Satisfiability Checking

Interval Constraint Propagation

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Non-linear real arithmetic

We consider input formulae φ from the theory of quantifier-free nonlinear real arithmetic (QFNRA):

multiplication makes the difference to linear arithmetic

p	$:= const \mid x \mid (p + p) \mid (p - p) \mid \boxed{(p \cdot p)}$	polynomials
c	$:= p < 0 \mid p = 0$	(polynomial) constraints
φ	$:= c \mid (\varphi \land \varphi) \mid \neg\varphi$	QFNRA formulas

where constants *const* and variables x take real values from $\mathbb{R}$.

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where constants *const* and variables x take real values from $\mathbb{R}$.

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\end{array}
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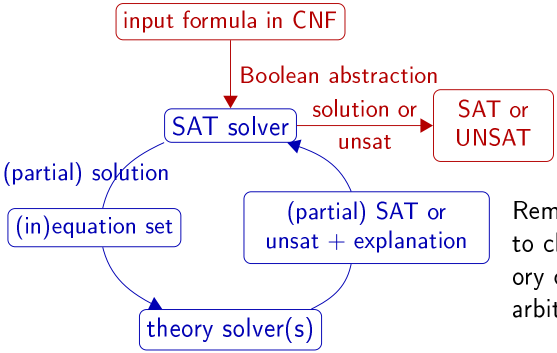
where constants *const* and variables x take real values from $\mathbb{R}$.

- Best known methods for checking the satisfiability of QFNRA formulas have exponential complexity $\rightarrow$ hard to solve
- Approaches we learn for solving QFNRA:
  - Interval constraint propagation (ICP) incomplete
  - Virtual substitution (VS) incomplete
  - Cylindrical algebraic decomposition (CAD) complete

Interval constraint propagation (ICP) in SMT



Remember: the theory solvers needs to check sets/conjunctions of theory constraints only (in contrast to arbitrary Boolean combinations)

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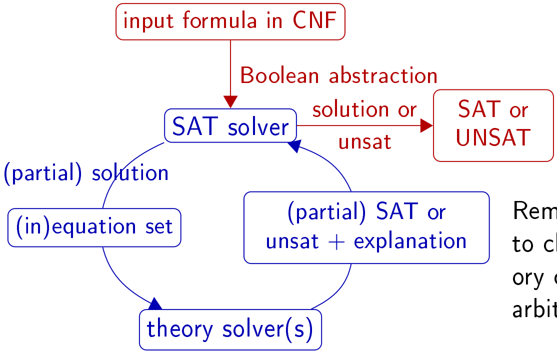

Remember: the theory solvers needs to check sets/conjunctions of theory constraints only (in contrast to arbitrary Boolean combinations)

We first use interval constraint propagation (ICP) in a theory solver module:

- Incomplete: ICP always terminates but it might return “unknown” → we extend it with a backend implementing a complete procedure.
- Relatively cheap reduction of the search space: Even if the answer is “unknown”, ICP might still be helpful because it returns a smaller search space (a set of subsets of the original search space) without loosing any solution.

In the following we consider closed intervals only ($\mathbb{R}$ denotes the real numbers).

Definition (Interval)

An interval $A = [\underline{A}, \overline{A}] = \{v \in \mathbb{R} \mid \underline{A} \leq v \leq \overline{A}\}$ is a closed and connected subset of $\mathbb{R}$, defined by its

- lower bound $\underline{A} \in \mathbb{R} \cup \{-\infty\}$ and its
- upper bound $\overline{A} \in \mathbb{R} \cup \{+\infty\}$,

where $-\infty \leq v \leq +\infty$ for all real numbers $v \in \mathbb{R}$.
We call A bounded if both of its bounds are real-valued ($\underline{A} \neq -\infty$ and $\overline{A} \neq +\infty$), and unbounded otherwise. Let $\mathbb{I}$ be the set of all intervals.

Note: $\mathbb{R} = [-\infty, +\infty]$

Remark: we write $[-\infty, +\infty]$ instead of $(-\infty, +\infty)$, because this way we do not need to make syntactical case distinctions, but of course $+\infty$ and $-\infty$ are not included in $[-\infty, +\infty]$.

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For point intervals $[v, v] = \{v\}$ for some $v \in \mathbb{R}$ we also write v.

Intervals and boxes

Definition (Interval diameter)

The width/diameter D_A of an interval $A = [\underline{A}, \overline{A}] \in \mathbb{I}$ is $D_A = +\infty$ if A is unbounded and $D_A = \overline{A} - \underline{A}$ otherwise.

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A: An interval is empty iff its width is negative.

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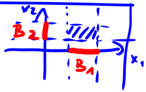
Two-dimensional box example:



Definition (Interval box)

An *n*-dimensional box is a cross product $B = B_1 \times \ldots \times B_n \in \mathbb{I}^n$ of n intervals.

Interval arithmetic

First we extend real arithmetic operations to intervals. Besides the interval-adaptations $+, -, \cdot : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}$ of the QFNRA operators $+, -, \cdot : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, we will also need division $\div : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}$ as the inverse of the multiplication, and square and square root operations $^2, \pm\sqrt{\ } : \mathbb{I} \rightarrow \mathbb{I}$ (we will see later why).

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- Constants and variables are now interval-valued
- Interval operations are conservatively over-approximating with respect to their real-valued counterparts, i.e.,

$$op\ A \supseteq \{\ op\ a \mid a \in A\}$$

for $op \in \{\ ^2, \pm\sqrt{\ }\}$, and

$$A\ op\ B \supseteq \{a\ op\ b \mid a \in A \wedge b \in B\}$$

for $op \in \{+, -, \times, \div\}$.

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Motivating example:

Real arithmetic	Interval arithmetic
$z = x \cdot y$, $x = 2$ $\Rightarrow$ $z = 2y$	$z = x \cdot y$, $x = [1;2]$ $\Rightarrow$ $z = [1;2] \cdot y$
$z = x \cdot y$, $x = 2$, $y = 3$ $\Rightarrow$ $z = 2 \cdot 3$	$z = x \cdot y$, $x = [1;2]$, $y = [2;3]$ $\Rightarrow$ $z = [1;2] \cdot [2;3]$

- The approach introduced in this lecture can be naturally extended to further operators like *sin*, *cos*, *exp*, ...

