Satisfiability Checking Fourier-Motzkin Variable Elimination

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The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

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$$(p_1 = 0 \lor p_2 = 0 \lor p_3 = 0) \land p_1 + p_2 + p_3 \ge 100 \land (p_1 \ge 5 \lor p_2 \ge 5) \land p_3 \ge 10 \land p_1 + 2p_2 + 5p_3 \le 180 \land 3p_1 + 2p_2 + p_3 \le 300$$

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Terms: $t := 0 \mid 1 \mid x \mid t+t$

Constraints: c ::= t < t

Formulas: $\varphi ::= c \mid \neg \varphi \mid \varphi \wedge \varphi \mid \exists x. \varphi$

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where x is a variable.

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- Linear real arithmetic is also called linear real algebra.
- We consider the satisfiability problem for the quantifier-free fragment QFLRA (equivalently, we consider the existential fragment, i.e., no negation of expressions containing quantifiers).

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Assume that the *i*th constraint is an equation containing a variable x_j with a non-zero coefficient $a_{ij} \neq 0$:

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- Replace x_i by β_i in all constraints.
- This substitutiton leads to an equisatisfiable problem in n-1 variables.

Linear arithmetic over the reals

 Goal: decide satisfiability of conjunctions of linear inequalities over the reals

$$\bigwedge_{1 \le i \le m} \sum_{1 \le j \le n} a_{ij} x_j \le b_i$$

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■ Input in matrix form: $A\overline{x} \leq \overline{b}$

$$\begin{array}{c} \textit{m} \text{ constraints} & \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{22} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}$$

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 - Continue until all variables are eliminated.

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- Basic idea of variable elimination:
 - Pick a variable and eliminate it, yielding an equisatisfiable formula that does not refer to the eliminated variable any more.
 - Continue until all variables are eliminated.
- Fourier-Motzkin: Put requirements on the lower an upper bounds on the variable we want to eliminate.

- For a variable x_n , we can partition the constraints according to the coefficient a_{in} :
 - $a_{in} > 0$: upper bound β_i on x_n
 - $a_{in} < 0$: lower bound β_i on x_n

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$$\sum_{j=1}^{n} a_{ij} \cdot x_{j} \leq b_{i}$$

$$\Rightarrow a_{in} \cdot x_{n} \leq b_{i} - \sum_{j=1}^{n-1} a_{ij} \cdot x_{j}$$

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$$\Rightarrow a_{in} \cdot x_{n} \leq b_{i} - \sum_{j=1}^{n-1} a_{ij} \cdot x_{j}$$

$$(a) \stackrel{a_{in} \geq 0}{\Rightarrow} x_{n} \leq \frac{b_{i}}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_{j} =: \beta_{I} \text{ upper bound}$$

$$(b) \stackrel{a_{in} \leq 0}{\Rightarrow} x_{n} \geq \frac{b_{i}}{a_{in}} - \sum_{i=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_{j} =: \beta_{u} \text{ lower bound}$$

Category for x_1 ?

- (1) $x_1 x_2 \leq 0$
- (2) $x_1 x_3 \leq 0$
- (3) $-x_1 + x_2 + 2x_3 \le 0$
- (4) $-x_3 \leq -1$

Category for
$$x_1$$
?

(1)
$$x_1 - x_2 \leq 0$$

(2)
$$x_1 - x_3 \leq 0$$

(3)
$$-x_1 + x_2 + 2x_3 \le 0$$

(4)
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Upper bound

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Upper bound

Upper bound

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Category for x₁?
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(3)
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 Lower bound

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$$-x_3 \leq -1$$

Category for x_1 ?

Upper bound Upper bound

No bound

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them).
- The new problem has a solution iff the old problem has one!

$$8x \geq 7y$$

$$x \geq 3$$

$$y \geq z$$

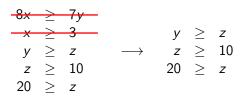
$$z \geq 10$$

$$20 > z$$

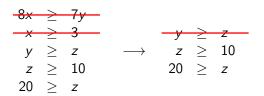
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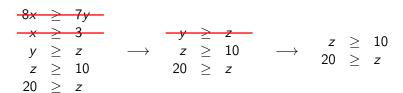
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■ For each pair of a lower bound β_I and an upper bound β_u , we have

$$\beta_l \leq x_n \leq \beta_u$$

Fourier-Motzkin variable elimination

■ For each pair of a lower bound β_I and an upper bound β_u , we have

$$\beta_l \leq x_n \leq \beta_u$$

■ For each such pair, add the constraint

$$\beta_I \leq \beta_u$$

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$$x_1 - x_2 \leq 0$$

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Category for x_1 ?

Upper bound

Upper bound

Lower bound

eliminate x_1

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(2)
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(3)
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(4)
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(5)
$$2x_3 \le 0$$
 (from 1,3)

Category for x₁?
Upper bound
Upper bound
Lower bound

eliminate x_1

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$$x_1 - x_2 \leq 0$$

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(3)
$$-x_1 + x_2 + 2x_3 \leq 0$$

(4)
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(5)
$$2x_3 \le 0$$
 (from 1,3)
(6) $x_2 + x_3 \le 0$ (from 2,3)

(6)
$$x_2 + x_3 \leq 0$$

Category for x_1 ? Upper bound Upper bound Lower bound

eliminate x_1

Category for x_1 ?

eliminate x_1

- (5) $2x_3 \le 0$ (from 1,3)
- (6) $x_2 + x_3 \le 0$

(from 2,3)

$$\frac{(1)}{(2)} \frac{x_1 - x_2 \le 0}{x_1 - x_3 \le 0}$$

$$\frac{(3)}{(3)} \frac{x_1 + x_2 + 2x_3 < 0}{(3)}$$

$$(4) -x_3 < -1$$

eliminate x_1

(5)
$$2x_3 \le 0$$
 (from 1,3)

(6)
$$x_2 + x_3 \le 0$$
 (from 2,3)

we eliminate x_3

Category for x_1 ?

$$\begin{array}{c|cccc} -(1) & x_1 & x_2 \le 0 \\ \hline -(2) & x_1 & x_3 \le 0 \\ \hline -(3) & x_1 + x_2 + 2x_3 \le 0 \\ \hline -(4) & -x_3 \le -1 \end{array}$$

(5)
$$2x_3 \le 0$$
 (from 1,3)
(6) $x_2 + x_3 \le 0$ (from 2,3)

Category for x_1 ?

Lower bound eliminate x_1 (from 1,3) Upper bound (from 2,3) Upper bound we eliminate x_3

Category for
$$x_1$$
?

$$\begin{array}{c|ccccc}
\hline (1) & x_1 & x_2 \leq 0 \\
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\hline (5) & 2x_3 \leq 0 \\
\hline (6) & x_2 + x_3 \leq 0 \\
\hline (7) & 1 \leq 0 \\
\hline (7) & 1 \leq 0
\end{array}$$

$$\begin{array}{c|ccccc}
\hline (from 1,3) & Upper bound \\
Upper bound \\
\hline (from 2,3) & Upper bound \\
\hline (from 4,5) \\
\hline (from 4,$$

■ Worst-case complexity:

$$m \rightarrow m^2$$

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$$m \rightarrow m^2 \rightarrow (m^2)^2$$

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■ The bottleneck: case-splitting