Satisfiability Checking

Simplex as a Theory Module in SMT

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Outline

1. Full lazy SMT-solving with Simplex

2. Less lazy SMT-solving with Simplex

Outline

The Xmas problem

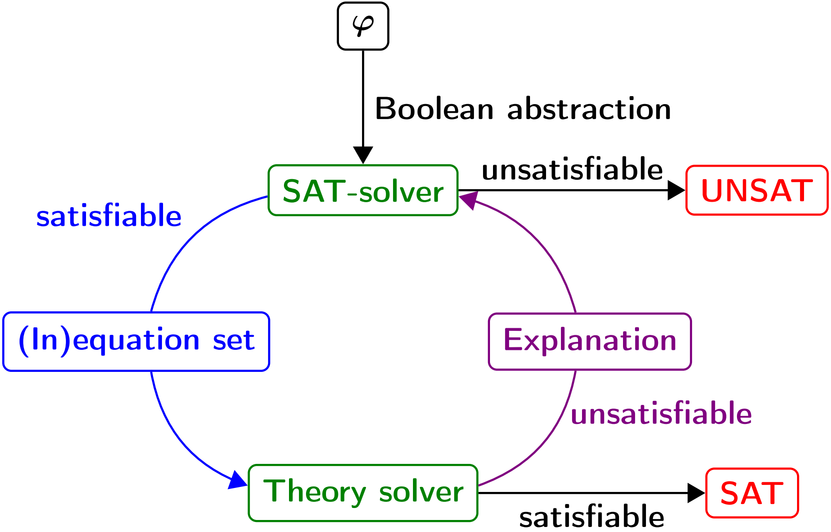
There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

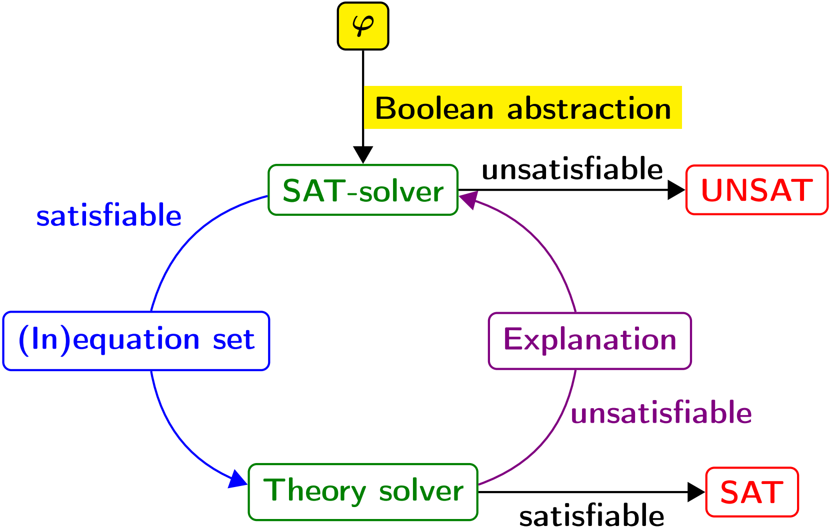
$$
\begin{aligned}
(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\
(p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\
3p_1 + 2p_2 + p_3 \leq 300
\end{aligned}
$$

For the moment we relax the integrality constraints, i.e., we search for a real-valued solution.

Full lazy SMT-solving



Full lazy SMT-solving



Boolean abstraction

Arithmetic formula:

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

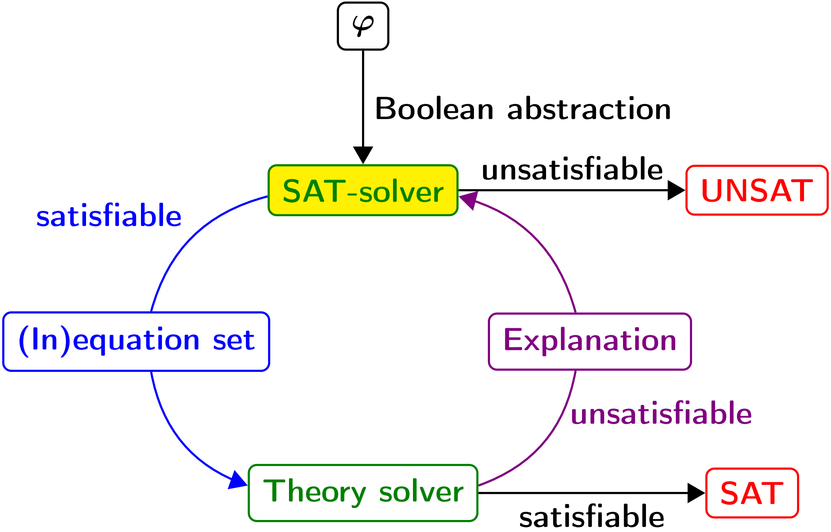
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$

$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Boolean abstraction:

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Full lazy SMT-solving



Full lazy SAT-solving

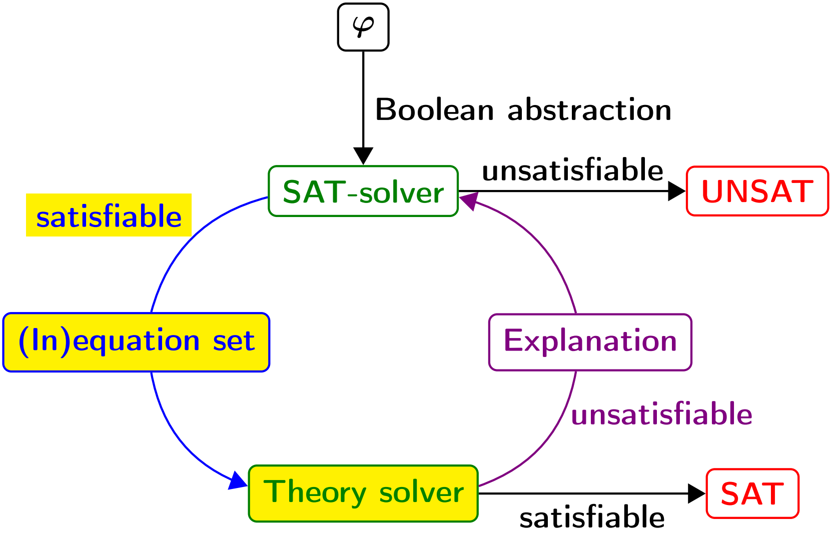
Boolean abstraction:

$$(a_1 \lor a_2 \lor a_3) \land a_4 \land (a_5 \lor a_6) \land a_7 \land a_8 \land a_9$$

Assume a fixed variable order: $a_1, \ldots, a_9$
Assignment to decision variables: false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

Full lazy SMT-solving



Full lazy theory solving

Current assignment:

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, \quad DL1 : a_1 : 0,$
$DL2 : a_2 : 0, a_3 : 1, \quad DL3 : a_5 : 0, a_6 : 1$

True theory constraints: a_4, a_7, a_8, a_9, a_3, a_6

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Encoding:
$p_1 + p_2 + p_3 \geq 100,\ p_3 \geq 10,$
$p_1 + 2p_2 + 5p_3 \leq 180,\ 3p_1 + 2p_2 + p_3 \leq 300,\ p_3 = 0,\ p_2 \geq 5$

Full lazy theory solving

$$\begin{array}{llrrrrl}
p_1 + p_2 + p_3 \geq 100 & \rightarrow & s_1 = & p_1 + & p_2 + & p_3 & s_1 \geq 100 \\
p_3 \geq 10 & \rightarrow & s_2 = & & & p_3 & s_2 \geq 10 \\
p_1 + 2p_2 + 5p_3 \leq 180 & \rightarrow & s_3 = & p_1 + & 2p_2 + & 5p_3 & s_3 \leq 180 \\
3p_1 + 2p_2 + p_3 \leq 300 & \rightarrow & s_4 = & 3p_1 + & 2p_2 + & p_3 & s_4 \leq 300 \\
p_3 = 0 & \rightarrow & s_5 = & & & p_3 & s_5 = 0 \\
p_2 \geq 5 & \rightarrow & s_6 = & & p_2 & & s_6 \geq 5
\end{array}$$

