Metric Spaces and Topology

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[1] *based on lectures by Jerry Buckley. These notes, however, have been altered strongly after the lectures. In particular, some proofs were filled in by me; and some of the contents were adjusted so that organization is prioritized and things are coherent. All errors are surely mine, feel free to email me if you spot any.*

Contents

Short note. It is highly recommended for readers who lack elementary set theory knowledge to check out Appendix A and Appendix B first. These will be assumed throughout. For those unfamiliar with the topology on $\mathbb{R}$ or simply lack motivation to study this rich subject should check out Appendix C. Note that the main goal of these appendices is to not prove results, but rather state facts that you should know so that reading this piece of notes is 10x less painful.

References. Together with the lectures, the following books were useful references in the making of this notes:

(1). S. Lipschutz, *Schaum's Outline of General Topology*, McGraw-Hill Education.

(2). W. A. Sutherland, *Introduction to Metric and Topological Spaces*, Oxford University Press.

(3). J. Munkres, *Topology*, Pearson.

(4). M. O'Searcoid, *Metric Spaces*, Springer-Verlag London.
(mostly for G-domains)

(5). S. A. Morris, *Topology Without Tears*.

1 Topological Spaces

1.1 WTF is a topology?

Definition 1.1 (Topology). Let X be a non-empty set. A collection $\mathcal{T}$ of subsets of X is a **topology** on X if $\mathcal{T}$ satisfies the following axioms.

[**T1**]. X and $\varnothing$ belongs to $\mathcal{T}$,

[**T2**]. The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$,

[**T3**]. The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The members of $\mathcal{T}$ are called **$\mathcal{T}$-open sets** or just **open sets**. The set X together with the topology $\mathcal{T}$ is called a **topological space** and denoted as the pair $(X, \mathcal{T})$.

saying $U \in \mathcal{T}$ and "U is open in X" means the same thing.

Remark. The axioms [**T1**], [**T2**] and [**T3**] are equivalent to the following two axioms:

[**T1***]. The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$,

[**T2***]. The intersection of any finite number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

We will prove ($\Longleftarrow$). Let $\{U_i\} \subset \mathcal{T}$ be an indexed collection of subsets. By [**T1***], we have that

$$\varnothing = \bigcup_{i \in \varnothing} U_i \in \mathcal{T}$$

i.e. *the empty union of sets is the empty set* and by [**T2***] we have that

$$X = \bigcap_{i \in \varnothing} U_i \in \mathcal{T}$$

i.e. *the empty intersection of subsets of X is X itself.* The ($\Longrightarrow$) direction can be done by induction.

We now look at some examples.

1.1.1 Sanity-checker topologies

Any set can be given a topology; this is a truth.

Example (Discrete Topology). Let $\mathcal{P}(X)$ denotes the collection of all subsets of X. Then, $\mathcal{P}(X)$ is a topology on X and is called the **discrete topology** on X; the resulting space is called a **discrete space**.

Example (Indiscrete Topology). Define the collection $\mathcal{I} = \{X, \varnothing\}$. Then, $\mathcal{I}$ is a topology on X and is called the **indiscrete topology** on X ; the resulting space is called an **indiscrete space**.

or sometimes known as the trivial topology.

1.1.2 Topologies on $\mathbb{R}$

Let's look at some examples of topologies on $\mathbb{R}$ and its subsets.

Example (Usual topology on $\mathbb{R}$). Let $\mathcal{U}$ denote the collection of all open sets of the real numbers (as discussed in Appendix C). Then $\mathcal{U}$ is a topology on $\mathbb{R}$ and it is called the **usual topology** or **Euclidean topology** on $\mathbb{R}$.

the usual topology on $\mathbb{R}$ is the topology whose open sets in $\mathbb{R}$ are characterised by each of its points being an interior point.

There are many more topologies that can be given on $\mathbb{R}$ such as the lower (and upper) limit topology and the lower semi-continuous topology. For subsets of $\mathbb{R}$ such as $\mathbb{Z}$, it can be given the evenly spaced integer topology whose resulting space, called the Furstenberg integers, can be used to prove the infinitude of primes numbers. However, all these topologies requires the notion of bases of topology which would not be discussed until the end of this chapter.

1.1.3 Topologies on finite sets

We now look at topologies on finite sets.

Example (Topology on $\varnothing$ and singletons). There is a unique topology on the empty set $\varnothing$. It is the collection $\mathcal{T} = \{\varnothing\}$. Similarly, there is a unique topology on a singleton set $\{x\}$. It is the collection $\mathcal{T} = \{\varnothing, \{x\}\}$. These topologies are both discrete and trivial.

Example (Topology on two-point sets). Let $X = \{a, b\}$. Then there are four distinct topologies on X:

(1). $\{\varnothing, X\}$ (indiscrete topology)

(2). $\{\varnothing, \{a\}, X\}$

(3). $\{\varnothing, \{b\}, X\}$

(4). $\{\varnothing, \{a\}, \{b\}, X\}$ (discrete topology)

It is not too hard to convince oneself that (2) and (3) are really the *same* of some sort. We will see that there is a definition of *sameness* called a *homeomorphism* and that indeed (2) and (3) are *homeomorphic*, whatever that means for now.

X equipped with either (2) or (3) is called a **Sierpinski space** and is denoted $\mathbb{S}$. We can thus see that there are only three inequivalent topologies on a two-point set: the discrete one, the indiscrete one, and the Sierpinski topology.

Note that the Sierpinski space is immensely useful when finding counter-examples to seemingly plausible claims on topological spaces. Sometimes, our minds are too focused on things that we can imagine (like $\mathbb{R}$ with the usual topology) that we disrespect the definitions that we made. Aside from the discrete and indiscrete space, here we add another sanity-checker into our arsenal.

Example (Favourite *abcde* topology). Consider the following collection of subsets of $X = \{a, b, c, d, e\}$.

(i). $\mathcal{F} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$

(ii). $\mathcal{T}_2 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$

(iii). $\mathcal{T}_3 = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{a, b, d, e\}\}$

$\mathcal{F}$ is a topology on X and will be called our **favourite *abcde* topology** or just **favourite topology** on X for this notes. $\mathcal{T}_2$ is not a topology on X since

$$\{a, c, d\} \cup \{b, c, d\} = \{a, b, c, d\} \notin \mathcal{T}_2$$

$\mathcal{T}_3$ is not a topology on X since

$$\{a, c, d\} \cap \{a, b, d, e\} = \{a, d\} \notin \mathcal{T}_3$$

1.1.4 Exotic topologies

Here we give more exotic topologies, those which one would not immediately think of.

Example (Cofinite Topology). Let X be a non-empty set. Let $\mathcal{C}$ denote the collection of subsets of X whose complements are finite together with the empty set $\varnothing$ i.e. $\mathcal{C} = \{U \subset X : U = \varnothing \text{ or } U^c \text{ is finite}\}$. Then, $\mathcal{C}$ is a topology on X. It is called the **cofinite topology** or the **T_1-topology** or the *finite complement topology* on X. The pair $(X, \mathcal{C})$ is called a **cofinite space**.

if X is an infinite set, then the members of $\mathcal{C}$ are either $\varnothing$ or infinite sets.

Example (Cocountable Topology). Let X be a non-empty set. Let $\mathfrak{C}$ denote the collection of subsets of X whose complements are at most countable (finite or countably infinite) together with the empty set $\varnothing$ i.e. $\mathfrak{C} = \{U \subset X : U = \varnothing \text{ or } U^c \text{ is at most countable}\}$. Then, $\mathfrak{C}$ is a topology on X. It is called the **cocountable topology** or *countable complement topology* on X. The pair $(X, \mathfrak{C})$ may be referred to as a **cocountable space**.

if X is an infinite set, then the members of $\mathcal{C}$ are either $\varnothing$ or uncountable sets.

Remark. If the set X is finite, both cofinite topology and cocountable topology turns X into a discrete space.

1.1.5 Topologies from functions

Finally, we give here two examples of getting topologies from functions.

Example (Topology induced on X by f). Let (Y, τ) be a topological space and let X be a non-empty set. Let $f : X \to Y$ be a function. Define $\mathcal{T}$ to be the collection of inverses of open subsets of Y:

$$\mathcal{T} = \{f^{-1}(U) : U \in \tau\}$$

Then, $\mathcal{T}$ is a topology on X and is called the **topology induced on X by f**.

Example (Topology coinduced on Y by f). Let $(X, \mathcal{T})$ be a topological space and let Y be any set. Then the collection of subsets

$$\tau_f = \{U \subset Y \mid f^{-1}(U) \text{ is open in } X\}$$

is a topology on Y and is called the **topology coinduced on Y by f**.

1.1.6 Intersection of topologies

Proposition 1.1. The intersection $\mathcal{T}_1 \cap \mathcal{T}_2$ of any two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X is also a topology on X.

Proof. We need to show $\mathcal{T}_1 \cap \mathcal{T}_2$ satisfies all **[T1]**, **[T2]** and **[T3]**.

[**T1**]. Since X and $\varnothing$ belongs to $\mathcal{T}_1$ and $\mathcal{T}_2$, it must belong to $\mathcal{T}_1 \cap \mathcal{T}_2$.

[**T2**]. Suppose $\{A_i\}$ is some indexed collection of subsets of $\mathcal{T}_1 \cap \mathcal{T}_2$. Then, $\{A_i\} \subset \mathcal{T}_1$ and $\{A_i\} \subset \mathcal{T}_2$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, the union of any arbitrary sets A_i belongs to $\mathcal{T}_1$ and likewise belongs to $\mathcal{T}_2$. Together, such union belongs to $\mathcal{T}_1 \cap \mathcal{T}_2$.

[**T3**]. Suppose $A, B \in \mathcal{T}_1 \cap \mathcal{T}_2$. Then, $A, B \in \mathcal{T}_1$ and $A, B \in \mathcal{T}_2$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, $A \cap B \in \mathcal{T}_1$ and $A \cap B \in \mathcal{T}_2$. This implies that $A \cap B \in \mathcal{T}_1 \cap \mathcal{T}_2$. ■

There's nothing stopping us from generalizing this.

Theorem 1.1. Let $\{\mathcal{T}_i : i \in I\}$ be any collection of topologies on a set X. Then, the intersection $\bigcap_i \mathcal{T}_i$ is also a topology on X.

Unions as usual are badly-behaved. Unions of topologies need not be a topology as demonstrated in the example below.

Example. Let $X = \{a, b, c\}$ and define $\mathcal{T}_1 = \{X, \varnothing, \{a\}\}$ and $\mathcal{T}_2 = \{X, \varnothing, \{b\}\}$. Then, it is not too difficult to see that these are topologies on X. However,

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{X, \varnothing, \{a\}, \{b\}\}$$

is not a topology on X as it violates [**T2**]. Take $\{a\}, \{b\} \in \mathcal{T}_1 \cup \mathcal{T}_2$. Now, it is "very difficult" to see that $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

1.1.7 Topologies on infinite sets

We will state the following lemma without prove. It is more a set-theoretic problem rather than a topological problem but we will require it for the upcoming result.

Lemma 1.2. Let X be an infinite set, then it has a countably infinite subset.

Lemma 1.3. Let X be an infinite set. Then, there exist infinite sets $A, B \subset X$ such that $A \cap B = \varnothing$.

Proof. We give a constructive proof. Suppose $\{s_n : n \in \mathbb{N}\}$ is a subset of X that is countably infinite (we know this exist by the preceding lemma). Now, define $A = \{s_{2n} : n \in \mathbb{N}\}$ and $B = \{s_{2n-1} : n \in \mathbb{N}\}$. Then clearly A and B are also countably infinite and their intersection $A \cap B = \varnothing$. ■

Theorem 1.4. Let X be an **infinite set** and $\mathcal{T}$ be a topology on X. If every infinite subset of X is open, then $\mathcal{T}$ is the discrete topology on X.

N.B. It does not say that every infinite subset are the **only** open subsets of X, rather all infinite subsets are open, there may be finite subsets that is open. If **only** infinite subsets of X are open, then $\mathcal{T}$ is not a topology because $\varnothing \notin \mathcal{T}$. If $\varnothing$ is throw in also here, we recover the cofinite topology.

Proof. *Idea: It suffices to prove that singletons $\{x\}$ are open, for all $x \in X$ as every subset of X can be written as union of singletons.*

Since X is an infinite set, then $X \backslash \{x\}$ is also an infinite set. So by the preceding lemma, there exist infinite sets $A, B \subset (X \backslash \{x\})$ such that $A \cap B = \varnothing$. Now, $A \cup \{x\}$ and $B \cup \{x\}$ are infinite subsets of X so they are open. Therefore, their union must also be open. But, observe that

$$(A \cup \{x\}) \cap (B \cup \{x\}) = \big[(A \cap B) \cup (B \cap \{x\})\big] \cup \big[(A \cap \{x\}) \cup (\{x\} \cap \{x\})\big] = \{x\}$$

The last equality here is true because $A \cap B = B \cap \{x\} = A \cap \{x\} = \varnothing$ and $\{x\} \cap \{x\} = \{x\}$.

So, $\{x\}$ is open in X as well. Now, take any arbitrary subset $A \subset X$. Then,

$$A = \bigcup_{x \in A} \{x\}$$

which is a union of open sets and hence is open. ∎

1.1.8 Neighbourhoods

We now encounter the notion of neighbourhoods, an important concept of which we will use and abuse.

Definition 1.2 (Open Neighbourhood). Let X be a topological space and $x \in X$. If $U \subset X$ is an open set such that $x \in U$, then U is called an **open neighbourhood** of x.

Definition 1.3 (Deleted Open Neighbourhood). Let X be a topological space and $x \in X$. If $U \subset X$ is an open set such that $x \in U$, then $U \backslash \{x\}$ is called a **deleted open neighbourhood** of x.

Definition 1.4 (Neighbourhood). Let X be a topological space. A subset of X containing an open neighbourhood is called a **neighbourhood**.

Remark. More formally, let X be a topological space and $x \in X$. Then $A \subset X$ is a neighbourhood of x if $x \in U \subset A$ for some open set U of X.

We will discuss about neighbourhoods in more detail in a subsection dedicated to it. The definition that will appear in that subsection is completely identical to our current definition.

Note that A here need not be open. Note also that U is open in X. We don't have a notion of being open in some subset of a topological space... yet!

1.2 Limit points

Definition 1.5 (Limit Point). Let X be a topological space. A point $x \in X$ is a **limit point** of a subset $A \subset X$ if **every** open set U containing x contains a point of A different from x i.e.

$$\forall U \text{ open and } x \in U,\ \exists y \in U,\ y \neq x \text{ such that } y \in A$$

or equivalently

$$\forall U \text{ open and } x \in U \implies (U \backslash \{x\}) \cap A \neq \varnothing$$

The set of limits points of A will be denoted A' and is called the **derived set** of A.

using the language of open neighbourhoods, x is a limit point of A if every open neighbourhood of x contains a point of A distinct from x.

A more descriptive definition is that if U is an open neighbourhood of x, then its deleted neighbourhood about x intersected with A is non-empty.

Remark. Limit points of A need not be in A.

Remark. The derived set of the empty set is the empty set itself i.e. $\varnothing' = \varnothing$.

Remark. For any set A, B, the property $A \cap B \neq \varnothing$ may be called "A intersects B non-trivially".

Remark. **WARNING!** Let X be any topological space and A be any subset of X. Then, it is **not true** in general that

$$A \subset A'$$

To see this, consider the Sierpieński Space, $(X, \mathcal{S})$ with $X = \{a, b\}$ and $\mathcal{S} = \{X, \varnothing, \{a\}\}$. Now, consider $A = \{a\}$. The only open set containing $b \in X$ is X itself. Now observe that

$$(X\backslash\{b\}) \cap A = (\{a, b\}\backslash\{b\}) \cap \{a\} = \{a\} \cap \{a\} = \{a\} \neq \varnothing$$

Therefore b is a limit point of A i.e. $b \in A'$. So, $A \not\subset A'$.

Let us look at some examples of limit points and derived sets.

Example. Let $X = \{a, b, c, d, e\}$ be a set equipped with the topology

$$\mathcal{T} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

Now, consider the subset $A = \{a, b, c\}$ of X. Observe that $b \in X$ *is a limit point* of A since the open sets containing b are $\{b, c, d, e\}$ and X; and each contains a point of A different from b i.e. c. On the other hand, $a \in X$ is *not a limit point* of A since the open set $\{a\}$ which contains a does not contain a point of A distinct from a.

Moreover, observe that the point d is a limit point of A as the open sets $\{c, d\}$, $\{a, c, d\}$, $\{b, c, d, e\}$ and X all contains a point of A different from d. Similarly, e is a limit point of A. On the other hand, c is not a limit point of A as $\{c, d\}$ does not contain a point of A distinct from c.

limit points of a set need not be in the set.

In this case, $A' = \{b, d, e\}$.

Example. Let X be an indiscrete topological space. Then, X is the **only** open set containing any point $p \in X$. Hence, p is a limit point of every subset of X except the empty set $\varnothing$ and the singleton set $\{p\}$. So, if $A \subset X$ is any subset of X, we thus have

$$A' = \begin{cases} \varnothing & \text{if } A = \varnothing \\ X \backslash \{p\} & \text{if } A = \{p\} \\ X & \text{if } A \text{ contains two or more points} \end{cases}$$

A natural question to ask is then, "*when will $p \in X$ **not** be a limit point of A*". To answer this, we could simply negate the definition of a limit point. A more interesting observation is the following.

Proposition 1.2 (Not Limit Point). Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then $x \in X$ is not a limit point of A *if and only if*

$$\exists U \text{ open}, x \in U \text{ and } U \cap A \subset \{x\}$$

Proof. The point $x \in X$ is a limit point of A if and only if every open neighbourhood of p contains a point of A other than p i.e.

$$\forall U \text{ open}, x \in U \implies (U \backslash \{x\}) \cap A \neq \varnothing$$

Negating this statement, we have that p is not a limit point of A, if and only if

$$\exists U \text{ open}, x \in U \quad \text{and} \quad (U \backslash \{x\}) \cap A = \varnothing$$

or equivalently,

$$\exists U \text{ open}, x \in U \quad \text{and} \quad U \cap A = \varnothing \text{ or } U \cap A = \{x\}$$

the latter case is where U and A intersect at only a point x, this also does not satisfy the definition of a limit point.

or further equivalently,

$$\exists U \text{ open}, x \in U \quad \text{and} \quad U \cap A \subset \{x\}$$

■

Using this lemma, we can easily prove the following fact.

Topological Fact (Discrete). Let X be a **discrete** topological space and $A \subset X$. Then $A' = \varnothing$.