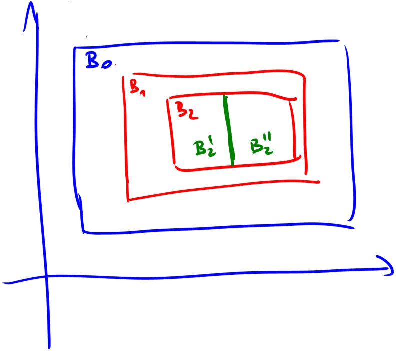
Basic idea of ICP: check whether a set of QFNRA constraints have a common solution in a box B_0.

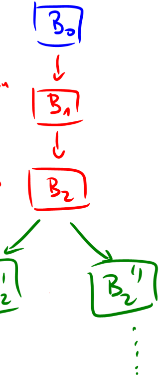
1. Start with initial box B_0
2. Contract B_0 (make smaller) without loosing any solution
3. Split box if contraction is not helpful

Example:

$B_0 = \underbrace{[1,2]}_{\text{domain for } x} \times \underbrace{[0,4]}_{\text{domain for } y}$

$\left.\begin{array}{l} x = y \\ x \in [1,2] \end{array}\right\} \Rightarrow y \in [0,4] \cap [1,2] = [1,2]$





(As we will see, contraction might also result in two boxes)

Computing with infinity

We first partially extend the operations $+, -, \cdot, \div : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ from $\mathbb{R}$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ as follows. Let $a, b \in \mathbb{R}$. The following tables define the extensions, where rows constain the first and columns the second operands.

Addition	$-\infty$	b	$+\infty$
$-\infty$	$-\infty$	$-\infty$	
a	$-\infty$	$a+b$	$+\infty$
$+\infty$		$+\infty$	$+\infty$

Subtraction	$-\infty$	b	$+\infty$
$-\infty$		$-\infty$	$-\infty$
a	$+\infty$	$a-b$	$-\infty$
$+\infty$	$+\infty$	$+\infty$	

Multiplication	$-\infty$	$b<0$	0	$b>0$	$+\infty$
$-\infty$	$+\infty$	$+\infty$	0	$-\infty$	$-\infty$
$a<0$	$+\infty$	$a \cdot b$	0	$a \cdot b$	$-\infty$
0	0	0	0	0	0
$a>0$	$-\infty$	$a \cdot b$	0	$a \cdot b$	$+\infty$
$+\infty$	$-\infty$	$-\infty$	0	$+\infty$	$+\infty$

Division	$-\infty$	$+\infty$
a	0	0

Note: The above tables define the arithmetic operations only partially (e.g., division $A \div B$ is not defined for $A \in \{-\infty, +\infty\}$). It is important to mention that the undefined cases (for which a meaningful definition cannot be given) will not be needed.

Interval arithmetic: Addition

Example (Interval addition)

$[-1;5] + [1;4] =$ [0 ; 9]

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Interval arithmetic: Addition

Example (Interval addition)

$[-1;5] + [1;4] = [0;9]$
$[-2;3] + 4 = [-2;3] + \underbrace{[4;4]}_{4} = [2;7]$

Interval arithmetic: Addition

Example (Interval addition)

$[-1;5] + [1;4] = [0;9]$
$[-2;3] + 4 = [-2;3] + \underbrace{[4;4]}_{4} = [2;7]$

Definition (Interval addition)

We define $A + B =$

Interval arithmetic: Addition

Example (Interval addition)

$[-1;5] + [1;4] = [0;9]$
$[-2;3] + 4 = [-2;3] + \underbrace{[4;4]}_{4} = [2;7]$

Definition (Interval addition)

We define $A + B = [\underline{A} + \underline{B}; \overline{A} + \overline{B}]$ for all non-empty $A = [\underline{A}, \overline{A}] \in \mathbb{I}$ and $B = [\underline{B}, \overline{B}] \in \mathbb{I}$, and $A + B = \emptyset$ otherwise (if either A or B is empty).

Interval arithmetic: Subtraction

Example (Interval subtraction)

$[-1;5] - [1;4] =$ $[-1-4;\ 5-1] = [-5;\ 4]$

Interval arithmetic: Subtraction

Example (Interval subtraction)

$[-1;5] - [1;4] = [-5;4]$

Interval arithmetic: Subtraction

Example (Interval subtraction)

$[-1;5] - [1;4] = [-5;4]$
$[-2;3] - 4 =$ $[-2;3] - [4;4] = [-2-4;3-4] = [-6;-1]$

Interval arithmetic: Subtraction

Example (Interval subtraction)

$[-1;5] - [1;4] = [-5;4]$
$[-2;3] - 4 = [-2;3] + [4;4] = [-6;-1]$

Interval arithmetic: Subtraction

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Definition (Interval subtraction)

We define $A - B =$

Interval arithmetic: Subtraction

Example (Interval subtraction)

$[-1;5] - [1;4] = [-5;4]$
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Definition (Interval subtraction)

We define $A - B = [\underline{A} - \overline{B}; \overline{A} - \underline{B}]$ for all non-empty $A = [\underline{A}, \overline{A}] \in \mathbb{I}$ and $B = [\underline{B}, \overline{B}] \in \mathbb{I}$, and $A - B = \emptyset$ otherwise.

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We can also define unary minus as syntactic sugar:

Definition (Unary interval minus)

We define $-A =$

Interval arithmetic: Subtraction

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We can also define unary minus as syntactic sugar:

Definition (Unary interval minus)

We define $-A = 0 - A = [-\overline{A}; -\underline{A}]$ for all $A = [\underline{A}, \overline{A}] \in \mathbb{I}$.