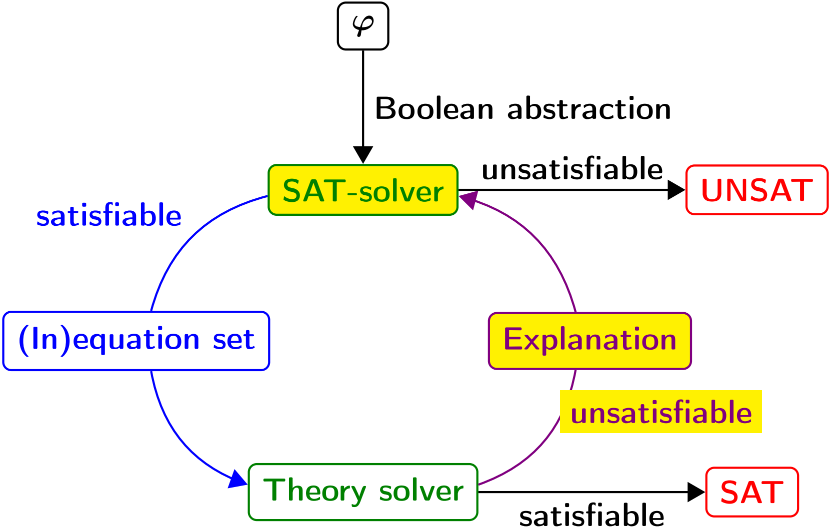
Variable order: $s_1 < \ldots < s_6 < p_1 < p_2 < p_3$, the values of the variables are given in parentheses

	$p_1(0)$	$p_2(0)$	$p_3(0)$		$s_1(100)$	$p_2(0)$	$p_3(0)$		$s_1(100)$	$p_2(0)$	$s_2(10)$
$s_1(0)$	1	1	1	$p_1(100)$	1	-1	-1	$p_1(90)$	1	-1	-1
$s_2(0)$	0	0	1	$s_2(0)$	0	0	1	$p_3(10)$	0	0	1
$s_3(0)$	1	2	5	$s_3(100)$	1	1	4	$s_3(140)$	1	1	4
$s_4(0)$	3	2	1	$s_4(300)$	3	-1	-2	$s_4(280)$	3	-1	-2
$s_5(0)$	0	0	1	$s_5(0)$	0	0	1	$s_5(10)$	0	0	1
$s_6(0)$	0	1	0	$s_6(0)$	0	1	0	$s_6(0)$	0	1	0

Conflict: the constraints for the basic variable of the conflicting row and all non-basic variables with non-zero coefficients in the conflicting row together are unsatisfiable.

Thus $\underbrace{p_3 = 0}_{a_3} \wedge \underbrace{p_3 \geq 10}_{a_7}$ is not satisfiable.

Full lazy SMT-solving



Full lazy SAT-solving

Current assignment:

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$
$DL3 : a_5 : 0, a_6 : 1$

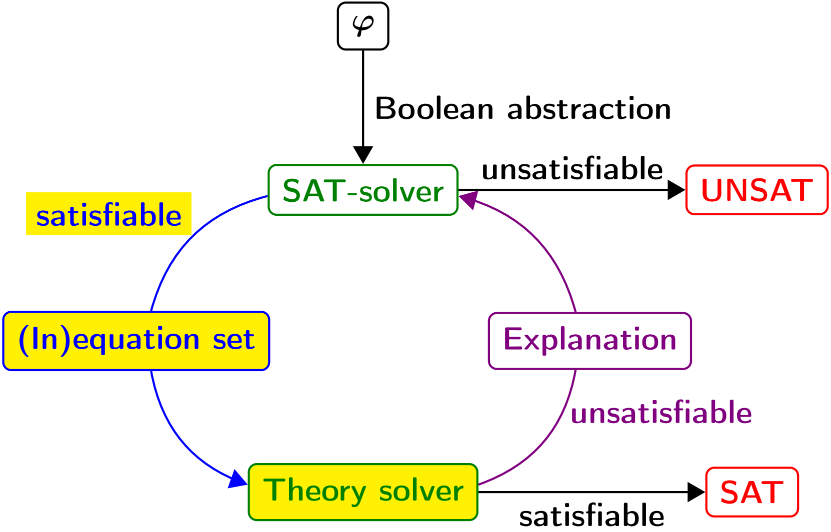
Learn new clause: $(\neg a_3 \vee \neg a_7)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

No conflict resolution needed, since the new clause is already asserting.
Backtrack to decision level $DL0$ and use the new clause for propagation.

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Full lazy SMT-solving



Full lazy theory solving

$DL0: a_4: 1, a_7: 1, a_8: 1, a_9: 1, a_3: 0, \quad DL1: a_1: 0, a_2: 1,$
$DL2: a_5: 0, a_6: 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_6$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7)$$

Encoding:

$p_1 + p_2 + p_3 \geq 100, \ p_3 \geq 10,$
$p_1 + 2p_2 + 5p_3 \leq 180, \ 3p_1 + 2p_2 + p_3 \leq 300, \ p_2 = 0, \ p_2 \geq 5$

Full lazy theory solving

$p_1 + p_2 + p_3 \geq 100$	$\to$	$s_1 =$	p_1+	p_2+	p_3	$s_1 \geq 100$
$p_3 \geq 10$	$\to$	$s_2 =$			p_3	$s_2 \geq 10$
$p_1 + 2p_2 + 5p_3 \leq 180$	$\to$	$s_3 =$	p_1+	$2p_2+$	$5p_3$	$s_3 \leq 180$
$3p_1 + 2p_2 + p_3 \leq 300$	$\to$	$s_4 =$	$3p_1+$	$2p_2+$	p_3	$s_4 \leq 300$
$p_2 = 0$	$\to$	$s_5 =$		p_2		$s_5 = 0$
$p_2 \geq 5$	$\to$	$s_6 =$		p_2		$s_6 \geq 5$

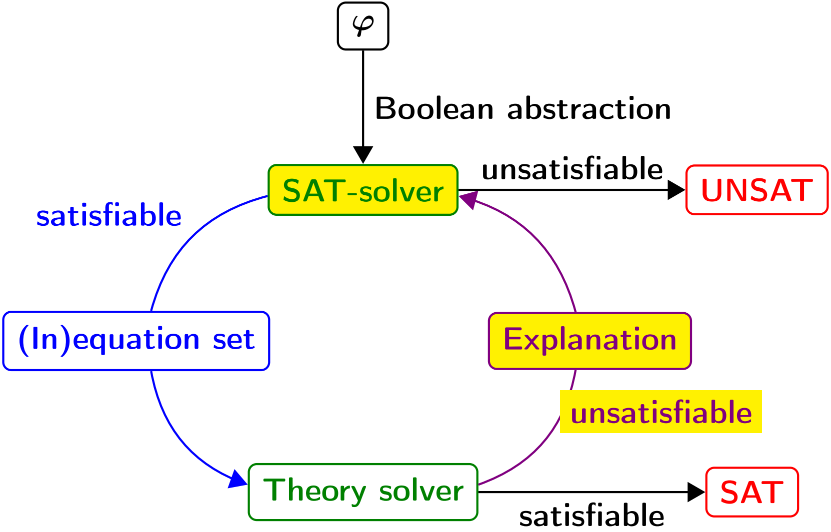
Variable order: $s_1 < \ldots < s_6 < p_1 < p_2 < p_3$, the values of the variables are given in parentheses

	$p_1(0)$	$p_2(0)$	$p_3(0)$		$s_1(100)$	$p_2(0)$	$p_3(0)$		$s_1(100)$	$p_2(0)$	$s_2(10)$		$s_1(100)$	$s_6(5)$	$s_2(10)$
$s_1(0)$	1	1	1	$p_1(100)$	1	-1	-1	$p_1(90)$	1	-1	-1	$p_1(85)$	1	-1	-1
$s_2(0)$	0	0	1	$s_2(0)$	0	0	1	$p_3(10)$	0	0	1	$p_3(10)$	0	0	1
$s_3(0)$	1	2	5	$s_3(100)$	1	1	4	$s_3(140)$	1	1	4	$s_3(145)$	1	1	4
$s_4(0)$	3	2	1	$s_4(300)$	3	-1	-2	$s_4(280)$	3	-1	-2	$s_4(275)$	3	-1	-2
$s_5(0)$	0	1	0	$s_5(0)$	0	1	0	$s_5(0)$	0	1	0	$s_5(5)$	0	1	0
$s_6(0)$	0	1	0	$s_6(0)$	0	1	0	$s_6(0)$	0	1	0	$p_2(5)$	0	1	0

Conflict: the constraints for the basic variable of the conflicting row and all non-basic variables with non-zero coefficients in the conflicting row together are unsatisfiable.

