

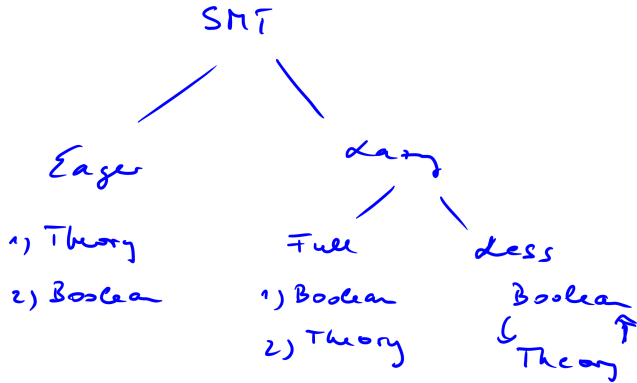
Satisfiability Checking

Lazy SAT-Modulo-Theories (SMT) Solving

Prof. Dr. Erika Ábrahám

RWTH Aachen University
Informatik 2
LuFG Theory of Hybrid Systems

WS 16/17



The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

$p_i \sim \# \text{ presents of type } i$

$$1) \quad p_1 = 0 \vee p_2 = 0 \vee p_3 = 0$$

$$2) \quad p_1 + p_2 + p_3 \geq 100$$

$$3) \quad p_1 \geq 5 \vee p_2 \geq 5$$

$$4) \quad p_3 \geq 10$$

$$5) \quad p_1 + 2p_2 + 5p_3 \leq 180$$

$$6) \quad 3p_1 + 2p_2 + p_3 \leq 300$$

The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

$$(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge$$
$$(p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge$$
$$3p_1 + 2p_2 + p_3 \leq 300$$

$$(N, 0, 1, +, <)$$

The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

$$\begin{aligned} & (p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\ & (p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\ & 3p_1 + 2p_2 + p_3 \leq 300 \end{aligned}$$

Logic:

The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

Sol: $p_1 = 50$ $p_2 = 0$ $p_3 = 10$

$$\begin{aligned} & (p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\ & (p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\ & 3p_1 + 2p_2 + p_3 \leq 300 \end{aligned}$$

Logic: First-order logic over the integers with addition.

Full lazy SMT-solving

$$(p_1 = 0 \vee p_2 = 0) \wedge p_1 < 0$$

\downarrow F T

$$(a \vee b) \wedge c$$

$$a = 1 \quad b = 0 \quad c = 1$$

φ

Boolean abstraction

$$\neg(\neg a \vee \neg c)$$

SAT-solver

unsatisfiable

UNSAT

satisfiable

$$p_1 = 0 \quad p_2 \neq 0 \quad p_1 < 0$$

(In)equation set

Explanation

$$p_1 = 0 \quad p_1 < 0$$

Theory solver

unsatisfiable

satisfiable

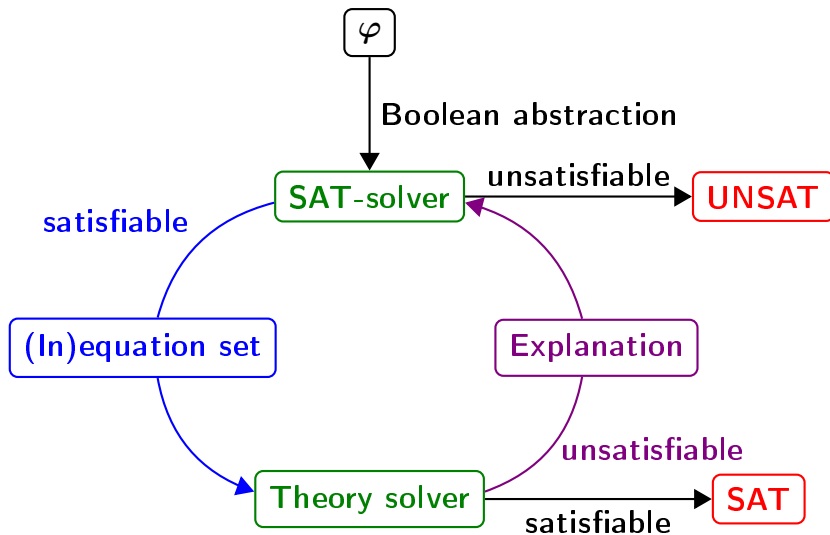
SAT

$$\begin{aligned} & (\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & (\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \end{aligned}$$

$$\begin{aligned} & (\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & (\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \end{aligned}$$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Full lazy SMT-solving



