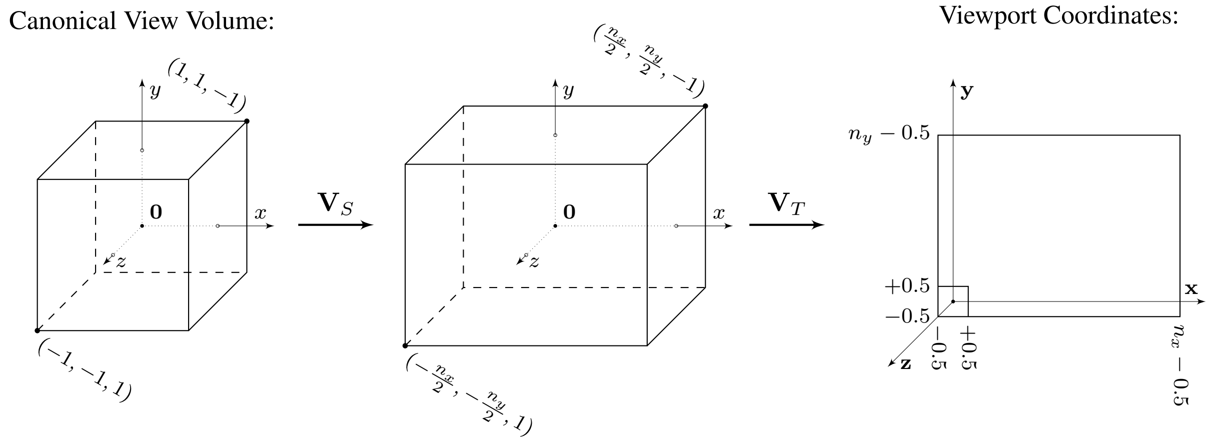
LECTURE NOTES
CG RECAP – PROJECTION

1 Viewport Transform



- n_x: horizontal number of pixels
  n_y: vertical number of pixels
- Split into two steps
- Scale view volume to window dimensions:

$$\mathbf{V}_S = \begin{bmatrix} \frac{n_x}{2} & 0 & 0 & 0 \\ 0 & \frac{n_y}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Translate the scaled view volume to pixel coordinates:

$$\mathbf{V}_T = \begin{bmatrix} 1 & 0 & 0 & \frac{n_x-1}{2} \\ 0 & 1 & 0 & \frac{n_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

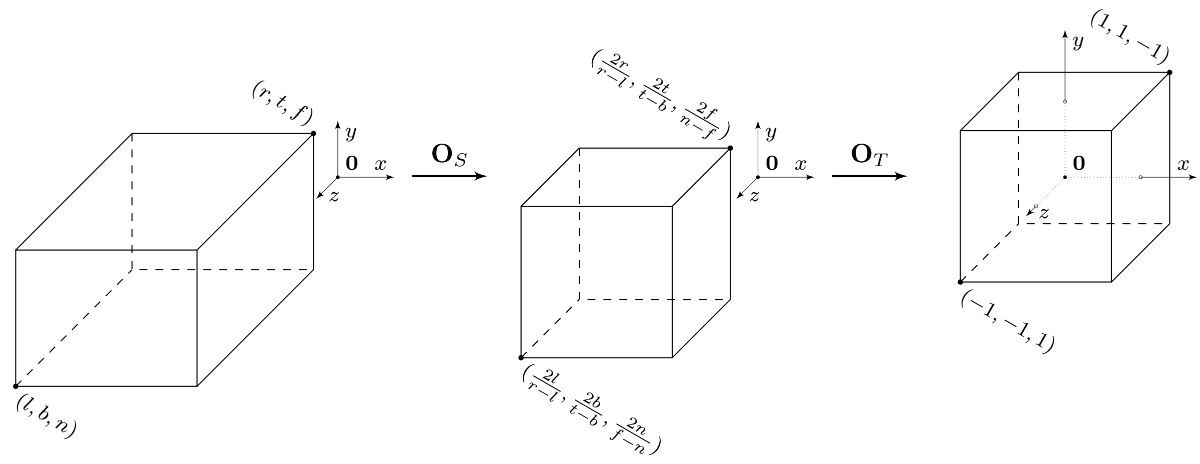
  Note: the coordinates of a pixel refer to its center.

