Satisfiability Checking - WS 2016/2017

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Series 5

Exercise 1

a) Give a formula describing the Unequal game instance of Series 3 Exercise 2 in *equality logic with uninterpreted functions*. Remember, that the formula shall be satisfiable iff the game instance has a solution. You must not use propositional variables in your solution!

b) Compare the resulting formula to the propositional encoding. More precisely, compare the number of literals and clauses using the big $\mathcal{O}$ notation. Draw a conclusion.

c) Does the usage of equality logic with uninterpreted functions improve the runtime complexity of solving the problem? Give an explanation!

Solution:

a) In the following, $g_{i,j}$ $(1 \leq i, j \leq n)$ is an uninterpreted variable and we introduce the constants given by $N := \{1, \ldots, n\}$. Then $g_{i,j}$ represents the number in the grid at the coordinates $(i,\ j)$. We define the uninterpreted function $greater(i,j)$ on $N \times N$ mapping to the constants T and F.

$$\varphi_{grid} := \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \bigvee_{k=1}^{n} g_{i,j} = k$$

$$\varphi_{row} := \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \bigwedge_{k=j+1}^{n} g_{i,j} \neq g_{i,k}$$

$$\varphi_{column} := \bigwedge_{j=1}^{n} \bigwedge_{i=1}^{n} \bigwedge_{k=i+1}^{n} g_{i,j} \neq g_{k,j}$$

$$\varphi_{>} := \bigwedge_{\substack{r=1 \\ s=1 \\ r>s}}^{n} greater(r,s) = T \quad \wedge \quad \bigwedge_{\substack{r=1 \\ s=1 \\ r<=s}}^{n} greater(r,s) = F$$

Hence, we get the formula representing the game instance of Series 3 Exercise 2:

$$\begin{aligned} \varphi^{UF} \;:=\; & g_{1,1} = 3 \quad \wedge \quad g_{4,1} = 1 \quad \wedge \quad greater(g_{1,2},\ g_{1,3}) = T \\ & greater(g_{2,3},\ g_{2,2}) = T \quad \wedge \quad greater(g_{1,4},\ g_{2,4}) = T \\ & greater(g_{3,4},\ g_{2,4}) = T \quad \wedge \quad greater(g_{4,3},\ g_{4,4}) = T \\ & \varphi_{grid} \quad \wedge \quad \varphi_{row} \quad \wedge \quad \varphi_{column} \quad \wedge \quad \varphi_{>} \end{aligned}$$

b) By comparing the literals/constraints and clauses of the result of part a) with the result of Series 3 Exercise 2 you see:

- As the number of variables in the formula using propositional logic of Series 3 Exercise 2 has been $\lfloor log(n) \rfloor^3$, the number of literals is:

$$\mathcal{O}(log(n)^3)$$

- The number of clauses is:

$$\mathcal{O}(n^2) + 2 \cdot \mathcal{O}(n^4) + 5 \cdot \mathcal{O}(n^2) + 2 \;=\; \mathcal{O}(n^4)$$

- The number of literals (different equations) in φ^{UF} is:

$$3 \cdot \mathcal{O}(n^3) + \mathcal{O}(n^2) + 5 + 2 \;=\; \mathcal{O}(n^3)$$

- The number of clauses in φ^{UF} is:

$$\mathcal{O}(n^2) + 2 \cdot \mathcal{O}(n^3) \;=\; \mathcal{O}(n^3)$$

Conclusion: The formula using propositional logic needs less literals but $\mathcal{O}(n)$ times more clauses than the formula using equality logic with uninterpreted functions.

c) Possible solving approach by reduction:

(a) **Eliminate constants** contained in φ_n^{UF}: Using the algorithm of the next exercise this increases the number of constraints quadratically in the number of constants. The number of constants ($= n+2$) produces $\mathcal{O}(n^2)$ new clauses which is negligible compared to the overall number of clauses.

(b) **Ackermann reduction** to eliminate the function symbols: There are $\mathcal{O}((n^2)^2) \;=\; \mathcal{O}(n^4)$ constraints added describing the functional congruence of greater.

(c) **Graph-based reduction** to propositional logic: The number of additional constraints is quadratic in the number of constraints contained in the reduced formula of step (b), thus quadratic in $\mathcal{O}(n^4)$. This yields a blowup of magnitude $\mathcal{O}(n^8)$ in the number of constraints. If we skip adding the functional congruence, it would be $\mathcal{O}(n^6)$ (careful analysis yields that the congruence is actually implied by our definition of $\varphi_{>}$).

Concluding, after all reductions the input formula size is in worst case of magnitude $\mathcal{O}(n^8)$ or $\mathcal{O}(n^6)$, compared to the original $\mathcal{O}(n^4)$ in the pure propositional modeling approach. Thus, it is likely that the runtime complexity of solving the Unequal game instance is worse when solving φ_n^{UF}, of course depending on the actual implementation of the SAT-solver.

Exercise 2

In the lecture “Equalities and Uninterpreted Functions”, it was mentioned that for each formula of equality logic with uninterpreted functions φ^{UF} there is an equisatifiable formula $\hat{\varphi}^{UF}$ without constants. Define a general constant elimination procedure for a formula in equality logic with uninterpreted functions containing constants.

Solution:

Constant elimination procedure:

Input: equality logic formula with unterpreted functions φ^{UF}

Output: equality logic formula with uninterpreted functions $\varphi^{UF'}$ without constants

Procedure: In pseudo-code.

