Satisfiability Checking - WS 2016/2017

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Series 8

Exercise 1

Consider the first-order logical formula over the reals with addition:

$$\begin{array}{rl} \varphi^{LRA} := & 2x_1 + x_3 \leq 2 \ \wedge\ (-x_1 + 2x_3 \leq 3 \ \vee\ x_3 \leq 0) \\ & \wedge\ x_2 + x_3 \leq -5 \ \wedge\ -x_1 + 4x_2 + x_3 \leq 0 \ \wedge\ -x_2 - 5x_3 \leq 1 \end{array}$$

The Boolean abstraction of this formula is

$$a_1 \wedge (a_2 \vee a_3) \wedge a_4 \wedge a_5 \wedge a_6.$$

Simulate how a less-lazy SMT solver solves φ^{LRA} for satisfiability as presented in the lecture. Show the progress in the SAT solver and the theory solver, which implements the incremental and infeasible subset generating version of the Fourier-Motzkin method, which was presented in the lecture. If the SAT solver makes a decision, it chooses the unassigned variable a_i with the lowest index and assigns it to false. If the Fourier-Motzkin method chooses a variable to eliminate, it chooses the variable x_i with the lowest index. Show how the theory solver benefits from its incrementality support, both when adding and removing constraints, and show how the infeasible subset(s) are computed.

Compared to Exercise 1, Series 7, here, the SMT formula is in negation normal form as it contains no negations. Therefore, a constraint is only added to the theory solver if the corresponding Boolean abstraction variable is assigned to true.

Solution:

In the SAT solver:
We apply Boolean constraint propagation.

$$DL0: \quad a_1 : 1, a_4 : 1, a_5 : 1, a_6 : 1$$

In the theory solver:
The set of constraints to check for consistency are those whose Boolean abstraction is assigned to true.

$$\begin{array}{ll} (1) & 2x_1 + x_3 \leq 2 \\ (2) & x_2 + x_3 \leq -5 \\ (3) & -x_1 + 4x_2 + x_3 \leq 0 \\ (4) & -x_2 - 5x_3 \leq 1 \end{array}$$

We eliminate x_1 by combining its lower bound (3) and its upper bound (1) to

$$\begin{array}{rrcl} & 4x_2 + x_3 & \leq & 1 - \frac{1}{2}x_3 \\ \Leftrightarrow & 4x_2 + \frac{3}{2}x_3 & \leq & 1 \\ \Leftrightarrow & 8x_2 + 3x_3 & \leq & 2 \end{array}$$

Hence, we now consider the constraints:

$$\begin{array}{lll} (2) & x_2 + x_3 \leq -5 & \\ (4) & -x_2 - 5x_3 \leq 1 & \\ (5) & 8x_2 + 3x_3 \leq 2 & \text{from (1) and (3)} \end{array}$$

We eliminate x_2 by combining its lower bound (4) and its upper bound (2) to

$$\begin{array}{rrcl} & -1 - 5x_3 & \leq & -5 - x_3 \\ \Leftrightarrow & -4x_3 & \leq & -4 \\ \Leftrightarrow & -x_3 & \leq & -1 \end{array}$$

and its lower bound (4) and its upper bound (5) to

$$\begin{array}{rll} & -1 - 5x_3 & \leq \frac{1}{4} - \frac{3}{8}x_3 \\ \Leftrightarrow & -\frac{37}{8}x_3 & \leq \frac{5}{4} \\ \Leftrightarrow & -37x_3 & \leq 10 \end{array}$$

Hence, we now consider the constraints:

$$\begin{array}{lll} (6) & -x_3 \leq -1 & \text{from } (2) \text{ and } (4) \\ (7) & -37x_3 \leq 10 & \text{from } (4) \text{ and } (5) \end{array}$$

Eliminating x_3 leads to no new constraint, as it has only lower bounds. All variables are eliminated and no conflicting constraint was created, therefore the theory solver returns that the given set of constraints is consistent.

