Virtual Substition: Substitution Rules

The following shows all cases occuring when a substitution $[e//x]$ is applied to a constraint $p(x) \ \sim\ 0$, with $p(x)$ a polynomial in x. The maximum degree of x in $p(x)$ is k and

$$\delta \ := \ \begin{cases} 1 & \text{, k is odd} \\ 0 & \text{, k is even} \end{cases} .$$

1 Substitution by a fraction

$$e \ = \ \frac{q}{r} \qquad \text{with } q,\ r \text{ polynomials}$$

$p(x) \ = \ 0$**:**

$$p(e) * r^k = 0$$

$p(x) \ \neq \ 0$**:**

$$p(e) * r^k \neq 0$$

$p(x) \ < \ 0$**:**

$$\begin{array}{lllll} & (& r^\delta > 0 & \wedge & p(e) * r^k < 0 &) \\ \vee & (& r^\delta < 0 & \wedge & p(e) * r^k > 0 &) \end{array}$$

$p(x) \ > \ 0$**:**

$$\begin{array}{lllll} & (& r^\delta > 0 & \wedge & p(e) * r^k > 0 &) \\ \vee & (& r^\delta < 0 & \wedge & p(e) * r^k < 0 &) \end{array}$$

$p(x) \ \leq \ 0$**:**

$$\begin{array}{lllll} & (& r^\delta > 0 & \wedge & p(e) * r^k \leq 0 &) \\ \vee & (& r^\delta < 0 & \wedge & p(e) * r^k \geq 0 &) \end{array}$$

$p(x) \ \geq \ 0$**:**

$$\begin{array}{lllll} & (& r^\delta > 0 & \wedge & p(e) * r^k \geq 0 &) \\ \vee & (& r^\delta < 0 & \wedge & p(e) * r^k \leq 0 &) \end{array}$$

2 Substitution by a square root term

Considering e as a square root term, it has the form

$$e \ = \ \frac{q + r * \sqrt{t}}{s} \qquad \text{with } q,\ r,\ s,\ t \text{ polynomials.}$$

Theorem: Given are a polynomial $f(x)$ and an expression e of the form

$$e \ := \ \frac{q + r\sqrt{t}}{s} \qquad (*).$$

Then $f(e)$ is of the form $(*)$.

Proof: Polynomials have just the two operators plus and times. We show that both operations will map two expressions of the form $(*)$ to another expression, which again has this form. Keep in mind, that the radicand of both operands must be the same.

1. Addition of two expressions of the form $(*)$:

$$\begin{aligned} & \tfrac{q_1+r_1\sqrt{t}}{s_1} + \tfrac{q_2+r_2\sqrt{t}}{s_2} \\ = \quad & \tfrac{s_2(q_1+r_1\sqrt{t})+s_1(q_2+r_2\sqrt{t})}{s_1 s_2} \\ = \quad & \tfrac{s_2q_1+s_2r_1\sqrt{t}+s_1q_2+s_1r_2\sqrt{t}}{s_1s_2} \\ = \quad & \tfrac{(s_2q_1+s_1q_2)+(s_2r_1+s_1r_2)\sqrt{t}}{(s_1s_2)} \end{aligned}$$

2. Multiplication of two expressions of the form $(*)$:

$$\begin{aligned} & \tfrac{q_1+r_1\sqrt{t}}{s_1} * \tfrac{q_2+r_2\sqrt{t}}{s_2} \\ = \quad & \tfrac{(q_1+r_1\sqrt{t})(q_2+r_2\sqrt{t})}{s_1s_2} \\ = \quad & \tfrac{q_1q_2+r_1\sqrt{t}q_2+q_1r_2\sqrt{t}+r_1\sqrt{t}r_2\sqrt{t}}{s_1s_2} \\ = \quad & \tfrac{(q_1q_2+r_1r_2t)+(r_1q_2+q_1r_2)\sqrt{t}}{(s_1s_2)} \end{aligned}$$

