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Recitation # 3 Forced Oscillators.

Exo # 1

$m = 0{,}2\,kg$, $\omega_D = 20\,rd/s$, $F_0 = 2N$, $\gamma = 4\,s^{-1}$

a) $\omega_0 = ?$

$$\omega_D = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \Rightarrow \omega_0 = \sqrt{\omega_D^2 + \frac{\gamma^2}{4}}$$

N.A. $\omega_0 = 20{,}10\,rd/s$

b) $Q = ?$: $Q = \frac{\omega_0}{\gamma} \Rightarrow Q \simeq 5$

c) $A(\omega_0) = ?$

$$A(\omega_0) = a.Q = \frac{F_0}{k}.Q = \frac{F_0}{m\omega_0^2}.Q \text{ or}$$

$$A(\omega_0) = \frac{F_0}{m} \cdot \frac{1}{\omega_0 \gamma} \Rightarrow A(\omega_0) = 0{,}12\,m$$

d) $A(\omega_f \ll \omega_0) = a = \frac{F_0}{k} = \frac{F_0}{m\omega_0^2} \Rightarrow A(\omega_f \ll \omega_0) = 0{,}025\,m$

Exo # 2

$$\ddot{x} + 4\dot{x} + 8x = 20\cos(2t)$$

so, we have $\gamma = 4\,s^{-1}$, $\omega_0^2 = 8\,rd^2/s^2$, $F_0 = 20\,N$ and $\omega_f = 2\,rd/s$.

a) the general solution is: $x_G(t) = x_H(t) + x_p(t)$

$$\Rightarrow x_G(t) = A_0 e^{-\frac{\gamma}{2}t}\cos(\omega_D t + \phi) + A(\omega_f).\cos(\omega_f t - \delta)$$

A_0 and ϕ are determined from initial conditions and we have:

$$A(\omega_f) = \frac{F_0/m}{[(\omega_0^2 - \omega_f^2)^2 + \gamma^2\omega_f^2]^{1/2}} \quad \text{and } \tan(\delta) = \frac{\gamma\omega_f}{(\omega_0^2 - \omega_f^2)}$$

A.N: we get $A(\omega_f) = 2{,}24\,m$, $\omega_D = 2\,rd/s$ and

$\tan(\delta) = 2 \Rightarrow \delta = 63{,}4°$ or $243{,}4°$ but

$0 < \delta < \pi \Rightarrow \delta = 63{,}4° \Rightarrow$

$$\omega_f = 2\,rd/s < \omega_0 = 2{,}83\,rd/s \Rightarrow$$

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so:

$$x_G(t) = A_0 e^{-2t} \cdot \cos(2t+\phi) + 2{,}24 \cdot \cos(2t - 1{,}11)$$

$$\dot{x}_G(t) = -2A_0 e^{-2t} \cdot \cos(2t+\phi) - 2A_0 e^{-2t} \sin(2t+\phi) - 2 \times 2{,}24 \sin(2t - 1{,}11).$$

$$\begin{cases} x_G(0) = A_0 \cdot \cos(\phi) + 2{,}24 \cdot \cos(-1{,}11) = 0 \quad \ldots (1) \\ \dot{x}_G(0) = -2A_0 \cos(\phi) - 2A_0 \sin(\phi) + 4{,}48 \cdot \sin(-1{,}11) = \quad (2) \end{cases}$$

from (1) and (2) we get $A_0 = 3{,}15\,m$, and $\phi = 108{,}5° \Rightarrow$

$$x_G(t) = 3{,}15 e^{-2t} \cdot \cos(2t + 1{,}89) + 2{,}24 \cos(2t - 1{,}11)\,m$$

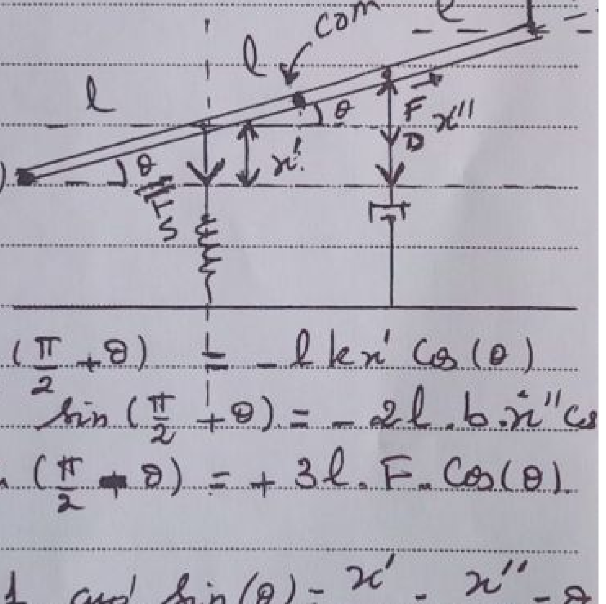
b) After a long time $\Rightarrow$ steady state $\Rightarrow$