Thus $\underbrace{p_2 = 0}_{a_2} \wedge \underbrace{p_2 \geq 5}_{a_6}$ is not satisfiable.

Full lazy SMT-solving



Full lazy SAT-solving

Current assignment:

$DL0: a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1: a_1 : 0, a_2 : 1$
$DL2: a_5 : 0, a_6 : 1$

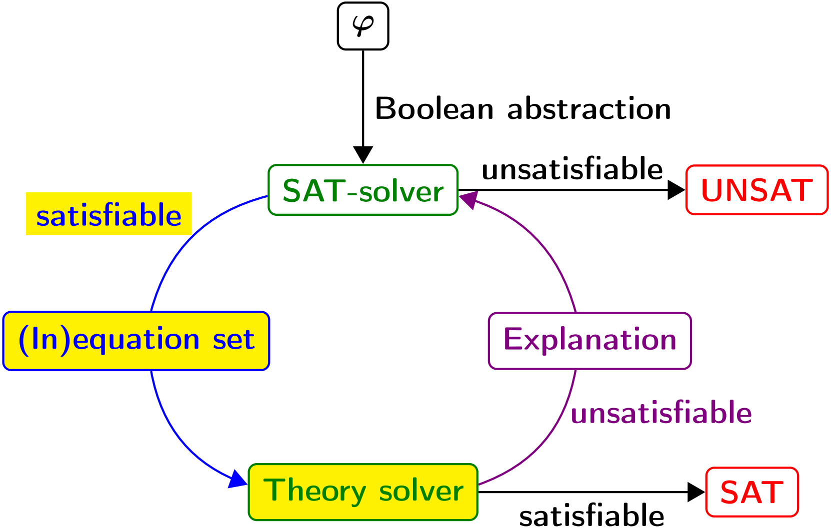
Learn new clause: $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

No conflict resolution needed, since the new clause is already asserting.
Backtrack to decision level $DL1$ and apply propagation.

$DL0: a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1: a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Full lazy SMT-solving



Full lazy theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0, \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

True theory constraints: $a_4, a_7, a_8, a_9, a_2, a_5$

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

Encoding:

$p_1 + p_2 + p_3 \geq 100,\ p_3 \geq 10,$
$p_1 + 2p_2 + 5p_3 \leq 180,\ 3p_1 + 2p_2 + p_3 \leq 300,\ p_2 = 0,\ p_1 \geq 5$

Full lazy theory solving

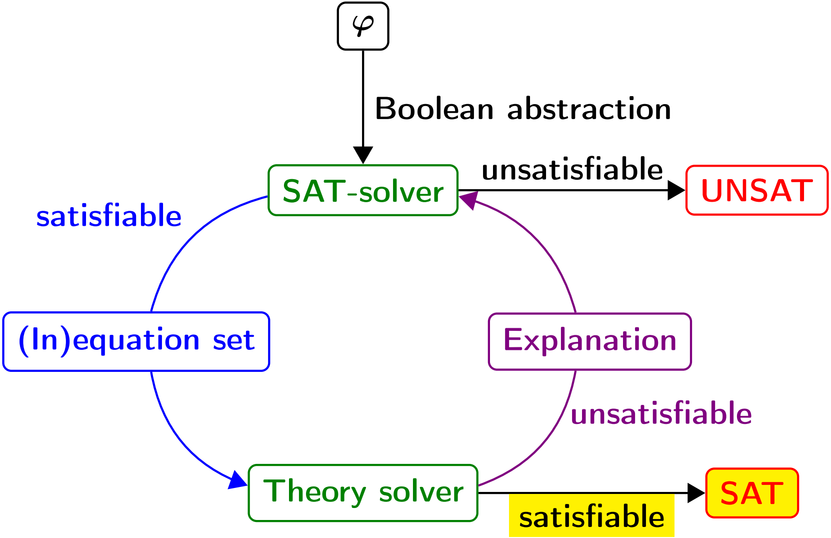
$p_1 + p_2 + p_3 \geq 100$	$\rightarrow$	$s_1 =$	p_1+	p_2+	p_3	$s_1 \geq 100$
$p_3 \geq 10$	$\rightarrow$	$s_2 =$			p_3	$s_2 \geq 10$
$p_1 + 2p_2 + 5p_3 \leq 180$	$\rightarrow$	$s_3 =$	p_1+	$2p_2+$	$5p_3$	$s_3 \leq 180$
$3p_1 + 2p_2 + p_3 \leq 300$	$\rightarrow$	$s_4 =$	$3p_1+$	$2p_2+$	p_3	$s_4 \leq 300$
$p_2 = 0$	$\rightarrow$	$s_5 =$		p_2		$s_5 = 0$
$p_1 \geq 5$	$\rightarrow$	$s_6 =$	p_1			$s_6 \geq 5$

Variable order: $s_1 < \ldots < s_6 < p_1 < p_2 < p_3$, the values of the variables are given in parentheses

	$p_1(0)$	$p_2(0)$	$p_3(0)$		$s_1(100)$	$p_2(0)$	$p_3(0)$		$s_1(100)$	$p_2(0)$	$s_2(10)$
$\mathbf{s_1(0)}$	1	1	1	$p_1(100)$	1	-1	-1	$p_1(90)$	1	-1	-1
$\mathbf{s_2(0)}$	0	0	1	$\mathbf{s_2(0)}$	0	0	1	$p_3(10)$	0	0	1
$s_3(0)$	1	2	5	$s_3(100)$	1	1	4	$s_3(140)$	1	1	4
$s_4(0)$	3	2	1	$s_4(300)$	3	-1	-2	$s_4(280)$	3	-1	-2
$s_5(0)$	0	1	0	$s_5(0)$	0	1	0	$s_5(0)$	0	1	0
$\mathbf{s_6(0)}$	1	0	0	$s_6(100)$	1	-1	-1	$s_6(90)$	1	-1	-1

Solution: $p_1 = 90, \quad p_2 = 0, \quad p_3 = 10.$

Full lazy SMT-solving



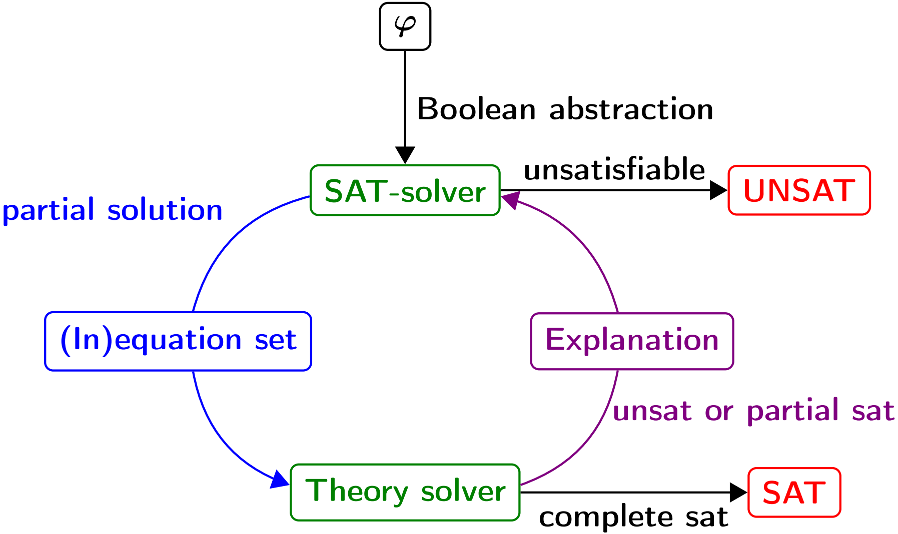
Outline

1. Full lazy SMT-solving with Simplex
2. Less lazy SMT-solving with Simplex

Less lazy SMT-solving

- In full lazy SMT-solving, the SAT solver asks the theory solver whether found complete satisfying assignments for the abstraction are consistent in the theory.
- In less lazy SMT-solving, the SAT solver asks for consistency checks in the theory more frequently, also for partial assignments.
- Usually, this happens after each completed decision level.