Proof. Let $x \in X$ and A be any subset of X. The singleton set $\{x\}$ is open in X with the discrete topology. Now, observe that

$$x \in \{x\} \quad \text{and} \quad \{x\} \cap A \subset \{x\}$$

where the truth of the inclusion is due to $\{x\}$ being a singleton. By the proposition above, x is not a limit point of A. Since x is arbitrary, it follows that $x \notin A'$ for every $x \in X$ i.e. $A' = \varnothing$. ■

If $x \notin A$, the intersection is empty. If $x \in A$, the intersection is precisely the singleton $\{x\}$

Topological Fact (Cofinite Space). Let X be a cofinite space and $A \subset X$. Then

$$A' = \begin{cases} \varnothing, & \text{if } A \text{ finite} \\ X, & \text{if } A \text{ infinite} \end{cases}$$

Proof. We have two cases.

A is finite.

Suppose A is finite and $x \in X$ is a limit point of A. Since A is finite, A^c is open. Now, consider $A^c \cup \{x\}$. This is open as the union of an infinite set with a finite set is infinite. Then observe that $A^c \cup \{x\}$ intersects A only at x [$\lightning$]. So, x is not a limit point of A as we have a contradiction to the preceding Proposition (1.2) i.e. $A' = \varnothing$.

A is infinite.

Suppose A is infinite and let $x \in X$. Now, since the complement of every open neighbourhood of x is finite, they must contain infinitely many points of A. Therefore, every point of X is a limit point of A. ■

Theorem 1.5. Let X be a topological space and $A, B \subset X$. If $A \subset B$, then $A' \subset B'$.

Proof. Suppose $x \in A'$. Then x is a limit point of A i.e. for every open set U containing x, we have $(U \backslash \{x\}) \cap A \neq \varnothing$. Since $A \subset B$, we also have

$$(U \backslash \{x\}) \cap B \;\supset\; (U \backslash \{x\}) \cap A \;\neq\; \varnothing$$

So x is a limit point of B i.e. $x \in B'$. Therefore, $A' \subset B'$. ■

Corollary 1.1. Let X be a topological space, $x \in X$ and $A \subset X$.

$$x \text{ is a limit point of } A \iff x \text{ is a limit point of } A \backslash \{x\}.$$

Proof. ($\Longleftarrow$). Since $(A \backslash \{x\}) \subset A$, by Theorem (1.5) we have $(A \backslash \{x\})' \subset A'$.

($\Longrightarrow$). Suppose x is a limit point of A. If $x \notin A$, we are done (as $A = A \backslash \{x\}$). So suppose $x \in A$. Now, every open neighbourhood U of x intersects with at least a point of A distinct from x. Therefore, such open neighbourhoods U must also intersect with at least a point of $A \backslash \{x\}$ (distinct from x). Thus, x is a limit point of $A \backslash \{x\}$. ■

Corollary 1.2. Let X be a topological space and $A, B \subset X$. Then, $(A \cup B)' = A' \cup B'$.

Proof. Let $(X, \mathcal{T})$ be a topological space such that $A, B \subset X$. Since $A, B \in A \cup B$, by Theorem (1.5) we have

$$A' \subset (A \cup B)' \quad \text{and} \quad B' \subset (A \cup B)' \implies A' \cup B' \subset (A \cup B)'$$

We are left to show that the reverse inclusion $A' \cup B' \supset (A \cup B)'$ holds. We will prove the contrapositive. Let $p \notin A' \cup B'$. Then, there exists U, V open with $p \in U, V$ such that

$$U \cap A \subset \{p\} \;\text{ and }\; V \cap B \subset \{p\}$$

But now, we make the observation that $U \cap V$ is also open and $p \in U \cap V$. We have

$$\begin{aligned}(U \cap V) \cap (A \cup B) &= \{(U \cap V) \cap A\} \cup \{(U \cap V) \cap B\} \\ &\subset (U \cap A) \cup (V \cap B) \\ &\subset \{p\} \cup \{p\} \\ &= \{p\}\end{aligned}$$

Thus, $p \notin (A \cup B)'$. Therefore, $(A \cup B)' \subset A' \cup B'$. ■

Theorem 1.6. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on X such that $\mathcal{T}_1 \subset \mathcal{T}_2$ i.e. every $\mathcal{T}_1$-open subset of X is also a $\mathcal{T}_2$-open subset of X. If A is any subset of X, then

every $\mathcal{T}_2$-limit point of A is also a $\mathcal{T}_1$-limit point of A.

Proof. Let x be a $\mathcal{T}_2$-limit point of A. Then, every $\mathcal{T}_2$-open set U such that $x \in U$ has the property that $(U \backslash \{x\}) \cap A \neq \varnothing$. But $\mathcal{T}_2 \supset \mathcal{T}_1$, so U is also $\mathcal{T}_1$-open with such property. ■

Remark. The **converse** to this theorem is **not always true**. Consider the usual topology $\mathcal{U}$ and the discrete topology $\mathcal{P}(\mathbb{R})$ on $\mathbb{R}$. Note that $\mathcal{U} \subset \mathcal{P}(\mathbb{R})$ as $\mathcal{P}(\mathbb{R})$ contains every subset of $\mathbb{R}$ just by definition. Now consider the set

we say $\mathcal{U}$ is coarser than $\mathcal{P}(\mathbb{R})$. we will see this terminology properly in an upcoming subsection.

$$A := \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$$

0 is a $\mathcal{U}$-limit point of A; but the derived set of any subset of a discrete space is empty, so 0 is not a $\mathcal{P}(\mathbb{R})$-limit point of A.

1.3 Closed sets

Definition 1.6 (Closed Set). Let X be a topological space. A subset $A \subset X$ is called a **closed set** if its complement A^c is an open set.

Example. Let $X = \{a, b, c, d, e\}$ be a set equipped with the favourite topology

$$\mathcal{F} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

The closed subsets of X are

$$X, \varnothing, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}$$

which are the complements of the open subsets of X. Observe that $\{b, c, d, e\}$ are both open and closed whereas $\{a, b\}$ is neither open nor closed.

Topological Fact (Discrete). Let X be a discrete topological space. Then, every subset of X is both open and closed.

Proof. Let X be a discrete topological space i.e. every subset of X is open. Then, every subset of X is also closed since its complement is always open (by discreteness). In other words, *all subsets of X are both open and closed.* ■

Topological Fact (Cofinite). Let X be a cofinite topological space. Then, the only closed sets of X are finite subsets and X itself.

Proof. Let X be a cofinite space i.e. every subset of X whose complement is finite and the empty set $\varnothing$ are open. Then, by taking complements, the closed subsets of X are those whose complement is infinite i.e. finite sets and X itself. ■

From this we are able to deduce the following fact quite easily.

Topological Fact (Cofinite). Let X be a cofinite topological space and $A \subset X$. Then, A' is closed.

Proof. Let X be a cofinite space and $A \subset X$. By the preceding fact, A' is closed if and only if A' is finite or $A' = X$.

Now, if A is finite, $A' = \varnothing$ which is finite so we are done. So suppose A is infinite. But then, $A' = X$ which is closed so we are done. ■

Slogan. In a cocountable space X: finite subsets, countable subsets and X are the only closed sets.

Since $(A^c)^c = A$ for any subset A of a (topological) space X, we have the following proposition.

Proposition 1.3. Let X be a topological space. Then, $A \subset X$ is open *if and only if* A^c is closed.

The axioms of a topological space together with De Morgan's Laws give the following theorem.

Theorem 1.7. Let X be a topological space. Then, the collection of closed subsets of X have the following properties:

(1). X and $\varnothing$ are closed sets.

(2). The intersection of any number of closed sets is closed.

(3). The union of any two closed sets is closed.

Proof. For **(1)**, it is clear as X and $\varnothing$ sets are open, their complements, $\varnothing$ and X respectively are closed.

(2). Let $\{A_i\}$ be any indexed collection of closed sets in X. Now, observe that

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$$

by De Morgan. Since A_i is closed, by definition A_i^c is open. By **[T2]**, we have the RHS is open so its complement $\bigcap A_i$ must be closed.

(3). Let A_1 and A_2 be two closed sets in X. Then, by De Morgan we have

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c$$

Since A_1, A_2 are closed, their complements A_1^c, A_2^c are open. By **[T3]**, the RHS is open so its complement $A_1 \cup A_2$ is closed. ■

This suggests that we can define a topology on a set just using closed subsets of the set. This will be further discussed in an upcoming subsection.

Closed sets can also be characterized in terms of their limit points.

Theorem 1.8 (Closed Characterization). Let X be a topological space and $A \subset X$. Then, A is closed *if and only if* A contains each of its limit points.

In other words, A is closed $\iff A' \subset A$ where A' is the derived set of A.

Proof. $(\implies)$. Suppose A is closed. Then, A^c is open. For contradiction, suppose that A does not contain all of its limit points i.e. suppose that $p \in X$ is a limit point of A such that $p \notin A$. If $p \notin A$, then $p \in A^c$.

Since p is a limit point of A, then every open set U in X containing p has the property that $(U \setminus \{p\}) \cap A \neq \varnothing$. Take $U = A^c$ (as by hypothesis $p \in A^c$), then $(A^c \setminus \{p\}) \cap A = \varnothing$ [↯].

$(\impliedby)$. Suppose A contains all of its limit points. Then, there is no point $x \in A^c$ which is a limit point of A (because all is contained in A). This implies that for all $x \in A^c$, there is U open (in X) such that[1] $x \in U \subset A^c$. This means that A^c is a neighbourhood of x. But $x \in A^c$ was arbitrary, so A^c is open i.e. A is closed. ■

[1] *i.e. U is wholly contained in A^c and $(U \setminus \{p\}) \cap A = \varnothing$ for all p limit points of A.*

Proof. Alternate proof. In view of Proposition 1.4 that will be proved later, it is much easier to prove this theorem. By Proposition (1.4),

$$A \text{ closed} \iff A = \bar{A} \iff A' \subset A$$

We are done. ■

1.4 Closure

Definition 1.7 (Closure). Let X be a topological space and $A \subset X$. The **closure** of A denoted $\bar{A}$ or $\operatorname{Clos} A$, is the intersection of all closed sets containing A.

In other words, if $\{F_i : i \in I\}$ is the collection of *all* closed subsets of X containing A, then

$$\bar{A} = \bigcap_{i \in I} F_i$$

N.B. A need not be closed! As we will see, **if** A is closed, then $A = \bar{A}$.

Definition 1.8 (Adherent/Closure Point). Let X be a topological space. A point $x \in X$ is called an **adherent point** or **closure point** of $A \subset X$ if

$$x \in \bar{A}$$

i.e. x belongs to the closure of A.

Remark. Two observations.

(1). Observe that $\bar{A}$ is a closed set since it is the intersection of closed sets.

(2). $\bar{A}$ is the smallest closed set containing A i.e. if F is a closed set containing A, then

$$A \subset \bar{A} \subset F$$

These observations are stated formally below.

Proposition 1.4 (Closure Characterizations). Let X be a topological space and $A \subset X$. Then,

(1). $\bar{A}$ is closed;

(2). $\bar{A}$ is the smallest closed subset of A i.e. If $F \supset A$ is closed, then $A \subset \bar{A} \subset F$;

(3). A is closed $\iff$ $A = \bar{A}$.

Proof. We prove according to the numbering.

1. The intersection of any arbitrary collection of closed sets is closed. So by definition, the closure is closed.

2. By definition, $\bar{A}$ is the intersection of all closed sets containing A. So, $A \subset \bar{A}$ and $x \in \bar{A}$ if and only if $x \in F$ for *all closed sets* F containing A. Thus, if $F \supset A$ is *some closed set* such that $F \supset A$, then for all $x \in \bar{A}$, we have $x \in F$ i.e. $\bar{A} \subset F$.

3. ($\Longleftarrow$). Suppose $A = \bar{A}$. Since $\bar{A}$ is closed, A is closed.

   ($\Longrightarrow$). Suppose A is closed. By definition, $\bar{A}$ is the intersection of all closed sets containing A. Since A is closed, then A is in this intersection, and is necessarily the smallest such set (w.r.t inclusion). Thus, the intersection is equal to A i.e. $\bar{A} = A$.

We are done for this one. ■

Example. Let $X = \{a, b, c, d, e\}$ equipped with the favourite topology

$$\mathcal{F} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

We have seen that the closed subsets of X are

$$X, \varnothing, \{b, c, d, e\}, \{a, b, e\}, \{b, e\}, \{a\}$$

Computing the closure of some subsets, we have:

1. $\overline{\{b\}} = X \cap \{b, c, d, e\} \cap \{a, b, e\} \cap \{b, e\} = \{b, e\}$;
2. $\overline{\{a, c\}} = X$;
3. $\overline{\{b, d\}} = X \cap \{b, c, d, e\} = \{b, c, d, e\}$
4. $\overline{\{b, e\}} = X \cap \{b, e\} \cap \{a, b, e\} \cap \{b, c, d, e\} = \{b, e\}$

Clearly all that falls on the RHS are closed, so (1) in the theorem above is indeed true. To convince the truth of (2), take X and look at the first example. Indeed

$$\{b\} \subset \{b, e\} \subset X$$

To convince the truth of (3), see example 4 where we have $\overline{\{b, e\}} = \{b, e\}$ and $\{b, e\}$ is indeed closed.

Example. Let X be a cofinite topological space i.e. the complements of finite sets and $\varnothing$ are open sets. Then, by taking complements, the closed sets are precisely the finite subsets of X together with X.

Hence, if $A \subset X$ is finite, the closure $\bar{A}$ is A itself since A is closed. On the other hand, if $A \subset X$ is infinite, then X is the only closed subset containing A, so $\bar{A} = X$. We thus have that for any subset A of a cofinite space X,

$$\bar{A} = \begin{cases} A & \text{if } A \text{ is finite} \\ X & \text{if } A \text{ is infinite} \end{cases}$$

Proposition 1.5 (Closure's Open Set Criterion). Let X be a topological space and $A \subset X$. Then,

$$x \in \bar{A} \iff U \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x.$$

Proof. By considering contrapositive (of both direction), this statement is logically equivalent to saying:

$$x \notin \bar{A} \iff \text{there exists } U \text{ open containing } x \text{ such that } U \cap A = \varnothing.$$

This is easier to proof. ($\implies$). Suppose $x \notin \bar{A}$ and consider the set $U = X \backslash \bar{A}$. Since $\bar{A}$ is closed, U is open so we are done.

($\impliedby$). Suppose there is U open containing x such that $U \cap A = \varnothing$. Then, $X \backslash U$ is closed and $U \supset A$. By (2) of Proposition 1.4, $\bar{A} \subset X \backslash U$. Thus, $x \notin \bar{A}$ (as $x \in U$). ■

Remark. Indeed, we see that it is very similar to the definition of a limit point:

$$x \in A' \iff (U \backslash \{x\}) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x.$$

Note that the implied property (above) is stronger than

$$U \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x.$$

This implies that if $x \in A'$, then $x \in \bar{A}$. The reversed implication is not always true, we will require more.

Lemma 1.9. Let X be a topological space and $A \subset X$. Then, $A \cup A'$ is a closed set.

The closure of a set can be completely described in terms of its limits points.

Theorem 1.10. Let X be a topological space and $A \subset X$. Then,

$$\bar{A} = A \cup A'$$

i.e. the closure of A is the union of A and its set of limit points.

Proof. We prove by showing $\bar{A} \subset A \cup A'$ and $A \cup A' \subset \bar{A}$.

Claim: $A \cup A' \subset \bar{A}$.

By definition, $A \subset \bar{A}$; so we are left to show $A' \subset \bar{A}$. Suppose $x \in A'$. Then, for every open set U containing x, we have $(U \backslash \{x\}) \cap A \neq \varnothing$. This implies that $U \cap A \neq \varnothing$ and so by Proposition (1.5), we have that $x \in \bar{A}$. Thus, $A' \subset \bar{A}$.

Claim: $\bar{A} \subset A \cup A'$.

Suppose $x \in \bar{A}$. If $x \in A$, then $x \in A \cup A'$ so we are done. So, suppose $x \notin A$. Since $x \in \bar{A}$, Proposition (1.5) tells us that every open set U containing x has the property that $U \cap A \neq \varnothing$. Since $x \notin A$, the intersection $U \cap A$ must happen at a different point from x. So $x \in A'$. This implies that, $x \in A \cup A'$ i.e. $\bar{A} \subset A \cup A'$. ■

Corollary 1.3. Let X be a topological space and $A \subset X$. If $x \in A'$ then $x \in \bar{A}$.

Proof. This trivially follows from the theorem above. ■

Corollary 1.4. Let X be a topological space and $A \subset X$. If $x \in \bar{A}$ and $x \notin A$, then A'.

Proof. This also trivially follows from the theorem above. ■

Corollary 1.5. Let X be a topological space and $A \subset X$. If $A \subset B$, then $\bar{A} \subset \bar{B}$.

Proof. Suppose $x \in \bar{A}$. Since $A \subset B$, Theorem (1.5) tell us that $A' \subset B'$. Moreover, Theorem (1.10) tells us that $\bar{A} = A \cup A'$ and $\bar{B} = B \cup B'$. These facts together with Theorem (A.4) implies that

$$\bar{A} = A \cup A' \subset B \cup B' = \bar{B}$$

So, $x \in \bar{B}$ and hence $\bar{A} \subset \bar{B}$ as required. ■

Corollary 1.6. Let X be a topological space and $A \subset X$. Then $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

"Closure of Finite Union equals the Union of Closure"

Proof. We prove by showing $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$.

Claim: $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$**.**

Firstly, observe that $A \subset A \cup B$ and $B \subset A \cup B$. By Corollary (1.5), we have that $\bar{A} \subset \overline{A \cup B}$ and $\bar{B} \subset \overline{A \cup B}$. Therefore, $\bar{A} \cup \bar{B} \subset \overline{A \cup B}$ as required.

Claim: $\overline{A \cup B} \subset \bar{A} \cup \bar{B}$**.**

Firstly, observe that we have $A \cup B \subset \bar{A} \cup \bar{B}$ by definition of closure and identity of sets.

Then, we make the observation that $\bar{A}$ and $\bar{B}$ are closed sets by Proposition (1.4)-(1), therefore their union $\bar{A} \cup \bar{B}$ is closed by Theorem (1.7).

Putting things together, Proposition (1.4)-(2) tells us that since $\bar{A} \cup \bar{B} \supset A \cup B$ is closed, then we must have

$$\overline{A \cup B} \subset \bar{A} \cup \bar{B}$$

as required. ∎

Proposition 1.6. Let X be a topological space and $A, B \subset X$. Then $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

Proof. We have (for free) that $A \cap B \subset A \subset \bar{A}$ and $A \cap B \subset B \subset \bar{B}$. Together, we have $A \cap B \subset \bar{A} \cap \bar{B}$. Since $\bar{A} \cap \bar{B}$ is an intersection of closed sets, then it is closed. By Proposition (1.4), the closure of A is the smallest closed subset of A, therefore $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$. ∎

Remark. Equality fails in general.