$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 :

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : a_4 : 1

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 :

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

$DL1 : a_1 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 :

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 : $a_2 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

DL3 :

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

DL3 : $a_5 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

DL3 : $a_5 : 0, a_6 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: a_1, \dots, a_9

Assignment to decision variables: false

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

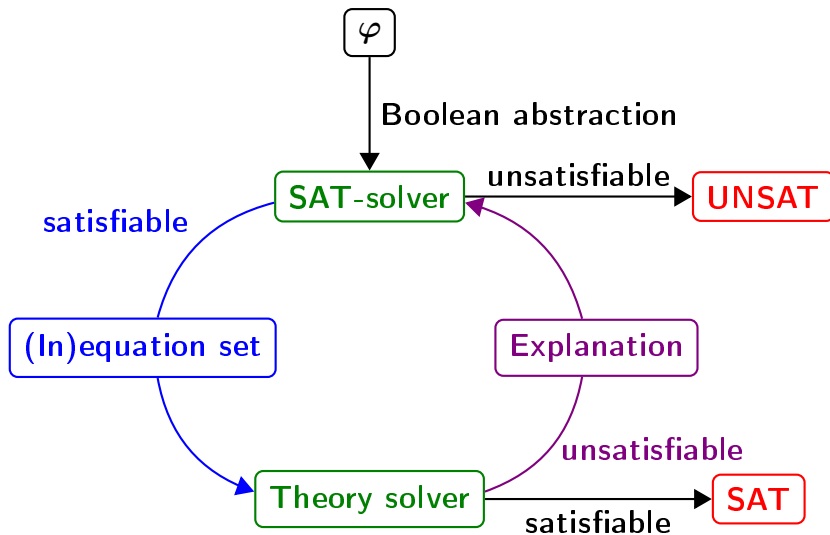
DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

DL3 : $a_5 : 0, a_6 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT-solving



Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $DL1 : a_1 : 0$
 $DL2 : a_2 : 0, a_3 : 1$ $DL3 : a_5 : 0, a_6 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $DL1 : a_1 : 0$

$DL2 : a_2 : 0, a_3 : 1$ $DL3 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $DL1 : a_1 : 0$

$DL2 : a_2 : 0, a_3 : 1$

$DL3 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

$$\begin{aligned} & (\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & (\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \quad \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \end{aligned}$$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$ $DL1 : a_1 : 0$

$DL2 : a_2 : 0, a_3 : 1$

$DL3 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_3, a_6$

$$\begin{aligned} & (\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & (\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \quad \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \end{aligned}$$

Encoding:

$$\begin{aligned} a_4 : p_1 + p_2 + p_3 &\geq 100 & a_7 : p_3 &\geq 10 & a_8 : p_1 + 2p_2 + 5p_3 &\leq 180 \\ a_9 : 3p_1 + 2p_2 + p_3 &\leq 300 & a_3 : p_3 &= 0 & a_6 : p_2 &\geq 5 \end{aligned}$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_3 : p_3 = 0$$

$$a_6 : p_2 \geq 5$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_3 : p_3 = 0$$

$$a_6 : p_2 \geq 5$$

No.

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_3 : p_3 = 0$$

$$a_6 : p_2 \geq 5$$

No.

Reason:

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

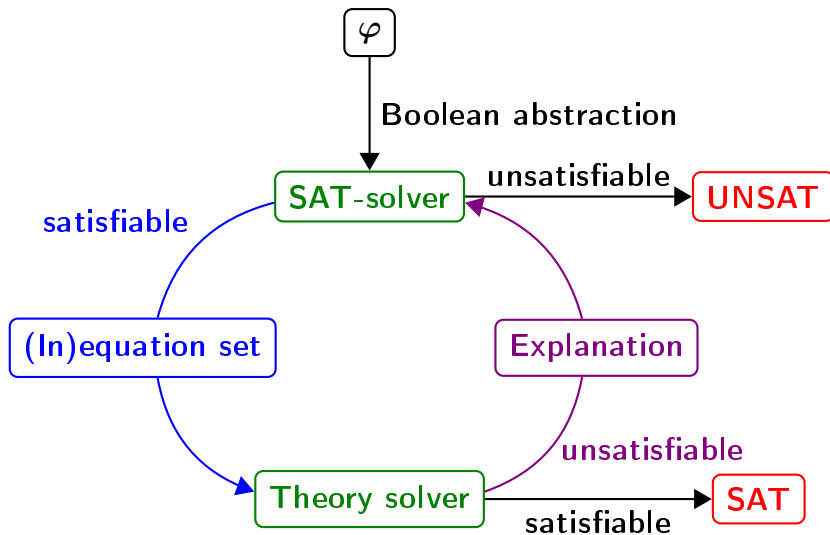
$$a_3 : p_3 = 0$$

$$a_6 : p_2 \geq 5$$

No.

Reason: $\underbrace{p_3 = 0}_{a_3} \wedge \underbrace{p_3 \geq 10}_{a_7}$ are conflicting.

Full lazy SMT-solving



Add clause $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

DL3 : $a_5 : 0, a_6 : 1$

Add clause $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

DL1 : $a_1 : 0$

DL2 : $a_2 : 0, a_3 : 1$

DL3 : $a_5 : 0, a_6 : 1$

Conflict resolution is simple, since the new clause is already an asserting one.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 :

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

DL2 :

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

DL2 : $a_5 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

DL2 : $a_5 : 0, a_6 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

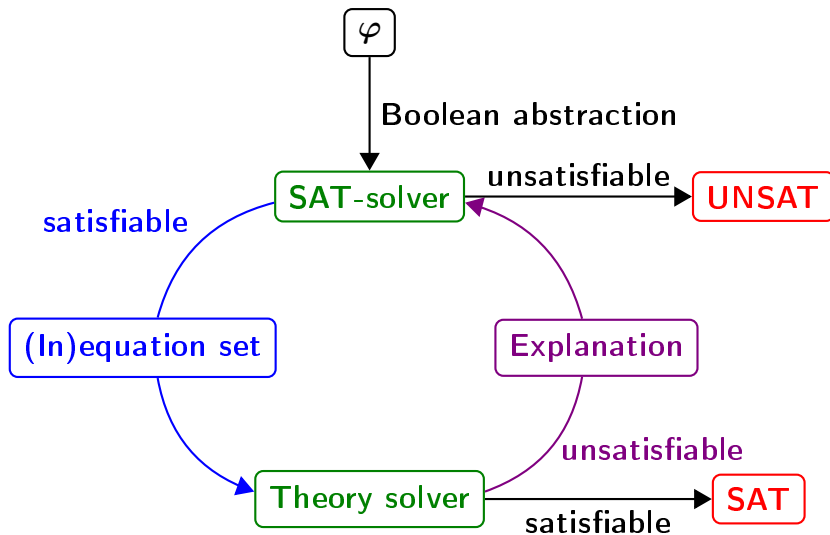
DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

DL2 : $a_5 : 0, a_6 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT-solving



Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1$

$DL2 : a_5 : 0, a_6 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1$

$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1$

$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$\begin{aligned} & (\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & (\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \quad \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \end{aligned}$$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1$

$DL2 : a_5 : 0, a_6 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$\begin{aligned} & (\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & (\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \end{aligned}$$

Encoding:

$$\begin{array}{lll} a_4 : p_1 + p_2 + p_3 \geq 100 & a_7 : p_3 \geq 10 & a_8 : p_1 + 2p_2 + 5p_3 \leq 180 \\ a_9 : 3p_1 + 2p_2 + p_3 \leq 300 & a_2 : p_2 = 0 & a_6 : p_2 \geq 5 \end{array}$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_6 : p_2 \geq 5$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_6 : p_2 \geq 5$$

No.

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_6 : p_2 \geq 5$$

No.