Less lazy SMT-solving



Requirements on the theory solver

- (Minimal) infeasible subsets (to explain infeasibility)
- Incrementality (to add constraints stepwise)
- Backtracking (to mimic backtracking in the SAT solver)

Minimal infeasible subsets in Simplex:

- As seen in full lazy SMT solving
- The constraints corresponding to the basic variable of the contradictory row and all non-basic variables with non-zero coefficients in this row are together unsatisfiable.

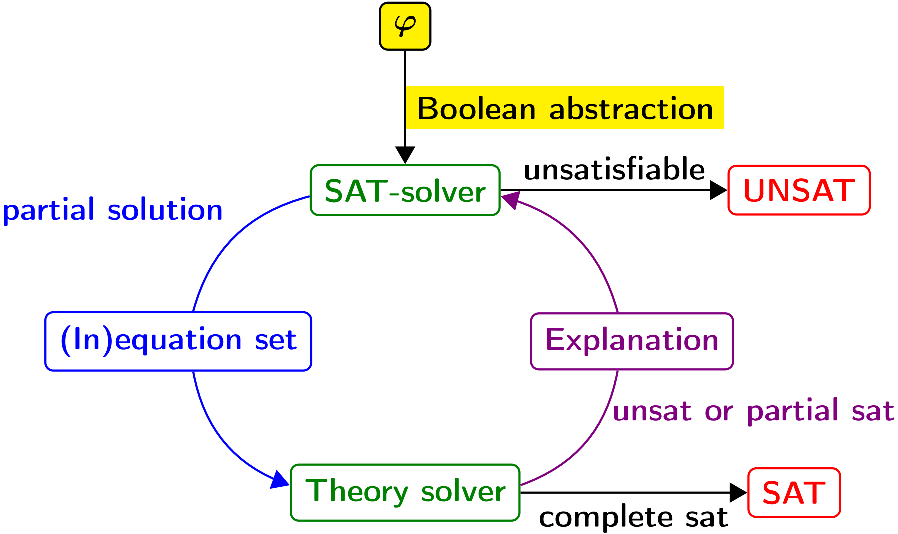
Incrementality in Simplex:

- Add all constraints but without bounds on non-active constraints.
- If a constraint becomes true, activate its bound.

Backtracking in Simplex:

- Remove bounds of unassigned constraints

Less lazy SMT-solving



Boolean abstraction

Arithmetic formula:

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$

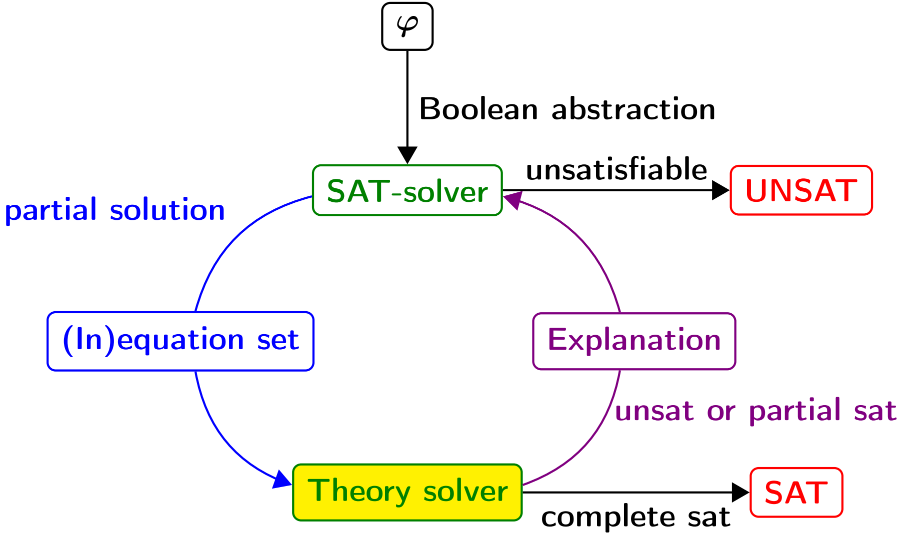
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$

$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Boolean abstraction:

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Less lazy SMT-solving



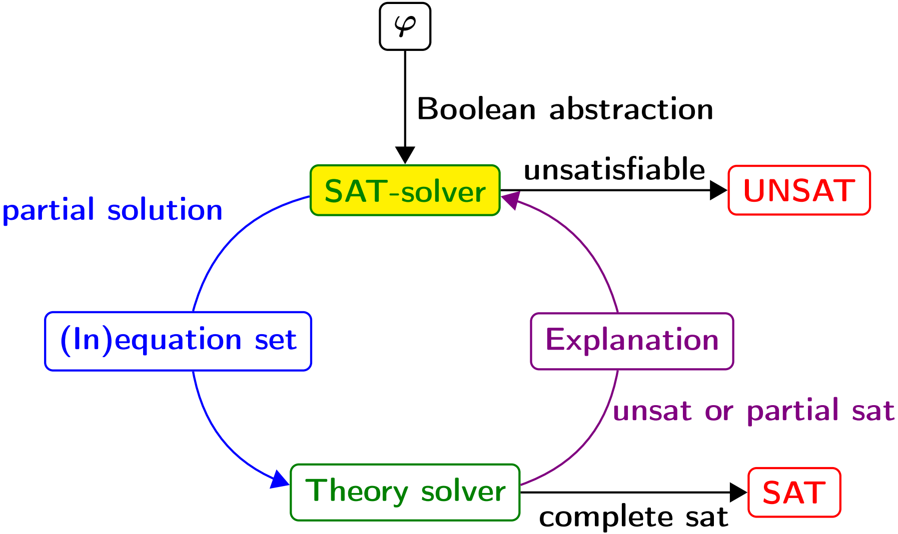
Less lazy theory solving

Initialize the Simplex tableau with all equalities but without any bounds.

$$\begin{array}{llcrrrl}
p_1 = 0 & \rightarrow & s_1 = & p_1 & & & s_1 = 0 \\
p_2 = 0 & \rightarrow & s_2 = & & p_2 & & s_2 = 0 \\
p_3 = 0 & \rightarrow & s_3 = & & & p_3 & s_3 = 0 \\
p_1 + p_2 + p_3 \geq 100 & \rightarrow & s_4 = & p_1 + & p_2 + & p_3 & s_4 \geq 100 \\
p_1 \geq 5 & \rightarrow & s_5 = & p_1 & & & s_5 \geq 5 \\
p_2 \geq 5 & \rightarrow & s_6 = & & p_2 & & s_6 \geq 5 \\
p_3 \geq 10 & \rightarrow & s_7 = & & & p_3 & s_7 \geq 10 \\
p_1 + 2p_2 + 5p_3 \leq 180 & \rightarrow & s_8 = & p_1 + & 2p_2 + & 5p_3 & s_8 \leq 180 \\
3p_1 + 2p_2 + p_3 \leq 300 & \rightarrow & s_9 = & 3p_1 + & 2p_2 + & p_3 & s_9 \leq 300
\end{array}$$

