Satisfiability Checking

Propositional Logic

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Informatik 2
LuFG Theory of Hybrid Systems

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Propositional logic

The slides are partly taken from:

www.decision-procedures.org/slides/

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

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Formulae

- Examples of well-formed formulae:
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- We omit parentheses whenever we may restore them through operator precedence:

binds stronger

$$\xleftarrow{\hspace{6em}}$$

$$\neg \quad \land \quad \lor \quad \rightarrow \quad \leftrightarrow$$

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Semantics: Assignments

Structures for predicate logic:

- The domain is $\mathbb{B} = \{0, 1\}$.
- The interpretation assigns Boolean values to the variables:

$$\alpha : \mathit{AP} \to \{0, 1\}$$

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Only the projected assignment matters...

- Let $\alpha_1, \alpha_2 \in$ *Assign* and $\varphi \in$ *APForm*.
- Let $AP(\varphi)$ be the atomic propositions in φ.
- Clearly $AP(\varphi) \subseteq AP$.
- Lemma: if $\alpha_1|_{AP(\varphi)} = \alpha_2|_{AP(\varphi)}$, then

Projection

$$(\alpha_1 \textit{ satisfies } \varphi) \quad \text{iff} \quad (\alpha_2 \textit{ satisfies } \varphi)$$

- We will assume, for simplicity, that $AP = AP(\varphi)$.

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p	q	$\neg p$	$p \land q$	$p \lor q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \bigoplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
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α satisfies φ iff in the line for α and the column for φ the entry is 1.

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- Q: Does α satisfy φ?
- A1: Compute with truth table:

a	b	c	$b \rightarrow c$	$a \vee (b \rightarrow c)$
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Semantics II: Satisfaction relation

Satisfaction relation: $\models \subseteq \mathit{Assign} \times \mathit{APForm}$

Instead of $(\alpha, \varphi) \in \models$ we write $\alpha \models \varphi$ and say that

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- φ holds for α or
- α is a model of φ.

$\models$ is defined recursively:

$$
\begin{array}{lll}
\alpha \models p & \textit{iff} & \alpha(p) = \mathit{true} \\
\alpha \models \neg\varphi & \textit{iff} & \alpha \not\models \varphi \\
\alpha \models \varphi_1 \wedge \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ and } \alpha \models \varphi_2 \\
\alpha \models \varphi_1 \vee \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ or } \alpha \models \varphi_2 \\
\alpha \models \varphi_1 \rightarrow \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ implies } \alpha \models \varphi_2 \\
\alpha \models \varphi_1 \leftrightarrow \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ iff } \alpha \models \varphi_2
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Note: More elegant but semantically equivalent to truth tables.

Semantics II: Example

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A2: Compute with the satisfaction relation:

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- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \rightarrow \{0, 1\}$ is a model of a propositional logic formula $\varphi \in \mathit{APForm}$:

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- Q: Complexity? A: Polynomial (time and space).

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- Recall our example
  - $\varphi = (a \vee (b \rightarrow c))$
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- Hence, $\alpha \models \varphi$.

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- For $\varphi \in APForm$ and $\alpha \in Assign$ it holds that

$$\alpha \models \varphi \qquad \textit{iff} \qquad \alpha \in sat(\varphi)$$

Satisfying assignments: Example

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Extensions of $\models$

- We define $\models \subseteq 2^{Assign} \times APForm$ by

$$T \models \varphi \textit{ iff } T \subseteq sat(\varphi)$$

  for formulae $\varphi \in APForm$ and assignment sets $T \subseteq 2^{Assign}$.

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  Examples: $a \wedge c \models a \vee (b \rightarrow c)$
  $x_1 \models x_1 \vee x_2$

Short summary for propositional logic

- Syntax of propositional formulae $\varphi \in APForm$:

$$\varphi \ := \ AP \mid (\neg\varphi) \mid (\varphi \wedge \varphi)$$

- Semantics:
  - Assignments $\alpha \in Assign$:

$$\begin{array}{l} \alpha : AP \rightarrow \{0,1\} \\ \alpha \in 2^{AP} \\ \alpha \in \{0,1\}^{AP} \end{array}$$

  - Satisfaction relation:

$$\begin{array}{lll} \models \ \subseteq Assign \times APForm & , & (\text{e.g., } \alpha \models \varphi\ ) \\ \models \ \subseteq 2^{Assign} \times APForm & , & (\text{e.g., } \{\alpha_1, \ldots, \alpha_n\} \models \varphi\ ) \\ \models \ \subseteq APForm \times APForm & , & (\text{e.g., } \varphi_1 \models \varphi_2) \\ sat : \ APForm \rightarrow 2^{Assign} & , & (\text{e.g., } \ sat(\varphi)\ ) \end{array}$$