Hence, substituting all x in $p(x)$ by e leads according the above theorem to a square root term

$$p(e) \ = \ \frac{\hat{q} + \hat{r} * \sqrt{t}}{\hat{s}} \qquad \text{with } \hat{q},\ \hat{r},\ \hat{s} \ \text{ polynomials}$$

or, if $\hat{r} = 0$, to a fraction

$$p(e) \ = \ \frac{\hat{q}}{\hat{s}} \qquad \text{with } \hat{q},\ \hat{s} \ \text{ polynomials.}$$

In the latter case the substitution rules of Section 1 hold; Otherwise the following rules define an equivalent real algebraic formula:

$p(x) \; = \; 0$**:**

$$\hat{q} * \hat{r} \leq 0 \quad \wedge \quad \hat{q}^2 - \hat{r}^2 * t = 0$$

$$\begin{array}{rlcccccl}
= & (& \hat{r} = 0 & \wedge & \hat{q} = 0 & & &) \\
\vee & (& \hat{q} = 0 & \wedge & t = 0 & & &) \\
\vee & (& \hat{q} < 0 & \wedge & \hat{r} > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t = 0 &) \\
\vee & (& \hat{q} > 0 & \wedge & \hat{r} < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t = 0 &)
\end{array}$$

$p(x) \; \neq \; 0$**:**

$$\begin{array}{rlcl}
 & (& \hat{q} * \hat{r} > 0 &) \\
\vee & (& \hat{q}^2 - \hat{r}^2 * t \neq 0 &)
\end{array}$$

$$\begin{array}{rlcccl}
= & (& \hat{r} > 0 & \wedge & \hat{q} > 0 &) \\
\vee & (& \hat{r} < 0 & \wedge & \hat{q} < 0 &) \\
\vee & (& \hat{q}^2 - \hat{r}^2 * t \neq 0 & & &)
\end{array}$$

$p(x) \; < \; 0$**:**

$$\begin{array}{rlcccl}
 & (& \hat{q} * \hat{s}^\delta < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t > 0 &) \\
\vee & (& \hat{r} * \hat{s}^\delta \leq 0 & \wedge & \hat{q} * \hat{s}^\delta < 0 &) \\
\vee & (& \hat{r} * \hat{s}^\delta \leq 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t < 0 &)
\end{array}$$

$$\begin{array}{rlcccccl}
= & (& \hat{q} < 0 & \wedge & \hat{s}^\delta > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t > 0 &) \\
\vee & (& \hat{q} > 0 & \wedge & \hat{s}^\delta < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t > 0 &) \\
\vee & (& \hat{r} \geq 0 & \wedge & \hat{q} < 0 & \wedge & \hat{s}^\delta > 0 &) \\
\vee & (& \hat{r} \leq 0 & \wedge & \hat{q} > 0 & \wedge & \hat{s}^\delta < 0 &) \\
\vee & (& \hat{r} > 0 & \wedge & \hat{s}^\delta < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t < 0 &) \\
\vee & (& \hat{r} < 0 & \wedge & \hat{s}^\delta > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t < 0 &)
\end{array}$$

$p(x) \; > \; 0$**:**

$$\begin{array}{rlcccl}
 & (& \hat{q} * \hat{s}^\delta > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t > 0 &) \\
\vee & (& \hat{r} * \hat{s}^\delta \geq 0 & \wedge & \hat{q} * \hat{s}^\delta > 0 &) \\
\vee & (& \hat{r} * \hat{s}^\delta \geq 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t < 0 &)
\end{array}$$

$$\begin{array}{rlcccccl}
= & (& \hat{q} > 0 & \wedge & \hat{s}^\delta > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t > 0 &) \\
\vee & (& \hat{q} < 0 & \wedge & \hat{s}^\delta < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t > 0 &) \\
\vee & (& \hat{r} \leq 0 & \wedge & \hat{q} < 0 & \wedge & \hat{s}^\delta < 0 &) \\
\vee & (& \hat{r} \geq 0 & \wedge & \hat{q} > 0 & \wedge & \hat{s}^\delta > 0 &) \\
\vee & (& \hat{r} > 0 & \wedge & \hat{s}^\delta > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t < 0 &) \\
\vee & (& \hat{r} < 0 & \wedge & \hat{s}^\delta < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t < 0 &)
\end{array}$$