- Concatenate both matrices:

$$\mathbf{V} = \mathbf{V}_T \cdot \mathbf{V}_S = \begin{bmatrix} \frac{n_x}{2} & 0 & 0 & \frac{n_x-1}{2} \\ 0 & \frac{n_y}{2} & 0 & \frac{n_y-1}{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2 Orthographic Projection

Orthographic View Volume: Canonical View Volume:



- l: x-coordinate of left clipping plane , r: x-coordinate of right clipping plane
  b: y-coordinate of bottom clipping plane , t: y-coordinate of top clipping plane
  n: z-coordinate of near clipping plane , f: z-coordinate of far clipping plane
- Split into two steps
- Scale view volume:

$$\mathbf{O}_S = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Find center of scaled view volume:

$$\left(\frac{1}{2} \cdot \frac{2r+2l}{r-l}, \frac{1}{2} \cdot \frac{2t+2b}{t-b}, \frac{1}{2} \cdot \frac{2n+2f}{n-f}\right) = \left(\frac{r+l}{r-l}, \frac{t+b}{t-b}, \frac{n+f}{n-f}\right)$$

- Translate the scaled view volume:

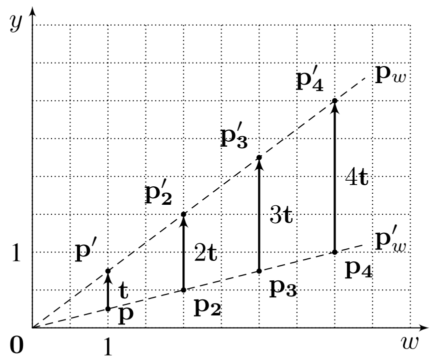
$$\mathbf{O}_T = \begin{bmatrix} 1 & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & 1 & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & 1 & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Concatenate both matrices:

$$\mathbf{O} = \mathbf{O}_T \cdot \mathbf{O}_S = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3 Homogeneous Coordinates Revisited

- $\begin{bmatrix} p_x & p_y & p_z & 1 \end{bmatrix}^\top$ is the position vector representing the point (p_x, p_y, p_z).
- $\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \\ p_z + t_z \\ 1 \end{bmatrix}$
- What about $\mathbf{p_w} = \begin{bmatrix} w \cdot p_x \\ w \cdot p_y \\ w \cdot p_z \\ w \end{bmatrix}^\top$?
- $\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} w \cdot p_x \\ w \cdot p_y \\ w \cdot p_z \\ w \end{bmatrix} = \begin{bmatrix} w \cdot p_x + w \cdot t_x \\ w \cdot p_y + w \cdot t_y \\ w \cdot p_z + w \cdot t_z \\ w \end{bmatrix}$