In the SAT solver:
We decide that a_2 is assigned to false and propagate this decision.

$$\begin{array}{ll} DL0: & a_1:1, a_4:1, a_5:1, a_6:1 \\ DL1: & a_2:0, a_3:1 \end{array}$$

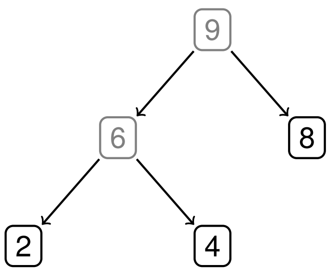
In the theory solver:
We add the constraint $x_3 \leq 0$. As it neither contains x_1 nor x_2 we can start with the previous result after the elimination of x_1 and x_2.

$$\begin{array}{lll} (6) & -x_3 \leq -1 & \text{from } (2) \text{ and } (4) \\ (7) & -37x_3 \leq 10 & \text{from } (4) \text{ and } (5) \\ (8) & x_3 \leq 0 & \end{array}$$

We have two new lower-upper bound pairs to consider, resulting in the following constraints.

$$\begin{array}{lll} (9) & 1 \leq 0 & \text{from } (6) \text{ and } (8) \\ (10) & -\frac{10}{37} \leq 0 & \text{from } (7) \text{ and } (8) \end{array}$$

The constraint (9) is conflicting, thus, the theory solver determines the inconsistency of the currently considered set of constraints. We find a reason for the conflict by use of tracing back the origins of (9):



Hence, the theory solver detects the infeasible subset

$$\{x_2 + x_3 \leq -5,\ -x_2 - 5x_3 \leq 1,\ x_3 \leq 0\}.$$

In the SAT solver:
The infeasible subset leads to the clause $(\neg a_3 \vee \neg a_4 \vee \neg a_6)$ which contains only one literal of the current decision level, thus, forming an asserting clause. We add the clause to the SAT solver's set of clauses, backtrack to decision level 0, assign a_3 to false and apply Boolean constraint propagation leading to:

$$DL0: \quad a_1:1, a_4:1, a_5:1, a_6:1, a_3:0, a_2:1$$

In the theory solver:
We remove the constraint (8) and the constraints having it as origin, which are the constraints (9) and (10). Then we add the constraint $-x_1 + 2x_3 \leq 3$ resulting in:

$$\begin{array}{ll} (1) & 2x_1 + x_3 \leq 2 \\ (2) & x_2 + x_3 \leq -5 \\ (3) & -x_1 + 4x_2 + x_3 \leq 0 \\ (4) & -x_2 - 5x_3 \leq 1 \\ (11) & -x_1 + 2x_3 \leq 3 \end{array}$$

We reuse the former results of the elimination of x_1 and add the combination of its new lower bound (11) with the upper bound (1):

$$\begin{array}{llll} & 2x_3 - 3 & \leq & 1 - \frac{1}{2}x_3 \\ \Leftrightarrow & \frac{5}{2}x_3 & \leq & 4 \\ \Leftrightarrow & 5x_3 & \leq & 8 \end{array}$$

Hence, we now consider the constraints:

$$\begin{array}{lll} (2) & x_2 + x_3 \leq -5 & \\ (4) & -x_2 - 5x_3 \leq 1 & \\ (5) & 8x_2 + 3x_3 \leq 2 & \text{from (1) and (3)} \\ (12) & 5x_3 \leq 8 & \text{from (1) and (11)} \end{array}$$

We just reuse the former results of the elimination of x_2 as no new constraint with this variable were added.

$$\begin{array}{lll} (6) & -x_3 \leq -1 & \text{from (2) and (4)} \\ (7) & -37x_3 \leq 10 & \text{from (4) and (5)} \\ (12) & 5x_3 \leq 8 & \text{from (1) and (11)} \end{array}$$

There are two new combination to consider for the elimination of x_3 resulting in:

$$\begin{array}{lll} (13) & 1 \leq \frac{8}{5} & \text{from (6) and (12)} \\ (14) & -\frac{10}{37} \leq \frac{8}{5} & \text{from (7) and (12)} \end{array}$$

All variables are eliminated and no conflicting constraint was created, therefore the theory solver returns that the given set of constraints is consistent.
As the complete Boolean assignment of the abstraction of φ^{LRA} is consistent, the SMT solver returns SAT.

