent.
2. Let Φ be a set of sentences. Φ is consistent iff $\Phi^{\models}$ is a theory.
3. The set $\mathrm{Th}(\mathcal{S}) := \{\varphi \mid \mathcal{S} \models \varphi\}$ is a theory. It is called the **theory of** $\mathcal{S}$.
4. $\mathrm{Th}(\mathcal{S})$ is complete.
5. Let $\Sigma = \{+, \cdot, \leq, =\}$. Give one example each:
   (a) a complete Σ-theory T_1,
   (b) an incomplete Σ-theory T_2.

   Hint: You can use different ways to define a theory.

2 + 2 + 2 + 2 + 4 points

Solution:

[1] Imagine Φ to be a (finite) set of axioms.

1. Suppose there is a sentence φ with $T \models \varphi$ and $T \models \neg\varphi$. Let $\mathcal{S} \models T$. Because T is a theory, $\varphi \in T$ and $\mathcal{S} \models \varphi$. Likewise, $\neg\varphi \in T$ and $\mathcal{S} \not\models \varphi$. $\lightning$ $\mathcal{S} \models \varphi$.

2. "$\Leftarrow$": Since $\Phi^{\models}$ is consistent as a theory, $\Phi \subseteq \Phi^{\models}$ is consistent.

   "$\Rightarrow$": Let Φ be consistent. Because of the construction of $\Phi^{\models}$ and $\models$ is transitive, it holds that

   $$\Phi^{\models} \models \varphi \;\Leftrightarrow\; \varphi \in \Phi^{\models}.$$

   It remains to prove that all sentences in $\Phi^{\models}$ are satisfiable. Assume there is an unsatisfiable sentence $\varphi \in \Phi^{\models}$. Hence, there is no $\mathcal{S} \models \varphi$. Because $\Phi \models \varphi$, there is also no $\mathcal{S} \models \Phi$, i.e., Φ has no models. Therefore, $\Phi \models \psi$ for each Σ sentence ψ; in particular, $\Phi \models \neg\varphi$. $\lightning$ Φ consistent.

3. Let $T := \mathsf{Th}(\mathcal{S})$. Each $\varphi \in T$ is indeed satisfiable since $\mathcal{S} \models \varphi$. It holds that

   $$T \models \varphi \Longleftrightarrow \mathcal{S} \models \varphi \Longleftrightarrow \varphi \in T.$$

4. Let $T := \mathsf{Th}(\mathcal{S})$. We assume that T is not complete. Therefore, a Σ sentence φ exists such that (1) $\varphi \notin T$ and (2) $\neg\varphi \notin T$. (A) implies that $\mathcal{S} \not\models \varphi$, i.e., there is no assignment from the domain of $\mathcal{S}$ to the variables of φ so that φ evaluates to true by the given interpretation of Σ in $\mathcal{S}$. Consequently, any such assignment satisfies $\neg\varphi$. Hence $\mathcal{S} \models \neg\varphi$. Thus, $\neg\varphi \in T$. $\lightning$ (B).

5. (a) We define the theory of a structure, e.g., $T_1 = \mathsf{Th}(\ (\mathbb{N}, +, \cdot, \leq, =)\ )$, as proven in 4..

   (b) We define a theory by a set of axioms, e.g.,

   $$\begin{aligned} T_2 \;=\; \{ & \forall x\, \forall y\; x \leq x, \\ & \forall x\, \forall y\; x \leq y \wedge y \leq x \rightarrow x = y, \\ & \forall x\, \forall y\, \forall z\; x \leq y \wedge y \leq z \rightarrow x \leq z \ \}^{\models} \end{aligned}$$

   the theory of linear orders. It is true that, e.g., $(\mathbb{N}, +, \cdot, \leq, =) \models T_2$, but $\mathsf{Th}(\mathbb{N}, +, \cdot, \leq, =) \supsetneq T_2$. A witness for this issue is, e.g., the sentence

   $$\varphi \;=\; \forall x\; x \leq x \cdot x.$$

   $\varphi \in T_2$, but neither $\varphi \in T_2$ nor $\neg\varphi \in T_2$, because multiplication $\cdot$ is not FO-definable within linear orderings.

   Another solution for T_2 is Presburger arithmetic, $T_2 = \mathsf{Th}(\ (\mathbb{N}, +, \leq, =)\ )$. Our incompleteness proof is based on decidability results:

   (1) satisfiability for Presburger arithmetic is decidable,

   (2) satisfiability for $T_1 = \mathsf{Th}(\ (\mathbb{N}, +, \cdot, \leq, =)\ )$ is undecidable.

   Firstly, we note that $T_2 \subseteq T_1$ because of the same interpretation of the signature.

   If we assume that T_2 is complete, we can define a formula $\varphi_{\cdot}(x, y, z)$ in T_2 so that $\varphi_{\cdot}(x, y, z) \equiv (x \cdot y = z)$ in T_1, i.e., we define multiplication in Presburger arithmetic. Thus, $T_2 \models T_1$, i.e., $T_2 = T_1$ because T_1 is complete due to part (a). This fact and (1) entail that T_1 must be decidable. Contradiction to (2).