	$p_1(0)$	$p_2(0)$	$p_3(0)$
$s_1(0)$	1	0	0
$s_2(0)$	0	1	0
$s_3(0)$	0	0	1
$s_4(0)$	1	1	1
$s_5(0)$	1	0	0
$s_6(0)$	0	1	0
$s_7(0)$	0	0	1
$s_8(0)$	1	2	5
$s_9(0)$	3	2	1

Less lazy SMT-solving



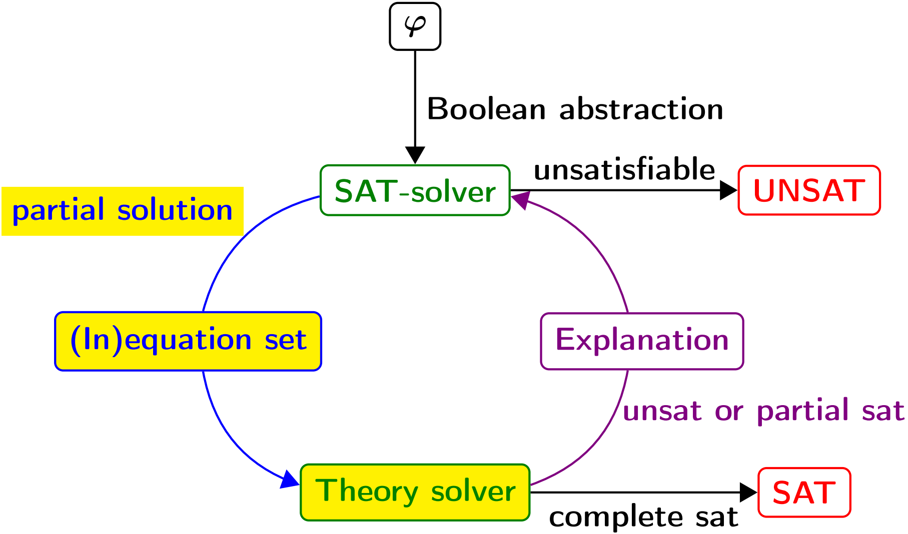
Less lazy SAT-solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: $a_1, \ldots, a_9$
Assignment to decision variables: false

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

Less lazy SMT-solving



$DL0$: $a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$

New true theory constraints: a_4, a_7, a_8, a_9

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Encoding:
$s_4 \geq 100$, $s_7 \geq 10$, $s_8 \leq 180$, $s_9 \leq 300$

Less lazy theory solving

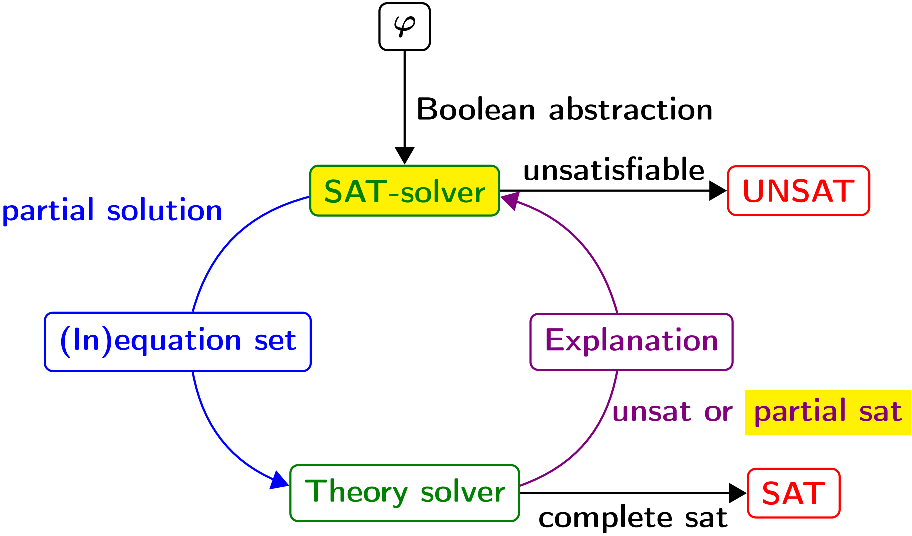
$p_1 = 0$	$\rightarrow$	$s_1 =$	p_1			$s_1 = 0$
$p_2 = 0$	$\rightarrow$	$s_2 =$		p_2		$s_2 = 0$
$p_3 = 0$	$\rightarrow$	$s_3 =$			p_3	$s_3 = 0$
$p_1 + p_2 + p_3 \geq 100$	$\rightarrow$	$s_4 =$	p_1+	p_2+	p_3	$s_4 \geq 100$
$p_1 \geq 5$	$\rightarrow$	$s_5 =$	p_1			$s_5 \geq 5$
$p_2 \geq 5$	$\rightarrow$	$s_6 =$		p_2		$s_6 \geq 5$
$p_3 \geq 10$	$\rightarrow$	$s_7 =$			p_3	$s_7 \geq 10$
$p_1 + 2p_2 + 5p_3 \leq 180$	$\rightarrow$	$s_8 =$	p_1+	$2p_2+$	$5p_3$	$s_8 \leq 180$
$3p_1 + 2p_2 + p_3 \leq 300$	$\rightarrow$	$s_9 =$	$3p_1+$	$2p_2+$	p_3	$s_9 \leq 300$

Variable order: $s_1 < \ldots < s_9 < p_1 < p_2 < p_3$, the values of the variables are given in parentheses

	$p_1(0)$	$p_2(0)$	$p_3(0)$		$s_4(100)$	$p_2(0)$	$p_3(0)$		$s_4(100)$	$p_2(0)$	$s_7(10)$
$s_1(0)$	1	0	0	$s_1(100)$	1	-1	-1	$s_1(90)$	1	-1	-1
$s_2(0)$	0	1	0	$s_2(0)$	0	1	0	$s_2(0)$	0	1	0
$s_3(0)$	0	0	1	$s_3(0)$	0	0	1	$s_3(10)$	0	0	1
$\mathbf{s_4(0)}$	1	1	1	$p_1(100)$	1	-1	-1	$p_1(90)$	1	-1	-1
$s_5(0)$	1	0	0	$s_5(100)$	1	-1	-1	$s_5(90)$	1	-1	-1
$s_6(0)$	0	1	0	$s_6(0)$	0	1	0	$s_6(0)$	0	1	0
$\mathbf{s_7(0)}$	0	0	1	$\mathbf{s_7(0)}$	0	0	1	$p_3(10)$	0	0	1
$s_8(0)$	1	2	5	$s_8(100)$	1	1	4	$s_8(140)$	1	1	4
$s_9(0)$	3	2	1	$s_9(300)$	3	-1	-2	$s_9(280)$	3	-1	-2

Return partial SAT.

Less lazy SMT-solving

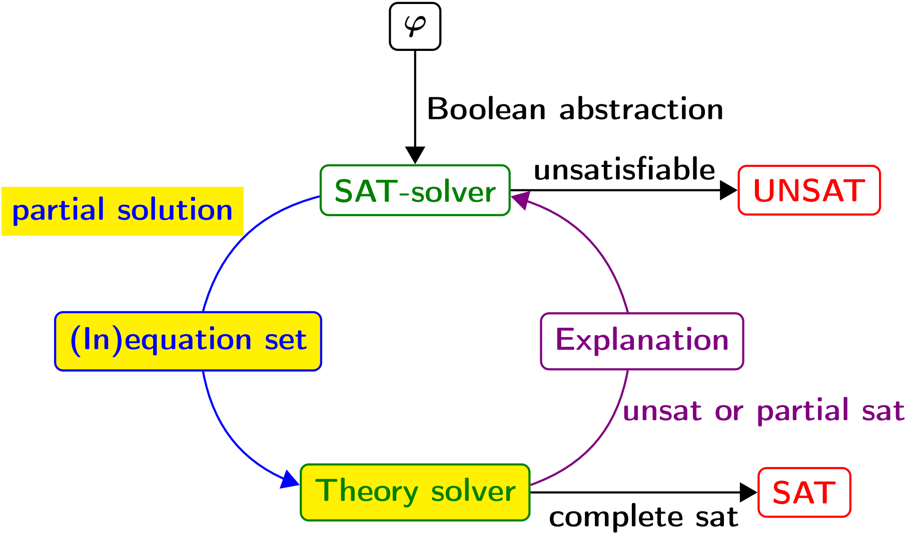


Less lazy SAT-solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9$$

Assume a fixed variable order: $a_1, \ldots, a_9$
Assignment to decision variables: false
$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$

