

Satisfiability Checking

SAT Solving

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Informatik 2
LuFG Theory of Hybrid Systems

WS 16/17

Given:

- Propositional logic formula φ in CNF.

Question:

- Is φ satisfiable?
(Is there a model for φ ?)

- Decision (enumeration)
- Boolean constraint propagation (BCP)
- Conflict resolution and backtracking

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Enumeration algorithm

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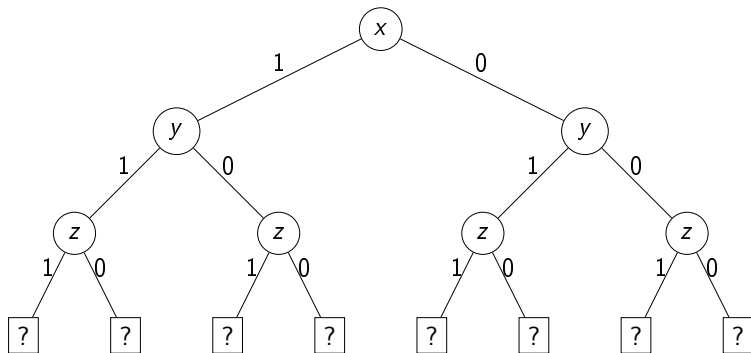
```
bool Enumeration(CNF_Formula  $\varphi$ ){
    trail.clear(); //trail is a stack
    while (true) {
        if there are unassigned variables then {
            choose unassigned variable  $x$ 
            choose value  $v \in \{0, 1\}$ 
            trail.push( $x, v, false$ )
        } else {
            if all clauses of  $\varphi$  are satisfied then return SAT
            while (true){
                if (!trail.empty()) then ( $x, v, b$ )=trail.pop()
                else return UNSAT;
                if (!b) {
                    trail.push( $x, \neg v, true$ )
                    break
                }
            }
        }
    }
}
```

Example CNF: Decision heuristics

$$\underbrace{(\neg x \vee y \vee z)}_{c_1} \wedge \underbrace{(y \vee \neg z)}_{c_2} \wedge \underbrace{(\neg x \vee \neg y)}_{c_3}$$