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

Semantic classification of formulae

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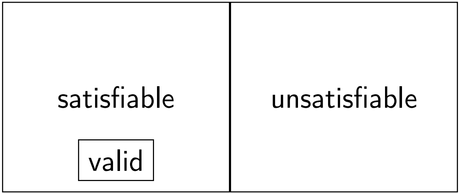

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- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
  - $\models \neg(a \vee b) \leftrightarrow (\neg a \wedge \neg b)$

The satisfiability problem for propositional logic

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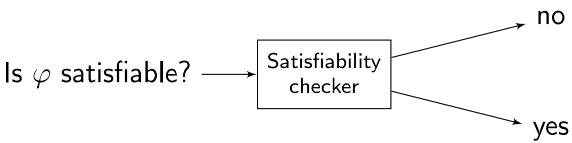

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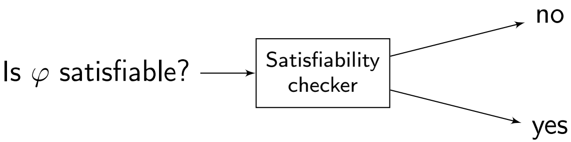

Note: A formula φ is valid iff $\neg\varphi$ is unsatisfiable.

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
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- There are numerous problems in the industry that are solved via the satisfiability problem of propositional logic
  - Logistics
  - Planning
  - Electronic Design Automation industry
  - Cryptography
  - ...

Example 1: Placement of wedding guests

- Three chairs in a row: $1, 2, 3$
- We need to place Aunt, Sister and Father.
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  - Aunt doesn't want to sit near Father
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- Q: Can we satisfy these constraints?

Example 1 (continued)

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- n radio stations
- For each station assign one of k transmission frequencies, $k < n$.
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- Q: Can we assign to each station a frequency, such that no station pairs from E have the same frequency?

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Example 3: Seminar topic assignment

- n participants
- n topics
- Set of preferences $E \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}$
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Example 3 (continued)

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Negation Normal Form (NNF)

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- $\varphi_1 = \neg(a \vee \neg b)$ is not in NNF
- $\varphi_2 = \neg a \wedge b$ is in NNF

Converting to NNF

- Every formula can be converted to NNF in linear time:
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- DNF is a special case of NNF.

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  A: 2^n

Satisfiability of DNF

- Q: Is the following DNF formula satisfiable?

$$(a_1 \wedge a_2 \wedge \neg a_1) \vee (a_2 \wedge a_1) \vee (a_2 \wedge \neg a_3 \wedge a_3)$$

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  A: Yes, because the term $a_2 \land a_1$ is satisfiable.

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  A: Yes, because the term $a_2 \wedge a_1$ is satisfiable.
- Q: What is the complexity of the satisfiability check of DNF formulae?

Satisfiability of DNF

- Q: Is the following DNF formula satisfiable?

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- A: No, it would violate the NP-completeness of the problem.

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- Also CNF is a special case of NNF.

Converting to CNF

- Every formula can be converted to CNF in exponential time and space:
  1. Convert to NNF
  2. Distribute disjunctions following the rule:
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- Now consider $\varphi_n = (a_1 \wedge b_1) \vee (a_2 \wedge b_2) \vee \ldots \vee (a_n \wedge b_n)$.

  Q: How many clauses does the resulting CNF have?

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  Q: How many clauses does the resulting CNF have?

  A: 2^n

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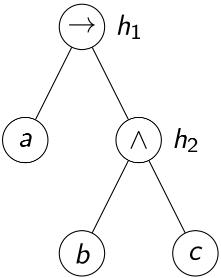
Parse tree:



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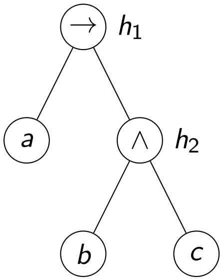

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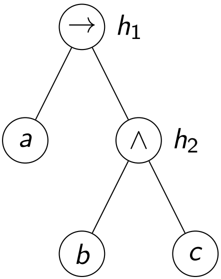

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- Consider the formula
  $\varphi = (a \rightarrow (b \wedge c))$
- Associate a new auxiliary variable with each gate.
- Add constraints that define these new variables.
- Finally, enforce the root node.