$p(x) \ \leq \ 0$:

$$
\begin{array}{rllll}
 & (& \hat{q} * \hat{s}^{\delta} \leq 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \geq 0 \quad) \\
\vee & (& \hat{r} * \hat{s}^{\delta} \leq 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \leq 0 \quad)
\end{array}
$$

$$
\begin{array}{rrcccccl}
= & (& \hat{q} < 0 & \wedge & \hat{s}^{\delta} > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \geq 0 &) \\
\vee & (& \hat{q} > 0 & \wedge & \hat{s}^{\delta} < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \geq 0 &) \\
\vee & (& \hat{r} = 0 & \wedge & \hat{q} = 0 & & &) \\
\vee & (& \hat{q} = 0 & \wedge & t = 0 & & &) \\
\vee & (& \hat{r} > 0 & \wedge & \hat{s}^{\delta} < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \leq 0 &) \\
\vee & (& \hat{r} < 0 & \wedge & \hat{s}^{\delta} > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \leq 0 &)
\end{array}
$$

$p(x) \ \geq \ 0$:

$$
\begin{array}{rllll}
 & (& \hat{q} * \hat{s}^{\delta} \geq 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \geq 0 \quad) \\
\vee & (& \hat{r} * \hat{s}^{\delta} \geq 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \leq 0 \quad)
\end{array}
$$

$$
\begin{array}{rrcccccl}
= & (& \hat{q} > 0 & \wedge & \hat{s}^{\delta} > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \geq 0 &) \\
\vee & (& \hat{q} < 0 & \wedge & \hat{s}^{\delta} < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \geq 0 &) \\
\vee & (& \hat{r} = 0 & \wedge & \hat{q} = 0 & & &) \\
\vee & (& \hat{q} = 0 & \wedge & t = 0 & & &) \\
\vee & (& \hat{r} > 0 & \wedge & \hat{s}^{\delta} > 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \leq 0 &) \\
\vee & (& \hat{r} < 0 & \wedge & \hat{s}^{\delta} < 0 & \wedge & \hat{q}^2 - \hat{r}^2 * t \leq 0 &)
\end{array}
$$

3 Substitution by a term plus an infinitesimal

Substitution by $[e + \epsilon // x]$:

$bx + c \ = \ 0$:

$$b = 0 \quad \wedge \quad c = 0$$

$bx + c \ \neq \ 0$:

$$
\begin{array}{rl}
 & b \neq 0 \\
\vee & c \neq 0
\end{array}
$$

$bx + c \ < \ 0$:

$$
\begin{array}{rllll}
 & (& (bx + c < 0)[e//x] & &) \\
\vee & (& (bx + c = 0)[e//x] & \wedge \quad (b < 0)[e//x] &)
\end{array}
$$

$bx + c \ > \ 0$:

$$
\begin{array}{rllll}
 & (& (bx + c > 0)[e//x] & &) \\
\vee & (& (bx + c = 0)[e//x] & \wedge \quad (b > 0)[e//x] &)
\end{array}
$$

$bx + c \ \leq \ 0$**:**

$$\begin{array}{lllll} & (& (bx+c<0)[e//x] & & &) \\ \vee & (& (bx+c=0)[e//x] & \wedge & (b<0)[e//x] &) \\ \vee & (& b=0 & \wedge & c=0 &) \end{array}$$

$bx + c \ \geq \ 0$**:**

$$\begin{array}{lllll} & (& (bx+c>0)[e//x] & & &) \\ \vee & (& (bx+c=0)[e//x] & \wedge & (b>0)[e//x] &) \\ \vee & (& b=0 & \wedge & c=0 &) \end{array}$$

$ax^2 + bx + c \ = \ 0$**:**

$$a = 0 \quad \wedge \quad b = 0 \quad \wedge \quad c = 0$$

$ax^2 + bx + c \ \neq \ 0$**:**

$$\begin{array}{ll} & a \neq 0 \\ & b \neq 0 \\ \vee & c \neq 0 \end{array}$$

$ax^2 + bx + c \ < \ 0$**:**

$$\begin{array}{lllllll} & (& (ax^2+bx+c<0)[e//x] & & & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b<0)[e//x] & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b=0)[e//x] & \wedge & (2a<0)[e//x] &) \end{array}$$