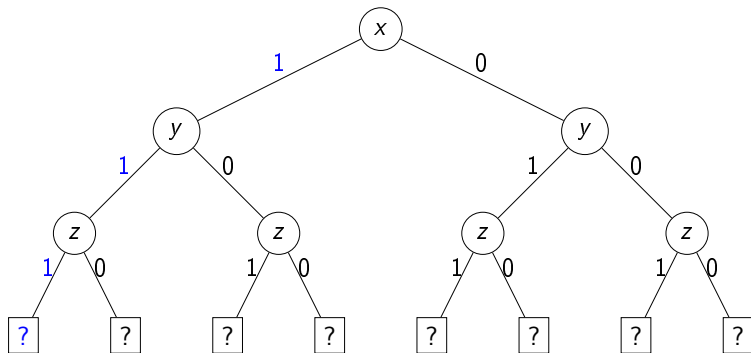
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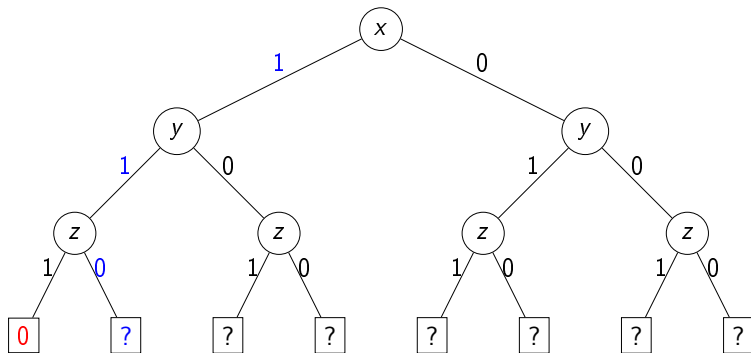
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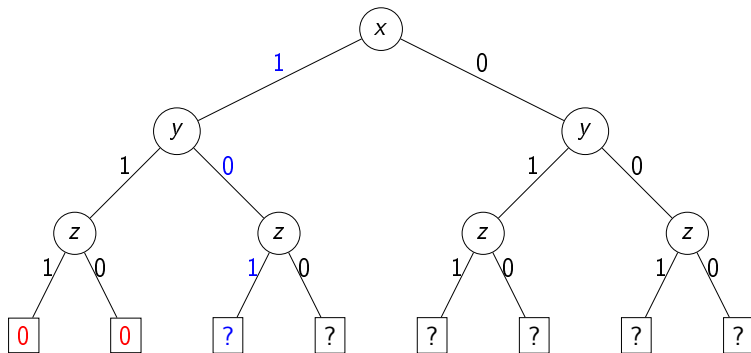
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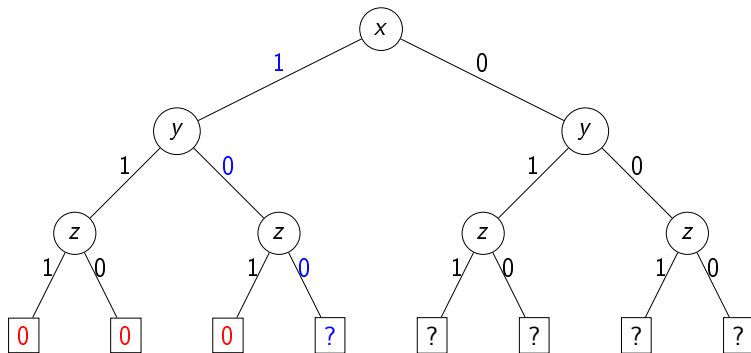
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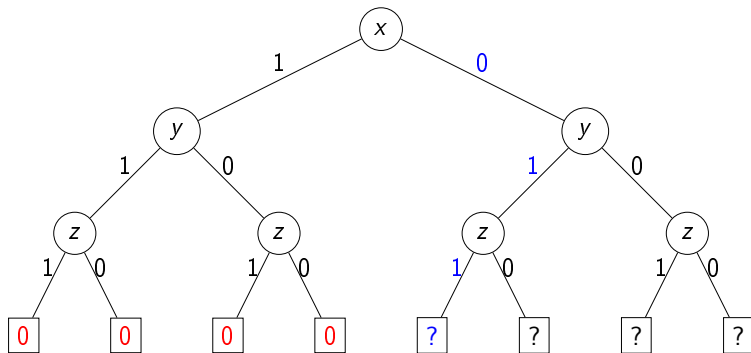
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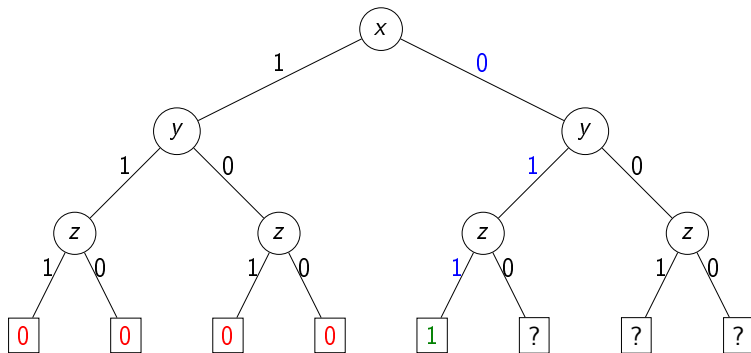
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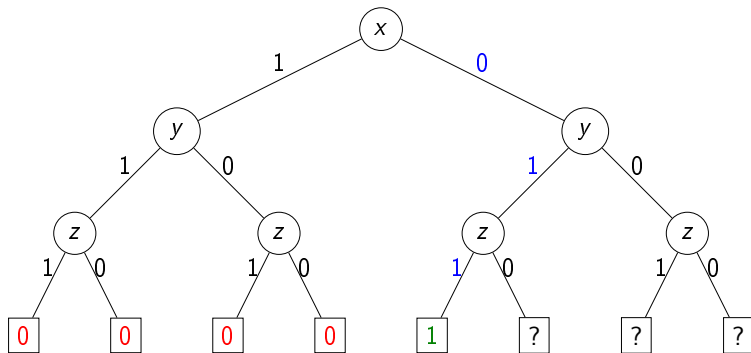
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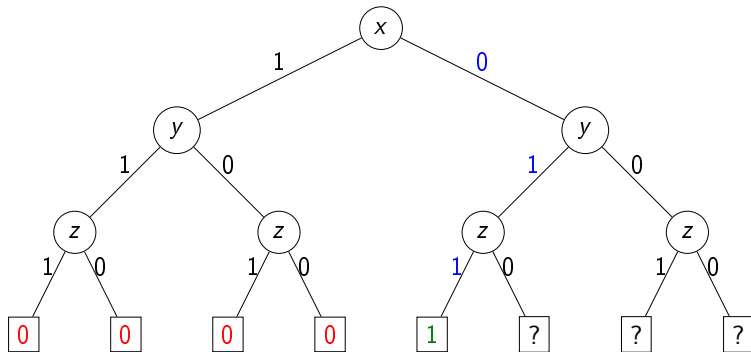
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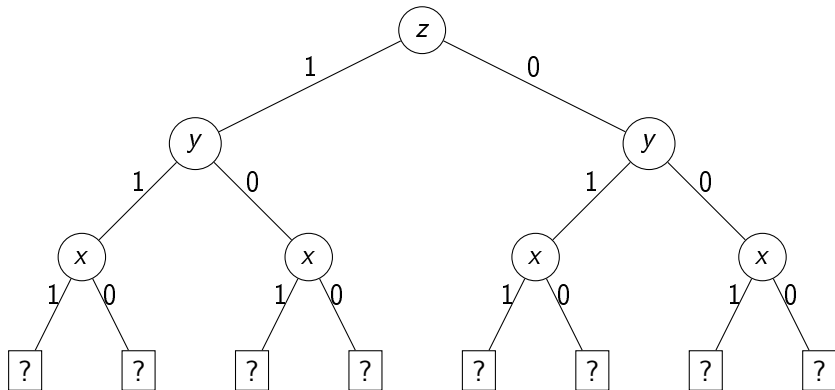
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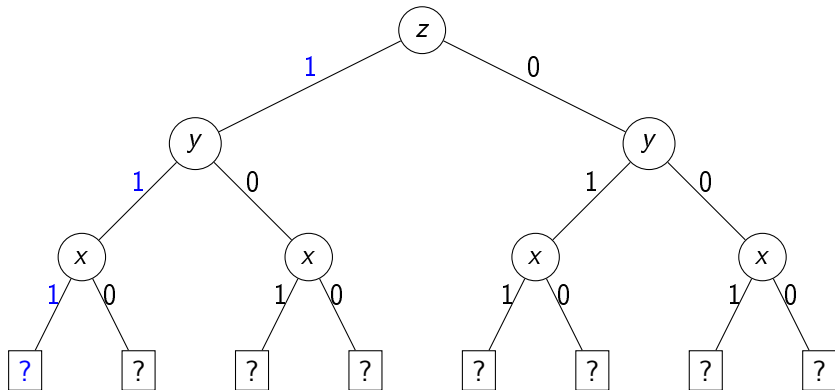
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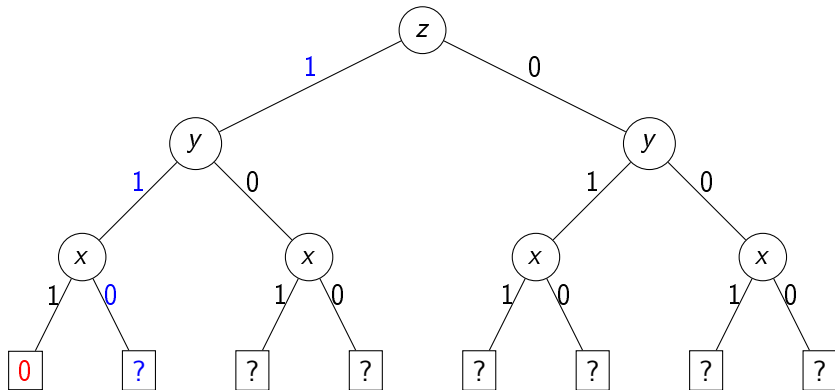
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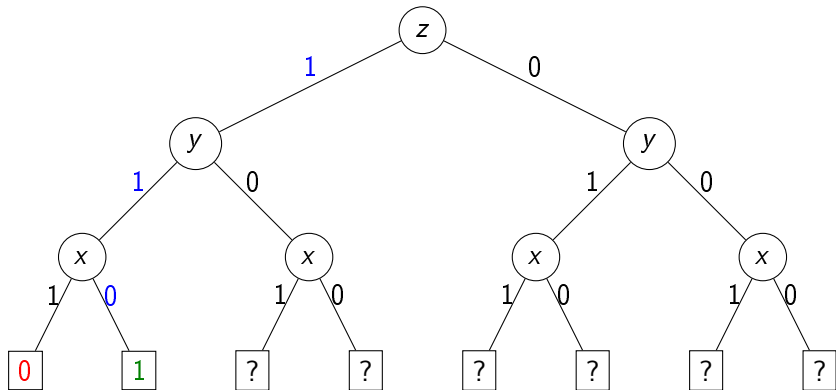
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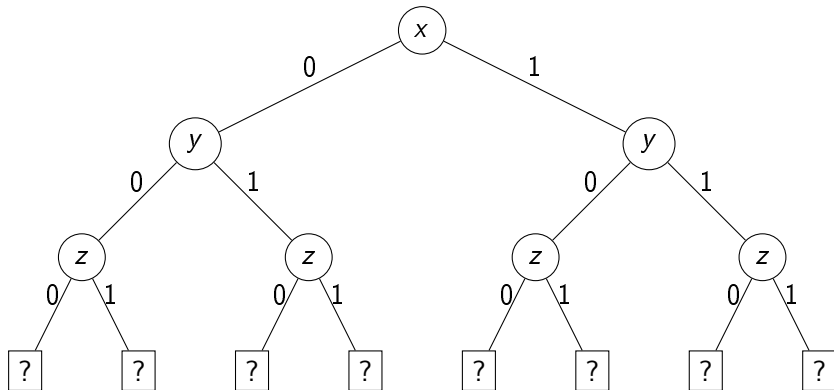
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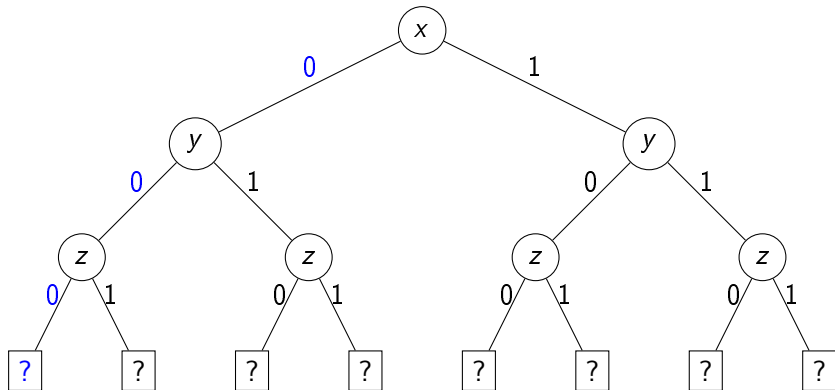
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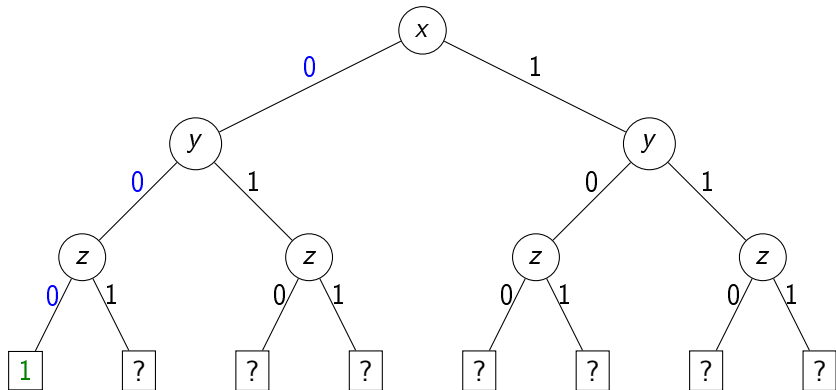
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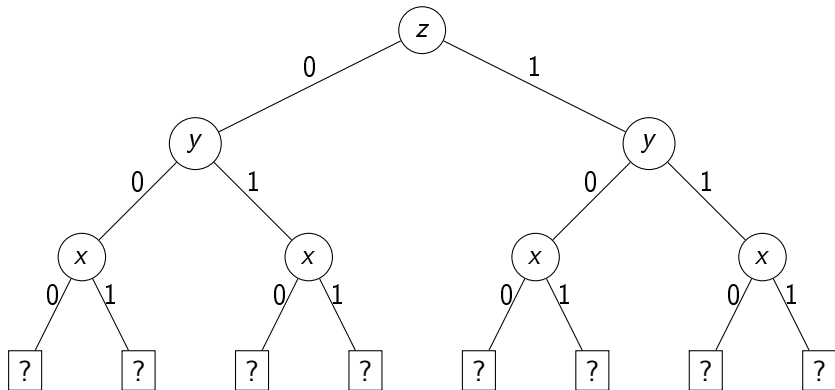
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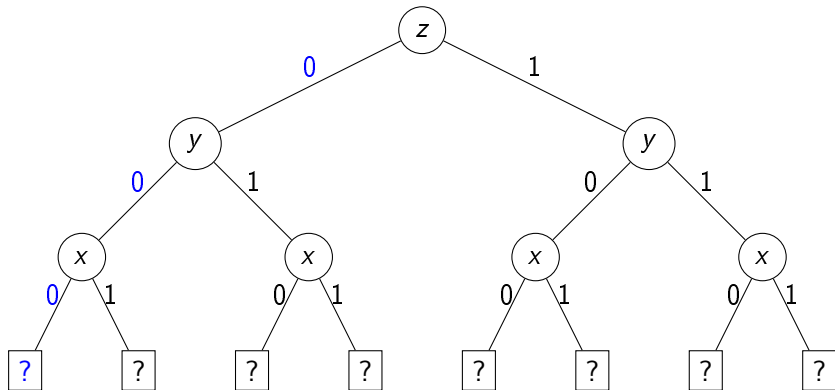
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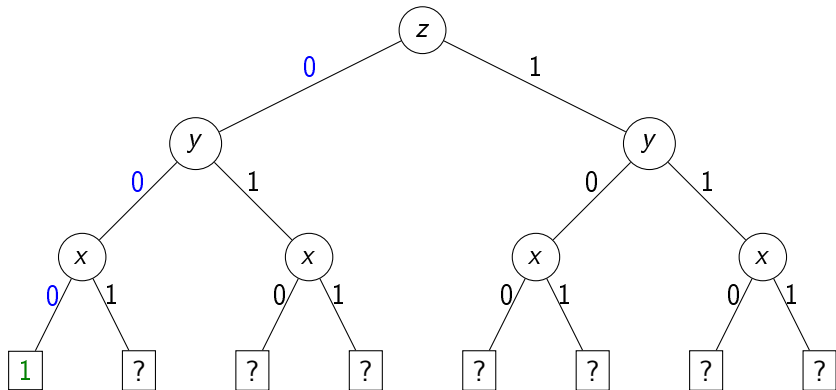
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Dynamic Largest Individual Sum (DLIS): Choose an assignment that increases the most the number of satisfied clauses