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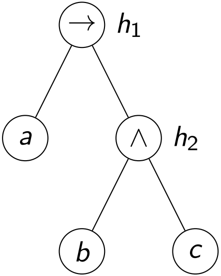
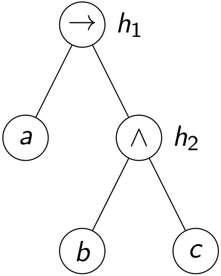


Converting to CNF: Tseitin's encoding

- Need to satisfy:
  $(h_1 \leftrightarrow (a \;\rightarrow\; h_2))\wedge$
  $(h_2 \leftrightarrow (b \wedge c))\wedge$
  (h_1)



- Each gate encoding has a CNF representation with 3 or 4 clauses.

Converting to CNF: Tseitin's encoding

- Need to satisfy:

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Converting to CNF: Tseitin's encoding

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- Second: $(\neg h_2 \lor b) \land (\neg h_2 \lor c) \land (h_2 \lor \neg b \lor \neg c)$

Converting to CNF: Tseitin's encoding

- Let's go back to

  $\varphi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$

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- With Tseitin's encoding we need:
  - n auxiliary variables $h_1, \ldots, h_n$.
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- Hence, we have
  - $3n + 1$ clauses, instead of 2^n.
  - $3n$ variables rather than $2n$.

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Modeling with propositional logic
- Normal forms
- Enumeration and deduction

Two classes of algorithms for validity

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- Two classes of algorithms for finding out:
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- More generally (beyond propositional logic):
  - Enumeration is possible only in some logics.
  - Deduction cannot necessarily be fully automated.

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- Q: What is the difference?
  A: Branching on complete vs. partial assignments.

Deduction requires axioms and inference rules

- Inference rules:

$$\frac{\textit{Antecedents}}{\textit{Consequents}} \qquad (\textit{rule name})$$

Meaning: If all antecedents hold then at least one of the consequents can be derived.

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Meaning: If all antecedents hold then at least one of the consequents can be derived.

- Examples:

$$\frac{a \rightarrow b \qquad b \rightarrow c}{a \rightarrow c} \qquad (\textit{Trans})$$

$$\frac{a \rightarrow b \qquad a}{b} \qquad (M.P.)$$

Axioms

- Axioms are inference rules with no antecedents, e.g.,

$$\frac{}{a \rightarrow (b \rightarrow a)} \quad (H1)$$

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- A proof system consists of a set of axioms and inference rules.

Proofs

- Let $\mathcal{H}$ be a proof system.
- $\Gamma \vdash_{\mathcal{H}} \varphi$ means: There is a proof of φ in system $\mathcal{H}$ whose premises are included in Γ
- $\vdash_{\mathcal{H}}$ is called the provability (derivability) relation.

Example

- Let $\mathcal{H}$ be the proof system comprised of the rules Trans and M.P. that we saw earlier:

$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \qquad (Trans)$$

$$\frac{a \rightarrow b \quad a}{b} \qquad (M.P.)$$

- Does the following relation hold?

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

Deductive proof: Example

$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \quad (Trans) \qquad \frac{a \rightarrow b \quad a}{b} \quad (M.P.)$$

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

Deductive proof: Example

$$\frac{a \to b \quad b \to c}{a \to c} \quad (Trans) \qquad \frac{a \to b \quad a}{b} \quad (M.P.)$$

$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1. $a \to b$

Deductive proof: Example

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$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1. $a \rightarrow b$ *premise*

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1. $a \rightarrow b$ *premise*
2. $b \rightarrow c$

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2. $b \rightarrow c$ *premise*
3. $a \rightarrow c$

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2. $b \rightarrow c$ *premise*
3. $a \rightarrow c$ 1, 2, *Trans*

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2. $b \rightarrow c$ *premise*
3. $a \rightarrow c$ 1, 2, *Trans*
4. $c \rightarrow d$

Deductive proof: Example

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$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

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6. $c \to e$

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4. $c \rightarrow d$ *premise*
5. $d \rightarrow e$ *premise*
6. $c \rightarrow e$ 4, 5, *Trans*

Deductive proof: Example

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1.	$a \rightarrow b$	*premise*
2.	$b \rightarrow c$	*premise*
3.	$a \rightarrow c$	1, 2, *Trans*
4.	$c \rightarrow d$	*premise*
5.	$d \rightarrow e$	*premise*
6.	$c \rightarrow e$	4, 5, *Trans*
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6. $c \to e$ 4, 5, *Trans*
7. $a \to e$ 3, 6, *Trans*

Deductive proof: Example

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3. $a \rightarrow c$ 1, 2, *Trans*
4. $c \rightarrow d$ *premise*
5. $d \rightarrow e$ *premise*
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7. $a \rightarrow e$ 3, 6, *Trans*
8. a

