# Satisfiability Checking - WS 2016/2017

## Series 10

gereon.kremer@cs.rwth-aachen.de
https://ths.rwth-aachen.de/teaching/

### Exercise 1

Throughout the lecture, you have seen various logics that are all based on the quantifier-free fragment of first-order logic (QF):

- propositional logic (QF)
- propositional logic with equalities (QF_EQ)
- propositional logic with equalities and uninterpreted functions (QF_UF)
- QF with linear real arithmetic (QF_LRA)
- QF with linear integer arithmetic (QF_LIA)
- QF with linear real or integer arithmetic (QF_LIRA)

Answer the following questions for each of these logics:

1. What is the signature of the logic?
2. How can you solve the satisfiability problem for this logic? (name an algorithm)
3. Can the logic be reduced to another one? (i.e. they have the same expressive power)
4. Is the logic decidable?

Encode the following statements in each of these logics or argue why this is impossible:

1. You want to meet with three friends A, B and C. You can't make it on Monday and Friday, A is unavailable on Friday and Monday, B is on holiday until Wednesday and C visits his parents on Friday. You can meet this week.
2. There are three different colors.
3. A function $\varphi$ is a homomorphism, i.e. $\varphi(a \circ b) = \varphi(a) \circ \varphi(b)$.
4. Two snails start $10m$ apart towards the same direction. The second snail is twice as fast, catches up after $2h$ and can be slower than $10m/h$.
5. Given six coins of values $1, 2, 5, 10, 20, 50$, you can pay an amount of $83$.
6. There is a rectangle whose perimeter is less than $5$ and whose area is greater than $1$.
7. You have the one-time opportunity to get land of a rectangular shape for free, but you must build a fence for $10$ per meter. You can make a profit of $1$ per square meter per year and you have a capital of $1000$. The investment amortises within two years. (You can only get full square meters)

*Solution:*

| Logic | 1. | 2. | 3. | 4. |
|---|---|---|---|---|
| QF | $\emptyset$ | DPLL | | ✓ |
| QF_EQ | $\emptyset$ over $D$ | Equivalence classes, Sparse method | like QF | ✓ |
| QF_UF | $(F)$ over $D$ | Ackermann | like QF | ✓ |
| QF_LRA | $(+,<)$ over $\mathbb{R}$ | FM, Simplex | | ✓ |
| QF_LIA | $(+,<)$ over $\mathbb{Z}$ | B&B, Omega test | | ✓ |
| QF_LIRA | $(+,<)$ over $\mathbb{R}$ | B&B, Omega test | | ✓ |

1. For all logics:

$$(\neg Mo \wedge \wedge \neg Tu) \wedge (\neg Fr \wedge \neg Mo) \wedge (\neg Mo \wedge \neg Tu \wedge \neg We) \wedge (\neg Fr)$$
$$\wedge (Mo \vee Th \vee We \vee Th \vee Fr), Mo, Tu, We, Th, Fr \in \mathbb{B}$$

2. For QF_EQ / QF_UF:

$$\neg\left(x = y \vee x = z \vee y = z\right), x, y, z \in Colors$$

All other logics can not argue about arbitrary domains. (Reducible to QF)

3. For QF_UF:

$$\varphi(\circ(a, b)) = \circ(\varphi(a), \varphi(b))$$

All other logics can not argue about functions. (Reducible to QF_EQ and QF). Note that this only encode the property itself, for checking whether $\varphi$ is a homomorphism would require to universally quantify over $a, b$.

4. For QF_L* and QF_N*:

$$x_1 = 0 \wedge x_2 = 10 \wedge x_1 + 2 \cdot v = x_2 + 2 \cdot 0.5 \cdot v \wedge v < 10, x_1, x_2, v \in \mathbb{R}$$

All other logics can not argue about numerical relations.

5. For QF_LI* and QF_NI*:

$$0 \leq a \leq 1 \wedge 0 \leq b \leq 1 \wedge 0 \leq c \leq 1 \wedge 0 \leq d \leq 1 \wedge 0 \leq e \leq 1 \wedge 0 \leq f \leq 1$$
$$\wedge a + 2 \cdot b + 5 \cdot c + 10 \cdot d + 20 \cdot e + 50 \cdot f = 83$$

Due to the limited size of variables and constants, this can also be encoded in QF.

6. For QF_N*:

$$2 \cdot a + 2 \cdot b < 5 \wedge a \cdot b > 1, a, b \in \mathbb{R}$$

All other logics can not argue about multiplications of real variables.

7. For QF_NI*:

$$i = 10 \cdot 2 \cdot (a + b) \wedge i \leq 1000 \wedge p = a \cdot b \wedge p \cdot 2 \geq i$$

Due to the limited size of variables and constants, this can also be encoded in QF.