gereon.kremer@cs.rwth-aachen.de
https://ths.rwth-aachen.de/teaching/

Satisfiability Checking - WS 2016/2017

Series 4

Exercise 1

In this exercise, we give some more details on the concept of *logical theory* and how it is related to axioms.

We fix an arbitrary signature Σ and an arbitrary structure $\mathcal{S}$ over Σ. In the following, all sentences are over Σ and Φ[1] is a set of sentences. We use the following notation:

- $\mathcal{S} \models \varphi$: $\mathcal{S}$ is a model of a sentence φ.
- $\mathcal{S} \models \Phi$: $\mathcal{S}$ is a model of all sentences φ from the set Φ.

Definitions:

- A sentence φ **is a consequence of** Φ ($\Phi \models \varphi$) iff $\mathcal{S} \models \varphi$ for each model $\mathcal{S} \models \Phi$.
- $\Phi^{\models} := \{\varphi \mid \Phi \models \varphi\}$ denotes the **set of consequences of** Φ.
- Φ is called **consistent** if there is no sentence φ with $\Phi \models \varphi$ and $\Phi \models \neg\varphi$.
- A satisfiable set of sentences T is called a **theory** if for all sentences φ

$$T \models \varphi \quad \Longleftrightarrow \quad \varphi \in T.$$

- A theory T is **complete** iff for all sentences φ

$$\text{either} \quad \varphi \in T \quad \text{or} \quad \neg\varphi \in T.$$

Prove the following three statements.

1. Each theory T is consistent.
2. Let Φ be a set of sentences. Φ is consistent iff $\Phi^{\models}$ is a theory.
3. The set $\mathsf{Th}(\mathcal{S}) := \{\varphi \mid \mathcal{S} \models \varphi\}$ is a theory. It is called the **theory of** $\mathcal{S}$.
4. $\mathsf{Th}(\mathcal{S})$ is complete.
5. Let $\Sigma = \{+, \cdot, \leq, =\}$. Give one example each:
   (a) a complete Σ-theory T_1,
   (b) an incomplete Σ-theory T_2.

   Hint: You can use different ways to define a theory.

[1] Imagine Φ to be a (finite) set of axioms.