Satisfiability Checking

Fourier-Motzkin Variable Elimination

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The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

$$
\begin{aligned}
(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\
(p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\
3p_1 + 2p_2 + p_3 \leq 300
\end{aligned}
$$

Quantifier-free linear real arithmetic (QFLRA)

Linear real arithmetic is the first-order theory with signature $\{0, 1, +, <\}$ and the domain being the reals $\mathbb{R}$.

Syntax of linear real arithmetic

Terms:	t	$::=$	0	$\mid$	1	$\mid$	x	$\mid$	$t + t$
Constraints:	c	$::=$	$t < t$						
Formulas:	φ	$::=$	c	$\mid$	$\neg\varphi$	$\mid$	$\varphi \land \varphi$	$\mid$	$\exists x.\ \varphi$

where x is a variable.

- Syntactic sugar for constraints: $t_1 \leq t_2$, $t_1 = t_2$, $t_1 \neq t_2$.
- The semantics is standard.
- Linear real arithmetic is also called linear real algebra.
- We consider the satisfiability problem for the quantifier-free fragment QFLRA (equivalently, we consider the existential fragment, i.e., no negation of expressions containing quantifiers).

Linear real arithmetic: Eliminating equations

Reminder: In the SMT-solving context, we need a decision procedure for sets of linear real arithmetic constraints (equations and inequations).

- Assume that the ith constraint is an equation containing a variable x_j with a non-zero coefficient $a_{ij} \neq 0$:

$$\sum_{k=1}^{n} a_{ik} \cdot x_k = b_i \quad (a_{i,k}, b_i\text{: } \textit{integer/rational constants, } x_k\text{: } \textit{variables})$$

$$\Rightarrow \quad a_{ij} \cdot x_j \ = \ b_i - \sum_{k \in \{1,\ldots,j-1,j+1,\ldots,n\}} a_{ik} \cdot x_k$$

$$\Rightarrow \quad x_j \ = \ \frac{b_i}{a_{ij}} - \sum_{k \in \{1,\ldots,j-1,j+1,\ldots,n\}} \frac{a_{ik}}{a_{ij}} \cdot x_k := \beta_j$$

- Replace x_j by β_j in all constraints.
- This substitutiton leads to an equisatisfiable problem in $n-1$ variables.

Linear arithmetic over the reals

- Goal: decide satisfiability of
  conjunctions of linear inequalities over the reals

$$\bigwedge_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{ij} x_j \leq b_i$$

- Input in matrix form: $A\overline{x} \leq \overline{b}$

$$\begin{array}{cc} m \text{ constraints} & \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{22} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{pmatrix} \\ & n \text{ variables} \end{array}$$

Fourier-Motzkin variable elimination

- Earliest method for solving linear inequality systems:
  discovered in 1826 by Fourier, re-discovered by Motzkin in 1936

- Basic idea of variable elimination:
  - Pick a variable and eliminate it, yielding an equisatisfiable formula that does not refer to the eliminated variable any more.
  - Continue until all variables are eliminated.

- Fourier-Motzkin: Put requirements on the lower an upper bounds on the variable we want to eliminate.

Variable bounds

- For a variable x_n, we can partition the constraints according to the coefficient a_{in}:
  - $a_{in} > 0$: upper bound β_i on x_n
  - $a_{in} < 0$: lower bound β_i on x_n

$$\sum_{j=1}^{n} a_{ij} \cdot x_j \leq b_i$$

$$\Rightarrow \quad a_{in} \cdot x_n \ \leq \ b_i - \sum_{j=1}^{n-1} a_{ij} \cdot x_j$$

$$(a) \quad \stackrel{a_{in}>0}{\Rightarrow} \quad x_n \ \leq \ \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_l \quad \text{upper bound}$$

$$(b) \quad \stackrel{a_{in}<0}{\Rightarrow} \quad x_n \ \geq \ \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_u \quad \text{lower bound}$$

Example for upper and lower bounds

		Category for x_1?
(1)	$x_1 - x_2 \leq 0$	Upper bound
(2)	$x_1 - x_3 \leq 0$	Upper bound
(3)	$-x_1 + x_2 + 2x_3 \leq 0$	Lower bound
(4)	$-x_3 \leq -1$	No bound

Eliminating unbounded variables

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them).
- The new problem has a solution iff the old problem has one!

$$
\begin{array}{rcl}
\cancel{8x} & \cancel{\geq} & \cancel{7y} \\
\cancel{x} & \cancel{\geq} & \cancel{3} \\
y & \geq & z \\
z & \geq & 10 \\
20 & \geq & z
\end{array}
\longrightarrow
\begin{array}{rcl}
\cancel{y} & \cancel{\geq} & \cancel{z} \\
z & \geq & 10 \\
20 & \geq & z
\end{array}
\longrightarrow
\begin{array}{rcl}
z & \geq & 10 \\
20 & \geq & z
\end{array}
$$

Fourier-Motzkin variable elimination

- For each pair of a lower bound β_l and an upper bound β_u, we have

$$\beta_l \quad \leq \quad x_n \quad \leq \quad \beta_u$$

- For each such pair, add the constraint

$$\beta_l \quad \leq \quad \beta_u$$

Fourier-Motzkin: Example

			Category for x_1?
~~(1)~~	$\cancel{x_1 - x_2 \leq 0}$		Upper bound
~~(2)~~	$\cancel{x_1 - x_3 \leq 0}$		Upper bound
~~(3)~~	$\cancel{-x_1 + x_2 + 2x_3 \leq 0}$		Lower bound
(4)	$-x_3 \leq -1$		Lower bound
			eliminate x_1
(5)	$2x_3 \leq 0$	(from 1,3)	Upper bound
(6)	$x_2 + x_3 \leq 0$	(from 2,3)	Upper bound
			we eliminate x_3
(7)	$1 \leq 0$	(from 4,5)	

$\rightarrow$ Contradiction (the system is UNSAT)

Complexity

- Worst-case complexity:

$$m \rightarrow m^2 \rightarrow (m^2)^2 \rightarrow \ldots \rightarrow m^{2^n}$$

- Heavy!

- The bottleneck: case-splitting