- For each variable x , let C_x be the number of unresolved clauses in which x appears positively.
- For each variable x , let $C_{\neg x}$ be the number unresolved clauses in which x appears negatively.
- Let x be a variable for which C_x is maximal ($C_x \geq C_z$ for all variables z).
- Let y be a variable for which $C_{\neg y}$ is maximal ($C_{\neg y} \geq C_{\neg z}$ for all variables z).
- If $C_x > C_{\neg y}$ choose x and assign it TRUE.
- Otherwise choose y and assign it FALSE.

- Requires $\mathcal{O}(\#literals)$ queries for each decision.

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$C_x = 0$	$C_y = 2$	$C_z = 1$
$C_{\neg x} = 2$	$C_{\neg y} = 1$	$C_{\neg z} = 1$

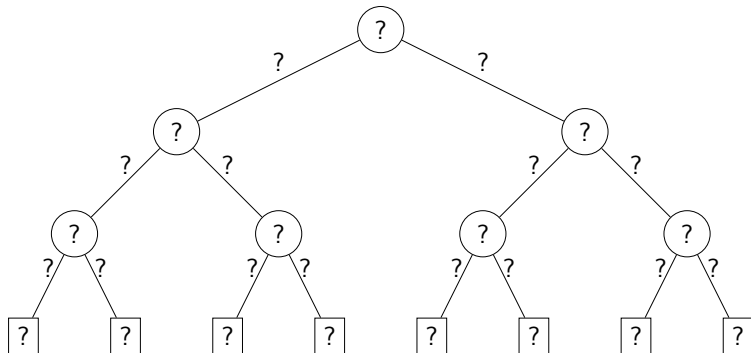
Dynamic Largest Individual Sum (DLIS) variable/sign order

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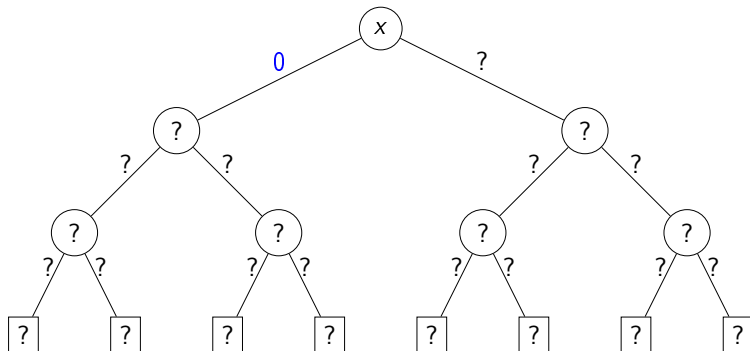


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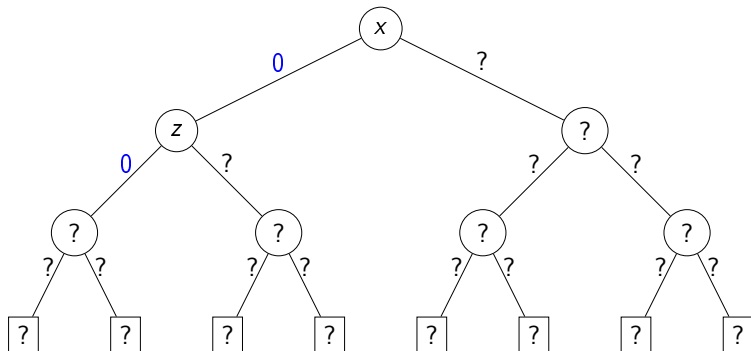


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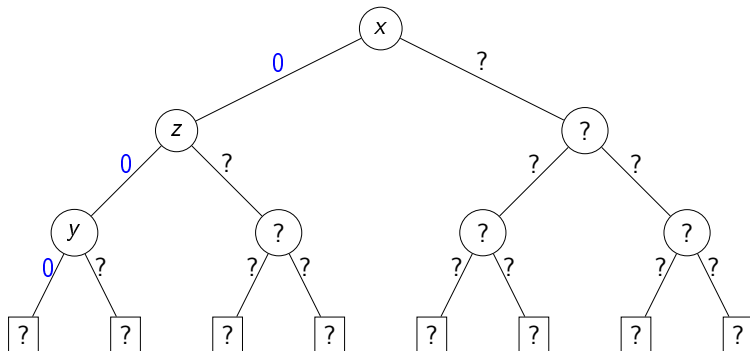


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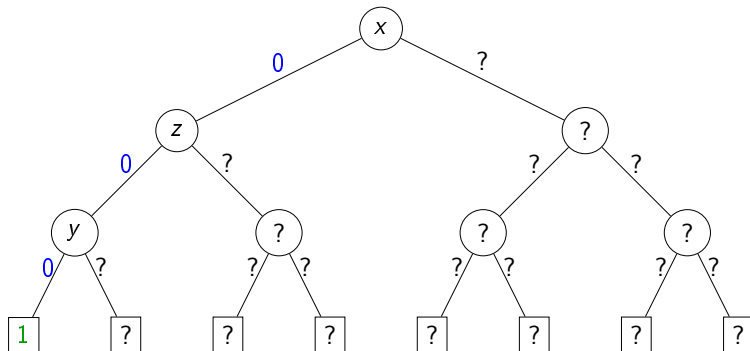


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Dynamic Largest Individual Sum (DLIS) variable/sign order



Jeroslow-Wang method

Compute for every literal l the following **static** value:

$$J(l) : \sum_{l \in c, c \in \phi} 2^{-|c|}$$

- Choose a literal l that maximizes $J(l)$.
- This gives an exponentially higher weight to literals in shorter clauses

Example CNF: Jersolow-Wang method

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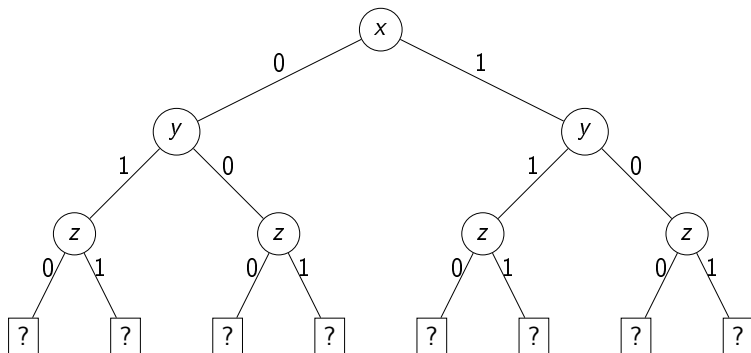
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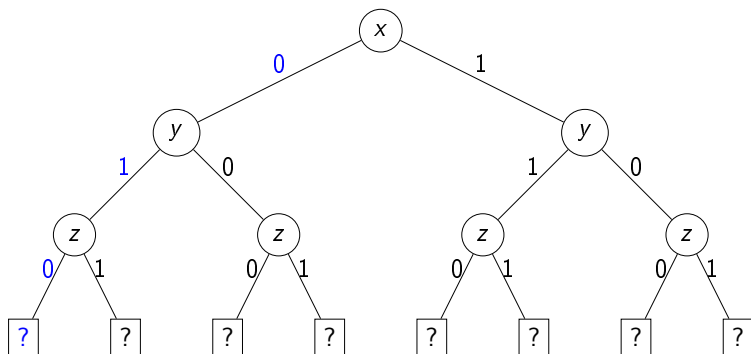