Interval arithmetic: Multiplication

Example (Interval multiplication)

$[-1; 5] \cdot [1; 4] =$ $[-1 \cdot 4; 5 \cdot 4] = [-4; 20]$

Interval arithmetic: Multiplication

Example (Interval multiplication)

$[-1;5]\cdot[1;4] = [-4;20]$

$[1;2]\cdot[2;3] = [1\cdot 2; 2\cdot 3] = [2;6]$
$[-1;2]\cdot[2;3] = [-1\cdot 3; 2\cdot 3] = [-3;6]$
$[-1;2]\cdot[-2;3] = [-4;6]$

Interval arithmetic: Multiplication

Example (Interval multiplication)

$[-1;5] \cdot [1;4] = [-4;20]$
$[-2;3] \cdot 4 = [-2;3] \cdot [4;4] = [-8;12]$

$[-2;3] \cdot (-4) = [-2;3] \cdot [-4;-4] = [3 \cdot (-4); (-2) \cdot (-4)]$
$= [-12;8]$

Interval arithmetic: Multiplication

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$[-1;5]\cdot[1;4] = [-4;20]$
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Definition (Interval multiplication)

We define $A \cdot B =$

Interval arithmetic: Multiplication

Example (Interval multiplication)

$[-1;5] \cdot [1;4] = [-4;20]$
$[-2;3] \cdot 4 = [-2;3] + [4;4] = [-8;12]$

Definition (Interval multiplication)

We define $A \cdot B =$
$[min(\underline{A} \cdot \underline{B}, \underline{A} \cdot \overline{B}, \overline{A} \cdot \underline{B}, \overline{A} \cdot \overline{B}); max(\underline{A} \cdot \underline{B}, \underline{A} \cdot \overline{B}, \overline{A} \cdot \underline{B}, \overline{A} \cdot \overline{B})]$ for all non-empty $A = [\underline{A}, \overline{A}] \in \mathbb{I}$ and $B = [\underline{B}, \overline{B}] \in \mathbb{I}$, and $A \cdot B = \emptyset$ if either A or B is empty.

Interval arithmetic: Multiplication

Example (Interval square)

Special case: Squaring an interval can only result in positive values.
$[-1;5]^2 =$

$$\underset{\substack{\uparrow \\ x}}{[-1;5]} \cdot \underset{\substack{\uparrow \\ y}}{[-1;5]} = \underset{\substack{\uparrow \\ x \cdot y}}{[-5;25]}$$

$$\Big(\underset{\substack{\uparrow \\ x}}{[-1;5]}\Big)^2 = \underset{\substack{\uparrow \\ x \cdot x}}{[0;25]}$$

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We define $A^2 =$

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Example (Interval square)

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$[-1;5]^2 = [0;25]$

Definition (Interval square)

We define $A^2 = (A \cdot A) \cap [0;+\infty)$ for all $A = [\underline{A}, \overline{A}] \in \mathbb{I}$.

Interval arithmetic: Multiplication

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$[-1; 5]^2 = [0; 25]$

Definition (Interval square)

We define $A^2 = (A \cdot A) \cap [0; +\infty)$ for all $A = [\underline{A}, \overline{A}] \in \mathbb{I}$.

Example (Interval square root)

$\pm\sqrt{[0; 4]} =$

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Example (Interval square root)

$\pm\sqrt{[0; 4]} = [-2; 2]$ $\quad$ $\pm\sqrt{[-4; 4]} = [-2; 2]$ $\quad$ $\pm\sqrt{[1; 4]} = [-2; -1] \cup [1; 2]$

Interval arithmetic: Multiplication

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Definition (Interval square root)

For all $A = [\underline{A}, \overline{A}] \in \mathbb{I}$ we define $[v_1, v_2] = A \cap [0, +\infty]$ and

$$\pm\sqrt{A} = \begin{cases} \emptyset & \textit{if } v_2 < v_1 \\ \left[-\sqrt{v_2}, +\sqrt{v_2}\right] & \textit{if } v_1 = 0 \ (\textit{with } \sqrt{+\infty} = +\infty) \\ \left[-\sqrt{v_2}, -\sqrt{v_1}\right] \cup \left[\sqrt{v_1}, \sqrt{v_2}\right] & \textit{else.} \end{cases}$$

This can be generalised to arbitrary powers A^k and roots $\sqrt[k]{A}$.

Interval arithmetic: Division

Example (Interval division for $0 \notin B$)

$[2;3] \div [4;5] = [2;3] \cdot \frac{1}{[4;5]} = [2;3] \cdot [\frac{1}{5};\frac{1}{4}] = [\frac{2}{5};\frac{3}{4}]$

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Definition (Interval division for $0 \notin B$)

We define $A \div B = A \cdot \frac{1}{B} = A \cdot [\frac{1}{\overline{B}}; \frac{1}{\underline{B}}]$ for all $A = [\underline{A}, \overline{A}] \in \mathbb{I}$ and $B = [\underline{B}, \overline{B}] \in \mathbb{I}$ with $0 \notin B$.

Note: works also if $0 \in A$ (but $0 \notin B$!)

$[-1;2] \div [2;4] = [-1;2] \cdot \frac{1}{[2;4]} = [-1;2] \cdot [\frac{1}{4}; \frac{1}{2}] = [-\frac{1}{2}; 1]$

Interval arithmetic: Division

Problem: B may contain 0, but division by 0 is not defined

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Example (Interval division for $0 \in B$)

If $0 \in B$ then the previous definition does not work correctly:

$\frac{1}{[-2;3]} = \underbrace{[\frac{1}{3}; -\frac{1}{2}]}_{\text{empty}} \rightarrow$ invalid bounds

empty, though it should contain, e.g., $\frac{1}{-2}$ and $\frac{1}{3}$!

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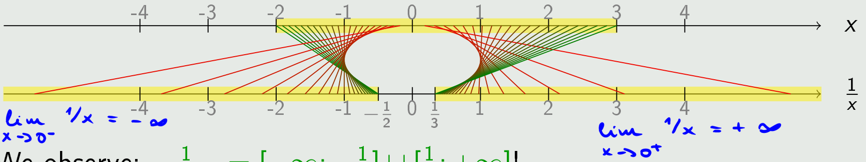
How should $\frac{1}{[-2;3]}$ be defined?



We observe: $\frac{1}{[-2;3]} = [-\infty; -\frac{1}{2}] \cup [\frac{1}{3}; +\infty]$!
Note: Resulting interval may contain a gap!