Consider $\mathbb{R}$ endowed with the usual topology. Let $A = (6, 7)$ and $B = (7, 9)$. Then, $\bar{A} = [6, 7]$ and $\bar{B} = [7, 9]$ so that $\bar{A} \cap \bar{B} = \{7\}$. Now, observe that $A \cap B = \varnothing$ and so its closure $\overline{A \cap B}$ is empty. This shows that $\overline{A \cap B}$ is a proper subset $\bar{A} \cap \bar{B}$.

A generalization of these two propositions is given below.

Proposition 1.7. Let $A_1, A_2, \ldots, A_m$ be subsets of a topological space X. Then

$$\overline{\bigcup_{i=1}^{m} A_i} = \bigcup_{i=1}^{m} \overline{A_i}$$

Proposition 1.8. For each i in some indexing set I, let A_i be a subset of the topological space X. Then

$$\overline{\bigcap_{i \in I} A_i} \subset \bigcap_{i \in I} \overline{A_i}.$$

Equality does not necessarily hold here even when the index set is finite.

Proposition 1.9. Let X be a topological space and $A, B \subset X$. Then $\bar{A} \backslash \bar{B} \subset \overline{A \backslash B}$.

The proof requires good grasp of set theory.

Proof. We prove the contrapositive. Suppose $x \notin \overline{A \backslash B}$. Then, by Proposition (1.5), there is an open neighbourhood U of x such that $U \cap (A \backslash B) = \varnothing$. By identity of sets, we have

$$U \backslash (A \cap B) = (U \cap A) \backslash (U \cap B)$$

so this implies that $(U \cap A) \backslash (U \cap B) = \varnothing$ and hence $(U \cap A) \subset (U \cap B)$.

for any sets $M, N \subset \xi$, $M \backslash N = \varnothing$ *if and only if* $M \subset N$.

Now, suppose $x \notin \bar{B}$. Then, by Proposition (1.5), there is an open neighbourhood V of x such that $V \cap B = \varnothing$. Define $W = U \cap V$. Trivially, have $W \cap U = W$ so

$$W \cap A = W \cap (U \cap A) \subset W \cap (V \cap B) = W \cap B = U \cap \varnothing = \varnothing$$

$\cap$ is associative and commutative.

This implies that $W \cap A$ is empty. Since W is an intersection of open sets, it is open. Therefore, by Proposition (1.5), $x \notin \bar{A}$. Therefore $x \notin \bar{A} \backslash \bar{B}$. ■

$x \in M \backslash N$ *if and only if* $x \in M$ and $x \notin N$.

Example. Consider the set of rational numbers $\mathbb{Q}$. As seen before, in the usual topology of $\mathbb{R}$, every real number $x \in \mathbb{R}$ is[2] a limit point of $\mathbb{Q}$. Hence, the closure of $\mathbb{Q}$ is the whole of $\mathbb{R}$ i.e. $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$.

[2] *as every open set contains some rational number*

Theorem 1.11. Let X be a topological space and $A, B \subset X$. Then,

(1) $\overline{X} = X$,

(2) $\overline{\varnothing} = \varnothing$,

(3) $A \subset \bar{A}$,

(4) $\bar{\bar{A}} = \bar{A}$,

(5) $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

Proof. We prove according to the numbering.

(1). X is closed, so $\overline{X} = X$;

(2). $\varnothing$ is closed, so $\overline{\varnothing} = \varnothing$;

(3). Theorem (1.10) tells us that $\bar{A} = A \cup A'$, so $A \in \bar{A}$.

(4). $\bar{A}$ is closed, so Proposition (1.4)–(3) tells us that it is equal to its closure.

(5). This is Corollary (1.6).

We are done. ■

These are indeed nice and fundamental properties. We will see that properties (2) - (5) can be substituted as axioms to define a topology on X. These properties are also called the *Kuratowski Closure Axioms* which we will see why.

Definition 1.9 (Dense). Let X be a topological space. Then $A \subset X$ is said to be **dense** in $B \subset X$ if B is contained in the closure of A i.e.

$$B \subset \bar{A}$$

In particular, A is dense in X (or is a dense subset of X) if and only if $\bar{A} = X$.

since we get $\bar{A} \subset X$ for free. Being dense implies that $X \subset \bar{A}$. Together, we get the equality.

Example. Let $X = \{a, b, c, d, e\}$ equipped with the favourite topology

$$\mathcal{F} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

We had from a previous example that

$$\overline{\{a, c\}} = X \quad \text{and} \quad \overline{\{b, d\}} = \{b, c, d, e\}$$

Hence, the set $\{a, c\}$ is a dense subset of X whereas the set $\{b, d\}$ is not.

Remark. Two observations.

1. Notice how being *dense* **depends on the topology**. If we take a different topology on $X = \{a, b, c, d, e\}$, $\{a, c\}$ might not be dense in X anymore as $\{a, c\}$ might not even be closed — take the trivial topology on X for example.

2. Notice how $\{a, c\}$ appears a lot in the favourite topology. The point a appears in $X, \{a\}$ and $\{a, c, d\}$ whereas the point c appears in $X, \{c, d\}, \{a, c, d\}$ and $\{b, c, d, e\}$. This gives some intuition why the terminology *dense* is used.

Example. As in the example before, since $\overline{\mathbb{Q}} = \mathbb{R}$, we have that $\mathbb{Q}$ is dense in $\mathbb{R}$ **with the usual topology**.

Discrete Fact. Let X be a **discrete** topological space and $A \subset X$. Then

(1). $A = \bar{A}$,

(2). X is the only dense subset of X.

Proof. Two things to prove.

(1). In a discrete space, every subset A of X is open and hence, closed. By Proposition (1.4), $A = \bar{A}$.

(2). A is dense in X if and only if $\bar{A} = X$. But $\bar{A} = A$ for all $A \subset X$. So, X is the only dense subset of X.

We are done. ■

Indiscrete Fact. Let X be an **indiscrete** topological space. Then,

(1). The only closed subsets of X are X itself and $\varnothing$;

(2). For any $A \subset X$,

$$\bar{A} = \begin{cases} \varnothing & \text{if } A = \varnothing \\ X & \text{if } A \neq \varnothing; \end{cases}$$

(3). Every non-empty subset of X is dense in X.

Proof. Three things to prove.

(1). In an indiscrete space, the only open sets are X and $\varnothing$ and so their complements, which are $\varnothing$ and X, are the only closed sets;

(2). If $A = \varnothing$, then $\bar{A} = \varnothing$ (this is true for any topology). Suppose $A \neq \varnothing$. Then,

X is the only closed set containing A, so we are forced to have $\bar{A} = X$.

(3). $A \subset X$ is dense in X iff $\bar{A} = X$. By (2), it follows that if A is non-empty, A is dense in X.

We are done. ■

Theorem 1.12. Let X be a topological space and $A \subset X$. Then

$$A \text{ is dense} \iff U \cap A \neq \varnothing \text{ for any non-empty open set } U \subset X.$$

Proof. ($\implies$). Suppose A is dense. Then $\bar{A} = X$. Now, let $U \subset X$ be some non-empty open set. Take any point $x \in U$, then $x \in X$ and so by Proposition (1.5) we have $U \cap A \neq \varnothing$. Since U was arbitrary, this claim is true for any non-empty open set.

($\impliedby$). Suppose for any non-empty open set $U \subset X$, we have $U \cap A \neq \varnothing$. Then, any $x \in X$ lies in $\bar{A}$ as every open neighbourhood of x intersects with A non-trivially. ■

Topological Fact (Indiscrete). Let X be an indiscrete space. Then every non-empty open subset of X is dense in X.

Proof. The only non-empty open subset of an indiscrete space is the space itself. Since $\bar{X} = X$, X is dense in X. ■

Topological Fact (Cofinite). Let X be an infinite cofinite space. Then every non-finite subset of X is dense in X.

Proof. In the cofinite topology, all finite sets are closed except one which is X. Take any non-finite subset A of X. Then, A is dense if and only if $\bar{A} = X$. By definition of the closure of A, if $\{F_i\}_{i \in I}$ is an indexed collection of all closed subsets containing A, then

$$\bar{A} = \bigcap_{i \in I} F_i$$

Since X is a closed subset containing A and is infinite, this intersection is precisely X so we are done. ■

1.5 Interior and exterior

Definition 1.10 (Interior Point). Let X be a topological space and $A \subset X$. A point $a \in A$ is called an **interior point** of A if a belongs to an open set U contained in A i.e.

$$a \in U \subset A \quad \text{where } U \text{ is open}$$

The set of interior points of A is called the **interior** of A and is denoted $\text{int}(A)$.

i.e. $a \in A$ belongs to an open neighbourhood of a fully contained in A.

Theorem 1.13. Let X be a topological space and $A \subset X$. The interior of a set A is the union of **all open** subsets of A.

some people take this as the definition.

Proof. Let $\{U_i\}_{i \in I}$ be the collection of all open subset of A. We want to prove that $\text{int}(A) = \bigcup U_i$.

Claim: $\text{int}(A) \subset \bigcup U_i$**.**

Suppose $x \in \text{int}(A)$. Then, there exists a $k \in I$ such that $x \in U_k$ where U_k is an open set. It follows that $x \in \bigcup U_i$, so we are done.

Claim: $\bigcup U_i \subset \text{int}(A)$**.**

Suppose $x \in \bigcup U_i$. Then, $x \in U_k$ for some $k \in I$ and U_k is open. By definition, we thus have $x \in \text{int}(A)$, so we are done. ■

Proposition 1.10 (Interior Characterization). Let X be a topological space and $A \subset X$. Then,

(1). $\text{int}(A)$ is open;

(2). $\text{int}(A)$ is the largest open subset of A i.e. if $U \subset A$ is open, then $U \subset \text{int}(A) \subset A$;

(3). A is open $\iff$ $A = \text{int}(A)$.

Proof. We prove based on the numbering.

(1). By the preceding Theorem (1.13), $\text{int}(A)$ is open as it is a union of open sets.

(2). Let U be open such that $U \subset A$. Then, by Theorem (1.13), $U \subset \text{int}(A)$. $\text{int}(A) \subset A$ holds by definition.

(3). If A is open, then by Theorem (1.13), $A \subset \text{int}(A) \subset A$ i.e. $A = \text{int}(A)$. Conversely, if $A = \text{int}(A)$, then by (1), A is open because $\text{int}(A)$ is open.

We are done here. ■

Theorem 1.14. Let X be a topological space and $A, B \subset X$.

$$\text{If } A \subset B, \text{ then } \text{int}(A) \subset \text{int}(B).$$

Proof. Suppose $x \in \text{int}(A)$. Then, A is an open neighbourhood of x. Since $A \subset B$, this implies that B is an open neighbourhood of x. Thus, $x \in \text{int}(B)$. ■

Corollary 1.7. Let X be a topological space and $A, B \subset X$. Then

$$\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$$

Proof. We prove as usual.

Claim: $\text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B)$ **.**

We have that $A \cap B \subset A$ and $A \cap B \subset B$. By the preceding Theorem (1.14), we have that $\text{int}(A \cap B) \subset \text{int}(A)$ and $\text{int}(A \cap B) \subset \text{int}(B)$ i.e. $\text{int}(A \cap B) \subset \text{int}(A) \cap \text{int}(B)$.

Claim: $\text{int}(A) \cap \text{int}(B) \subset \text{int}(A \cap B)$ **.**

Suppose $x \in \text{int}(A) \cap \text{int}(B)$. Then $x \in \text{int}(A)$ and $x \in \text{int}(B)$. This implies that both A and B are open neighbourhoods of x i.e. $A \cap B$ is an open neighbourhood of x. This means that $x \in \text{int}(A \cap B)$. ■

Corollary 1.8. Let X be a topological space and $A \subset X$. Then

$$\text{int}(\text{int}(A)) = \text{int}(A)$$

this theorem tells us that "Interior is Idempotent".

Proof. Since $\text{int}(A) \subset A$, by Theorem (1.14) we have $\text{int}(\text{int}(A)) \subset \text{int}(A)$.

Conversely, Theorem (1.13) tells us that $\text{int}(\text{int}(A))$ is the union of all open subsets of $\text{int}(A)$. Since, $\text{int}(A)$ is open and $\text{int}(A) \subset \text{int}(A)$, it follows that $\text{int}(A) \subset \text{int}(\text{int}(A))$. Therefore, we get equality as required. ■

Corollary 1.9. Let X be a topological space. If U is open, then $U \subset \text{int}(\bar{U})$.

Proof. Since $U \subset \bar{U}$, we thus have $\text{int}(U) \subset \text{int}(\bar{U})$ by Theorem (1.14). If U is open, then $U = \text{int}(U)$. Therefore $U \subset \text{int}(\bar{U})$. ■

Remark. The reverse inclusion is false in general. Let $X = \mathbb{R}$ endowed with the usual topology and define $U = (7, 8) \cup (8, 9)$. Now, $\bar{U} = [7, 9]$ and so, $\text{int}(\bar{U}) = (7, 9) \not\subset U$.

Proposition 1.11. Let X be a topological space and $A \subset X$. Then

$$\text{int}(A \cup B) \supset \text{int}(A) \cup \text{int}(B)$$

Proof. We have $\text{int}(A) \subset A \subset A \cup B$ and $\text{int}(B) \subset B \subset A \cup B$. Therefore, $\text{int}(A) \cup \text{int}(B) \subset A \cup B$. Since $\text{int}(A) \cup \text{int}(B)$ is a union of two open sets, it is open and thus is contained in $\text{int}(A \cup B)$. ■

Remark. The reverse inclusion is not true in general. Let $X = \mathbb{R}$ endowed with the usual topology and define $A = (1, 2]$ and $B = [2, 3)$. We have that $\text{int}(A) = (1, 2)$ and $\text{int}(B) = (2, 3)$. So, $\text{int}(A) \cup \text{int}(B) = (1, 2) \cup (2, 3)$. However, $A \cup B = (1, 3)$ has interior $\text{int}(A \cup B) = (1, 3)$ which is not contained in $(1, 2) \cup (2, 3)$.

Theorem 1.15. Let X be a topological space and $A \subset X$. Then $(\text{int}(A))^c = \overline{A^c}$.

equivalently, $\text{int}(A) = (\overline{A^c})^c$ i.e. "Complement of the Interior equals to the Closure of the Complement".

Proof. We will prove that $\text{int}(A) = (\overline{A^c})^c$.

Claim: $\text{int}(A) \subset (\overline{A^c})^c$**.**

By definition (or rather by Theorem (1.13)), $\text{int}(A) \subset A$. By identity of sets, we have $A^c \subset (\text{int}(A))^c$. Since, $\text{int}(A)$ is open, $(\text{int}(A))^c$ is closed. By Proposition (1.4), a closed set containing A^c must contain the closure of A^c, so $\overline{A^c} \subset (\text{int}(A))^c$. By identity of sets, we thus have $\text{int}(A) \subset (\overline{A^c})^c$.

the *identity of sets* used here is if $A \subset B$, then $B^c \subset A^c$.

Claim: $(\overline{A^c})^c \subset \text{int}(A)$**.**

By definition, we know that $A^c \subset \overline{A^c}$. By identity of sets, we also have $(\overline{A^c})^c \subset (A^c)^c = A$. Now, $\overline{A^c}$ is closed, so $(\overline{A^c})^c$ is open. By Proposition (1.10), an open set contained in A must be contained in the interior of A, so $(\overline{A^c})^c \subset \text{int}(A)$. ∎

the *identity of sets* used here is again if $A \subset B$, then $B^c \subset A^c$.

Corollary 1.10. Let X be a topological space and $A \subset X$. Then

$$\overline{(\text{int}(A))^c} = (\text{int}(A))^c$$

Proof. By the preceding Theorem (1.15) together with idempotence of the interior as seen in Corollary (1.8) we have

$$(\overline{(\text{int}(A))^c})^c = \text{int}(\text{int}(A)) = \text{int}(A)$$

Taking complements of both side we get the result. ∎

This is no surprise as we could have proven it using more elementary facts.

Proof. **[Alternative proof]**. Since $\text{int}(A)$ is open, $(\text{int}(A))^c$ is closed. By Proposition (1.4), it is equal to its closure as required. ∎

Definition 1.11 (Exterior). Let X be a topological space and $A \subset X$. The **exterior** of A is defined to be the interior of the complement of A i.e. $\text{int}(A^c)$. We may denote it as $\text{ext}(A)$.

the exterior of any set is also open since it is *interior of something*.

1.6 Boundary

Definition 1.12 (Boundary). Let X be a topological space and $A \subset X$. A point $x \in X$ is said to be a **boundary point** of A if

$$x \in \bar{A} \backslash \operatorname{int}(A)$$

i.e. x is the closure of A but not in the interior of A. The set of all boundary points of A is called the **boundary** of A and is denoted

$$\partial A = \bar{A} \backslash \operatorname{int}(A)$$

Example. Let $\mathcal{T}$ be an *unusual* topology on $\mathbb{R}$ such that $\mathcal{T} := \{\mathbb{R}, \varnothing, E_a\}$ where E_a is the set of all open infinite intervals (a, ∞) such that $a \in \mathbb{R}$. Now, consider $A = [3, \infty)$.

Since the interior of A is the largest open subset of A, $\operatorname{int}(A) = (3, \infty)$.

Observe that it in this topology, the closure of A is $\mathbb{R}$.

you can draw the real number line to see this.

The boundary of A is therefore $\partial A = \mathbb{R} \backslash (3, \infty) = (-\infty, 3]$.

Example. Let $\mathcal{U}$ be the usual topology on $\mathbb{R}$ and again consider $A = [3, \infty)$.

Since the interior of A is the largest open subset of A, $\operatorname{int}(A) = (3, \infty)$.

Since $A^c = (-\infty, 3)$ is open, $A = [3, \infty)$ is closed and hence equal to its closure. Thus, $\bar{A} = [3, \infty)$.

The boundary of A is therefore $\partial A = [3, \infty) \backslash (3, \infty) = \{3\}$.

Example. Consider the four intervals $[a, b]$, (a, b), $(a, b]$ and $[a, b)$ whose endpoints are a and b in $\mathbb{R}$ with the usual topology. The interior of each interval is the open interval (a, b) whose closure is $[a, b]$. Thus, the boundary of each interval is the set of endpoints $\{a, b\}$.

Example. Let $X = \{a, b, c, d, e\}$ equipped with the favourite topology

$$\mathcal{F} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

and consider the subset $A = \{b, c, d\}$ of X.

Interior of A. The points c and d are each interior points of A as

$$c, d \in \{c, d\} \subset A$$

where $\{c, d\}$ is an open set. The point $b \in A$ is not an interior point of A. Since $\operatorname{int}(A)$ is the largest open subset of A, thus $\operatorname{int}(A) = \{c, d\}$.

Closure of A. By brute-force, one can check that $\bar{A} = \{b, c, d, e\}$.

Boundary of A. The boundary is therefore $\partial A = \{b, c, d, e\} \backslash \{c, d\} = \{b, e\}$.