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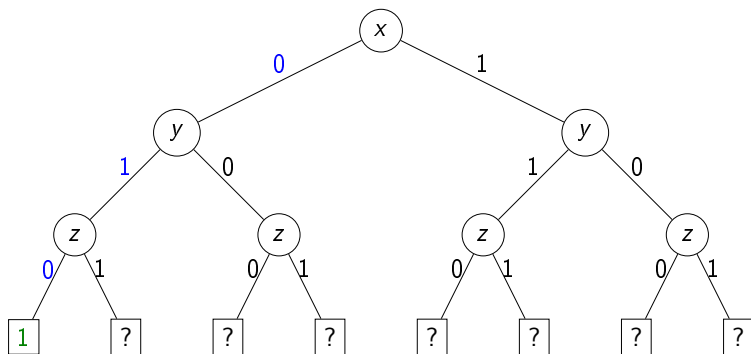


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- We will see other (more advanced) decision heuristics later.

- Decision (enumeration)
- Boolean constraint propagation (BCP)
- Conflict resolution and backtracking

Status of clause

- Given a (partial) assignment, a clause can be
 - satisfied**: at least one literal is satisfied
 - unsatisfied**: all literals are assigned but none are satisfied
 - unit**: all but one literals are assigned but none are satisfied
 - unresolved**: all other cases
- **Example**: $c = (x_1 \vee x_2 \vee x_3)$

x_1	x_2	x_3	c
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0	0	0	unsatisfied
0	0		unit
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BCP: Unit clauses are used to imply consequences of decisions.

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BCP: Unit clauses are used to imply consequences of decisions.

Some notations:

- **Decision Level (DL)** is a counter for decisions
- **Antecedent(l)**: unit clause implying the value of the literal l (nil if decision)

Example CNF: Boolean constraint propagation

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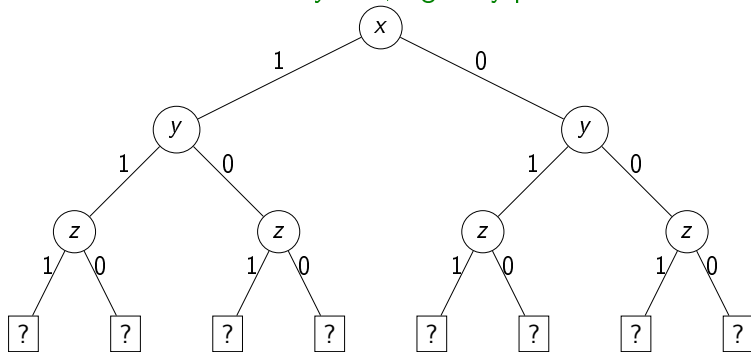
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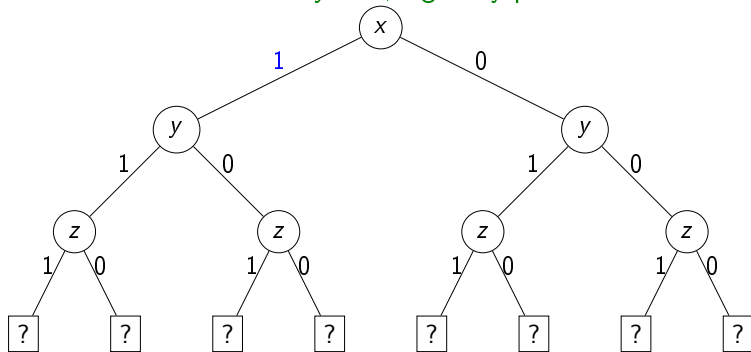
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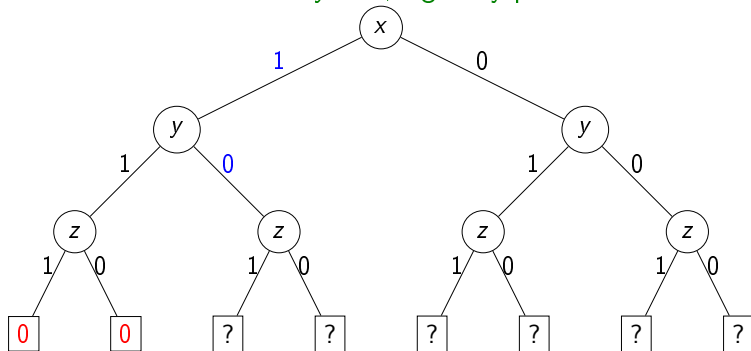
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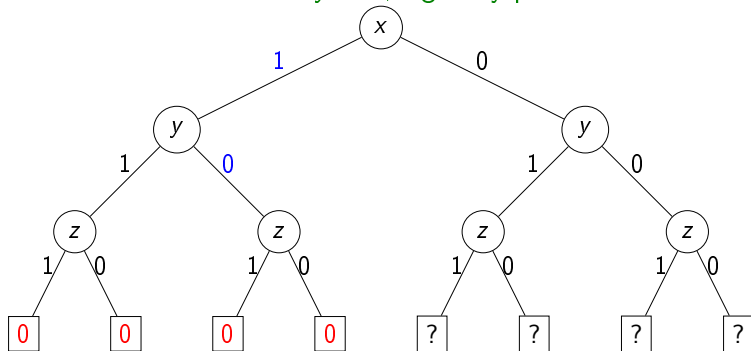
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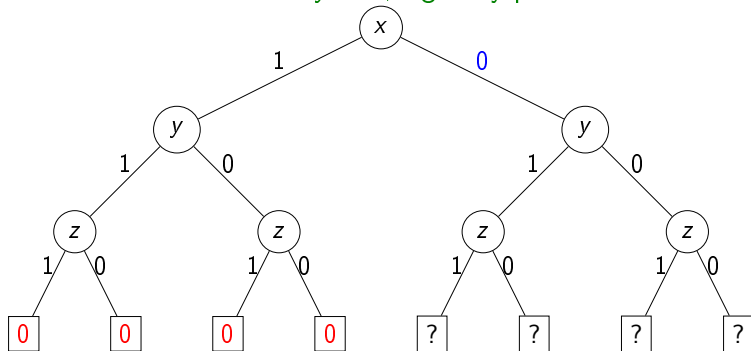
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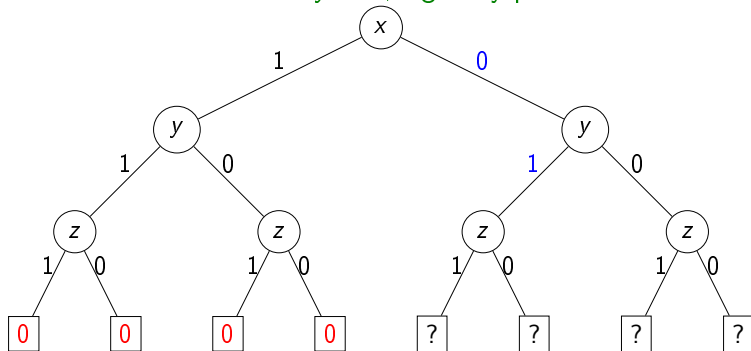
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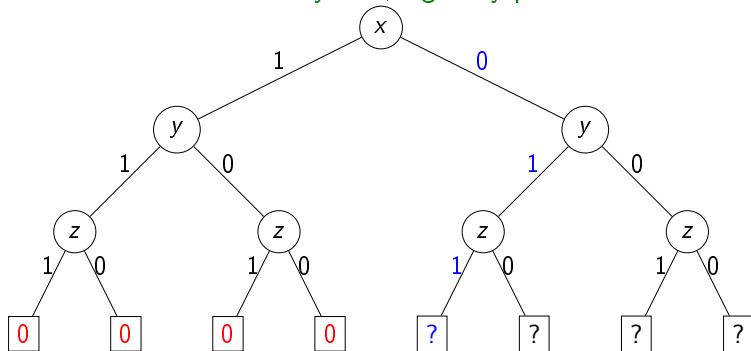
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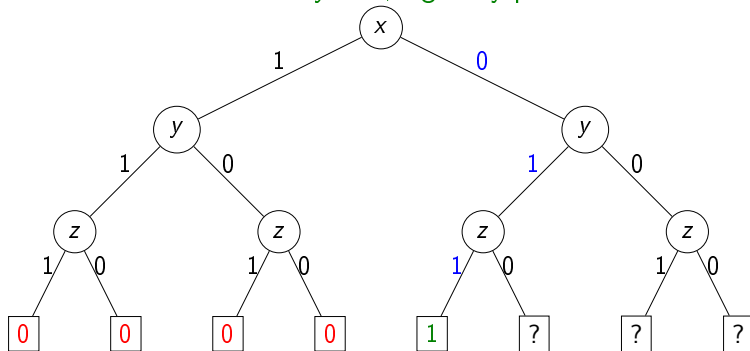
Static variable order $x < y < z$, sign: try positive first