Exercise 2

Consider the first-order logical formula over the reals with addition:

$$\begin{array}{lllllll} \varphi^{LRA} & := & 2x_1 + x_3 \leq 2 & \wedge & (-x_1 + 2x_3 \leq 3 & \vee & x_3 \leq 0) \\ & \wedge & x_2 + x_3 \leq -5 & \wedge & -x_1 + 4x_2 + x_3 \leq 0 & \wedge & -x_2 - 5x_3 \leq 1 \end{array}$$

The Boolean abstraction of this formula is

$$a_1 \wedge (a_2 \vee a_3) \wedge a_4 \wedge a_5 \wedge a_6.$$

Simulate how a less-lazy SMT solver solves φ^{LRA} for satisfiability as presented in the lecture. Show the progress in the SAT solver and the theory solver, which implements the incremental and infeasible subset generating version of the Simplex method, which was presented in the lecture. If the SAT solver makes a decision, it chooses the unassigned variable a_i with the lowest index and assigns it to false. Use the variable order

$$x_1 < x_2 < x_3 < s_1 < \ldots < s_6$$

for the Simplex method, where s_i corresponds to the slack variable introduced for the constraint abstracted by a_i ($1 \leq i \leq 6$). Show how the theory solver benefits from its incrementality support, both when adding and removing constraints, and show how the infeasible subset(s) are computed.

Compared to Exercise 1, Series 7, here, the SMT formula is in negation normal form as it contains no negations. Therefore, a constraint is only added to the theory solver if the corresponding Boolean abstraction variable is assigned to true.

Solution:
In the SAT solver:
We apply Boolean constraint propagation.

$$DL0: \quad a_1 : 1, a_4 : 1, a_5 : 1, a_6 : 1$$

In the theory solver:
The set of constraints to check for consistency are those whose Boolean abstraction is assigned to true:

$$2x_1 + x_3 \leq 2,\ x_2 + x_3 \leq -5,\ -x_1 + 4x_2 + x_3 \leq 0,\ -x_2 - 5x_3 \leq 1$$

We initialize the tableau for all constraints in φ^{LRA} and the variable bounds for the aforementioned constraints:

	x_1	x_2	x_3
s_1	2	0	1
s_2	-1	0	2
s_3	0	0	1
s_4	0	1	1
s_5	-1	4	1
s_6	0	-1	-5

$$\begin{array}{lcl} s_1 & \leq & 2 \\ s_4 & \leq & -5 \\ s_5 & \leq & 0 \\ s_6 & \leq & 1 \end{array}$$

The initial assignment for all variables is

$$\alpha(x_1) = \alpha(x_2) = \alpha(x_3) = \alpha(s_1) = \ldots = \alpha(s_6) = 0.$$

We apply the Simplex method. The first and only basic variable violating its bound is s_4. Both variables x_2 and x_3 are suitable to fix the assignment of s_4, as both variables are original variables not having any bound. We choose according to the variable order x_2 for pivoting. We solve the equation corresponding to the row of s_4 for x_2:

$$s_4 = x_2 + x_3 \quad \Leftrightarrow \quad x_2 = s_4 - x_3$$

Then we substitute x_2 by $s_4 - x_3$ in the equations of the other rows:

$$\begin{array}{lclcl} s_5 & = & -x_1 + 4(s_4 - x_3) + x_3 & = & -x_1 + 4s_4 - 3x_3 \\ s_6 & = & -(s_4 - x_3) - 5x_3 & = & -s_4 - 4x_3 \end{array}$$

Hence, we get the following tableau after pivoting:

	x_1	s_4	x_3
s_1	2	0	1
s_2	-1	0	2
s_3	0	0	1
x_2	0	1	-1
s_5	-1	4	-3
s_6	0	-1	-4

$$\begin{array}{lcl} s_1 & \leq & 2 \\ s_4 & \leq & -5 \\ s_5 & \leq & 0 \\ s_6 & \leq & 1 \end{array}$$