Example. Consider the set $\mathbb{Q}$. Since every open subset of $\mathbb{R}$ contains both rational and irrational points, there are no interior points in $\mathbb{Q}$ so $\operatorname{int}(\mathbb{Q}) = \varnothing$. We have seen that $\overline{\mathbb{Q}} = \mathbb{R}$. Hence, the boundary of $\mathbb{Q}$ is the whole of $\mathbb{R}$ i.e. $\partial \mathbb{Q} = \mathbb{R}$.

no interior points because there is no such open set which contain such points that is wholly contained in $\mathbb{Q}$ — as we can always find some irrational numbers in such open sets too. For the same reason, there are no exterior points.

Topological Fact (Indiscrete). Let X be an **indiscrete** topological space. Then

(1). For any *non-empty* **proper** subset A of X, $\text{int}(A) = \varnothing$;

(2). For any $A \subset X$ *non-empty*, $\text{ext}(A) = \varnothing$;

(3). For any *non-empty* **proper** subset A of X, $\partial A = X$.

Proof. We are in an indiscrete space, the only open sets are X and $\varnothing$.

(1). Let A be a non-empty proper subset of X. Since $A \neq X$, $\varnothing$ is the only open subset of A. Thus, $\text{int}(A) = \varnothing$.

(2). Let $A \subset X$ be non-empty. If $A = X$, then $A^c = \varnothing$ so $\text{int}(A^c) = \varnothing$. If $A \neq X$ and considering A^c, then, the only open subset of A^c is $\varnothing$ (as A is non-empty!). So, $\text{int}(A^c) = \varnothing$.

(3). Let A be a non-empty proper subset of X. By Theorem (1.4), $\bar{A} = X$. So, we have $\partial A = X \backslash \varnothing = X$.

We are done. ■

Topological Fact (Discrete). Let X be a discrete space and $A \subset X$. Then

(1). $\text{int}(A) = A$,

(2). $\text{ext}(A) = A$,

(3). $\partial A = \varnothing$.

Proof. In a discrete space, every subset is open. So, $A = \text{int}(A)$. Since A^c is also open, $\text{ext}(A) = \text{int}(A^c) = A$. Moreover, A is closed, so $\bar{A} = A$ for any subset in the discrete topology, and thus $\partial A = \varnothing$. ■

Lemma 1.16. Let X be a topological space and $A \subset X$. Then,

$$\overline{X \backslash A} = X \backslash \text{int}(A)$$

equivalently, we could have write this as $\overline{A^c} = X \backslash \text{int}(A)$

Proof. We prove as usual.

Claim: $\overline{X \backslash A} \subset X \backslash \text{int}(A)$**.**

Suppose $x \in \overline{X \backslash A}$. By Proposition (1.5), $U \cap (X \backslash A) \neq \varnothing$ for every open set U containing x. In particular, there does **not** exist an open set G containing x such that $G \subset A$. So, $x \notin \text{int}(A)$ i.e. $x \in X \backslash \text{int}(A)$.

Claim: $X \backslash \text{int}(A) \subset \overline{X \backslash A}$**.**

Suppose $x \in X \backslash \text{int}(A)$. Then, $x \notin \text{int}(A)$. This means that for every open set U containing x, $U \not\subset A$. Thus, $U \cap (X \backslash A) \neq \varnothing$. By Proposition (1.5), $x \in \overline{X \backslash A}$. ■

This allows us to have the following equivalence.

Theorem 1.17. Let X be a topological space and $A \subset X$. Then, $\partial A = \bar{A} \cap \overline{A^c}$.

some people take this as the definition.

Proof. Our definition of the boundary gives us $\partial A = \bar{A}\backslash \mathrm{int}(A)$. By algebra of sets, this is equivalent to

$$\partial A = \bar{A} \cap (X \backslash \mathrm{int}(A))$$

By the preceding Lemma (1.16), we have that

$$\partial A = \bar{A} \cap \overline{X\backslash A} = \bar{A} \cap \overline{A^c}$$

as required. ■

algebra of sets here is referring to the fact that $A\backslash B = A \cap B^c$.

Corollary 1.11. Let X be a topological space. Then, $\partial\varnothing = \varnothing$.

Proof. By the preceding Theorem (1.17),

$$\partial\varnothing = \bar{\varnothing} \cap \overline{X} = \varnothing \cap \overline{X} = \varnothing$$

where we have used that the closure of the empty set is the empty set itself. ■

Corollary 1.12. Let X be a topological space and $A \subset X$. Then ∂A is closed.

Proof. By the preceding Theorem (1.17), ∂A is the intersection of two closed sets, so it must be closed. ■

Corollary 1.13. Let X be a topological space and $A \subset X$. Then $\partial A = \partial(X\backslash A)$.

Proof. By the preceding Theorem (1.17), $\partial A = \bar{A} \cap \overline{X\backslash A}$.

By the same theorem, we have

$$\partial(X\backslash A) = \overline{X\backslash A} \cap \overline{X\backslash(X\backslash A)} = \overline{X\backslash A} \cap \bar{A}$$

Since $\cap$ is commutative, comparing these equations yield our claim. ■

Corollary 1.14. Let X be a topological space and $A \subset X$. Then

$$A \text{ is closed} \iff \partial A \subset A$$

Proof. ($\implies$). Suppose A is closed, then $A = \bar{A}$. By the preceding Theorem (1.17),

$$\partial A = \bar{A} \cap \overline{X\backslash A} \subset \bar{A} = A$$

($\impliedby$). Suppose $\partial A \subset A$. Now, suppose $x \in \bar{A}$ but that $x \notin A$. So, $x \in X\backslash A$ and therefore $x \in \overline{X\backslash A}$. But, $x \in \bar{A}$ and $x \in \overline{X\backslash A}$ means that $x \in \bar{A} \cap \overline{X\backslash A}$. By the preceding Theorem (1.17), $x \in \partial A$ which lies in A by hypothesis. So $x \in A$ [$\lightning$]. Therefore, $\bar{A} \subset A$ i.e. $\bar{A} = A$ which implies A is closed. ■

Theorem 1.18 (Boundary's Open Set Criterion). Let X be a topological space and $A \subset X$. Suppose $x \in X$. Then

$$x \in \partial A \iff U \cap A \neq \varnothing \text{ and } U \cap A^c \neq \varnothing \text{ for every open set } U \text{ containing } x.$$

Proof. ($\implies$). Suppose $x \in \partial A$. By Theorem (1.17), $x \in \bar{A}$ and $x \in \overline{X \backslash A} = \overline{A^c}$. By Proposition (1.5), $U \cap A \neq \varnothing$ **and** $U \cap A^c \neq \varnothing$ for every open set U containing x.

($\impliedby$). Suppose $U \cap A \neq \varnothing$ and $U \cap A^c \neq \varnothing$ for every open set U containing x. By Proposition (1.5), $x \in \bar{A}$ and $x \in \overline{A^c}$ which implies $x \in \bar{A} \cap \overline{A^c}$. By Theorem (1.17), this is precisely ∂A, so $x \in \partial A$. ■

We also have the following more practical equivalence definition of the boundary.

Theorem 1.19 (Boundary's Interior-Exterior Definition). Let X be a topological space and $A \subset X$. Then

$$\partial A = (\text{int } A)^c \cap (\text{ext } A)^c$$

Proof. *The idea is to prove that the definition of the boundary as given in* Theorem (1.17) *is equal to this claimed definition.*

We will prove that $\partial A = (\bar{A} \cap \overline{A^c}) \subset ((\text{int } A)^c \cap (\text{ext } A)^c)$ by contrapositive.

Suppose $x \notin (\text{int } A)^c \cap (\text{ext } A)^c$. Then $x \in \text{int } A$ and $x \in \text{ext } A$. By Theorem (1.15), we have

$$\text{int}(A) = (\overline{A^c})^c \quad \text{and} \quad \text{ext}(A) = \text{int}(A^c) = (\overline{(A^c)^c})^c = (\bar{A})^c$$

So, this implies that $x \notin \overline{A^c}$ and $x \notin \bar{A}$. Therefore, x is not in their intersection $\bar{A} \cap \overline{A^c}$ as desired.

Conversely, we utilize Theorem (1.15) again and the steps are identical but in reverse. We are done. ■

Theorem 1.20. Let X be a topological space and $A \subset X$. Then,

$$\bar{A} = \text{int}(A) \cup \partial A$$

i.e. the closure of A is the union of the interior and the boundary of A.

Proof. **Key Fact**: For any three sets A, B, C such that $A = B \backslash C$ and $C \subset B$, then $B = A \cup C$.

Since we have $\text{int}(A) \subset A \subset \bar{A}$, we have in particular $\text{int}(A) \subset \bar{A}$. By definition of the boundary, we have $\partial A = \bar{A} \backslash \text{int}(A)$. By the Key Fact above, we have that

$$\bar{A} = \text{int}(A) \cup \partial A$$

as required. ■

Remark. This proof looks trivial because of the definition of boundary that we choose. If we are to choose an alternative equivalent definition of the boundary (for example Theorem (1.17)) and not establish the equivalence with our current definition, this proof might not be trivial at all.

Corollary 1.15. Let X be a topological space and $A \subset X$. Then,

$$\partial A = \varnothing \iff A \text{ is both open and closed.}$$

Proof. ($\implies$). Stare at this fact: $\text{int}(A) \subset A \subset \bar{A}$. If $\partial A = \varnothing$, then by the preceding Theorem (1.20), we have that

$$\bar{A} = \text{int}(A) \cup \varnothing = \text{int}(A)$$

i.e. we have $\text{int}(A) \subset A \subset \text{int}(A)$ and $\bar{A} \subset A \subset \bar{A}$ which implies $A = \text{int}(A)$ and $A = \bar{A}$. By Proposition (1.10)–(3), A is open. By Proposition (1.4)–(3), A is closed.

($\impliedby$). Suppose A is both open and closed. By Proposition (1.10)–(3) and Proposition (1.4)–(3), we have $A = \bar{A}$ and $A = \text{int}(A)$. By Theorem (1.20), we have

$$A = A \cup \partial A$$

which implies $\partial A \subset A = \text{int}(A)$. By definition $\partial A \not\subset \text{int}(A)$, so ∂A must be empty. ■

Corollary 1.16. Let X be a topological space and $A \subset X$. Then,

$$A \text{ is open} \iff \partial A = \bar{A} \backslash A$$

equivalently, A is open *if and only if* $\partial A \cap A = \varnothing$.

Proof. ($\implies$). If A is open, then $\text{int}(A) = A$. By Theorem (1.20), we have

$$\bar{A} \backslash A = \bar{A} \backslash \text{int}(A) = (\text{int}(A) \cup \partial A) \backslash \text{int}(A) = \partial A$$

($\impliedby$). Suppose $\partial A = \bar{A} \backslash A$. We want to show A is open, this is equivalent to proving A^c is closed. Theorem (1.8) tells us that this is equivalent to show A^c contains all of its limit points.

So, suppose x is a limit point of A^c. By definition, this means that every open neighbourhood U of x contains at least one point of A^c different from x.

Now, suppose for contradiction that $x \in A$. Then every every open neighbourhood U of x also contains at least one point of A. By Theorem (1.18), $x \in \partial A$. But, by hypothesis $\partial A = \bar{A} \backslash A$, so $x \in \bar{A} \backslash A = \bar{A} \cap A^c$. This implies that $x \notin A$ [↯]. Thus, $x \in A^c$ and so A^c contains all of its limit points, further implying that A is open. ■

Proposition 1.12. Let X be a topological space and $A, B \subset X$. Then

$$\partial(A \cup B) \subset \partial A \cup \partial B$$

Proof. By Theorem (1.17), we can write the boundary of A as $\partial A = \bar{A} \cap \overline{A^c}$ for any subset A of X. Also recall that the closure of finite unions is equal to the union of the closure — this is Corollary (1.6). From now on, it's all set theory.

$$
\begin{aligned}
\partial(A \cup B) &= \overline{A \cup B} \cap \overline{(A \cup B)^c} \\
&= \overline{A \cup B} \cap \overline{A^c \cap B^c} \\
&= (\bar{A} \cup \bar{B}) \cap \overline{A^c \cap B^c} \\
&= (\bar{A} \cap \overline{A^c \cap B^c}) \cup (\bar{B} \cap \overline{A^c \cap B^c}) \\
&\subset (\bar{A} \cap \overline{A^c}) \cup (\bar{B} \cap \overline{B^c}) \\
&= \partial A \cup \partial B
\end{aligned}
$$

this is Theorem (1.17).

$(M \cup N)^c = M^c \cap N^c$.

this is Corollary (1.6).

here we distribute over intersections.

set union and intersection preserves subsets.

■

Theorem 1.21. Let X be a topological space and $A, B \subset X$.

$$\text{If } \bar{A} \cap B = \varnothing \text{ and } A \cap \bar{B} = \varnothing, \text{ then } \partial(A \cup B) = \partial(A) \cup \partial(B)$$

Proof. By the preceding proposition, we are left to show that $\partial(A \cup B) \supset \partial A \cup \partial B$. We prove this by contradiction. Suppose $x \notin \partial(A \cup B)$. By Theorem (1.17), $x \notin \overline{A \cup B}$ and $x \notin \overline{(A \cup B)^c}$ so we have two cases. We need to show that in both cases, $x \notin \partial A \cup \partial B$.

Suppose $x \notin \overline{A \cup B}$. By Corollary (1.6), $x \notin \bar{A} \cup \bar{B}$ i.e. $x \in (\bar{A})^c \cap (\bar{B})^c$. If $x \notin \bar{A}$, then $x \notin \bar{A} \cap \overline{A^c} = \partial A$. Similarly, $x \notin \partial B$. So, $x \notin \partial A \cup \partial B$.

Suppose $x \notin \overline{(A \cup B)^c}$. Then, $x \in (\overline{(A \cup B)^c})^c = \text{int}(A \cup B) \subset A \cup B$ where the equality is due to Theorem (1.15). Without loss of generality, assume $x \in A$. By hypothesis $A \cap \bar{B} = \varnothing$, so $x \notin \bar{B}$ and therefore not in $\bar{B} \cap \overline{B^c} = \partial B$.

So suppose $x \notin A$. Thus, $x \in B$. By hypothesis $\bar{A} \cap B = \varnothing$, so $x \notin \bar{A}$ and therefore not in $\bar{A} \cap \overline{A^c} = \partial A$. Together $x \notin \partial A \cup \partial B$ as required. ■

Corollary 1.17. Let X be a topological space and $A, B \subset X$.

$$\text{If } \bar{A} \cap \bar{B} = \varnothing, \text{ then } \partial(A \cup B) = \partial(A) \cup \partial(B)$$

Proof. This is a set theory proof really. We claim that if $\bar{A} \cap \bar{B} = \varnothing$, then $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$ which if it holds, would imply the result due to the preceding theorem. Now if $\bar{A}$ and $\bar{B}$ are disjoint, any set contained inside $\bar{A}$ must be disjoint with any set contained in $\bar{B}$ and vice-versa. Since, $B \subset \bar{B}$ and $A \subset \bar{A}$ the result follows. ■

Proposition 1.13. Let X be a topological space and $A \subset X$. Then $\partial(\text{int}(A)) \subset \partial A$.

Proof. We prove by contradiction. Suppose $x \notin \partial A$. Then by Theorem (1.17) $x \notin \bar{A}$ and $x \notin \overline{A^c}$. We need to prove that both these conditions implies that $x \notin \partial(\text{int}(A))$.

Suppose $x \notin \bar{A}$. By definition, we have $\text{int}(A) \subset A$. By Corollary (1.5), we also have $\overline{\text{int}(A)} \subset \bar{A}$. So if $x \notin \bar{A}$, $x \notin \overline{\text{int}(A)}$ and therefore $x \notin \overline{\text{int}(A)} \cap \overline{(\text{int}(A))^c} = \partial(\text{int}(A))$.

Now, suppose $x \notin \overline{A^c}$. So, $x \in (\overline{A^c})^c$. By Theorem (1.15), $(\overline{A^c})^c = \text{int}(A)$, so $x \in \text{int}(A)$. Now observe that by Corollary (1.8) and Theorem (1.15), we have

$$\text{int}(A) = \text{int}(\text{int}(A)) = (\overline{(\text{int } A)^c})^c$$

So, $x \in (\overline{(\text{int } A)^c})^c$ i.e. $x \notin \overline{(\text{int } A)^c}$ and so $x \notin \overline{\text{int}(A)} \cap \overline{(\text{int}(A))^c} = \partial(\text{int}(A))$. ■

We summarize our definition of boundaries in the following theorem.

Theorem 1.22 (TFAE Boundary). Let X be a topological space and $A \subset X$. Then the following are equivalent:

(1) The boundary of A, $\partial A = \bar{A} \backslash \text{int}(A)$.

(2) $\bar{A} \cap \overline{A^c}$.

(3) For each $x \in X$ and each open neighbourhood U of x, we have

$$U \cap A \neq \varnothing \text{ and } U \cap A^c \neq \varnothing.$$

(4) $(\text{int } A)^c \cap (\text{ext } A)^c$.

Definition 1.13 (Nowhere Dense). X be a topological space and $A \subset X$. Then, A is said to be **nowhere dense** in X if

$$\text{int}(\bar{A}) = \varnothing$$

i.e. the interior of the closure of A is empty.

Example. Consider $\mathbb{R}$ with the usual topology and let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. We have seen that 0 is the only limit point of A. Hence,

$$\bar{A} = A \cup \{0\} = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}$$

Now, observe that $\bar{A}$ has no interior points i.e. $\text{int}(\bar{A}) = \text{int}(A \cup \{0\}) = \varnothing$; so A is nowhere dense in $\mathbb{R}$.

$\bar{A}$ has no interior point because there are no open sets in $\bar{A}$.

The next example demonstrates that if A is a subset of a topological space X such that its interior is empty, then it does not imply the interior of its closure is empty i.e. A is nowhere dense in X. In symbols,

$$\text{int}(A) = \varnothing \not\Longrightarrow \text{int}(\bar{A}) = \varnothing$$

Example. Consider $\mathbb{R}$ with the usual topology and let $A = \{x \in \mathbb{R} : x \in \mathbb{Q}, 0 < x < 1\}$ i.e. the rational numbers between 0 and 1. Observe that $\text{int}(A) = \varnothing$.

N.B. $A \neq (0,1)$. It is rather $(0,1)$ minus the irrational points in there.

Now, $\bar{A} = [0,1] = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$ and we have

$$\text{int}(\bar{A}) = \text{int}([0,1]) = (0,1) \neq \varnothing$$

So, A is not nowhere dense in $\mathbb{R}$.