Reason:

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

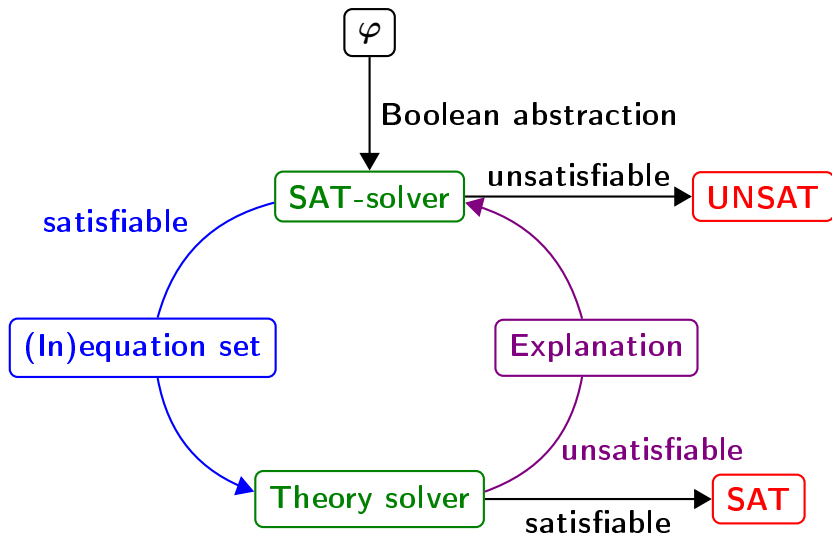
$$a_2 : p_2 = 0$$

$$a_6 : p_2 \geq 5$$

No.

Reason: $\underbrace{p_2 = 0}_{a_2} \wedge \underbrace{p_2 \geq 5}_{a_6}$ are conflicting.

Full lazy SMT-solving



Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

DL2 : $a_5 : 0, a_6 : 1$

Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

$DL1 : a_1 : 0, a_2 : 1$

$DL2 : a_5 : 0, a_6 : 1$

Conflict resolution is simple, since the new clause is already an asserting one.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1, a_6 : 0$

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

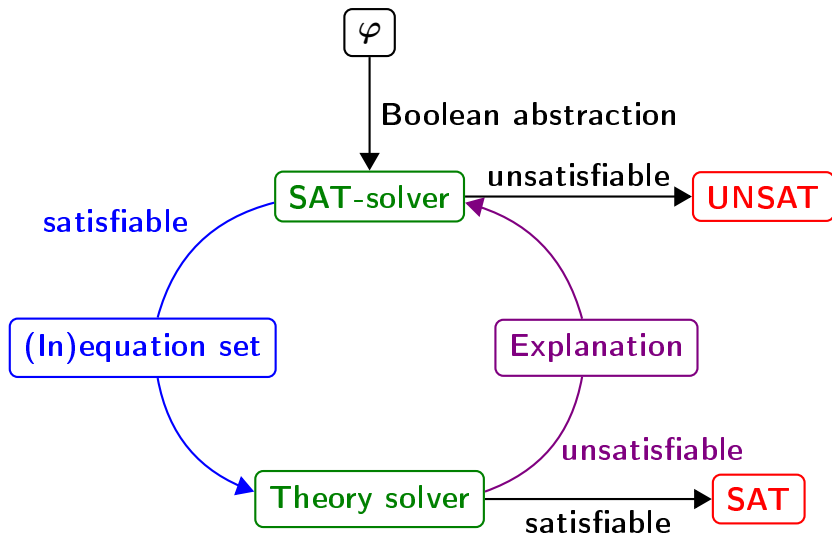
$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge$$
$$(\neg a_2 \vee \neg a_6)$$

DL0 : $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$

DL1 : $a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Solution found for the Boolean abstraction.

Full lazy SMT-solving



$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$\begin{aligned} & \underbrace{(p_1 = 0)}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3} \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge \\ & \underbrace{(p_1 \geq 5)}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6} \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge \\ & \underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6) \end{aligned}$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$ $DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$\underbrace{(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0)}_{a_1} \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$\underbrace{(p_1 \geq 5 \vee p_2 \geq 5)}_{a_5} \wedge \underbrace{p_3 \geq 10}_{a_6} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_7} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_8} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

a_9

Encoding:

$$\begin{array}{lll} a_4 : p_1 + p_2 + p_3 \geq 100 & a_7 : p_3 \geq 10 & a_8 : p_1 + 2p_2 + 5p_3 \leq 180 \\ a_9 : 3p_1 + 2p_2 + p_3 \leq 300 & a_2 : p_2 = 0 & a_5 : p_1 \geq 5 \end{array}$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_5 : p_1 \geq 5$$

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_5 : p_1 \geq 5$$

Yes.

