Satisfiability Checking - WS 2016/2017

Series 13

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Exercise 1

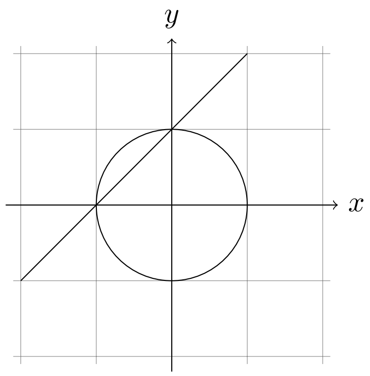
The Cylindrical Algebraic Decomposition aims at decomposing the whole solution space into *sign-invariant regions*. Each such region is represented by a single sample point.

a) Why can you decide satisfiability using only a few sample points, although the solution space is infinitely large?

b) The notion of *delineability* of intervals is crucial for the CAD. Explain why the CAD method relies on this notion. Are sign-invariant regions delineable in general, or can you give regions that are not delineable?

c) Consider the following problem. Give a minimal selection of sample points that could be used to solve this example. You can give the sample points as dots in the diagram.

$$p_1 : x^2 + y^2 - 1 = 0 \land p_2 : x - y + 1 = 0$$



d) Due to the way how the CAD algorithm determines the sample points, the set of sample points that will actually be used is much larger. Try to give a set of samples points that may be used by an actual implementation in the above example and argue why the additional sample points are included.

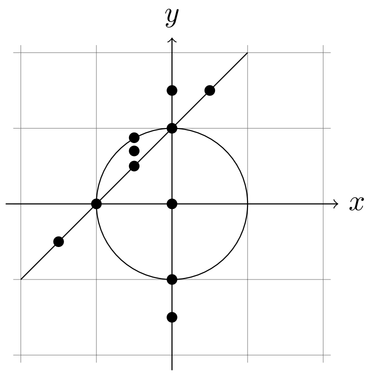
Solution:

a) The inequalities are in a canonical form and thus the polynomials only compare to zero. Within a region, no polynomial changes its sign and thus, all points within a region are equivalent with respect to the input inequalities – they fulfill and conflict with the same inequalities. Hence, it suffices to select (at least) one sample point per region. As there are only finitely many polynomials of finite degree, the number of regions is finite.

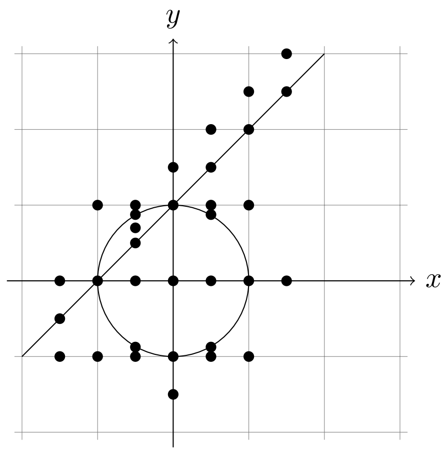
b) Delineability is crucial because the CAD handles the individual variables separately. This property guarantees, that the ordering of the different roots (and thus the sign changes) do not depend on a certain variable, once an interval (and thus a range of values) has been selected for this variables. Thereby, all sample points in this range are equivalent with respect to the samples generated afterwards when lifting.

In general, a region may not be delineable. In the example of Task c, everything below the line and the circle might be in a single region. However, this region is clearly not delineable.

c) Minimum selection of sample points:



d) An actual selection of sample points:



The CAD algorithm first selects a set of sample points in one dimension only. Let's assume that this is the x dimension. For this, every point *where something happens* is selected, i.e. at least $-1, 0, 1$ as well as points smaller, inbetween and greater, for example $-1.5, -0.5, 0.5, 1.5$. Then, each of these sample points is plugged into each polynomial and for the next dimension, once again all points *where something happens* are selected. However, the lifting is done independently for different values of x and thus multiple samples that describe the same region are constructed.