Satisfiability Checking

Summary II

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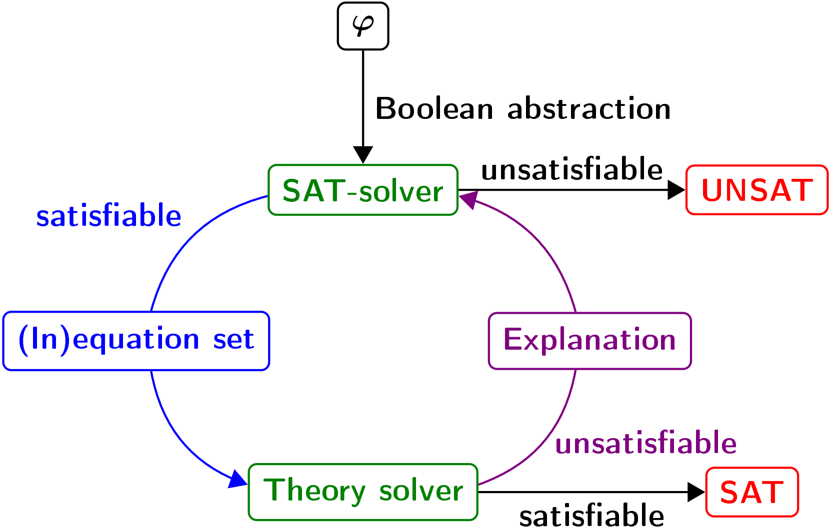
WS 16/17

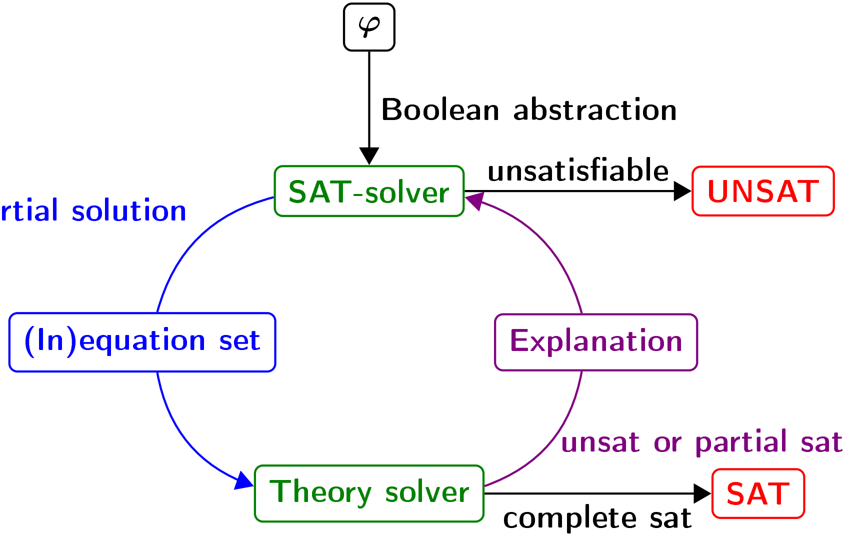
Outline

Outline

Full lazy SMT-solving



Less lazy SMT-solving



Requirements on the theory solver

1. Incrementality: In less lazy solving we incrementally extend a set of constraints, whose satisfiability check should be carried out by the theory solver. The theory solver should make use of the previous satisfiability check for the analysis of the extended set.
2. (Preferably minimal) infeasible subsets: If the constraint set is infeasible then compute a reason for unsatisfaction.
3. Backtracking: The theory solver should be able to remove constraints (in inverse chronological order).

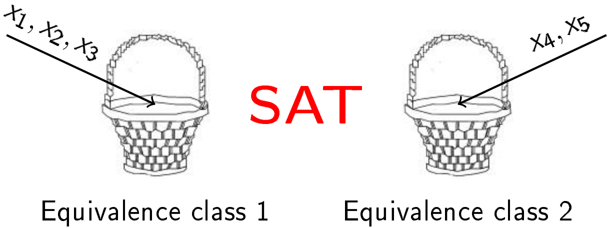
Outline

1. Full and less lazy SMT solving
2. Equality logic with uninterpreted functions
3. Gaussian and Fourier-Motzkin variable elimination
4. The Simplex method
5. Branch and bound
6. The Omega test

$$\varphi^E : \quad x_1 = x_2 \wedge x_2 = x_3 \wedge x_4 = x_5 \wedge x_5 \neq x_1$$

Transitive closure:

For each equality, merge the classes of the respective variables!



