Satisfiability Checking

Fourier-Motzkin Variable Elimination

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Informatik 2
LuFG Theory of Hybrid Systems

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The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

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$$\begin{aligned}(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\ (p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\ 3p_1 + 2p_2 + p_3 \leq 300\end{aligned}$$

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Syntax of linear real arithmetic

Terms:	t	$::=$	0	$\mid$	1	$\mid$	x	$\mid$	$t + t$
Constraints:	c	$::=$	$t < t$						
Formulas:	φ	$::=$	c	$\mid$	$\neg\varphi$	$\mid$	$\varphi \land \varphi$	$\mid$	$\exists x.\ \varphi$

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- The semantics is standard.
- Linear real arithmetic is also called linear real algebra.
- We consider the satisfiability problem for the quantifier-free fragment QFLRA (equivalently, we consider the existential fragment, i.e., no negation of expressions containing quantifiers).

Linear real arithmetic: Eliminating equations

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- Assume that the ith constraint is an equation containing a variable x_j with a non-zero coefficient $a_{ij} \neq 0$:

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- This substitutiton leads to an equisatisfiable problem in $n-1$ variables.

Linear arithmetic over the reals

- Goal: decide satisfiability of
  conjunctions of linear inequalities over the reals

$$\bigwedge_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{ij} x_j \leq b_i$$

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- Input in matrix form: $A\overline{x} \leq \overline{b}$

$$m \text{ constraints} \quad \underset{n \text{ variables}}{\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{22} & \cdots & \cdots & a_{mn} \end{pmatrix}} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}$$

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  - Continue until all variables are eliminated.
- Fourier-Motzkin: Put requirements on the lower an upper bounds on the variable we want to eliminate.

Variable bounds

- For a variable x_n, we can partition the constraints according to the coefficient a_{in}:
  - $a_{in} > 0$: upper bound β_i on x_n
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$$(a) \quad \overset{a_{in}>0}{\Rightarrow} \quad x_n \;\leq\; \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_l \quad \text{upper bound}$$

$$(b) \quad \overset{a_{in}<0}{\Rightarrow} \quad x_n \;\geq\; \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \quad =: \beta_u \quad \text{lower bound}$$

Example for upper and lower bounds

Category for x_1?

$$
\begin{array}{ll}
(1) & x_1 - x_2 \leq 0 \\
(2) & x_1 - x_3 \leq 0 \\
(3) & -x_1 + x_2 + 2x_3 \leq 0 \\
(4) & -x_3 \leq -1
\end{array}
$$

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Eliminating unbounded variables

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them).
- The new problem has a solution iff the old problem has one!

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\begin{array}{rcl}
8x & \geq & 7y \\
x & \geq & 3 \\
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- For each such pair, add the constraint

$$\beta_l \quad \leq \quad \beta_u$$

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(6)	$x_2 + x_3 \leq 0$	(from 2,3)	

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eliminate x_1

Fourier-Motzkin: Example

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eliminate x_1

(5) $\quad 2x_3 \leq 0$ (from 1,3)

(6) $\quad x_2 + x_3 \leq 0$ (from 2,3)

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we eliminate x_3

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(4)	$-x_3 \leq -1$		Lower bound
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(5)	$2x_3 \leq 0$	(from 1,3)	Upper bound
(6)	$x_2 + x_3 \leq 0$	(from 2,3)	Upper bound
			we eliminate x_3
(7)	$1 \leq 0$	(from 4,5)	

$\rightarrow$ Contradiction (the system is UNSAT)

Complexity

- Worst-case complexity:

$$m \rightarrow m^2$$

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$$m \rightarrow m^2 \rightarrow (m^2)^2$$

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$$m \to m^2 \to (m^2)^2 \to \ldots \to m^{2^n}$$

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- The bottleneck: case-splitting