Interval arithmetic: Division

Definition (Interval division $A \div B$ for $0 \in B$)

The following table defines the result of $A \div B$ for $0 \in B$; rows define case distinctions on A, columns on B:

$A \div B$	$B = [0,0]$	$\underline{B} < \overline{B} = 0$	$\underline{B} < 0 < \overline{B}$	$0 = \underline{B} < \overline{B}$
$0 \in A$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$	$(-\infty, +\infty)$
$\overline{A} < 0$	$\emptyset$	$[\overline{A}/\underline{B}, +\infty]$	$[-\infty, \overline{A}/\overline{B}] \cup [\overline{A}/\underline{B}, +\infty]$	$[-\infty, \overline{A}/\overline{B}]$
$0 < \underline{A}$	$\emptyset$	$[-\infty, \underline{A}/\underline{B}]$	$[-\infty, \underline{A}/\underline{B}] \cup [\underline{A}/\overline{B}, +\infty]$	$[\underline{A}/\overline{B}, +\infty]$

Examples:

row 1 $\rightarrow [-1;2] \div [-1;1] = [-\infty;+\infty]$

row 3 column 3 $[1;2] \div [-1;1] = [-\infty;-1] \cup [1;+\infty]$

row 3 column 2 $[1;2] \div [-1;0] = [-\infty;-1]$

row 3 column 4 $[1;2] \div [0;1] = [1;+\infty]$

Note: $\dfrac{1}{[-1;1]} = [-\infty;-1] \cup [1;+\infty]$

How to strenthen bounds using interval arithmetic

- Now we can compute with intervals.
- Remember that the input of ICP (as a theory solver in an SMT solver) is a set C of QFNRA constraints in n ordered variables $x_1, \ldots, x_n$ and an initial box $B = A_1 \times \ldots \times A_n$ (interval domains A_i for the variables x_i in the constraints).
- Our goal is to decide whether the initial box B contains a common satisfying solution for the constraints in C.
- Let us first have a look at how we can make the initial box B smaller without loosing any solutions.

  This bound strengthening is done via propagation.
- We learn two different propagation methods.

Propagation I: Preprocessing

- The first propagation method requires that for each $c \in C$ and each variable x in c, we can bring c to an equivalent form $x \sim e$ with $\sim \in \{<, \leq, =, \geq, >\}$, where x does not appear in e.

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Preprocessing: Example

1. $x^2 \cdot y + z = 0 \quad \rightarrow \quad h + z = 0 \wedge h_1 = x^2 \cdot y = x \cdot x \cdot y$
2. Now the constraints satisfy the requirements:

$$h + z = 0 \quad \rightarrow \quad h = -z \qquad\qquad h_1 = x^2 \cdot y \quad \rightarrow \quad h_1 = x^2 \cdot y$$
$$\rightarrow \quad z = -h \qquad\qquad\qquad\qquad\qquad \rightarrow \quad x = \pm\sqrt{h_1 \div y}$$
$$\rightarrow \quad y = h_1 \div (x^2)$$

Propagation I: Preprocessing

- Set $C' := C$ and $C := \emptyset$.
- Repeat as long as C' is not empty:
  - Take a constraint $e_1 \sim e_2$, $\sim \in \{<, \leq, =, \geq, >\}$, from C'.
  - Bring $e_1 \sim e_2$ to the normal form $r_1 \cdot m_1 + \ldots + r_k \cdot m_k \sim 0$, where $r_i \in \mathbb{R}$ and m_i are monomials (either 1 or a product of variables) for each $i = 1, \ldots, k$.
  - Replace each non-linear monomial m_i in $r_1 \cdot m_1 + \ldots + r_k \cdot m_k \sim 0$ by a fresh variable h_i and add the result to C.
  - For each newly added variable h_i replacing m_i in the previous step, add an equation $h_i - m_i = 0$ to C, and initialize the bounds of h_i to the interval we get when we substitute the variable bounds in m_i and evaluate the result using interval arithmetic (note: the result will always be a single interval because there is no division or square root in m_i).

Propagation I: Method

- Choose a constraint $c \in C$ and a variable x appearing in c.

  We call such a pair (c, x) a contraction candidate (CC).
- Bring c to a form $x \sim e$, $\sim \in \{<, \leq, =, \geq, >\}$, where e does not contain x. (Note: due to preprocessing, if c is non-linear then it is of the form $h - m = 0$ with h a variable and m a monomial.)

  Example:
  $$x = e$$
  $$I_x := I_x \cap I_e$$
- Replace all variables in e by their current bounds.
- Apply interval arithmetic to evaluate the right-hand-side (e with the variables substituted by their bounds) to a union of intervals.
- Make a case distinction for each interval B in that union.
- For each case, derive from the current bound A for x and the computed bound B for e a new bound on x, depending on the type of $\sim$, as follows:

$$
\begin{array}{lll}
x < e & \textit{if } \underline{A} \geq \overline{B} \textit{ then } \emptyset \textit{ else} & [\underline{A}, \min\{\overline{A}, \overline{B}\}] \\
x \leq e & & [\underline{A}, \min\{\overline{A}, \overline{B}\}] \\
x = e & & [\max\{\underline{A}, \underline{B}\}, \min\{\overline{A}, \overline{B}\}] \\
x \geq e & & [\max\{\underline{A}, \underline{B}\}, \overline{A}] \\
x > e & \textit{if } \overline{A} \leq \underline{B} \textit{ then } \emptyset \textit{ else} & [\max\{\underline{A}, \underline{B}\}, \overline{A}]
\end{array}
$$

Propagation I: Method

Example (Propagation)

$x \in [1;3], y \in [1;2], c_1 : y = x, c_2 : y = x^2$

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$x \in [1;3], y \in [1;2], c_1 : y = x, c_2 : y = x^2$
$c_2, x : x = \pm\sqrt{y}$

Propagation I: Method

Example (Propagation)

$x \in [1;3], y \in [1;2], c_1 : y = x, c_2 : y = x^2$
$c_2, x : x = \pm\sqrt{y} \rightarrow x = \pm\sqrt{[1;2]} = [-\sqrt{2}; -1]] \cup [1; \sqrt{2}]$

Propagation I: Method

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$x \in [1;3], y \in [1;2], c_1 : y = x, c_2 : y = x^2$

$c_2, x : x = \pm\sqrt{y} \to x = \pm\sqrt{[1;2]} = [-\sqrt{2}; -1]] \cup [1; \sqrt{2}] \to$

$\quad x \in [1;3] \cap ([-\sqrt{2}; -1] \cup [1; \sqrt{2}]) = [1; \sqrt{2}]$

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$c_1, y : y = x \to y = [1;\sqrt{2}]$