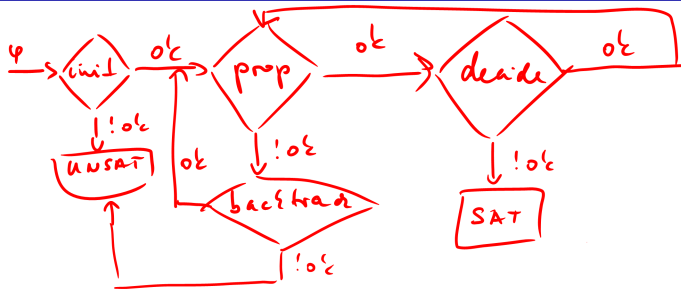
Example CNF: Boolean constraint propagation

$$\underbrace{(\neg x \vee y \vee z)}_{c_1} \wedge \underbrace{(y \vee \neg z)}_{c_2} \wedge \underbrace{(\neg x \vee \neg y)}_{c_3}$$

Static variable order $x < y < z$, sign: try positive first



The DPLL algorithm: Enumeration+propagation



The DPLL algorithm: Enumeration+propagation

```
bool DPLL(CNF_Formula  $\varphi$ ){
    trail.clear(); //trail is a global stack of assignments
    if (!BCP()) then return UNSAT;
    while (true) {
        if (!decide()) then return SAT;
        while (!BCP())
            if (!backtrack()) then return UNSAT;
    }
}

bool BCP() { //more advanced implementation: return false as soon as an unsatisfied clause is detected
    while (there is a unit clause implying that a variable  $x$  must be set to a value  $v$ )
        trail.push( $x, v, true$ );
    if (there is an unsatisfied clause) then return false;
    return true;
}
```

/

The DPLL algorithm: Enumeration+propagation (cont)

The DPLL algorithm: Enumeration+propagation (cont)

```
bool decide() {  
    if (all variables are assigned) then return false;  
    choose unassigned variable  $x$ ;  
    choose value  $v \in \{0, 1\}$ ;  
    trail.push( $x, v, false$ );  
    return true  
}  
  
bool backtrack() {  
    while (true){  
        if (trail.empty()) then return false;  
        ( $x, v, b$ )=trail.pop()  
        if (! $b$ ) {  
            trail.push( $x, \neg v, true$ );  
            return true  
        }  
    }  
}
```

$$(x)_{c_1} \wedge (\neg x \vee y)_{c_2} \wedge (\neg y \vee \frac{z}{c_3} \vee \underline{u}) \wedge (\frac{z}{c_4} \vee \neg u)$$

t.a.e

$$\begin{aligned} & (x, 1, t) \\ \Rightarrow & (y, 1, t) \end{aligned} \} \text{DL0}$$

$$\begin{aligned} \Rightarrow & (z, 0, \textcircled{f}) \quad (z, 1, t) \\ & \underline{(u, 1, \textcircled{f})} \quad (u, 1, f) \end{aligned} \} \text{DL1}$$

- For BCP, it would be a large effort to check for each propagation the value of each literal in each clause.

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- One could keep for each literal a list of clauses in which it occurs.

- For BCP, it would be a large effort to check for each propagation the value of each literal in each clause.
- One could keep for each literal a list of clauses in which it occurs.
- It is even enough to **watch two literals** in each clause such that either one of them is true or both are unassigned.

If a literal l gets true, we check each clause in which $\neg l$ is a watched literal (which is now false).

- If the other watched literal is true, the clause is satisfied.
- Else, if we find a new literal to watch, we are done.
- Else, if the other watched literal is unassigned, the clause is unit.
- Else, if the other watched literal is false, the clause is conflicting.

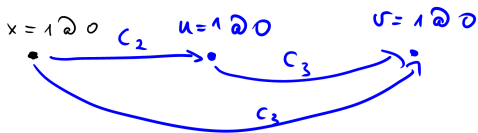
SAT-solving: Components

- Decision (enumeration)
 - Boolean Constraint Propagation (BCP)
 - Conflict resolution and backtracking
- $\left. \begin{array}{l} \Rightarrow \text{DPLL} \end{array} \right\} \Rightarrow \begin{array}{c} \text{DPLL} \\ + \\ \text{CDCL} \end{array}$

Implication graph

We represent (partial) variable assignments in the form of an **implication graph**.

$$\overset{c_1}{(\underline{x})} \wedge \overset{c_2}{(\underline{\neg x} \vee \underline{u})} \wedge \overset{c_3}{(\underline{\neg x} \vee \underline{\neg u} \vee \underline{v})}$$



Implication graph

We represent (partial) variable assignments in the form of an **implication graph**.

Definition

An **implication graph** is a labeled directed acyclic graph $G = (V, E, L)$, where

- V is a set of nodes, one for each currently assigned variable and an additional conflict node κ if there is a currently conflicting clause c_{conf} .
- L is a labeling function assigning a label to each node. The conflict node (if any) is labelled by $L(\kappa) = \kappa$. Each other node n , representing that x is assigned $v \in \{0, 1\}$ at decision level d , is labeled with $L(n) = (x = v@d)$; we define $literal(n) = x$ if $v = 1$ and $literal(n) = \neg x$ if $v = 0$.
- $E = \{(n_i, n_j) | n_i, n_j \in V, n_i \neq n_j, \neg literal(n_i) \in \text{Antecedent}(literal(n_j))\} \cup \{(n, \kappa) | n, \kappa \in V, \neg literal(n) \in c_{conf}\}$ is the set of directed edges where each edge (n_i, n_j) is labeled with $\text{Antecedent}(literal(n_j))$ if $n_j \neq \kappa$ and with c_{conf} otherwise.

Implication graph: Example

$$\underbrace{(\neg x \vee y \vee z)}_{c_1} \wedge \underbrace{(y \vee \neg z)}_{c_2} \wedge \underbrace{(\neg x \vee \neg y)}_{c_3}$$

Implication graph: Example

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Static variable order $x < y < z$, sign: try positive first

Implication graph: Example

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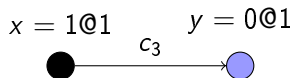
$$x = 1@1$$



Implication graph: Example

$$\underbrace{(\neg x \vee y \vee z)}_{c_1} \wedge \underbrace{(y \vee \neg z)}_{c_2} \wedge \underbrace{(\neg x \vee \neg y)}_{c_3}$$

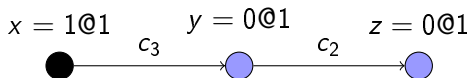
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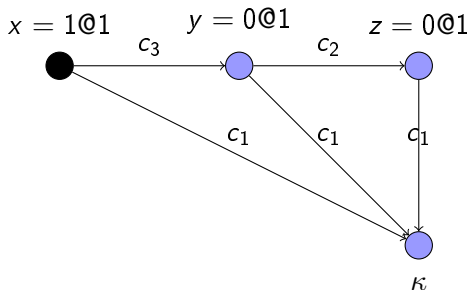
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Implication graph: Example

$$\underbrace{(\neg x \vee y \vee z)}_{c_1} \wedge \underbrace{(y \vee \neg z)}_{c_2} \wedge \underbrace{(\neg x \vee \neg y)}_{c_3}$$

Static variable order $x < y < z$, sign: try positive first



Implication graph: Example

Decisions: { }

$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_7)$$

$$c_3 = (\neg x_2 \vee \neg x_3 \vee x_4)$$

$$c_4 = (\neg x_4 \vee x_5 \vee x_8)$$

$$c_5 = (\neg x_4 \vee x_6 \vee x_9)$$

$$c_6 = (\neg x_5 \vee \neg x_6)$$

Implication graph: Example

Decisions: $\{x_7 = 0@1\}$

$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_7)$$

$$c_3 = (\neg x_2 \vee \neg x_3 \vee x_4)$$

$$c_4 = (\neg x_4 \vee x_5 \vee x_8)$$

$$c_5 = (\neg x_4 \vee x_6 \vee x_9)$$

$$c_6 = (\neg x_5 \vee \neg x_6)$$



$x_7 = 0@1$

Implication graph: Example

Decisions: $\{x_7 = 0@1, x_8 = 0@2\}$

$$\begin{aligned}c_1 &= (\neg x_1 \vee x_2) \\c_2 &= (\neg x_1 \vee x_3 \vee x_7) \\c_3 &= (\neg x_2 \vee \neg x_3 \vee x_4) \\c_4 &= (\neg x_4 \vee x_5 \vee x_8) \\c_5 &= (\neg x_4 \vee x_6 \vee x_9) \\c_6 &= (\neg x_5 \vee \neg x_6)\end{aligned}$$