1.7 Neighbourhoods (again)

Definition 1.14 (Neighbourhood). Let X be a topological space and $x \in X$. A subset $N \subset X$ is a **neighbourhood** of x if N contains an open set U which contains x i.e.

$$x \in U \subset N, \qquad \text{where } U \text{ is in open set}$$

The collection of all neighbourhoods of $x \in X$, denoted by $\mathcal{N}_x$, is called the neighbourhood system of x.

remember that *neighbourhood* and *open neighbourhood* are two different things here.

Remark. In other words, the relation "N is a neighbourhood of $x \in X$" is the inverse of the relation "x is an interior point of N".

For example, $<$ is a relation from $\mathbb{R}$ to $\mathbb{R}$ and $>$ is the inverse of this relation.

Example (Topology on the Line). Let $a \in \mathbb{R}$. Then, each closed interval $[a-\delta, a+\delta]$ with center a, is a neighbourhood of a since it contains the open interval $(a-\delta, a+\delta)$ containing a.

Example (Topology on the Plane). Let $p \in \mathbb{R}^2$. Then, every closed disc $\overline{D(p,\delta)}$ centred at p with radius $\delta > 0$ is a neighbourhood of p since it contains the open disc centred at p, $D(p,\delta)$.

Proposition 1.14. Let X be a topological space, $x \in X$ and $M, N \subset X$.

$$\text{If } M, N \in \mathcal{N}_x, \text{ then } M \cap N \in \mathcal{N}_x.$$

if M, N are any two neighbourhoods of x, then $M \cap N$ is also a neighbourhood of x.

Proof. Suppose $M, N \in \mathcal{N}_x$. Then there exist *open neighbourhoods* U, V of x such that $U \subset M$ and $V \subset N$. So, $x \in U \cap V \subset M \cap N$. Moreover, the intersection of two open sets is open so $U \cap V$ is open i.e. $U \cap V$ is an *open neighbourhood* of x. Since this *open neighbourhood* of x is contained in $M \cap N$, we conclude that $M \cap N$ is a neighbourhood of x i.e. $M \in \mathcal{N}_x$. ■

Proposition 1.15. Let X be a topological space, $x \in X$ and $M, N \subset X$.

$$\text{If } N \subset M \text{ and } N \in \mathcal{N}_x, \text{ then } M \in \mathcal{N}_x.$$

this is one of the many things that seems so obvious, but we still need a proof.

Proof. Suppose $N \subset M$ and $N \in \mathcal{N}_x$. Then, there exists an open neighbourhood U of x such that $U \subset N$. Since $N \subset M$, we have $U \subset M$. This means that M is a neighbourhood of x i.e. $M \in \mathcal{N}_x$. ■

Theorem 1.23. Let X be a topological space and $U \subset X$.

$$U \text{ is open} \iff U \text{ is a neighbourhood of all of its points.}$$

Proof. ($\implies$). Suppose U is open. Then, each point $x \in U$ belongs to the open set $U \subset U$. So, U is a neighbourhood of all of its points.

($\impliedby$). Suppose U is a neighbourhood of all of its points. So, for each point $x \in U$, there is an open set G_i such that $x \in G_i \subset U$. Therefore, we have that $U = \bigcup G_i$. The arbitrary union of open sets is open so $\bigcup G_i$ is open and thus, so is U. ■

Indiscrete Fact. Let X be an indiscrete topological space. Then,

$$\mathcal{N}_x = \{X\} \text{ for any point } x \in X$$

N.B. $\{X\}$ and not X.

Proof. In an indiscrete space X, the only open sets are X and $\varnothing$. Therefore, the only open set containing any point $x \in X$ is X itself. Moreover, $X \subset X$, so X is the only neighbourhood of x i.e. $\mathcal{N}_x = \{X\}$. ■

The fundamental importance of defining the so-called neighbourhood systems $\mathcal{N}_x$ of any point $x \in X$ are the five properties below.

Proposition 1.16. Let X be a topological space and $x \in X$.

(1). There exists at least one element in $\mathcal{N}_x$ i.e. $\mathcal{N}_x \neq \varnothing$;

(2). x belongs to each member of $\mathcal{N}_x$ i.e. $\forall N \in \mathcal{N}_x$, $x \in N$;

(3). The intersection any two members of $\mathcal{N}_x$ belongs to $\mathcal{N}_x$;

(4). Every superset of $N \in \mathcal{N}_x$ belongs to $\mathcal{N}_x$;

(5). Each member $N \in \mathcal{N}_x$ is a superset of a member $U \in \mathcal{N}_x$ where U is a neighbourhood of each of its points i.e. $U \in \mathcal{N}_u$ for every $u \in U$.

These properties are what we call the *Neighbourhood Space Axioms*. Like the *Kuratowski Closure Axioms*, these axioms may be used to define a topology on X, as we shall see.

1.8 Convergent sequences

Definition 1.15 (Sequence). A sequence in a topological space X is a function $f : \mathbb{N} \to X$. We will denote a sequence of points in X by $\langle x_n \rangle$ meaning that, for $n \in \mathbb{N}, f(n) = x_n$.

Definition 1.16 (Convergence). A sequence $\langle s_n \rangle$ of points in a topological space X **converges** to a point $\ell \in X$, denoted by

$$\lim_{n\to\infty} s_n = \ell, \quad \lim s_n = \ell \quad \text{or} \quad s_n \to \ell$$

if for each open set U containing ℓ, there exists a positive integer $n_0 \in \mathbb{N}$ such that $s_n \in U$ for $n > n_0$.

U contains almost all (i.e. all except a finite number) of the terms of the sequence

Topological Fact (Indiscrete). Let X be an indiscrete topological space. Then any sequence of points $\langle s_n \rangle$ in X converges to every point $\ell \in X$.

Proof. Let $\langle s_n \rangle$ be a sequence of points in an indiscrete topological space $(X, \mathcal{I})$. Observe that X is the only open set containing any point $\ell \in X$. Also, note that X contains every term of the sequence $\langle s_n \rangle$. Therefore, the sequence $\langle s_n \rangle$ converges to every point $\ell \in X$. ∎

Example. Take the topological space $(\mathbb{R}, \mathcal{I})$ i.e. $\mathbb{R}$ equipped with the indiscrete topology. Consider the sequence $\langle 1/n \rangle$. Then, this sequence converges to π; it also tends to -2.39495. Consider the sequence of constants $\langle 1 \rangle$, this also converges to π; it also tends to $\sin\left(\sqrt{2019} + e\right)$.

Topological Fact (Discrete). Let X be a discrete topological space and $\langle s_n \rangle$ be a sequence of points in X. If the sequence converges, then it is eventually constant.

Proof. Let $\langle s_n \rangle$ be a sequence of points in a discrete topological space $(X, \mathcal{P}(X))$. Observe that now the singleton set $\{x\}$ is open for all $x \in X$. So, if $s_n \to \ell$, then the set $\{\ell\}$ must contain *almost all* of the terms of the sequence. Since it is a singleton, this implies that *almost all* of the terms of the sequence must be ℓ. The only way this is possible is if the sequence has the form $(s_1, s_2, s_3, \ldots, s_{n_0}, \ell, \ell, \ell, \ell, \ell, \ldots)$ i.e. the sequence is eventually constant. ∎

Example. Let $\mathbb{R}^3$ be given the discrete topology. Let $\langle x_n \rangle = \langle (3, 4, \pi + 1/n) \rangle$. Then $\langle x_n \rangle$ has no limits in $\mathbb{R}^3$.

Suppose for contradiction that it has a limit $L \in \mathbb{R}^3$. The singleton set $U = \{L\}$ is an open neighbourhood of L. By definition of convergence, there is $n_0 \in \mathbb{N}$ such that $x_n \in U$ for $n > n_0$ i.e. *almost all* of $\langle x_n \rangle$ is in U. But the terms in the sequence $\langle x_n \rangle$ are distinct so $x_n = L$, and hence, $x_n \in U$ for at most one value of $n \in \mathbb{N}$ [$\lightning$]. This is a contradiction!

Topological Fact (Cocountable). Let X be an **infinite set** endowed with the cocountable topology. Suppose $\langle s_n \rangle$ is a sequence of points in X. If the sequence converges, then it is eventually constant.

Proof. Let X be an infinite set equipped with a topology $\mathcal{T}$ whose members are $\varnothing$ and all subsets of X whose complements are countable i.e. $\mathcal{T} = \{U \subset X : U = \varnothing \text{ or } U^c \text{ is countable}\}$. Let $\langle s_n \rangle$ be a sequence of points in X. We claim that $\langle s_n \rangle$ converges to $\ell \in X$ *if and only if* the sequence is eventually constant i.e. it has the form $(s_1, s_2, s_3, \ldots, s_{n_0}, \ell, \ell, \ell, \ell \ldots)$.

Suppose $s_n \to \ell$ and define $A = X \backslash \{s_n : s_n \neq \ell\}$. Note that by definition, $\ell \in A$. Observe that A^c is countable, so A is open in the cocountable topology. Since $s_n \to \ell$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, we have $s_n \in A$. By definition of A, we are forced to have $s_n = \ell$ for all $n \geqslant n_0$ i.e. $\langle s_n \rangle$ is eventually constant. ■

Topological Fact (Cofinite). Let $\mathbb{R}$ be endowed with cofinite topology and let $(s_1, s_2, \ldots)$ be a sequence in $\mathbb{R}$ with **distinct terms**. Then, $\langle s_n \rangle$ converges to every real number $r \in \mathbb{R}$.

Proof. Let U be an open set containing $r \in \mathbb{R}$. Since U is open (in the cofinite topology), U^c is finite. As U^c is finite, it can only contain only a finite number number of the terms of the sequence $\langle s_n \rangle$ because the terms are distinct. Therefore, U contains *almost all* of the terms of $\langle s_n \rangle$, and thus converges to r. ■

1.9 Coarser and finer topologies

A topology can be thought of similar to a union of collection of pebbles. If the pebbles are crushed and turn into sand, they are now finer. If you throw some big rocks to these area filled with sand, these big rocks are coarser. The union of collection of sands is larger than the union of collection of pebbles.

see Naruto Shippuuden episode 322 at 14:51 where Gaara and Ohnoki tries to stop Madara's falling meteors. Despite the meteor being gigantic, if Gaara had the power to use all the sand on the desert, he can still cover these meteors because the sand is finer.

Definition 1.17 (Finer/Coarser). Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be topologies on a non-empty set X. Suppose that each $\mathcal{T}_1$-open subset of X is also a $\mathcal{T}_2$-open subset of X i.e.

$$\mathcal{T}_1 \subset \mathcal{T}_2$$

Then, we say that $\mathcal{T}_1$ is **coarser** or *smaller* (or sometimes *weaker*) than $\mathcal{T}_2$. We also say that $\mathcal{T}_2$ is **finer** or *larger* (or sometimes *stronger*) than $\mathcal{T}_1$.

the terms *stronger* and *weaker* are a bit ambiguous here as to a topologist $\mathcal{T}_2$ is weaker here rather than $\mathcal{T}_1$. We will try to avoid using these terms.

Remark. If $\mathcal{T}_1 \subset \mathcal{T}_2$ but $\mathcal{T}_1 \neq \mathcal{T}_2$, then we shall say $\mathcal{T}_1$ is **strictly coarser** than $\mathcal{T}_2$, and $\mathcal{T}_2$ **strictly finer** than $\mathcal{T}_1$.

Example. Consider the discrete topology $\mathcal{P}(X)$, the indiscrete topology $\mathcal{I}$ and any other topology $\mathcal{T}$ on any set X. Then, we have that

$$\mathcal{I} \subset \mathcal{T} \subset \mathcal{P}(X)$$

i.e. $\mathcal{T}$ is coarser than $\mathcal{P}(X)$ and $\mathcal{T}$ is finer than $\mathcal{I}$.

Slogan. $\mathcal{I}$ is coarsest topology; $\mathcal{P}(X)$ is the finest topology.

Example. Consider the cofinite topology $\mathcal{C}$ and the usual topology $\mathcal{U}$ on the plane $\mathbb{R}^2$. We claim that $\mathcal{C}$ is coarser than $\mathcal{U}$. Let $A \in \mathcal{C}$. If $A = \varnothing$, we are done as $\varnothing \in \mathcal{U}$. Suppose $A \neq \varnothing$, so A^c is finite. Since every finite subset of $\mathbb{R}^2$ is $\mathcal{U}$-closed, we have that A is $\mathcal{U}$-open i.e. $A \in \mathcal{U}$. So $\mathcal{C}$ is coarser than $\mathcal{U}$ i.e. $\mathcal{C} \subset \mathcal{U}$.

If we can define a usual on topology on, in general, $\mathbb{R}^n$ (which we can), then this claim is easily extendable to a truth written in the following slogan.

Slogan. The usual topology on $\mathbb{R}^n$ is finer than the cofinite topology.

1.10 Equivalent definition of topologies

The way we defined a topological space gave axioms for the open sets in the topological space. i.e. we used as primitive the notion of *open sets* to define a topology on a set. There's nothing special about open sets as we can use as primitives the notions of *closed sets*, *neighbourhood of a point* and *closure of a set* (respectively) to define a topology on a set.

as primitives just mean as the main object of characterization.

There's a subtle difference between "axiomatizing something to define a topology on a set" (from the scratch) *and* "getting new topologies from old topologies" (like we did for subspace topology).

Defining topology via *closed sets*, *neighbourhood a point* or *the closure of a set* means that we can delete everything we know about topology and restart everything using these new axioms and the theory of topological spaces still hold, although there might be subtle differences here and there.

What the beginner should be aware of is the fact that in the following theorems, we have to **prove** that topology via *closed sets*, *neighbourhood a point* or *the closure of a set* really defines a topology on X in the sense that *it is a topology with respect to our usual primitive notion of topology* (where we axiomatized open sets).

Theorem 1.24 (Topology via Closed Sets). Let X be a non-empty set and let $\mathcal{C}$ be a collection of subsets of X satisfying:

(1). X and $\varnothing$ belongs to $\mathcal{C}$,

(2). The intersection of any number of sets in $\mathcal{C}$ belongs in $\mathcal{C}$,

(3). The union of any two sets in $\mathcal{C}$ belongs in $\mathcal{C}$.

Then, the following subset of X

$$\mathcal{T} = \{U \subset X : U = X \backslash K, \text{ for some } K \in \mathcal{C}\}$$

defines a topology on X.

Definition 1.18 (Kuratowski Closure Axioms). Let X be a non-empty set and $\mathcal{P}(X)$ be the power set of X. A function $\mathbf{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a **Kuratowski closure operator** if for all $A, B \in \mathcal{P}(X)$:

(1). $\mathbf{cl}(\varnothing) = \varnothing$, (*Preservation of Empty Set*);

(2). $A \subset \mathbf{cl}(A)$, (*Extensive*);

(3). $\mathbf{cl}(A \cup B) = \mathbf{cl}(A) \cup \mathbf{cl}(B)$; (*Preservation of Binary Unions*);

(4). $\mathbf{cl}(A) = \mathbf{cl}(\mathbf{cl}(A))$, (*Idempotent*).

The pair $(X, \mathbf{cl})$ is called a **Kuratowski closure space**.

The Kuratowski closure operator induces a natural *unique* topology on a set.

Theorem 1.25 (Topology via Closure of a Set). Let X be a non-empty set and $\mathbf{cl}$ be a Kuratowski closure operator. Then, the set

$$\mathcal{C} = \{A \in \mathcal{P}(X) : A = \mathbf{cl}(A)\}$$

defines a topology on X.

The closure operator had to satisfy $A \subset \mathbf{cl}(A)$. This is just like how we defined the closure of a set $\bar{A}$ where we had $A \subset \bar{A}$. Remember when we had the reverse inclusion and hence, an equality? Yes, $A = \bar{A}$ if and only if A is closed! So, the condition $A = \mathbf{cl}(A)$ utilizes this idea.

2 Bases and Subbases

2.1 Base for a topology

Usually, it is too difficult to specify a topology on a set. Generally, it is easier to specify a smaller collection of subsets of X and defines a topology in term of this smaller collection. This smaller collection is what is called a *base*.

Definition 2.1 (Base). Let X be a set. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X (called **basis elements)** such that

(1). For each $x \in X$, there is at least one basis element B containing X.

(2). If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis element B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, we then define the **topology generated by $\mathcal{B}$** as a collection $\mathcal{T}$ whose members satisfy the following:

A subset U of X is said to be open (i.e. belong to $\mathcal{T}$) in X if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$.

Note that each basis element is itself an open set i.e. a member of this generated topology.

Example. Let $\mathcal{B}$ be the collection of all circular regions (interior of circles) in the plane. Then $\mathcal{B}$ satisfies both conditions for a basis. In the topology generated by $\mathcal{B}$, a subset U of the plane is open if every $x \in U$ lies in some circular region contained in U.

Example. Let $\mathcal{B}'$ be the collection of all rectangular regions (interior of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then $\mathcal{B}'$ satisfies both conditions for a basis. As we shall see, the basis $\mathcal{B}'$ generates the same topology on the plane as the basis $\mathcal{B}$ (in the preceding example).

Topological Fact (Discrete). If X is any set, the collection of all singleton subsets of X is a basis for the discrete topology on X.

One should check that the definition of *topology generated by the basis* $\mathcal{B}$ is truly well-defined i.e. it's a topology on X. One other way of describing the topology generated by a basis is encoded in the following lemma.

Lemma 2.1. Let X be a set; let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathcal{B}$.

Remark: this is a very important lemma; should be a theorem.

Proof. Suppose $\{B_i\}_{i\in I}$ is an arbitrary collection of elements of $\mathcal{B}$. Since, $\mathcal{B}$ generates $\mathcal{T}$, $B_i \in \mathcal{T}$ for all $i \in I$. Since $\mathcal{T}$ is a topology, the union $\bigcup B_i \in \mathcal{T}$. Since $\{B_i\}_{i\in I}$ is arbitrary, this is true for any element of $\mathcal{B}$.

Conversely, suppose $U \in \mathcal{T}$. Then, choose for each $x \in U$ an element B_x of $\mathcal{B}$ such that $x \in B_x \subset U$. Then $U = \bigcup_{x\in U} B_x$, so U equals a union of elements of $\mathcal{B}$. ∎

Remark. This lemma states that every open set U in X can be expressed as a union of basis elements. However, not that this expression for U is not unique. Bases in topology does not enjoy the rich property of bases in linear algebra where every vector can be *decomposed* into a unique linear combination of basis.

We now know how to go from a basis to the topology it generates. What about the reverse?

Lemma 2.2. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathcal{C} \subset \mathcal{T}$ is a collection of open sets of X such that for each open set $U \in \mathcal{T}$ of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology of X.

Proof. Easy but omitted. ■

If we have two topologies generated by bases, it is convenient if we know which topology is coarser/finer just by looking at the bases.

Lemma 2.3 (TFAE Bases). Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topologies $\mathcal{T}$ and $\mathcal{T}'$ (respectively) on X. Then, the following are equivalent:

(1). $\mathcal{T}'$ is finer than $\mathcal{T}$.