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$x \in [1;3], y \in [1;2], c_1 : y = x, c_2 : y = x^2$

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$c_1, y : y = x \rightarrow y = [1; \sqrt{2}] \rightarrow y \in [1;2] \cap [1; \sqrt{2}] = [1; \sqrt{2}]$

Propagation II: Preprocessing

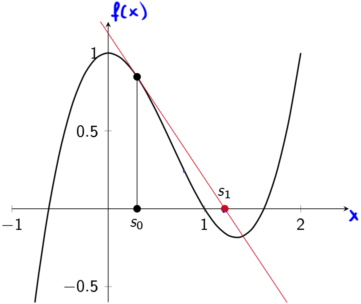
- Now we look at an alternative method for propagation.
- This method is called the interval Newton method.
- Also this second propagation method needs some lightweight preprocessing:
  - Transform each constraint $e_1 \sim e_2$ in C to $e_1 - e_2 \sim 0$.
  - For each inequation $p \sim 0$ with $\sim \in \{<, \leq, \geq, >\}$ in C, replace p by a fresh variable h, add an equation $h - p = 0$ to C, and initialize the bounds of h to the interval we get when we substitute the variable bounds in p and evaluate the result using interval arithmetic (note: the result will always be a single interval because there is no division or square root in p).
- After this preprocessing, the constraint set contains equations $p = 0$ stating that a polynomial equals to zero, and inequations of the form $x \sim 0$ with x a variable and $\sim \in \{<, \leq, \geq, >\}$.
- Assume in the following a constraint $c \in C$ and a variable x in c as a contraction candidate.
- Next we see how we can reduce the domain of x using c via the interval Newton method.

Propagation II: Method

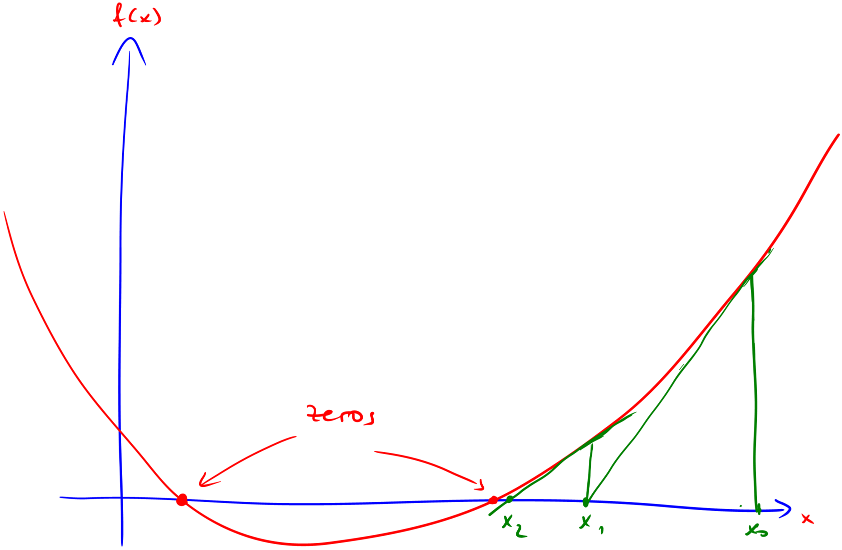
Due to the preprocessing, if the constraint c is an inequation then it has the form $x \sim 0$ (where x is a variable). In this case we propagate similarly as with the first method, assuming that the current interval for x is A:

$$
\begin{array}{lll}
x < 0 & \textit{if } \underline{A} \geq 0 \textit{ then } \emptyset \textit{ else} & [\underline{A}, \min\{\overline{A}, 0\}] \\
x \leq 0 & & [\underline{A}, \min\{\overline{A}, 0\}] \\
x \geq 0 & & [\max\{\underline{A}, 0\}, \overline{A}] \\
x > 0 & \textit{if } \overline{A} \leq 0 \textit{ then } \emptyset \textit{ else} & [\max\{\underline{A}, 0\}, \overline{A}]
\end{array}
$$

Assume now that the constraint c is $f(x) = 0$, where $f(x) : \mathbb{R} \to \mathbb{R}$ is an univariate polynomial in x, and let $f'(x) : \mathbb{R} \to \mathbb{R}$ be the first derivative of $f(x)$.
Reminder for Newton method for root finding (univariate case): Compute a sequence of real values $s_0, s_1, \ldots$ such that $s_0 \in \mathbb{R}$ is an initial guess, and $s_{i+1} = s_i - \frac{f(s_i)}{f'(s_i)}$ for all $i \geq 0$.
For a "good enough" initial guess s_0, the sequence converges to a zero $r \in \mathbb{R}$ of $f(x)$, i.e., to a value r for which $f(r) = 0$. If it converges then it does so quadratically. Unfortunately, this procedure can be unstable near a horizontal asymptote or a local extremum.



$$
\begin{aligned}
f(x) &= x^3 - 2x^2 + 1 \\
f'(x) &= 3x^2 - 4x \\
s_0 &= 0.3 \\
s_1 &= s_0 - \frac{f(s_0)}{f'(s_0)} \\
&= 0.3 - \frac{f(0.3)}{f'(0.3)} \\
&= 0.3 - \frac{0.847}{-0.93} \\
&\approx 1.2107
\end{aligned}
$$



f(x)
zeros
x_2
x_1
x_0
x

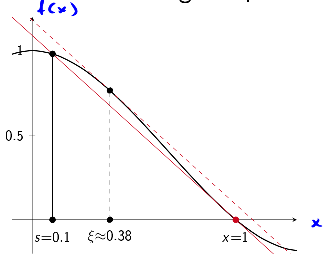
Propagation II: Taylor's Theorem

The interval Newton method is an extension of the Newton method. It takes a function $f : \mathbb{R} \to \mathbb{R}$ which is continuously differentiable on an interval A (polynomials satisfy this condition) and a sample point $s \in A$, and uses information about $f(s)$ and the range of f' on A to contract the set of possible zeros of f within A.