$x_8 = 0@2$



$x_7 = 0@1$

Implication graph: Example

Decisions: $\{x_7 = 0@1, x_8 = 0@2, x_9 = 0@3\}$

$$c_1 = (\neg x_1 \vee x_2)$$

$$c_2 = (\neg x_1 \vee x_3 \vee x_7)$$

$$c_3 = (\neg x_2 \vee \neg x_3 \vee x_4)$$

$$c_4 = (\neg x_4 \vee x_5 \vee x_8)$$

$$c_5 = (\neg x_4 \vee x_6 \vee x_9)$$

$$c_6 = (\neg x_5 \vee \neg x_6)$$

$$x_8 = 0@2$$



$$x_7 = 0@1$$



$$x_9 = 0@3$$

Implication graph: Example

Decisions: $\{x_7 = 0@1, x_8 = 0@2, x_9 = 0@3, x_1 = 1@4\}$

$$\begin{aligned}c_1 &= (\neg x_1 \vee x_2) \\c_2 &= (\neg x_1 \vee x_3 \vee x_7) \\c_3 &= (\neg x_2 \vee \neg x_3 \vee x_4) \\c_4 &= (\neg x_4 \vee x_5 \vee x_8) \\c_5 &= (\neg x_4 \vee x_6 \vee x_9) \\c_6 &= (\neg x_5 \vee \neg x_6)\end{aligned}$$

$$x_8 = 0@2$$



$$x_1 = 1@4$$



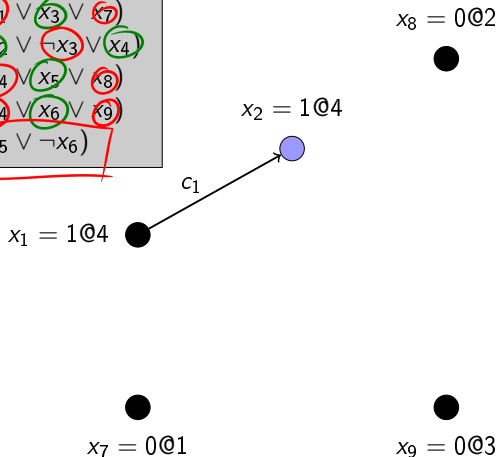
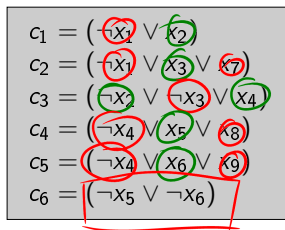
$$x_7 = 0@1$$



$$x_9 = 0@3$$

Implication graph: Example

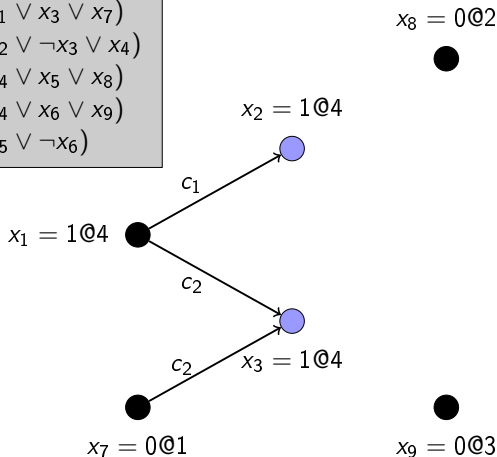
Decisions: $\{x_7 = 0@1, x_8 = 0@2, x_9 = 0@3, x_1 = 1@4\}$



Implication graph: Example

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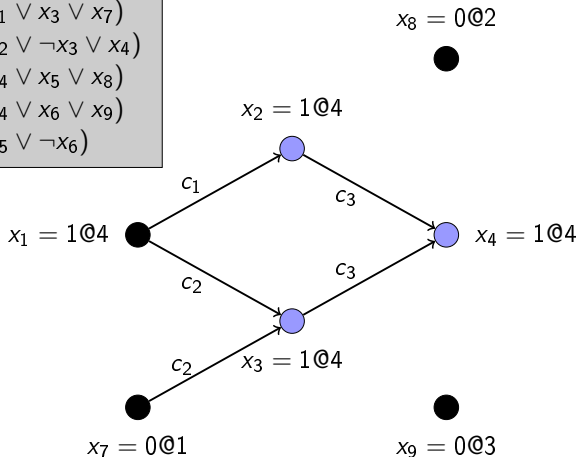
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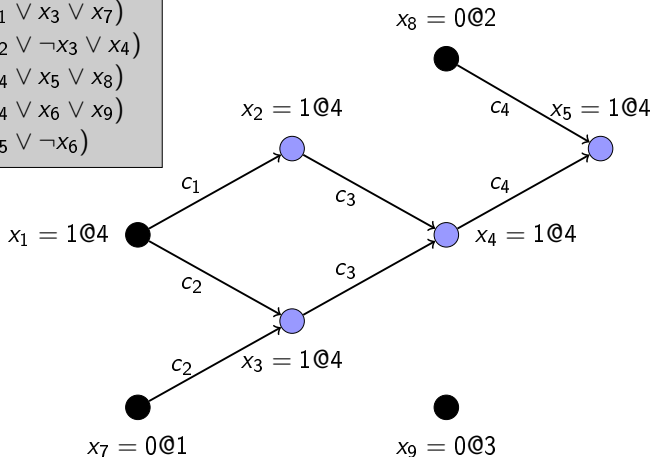
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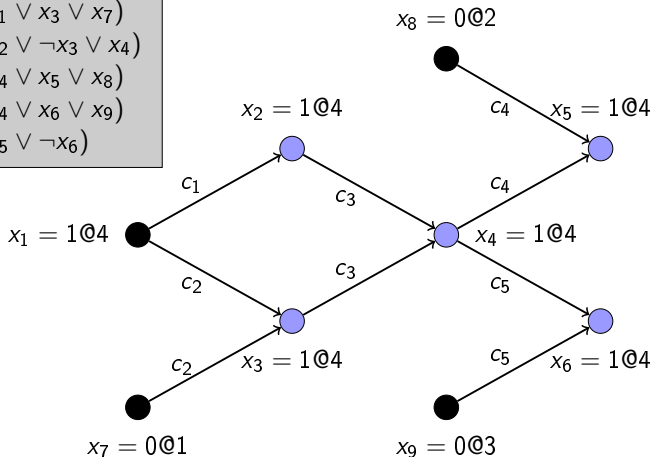
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Implication graph: Example