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

$$a_5 : p_1 \geq 5$$

Yes. E.g.,

Is the conjunction of the following constraints satisfiable?

$$a_4 : p_1 + p_2 + p_3 \geq 100$$

$$a_7 : p_3 \geq 10$$

$$a_8 : p_1 + 2p_2 + 5p_3 \leq 180$$

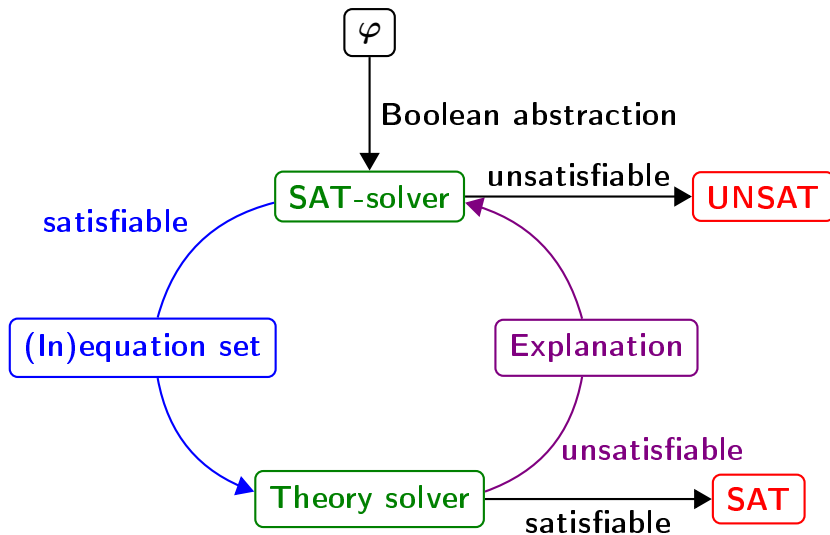
$$a_9 : 3p_1 + 2p_2 + p_3 \leq 300$$

$$a_2 : p_2 = 0$$

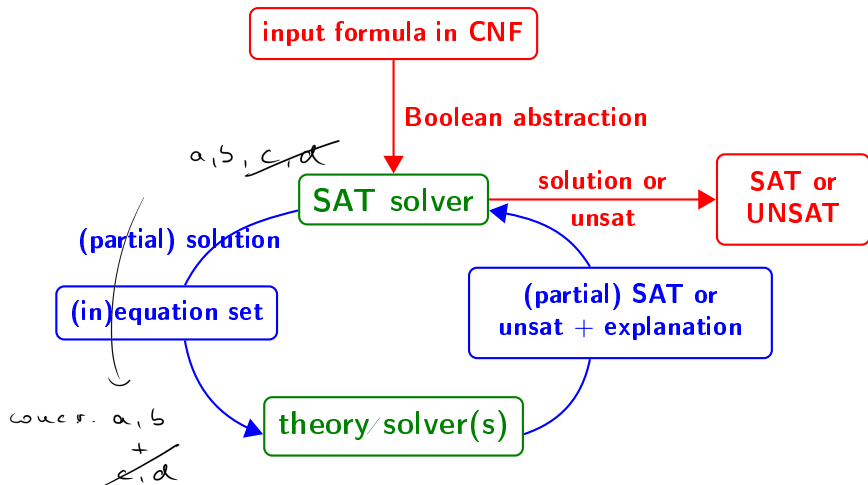
$$a_5 : p_1 \geq 5$$

Yes. E.g., $p_1 = 90$, $p_2 = 0$, $p_3 = 10$ is a solution.

Full lazy SMT-solving



Less lazy SMT-solving



Requirements on the theory solver

- 1 **Incrementality**: In less lazy solving we extend the set of constraints. The solver should make use of the previous satisfiability check for the check of the extended set.
- 2 **(Preferably minimal) infeasible subsets**: Compute a reason for unsatisfaction
- 3 **Backtracking**: The theory solver should be able to remove constraints in inverse chronological order.

- This approach strictly divides between logical (Boolean) structure and theory constraints.
- There are other approaches, which do not divide Boolean and theory solving so strictly.
- One idea: Propagate in the SAT-solver **bounds** on theory variables.