We update the assignments:

$$
\begin{array}{llllll}
\alpha(s_4) & := & -5 & & & \\
\alpha(x_2) & := & \alpha(x_2)+\theta & = & 0+\frac{(-5)-0}{1} & = & -5
\end{array}
$$

$$
\begin{array}{lllllllll}
\alpha(s_5) & := & (-1)\cdot 0 & + & 4\cdot(-5) & + & (-3)\cdot 0 & = & -20 \\
\alpha(s_6) & := & & & (-1)\cdot(-5) & + & (-4)\cdot 0 & = & 5
\end{array}
$$

Now the basic variable s_6 conflicts its upper bound and, hence, we must try to decrease its assignment. The non-basic variable s_4 is not suitable, as it is at its upper bound and we would need to increase its assignment in order to decrease the assignment of s_6. Even if it was suitable, we must consider the variable x_3 first, as it comes first in the fixed variable order. The variable x_3 has no bounds and is therefore suitable. We solve the equation corresponding to the row of s_6 for x_3:

$$
s_6 = -s_4 - 4x_3 \quad \Leftrightarrow \quad x_3 = -\frac{1}{4}s_4 - \frac{1}{4}s_6
$$

Then we substitute x_3 by $-\frac{1}{4}s_4 - \frac{1}{4}s_6$ in the equations of the other rows:

$$
\begin{array}{lllll}
s_1 & = & 2x_1 + (-\frac{1}{4}s_4 - \frac{1}{4}s_6) & = & 2x_1 - \frac{1}{4}s_4 - \frac{1}{4}s_6 \\
s_2 & = & -x_1 + 2(-\frac{1}{4}s_4 - \frac{1}{4}s_6) & = & -x_1 - \frac{1}{2}s_4 - \frac{1}{2}s_6 \\
s_3 & = & -\frac{1}{4}s_4 - \frac{1}{4}s_6 & & \\
x_2 & = & s_4 - (-\frac{1}{4}s_4 - \frac{1}{4}s_6) & = & \frac{5}{4}s_4 + \frac{1}{4}s_6 \\
s_5 & = & -x_1 + 4s_4 - 3(-\frac{1}{4}s_4 - \frac{1}{4}s_6) & = & -x_1 + \frac{19}{4}s_4 + \frac{3}{4}s_6
\end{array}
$$

Hence, we get the following tableau after pivoting:

	x_1	s_4	s_6
s_1	2	$-\frac{1}{4}$	$-\frac{1}{4}$
s_2	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
s_3	0	$-\frac{1}{4}$	$-\frac{1}{4}$
x_2	0	$\frac{5}{4}$	$\frac{1}{4}$
s_5	-1	$\frac{19}{4}$	$\frac{3}{4}$
x_3	0	$-\frac{1}{4}$	$-\frac{1}{4}$

$$
\begin{array}{lll}
s_1 & \leq & 2 \\
s_4 & \leq & -5 \\
s_5 & \leq & 0 \\
s_6 & \leq & 1
\end{array}
$$

We update the assignments:

$$
\begin{array}{llllll}
\alpha(s_6) & := 1 & & & & \\
\alpha(x_3) & := & \alpha(x_3)+\theta & = & 0+\frac{1-5}{-4} & = & 1
\end{array}
$$

$$
\begin{array}{lllllllll}
\alpha(s_1) & := & 2\cdot 0 & + & (-\frac{1}{4})\cdot(-5) & + & (-\frac{1}{4})\cdot 1 & = & 1 \\
\alpha(s_2) & := & (-1)\cdot 0 & + & (-\frac{1}{2})\cdot(-5) & + & (-\frac{1}{2})\cdot 1 & = & 2 \\
\alpha(s_3) & := & & & (-\frac{1}{4})\cdot(-5) & + & (-\frac{1}{4})\cdot 1 & = & 1 \\
\alpha(x_2) & := & & & \frac{5}{4}\cdot(-5) & + & \frac{1}{4}\cdot 1 & = & -6 \\
\alpha(s_5) & := & (-1)\cdot 0 & + & \frac{19}{4}\cdot(-5) & + & \frac{3}{4}\cdot 1 & = & -23
\end{array}
$$

No variable conflicts its bound, therefore the theory solver returns that the given set of constraints is consistent.