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8. a *premise*

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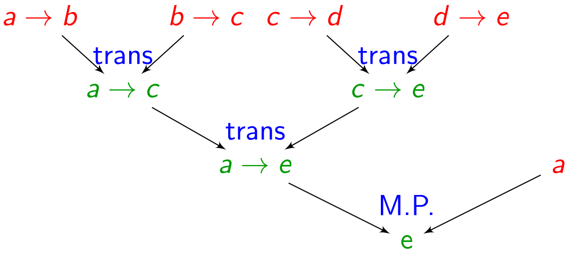
$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \quad (Trans) \qquad \frac{a \rightarrow b \quad a}{b} \quad (M.P.)$$

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1.	$a \rightarrow b$	*premise*
2.	$b \rightarrow c$	*premise*
3.	$a \rightarrow c$	1, 2, *Trans*
4.	$c \rightarrow d$	*premise*
5.	$d \rightarrow e$	*premise*
6.	$c \rightarrow e$	4, 5, *Trans*
7.	$a \rightarrow e$	3, 6, *Trans*
8.	a	*premise*
9.	e	7, 8, *M.P.*

Proof graph


$a \rightarrow b$
$b \rightarrow c$
$c \rightarrow d$
$d \rightarrow e$
trans
trans
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a
M.P.
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  With respect to the semantic definition of the logic. In the case of propositional logic truth tables give us this.

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- How to prove soundness and completeness?

Example: Hilbert axiom system (H)

- Let H be (M.P.) together with the following axiom schemes:

$$\frac{}{a \to (b \to a)} \quad (H1)$$

$$\frac{}{((a \to (b \to c)) \to ((a \to b) \to (a \to c)))} \quad (H2)$$

$$\frac{}{(\neg b \to \neg a) \to (a \to b)} \quad (H3)$$

- H is sound and complete for propositional logic.

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0	0	1
0	1	1
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- Completeness: harder, but possible.

The resolution proof system

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- The resolution inference rule for CNF:

$$\frac{(l \vee l_1 \vee l_2 \vee \ldots \vee l_n) \quad (\neg l \vee l'_1 \vee \ldots \vee l'_m)}{(l_1 \vee \ldots \vee l_n \vee l'_1 \vee \ldots \vee l'_m)} \; \textit{Resolution}$$

- Example:

$$\frac{(a \vee b) \quad (\neg a \vee c)}{(b \vee c)}$$

- We first see some example proofs, before proving soundness and completeness.

Proof by resolution

- Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2 \vee a_5) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4)$
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$$\begin{tikzcd}
 & (\neg a_1 \vee a_4) \arrow[dr] & & (\neg a_1 \vee \neg a_4) \arrow[dl] \\
(a_1 \vee a_3) \arrow[dr] & & (\neg a_1) \arrow[dl] & \\
 & (a_3) & &
\end{tikzcd}$$

Resolution

- Resolution is a sound and complete proof system for CNF.
- If the input formula is unsatisfiable, there exists a proof of the empty clause.

Example

Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4) \wedge (\neg a_3)$.

$$
\begin{tikzcd}
(\neg a_1 \vee a_4) \arrow[dr] & & (\neg a_1 \vee \neg a_4) \arrow[dl] & & \\
& (\neg a_1) \arrow[dr] & & (a_1 \vee a_3) \arrow[dl] & \\
& & a_3 \arrow[dr] & & \neg a_3 \arrow[dl] \\
& & & () &
\end{tikzcd}
$$

Soundness and completeness of resolution

- Soundness

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- Soundness is straightforward. Just prove by truth table that

  $\models ((\varphi_1 \vee a) \wedge (\varphi_2 \vee \neg a)) \rightarrow (\varphi_1 \vee \varphi_2)$.

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  $$\models ((\varphi_1 \vee a) \wedge (\varphi_2 \vee \neg a)) \rightarrow (\varphi_1 \vee \varphi_2).$$

- Completeness is a bit more involved.
  Basic idea: Use resolution for variable elimination.

$$
\begin{array}{c}
(a \vee \varphi_1) \wedge \ldots \wedge (a \vee \varphi_n) \wedge \\
(\neg a \vee \psi_1) \wedge \ldots (\neg a \vee \psi_m) \wedge \\
R \\
\Leftrightarrow \\
(\varphi_1 \vee \psi_1) \wedge \ldots \wedge (\varphi_1 \vee \psi_m) \wedge \\
\ldots \\
(\varphi_n \vee \psi_1) \wedge \ldots (\varphi_n \vee \psi_m) \wedge \\
R
\end{array}
$$

where φ_i $(i = 1, \ldots, n)$, ψ_j $(j = 1, \ldots, m)$, and R contains neither a nor $\neg a$.