Decisions: $\{x_7 = 0@1, x_8 = 0@2, x_9 = 0@3, x_1 = 1@4\}$ *DLO*

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$$x_{11} = 0@0$$

$$x_8 = 0@2$$

$$x_2 = 1@4$$

$$x_5 = 1@4$$

$$x_4 = 1@4$$

$$x_3 = 1@4$$

$$x_6 = 1@4$$

$$x_1 = 1@4$$

$$x_9 = 0@3$$

$$x_7 = 0@1$$

$$c_2 = (x_7 \vee x_8 \vee x_9 \vee \neg x_1)$$

$$c_8 = (x_9 \vee x_8 \vee \neg x_4)$$

$$(\neg x_5 \vee \neg x_4 \vee x_5)$$

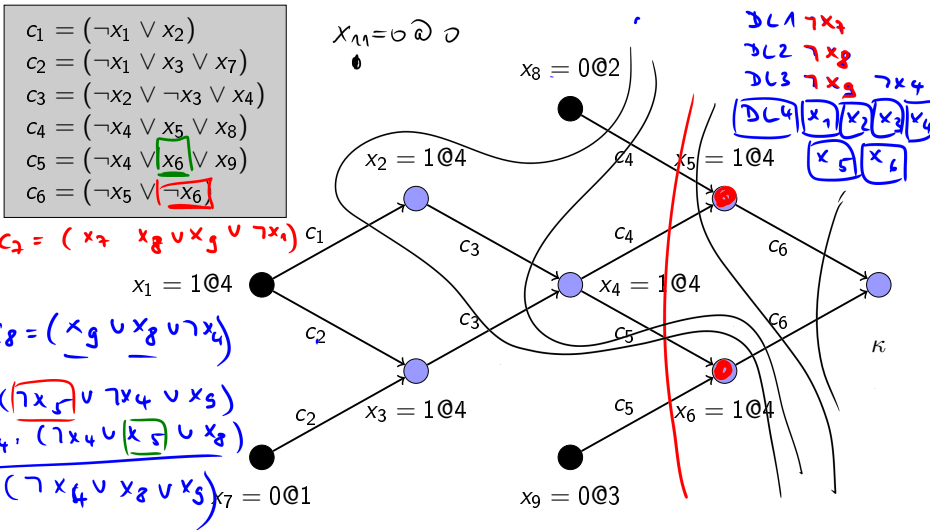
$$c_4: (\neg x_4 \vee x_5 \vee x_8)$$

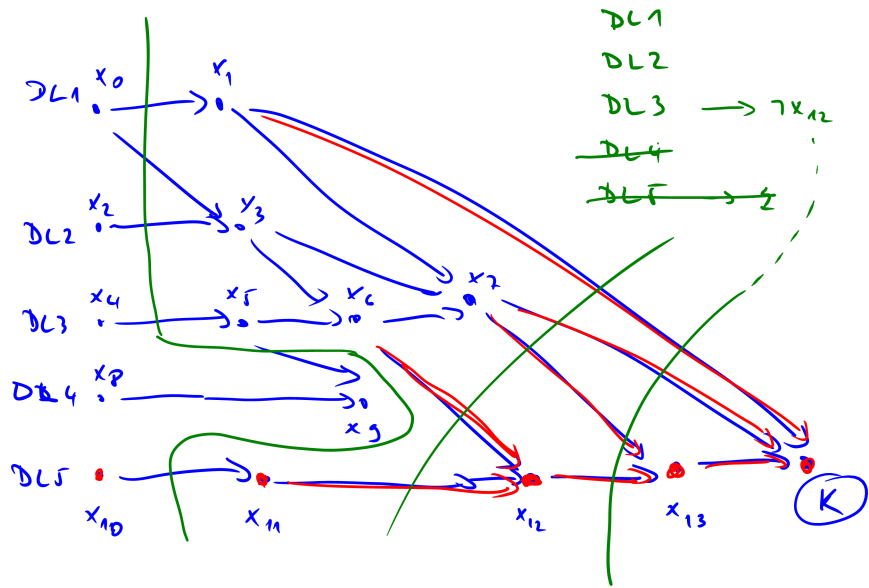
$$(\neg x_4 \vee x_8 \vee x_5)$$

DLO

$\neg x_1 \neg x_2 \neg x_3 \neg x_4$

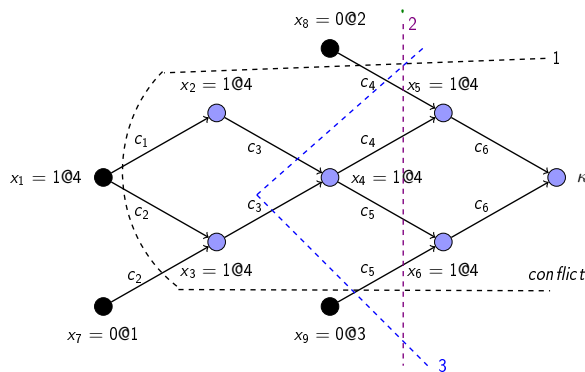
$\neg x_5 \neg x_6$





Conflict resolution

- Assume that the current (partial) assignment doesn't satisfy our formula.
- Let L be a set of literals labeling nodes that form a cut in the implication graph, separating a conflict node from the roots.
- $\bigvee_{l \in L} \neg l$ is called a **conflict clause**: its satisfaction is necessary for the satisfaction of the formula.



$$1. (x_8 \vee \neg x_1 \vee x_7 \vee x_9)$$

$$2. (x_8 \vee \neg x_4 \vee x_9)$$

$$3. (x_8 \vee \neg x_2 \vee \neg x_3 \vee x_9)$$

...

Conflict resolution

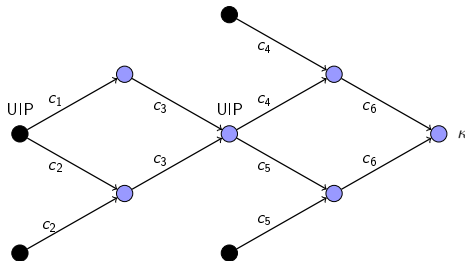
- Which conflict clauses should we consider?

Conflict resolution

- Which conflict clauses should we consider?
- An **asserting clause** is a conflict clause with a single literal from the current decision level.
Backtracking (to the right level) makes it a **unit clause**.
- Modern solvers consider only asserting clauses.

Conflict resolution

- Which conflict clauses should we consider?
- An **asserting clause** is a conflict clause with a single literal from the current decision level.
Backtracking (to the right level) makes it a **unit clause**.
- Modern solvers consider only asserting clauses.
- A **unique implication point (UIP)** is an internal node in the implication graph such that **all paths from the last decision to the conflict node go through it**.
- The **first UIP** is the UIP closest to the conflict.



Conflict-driven backtracking

- Usually, the asserting conflict clause is **learnt** by adding it to the clause set. However, this is not necessary for completeness.

Conflict-driven backtracking

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- Backtrack to the **second** highest decision level d_l in the asserting conflict clause (but do not erase it).
- This way the literal with the currently highest decision level will be implied at decision level d_l .
- Propagate all new assignments.

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For example, from $(x \vee \neg y) \wedge (x \vee y)$ and decision $x = 0$, we get (x) .

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Conflict-driven backtracking

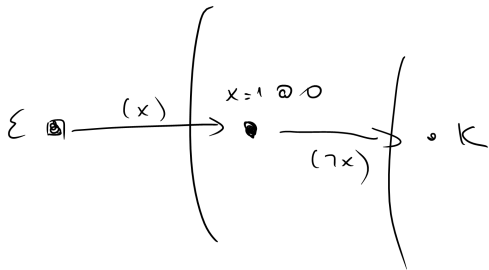
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$$\frac{(x) \quad (x)}{()}$$

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Q: What happens if the conflict appears at decision level 0?

A: The formula is unsatisfiable.

The CDCL algorithm

```
if (!BCP()) return UNSAT;
while (true)
{
    if (!decide()) return SAT;
    while (!BCP())
        if (!resolve_conflict()) return UNSAT;
}
```

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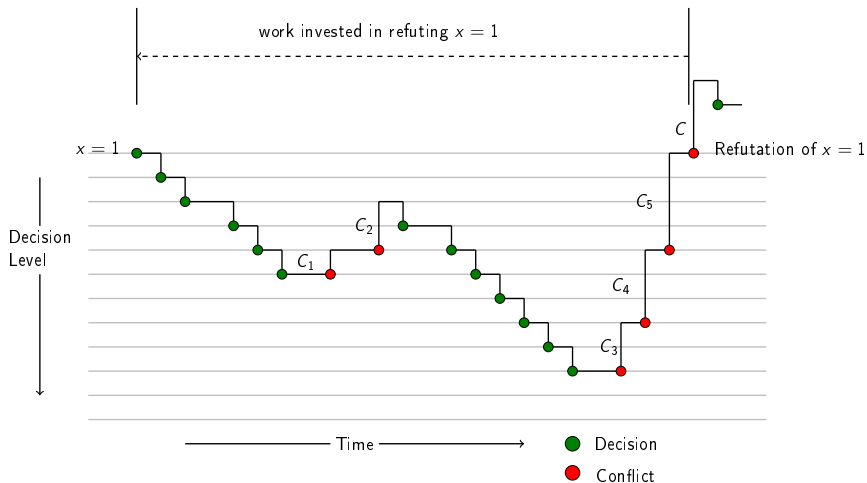
Choose the next variable and value.

Return false if all variables are assigned.

Boolean constraint propagation.
Return false if reached a conflict

Conflict resolution and
backtracking. Return false
if impossible.

Progress of a DPLL+CDCL-based SAT solver



- The binary resolution is a sound (and complete) inference rule:

$$\frac{(\beta \vee a_1 \vee \dots \vee a_n) \quad (\neg\beta \vee b_1 \vee \dots \vee b_m)}{(a_1 \vee \dots \vee a_n \vee b_1 \vee \dots \vee b_m)} \text{ (Binary Resolution)}$$

- Example:

$$\frac{(x_1 \vee x_2) \quad (\neg x_1 \vee x_3 \vee x_4)}{(x_2 \vee x_3 \vee x_4)}$$

Conflict clauses and resolution

- The binary resolution is a sound (and complete) inference rule:

$$\frac{(\beta \vee a_1 \vee \dots \vee a_n) \quad (\neg\beta \vee b_1 \vee \dots \vee b_m)}{(a_1 \vee \dots \vee a_n \vee b_1 \vee \dots \vee b_m)} \text{ (Binary Resolution)}$$

- Example:

$$\frac{(x_1 \vee x_2) \quad (\neg x_1 \vee x_3 \vee x_4)}{(x_2 \vee x_3 \vee x_4)}$$

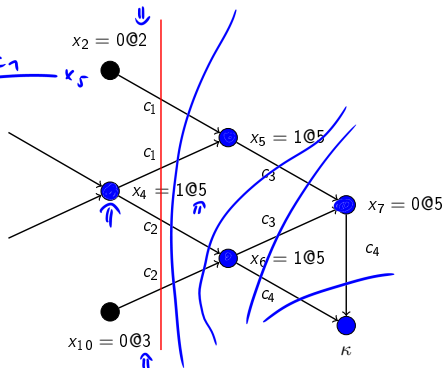
What is the relation of resolution and conflict clauses?

