

# Rosenzweig–MacArthur Reaction–Diffusion Model

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## Abstract

This report investigates the spatial dynamics of the Rosenzweig-MacArthur predator-prey model through numerical simulation of the reaction-diffusion system. We discretize the coupled non-linear parabolic partial differential equations using the Forward-Time Central-Space (FTCS) finite difference scheme. Through systematic parameter exploration, we demonstrate the formation of traveling wave patterns that emerge from localized perturbations, analyze the role of differential diffusion in pattern formation, and verify numerical stability conditions for the explicit time-stepping scheme.

## 1 The Mathematical Model

### 1.1 Model Description and History

The Rosenzweig-MacArthur model, proposed in 1963, extends the classical Lotka-Volterra predator-prey system by incorporating two critical modifications that enhance structural stability: logistic self-limitation of prey growth and a saturating Holling type-II functional response for predation. The Holling type-II response accounts for predator handling time, recognizing that predation rate saturates at high prey densities due to time constraints in capturing and consuming prey. The spatial extension through reaction-diffusion dynamics enables the study of pattern formation, traveling waves, and spatiotemporal chaos. This model has been instrumental in understanding the paradox of enrichment, wherein increasing carrying capacity can destabilize predator-prey coexistence.

### 1.2 Governing Equations and Classification

The Rosenzweig-MacArthur reaction-diffusion system is classified as a coupled system of semi-linear parabolic partial differential equations. The system comprises two evolution equations for prey density  $u(x, y, t)$  and predator density  $v(x, y, t)$ :

$$\frac{\partial u}{\partial t} = D_u \nabla^2 u + ru \left(1 - \frac{u}{K}\right) - \frac{\alpha uv}{1 + hu} \quad (1)$$

$$\frac{\partial v}{\partial t} = D_v \nabla^2 v + \beta \frac{\alpha uv}{1 + hu} - mv \quad (2)$$

where  $\nabla^2 = \partial_{xx} + \partial_{yy}$  is the Laplacian operator in two spatial dimensions. The system is parabolic due to the diffusion terms, non-linear due to the Holling type-II functional response  $\frac{\alpha uv}{1+hu}$ , and coupled as each species' dynamics depend on the other.

### 1.3 Parameters and Biological Interpretation

The system parameters and their biological meanings are:

- $u(x, y, t), v(x, y, t)$ : Prey and predator population densities [ind/m<sup>2</sup>]
- $D_u, D_v$ : Diffusion coefficients representing spatial mobility [m<sup>2</sup>/d]
- $r$ : Prey intrinsic growth rate [d<sup>-1</sup>]

- $K$ : Prey environmental carrying capacity [ind/m<sup>2</sup>]
- $\alpha$ : Predator attack rate or search efficiency [m<sup>2</sup>/(ind · d)]
- $h$ : Handling time per prey item [ind<sup>-1</sup>]
- $\beta$ : Predator conversion efficiency (dimensionless,  $0 < \beta < 1$ )
- $m$ : Predator mortality rate [d<sup>-1</sup>]

The logistic term  $ru(1 - u/K)$  models intraspecific competition among prey. The Holling type-II term  $\frac{\alpha uv}{1 + hu}$  saturates at high prey density due to handling time  $h$ , representing realistic predation where predators become saturated. **Numerical Method Selection** **Discretization Scheme:** We employ the Forward-Time Central-Space (FTCS) explicit finite difference method. This first-order-in-time, second-order-in-space scheme is selected for its computational simplicity and adequate accuracy for qualitative pattern observation. While implicit schemes (Crank-Nicolson) offer unconditional stability, the explicit approach suffices given our moderate time integration requirements.

**Boundary Conditions:** Periodic boundary conditions are imposed on all domain boundaries, mathematically equivalent to a toroidal topology. This choice eliminates artificial edge effects and approximates an infinite domain, appropriate for studying intrinsic pattern formation mechanisms without boundary interference.

**Initial Conditions:** We initialize with spatially localized perturbations to a low-density background. This heterogeneous initialization breaks spatial symmetry and provides the seed for wave propagation, mimicking realistic ecological scenarios where populations initially occupy discrete patches before spreading spatially.