Conjunctions of equalities: Transitive closure

$$\varphi^E : \quad x_1 = x_2 \wedge x_2 = x_3 \wedge x_4 = x_5 \wedge x_5 \neq x_1$$

Transitive closure:

For each equality, merge the classes of the respective variables!

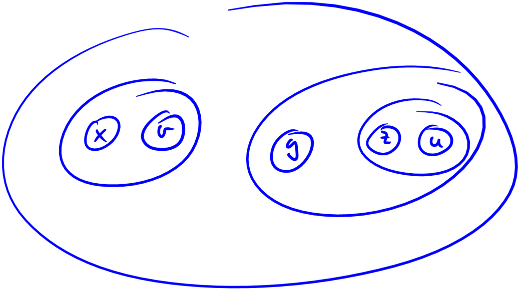
Equivalence class 1 Equivalence class 2

How to achieve incrementality? How to compute infeasible subsets?

$x \neq y$ + extension with "="

$\searrow$

H-check $x \neq y$



$x = v$

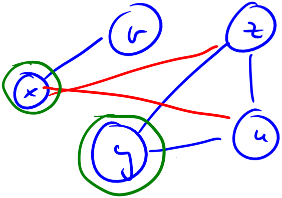
$z = u$

$y = z$

$y = u$

$x = z$

$x = u$



Explanation: e.g. $x = z \wedge z = y$

or $x = u \wedge u = y$

Conjunction of equalities: Algorithm

Input: A conjunction φ of equalities and disequalities without UF

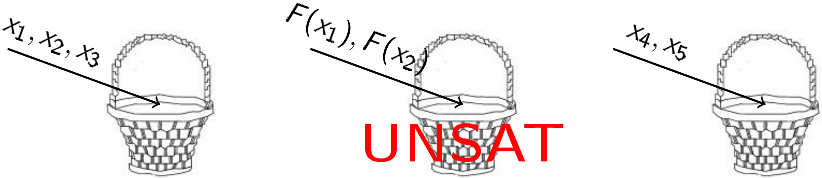
Algorithm

1. Define an equivalence class for each variable in φ.
2. For each equality $x = y$ in φ: merge the equivalence classes of x and y.
3. For each disequality $x \neq y$ in φ:
   if x is in the same class as y, return 'UNSAT'.
4. Return 'SAT'.

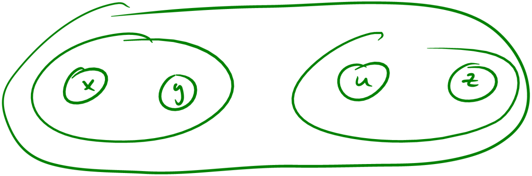
$$\varphi^E: \quad x_1 = x_2 \wedge x_2 = x_3 \wedge x_4 = x_5 \wedge x_5 \neq x_1 \wedge F(x_1) \neq F(x_2)$$

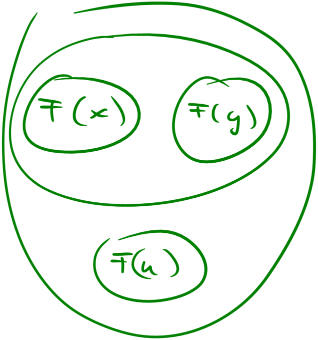
Congruence closure:

If all the arguments of two function applications are in the same class, merge the classes of the applications!



Equivalence class 1 Equivalence class 2 Equivalence class 3





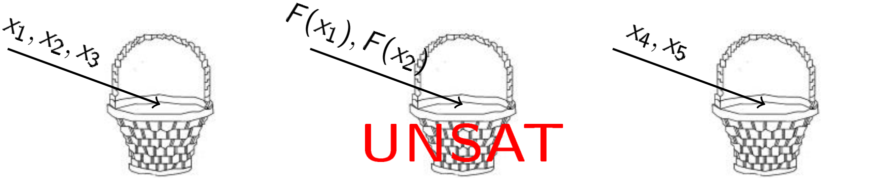
1. $x = y$ $\quad u = z$
2. $x = u$
3. $F(x) \neq F(u) \to$ UNSAT

Uninterpreted functions: Congruence closure

$$\varphi^E : \quad x_1 = x_2 \land x_2 = x_3 \land x_4 = x_5 \land x_5 \neq x_1 \land F(x_1) \neq F(x_2)$$

Congruence closure:

If all the arguments of two function applications are in the same class, merge the classes of the applications!