We make use of the first-order version of Taylor's theorem which states that

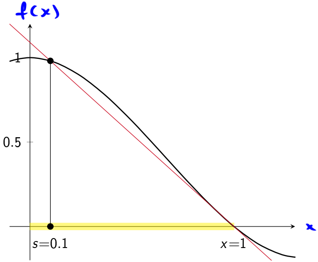
$$\forall s, x \in A. \exists \xi \in A. f(x) = f(s) + (x - s) f'(\xi).$$

That means, if we take an arbitrary point $s \in A$ then for any $x \in A$ with $f(x) = 0$, the gradient of the line connecting the points $(s, f(s))$ and $(x, 0)$ is in the interval $f'(I)$.



Propagation II: Interval Newton method

Interval extension of Newton's method:



Function: $f(x) = x^3 - 2x^2 + 1$, $f'(x) = 3x^2 - 4x$

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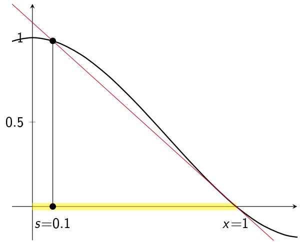

Function: $f(x) = x^3 - 2x^2 + 1$, $f'(x) = 3x^2 - 4x$

Starting interval: $A = [0; 1]$

Sample point: $s = 0.1$

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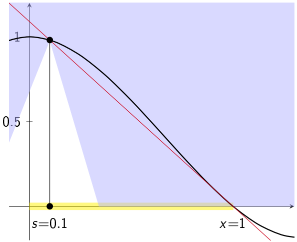

Function: $f(x) = x^3 - 2x^2 + 1$, $f'(x) = 3x^2 - 4x$
Starting interval: $A = [0; 1]$
Sample point: $s = 0.1$
Derivatives in A: $f'(A) = 3 \cdot [0; 1]^2 - 4 \cdot [0; 1] = [-4; 3]$

Propagation II: Interval Newton method

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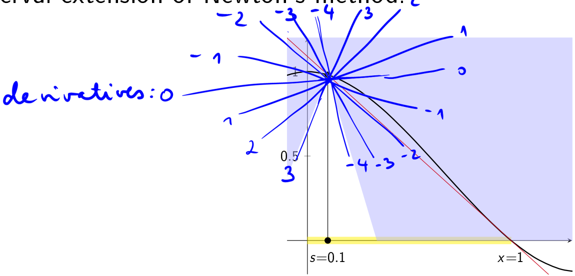

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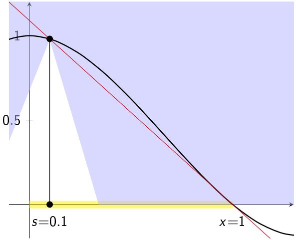

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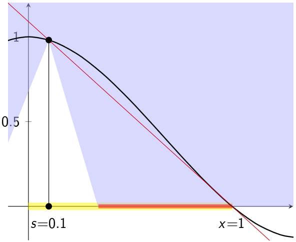
Possible zeros in A at: $\underbrace{s - \frac{f(s)}{f'(A)}}_{=:N(s,f)} = [-\infty; -0.227] \cup [0.34525; +\infty]$

Propagation II: Interval Newton method

Interval extension of Newton's method:



Function: $f(x) = x^3 - 2x^2 + 1$, $f'(x) = 3x^2 - 4x$

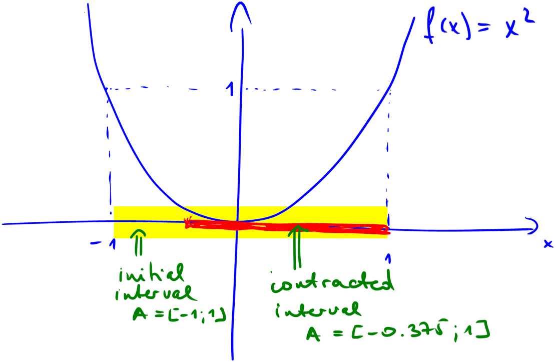
Starting interval: $A = [0;1]$

Sample point: $s = 0.1$

Derivatives in A: $f'(A) = 3 \cdot [0;1]^2 - 4 \cdot [0;1] = [-4;3]$

Possible zeros in A at: $s - \frac{f(s)}{f'(A)} = [-\infty; -0.227] \cup [0.34525; +\infty]$

New interval: $A = [0;1] \cap ([-\infty; -0.227] \cup [0.34525; +\infty]) = [0.34525; 1]$



$A = [-1;1]$

$s_0 = -\frac{1}{2}$ ⇐ Note: arbitrary sampled from A!

$f(x) = x^2$

$f'(x) = 2x$

$$N(s_0, x^2) = s_0 - \frac{f(s_0)}{f'(A)} = -\frac{1}{2} - \frac{1/4}{2\cdot[-1;1]} = -\frac{1}{2} - \frac{1/4}{[-2;2]} =$$

$$= -\frac{1}{2} - \left([-\infty;\tfrac{1/4}{-2}] \cup [\tfrac{1/4}{2};+\infty]\right)$$

$$= -\frac{1}{2} - \left([-\infty;-\tfrac{1}{8}] \cup [\tfrac{1}{8};+\infty]\right)$$

$$= [-\tfrac{3}{8};+\infty] \quad \cup [-\infty;-\tfrac{5}{8}]$$

$$= [-0.375;+\infty] \quad \cup [-\infty;-0.625]$$

$$A := A \cap N(s_0, x^2) = [-1;1] \cap \left([-0.375;+\infty] \cup [-\infty;-0.625]\right)$$

$$= [-0.375;1]$$

Propagation II: Componentwise multivariate interval Newton

Reminder: Componentwise Multivariate Newton

Variables $x = (x_1, \ldots, x_n)$, function $f(x)$, sample $s_i = (s_{i,1}, \ldots, s_{i,n})$

$s_{i+1} = N_{cmp}(s, f(x), x_j) = s_i - \frac{f(s_i)}{\frac{\partial f}{\partial x_j}(s_i)}$

Propagation II: Componentwise multivariate interval Newton

Reminder: Componentwise Multivariate Newton
Variables $x = (x_1, \ldots, x_n)$, function $f(x)$, sample $s_i = (s_{i,1}, \ldots, s_{i,n})$
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Componentwise multivariate interval Newton:
interval $A = A_1 \times \ldots A_n$, $mid(A_j)$ is the middle point of A_j

$$N_{cmp}(A, \overbrace{f(x), x_j}^{CC}) :=$$
$$mid(A_j) - \frac{f(A_1, \ldots, A_{j-1}, mid(A_j), A_{j+1}, \ldots, A_n)}{\frac{\partial f}{\partial x_j}(A_1, \ldots, A_n)}$$

Propagation II: Componentwise multivariate interval Newton

Reminder: Componentwise Multivariate Newton
Variables $x = (x_1, \ldots, x_n)$, function $f(x)$, sample $s_i = (s_{i,1}, \ldots, s_{i,n})$
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Componentwise multivariate interval Newton:
interval $A = A_1 \times \ldots A_n$, $mid(A_j)$ is the middle point of A_j

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The operator N_{cmp} has two important properties:

- If $f(x^*) = 0$ and $x^* \in A$, then $x^* \in N_{cmp}(A, f(x), x_j)$.
- If $A \cap N_{cmp}(A, f(x), x_j) = \varnothing$ then $f(x) \neq 0$ for all $x \in A$.