Less lazy SMT-solving



Less lazy theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, \quad DL1 : a_1 : 0,$
$DL2 : a_2 : 0, a_3 : 1$

Incrementality: add a_3

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Encoding:
$s_3 = 0$

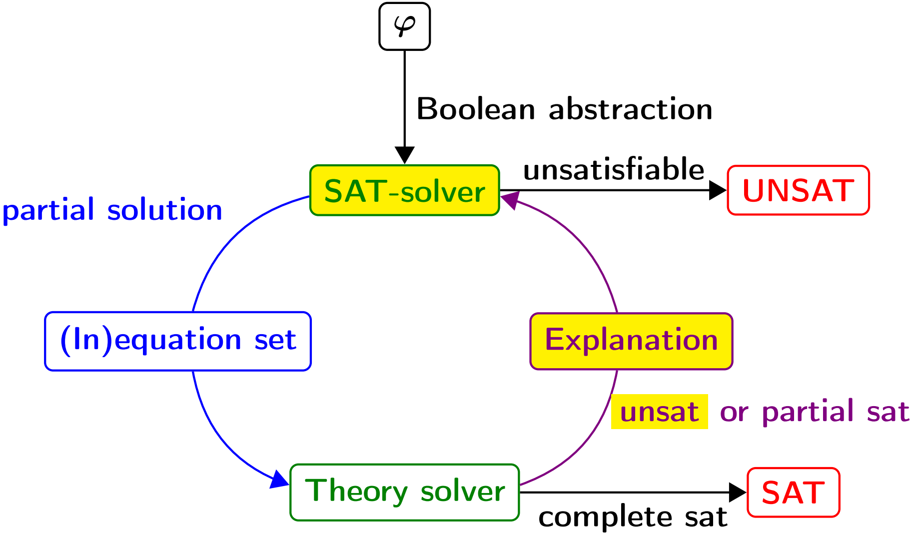
Less lazy theory solving

$$\begin{array}{lclllll}
p_1 = 0 & \rightarrow & s_1 = & p_1 & & & s_1 = 0 \\
p_2 = 0 & \rightarrow & s_2 = & & p_2 & & s_2 = 0 \\
p_3 = 0 & \rightarrow & s_3 = & & & p_3 & \boxed{s_3 = 0} \\
p_1 + p_2 + p_3 \geq 100 & \rightarrow & s_4 = & p_1 + & p_2 + & p_3 & s_4 \geq 100 \\
p_1 \geq 5 & \rightarrow & s_5 = & p_1 & & & s_5 \geq 5 \\
p_2 \geq 5 & \rightarrow & s_6 = & & p_2 & & s_6 \geq 5 \\
p_3 \geq 10 & \rightarrow & s_7 = & & & p_3 & s_7 \geq 10 \\
p_1 + 2p_2 + 5p_3 \leq 180 & \rightarrow & s_8 = & p_1 + & 2p_2 + & 5p_3 & s_8 \leq 180 \\
3p_1 + 2p_2 + p_3 \leq 300 & \rightarrow & s_9 = & 3p_1 + & 2p_2 + & p_3 & s_9 \leq 300
\end{array}$$

	$s_4(100)$	$p_2(0)$	$s_7(10)$
$s_1(90)$	1	-1	-1
$s_2(0)$	0	1	0
$s_3(10)$	0	0	1
$p_1(90)$	1	-1	-1
$s_5(90)$	1	-1	-1
$s_6(0)$	0	1	0
$p_3(10)$	0	0	1
$s_8(140)$	1	1	4
$s_9(280)$	3	-1	-2

Conflict: $\underbrace{p_3 = 0}_{a_3} \wedge \underbrace{p_3 \geq 10}_{a_7}$ is not satisfiable.

Less lazy SMT-solving



Less lazy SAT-solving

Current assignment:

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1$
$DL1 : a_1 : 0$
$DL2 : a_2 : 0, a_3 : 1$

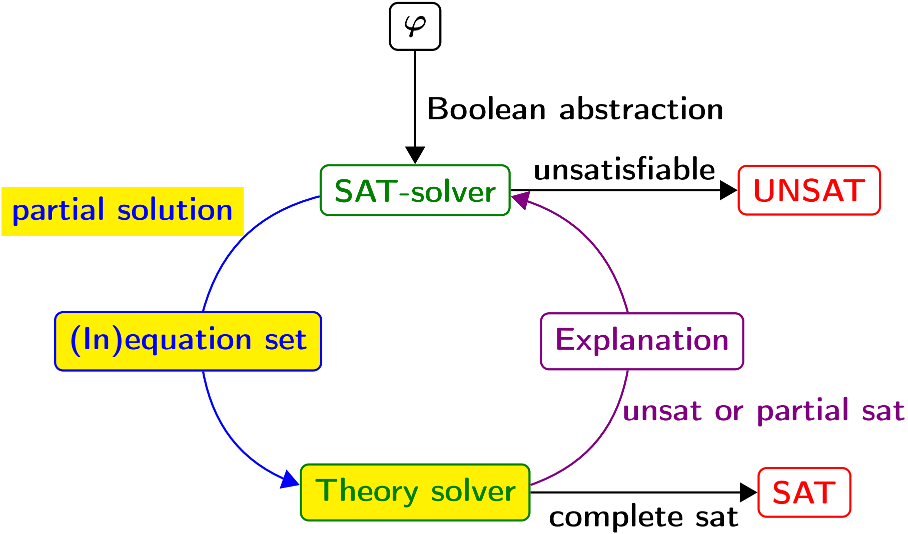
Add clause $(\neg a_3 \vee \neg a_7)$:

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

No conflict resolution needed, since the new clause is already asserting. Backtracking removes $DL1$ and $DL2$ first, then propagation is applied.

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$

Less lazy SMT-solving



Less lazy theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0 \quad DL1 : a_1 : 0, a_2 : 1$

Backtracking: remove a_3, Incrementality: add a_2

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9}$$

Encoding:
remove $s_3 = 0$, add $s_2 = 0$

Less lazy theory solving

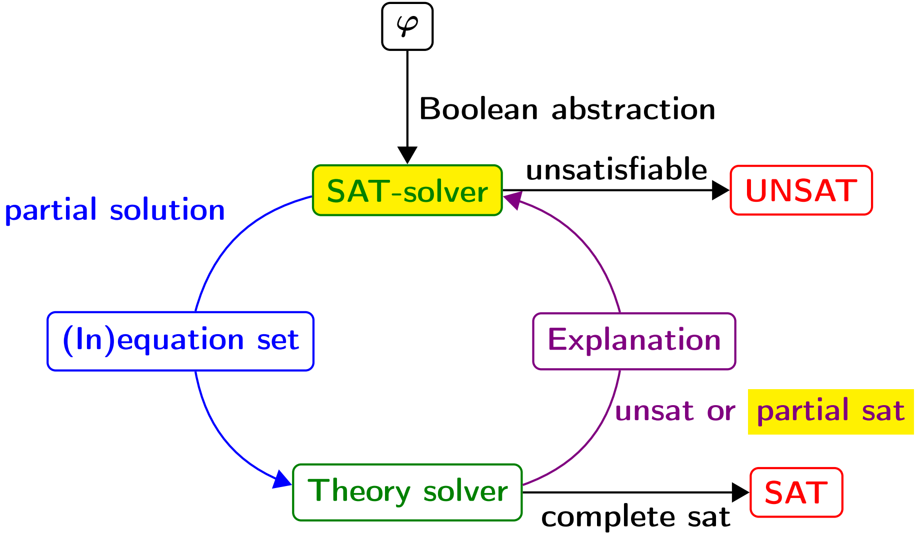
Backtracking: remove bound $s_3 = 0$, add bound $s_2 = 0$