Equivalence class 1 Equivalence class 2 Equivalence class 3

How to achieve incrementality? How to compute infeasible subsets?

Conjunction of equalities with UF: Algorithm

Input: A conjunction φ of equalities and disequalities with UFs of type $D \to D$

Output: Satisfiability of φ

Algorithm

1. $\mathcal{C} := \{\{t\} \mid t$ occurs as subexpression in an (in)equation in $\varphi\}$;
2. for each equality $t = t'$ in φ with $[t] \neq [t']$

   $\mathcal{C} := (\mathcal{C} \setminus \{[t], [t']\}) \cup \{[t] \cup [t']\}$;

   while exists $F(t), F(t')$ in φ with $[t] = [t']$ and $[F(t)] \neq [F(t')]$

   $\mathcal{C} := (\mathcal{C} \setminus \{[F(t)], [F(t')]\}) \cup \{[F(t)] \cup [F(t')]\}$;
3. for each inequality $t \neq t'$ in φ

   if $[t] = [t']$ return "UNSAT";
4. return "SAT";

($[t] \in \mathcal{C}$ denotes the unique set in $\mathcal{C}$ that contains t)

Outline

$\varphi : \varphi' \wedge$

gaup : $\ldots + ax + \ldots = b$ $\quad a, b \in \mathbb{Q}$

$$x = \frac{b}{a} - \frac{\ldots}{a}$$

$$\exists x . \varphi \longleftrightarrow \varphi[{}^{a}/_{x}]$$

Not directly applicable for the integer domain

E.g. $2x = y \wedge y = 1$

$$x = \frac{1}{2} y \wedge y = 1 \xrightarrow[\text{elim. } x]{} y = 1$$

$$x = \frac{1}{2} y = \frac{1}{2}$$

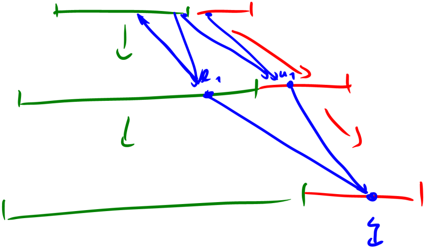
Note: φ is a CONJUNCTION of constraints!

Fourier-Motzkin:

$$\exists x.\ \begin{matrix} l_1 \\ \vdots \\ l_n \end{matrix} \le x \le \begin{matrix} u_1 \\ \vdots \\ u_m \end{matrix} \iff \forall 1 \le i \le n\ \forall 1 \le j \le m.\ l_i \le u_j$$

Incrementality:

$l \le x \to$ add $\forall 1 \le j \le m.\ l \le u_j$



$l_1 \le x \wedge x \le u_1$

$[l_1, u_1] = \emptyset$

Gaussian and Fourier-Motzkin variable elimination

- Goal: decide satisfiability of conjunctions of linear constraints over the reals (n variables, k inequations, l equations)

$$\bigwedge_{1\leq i\leq k} \sum_{1\leq j\leq n} a_{ij}x_j \leq b_i \wedge \bigwedge_{1\leq i\leq l} \sum_{1\leq j\leq n} c_{ij}x_j = d_i$$

- Eliminate variable x_n:

  Gauss: If there exists an equation $\sum_{1\leq j\leq n} c_{ij}x_j = d_i$ with $c_{in} \neq 0$ then remove this equation and replace x_n by $\frac{d_i}{c_{in}} - \sum_{j=1}^{n-1} \frac{c_{ij}}{c_{in}}x_j$ in all remaining constraints.

  Fourier-Motzkin: Otherwise partition the inequations according to the coefficients a_{in}:

  - $a_{in} = 0$: no bound on x_n
  - $a_{in} > 0$: upper bound $\beta_i = \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}}x_j$ on x_n
  - $a_{in} < 0$: lower bound $\beta_i = \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}}x_j$ on x_n

  Remove all inequalitites defining a bound on x_n and add for each pair of a lower bound β_l and upper bound β_u the constraint $\beta_l \leq \beta_u$.

Gaussian and Fourier-Motzkin variable elimination

How to achieve incrementality?
How to compute infeasible subsets?