$ax^2 + bx + c \ > \ 0$**:**

$$\begin{array}{lllllll} & (& (ax^2+bx+c>0)[e//x] & & & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b>0)[e//x] & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b=0)[e//x] & \wedge & (2a>0)[e//x] &) \end{array}$$

$ax^2 + bx + c \ \leq \ 0$**:**

$$\begin{array}{lllllll} & (& (ax^2+bx+c<0)[e//x] & & & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b<0)[e//x] & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b=0)[e//x] & \wedge & (2a<0)[e//x] &) \\ \vee & (& a=0 & \wedge & b=0 & \wedge & c=0 &) \end{array}$$

$ax^2 + bx + c \ \geq \ 0$**:**

$$\begin{array}{lllllll} & (& (ax^2+bx+c>0)[e//x] & & & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b>0)[e//x] & & &) \\ \vee & (& (ax^2+bx+c=0)[e//x] & \wedge & (2ax+b=0)[e//x] & \wedge & (2a>0)[e//x] &) \\ \vee & (& a=0 & \wedge & b=0 & \wedge & c=0 &) \end{array}$$

4 Substitution by minus infinity

Substitution by $[-\infty//x]$:

$bx + c \ = \ 0$**:**

$$b = 0 \quad \wedge \quad c = 0$$

$bx + c \ \neq \ 0$**:**

$$\begin{array}{ll} & b \neq 0 \\ \vee & c \neq 0 \end{array}$$

$bx + c \ < \ 0$**:**

$$\begin{array}{lllll} & (& b > 0 & &) \\ \vee & (& b = 0 & \wedge \quad c < 0 &) \end{array}$$

$bx + c \ > \ 0$**:**

$$\begin{array}{lllll} & (& b < 0 & &) \\ \vee & (& b = 0 & \wedge \quad c > 0 &) \end{array}$$

$bx + c \ \leq \ 0$**:**

$$\begin{array}{lllll} & (& b > 0 & &) \\ \vee & (& b = 0 & \wedge \quad c \leq 0 &) \end{array}$$

$bx + c \ \geq \ 0$**:**

$$\begin{array}{lllll} & (& b < 0 & &) \\ \vee & (& b = 0 & \wedge \quad c \geq 0 &) \end{array}$$

$ax^2 + bx + c \ = \ 0$**:**

$$a = 0 \quad \wedge \quad b = 0 \quad \wedge \quad c = 0$$

$ax^2 + bx + c \ \neq \ 0$**:**

$$\begin{array}{ll} & a \neq 0 \\ \vee & b \neq 0 \\ \vee & c \neq 0 \end{array}$$

$ax^2 + bx + c \ < \ 0$**:**

$$\begin{array}{llllll} & (& a < 0 & & &) \\ \vee & (& a = 0 & \wedge \quad b > 0 & &) \\ \vee & (& a = 0 & \wedge \quad b = 0 & \wedge \quad c < 0 &) \end{array}$$

$ax^2 + bx + c \ > \ 0$**:**

$$\begin{array}{llllll} & (& a > 0 & & &) \\ \vee & (& a = 0 & \wedge \quad b < 0 & &) \\ \vee & (& a = 0 & \wedge \quad b = 0 & \wedge \quad c > 0 &) \end{array}$$

$ax^2 + bx + c \ \leq \ 0$**:**

$$\begin{array}{cclcccccl}
 & (& a < 0 & & & & &) \\
\vee & (& a = 0 & \wedge & b > 0 & & &) \\
\vee & (& a = 0 & \wedge & b = 0 & \wedge & c \leq 0 &)
\end{array}$$

$ax^2 + bx + c \ \geq \ 0$**:**

$$\begin{array}{cclcccccl}
 & (& a > 0 & & & & &) \\
\vee & (& a = 0 & \wedge & b < 0 & & &) \\
\vee & (& a = 0 & \wedge & b = 0 & \wedge & c \geq 0 &)
\end{array}$$