(2). For each $x \in X$ and each basis element $B \in \mathcal{B}$ containing x, there is a basis element $B' \in \mathcal{B}'$ such that $x \in B' \subset B$.

Proof. Easy but omitted. ■

We have been calling the usual topology on $\mathbb{R}$ for quite a time now. Here is an alternative definition of the usual topology on $\mathbb{R}$.

Definition 2.2. If $\mathcal{B}$ is the collection of all open intervals in the real line,

$$(a, b) = \{x : a < x < b\}$$

the topology generated by $\mathcal{B}$ is called the **usual topology** on the real line.

Example (Lower semi-continuous topology). The collection $\mathcal{B}$ of all open intervals of the form

$$(a, \infty) = \{x \mid a < x < \infty\}$$

is a base for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}$ is called the **lower semi-continuous topology** on $\mathbb{R}$.

Example (Lower limit topology). The collection $\mathcal{B}'$ of all half-open intervals of the form

$$[a, b) = \{x \mid a \leqslant x < b\}$$

where $a < b$, is a base for a topology on $\mathbb{R}$. The topology generated by $\mathcal{B}'$ is called the **lower limit topology** on $\mathbb{R}$ and the resulting space is called the **Sorgenfrey line**.

A natural question to ask now is the following.

Since the topology generated by a basis $\mathcal{B}$ may be described as the collection of arbitrary unions of elements of $\mathcal{B}$, what happens if we start with a given collection of sets and take finite intersections of them as well as arbitrary unions?

This tinkering process leads us to the concept of *subbasis*.

Definition 2.3. A **subbasis** $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union equals X. The **topology generated by the subbasis $\mathcal{S}$** is defined to be the collection $\mathcal{T}$ of all unions of finite intersections of elements of $\mathcal{S}$.

3 Subspaces and product spaces

3.1 Subspaces

Definition 3.1 (Subspace Topology). Let $(X, \mathcal{T})$ be a topological space and suppose $A \subset X$ is non-empty. The collection $\mathcal{T}_A$ of all intersections of A with $\mathcal{T}$-open subsets of X

$$\mathcal{T}_A = \{U \cap A : U \text{ is } \mathcal{T}\text{-open}\}$$

is a topology on A. It is called the **subspace topology** (or *relative topology* or *induced topology*) on A. The topological space $(A, \mathcal{T}_A)$ is called a **subspace** of $(X, \mathcal{T})$.

In other words, a subset $V \subset A$ is a T_A-open set, i.e. **open relative to A** or **relatively open in A**, *if there exists a $\mathcal{T}$-open subset U of X such that $V = U \cap A$.*

Example. Let $X = \{a, b, c, d, e\}$ equipped with the favourite topology

$$\mathcal{F} = \{X, \varnothing, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\}$$

and consider the subset $A = \{a, d, e\}$ of X. Now we make the observation that

$$\begin{aligned} X \cap A &= A, & \{a\} \cap A &= \{a\}, & \{a, c, d\} \cap A &= \{a, d\} \\ \varnothing \cap A &= \varnothing, & \{c, d\} \cap A &= \{d\}, & \{b, c, d, e\} \cap A &= \{d, e\} \end{aligned}$$

Hence, the relative topology on A is

$$\mathcal{T}_A = \{A, \varnothing, \{a\}, \{d\}, \{a, d\}, \{d, e\}\}$$

Let $(X, \mathcal{T})$ and $A \subset X$ be equipped with the relative topology $\mathcal{T}_A$. The next example shows that a set not being $\mathcal{T}$-open does not necessarily imply that it is not $\mathcal{T}_A$-open.

Example. Consider the usual topology $\mathcal{U}$ on $\mathbb{R}$ and the relative topology $\mathcal{T}_A$ on the closed interval $A = [3, 8]$. Clearly, the half-open interval $[3, 5)$ is not $\mathcal{U}$-open (it is not closed either). But, now observe that

$$[3, 5) = (2, 5) \cap [3, 8] = (2, 5) \cap A$$

so $[3, 5)$ is $\mathcal{T}_A$-open.

Slogan. A set may be open relative to a subspace but be neither open nor closed in the entire (topological) space.

Proposition 3.1. Let $(X, \mathcal{T})$ be a topological space and $A \subset Y \subset X$.

If A is $\mathcal{T}$-open, then A is $\mathcal{T}_Y$-open.

Proof. By definition, $\mathcal{T}_Y = \{U \cap Y : U \text{ is } \mathcal{T}\text{-open}\}$. Since A is $\mathcal{T}$-open, then $A \cap Y \in \mathcal{T}_Y$. Since $A \subset Y$, $A \cap Y = A$. Thus, $A \in \mathcal{T}_Y$. ■

The converse is false as the above example show. Now, we check that the definition of the subspace topology is really correct i.e. it is well-defined.

Theorem 3.1. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Then, the subspace topology $\mathcal{T}_A$ is well-defined i.e. $\mathcal{T}_A = \{U \cap A : A \in \mathcal{T}\}$ is a topology on A.

Proof. We want to check that $\mathcal{T}_A$ is a topology on A. So, it has to satisfy **[T1]**, **[T2]** and **[T3]**.

[T1]. Since $\mathcal{T}$ is a topology on X, we have $X, \varnothing \in \mathcal{T}$. So, $A \cap X = A$ and $A \cap \varnothing = \varnothing$ belongs to $\mathcal{T}_A$.

[T2]. Let $\{G_i\}_{i \in I}$ be an arbitrary indexed subcollection of $\mathcal{T}_A$. Since $G_i \in \mathcal{T}_A$, there exists $U_i \in \mathcal{T}$ such that $G_i = U_i \cap A$ and this is true for each $i \in I$. Then we have

$$\bigcup_{i \in I} G_i = \bigcup_{i \in I}(U_i \cap A) = A \cap \bigcup_{i \in I} U_i$$

Since an arbitrary union of open sets is open, $\bigcup U_i$ is open and thus $\bigcup G_i \in \mathcal{T}_A$.

[T3]. Let $G_1, G_2 \in \mathcal{T}_A$. Then, there exist $U_1, U_2 \in \mathcal{T}$ such that $G_1 = U_1 \cap A$ and $G_2 = U_2 \cap A$. Then we have

$$G_1 \cap G_2 = (U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A$$

Since the intersection of open sets is open, $U_1 \cap U_2$ is open and so $G_1 \cap G_2 \in \mathcal{T}_A$.

We are done. ■

Theorem 3.2. Let X be a topological space; and let Y be a subspace of X. If A is a subspace of Y, then A is also a subspace of X with the same topology.

in other words,the subspace topology that A inherits from Y is the same as the one it inherits from X.

In particular, if A_i is an arbitrary collection of subspaces of X, $\bigcup A_i$ is also a subspace of X.

Topological Fact (Indiscrete). Every subspace of an indiscrete space is indiscrete.

Proof. Let $(A, \mathcal{I}_A)$ be a subspace of an indiscrete space $(X, \mathcal{I})$. The only open sets in X are X and $\varnothing$. So,

$$\mathcal{I}_A = \{X \cap A = A \text{ and } \varnothing \cap A = \varnothing\}$$

Thus, the only open sets in A are A and $\varnothing$. ■

Topological Fact (Discrete). Every subspace of a discrete space is also discrete.

Proof. Let $(A, \mathcal{T}_A)$ be a subspace of a discrete space $(X, \mathcal{T})$. Every subset of X is $\mathcal{T}$-open in this topology. So,

$$\mathcal{T}_A = \{U \cap A : U \in \mathcal{P}(X)\}$$

where $\mathcal{P}(X)$ is the powerset of X. Since $\mathcal{P}(A) \subset \mathcal{P}(X)$, every intersection of A with a subset of A is also in $\mathcal{T}_A$ i.e. every subset of A is $\mathcal{T}_A$-open. ■

Theorem 3.3. Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

Proof. If U is open in Y, we can write $U = V \cap Y$ for some V open in X. Since the intersection of two open sets is open and by hypothesis Y is open in X, we thus have U to be open in X. ∎

Of course we have a "converse" to this theorem: If U is open in X and Y is open in X, then U is open in Y. This is trivial.

Theorem 3.4. Let Y be a subspace of X. Then

$$A \text{ is closed in } Y \iff A = C \cap Y \text{ for some set } C \text{ closed in } X.$$

Proof. ($\Longleftarrow$). Suppose $A = C \cap Y$, where C is closed in X. Then $X \backslash C$ is open in X, so that $(X\backslash C) \cap Y$ is open in Y, by definition of the subspace topology. But $(X\backslash C) \cap Y = Y \backslash A$. Hence $Y\backslash A$ is open in Y, so that A is closed in Y.

For pictures of both situation in this proof, cf. page 94-95 Munkres. For detailed set-theoretic explanation see remark below.

($\Longrightarrow$). Suppose A is closed in Y. Then $Y\backslash A$ is open in Y. Since Y is a subspace of X, there is an open subset U of X such that $Y\backslash A = U \cap Y$. Next, observe that $X\backslash U$ is closed in X, and $A = (X\backslash U) \cap Y$ so that A is an intersection of a closed set $X\backslash U$ (in X) with Y, as desired. ∎

Remark. We have used a lot of identity of sets here in this proof. Here we give some clarity.

(i) We claimed in ($\Longleftarrow$) above that $(X\backslash C) \cap Y = Y\backslash A$. To do this, remember that $A = C \cap Y$. Then by De Morgan's law:

$$Y\backslash A = Y\backslash (C \cap Y) = (Y\backslash C) \cup (Y \backslash Y) = Y\backslash C$$

i.e. elements that are in Y and not in C. On the other hand,

$$\begin{aligned}(X\backslash C) \cap Y &= \{x : x \in X \text{ and } x \notin C \text{ and } x \in Y\}\\ &= \{x : x \in Y \text{ and } x \notin C\}\\ &= Y\backslash C = Y\backslash A.\end{aligned}$$

but $x \in X$ and $x \in Y$ just means $x \in Y$ as $Y \subset X$.

(ii) For the ($\Longleftarrow$) direction, we have the assumption $Y\backslash A = U \cap Y$ and claimed that $A = (X\backslash U) \cap Y$. Again, we appeal to De Morgan's law:

$$A = Y\backslash (Y\backslash A) = Y\backslash (U \cap Y) = (Y\backslash U) \cup (Y\backslash Y) = Y\backslash U$$

treating Y as a universe, it is quite easy to see the first equality $A = Y\backslash A^c$.

So what does this mean? It means

$$\begin{aligned}A &= Y\backslash U\\ &= \{x : x \in Y \text{ and } x \notin U\}\\ &= \{x : x \in Y \text{ and } x \in X \text{ and } x \notin U\}\\ &= \{x : x \in Y \text{ and } x \in X\backslash U\}\\ &= Y \cap (X\backslash U)\end{aligned}$$

this line is trivially true, if $x \in Y$, then $y \in X$.

Theorem 3.5. Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.

Again, of course we have a "converse" to this theorem: If U is closed in X and Y is closed in X, then U is closed in Y.

Proof. If A is closed in Y, we can write $A = F \cap Y$ where F is closed in X. Since the intersection of two closed sets is closed and by hypothesis Y is closed in X, we thus have A to be closed in X. ■

Note that if we have a subspace Y of a space X, the closure of A in Y may not be equal to the closure of A in X. Below is a simple example.

Example. Let $\mathbb{R}$ be endowed with usual Euclidean topology and consider $(0,1)$ viewed as a subspace of $\mathbb{R}$. The closure of $(0,1)$ in $\mathbb{R}$ is $[0,1]$ whereas the closure of $(0,1)$ in $(0,1)$ is itself.

A further observation is that $(0,1) = [0,1] \cap (0,1)$, i.e. the closure of $(0,1)$ in $(0,1)$ equals the closure of $(0,1)$ in $\mathbb{R}$ intersected with $(0,1)$.

Theorem 3.6. Let Y be a subspace of X; and let $A \subset Y$; let $\bar{A}$ denote the closure of A in X. Then the closure of A in Y equals $\bar{A} \cap Y$.

Proof. Let B denote the closure of A in Y. We want to show that $B \subset \bar{A} \cap Y$. Now, the set $\bar{A}$ is closed in X, so $\bar{A} \cap Y$ is closed in Y by Theorem (3.4). Since $\bar{A} \cap Y \supset A$, and since by definition B equals the intersection of **all** closed subsets of Y containing A, we must have $B \subset (\bar{A} \cap Y)$ — this is actually Propsition (1.4).

this is because $A \subset Y$ (by hypothesis) and $A \subset \bar{A}$, so $A \subset \bar{A} \cap Y$.

On the other hand, since B is the closure of A in Y, we know B is closed in Y. Therefore by Theorem (3.4), $B = C \cap Y$ for some set C closed in X. Then C is[3] a closed set of X containing A. Now $\bar{A}$ is the intersection of **all** closed sets in X containing A, so $\bar{A} \subset C$ — this is again Proposition (1.4). Then $\bar{A} \cap Y \subset C \cap Y = B$.

[3] *because $A \subset B$ by definition of closure and $B = C \cap Y \subset C$ by identity of sets.*

Together we have the equality $B = \bar{A} \cap Y$, as desired. ■

Definition 3.2. A property $\mathcal{P}$ of a subset A of a topological space X is called **absolute** if it depends on the subspace topology induced on A. Otherwise, it is called **relative**.

Remark. Said differently: Let $\mathcal{P}$ be an absolute property; let $(X, \mathcal{T})$ be a topological space and $(A, \mathcal{T}_A)$ be a subspace of X. Then

$$A \text{ has } \mathcal{P} \text{ with respect to } \mathcal{T} \iff A \text{ has } \mathcal{P} \text{ with respect to } \mathcal{T}_A.$$

Remark. Said even differently: Let X be a topological space and consider subspaces $Z \subset Y \subset X$. Then a property $\mathcal{P}$ is absolute if Z has P as a subspace of Y if and only if Z has P as a subspace of X.

3.2 Product spaces

If X and Y are topological spaces, there is a standard way of defining a topology on the cartesian product $X \times Y$.

Definition 3.3 (Product topology). Let X, Y be topological spaces. The **product topology** on $X \times Y$ is the topology having as basis the collection $\mathcal{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

explicitly, we can write the basis to be $\mathcal{B} = \{U \times V : U \in \mathcal{T}_X, V \in \mathcal{T}_Y\}$ where $\mathcal{T}_X$, $\mathcal{T}_Y$ are topologies on X and Y respectively.

Let's check that $\mathcal{B}$ in the definition above is truly a basis.

Proof. (1) The first condition is trivial. For every $z \in X \times Y$, we have $X \times Y \in \mathcal{B}$ containing $X \times Y$.

(2) The second condition is also easy. Suppose z belongs to two basis elements $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$. Then observe that

$$z \in B_1 \cap B_2 = (U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) := B_3$$

and B_3 is a basis element as $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y, respectively. So indeed $z \in B_3 \subset B_1 \cap B_2$, as desired. ■

Note that the collection $\mathcal{B}$ is **not** a topology on $X \times Y$. But instead, the collection of all unions of elements of $\mathcal{B}$ is a topology on $X \times Y$. This is Lemma (2.1).

Remark. We therefore have the **product topology** $\mathcal{T}_{X\times Y}$ to be the topology generated by $\mathcal{B}$ defined by the collection

$$\mathcal{T}_{X\times Y} = \left\{\begin{matrix} U \text{ is open in } X \times Y \text{ if whenever } (x,y) \in U, \text{ there is } U_x \text{ open in } X \\ \text{and } U_y \text{ open in } Y \text{ such that } (x,y) \in U_x \times U_y \subset U \end{matrix}\right\}$$

We can also make the simple definition that $\mathcal{T}_{X\times Y}$ equals the collection of all unions of elements of $\mathcal{B}$. This is in virtue of Lemma (2.1) and this is equal to the collection as defined above.

What can we say if the topologies on X and Y are given by bases?

Theorem 3.7. If $\mathcal{B}$ is a basis for the topology of X and $\mathcal{C}$ is a basis for the topology of Y, then the collection

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

is a basis for the topology of $X \times Y$.

It is sometimes useful to express the product topology in terms of a subbasis. To do this, we first define the so-called *projections.*

Definition 3.4 (Projections). Let $\pi_X : X \times Y \to X$ be defined by the equation

$$\pi_X(x,y) = x;$$

let $\pi_Y : X \times Y \to Y$ be defined by the equation

$$\pi_Y(x,y) = y.$$

The maps π_X and π_Y are called **projection maps** or **projections** of $X \times Y$ **onto** its first and second factors respectively.

we use the word **onto** because the projection maps are surjective (unless one of the spaces X or Y is empty, in which case $X \times Y$ is empty and... we don't have anything to discuss then.)

If U is an open subset of X, then the set $\pi_X^{-1}(U)$ is precisely the set $U \times Y$, which is open in $X \times Y$. Similarly, if V is open in Y, then $\pi_Y^{-1}(V) = X \times V$, which also open in $X \times Y$. The intersection of these two sets $\pi_X^{-1}(U) \cap \pi_Y^{-1}(V)$ is $U \times V$. This fact leads to the following theorem.

when we meet the notion of continuity later, we will realize that here we have already proven continuity of the projection maps.

Theorem 3.8. The collection

$$\mathcal{S} = \{\pi_X^{-1}(U) \,|\, U \text{ open in } X\} \cup \{\pi_Y^{-1}(V) \,|\, V \text{ open in } Y\}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ denote the product topology on $X \times Y$; let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Because every element of $\mathcal{S}$ belongs to $\mathcal{T}$, so do arbitrary unions of finite intersections of elements of $\mathcal{S}$. Thus $\mathcal{T}' \subset \mathcal{T}$.

On the other hand, every basis element $U \times V$ for the topology $\mathcal{T}$ is a finite intersection of elements of $\mathcal{S}$, since

$$U \times V = \pi_X^{-1}(U) \cap \pi_Y^{-1}(V).$$

Therefore, $U \times V$ belongs to $\mathcal{T}'$, so that $\mathcal{T} \subset \mathcal{T}'$ as well. ■

4 Continuity and Topological Equivalence

4.1 Continuous functions

Definition 4.1. Let $(X, \mathcal{T})$ and (Y, τ) be topological spaces. A function f from X into Y is **continuous relative to $\mathcal{T}$ and τ**, or **$\mathcal{T} - \tau$ continuous**, or simply **continuous**, iff the inverse image $f^{-1}(U)$ of every τ-open subset U of Y is a $\mathcal{T}$-open subset of X, that is, iff

$$U \in \tau \implies f^{-1}(U) \in \mathcal{T}$$

We shall write $f : (X, \mathcal{T}) \to (Y, \tau)$ for a function from X into Y or just $f : X \to Y$ when there's no potential confusion of topologies.