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φ^UF' := φ^UF;
V := ∅;
for each constant symbol c in φ^UF':
  x_c := new variable;
  φ^UF' := φ^UF'[c/x_c];
  V := V ∪ {x_c};
for each pair {x_c, x'_c} ∈ (V choose 2):
  φ^UF' := φ^UF' ∧ (x_c ≠ x'_c);
```

Correctness: φ^{UF} is satisfiable iff $\varphi^{UF'}$ is satisfiable.

Proof. Let $C = \{c_1, \dots, c_n\}$ and X be the sets of constant respectively variable symbols appearing in φ, and let $X_C = \{x_{c_1}, \dots, x_{c_n}\}$ be a set of fresh variable symbols with $X_C \cap (X \cup C) = \emptyset$. Let D be a domain and I, I' interpretations such that $I'(x_{c_i}) = I(c_i)$ for all $c_i \in C$ and $I'(x) = I(x)$ for all other signature elements $x \notin X_C$. Then we make the following observation:

$$\begin{aligned} [\![\varphi^{UF'}]\!]_{D,I'} &\overset{(1)}{=} [\![\varphi^{UF'}[x_{c_1}/c_1] \dots [x_{c_n}/c_n]]\!]_{D,I'} \\ &\overset{(2)}{=} [\![\varphi^{UF}]\!]_{D,I'} \\ &\overset{(3)}{=} [\![\varphi^{UF}]\!]_{D,I} \,. \end{aligned}$$

Remember that $\varphi[x/e]$ stays for the formula resulting from φ after having substituted e for x (i.e., after replacing each occurence of x by e). In Equations (1) and (2) we reverse the transformation from φ^{UF} to $\varphi^{UF'}$ by substituting c_i for each $x_{c_i} \in X_C$; by the definitions of I and I' this does not change the evaluation of the formula under I'. Because φ^{UF} does not have any symbols from X_C, Equation (3) holds by the definition of I'.

"$\Rightarrow$": Let $D, I \models \varphi^{UF}$. We define I' by $I'(x_{c_i}) = I(c_i)$ for all $c_i \in C$, and assign the same values as I to all other signature elements. By Equations (1)-(3) we get $[\![\varphi^{UF'}]\!]_{D,I'} = [\![\varphi^{UF}]\!]_{D,I}$, where the latter is true by assumption. Thus $D, I' \models \varphi^{UF'}$.

"$\Leftarrow$": Let $D, I' \models \varphi^{UF'}$. We define the interpretation I by $I(c_i) = I'(x_{c_i})$ for all $c_i \in C$, and $I(x) = I'(x)$ for all other signature elements $x \notin C$. By Equations (1)-(3) we get $[\![\varphi^{UF'}]\!]_{D,I'} = [\![\varphi^{UF}]\!]_{D,I}$, where the former is true by assumption. Thus $D, I \models \varphi^{UF}$. □

Complexity: Let n be the number of constants in φ^{UF}. Then the time and space complexity of removing the constants is

$$n + \binom{n}{2} = n + \frac{n(n-1)}{2} \in \mathcal{O}(n^2).$$

Exercise 3

Consider the following formula in equality logic:

$$\varphi \quad := \quad a = b \ \wedge \ (\ b = c \ \vee \ c = e\ ) \ \wedge \ (\ b = d \ \vee \ c = f\ ) \ \wedge \ a \neq e$$

a) Construct the equality graph with polarity for φ.

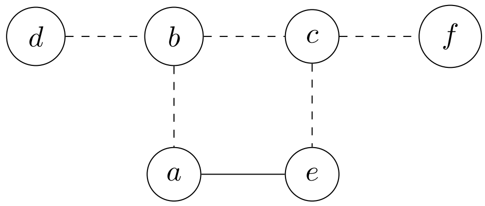
b) Simplify the constructed equality graph and the formula using the method presented in the lecture (slides 37-38).

c) Make the simplified equality graph without polarity chordal. What are the chord-free simple cycles?

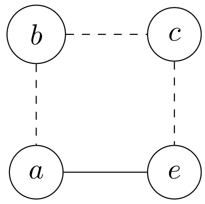
d) Construct the satisfiability-equivalent propositional logic formula for φ using the previous results.

Solution:

a) The equality graph with polarity for φ is given by:



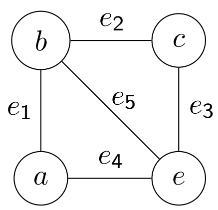
b) The edges (b, d) and (c, f) are not part of any contradictory cycle, thus we can remove them and replace in the formula both $b = d$ and $c = f$ by *true*. We get the following simplified equality graph:



The simplified formula is

$$a = b \wedge (b = c \vee c = e) \wedge a \neq e$$

c) A chordal completion of the simplified equality graph without polarity is:



The chord-free simple cycles are $(a,\ b,\ e,\ a)$ and $(b,\ c,\ e,\ b)$.

d) The satisfiability-equivalent propositional logic formula for φ using the previous results is:

$$\begin{aligned}
\varphi^{\mathit{prop}} &:= \varphi_{\mathit{sk}} \wedge \varphi_{\mathit{trans}} \\
\varphi_{\mathit{sk}} &:= e_1 \wedge (e_2 \vee e_3) \wedge \neg e_4 \\
\varphi_{\mathit{trans}} &:= ((e_1 \wedge e_4) \rightarrow e_5) \\
&\quad \wedge ((e_1 \wedge e_5) \rightarrow e_4) \\
&\quad \wedge ((e_4 \wedge e_5) \rightarrow e_1) \\
&\quad \wedge ((e_2 \wedge e_3) \rightarrow e_5) \\
&\quad \wedge ((e_2 \wedge e_5) \rightarrow e_3) \\
&\quad \wedge ((e_3 \wedge e_5) \rightarrow e_2)
\end{aligned}$$