Outline

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2. Equality logic with uninterpreted functions
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4. The Simplex method
5. Branch and bound
6. The Omega test

- Problem: Check the satisfiability of a linear inequation system

$$\bigwedge_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j \bowtie_i b_i \qquad \bowtie_i \in \{=, \leq, \geq\}$$

in the real domain

- General form:

Inv. 1

$$A\vec{x} = \vec{s} \quad \wedge \quad \bigwedge_{i=1}^{m} s_i \bowtie_i b_i$$

$s_1, \ldots, s_m$: additional/auxiliary/slack variables

- Tableau:

Inv. 2: these bounds are satisfied

bounds ok?

$$\begin{array}{c|ccc}
 & x_1 & x_j & x_n \\
\hline
s_1 & a_{11} & \ldots & a_{1n} \\
s_i & & \ldots & \\
s_m & a_{m1} & \ldots & a_{mn}
\end{array}$$

Basic variables $\mathcal{B}$ $\rightarrow$

$\leftarrow$ Non-basic variables $\mathcal{N}$

Data structures

- Simplex maintains:
  - The tableau,
  - an assignment α to all (original and additional) variables.

  Initially, $\alpha(x_i) = 0$ for $i \in \{1, ..., n + m\}$
- Two invariants are maintained throughout:
  - $A\vec{x} = \vec{s}$
  - All non-basic variables satisfy their bounds

1. If the bounds of all basic variables are satisfied by α, return "satisfiable".
2. Otherwise, find a basic variable x_i that violates its bounds. Suppose that $\alpha(x_i) < l_i$.
3. Find a suitable non-basic variable x_j such that
   - $a_{ij} > 0$ and $\alpha(x_j) < u_j$, or
   - $a_{ij} < 0$ and $\alpha(x_j) > l_j$.
4. If there is no suitable variable, return "unsatisfiable".
5. Otherwise, pivot.

Pivoting

$$
\begin{array}{c}
\begin{array}{cc}
 & \begin{array}{ccccc} y_1' & \cdots & y_j' & \cdots & y_n' \end{array} \\
\begin{array}{c} y_1 \\ \cdots \\ y_i \\ \cdots \\ y_m \end{array} &
\begin{pmatrix}
a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\
 & & \cdots & & \\
a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\
 & & \cdots & & \\
a_{m1} & \cdots & a_{mj} & \cdots & a_{mn}
\end{pmatrix}
\end{array} \\
\Downarrow \\
\begin{array}{cc}
 & \begin{array}{ccccc} y_1' & \cdots & y_i & \cdots & y_n' \end{array} \\
\begin{array}{c} y_1 \\ \cdots \\ y_j' \\ \cdots \\ y_m \end{array} &
\begin{pmatrix}
a_{11} - \frac{a_{1j} a_{i1}}{a_{ij}} & \cdots & \frac{a_{1j}}{a_{ij}} & \cdots & a_{1n} - \frac{a_{1j} a_{in}}{a_{ij}} \\
 & & \cdots & & \\
-\frac{a_{i1}}{a_{ij}} & \cdots & \frac{1}{a_{ij}} & \cdots & -\frac{a_{in}}{a_{ij}} \\
 & & \cdots & & \\
a_{m1} - \frac{a_{mj} a_{i1}}{a_{ij}} & \cdots & \frac{a_{mj}}{a_{ij}} & \cdots & a_{mn} - \frac{a_{mj} a_{in}}{a_{ij}}
\end{pmatrix}
\end{array}
\end{array}
$$

1. $\alpha(y_i) = l_i$ 2. $\alpha(y_k')$ unchanged for $k \neq j$ 3. rest according to the new matrix

Termination

To achieve completeness, we use Bland's rule:

1. Determine a total order on the variables
2. Choose the first basic variable that violates its bounds, and the first non-basic suitable variable for pivoting.

General simplex with Bland's rule

1. Transform the system into the general form

$$A\vec{x} = \vec{s} \quad \text{and} \quad \bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i \ .$$

2. Construct the tableau with the initial assignment.
3. Determine a fixed order on the variables.
4. If there is no basic variable that violates its bounds, return "satisfiable". Otherwise, let y_i be the first basic variable in the order that violates its bounds.
5. Search for the first suitable non-basic variable y'_j in the order for pivoting with y_i. If there is no such variable, return "unsatisfiable".
6. Perform the pivot operation on y_i and y'_j.
7. Go to step 4.