$$
\begin{array}{llllllll}
p_1 = 0 & \to & s_1 = & p_1 & & & s_1 = 0 \\
p_2 = 0 & \to & s_2 = & & p_2 & & s_2 = 0 \\
p_3 = 0 & \to & s_3 = & & & p_3 & s_3 = 0 \\
p_1 + p_2 + p_3 \geq 100 & \to & s_4 = & p_1 + & p_2 + & p_3 & s_4 \geq 100 \\
p_1 \geq 5 & \to & s_5 = & p_1 & & & s_5 \geq 5 \\
p_2 \geq 5 & \to & s_6 = & & p_2 & & s_6 \geq 5 \\
p_3 \geq 10 & \to & s_7 = & & & p_3 & s_7 \geq 10 \\
p_1 + 2p_2 + 5p_3 \leq 180 & \to & s_8 = & p_1 + & 2p_2 + & 5p_3 & s_8 \leq 180 \\
3p_1 + 2p_2 + p_3 \leq 300 & \to & s_9 = & 3p_1 + & 2p_2 + & p_3 & s_9 \leq 300
\end{array}
$$

	$s_4(100)$	$p_2(0)$	$s_7(10)$
$s_1(90)$	1	-1	-1
$s_2(0)$	0	1	0
$s_3(10)$	0	0	1
$p_1(90)$	1	-1	-1
$s_5(90)$	1	-1	-1
$s_6(0)$	0	1	0
$p_3(10)$	0	0	1
$s_8(140)$	1	1	4
$s_9(280)$	3	-1	-2

Return partial SAT.

Less lazy SMT-solving

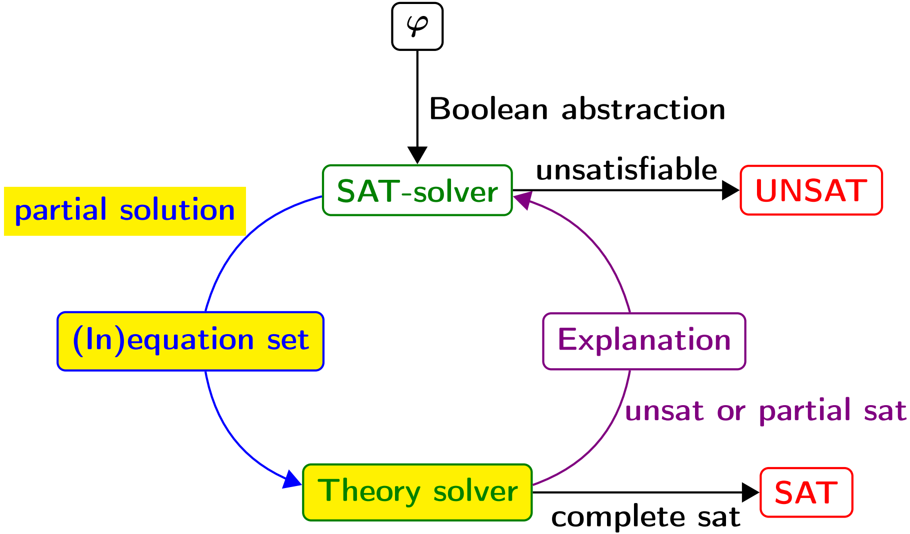


Less lazy SAT-solving

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7)$$

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1 : a_1 : 0, a_2 : 1$
$DL2 : a_5 : 0, a_6 : 1$

Less lazy SMT-solving



Less lazy theory solving

$DL0: a_4: 1, a_7: 1, a_8: 1, a_9: 1, a_3: 0, \quad DL1: a_1: 0, a_2: 1,$
$DL2: a_5: 0, a_6: 1$

Incrementality: add a_6

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7)$$

Encoding:
$s_6 \geq 5$

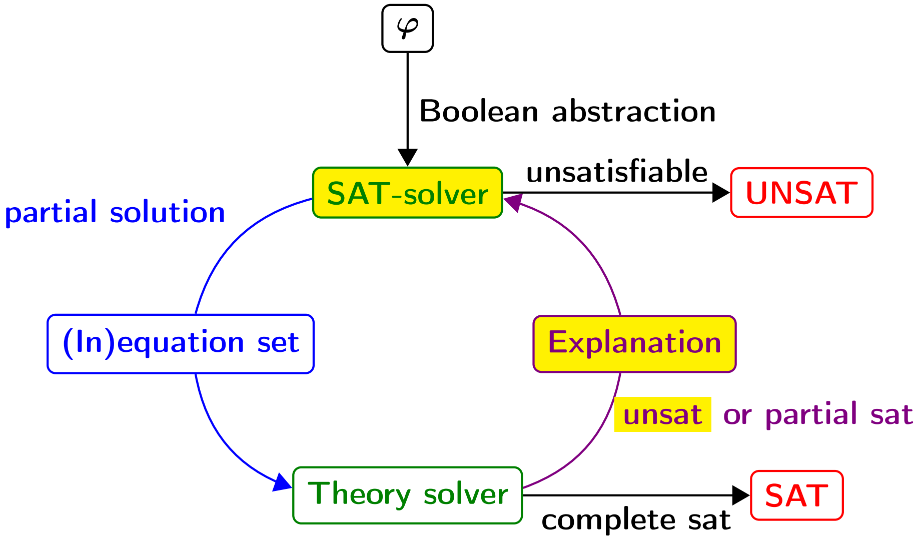
Less lazy theory solving

$$\begin{array}{lcllllll}
p_1 = 0 & \rightarrow & s_1 = & p_1 & & & s_1 = 0 \\
p_2 = 0 & \rightarrow & s_2 = & & p_2 & & s_2 = 0 \\
p_3 = 0 & \rightarrow & s_3 = & & & p_3 & s_3 = 0 \\
p_1 + p_2 + p_3 \geq 100 & \rightarrow & s_4 = & p_1 + & p_2 + & p_3 & s_4 \geq 100 \\
p_1 \geq 5 & \rightarrow & s_5 = & p_1 & & & s_5 \geq 5 \\
p_2 \geq 5 & \rightarrow & s_6 = & & p_2 & & \boxed{s_6 \geq 5} \\
p_3 \geq 10 & \rightarrow & s_7 = & & & p_3 & s_7 \geq 10 \\
p_1 + 2p_2 + 5p_3 \leq 180 & \rightarrow & s_8 = & p_1 + & 2p_2 + & 5p_3 & s_8 \leq 180 \\
3p_1 + 2p_2 + p_3 \leq 300 & \rightarrow & s_9 = & 3p_1 + & 2p_2 + & p_3 & s_9 \leq 300
\end{array}$$

	$s_4(100)$	$p_2(0)$	$s_7(10)$		$s_4(100)$	$s_6(5)$	$s_7(10)$
$s_1(90)$	1	-1	-1	$s_1(85)$	1	-1	-1
$s_2(0)$	0	1	0	$s_2(5)$	0	1	0
$s_3(10)$	0	0	1	$s_3(10)$	0	0	1
$p_1(90)$	1	-1	-1	$p_1(85)$	1	-1	-1
$s_5(90)$	1	-1	-1	$s_5(85)$	1	-1	-1
$s_6(0)$	0	1	0	$p_2(5)$	0	1	0
$p_3(10)$	0	0	1	$p_3(10)$	0	0	1
$s_8(140)$	1	1	4	$s_8(145)$	1	1	4
$s_9(280)$	3	-1	-2	$s_9(275)$	3	-1	-2

Conflict: $\underbrace{p_2 = 0}_{a_2} \wedge \underbrace{p_2 \geq 5}_{a_6}$ is not satisfiable.