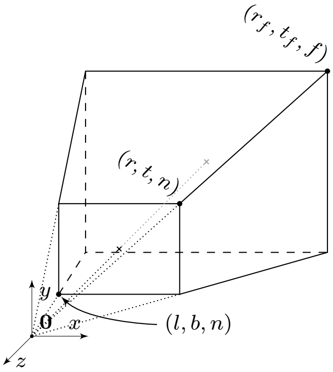


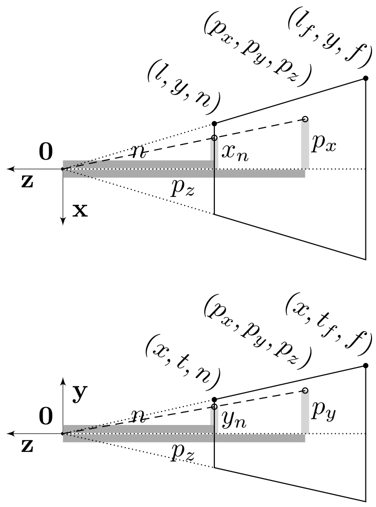
- The point $\mathbf{p} = (p_x, p_y, p_z)$ in Cartesian coordinates is equal to the line $\mathbf{p}_w = (w \cdot p_x, w \cdot p_y, w \cdot p_z, w)$ in homogeneous coordinates.
- Any point $\mathbf{p}_w = (p_x, p_y, p_z, w)$ in homogeneous coordinates is eaqual to the point $\mathbf{p} = (\frac{p_x}{w}, \frac{p_y}{w}, \frac{p_z}{w})$ in Cartesian coordinates.
- For convenience:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ w \end{bmatrix} \equiv \begin{bmatrix} \frac{p_z}{w} \\ \frac{p_z}{w} \\ \frac{p_z}{w} \\ 1 \end{bmatrix}$$

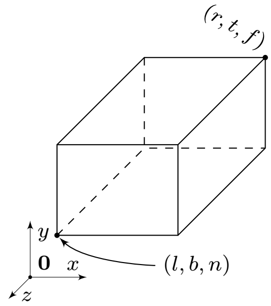
4 Perspective Projection

Perspective View Frustum:





Orthographic View Volume:



- From 2nd intercept theorem:

$$\frac{x_n}{p_x} = \frac{n}{p_z} \iff x_n = \frac{n}{p_z} \cdot p_x \qquad , \qquad \frac{y_n}{p_y} = \frac{n}{p_z} \iff y_n = \frac{n}{p_z} \cdot p_y$$

- Consequently:

$$\mathbf{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{n}{p_z} \cdot p_x \\ \frac{n}{p_z} \cdot p_y \\ p'_z \\ 1 \end{bmatrix} \equiv \begin{bmatrix} n \cdot p_x \\ n \cdot p_y \\ p'_z \cdot p_z \\ p_z \end{bmatrix} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

- How to find the c's? We want to map $p_z = n$ to $p'_z = n$ and $p_z = f$ to $p'_z = f$.

$$\begin{bmatrix} c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \end{bmatrix} \cdot \begin{bmatrix} x_n \\ y_n \\ n \\ 1 \end{bmatrix} = n^2 \quad \wedge \quad \begin{bmatrix} c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \end{bmatrix} \cdot \begin{bmatrix} x_f \\ y_f \\ f \\ 1 \end{bmatrix} = f^2$$

$$\begin{aligned}
&\iff c_{3,1} = c_{3,2} = 0 \quad \wedge \quad c_{3,3} \cdot n + c_{3,4} = n^2 \quad \wedge \quad c_{3,3} \cdot f + c_{3,4} = f^2 \\
&\implies c_{3,4} = n^2 - c_{3,3} \cdot n \quad \wedge \quad c_{3,3} \cdot f + n^2 - c_{3,3} \cdot n = f^2 \\
&\iff c_{3,4} = n^2 - c_{3,3} \cdot n \quad \wedge \quad c_{3,3} \cdot (f - n) = f^2 - n^2 \\
&\iff c_{3,4} = n^2 - c_{3,3} \cdot n \quad \wedge \quad c_{3,3} = f + n \\
&\iff c_{3,4} = n^2 - (f + n) \cdot n \quad \wedge \quad c_{3,3} = f + n \\
&\iff c_{3,4} = -nf \quad \wedge \quad c_{3,3} = f + n
\end{aligned}$$

- Finally, for the frustum to cuboid transform

$$\mathbf{F} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -nf \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

- Concatenating with the orthographic projection:

$$\mathbf{P} = \mathbf{O} \cdot \mathbf{F} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & \frac{2}{n-f} & -\frac{n+f}{n-f} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -nf \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{2n}{r-l} & 0 & -\frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & -\frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{n+f}{n-f} & -\frac{2nf}{n-f} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$