Requirements on the theory solver

- (Minimal) infeasible subsets (to explain infeasibility)
- Incrementality (to add constraints stepwise)
- Backtracking (to mimic backtracking in the SAT solver)

Minimal infeasible subsets in Simplex:

- The constraints corresponding to the basic variable of the contradictory row and all non-basic variables with non-zero coefficients in this row are together unsatisfiable.

Incrementality in Simplex:

- Add all constraints but without bounds on non-active constraints.
- If a constraint becomes true, activate its bound.

Backtracking in Simplex:

- Remove bounds of unassigned constraints

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Integer linear systems

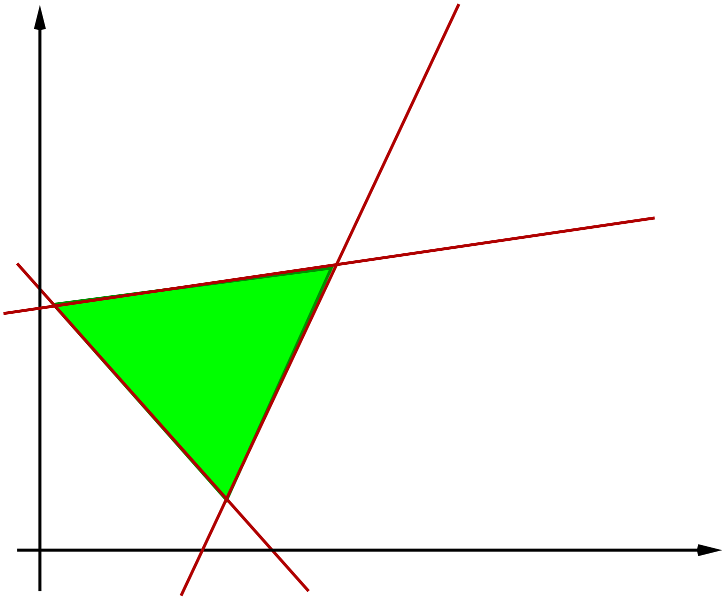
Definition

An integer linear system S is a linear system $Ax = 0$, $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$, with the additional integrality requirement that all variables are of type integer.

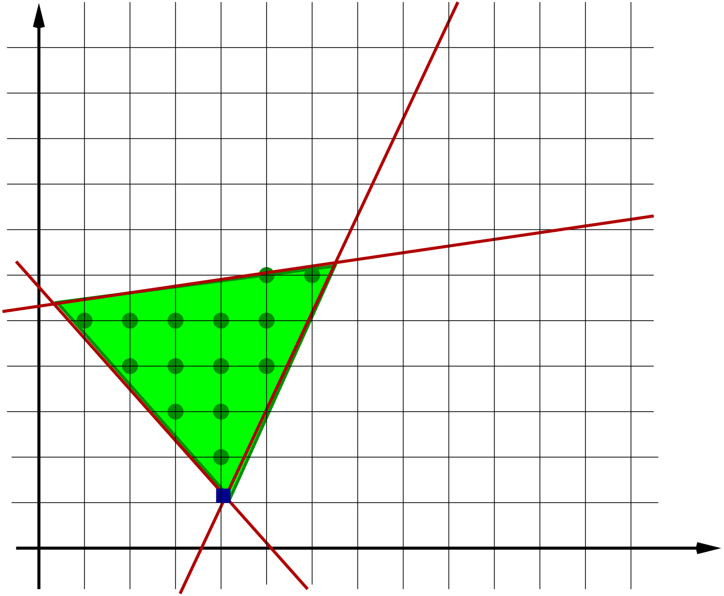
Definition (relaxed system)

Given an integer linear system S, its relaxation relaxed(S) is S without the integrality requirement.