Reminder: Componentwise Multivariate Newton
Variables $x = (x_1, \ldots, x_n)$, function $f(x)$, sample $s_i = (s_{i,1}, \ldots, s_{i,n})$
$s_{i+1} = N_{cmp}(s, f(x), x_j) = s_i - \frac{f(s_i)}{\frac{\partial f}{\partial x_j}(s_i)}$

Componentwise multivariate interval Newton:
interval $A = A_1 \times \ldots A_n$, $mid(A_j)$ is the middle point of A_j

$$N_{cmp}(A, \overbrace{f(x), x_j}^{CC}) :=$$
$$mid(A_j) - \frac{f(A_1, \ldots, A_{j-1}, mid(A_j), A_{j+1}, \ldots, A_n)}{\frac{\partial f}{\partial x_j}(A_1, \ldots, A_n)}$$

Note: instead of mid(A_j), one can take any point in the interval A_j.

The operator N_{cmp} has two important properties:

- If $f(x^*) = 0$ and $x^* \in A$, then $x^* \in N_{cmp}(A, f(x), x_j)$.
- If $A \cap N_{cmp}(A, f(x), x_j) = \emptyset$ then $f(x) \neq 0$ for all $x \in A$.

$\rightarrow$ Advantage: No diverging behavior like the original Newton method due to interval arithmetic.

$\rightarrow$ We can drop boxes when they contract to empty.

The global ICP algorithm

- Now we know how to reduce the bounds of a variable based on a constraint in which it appears.
- Next we look how to use these reduction methods iteratively in an algorithm, which can be used as a theory solver for QFNRA constraint sets in an SMT solver.

Algorithm

Input: Set of QFNRA constraints, non-empty initial box B_0
Box diameter threshold D, contraction condition for boxes (fix later)

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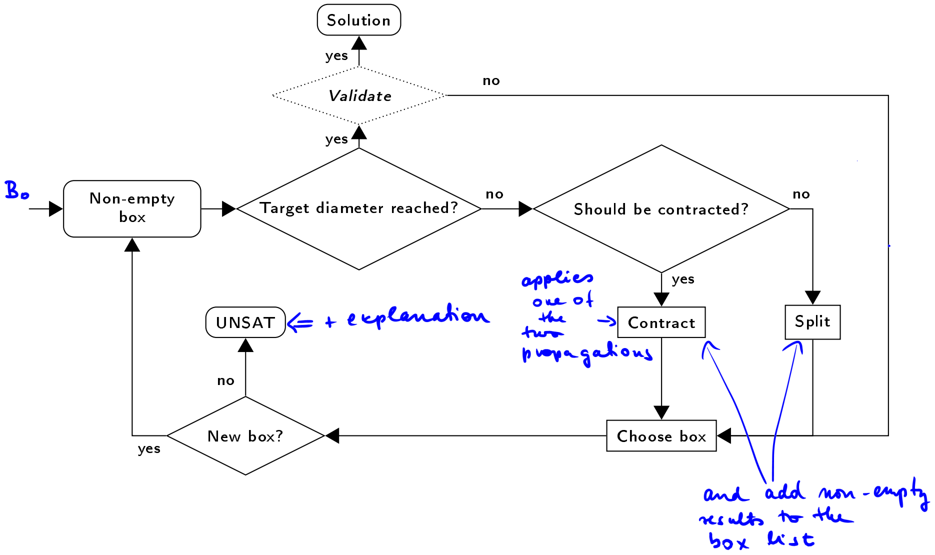
Compute a set of boxes B whose union contains all solutions from B_0 (if any) by iteratively executing one of the following steps:

1. Set $B := \{B_0\}$.
2. If B is empty then return unsatisfiable.
   Otherwise choose a box $B_i \in B$ and remove it from B.
3. If the diameter of B_i is at most D then pass on B_i to a complete procedure for satisfiability check; if B_i contains a solution then return SAT otherwise go to 2.
4. If the contraction condition for B_i holds then try to reduce this box, add the resulting box(es) to B, and go to 2. Note: Due to interval division or square root propagation may result in two boxes.
5. Otherwise split the box into two halves, add them to B, and go to 2.

Algorithm



Solution
yes
Validate
no
yes
B0
Non-empty box
Target diameter reached?
no
Should be contracted?
no
yes
applies one of the two propagations
Contract
Split
UNSAT
⇐ + explanation
no
New box?
yes
Choose box
and add non-empty results to the box list

Algorithm

Further algorithmic aspects:

- Heuristics to choose CCs (constraints and variables)
- Assure termination
- ICP does not behave well on linear constraints
- ICP needs to return an explanation for unsatisfiable problems

Heuristics to choose CCs

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At each step we can consider 4 contractions:

- $I_x \overset{c_1,x}{\rightarrow} [1;2]$ $\quad (gain_{rel} : 0.5)$
- $I_y \overset{c_1,y}{\rightarrow} [1;2]$ $\quad (gain_{rel} : 0)$
- $I_x \overset{c_2,x}{\rightarrow} [1;\sqrt{2}]$ $\quad (gain_{rel} : 0.793)$
- $I_y \overset{c_2,y}{\rightarrow} [1;2]$ $\quad (gain_{rel} : 0)$

Relative contraction:

$$gain_{rel} = \frac{D_{old} - D_{new}}{D_{old}} = 1 - \frac{D_{new}}{D_{old}}$$

$\rightarrow$ Contraction gain varies.