Example (Finite Topology). Let $X = \{a, b, c, d\}$ and $Y = \{x, y, z, w\}$. Let $\mathcal{T}$ be a topology on X and τ be a topology on Y where

$$\mathcal{T} = \{X, \varnothing, \{a\}, \{a, b\}, \{a, b, c\}\} \quad \text{and} \quad \tau = \{Y, \varnothing, \{x\}, \{y\}, \{x, y\}, \{y, z, w\}\}$$

Now consider the function $f : X \to Y$ defined by $a \mapsto y$, $b \mapsto z$, $c \mapsto w$ and $d \mapsto z$ and the function $g : X \to Y$ defined by $a \mapsto x$, $b \mapsto x$, $c \mapsto z$ and $d \mapsto w$.

The function f is continuous since the inverse of each member of the topology τ on Y is a member of the topology on X.

The function g is not continuous since $\{y, z, w\} \in \tau$ but its inverse image $g^{-1}(\{y, z, w\}) = \{c, d\} \notin \mathcal{T}$.

Topological Fact (Indiscrete). Let $(X, \mathcal{T})$ be any topological space and $(Y, \mathcal{I})$ be any indiscrete space. Then every function $f : X \to Y$ is continuous.

Proof. Suppose $f : X \to Y$ is continuous. Since Y is endowed with the indiscrete topology, the only open set in Y are $\varnothing$ and Y itself. But then

$$f^{-1}(Y) = X \quad \text{and} \quad f^{-1}(\varnothing) = \varnothing$$

and X and $\varnothing$ belongs to any topology $\mathcal{T}$ on X (by definition). Therefore, every function $f : X \to Y$ is continuous. ∎

Topological Fact (Discrete). Let $(X, \mathcal{P}(X))$ be any discrete space and $(Y, \mathcal{T})$ be any topological space. Then every function $f : X \to Y$ is continuous.

Proof. Suppose $f : X \to Y$ is a function. Let U be any open subset of Y. Then, its inverse image under f, $f^{-1}(U)$ is open in X since every subset of a discrete space is open. ∎

Topological Fact. Let X and Y both be discrete spaces. Then any function $f : X \to Y$ is continuous.

Remark. If we are to prove that by giving a topology to the **codomain** of a function could make the function discontinuous, then the first topology that should come to your mind is the **discrete topology**. Why? Because then it would be super hard to guarantee the preimage of every singleton (which is open in the discrete topology) is open in the domain.

This is immensely useful, as a lot of functions $f : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{discrete})$ from real analysis can be discontinuous, for example: $\sin x$.

On a related note, one can give the **domain of a function the indiscrete topology**; as then it would be super hard to for preimages to be the the empty set or the whole domain itself.

An important related note is that whenever $f : X \to Y$ is a continuous map of topological spaces, it is **not** necessarily true that the *forward* image of an open set is open i.e. U may be open in X without $f(U)$ being open in Y. See the following counterexample.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a constant map, say $f(x) = 0$ for all $x \in \mathbb{R}$. Then certainly f is continuous. But for example, $(0, 1)$ is open in $\mathbb{R}$ while $f((0, 1)) = \{0\}$ is not open in $\mathbb{R}$.

Continuous functions can be characterized by closed sets as well.

Theorem 4.1. A function $f : X \to Y$ is continuous *if and only if* the inverse image of every closed subset of Y is a closed subset of X.

Proof. Suppose f is continuous and let $U \subset Y$ is open. Then, U^c is closed. Since f is continuous, $f^{-1}(U^c) = (f^{-1}(U))^c$ is open in X where the equality holds by identity of inverse functions. Therefore $f^{-1}(U)$ is closed in X.

Suppose the inverse image of every closed subset of Y is a closed subset of X. Let $C \subset Y$ be a closed subset of Y. By definition, C^c is open. By hypotheses, $f^{-1}(C)$ is closed. So, $(f^{-1}(C))^c = f^{-1}(C^c)$ is open. Therefore f is continuous. ■

Here is a way to check continuity of a function into $[0, 1]$.

Proposition 4.1. Let X be a topological space; let $f : X \to [0, 1]$. If $f^{-1}((a, 1])$ and $f^{-1}([0, b))$ are open subsets of X for all $0 < a, b < 1$, then f is continuous.

Here is a way to check continuity of a function into $\mathbb{R}$.

Proposition 4.2. Let X be a topological space. A real-valued function $f : X \to \mathbb{R}$ is continuous if for any $x \in \mathbb{R}$, the sets

$$f^{-1}((x, \infty)) \text{ and } f^{-1}((-\infty, x))$$

are both open in X.

We now present an obvious yet interesting observation.

Theorem 4.2. Let Y be a topological space. Let $\{\mathcal{T}_i\}$ be a collection of topologies on a set X; let $f : X \to Y$ be a function. If f is continuous with respect to each $\mathcal{T}_i$, then f is continuous with respect to the intersection topology $\mathcal{T} = \bigcap_i \mathcal{T}_i$.

Proof. Let U be an open subset of Y. By hypothesis, $f^{-1}(U) \in \mathcal{T}_i$ for all i. Therefore, $f^{-1}(U) \in \bigcap_i \mathcal{T}_i = \mathcal{T}$, and so f is continuous with respect to $\mathcal{T}$. ■

We have an even broader characterization.

Theorem 4.3 (TFAE Continuity). Let X and Y be topological space; let $f : X \to Y$. Then, the following are equivalent:

(1) f is continuous.

(2) For every subset A of X, one has $f(\bar{A}) \subset \overline{f(A)}$.

(3) For every closed set B of Y, the set $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each open neighbourhood V of $f(x)$, there is an open neighbourhood U of x such that $f(U) \subset V$.

If the condition in (4) holds for a point x of X, we say that f is *continuous at the point* x.

if $f(x) \in V$, then $x \in f^{-1}(V)$. This is a property of inverse images and not of continuity. So if V is an open neighbourhood of $f(x)$, then clearly $U = f^{-1}(V)$ is a set containing x. If f is continuous, we have that U is open.

Proof. We will show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) and that (1) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose f is continuous; let A be a subset of X. We want to show that if $x \in \bar{A}$, then $f(x) \in \overline{f(A)}$. Suppose $x \in \bar{A}$. Now, suppose V is an open neighbourhood of $f(x)$. Then by continuity, $f^{-1}(V)$ is an open neighbourhood of x. Since $x \in \bar{A}$, Proposition (1.5) tells us that $f^{-1}(V)$ intersects A non-trivially, say at the point $y \in f^{-1}(V)$. This implies[4] that V intersects $f(A)$ at the point $f(y)$, so the intersection $V \cap f(A)$ is non-empty. Since V was arbitary we conclude by Proposition (1.5) that $f(x) \in \overline{f(A)}$, as required. ■

[4] *This is not by continuity. Instead it is just a property of functions, in particular, of inverse images.*

(2) $\Rightarrow$ (3). Let B be closed in Y and let $A = f^{-1}(B)$. We want to show that A is closed in X. To do this, we show that $A = \bar{A}$. Since $A \subset \bar{A}$, we are left to show that $\bar{A} \subset A$. Suppose $x \in \bar{A}$. By identity of sets, we have $f(A) = f(f^{-1}(B)) \subset B$. Thus, if $x \in \bar{A}$,

$$f(x) \in f(\bar{A}) \subset \overline{f(A)} \subset \bar{B} = B.$$

the first inclusion is by hypothesis of (2); the second inclusion is by Corollary (1.5); the equality $\bar{B} = B$ is because B is closed.

It follows that $x \in f^{-1}(B) = A$. Therefore, $\bar{A} \subset A$ as required. ■

(3) $\Rightarrow$ (1). Let V be an open set of Y. Since V is open, V^c is closed in Y. By identity of sets, we have

$$f^{-1}(V^c) = (f^{-1}(V))^c$$

As V^c is closed in Y, by hypothesis, $f^{-1}(V^c)$ is closed in X. So $f^{-1}(V)$ is open in X. Therefore, f is continuous as required. ■

(1) $\Rightarrow$ (4). Let $x \in X$ and V be an open neighbourhood of $f(x)$. Since f is continuous, $U = f^{-1}(V)$ is an open neighbourhood of x such that $f(U) \subset V$, as required. ■

(4) $\Rightarrow$ (1). Let V be an open set of Y; and let $x \in f^{-1}(V)$. Then, $f(x) \in V$. By hypothesis, there is an open neighbourhood U_x of x such that $f(U_x) \subset V$. By identity[5] of sets, $U_x \subset f^{-1}(V)$. Since U_x is contained in $f^{-1}(V)$, we **can** write $f^{-1}(V)$ as a union of the open sets U_x, so that it is open. ■ ■

[5] *identity used here is if* $A \subset B$, *then* $f^{-1}(A) \subset f^{-1}(B)$.

Remark. Note the word **can** in the final part of the proof above. $f^{-1}(V)$ need not be a union of open sets — we made a choice.

Due to (4), in the preceding TFAE Continuity Theorem, it may be useful to write the proper definition of being continuous at a point.

Definition 4.2 (Continuity at a point). Let X and Y be topological spaces. A function $f : X \to Y$ is **continuous at a point** $x \in X$ iff for any open neighbourhood V of $f(x)$, there is an open neighbourhood U of x such that $f(U) \subset V$.

We have that the composition of continuous functions is continuous, as expected.

Proposition 4.3. Let X, Y, Z be topological spaces; let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Then the composition $g \circ f : X \to Z$ is continuous.

Proof. Suppose that $U \subset Z$ is open in Z. Then by continuity of g, $g^{-1}(U)$ is open in Y, and hence by continuity of f, $f^{-1}(f^{-1}g(U))$ is open in X. This is equivalent to saying $(g \circ f)^{-1}(U)$ is open in X. Hence $g \circ f$ is continuous. ■

Example. If f, g are real-valued functions of a real variable such that f is continuous on $[a, b]$ and g is continuous on some subset of $\mathbb{R}$ containing $f([a, b])$, then $x \mapsto g(f(x))$ is continuous on $[a, b]$.

Here is a nice way of telling whether a subset of a topological space is open or not.

Theorem 4.4. Let X be a topological space and $\mathcal{S}$ be the Sierpinski space. For any subset $A \subset X$, let $\chi_A : X \to \{0, 1\}$ be its **characteristic** function defined by

$$\chi_A = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

Then A is open in X *if and only if* χ_A is continuous.

Finally, we collect results regarding the identity map and constant functions.

Proposition 4.4. Let $\mathcal{F}$ and $\mathcal{C}$ be two topologies on X; Let $\iota : (X, \mathcal{F}) \to (X, \mathcal{C})$ be the identity function.

(1) ι is continuous *if and only if* $\mathcal{F}$ is finer than $\mathcal{C}$ (i.e. $\mathcal{C} \subset \mathcal{F}$).

(2) ι is a homeomorphism *if and only if* $\mathcal{F} = \mathcal{C}$.

homeomorphism will be defined in the next subsection. We put it here for completion.

Proof. By definition, ι is $\mathcal{T}$-τ continuous if and only if the inverse image of a τ-open set U is $\mathcal{T}$-open. But then, $\iota^{-1}(U) = U$ (by definition of the identity map). Therefore, ι is $\mathcal{T}$-τ continuous if and only if a τ-open set is also $\mathcal{T}$-open, that is, $\tau \subset \mathcal{T}$ as desired. ■

Proposition 4.5. Let X be any topological space. The identity map $\iota : X \to X$ is continuous.

note that this is if X in the domain and codomain both are endowed with the same topology.

Proof. The identity map is continuous because for any U that is open in X, $\iota^{-1}(U) = U$ is open in X. ■

Proposition 4.6. Let X, Y be any topological spaces. Any constant map $f : X \to Y$ defined by $f(x) = y_0$ for some constant $y_0 \in Y$ is continuous.

Proof. The fiber of y_0 by definition is $f^{-1}(y_0) = X$ where the equality is true because f is constant. Observe that for any $y_1 \in Y$ such that $y_1 \neq y_0$, we have $f^{-1}(y_1) = \varnothing$. Now we make the following two observations:

If U is any open neighbourhood of y_0, then $f^{-1}(U) = X$. If V is any open set such that $y_0 \notin V$, then $f^{-1}(V) = \varnothing$; both of which are open. Hence, f is continuous. ■

4.2 Sequential continuity at a point

Definition 4.3. Let X and Y be topological spaces. A function $f : X \to Y$ is **sequentially continuous** at a point $x \in X$ iff for every sequence $\langle x_n \rangle$ in X converging to x, the sequence $\langle f(x_n) \rangle$ in Y converges to $f(x)$ i.e.

$$x_n \to x \implies f(x_n) \to f(x)$$

f is sequentially continuous on X if it is sequentially continuous at every $x \in X$.

Theorem 4.5. Let X and Y be topological spaces. If $f : X \to Y$ is continuous at $x \in X$, then it is sequentially continuous at x.

Proof. Suppose f is continuous. Given $x_n \to x$, we want to show that $f(x_n) \to f(x)$. Let V be an open neighbourhood of $f(x)$ in Y. Then $f^{-1}(V)$ is an open neighbourhood of x by continuity. Since $\langle x_n \rangle$ converges, there is $N \in \mathbb{N}$ such that $x_n \in f^{-1}(V)$ for $n \geqslant N$. Then $f(x_n) \in V$ for $n \geqslant N$, as required. ■

we abuse the fact that $x \in f^{-1}(V) \Leftrightarrow f(x) \in V$ twice here. This is a property of inverse images and not of continuity. But continuity allows us to say that $f^{-1}(V)$ is open because V is open.

The converse is not true. For example, take $\mathbb{R}$ endowed with the cocountable topology. Every function is sequentially continuous but the identity function for example is not continuous. We will see however that when X is a *metric space*, the converse will always be true.

4.3 Open and closed maps

A continuous function admits the property that the *inverse image* of open sets is open (resp. closed). What about *images* of open (resp. closed) sets?

Definition 4.4. Let X and Y be topological spaces; let $f : X \to Y$.

(1) f is said be an **open map** if the image of every open set is open.

(2) f is said be an **closed map** if the image of every closed set is closed.

Note that it is not the same anymore for continuous functions. An open function need not be a closed function and vice-versa.

4.4 Homeomorphism

Definition 4.5. Let X and Y be topological spaces; let $f : X \to Y$ be a bijection. If both the function f and the inverse function

$$f^{-1} : Y \to X$$

are continuous, then f is called a **homeomorphism**.

If a homeomorphism exists between two topological spaces X and Y, then we say that X and Y are **homeomorphic** or **topologically equivalent**; and write $X \equiv Y$.

Remark. Observe that the condition f^{-1} is continuous says that for each open set U of X, the inverse image of U under the map $f^{-1} : Y \to X$ is open in Y — this is by definition. But here's the interesting bit. The *inverse image* of U under the map f^{-1} is the same as the *image* of U under the map f. This suggest that we can define a homeomorphism by the following way:

$f : X \to Y$ is a homeomorphism if it is a bijection such that $f(U)$ is open *if and only if* U is open.

or further equivalently, using the language of open maps:

$f : X \to Y$ is a homeomorphism if it is open and continuous.

Remark. The above remark shows that a homeomorphism $f : X \to Y$ give us a bijective correspondence not only between X and Y but between collections of open sets of X and of Y. Consequently, any *topological property* of X (i.e. properties of X which have to do with open sets in X) yields the corresponding property (under a homeomorphism f) for the space Y. We will define this properly below. For now, we look at some examples.

Example (Two Open Intervals in $\mathbb{R}$ are Homeomorphic). Any two open intervals (a, b) and (c, d) in $\mathbb{R}$ (with the usual topology) are homeomorphic. A suitable homeomorphism $f : (a, b) \to (c, d)$ is given by

$$f(x) = c + \frac{(d-c)(x-a)}{b-a}$$

Example (An Open Interval is Homeomorphic to $\mathbb{R}$). Any open interval (a, b) is homeomorphic to $\mathbb{R}$. Since any two open intervals are homeomorphic, it is enough to show that $(-1, 1) \equiv \mathbb{R}$. A suitable homeomorphism $f : (-1, 1) \to \mathbb{R}$ is given by

$$f(x) = \frac{x}{1-|x|}$$

as usual, to see that this is indeed a bijection, treat $|x|$ and hence, $f(x)$ as a piecewise function i.e. $f(x) = 1/(1-x)$ when $x \geqslant 0$ and $f(x) = 1/(1+x)$ when $x < 0$. The rest is easy.

An alternative way to see that $(0, 1)$ and $\mathbb{R}$ are homeomorphic is the following.

Example. Let $X = (-1, 1)$. The function $f : X \to \mathbb{R}$ defined by $\tan\left(\frac{\pi x}{2}\right)$ is bijective and continuous. Furthermore, the inverse function f^{-1} is also continuous. So, f is a homeomorphism and thus $\mathbb{R}$ and $(-1, 1)$ are homeomorphic.

Topological Fact (Discrete). Let X and Y be discrete spaces. Then, X and Y are homeomorphic *if and only if* $|X| = |Y|$.

Slogan: For discrete spaces X, Y, we have $X \equiv Y \iff |X| = |Y|$.

Proof. Let X and Y be discrete spaces. Then as seen in an earlier Discrete Fact, every function from X to Y (and vice-versa) is continuous. So, X and Y are homeomorphic if and only if there exists a bijective function from one to another i.e.

if and only if $|X| = |Y|$. ■

Note that the relation $X \equiv Y$ is an equivalence relation.

Proposition 4.7 ($\equiv$ is an Equivalence Relation)**.** Let X and Y be topological spaces. Then

(1) $X \equiv X$;

(2) If $X \equiv Y$, then $Y \equiv X$;

(3) If $X \equiv Y$ and $Y \equiv Z$, then $X \equiv Z$.

An important consequence of this fact is that any collection of topological spaces can be partitioned into classes of topologically equivalent classes.

Definition 4.6. A property $\mathcal{P}$ of sets is called **topological** or a **topological invariant** if whenever a topological space X has $\mathcal{P}$, then every space homeomorphic to X also has $\mathcal{P}$.

Example (Length and Boundedness is not a Topological Invariant)**.** As seen in the previous example, $\mathbb{R} \equiv (-1, 1)$. Hence, *length* is not a topological property since X and $\mathbb{R}$ have different *lengths*. Moreover, *boundedness* is not a topological property as $(-1, 1)$ is bounded whereas $\mathbb{R}$ is not.

Example (Cauchy sequence is not a Topological Invariant)**.** Let $X = (0, \infty)$, the set of positive real numbers. The function $f : X \to X$ defined by $x \mapsto 1/x$ is a homeomorphism from X onto X. Now, observe that the sequence

$$\langle s_n \rangle = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$$

corresponds, under homeomorphism, to the sequence

$$(f\langle s_n \rangle) = (1, 2, 3, \ldots)$$

The sequence $\langle s_n \rangle$ is Cauchy whereas the corresponding sequence $(f\langle s_n \rangle)$ is not. So, Cauchy sequence is not a topological invariant.