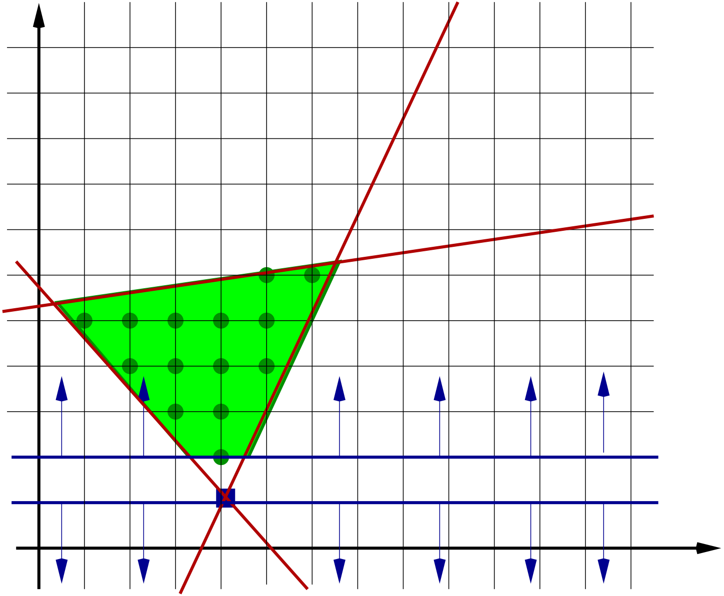
Geometric interpretation



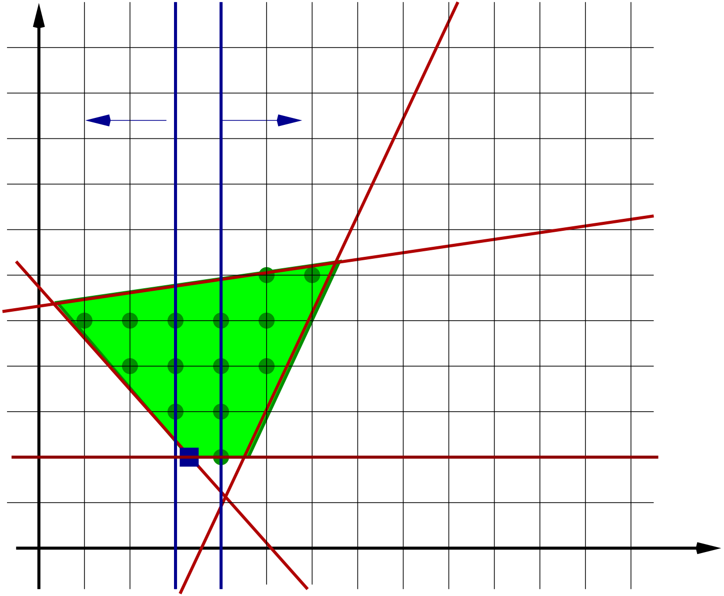
Geometric interpretation



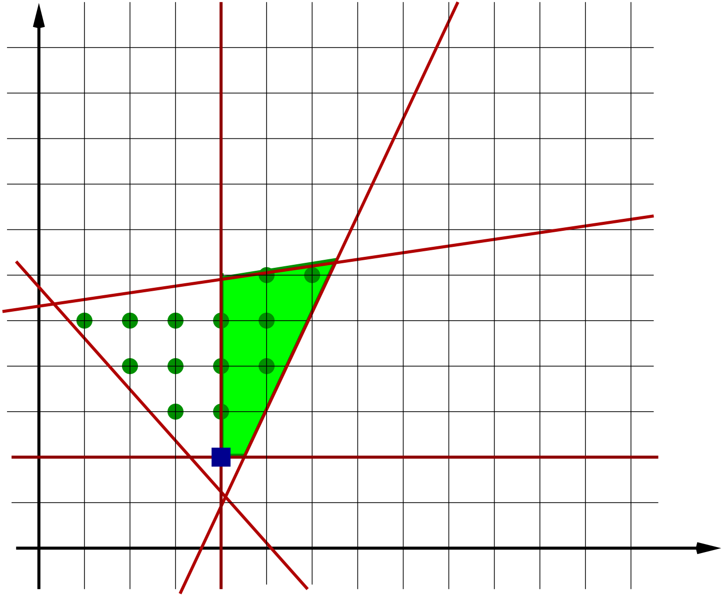
Geometric interpretation



Geometric interpretation



Geometric interpretation



Branch and bound algorithm

Input: An integer linear system S
Output: SAT if S is satisfiable, UNSAT otherwise

```
procedure Branch-and-Bound(S) {
  res = LP(relaxed(S));
  if (res==UNSAT) return UNSAT;
  else if (res is integral) return SAT;
  else {
    Select a variable v that is assigned a non-integral value r;
    if (Branch-and-Bound(S ∪ (v ≤ ⌊r⌋))==SAT) return SAT;
    else if (Branch-and-Bound(S ∪ (v ≥ ⌈r⌉))==SAT) return SAT;
    else return UNSAT;
  }
}
```