$$x_G(t) \simeq x_p(t) = A(\omega_f) \cdot \cos(\omega_f t - \delta) = 2{,}24 \cdot \cos(2t - 1{,}11)$$

$$\Rightarrow T_f = \frac{2\pi}{\omega_f} = \pi s \text{ and } f = \frac{2\pi}{\omega_f} = 0{,}32\,Hz, \ A(\omega_f) = 2{,}24\,m.$$

EXO # 3

$\overline{F}$: spring, $\overline{F}_D$: damping force



$$\sum \vec{\tau} = I_{/\Delta} \ddot{\vec{\theta}} \Rightarrow$$

$$\vec{\tau}_s + \vec{\tau}_D + \vec{\tau}_F = I_{/\Delta} \ddot{\vec{\theta}}$$

$$\vec{\tau}_s = \vec{R}_s \wedge \vec{F}_s \Rightarrow \tau_s = -\, l k x' \sin(\tfrac{\pi}{2} + \theta) = -\, l k x' \cos(\theta)$$

$$\vec{\tau}_D = \vec{R}_D \wedge \vec{F}_D \Rightarrow \tau_D = -\, 2l \cdot b \cdot \dot{x}'' \sin(\tfrac{\pi}{2} + \theta) = -\, 2l \cdot b \cdot \dot{x}'' \cos$$

$$\vec{\tau}_F = \vec{R}_F \wedge \vec{F} \Rightarrow \tau_F = +\, 3l \cdot F \cdot \sin(\tfrac{\pi}{2} + \theta) = +\, 3l \cdot F \cdot \cos(\theta)$$

for small oscillations $\Rightarrow \cos(\theta) \simeq 1$ and $\sin(\theta) = \frac{x'}{l} = \frac{x''}{2l} = \theta$

$$\Rightarrow \frac{\dot{x}''}{2l} = \dot{\theta} \Rightarrow \dot{x}'' = 2l\dot{\theta}$$

we replace in our diff-equ, we get:

$$-\, l k (l\theta) - 2l b (2l\dot{\theta}) + 3l F = m (3l)^2 \ddot{\theta}$$

where $I_{/\Delta} = m D^2 = m(3l)^2$

$$\ddot{\theta} + \frac{4}{9}\frac{b}{m}\dot{\theta} + \frac{1}{9}\frac{k}{m}\theta = \frac{F_0}{3lm}\cos(\omega_f t)$$

$$\ddot{\theta} + \gamma\dot{\theta} + \omega_0^2\theta = \frac{F_0}{3lm}\cos(\omega_f t)$$ / $\gamma = \frac{4}{9}\frac{b}{m}$ and $\omega_0 = \sqrt{\frac{1}{9}\frac{k}{m}}$

N.A: $\gamma = 0,44\,s^{-1}$, $\omega_0 = 2,98\,rd/s$, $\omega_f = 2\pi\,rd/s$

$\Rightarrow$ The steady state response is $\theta(t) = \theta(\omega_f)\cdot\cos(\omega_f t - \delta)$

$$\theta(\omega_f) = \frac{F_0/3ml}{[(\omega_0^2-\omega_f^2)^2+\gamma^2\omega_f^2]^{1/2}} = 0,44\,rd.$$

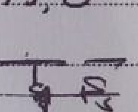
$\tan(\delta) = \frac{\gamma\omega_f}{(\omega_0^2-\omega_f^2)} \Rightarrow \delta = 5,16^\circ$ or $174,84^\circ$ but we have $\omega_f > \omega_0 \Rightarrow \delta = 174,84^\circ$.

$$\theta(t) = 0,44\cdot\cos(2\pi t - 3,05)\,rd$$

EXO#4 :

$m = 2kg$, $\Delta x_0 = 2,50\,cm$, $a = 1,00\,m$, $Q = 15,0$

a) $\omega_0 = ?$ $\omega_0 = \sqrt{\frac{k}{m}}$,



at equilibrium $\Sigma\vec{F} = 0 \Rightarrow \vec{F}_s + \vec{W} = 0 \Rightarrow -k\Delta x_0 + mg = 0$.