In the SAT solver:
We decide that a_2 is assigned to false and propagate this decision.

$$\begin{array}{rl} DL0: & a_1:1, a_4:1, a_5:1, a_6:1 \\ DL1: & a_2:0, a_3:1 \end{array}$$

In the theory solver:
We add the constraint $x_3 \leq 0$ by just adding the corresponding bound of s_3

	x_1	s_4	s_6
s_1	2	$-\frac{1}{4}$	$-\frac{1}{4}$
s_2	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
s_3	0	$-\frac{1}{4}$	$-\frac{1}{4}$
x_2	0	$\frac{5}{4}$	$\frac{1}{4}$
s_5	-1	$\frac{19}{4}$	$\frac{3}{4}$
x_3	0	$-\frac{1}{4}$	$-\frac{1}{4}$

$$\begin{array}{rcr} s_1 & \leq & 2 \\ s_3 & \leq & 0 \\ s_4 & \leq & -5 \\ s_5 & \leq & 0 \\ s_6 & \leq & 1 \end{array}$$

and keep the assignments as they have been before

$$\alpha(x_1) = 0, \quad \alpha(x_2) = -6, \quad \alpha(x_3) = 1,$$

$$\alpha(s_1) = 1, \quad \alpha(s_2) = 2, \qquad \alpha(s_3) = 1, \quad \alpha(s_4) = -5, \quad \alpha(s_5) = -23, \quad \alpha(s_6) = 1$$

The basic variable s_3 now conflicts its upper bound and none of the non-basic variables in its row is suitable (all have a negative coefficient and are at their upper bound, hence we cannot decrease the assignment of s_3 with their help).

Thus, the theory solver determines the inconsistency of the currently considered set of constraints. We find a reason for the conflict collecting the constraints corresponding to the slack variables s_3, s_4 and s_6 in the conflicting row resulting in the infeasible subset

$$\{x_3 \leq 0,\ x_2 + x_3 \leq -5,\ -x_2 - 5x_3 \leq 1\}.$$

In the SAT solver:
The infeasible subset leads to the clause $(\neg a_3 \vee \neg a_4 \vee \neg a_6)$ which contains only one literal of the current decision level, thus, forming an asserting clause. We add the clause to the SAT solver's set of clauses, backtrack to decision level 0, assign a_3 to false and apply Boolean constraint propagation leading to:

$$DL0: \quad a_1:1, a_4:1, a_5:1, a_6:1, a_3:0, a_2:1$$

In the theory solver:
We remove the constraint $x_3 \leq 0$ by removing its corresponding bound and add the constraint $-x_1 + 2x_3 \leq 3$ by adding its corresponding bound

	x_1	s_4	s_6
s_1	2	$-\frac{1}{4}$	$-\frac{1}{4}$
s_2	-1	$-\frac{1}{2}$	$-\frac{1}{2}$
s_3	0	$-\frac{1}{4}$	$-\frac{1}{4}$
x_2	0	$\frac{5}{4}$	$\frac{1}{4}$
s_5	-1	$\frac{19}{4}$	$\frac{3}{4}$
x_3	0	$-\frac{1}{4}$	$-\frac{1}{4}$

$$\begin{array}{lcl} s_1 & \leq & 2 \\ s_2 & \leq & 3 \\ s_4 & \leq & -5 \\ s_5 & \leq & 0 \\ s_6 & \leq & 1 \end{array}$$

and keep the assignments as they have been before

$$\alpha(x_1) = 0, \quad \alpha(x_2) = -6, \quad \alpha(x_3) = 1,$$

$$\alpha(s_1) = 1, \quad \alpha(s_2) = 2, \quad \alpha(s_3) = 1, \quad \alpha(s_4) = -5, \quad \alpha(s_5) = -23, \quad \alpha(s_6) = 1$$

No variable conflicts its bound, therefore the theory solver returns that the given set of constraints is consistent.
As the complete Boolean assignment of the abstraction of φ^{LRA} is consistent, the SMT solver returns SAT.