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We can improve the choice of CCs by heuristics:

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The factor $\alpha \in [0;1]$ decides how the importance of the events is rated:

- Large α (e.g. 0.9) $\rightarrow$ The last recent event is most important
- Small α (e.g. 0.1) $\rightarrow$ The initial weight is most important

CCs with a weight less than some threshold ε are not considered for contraction.

Assure termination

Example (Propagation)

$x \in [1;3], y \in [1;2], c_1 : y = x, c_2 : y = x^2$

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$c_1, y : y = x \to y = [1; \sqrt{2}] \to y \in [1;2] \cap [1; \sqrt{2}] = [1; \sqrt{2}]$

Contraction sequence:
$x : [1;3] \stackrel{c_2,x}{\to} [1; \sqrt{2}] \stackrel{c_2,x}{\to} [1; \sqrt[4]{2}] \stackrel{c_2,x}{\to} [1; \sqrt[8]{2}] \stackrel{c_2,x}{\to} \ldots \leadsto [1;1]$
$y : [1;2] \stackrel{c_1,y}{\to} [1; \sqrt{2}] \stackrel{c_1,y}{\to} [1; \sqrt[4]{2}] \stackrel{c_1,y}{\to} [1; \sqrt[8]{2}] \stackrel{c_1,y}{\to} \ldots \leadsto [1;1]$
$\to$ Propagation might not terminate!

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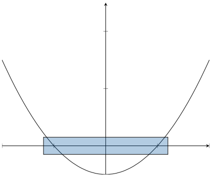

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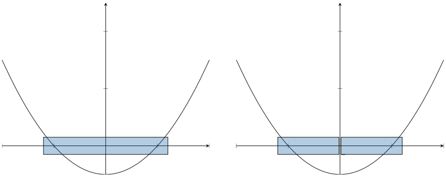

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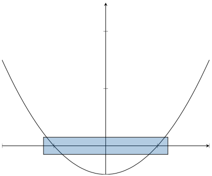

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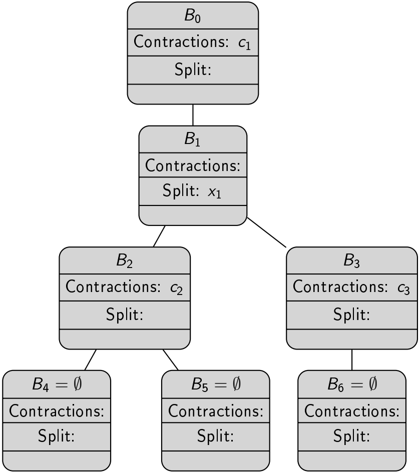
- In case box is linear infeasible: Add violated linear constraint for contraction

Explanations

To keep track of current status we utilize a tree-structure, which holds solver states:

- Search box
- Applied contractions or
- Dimension of applied splitting



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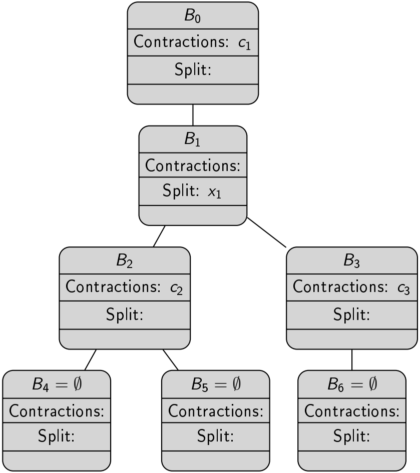

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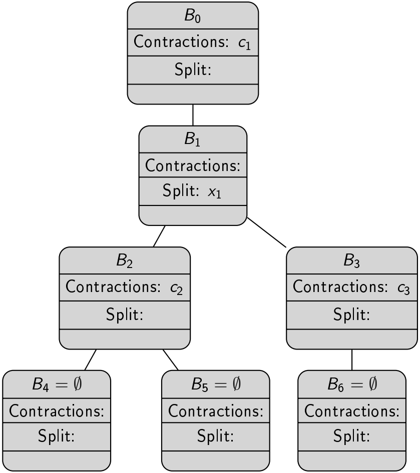

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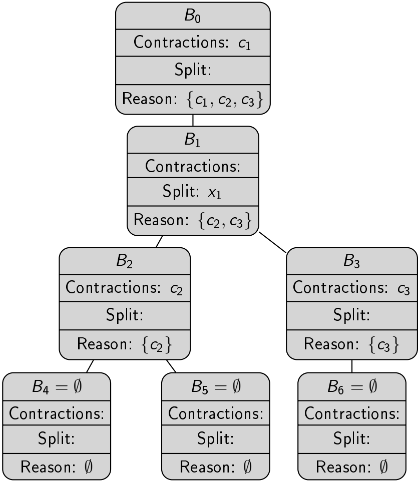

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If you like to see a video about ICP:

http://www-sop.inria.fr/coprin/logiciels/ALIAS/Movie/movie_undergraduate.mpg

A remark on <u>simplex</u>: Why do we call the variables for the rows „basic", and not those for the columns?

<u>Simplex tableau:</u>

$$
\begin{array}{c|c|c|c}
 & x_1 & x_j & x_n \\
\hline
s_1 & & & \\
\vdots & & & \\
\hline
s_i & & a_{ij} & \\
\hline
\vdots & & & \\
s_m & & &
\end{array}
\qquad
s_i = \sum_{j=1}^{n} a_{ij} x_j
$$

This is just a simplified representation of:

$$
\begin{array}{cccccc}
s_1 & \cdots & s_m & x_1 & \cdots & x_n
\end{array}
$$

$$
\left(\begin{array}{ccc|ccc}
-1 & & 0 & a_{11} & \cdots & a_{1n} \\
 & \ddots & & \vdots & & \\
0 & & -1 & a_{mn} & \cdots & a_{mn}
\end{array}\right)
\qquad
-s_i + \sum_{j=1}^{n} a_{ij} x_j = 0
$$

<u>Def.</u> An $m \times m$ sub-matrix S of an $m \times (m+n)$ matrix A is called a <u>basis of A</u> if the rows/columns of S are linearly independent.

The $-I$ submatrix, corresponding to the rows in the blue representation, is a basis.

$\Rightarrow$ The variables for the rows are called <u>basic</u>.