$\Rightarrow k = \frac{mg}{\Delta x_0} = 784\,N.m^{-1} \Rightarrow \omega_0 = 19,8\,rd/s$

b) The amplitude:

at resonance: $\Rightarrow A(\omega_0) = aQ = 10^{-3} \times 15 \Rightarrow A(\omega_0) = 1,5\,cm$

c) The power:

$\bar{P}(\omega_f)$ / $\omega_f = \omega_0 + 2\%\,\omega_0 = \omega_0 + 0,02\,\omega_0 = 20,2\,rd/s$

and so $\bar{P}(\omega_f) = 0,086\,W$

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Ex 04 ⇒

$m = 0{,}03\,kg$, $F_0 = 0{,}12\,N$, $\omega_f = 20\,rd/s$, $b = 0{,}06\,kg/s$

$x(t) = A\sin(20t)$

. after a certain time $x(t) = A(\omega_f).\cos(\omega_f t - \delta)$

$\Rightarrow x(t) = A(\omega).\cos(20t - \frac{\pi}{2})$.

a) $A(\omega) = ?$: from the equ of displacement $x(t)$ we get that $\omega_f = 20\,rd/s$ and $\delta = \frac{\pi}{2}$, so ⇒ the system is at resonance because $\delta = \frac{\pi}{2}$.

$\Rightarrow A(\omega_f = \omega_0) = a.Q = a.\frac{\omega_0}{\gamma} = \frac{F_0}{k}.\frac{\omega_0}{\gamma} = \frac{F_0}{m}.\frac{1}{\gamma\omega_0}$

$A(\omega) = 10\,cm$.

b) $E = ?$: $E(t) = \frac{1}{2}kx^2 + \frac{1}{2}m\dot{x}^2$

$= \frac{1}{2}kA^2(\omega_f)\cos^2(20t - \frac{\pi}{2}) + \frac{1}{2}mA^2(\omega)\omega_f^2\sin^2(20t - \frac{\pi}{2})$

$E(t) = \frac{1}{2}A^2(\omega).m\left[\omega_0^2\cos^2(20t - \frac{\pi}{2}) + \omega_f^2\sin^2(20t - \frac{\pi}{2})\right]$

we are in the case of resonance $\Rightarrow \omega_f = \omega_0 \Rightarrow$

$E(t) = \frac{1}{2}mA^2(\omega_0)\omega_0^2 = 6\,10^{-2}\,J$ → stored energy

c) $\overline{E}_{dis} = ?$ / $\overline{P} = \frac{1}{T}\int_0^T P.dt$ / $P_{dis} = -b\dot{x}^2 \Rightarrow$

$\overline{P}_{dis} = -\frac{bA^2(\omega).\omega^2}{T}\int_0^T \sin^2(20t - \frac{\pi}{2})dt = -\frac{bA^2(\omega).\omega_f^2}{T}.\frac{T}{2}$

$\overline{P}_{dis} = -\frac{bA^2(\omega_0).\omega_0^2}{2} = -0{,}12\,W$

or we use the expression of $\overline{P}(\omega) = \frac{F_0^2}{2m}.\frac{\gamma\omega_f^2}{[(\omega_0^2 - \omega_f^2)^2 + \gamma^2\omega_f^2]}$

at resonance: $\overline{P}(\omega_0) = \frac{F_0^2}{2m}.\frac{F_0^2}{\gamma} = 0{,}12\,W$

$\Rightarrow \overline{P}_{dis}(\omega_0) = -0{,}12\,W$ $\Rightarrow \overline{P}_{dis} = \frac{\overline{E}_{dis}}{T} \Rightarrow$

$\overline{E}_{dis} = \overline{P}.\frac{2\pi}{\omega_0} = -0{,}038\,J$ (at resonance)