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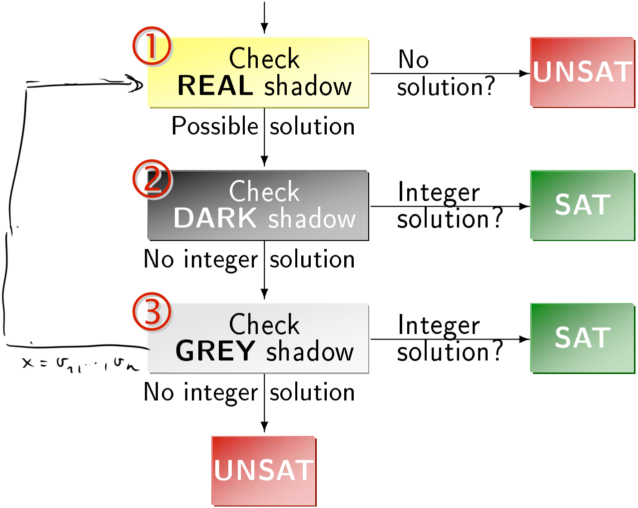
The Omega test

- Goal: Decide satisfiability for conjunctions of linear constraints of the form

  $$\sum_{0 \leq i \leq n} a_i x_i \geq b$$

  over integers.
- Extension of *Fourier-Motzkin* variable elimination:
  - Pick one variable and eliminate it.
  - Continue until all variables but one are eliminated.

Overview of the Omega test



The real shadow

$ax = t$

$\cancel{x = \frac{t}{a}}$

1 Check **REAL** shadow

- Assume we eliminate variable z
- For each pair of upper and lower bound:

$$\beta \leq \; bz \qquad cz \leq \; \gamma \qquad (b, c > 0)$$

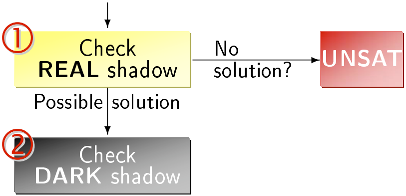
- make the factors of z equal without introducing non-integer terms:

$$c\beta \leq cbz \qquad cbz \leq b\gamma$$

- and add the following constraint to the real shadow:

$$c\beta \; \leq \; b\gamma$$

The dark shadow



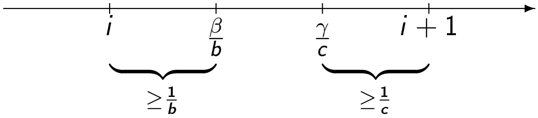
For the contraints

$$\beta \leq bz \qquad cz \leq \gamma$$

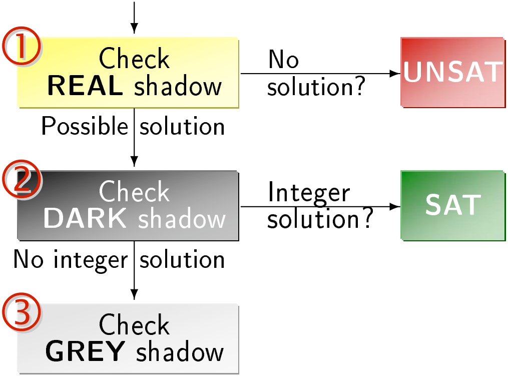
the following condition is sufficient to assure the existence of an integer value between β/b and γ/c:

$$b\gamma - c\beta \geq (c-1)(b-1)$$

This sufficient condition is derived assuming that there is no integer solution for z between β and γ and thus



The grey shadow



- No integer solution in the DARK shadow does not guarantee that there is no integer solution for the original problem.
- Thus, we check the GREY shadow next.

The grey shadow

If the real shadow R has integer solutions,
but the dark shadow D does not, search $R \setminus D$.

$$
\begin{array}{rl}
\text{In R:} & b\gamma \geq cbz \geq c\beta \\
\textbf{Not in D:} & cb - c - b \geq b\gamma - c\beta \\
\leftrightarrow & cb - c - b + c\beta \geq b\gamma \\
\Rightarrow & cb - c - b + c\beta \geq cbz \geq c\beta \qquad \mid : c \\
 & (cb - c - b)/c + \beta \geq bz \geq \beta
\end{array}
$$

- Try all values of z such that

$$(cb - c - b)/c + \beta \geq bz \geq \beta$$

- Optimization: find the largest coefficient c in any upper bound and try the following for each lower bound $bz \geq \beta$:

$$bz = \beta + i \quad \text{for } (cb - c - b)/c \geq i \geq 0$$

- As before, combine this with the original problem, and solve recursively.