Conflict clauses and resolution

- Consider the following example:

$$\begin{array}{l}
 \frac{c_4 \quad c_3}{(\neg x_5 \vee \neg x_6)} \quad c_2 \\
 \frac{(\neg x_5 \vee \neg x_4 \vee x_{10})}{(\neg x_4 \vee x_2 \vee x_{10})} \quad x_5 \quad c_1
 \end{array}$$

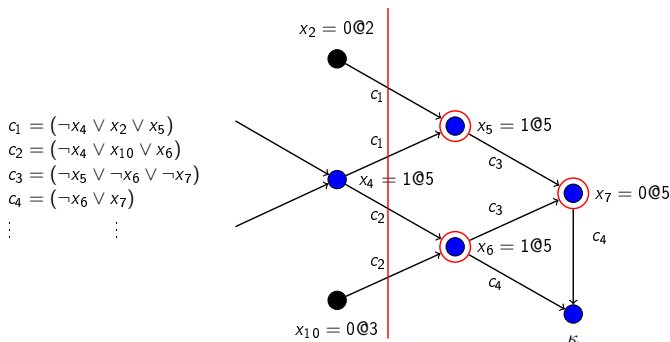
$$\begin{array}{l}
 c_1 = (\neg x_4 \vee x_2 \vee \underline{x_5}) \\
 c_2 = (\neg x_4 \vee x_{10} \vee \underline{x_6}) \\
 c_3 = (\neg x_5 \vee \neg x_6 \vee \neg x_7) \\
 c_4 = (\neg x_6 \vee \underline{x_7}) \\
 \vdots \qquad \qquad \vdots
 \end{array}$$



- Conflict clause: $c_5 : (\underline{x_2 \vee \neg x_4 \vee x_{10}})$

Conflict clauses and resolution

- Conflict clause: $c_5 : (x_2 \vee \neg x_4 \vee x_{10})$



- Assignment order: x_4, x_5, x_6, x_7
 - $T1 = \text{Res}(c_4, c_3, x_7) = (\neg x_5 \vee \neg x_6)$
 - $T2 = \text{Res}(T1, c_2, x_6) = (\neg x_4 \vee \neg x_5 \vee x_{10})$
 - $T3 = \text{Res}(T2, c_1, x_5) = (x_2 \vee \neg x_4 \vee x_{10})$

Finding the conflict clause

```
procedure analyze_conflict() {  
    if (current_decision_level = 0) return false;  
    cl := current_conflicting_clause;  
    while (not stop_criterion_met(cl)) do {  
        lit := last_assigned_literal(cl);  
        var := variable_of_literal(lit);  
        ante := antecedent(var);  
        cl := resolve(cl, ante, var);  
    }  
    add_clause_to_database(cl);  
    return true;  
}
```

Applied to our example:

name	<i>cl</i>	<i>lit</i>	<i>var</i>	<i>ante</i>
c_4	$(\neg x_6 \vee x_7)$	x_7	x_7	c_3
	$(\neg x_5 \vee \neg x_6)$	$\neg x_6$	x_6	c_2
	$(\neg x_4 \vee x_{10} \vee \neg x_5)$	$\neg x_5$	x_5	c_1
c_5	$(\neg x_4 \vee x_2 \vee x_{10})$			

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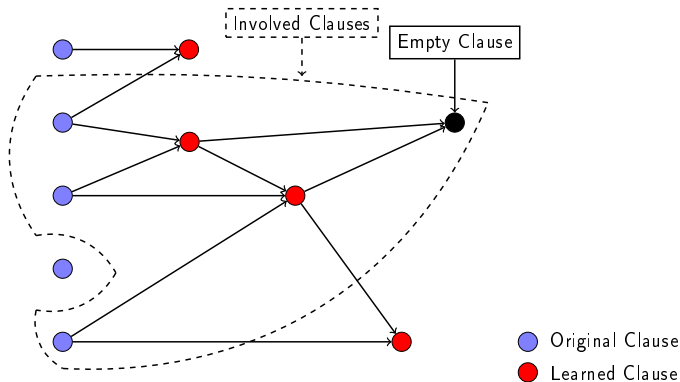
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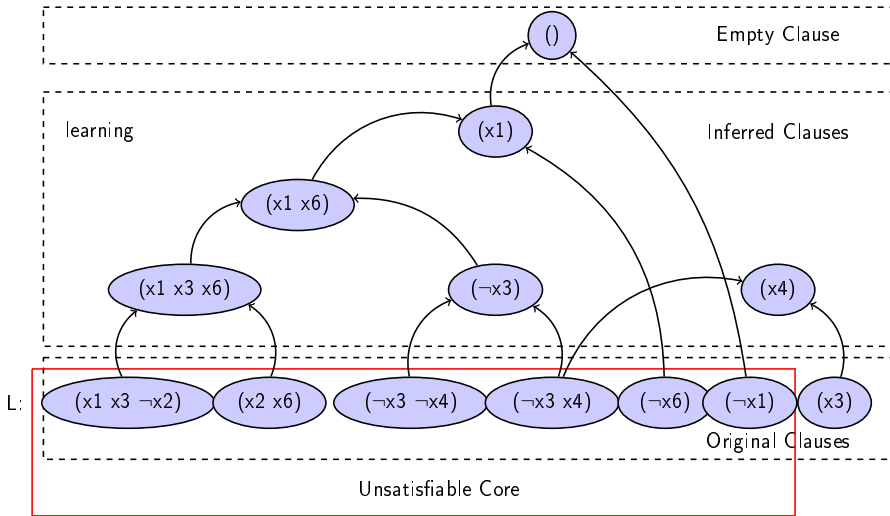
- The set of all original clauses is an unsatisfiable core.
- The set of those original clauses that were used for resolution in conflict analysis during SAT-solving (inclusively the last conflict at decision level 0) gives us an unsatisfiable core which is in general much smaller.
- However, this unsatisfiable core is still not always minimal (i.e., we can remove clauses from it still having an unsatisfiable core).

The resolution graph

A **resolution graph** gives us more information to get a minimal unsatisfiable core.



Resolution graph: Example



Theorem

It is never the case that the solver enters decision level dl again with the same partial assignment.

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Proof.

Define a partial order on partial assignments: $\alpha < \beta$ iff either α is an extension of β or α has more assignments at the smallest decision level at that α and β do not agree.

BCP decreases the order, conflict-driven backtracking also. Since the order always decreases during the search, the theorem holds. \square

DL0	x	→
DL1	✓	→
DL2	x	→
DL3	x	→
DL4	x	→
DL5	x	

x	→
x	→
x	→
x	→
x	→
x	→

x	→
x	→
x	→
x	→
x	→
x	→

● > ●

● > ●

● > ●

state > extended state



Back to decision heuristics...

- Decision (enumeration)
- Boolean Constraint Propagation (BCP)
- Conflict resolution and backtracking

Decision heuristics - VSIDS

- VSIDS (variable state independent decaying sum)
- Gives priority to variables involved in recent conflicts.
- “Involved” can have different definitions. We take those variables that occur in clauses used for conflict resolution.

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-
- 1 Each variable in each polarity has a **counter** initialized to 0.
 - 2 We define an **increment** value (e.g., 1).
 - 3 When a **conflict** occurs, we increase the counter of each variable, that occurs in at least one clause used for conflict resolution, by the increment value.
Afterwards we increase the increment value (e.g., by 1).
 - 4 For decisions, the unassigned variable with the **highest counter** is chosen.
 - 5 Periodically, all the counters and the increment value are **divided** by a constant.

- **Chaff** holds a list of unassigned variables sorted by the counter value.
- Updates are needed only when adding conflict causes.
- Thus - decision is made in constant time.

VSIDS is a 'quasi-static' strategy:

- **static** because it doesn't depend on current assignment
- **dynamic** because it gradually changes. Variables that appear in recent conflicts have higher priority.

This strategy is a **conflict-driven** decision strategy.

"...employing this strategy dramatically (i.e., an order of magnitude) improved performance..."