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d) $\frac{\bar{E}_{di}}{E} = ?$ we have $\bar{E}_{dis} = \bar{P}_{dis} \cdot T \Rightarrow$

$$\bar{E}_{dis} = \frac{bA^2(\omega_0)\cdot\omega_0^2}{2} \cdot \frac{2\pi}{\omega_0} = bA^2(\omega_0)\cdot\omega_0\pi$$

$$E = \frac{1}{2}A^2(\omega_0)\cdot\omega_0^2\cdot m \Rightarrow$$

$$\frac{\bar{E}_{di}}{E_{st}} = \frac{2b}{m}\cdot\frac{\pi}{\omega_0} = 2\pi\cdot\frac{\gamma}{\omega_0} = \frac{2\pi}{Q} = 0{,}2\pi$$

<u>Exo # 6</u>

$\Rightarrow$ lower frequencies $\Rightarrow \omega_f \ll \omega_0 \Rightarrow A(\omega_0) = a = 0{,}01\,mm$

At resonance (because the experiment is done for resonance

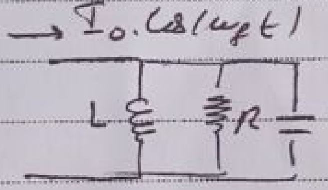
$\Rightarrow \omega_f \simeq \omega_0 \Rightarrow A(\omega_0) = aQ = 5\,mm$ and so;

$$f_0 = 250\,Hz \Rightarrow \omega_0 = 2\pi f_0 = 500\pi\ rad/s$$

a) $Q = ?$ $\quad A(\omega_0) = aQ = 5\,mm \Rightarrow Q = 500$

b). $BW = ?$ $\quad BW = \gamma = \frac{\omega_0}{Q} = \frac{500\pi}{500} \Rightarrow BW = \pi s^{-1}$

<u>Exo # 7</u>



$V(t) = ?$;

Kirchhoff's law for current:

$$I_L + I_R + I_C = I(t) = I_0\cos(\omega_f t)$$

$$\begin{cases} V_L = L\frac{dI_L}{dt} \Rightarrow I_L = \frac{1}{L}\int_0^t V_L\,dt \\ V_R = RI_R \Rightarrow I_R = \frac{V_R}{R} \\ V_C = \frac{q}{C} \text{ and } I_C = \frac{dq}{dt} = C\frac{d}{dt}(q/C) = C\frac{d}{dt}V_C \end{cases}$$

we replace in our equation, we get:

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$$\frac{1}{L}\cdot\int V_L\,dt + \frac{V_R}{R} + C\frac{d}{dt}V_c = I_0 \cdot \cos(\omega_f t) \quad (*)$$

$V_R = V_c = V_L = V$ (parallel circuit) , we differentiate the equ (*) we get

$$\frac{1}{L}V + \frac{1}{R}\frac{dV}{dt} + C\frac{d^2}{dt^2}V = -I_0\,\omega_f \sin(\omega_f t)$$

$$\Rightarrow C\ddot{V} + \frac{1}{R}\dot{V} + \frac{1}{L}V = -I_0 \cdot \omega_f \sin(\omega_f t).$$

$$\ddot{V} + \frac{1}{RC}\dot{V} + \frac{1}{LC}V = -\frac{I_0}{C}\cdot\omega_f \cdot \sin(\omega_f t)$$

b). from the diff - equ , we have:

$$\gamma = \frac{1}{RC}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \quad Q = \frac{\omega_0}{\gamma} \Rightarrow$$

$$Q = \frac{1}{\sqrt{LC}}\cdot RC = R\sqrt{\frac{C}{L}}$$

$$BW = \gamma = \frac{1}{RC}$$

c) $\overline{P}(\omega_0) = ?$

$$\overline{P}(\omega_f) = \frac{1}{2m}\frac{F_0^2 \cdot \gamma\omega_f^2}{[(\omega_0^2 - \omega_f^2)^2 + \gamma^2\omega_f^2]}$$ in mechanical system.

$$\overline{P}_{mca}(\omega_0) = \frac{1}{2m}\cdot\frac{F_0^2}{\gamma}$$ by identification with the electrical system.

$F_0 \equiv I_0$, $m \equiv C$, $k = \frac{1}{L}$, $b \equiv \frac{1}{R}$ $\Rightarrow$

$$\overline{P}_{mca}(\omega_0) = \frac{I_0^2}{2C}\cdot R \cdot C \Rightarrow \overline{P}_{mca}(\omega_0) = \frac{1}{2}I_0^2 R$$