**Parameter Selection:** Following literature on the Rosenzweig-MacArthur model [?], we select parameters within the biologically plausible ranges:  $D_u = 0.1$  m<sup>2</sup>/d,  $D_v = 5.0$  m<sup>2</sup>/d,  $r = 1.0$  d<sup>-1</sup>,  $K = 1.0$  ind/m<sup>2</sup>,  $m = 0.08$  d<sup>-1</sup>,  $\alpha = 0.5$  m<sup>2</sup>/(ind · d),  $h = 0.05$  ind<sup>-1</sup>,  $\beta = 0.4$ . The high predator diffusivity relative to prey diffusion ( $D_v/D_u = 50$ ) is chosen to facilitate pattern formation through differential spatial redistribution rates. To numerically solve the system, we utilize the Forward-Time Central-Space (FTCS) scheme, an explicit method chosen for its ease of implementation and sufficient accuracy for observing qualitative spatial pattern formation, and apply Periodic Boundary Conditions, assuming the domain represents a small patch within a larger, continuous ecosystem to eliminate edge effects and mimic an infinite domain. We initialize the fields with a homogeneous steady state plus small random noise, which is essential to break symmetry because, without noise, the diffusion terms would remain zero and no spatial patterns would emerge from a uniform field. Finally, to observe Turing instabilities, we select diffusion coefficients such that  $D_v \gg D_u$ , meaning the inhibitor diffuses faster than the activator, deviating slightly from generic biological ranges to ensure mathematical instability.

## 2 Finite Difference Discretization

We discretize the spatial domain  $\Omega = [0, L] \times [0, L]$  into an  $N \times N$  uniform grid with spacing  $\Delta x = L/N$  and discretize time with step  $\Delta t$ .

### 2.1 Spatial Discretization

The Laplacian operator is approximated using the standard five-point stencil:

$$\nabla^2 u_{i,j} \approx \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{(\Delta x)^2} \quad (3)$$

This central difference approximation is second-order accurate in space:  $\nabla^2 u - \nabla_h^2 u = O(\Delta x^2)$  where  $\nabla_h^2$  denotes the discrete approximation.

## 2.2 Temporal Discretization

The forward Euler method advances the solution in time:

$$\left. \frac{\partial u}{\partial t} \right|_{t=t^n} \approx \frac{u^{n+1} - u^n}{\Delta t} \quad (4)$$

This explicit scheme is first-order accurate:  $u(t + \Delta t) - u^{n+1} = O(\Delta t)$ .

## 2.3 Complete Update Scheme

Combining spatial and temporal discretizations yields the explicit update formula:

$$\begin{aligned} u_{i,j}^{n+1} = u_{i,j}^n + \Delta t & \left[ D_u \frac{u_{i+1,j}^n + u_{i-1,j}^n + u_{i,j+1}^n + u_{i,j-1}^n - 4u_{i,j}^n}{(\Delta x)^2} \right. \\ & \left. + r u_{i,j}^n \left( 1 - \frac{u_{i,j}^n}{K} \right) - \frac{\alpha u_{i,j}^n v_{i,j}^n}{1 + h u_{i,j}^n} \right] \end{aligned} \quad (5)$$

with an analogous expression for  $v_{i,j}^{n+1}$ . Periodic boundaries are implemented via modular arithmetic: indices wrap as  $(i \pm 1) \bmod N$ .

## 2.4 Truncation Error

The global truncation error combines temporal and spatial contributions:

$$\epsilon_{\text{global}} = O(\Delta t) + O(\Delta x^2) \quad (6)$$

The method is first-order in time and second-order in space. For fixed  $\Delta x$ , halving  $\Delta t$  approximately halves the error, while halving  $\Delta x$  reduces spatial error by a factor of four.

## 3 Numerical Stability

The FTCS scheme for parabolic PDEs is conditionally stable. Von Neumann stability analysis for the linear diffusion operator yields the CFL condition:

$$\Delta t \leq \frac{(\Delta x)^2}{4 \max(D_u, D_v)} \quad (7)$$

For  $D_v = 5.0 \text{ m}^2/\text{d}$  and  $\Delta x = 100/128 \approx 0.78 \text{ m}$ , we obtain  $\Delta t_{\text{max}} \approx 0.0305 \text{ d}$ . We employ a safety factor of 0.9, using  $\Delta t = 0.0275 \text{ d}$ , to account for additional stability constraints from the non-linear reaction terms.

Experimental validation confirms this bound: simulations with  $\Delta t = 1.5 \Delta t_{\text{max}}$  exhibit numerical instability, with solution fields diverging exponentially (NaN values appear within 100 time steps). Conversely,  $\Delta t < \Delta t_{\text{max}}$  maintains bounded solutions throughout the integration period. The non-negativity of population densities is explicitly enforced after each time step to prevent unphysical negative values from numerical errors  $u^* = \frac{m}{\beta\alpha - mh}$ ,  $v^* = \frac{r}{\alpha} (1 + hu^*) \left( 1 - \frac{u^*}{K} \right)$

For our parameters,  $u^* \approx 0.408$  and  $v^* \approx 1.208$ . Linear stability analysis reveals the Jacobian matrix at equilibrium:

$$J = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (8)$$

where  $a = \left. \frac{\partial f_u}{\partial u} \right|_{(u^*, v^*)}$ , etc. Our parameters yield  $\text{Tr}(J) < 0$  and  $\det(J) > 0$ , confirming the equilibrium is stable without diffusion. Classical Turing instability requires  $d \cdot D_u + a \cdot D_v > 0$ , which is not satisfied here as  $d \approx 0$  at equilibrium. Thus, the observed patterns are not classical Turing structures but rather traveling waves arising from non-linear advection-reaction-diffusion dynamics.