Less lazy SMT-solving



Less lazy SAT-solving

Current assignment:

$DL0: a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1: a_1 : 0, a_2 : 1$
$DL2: a_5 : 0, a_6 : 1$

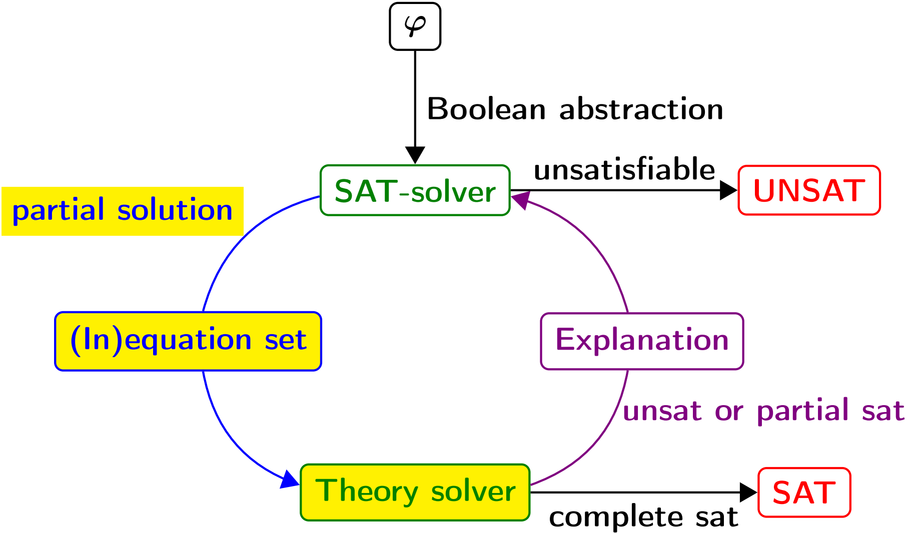
Add clause $(\neg a_2 \vee \neg a_6)$.

$$(a_1 \vee a_2 \vee a_3) \wedge a_4 \wedge (a_5 \vee a_6) \wedge a_7 \wedge a_8 \wedge a_9 \wedge (\neg a_3 \vee \neg a_7) \\ \wedge (\neg a_2 \vee \neg a_6)$$

No conflict resolution needed, since the new clause is already asserting. Backtracking removes $DL2$ first, then propagation is used to imply new assignments (first using the new learnt clause).

$DL0: a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0$
$DL1: a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Less lazy SMT-solving



Less lazy theory solving

$DL0 : a_4 : 1, a_7 : 1, a_8 : 1, a_9 : 1, a_3 : 0, \quad DL1 : a_1 : 0, a_2 : 1, a_6 : 0, a_5 : 1$

Backtracking: remove a_6, Incrementality: add a_5

$$(\underbrace{p_1 = 0}_{a_1} \vee \underbrace{p_2 = 0}_{a_2} \vee \underbrace{p_3 = 0}_{a_3}) \wedge \underbrace{p_1 + p_2 + p_3 \geq 100}_{a_4} \wedge$$
$$(\underbrace{p_1 \geq 5}_{a_5} \vee \underbrace{p_2 \geq 5}_{a_6}) \wedge \underbrace{p_3 \geq 10}_{a_7} \wedge \underbrace{p_1 + 2p_2 + 5p_3 \leq 180}_{a_8} \wedge$$
$$\underbrace{3p_1 + 2p_2 + p_3 \leq 300}_{a_9} \wedge (\neg a_3 \vee \neg a_7) \wedge (\neg a_2 \vee \neg a_6)$$

Encoding: remove $s_6 \geq 5$, add $s_5 \geq 5$

Less lazy theory solving

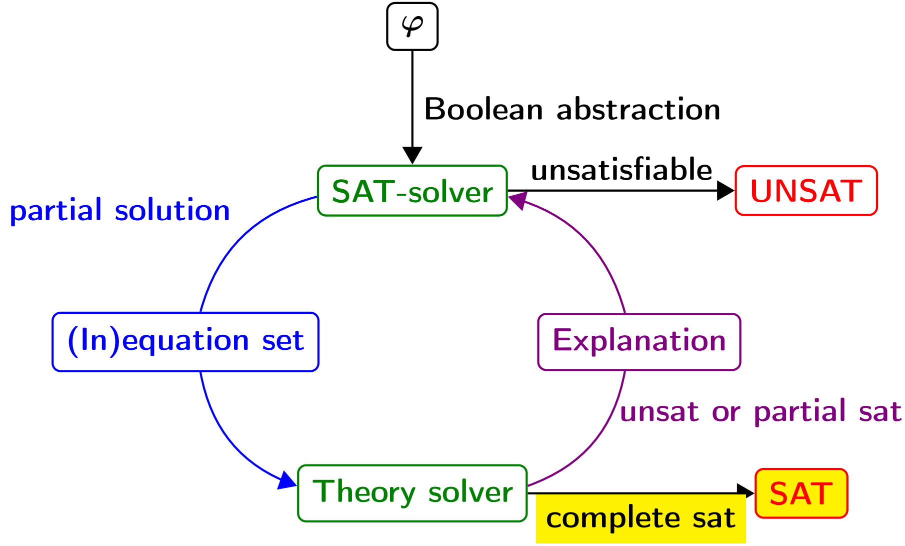
Backtracking: remove $s_6 \geq 5$, Incrementality: add $s_5 \geq 5$

$$
\begin{array}{lcrcrrrl}
p_1 = 0 & \rightarrow & s_1 = & p_1 & & & s_1 = 0 \\
p_2 = 0 & \rightarrow & s_2 = & & p_2 & & s_2 = 0 \\
p_3 = 0 & \rightarrow & s_3 = & & & p_3 & s_3 = 0 \\
p_1 + p_2 + p_3 \geq 100 & \rightarrow & s_4 = & p_1 + & p_2 + & p_3 & s_4 \geq 100 \\
p_1 \geq 5 & \rightarrow & s_5 = & p_1 & & & s_5 \geq 5 \\
p_2 \geq 5 & \rightarrow & s_6 = & & p_2 & & s_6 \geq 5 \\
p_3 \geq 10 & \rightarrow & s_7 = & & & p_3 & s_7 \geq 10 \\
p_1 + 2p_2 + 5p_3 \leq 180 & \rightarrow & s_8 = & p_1 + & 2p_2 + & 5p_3 & s_8 \leq 180 \\
3p_1 + 2p_2 + p_3 \leq 300 & \rightarrow & s_9 = & 3p_1 + & 2p_2 + & p_3 & s_9 \leq 300
\end{array}
$$

	$s_4(100)$	$s_6(5)$	$s_7(10)$		$s_4(100)$	$s_2(0)$	$s_7(10)$
$s_1(85)$	1	-1	-1	$s_1(90)$	1	-1	-1
$s_2(5)$	0	1	0	$s_6(0)$	0	1	0
$s_3(10)$	0	0	1	$s_3(10)$	0	0	1
$p_1(85)$	1	-1	-1	$p_1(90)$	1	-1	-1
$s_5(85)$	1	-1	-1	$s_5(90)$	1	-1	-1
$p_2(5)$	0	1	0	$p_2(0)$	0	1	0
$p_3(10)$	0	0	1	$p_3(10)$	0	0	1
$s_8(145)$	1	1	4	$s_8(140)$	1	1	4
$s_9(275)$	3	-1	-2	$s_9(280)$	3	-1	-2

Since the assignment is complete, return SAT for the original problem.

Less lazy SMT-solving



What could also happen...

- Problem: When working in the less lazy modus, in the Simplex theory solver a bound of a non-basic slack variable s could be activated. If the current value of this non-basic variable now violates its newly activated bound, our invariant (all non-basic variable values are within the corresponding bounds) would not hold!
- Solution: Since the result in the previous solver state was SAT, all the basic variables satisfy their bounds. Thus pivoting with an arbitrary basic variable s' whose row has a non-zero coefficient for s solves the problem. Note: there is always such a row.
- New problem: Now also the bounds of s' could be activated, leading to a similar problem. However, now it can happen that all basic variables are assigned values outside their bounds!
- Solution: After activating a bound, first check satisfiability and activate further bounds afterwards one by one.