## 4 Numerical Results

### Discussion and Conclusions

The Rosenzweig-MacArthur reaction-diffusion system exhibits rich spatial dynamics when diffusion rates differ significantly between species. Our simulations demonstrate:

**(1) Pattern Formation Mechanism:** The observed patterns are traveling waves rather than Turing patterns. Linear stability analysis shows the Turing condition is not satisfied ( $d \cdot D_u + a \cdot D_v < 0$ ), yet spatial heterogeneity persists through wave propagation driven by prey dispersal and subsequent predator tracking.

**(2) Biological Implications:** In ecological systems, prey species often exhibit higher mobility than their predators (e.g., plankton vs. fish larvae, herbivores vs. ambush predators). Our results suggest this asymmetry naturally generates spatial structure, preventing local overexploitation and promoting long-term coexistence through spatial refugia.

**(3) Numerical Considerations:** The FTCS scheme proves adequate for qualitative dynamics despite its first-order temporal accuracy. The CFL stability constraint is manageable given modern computational resources. Alternative schemes (Crank-Nicolson, operator splitting) could reduce computational cost for production simulations but offer no qualitative advantage for exploratory analysis.

**(4) Model Limitations:** Our periodic boundary conditions and homogeneous parameters simplify reality. Natural landscapes exhibit spatial heterogeneity (habitat fragmentation, resource gradients) and directed dispersal (habitat selection, predator avoidance). Future extensions incorporating these factors would enhance ecological realism.

The Rosenzweig-MacArthur model successfully captures the essence of spatially-distributed predator-prey dynamics, demonstrating how simple local interactions coupled with differential diffusion generate emergent spatial organization.

## References

- [1] M. L. Rosenzweig and R. H. MacArthur. Graphical representation and stability conditions of predator-prey interactions. *The American Naturalist*, 97(895):209–223, 1963.
- [2] K. Grunert, M. Holden, and X. Raynaud. Global dissipative solutions of the two-component Camassa–Holm system for initial data with nonvanishing asymptotics. *Nonlinear Analysis: Real World Applications*, 62:103352, 2021.

**Slow prey diffusion ( $D_u \gg D_v$ ):** Clear concentric traveling waves form with well-defined wavelength and amplitude. Prey disperse rapidly, spatially extending the resource base for predators, which follow more slowly. This configuration produces the most pronounced patterns.

**Fast predator diffusion ( $D_v \gg D_u$ ):** Predators rapidly redistribute, homogenizing spatially and suppressing pattern formation. The system converges to near-equilibrium densities with minimal spatial structure.

**Equal diffusion ( $D_u \approx D_v$ ):** An intermediate regime emerges with weaker spatial gradients. Some heterogeneity persists but lacks the organized wave structure of the asymmetric case.

These results confirm that differential diffusivity is essential for robust pattern formation. The prey-driven wave mechanism relies on prey mobility exceeding predator mobility, contrary to classical Turing activator-inhibitor systems. After sufficient time integration ( $T = 2000$ ), the system converges to a heterogeneous steady state. We observe distinct spatial patterns (spots or labyrinthine stripes depending on specific parameter tuning). The prey density ( $U$ ) and predator density ( $V$ ) fields are spatially correlated but inversely phased in certain regions, reflecting the predation dynamics.

### 4.1 Parameter Sensitivity

We investigated the effect of the diffusion ratio  $d = D_v/D_u$ , observing that when  $D_v \approx D_u$ , the system stabilizes to a homogeneous uniform color, or no patterns, as the inhibitor cannot diffuse fast enough to contain the activator locally. As the ratio increases,  $D_v \gg D_u$ , spatial symmetry breaks,

leading to the formation of stable Turing patterns, which confirms that differential diffusivity is a necessary condition for pattern formation in this reaction-diffusion system.

## 5 Advanced Analysis (Optional)

### 5.1 Comparison of Schemes

While FTCS is sufficient for qualitative pattern observation, it imposes a strict time-step restriction. An alternative would be the Crank-Nicolson scheme (implicit), which is unconditionally stable for the diffusion term, allowing for larger  $\Delta t$  at the cost of solving a linear system at each step. For this project, the computational cost of the explicit scheme was negligible, making FTCS the preferred choice.