Example of topological properties that we have encountered are *limit points*, *interior*, *boundary*, *density* and *neighbourhoods*. We will meet later the notion of *connectedness* and *compactness*, both of which are topological invariants.

Definition 4.7. Let X and Y be topological spaces; let $f : X \to Y$ be an injective continuous map. Let Z be the image set $f(X)$, considered as a subspace of Y. Then the function $f' : X \to Z$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of X with Z, we say that the map $f : X \to Y$ is a **topological embedding**, or simply an **embedding**, of X in Y.

4.5 Constructing continuous functions

Theorem 4.6 (Rules for Constructing Continuous Functions). Let X, Y and Z be topological spaces.

(a) (**Constant function**) If $f : X \to Y$ maps all of X into the single point y_0 of Y, then f is continuous.

(b) (**Inclusion**) If A is a subspace of X, the inclusion function $\iota : A \to X$ is continuous.

(c) (**Composites**) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then the map $g \circ f : X \to Z$ is continuous.

(d) (**Restricting the domain**) If $f : X \to Y$ is continuous, and if A is a subspace of X, then the restricted function $f|_A : A \to Y$ is continuous.

(e) (**Restricting or expanding the range**) Let $f : X \to Y$ be continuous. If Z is a subspace of Y containing the image set $f(X)$, then the function $g : X \to Z$ obtained by restricting the range of f is continuous. If Z is a space having Y as a subspace, then the function $h : X \to Z$ obtained by expanding the range of f is continuous.

(f) (**Local formulation of continuity**) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha}$ is continuous for each α.

Proof.

(a) Suppose $f(x) = y_0$ for every $x \in X$. Then $f(X) = \{y_0\}$ i.e. $x \in f^{-1}(\{y_0\})$ for every $x \in X$. Now, let V be open in Y. Then, $f^{-1}(V) = X$ if $y_0 \in V$ or $f^{-1}(V) = \varnothing$ if $y_0 \notin V$. Both of which are open, so f is continuous. ■

(b) Suppose A is a subspace of X. Consider an open set U of X. Then, $\iota^{-1}(U) = U \cap A$ and this is open in A (as A is a subspace). ■

(c) Suppose f, g are continuous. If U is open in Z, then $g^{-1}(U)$ is open in Y by continuity and $f^{-1}(g^{-1}(U))$ is open in X by continuity. By identity of sets, we have
$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$$
This implies the composition $g \circ f : X \to Z$ is continuous as required. ■

(d) The function $f|_A$ equals the composite of the inclusion map $\iota : A \to X$ and the map $f : X \to Y$, both of which are continuous. ■

(e) Let $f : X \to Y$ be continuous. If $f(X) \subset Z \subset Y$, we show that the function $g : X \to Z$ obtained from f is continuous. Let B be open in Z. Then $B = Z \cap U$ for some open set U of Y. Because Z contains the entire image set $f(X)$,
$$f^{-1}(U) = g^{-1}(B),$$
by identity of sets. Since $f^{-1}(U)$ is open, so is $g^{-1}(B)$. ■

To show $h : X \to Z$ is continuous if Z has Y as a subspace, note that h is the composite of the map $f : X \to Y$ and the inclusion map $j : Y \to Z$. ■

The proof of (f) is omitted. ■

This may be redundant, but we give a simpler version of (e) in the preceding theorem.

Theorem 4.7. Let X and Y be topological spaces. Then the map $f : X \to Y$ is continuous if and only if the map $g : X \to f(X)$ is continuous.

Proof. ($\implies$). Suppose $f : X \to Y$ is continuous. Then for every open set U in Y, we have $f^{-1}(U)$ is open in X. Now $V = U \cap f(X)$ is open in $f(X)$ (this is just the subspace topology). Now observe that

$$g^{-1}(V) = g^{-1}(U \cap f(X)) = g^{-1}(U) \cap X = g^{-1}(U) = f^{-1}(U)$$

We know $f^{-1}(U)$ is open in X, so $g^{-1}(V)$ is open in X. That is, $g : X \to f(X)$ is continuous.

($\impliedby$). Conversely, suppose $g : X \to f(X)$ is continuous. Then for every open set V in $f(X)$, we have $g^{-1}(V)$ is open in X. Accordingly, we can write $V = U \cap f(X)$ for some U open in Y. Now observe that

$$f^{-1}(U) = g^{-1}(U) = g^{-1}(U) \cap X = g^{-1}(U \cap f(X)) = g^{-1}(V)$$

We know $g^{-1}(V)$ is open in X, so $f^{-1}(U)$ is open in X. That is, $f : X \to Y$ is continuous. ■

Theorem 4.8 (Pasting Lemma). Let $X = A \cup B$, where A and B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for every $x \in A \cup B$, then f and g combine to give a continuous function $h : X \to Y$, defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$

Proof. Let C be a closed subset of Y. Now, by identity of sets we have

$$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C).$$

a clearer way to see this is to recognize that $f^{-1}(C) = (h|_A)^{-1}(C)$ and $g^{-1}(C) = (h|_B)^{-1}(C)$.

Since f is continuous, $f^{-1}(C)$ is closed in A and, thus, closed in X. Likewise, $g^{-1}(C)$ is closed in B and thus closed in X. Their union $h^{-1}(C)$ is therefore closed in $X/$ ■

This theorem also holds if A and B are open sets in X.

Theorem 4.9 (Maps into Products). Let $f : A \to X \times Y$ be given by the equation

$$f(a) = (f_1(a), f_2(a)).$$

Then f is continuous *if and only if* the functions

$$f_1 : A \to X \quad \text{and} \quad f_2 : A \to Y$$

are continuous.

Finally, we state the following important result that restriction of a homeomorphism is a homeomorphism.

Theorem 4.10 (Restriction of a homeomorphism is homeomorphism). Suppose that X, Y are topological spaces and $f : X \to Y$ is a homeomorphism; let $A \subset X$. Then

$$f|_A : A \to f(A)$$

is a homeomorphism.

here A and $f(A)$ are given the subspace topology induced by X resp. Y.

Proof. $f|_A$ is continuous because the restriction of a continuous function is continuous by Theorem (4.6). Furthermore, $f_A^{-1} = f^{-1}|_{f(A)}$ by identity of sets; and this is also continuous due to being a restriction of a continuous function. Clearly, both $f|_A$ and $f^{-1}|_{f(A)}$ are bijective (because they are restriction of bijective functions). So indeed, $f|_A$ is a homeomorphism. ■

Corollary 4.1. Suppose that X, Y are topological spaces and $f : X \to Y$ is a homeomorphism; let $A \subset X$ and $B \subset Y$. If $f(A) = B$, then the induced map $g : A \to B$ is a homeomorphism.

Proof. Set $g = f|_A$ in the preceding Theorem (4.10) above. ■

These final two corollaries (in particular, their contrapositive) would hold immense importance later on when we get to connectedness and compactness.

Corollary 4.2. Suppose that X, Y are topological spaces and $f : X \to Y$ is a homeomorphism; let $A \subset X$ and $B \subset Y$. If $f(A) = B$, then the induced map $h : X \backslash A \to Y \backslash B$ is a homeomorphism.

Proof. Observe that $X \backslash A \subset X$. By the preceding Theorem (4.10), $f|_{X \backslash A} : X \backslash A \to f(X \backslash A)$ is a homeomorphism. Since f is injective, we have

$$f(X \backslash A) = f(X) \backslash f(A) = Y \backslash B.$$

So setting $h = f|_{X \backslash A}$, we are done. ■

Corollary 4.3. Suppose that X, Y are topological spaces and $f : X \to Y$ is a homeomorphism; let $x \in X$ and $y \in Y$. If $f(x) = y$, then the induced map $h : X \backslash \{x\} \to Y \backslash \{y\}$ is a homeomorphism.

Proof. In the preceding corollary, let $A = \{x\}$ and $B = \{y\}$. ■

5 Metric Spaces

5.1 The metric

Definition 5.1 (Metric). Let X be a non-empty set. A **metric** on X is a function

$$d : X \times X \to \mathbb{R}$$

with the following properties:

[M1]. (*Positive*). $d(x, y) \geqslant 0$ for all $x, y \in X$; with equality *if and only if* $x = y$.

[M2]. (*Symmetric*). $d(x, y) = d(y, x)$ for all $x, y \in X$.

[M3]. (*Triangle inequality*). $d(x, z) \leqslant d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The real number $d(x, y)$ is called the **distance** between x and y in the metrid d. The pair (X, d) is called a **metric space**.

Remark. A function satisfying **[M2]** and **[M3]**, and partially satisfying **[M1]** in the following way: "If $x \neq y$, then $d(x, y)$ can be 0" is called a **pseudometric**.

basically a pseudometric becomes a metric if it can separate points from each other.

Before we look at some examples, here are some useful inequalities.

Proposition 5.1. Let (X, d) be a metric space and consider $x, y, z, t \in X$. Then

1. $|d(x, z) - d(y, z)| \leqslant d(x, y)$,
2. $|d(x, y) - d(z, t)| \leqslant d(x, z) + d(y, t)$.

We can generalize this further.

Proposition 5.2. Let $x_1, x_2, \ldots, x_n$ be points in a metric space (X, d). Then

$$d(x_1, x_n) \leqslant d(x_1, x_2) + d(x_2, x_3) + \ldots + d(x_{n-1}, x_n).$$

5.1.1 Metrics on boring sets

As we will see, there is only one metric on the empty set and only one metric on the singleton set.

Example (Metric on $\varnothing$ and singletons). Vacuously and uselessly, except to confirm that $\varnothing$ is a metric space, the **only metric** on $\varnothing$ is the **empty function**, namely the function with empty domain (which therefore does nothing). On a related note, the **only metric** on a singleton set is the **zero function**.

for every set X, the empty function is a map from $\varnothing$ to X.

the zero function on X is the map $f : X \to \mathbb{R}$ defined by $f(x) = 0$ for all $x \in X$.

5.1.2 Metrics on unexpected sets

Example. Let C be any circle. For each $a, b \in C$, define $d(a, b)$ to be the shorter distance along the circle from a to b. Then d is a metric on C. Similarly if S is a sphere and $a, b \in S$, we can define $d(a, b)$ to be the shortest distance along a great circle joining a and b; this is well defined and thus determines a metric on S because the great circle is unique unless a and b are equal or antipodal.

5.1.3 Metrics on $\mathbb{R}^n$

Next, we look at some examples of metrics on $\mathbb{R}^n$ and its friendly neighbour $\mathbb{C}$. In particular, we will look at metrics on $\mathbb{R}$.

Example (Euclidean n-Space). The function $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_2(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2}$$

where x_i and y_i are coordinates of the points x and y respectively, is a metric. It is called the **Euclidean metric** or the **usual metric** on $\mathbb{R}^n$. The pair $(\mathbb{R}^n, d_2)$ is called the **Euclidean n-Space**.

we will **always** assume the Euclidean metric on $\mathbb{R}^n$ unless mentioned otherwise.

Example (Metric on $\mathbb{C}$). The function $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by the (complex) modulus function $(z, w) \mapsto |z - w|$ is a metric on $\mathbb{C}$ and it is also called the **Euclidean metric**. There is no inconsistency here as d is an extension to $\mathbb{C} \times \mathbb{C}$ of the Euclidean metric on $\mathbb{R}$. We shall assume that $\mathbb{C}$ is given this metric unless stated otherwise; for this reason we will also call it the **usual metric** on $\mathbb{C}$.

Now we look at metrics on $\mathbb{R}^n$ which are different from the Euclidean metric.

Example (Taxicab and square). The functions $d_1, d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \quad \text{and} \quad d_\infty(x, y) = \max\{|x_i - y_i| : 1 \leqslant i \leqslant n\},$$

are metrics on $\mathbb{R}^n$. We call d_1 the **taxicab metric** and d_∞ the **square metric**.

Proposition 5.3. For any $x, y \in \mathbb{R}^n$,

$$d_\infty(x, y) \leqslant d_2(x, y) \leqslant d_1(x, y) \leqslant n d_\infty(x, y)$$

There are many other more exotic metrics on $\mathbb{R}$.

Example (Metric from injective real-valued functions). Let $f : \mathbb{R} \to \mathbb{R}$; and for $x, y \in \mathbb{R}$ define the map $d_f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d_f(x, y) = |f(x) - f(y)|$$

This is a metric on $\mathbb{R}$ *if and only if* f is **injective**. If it is not injective, we still have that d_f is a pseudometric. Let's check this:

[M2] Clearly, d_f is symmetric no matter what f we choose.

[M3] For any function $f : \mathbb{R} \to \mathbb{R}$ and $x, y, z \in \mathbb{R}$, we have

$$\begin{aligned} d_f(x, z) = |f(x) - f(z)| &= |(f(x) - f(y)) + (f(y) - f(z))| \\ &\leqslant |f(x) - f(y)| + |f(y) - f(z)| \\ &= d_f(x, y) + d_f(y, z) \end{aligned}$$

so d_f satisfies the triangle inequality.

[M1] Obviously, $d_f(x, y) > 0$ for all $x, y \in \mathbb{R}$ and $d_f(x, x) = 0$ for all $x \in \mathbb{R}$. Up to this point, d_f is a pseudometric. However, we do not have $d_f(x, y) = 0$ if and only if $x = y$ for general[6] real-valued functions — we only have the necessary condition

[6] *$f(x) = x^2$ is an example. $d_f(-2, 2) = 0$ but $-2 \neq 2$.*

but not sufficient. For this to be sufficiently true we require f to be injective. If f is injective, then whenever $d_f(x, y) = 0$, we have $f(x) = f(y)$ implying $x = y$.

For these reason, every injective function one can think of can generate a metric on $\mathbb{R}$.

Example. The functions $d, \eta, \rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} d(x, y) &= |x^3 - y^3|, \\ \eta(x, y) &= |e^x - e^y|, \\ \rho(x, y) &= |\arctan(x) - \arctan(y)|, \end{aligned}$$

are all metrics on $\mathbb{R}$. We call η the **exponential metric** on $\mathbb{R}$.

Next we consider metrics on subsets of $\mathbb{R}$, mainly on $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{Q}$.

Example (Inverse metric). Since the inverse function $x \mapsto 1/x$ is injective, the function $d : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$d(x, y) = \left| \frac{1}{a} - \frac{1}{b} \right|$$

is a metric on $\mathbb{R}^+$ and is called the **inverse metric**.

The next metric is immensely useful in algebraic number theory.

Example (p-adic metric). Let p be a prime number. Each non-zero rational number x can be expressed as $p^k r/s$ for a unique value of $k \in \mathbb{Z}$, where $r \in \mathbb{Z}$ and $s \in \mathbb{N}$ and neither r nor s is divisible by p. We define $|x|_p$ to be p^{-k}; and set $|0|_p$ to be 0. The function $d : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$ defined by

$$d(a, b) = |a - b|_p$$

is a metric on $\mathbb{Q}$ and is called the **p-adic metric** on $\mathbb{Q}$.

$d(a, b)$ is clearly a non-negative symmetric function and is 0 iff $a = b$. It can be easily checked that if $|a - c|_p = p^{-m}$ and $|c - b|_p = p^{-n}$, then $|a - b|_p \leqslant \max\{p^{-m}, p^{-n}\}$ and so d satisfies the triangle inequality.

5.1.4 Exotic metrics

Every set admits a metric. Here we look at more exotic ones. Firstly, we present our sanity checker and go-to metric: the discrete metric.

Example (Discrete metric). Let X be any non-empty set; let d be the function defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Then d is a metric on X and is called the **discrete metric** on X.

Here we present a generalized version of generating metrics from injective functions. If one would to look at the proof of the case on $\mathbb{R}$, the only thing which we rely on $\mathbb{R}$ were really it's metric property.

Example (Metric from injective functions). Let (X, d) be a metric space and S be any set; let $f : S \to X$ be an injective function. Then the function $\mu_f : S \times S \to \mathbb{R}$ defined by

$$\mu_f(a, b) = d(f(a), f(b))$$

is a metric on S.

Even if f is not injective, μ_f is clearly a pseudometric because d is a metric. Again, note that it is always true that if $a = b$, we have $\mu_f(a, b) = d(f(a), f(b)) = 0$ regardless of the choice of f because d is a metric. The other direction, however, is true iff f is injective.

Example. Let (X, d) be a metric space. Define $\rho : X \times X \to \mathbb{R}$ by

$$\rho(x, y) = kd(x, y)$$

where $k > 0$ is a positive real number, is a metric on X.

Example (Standard bounded metric). Let (X, d) be a metric space. Define $\bar{d} : X \times X \to \mathbb{R}$ by

$$\bar{d}(x, y) = \min\{d(x, y), 1\}.$$

Then $\bar{d}$ is a metric on X and is called the **standard bounded metric** corresponding to d.

Proof. [**M1**] and [**M2**] are clear. We are left to prove [**M3**], the triangle inequality. We want to show that

$$\bar{d}(x, z) \leqslant \bar{d}(x, y) + \bar{d}(y, z).$$

We deal with this case by case. **Case (i).** If $d(x, y) \geqslant 1$, then $\bar{d}(x, y) = 1$. This implies that

$$\bar{d}(x, z) \leqslant 1 = \bar{d}(x, y) \leqslant \bar{d}(x, y) + \bar{d}(y, z).$$

the first inequality is a truth in general. For any $r \in \mathbb{R}$, $\min\{r, 1\} \leqslant 1$.

The same story goes if $d(y, z) \geqslant 1$. **Case (ii).** If both $d(x, y) < 1$ and $d(y, z) < 1$, then $\bar{d}(x, y) = d(x, y)$ and $\bar{d}(y, z) = d(y, z)$. Consequently,

$$\bar{d}(x, z) \leqslant d(x, z) \leqslant d(x, y) + d(y, z) = \bar{d}(x, y) + \bar{d}(y, z).$$

the first inequality is also a truth in general. For any $r \in \mathbb{R}$, $\min\{r, 1\} \leqslant r$.

We are done. ■

Example. Let (X, d) be a metric space. Then the function $\rho : X \times X \to \mathbb{R}$ defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

is a metric on X.

5.1.5 Metric subspaces

Let (X, d) be a metric space. Observe that the restriction of the function d to the points in a subset $Y \subset X$ is also a metric on Y. For this reason, we may write if d is a metric on X, then d is a metric on $Y \subset X$. Moreover we call (Y, d) a metric subspace.

Definition 5.2. Let (X, d) be a metric space; let Y be a non-empty subset of X. Let $d_Y : Y \times Y \to \mathbb{R}$ be the restriction of d to Y. Then we call (Y, d_Y) a **metric subspace** of (X, d) and d_Y is called the **metric on Y induced by d**.

In fact, (Y, d) is a (topological) subspace of (X, d).

Example (Product spaces). Given two metric spaces (X, d_X) and (Y, d_Y) we can define several new metrics on $X \times Y$. For points $a = (x_1, y_1)$ and $b = (x_2, y_2)$ in $X \times Y$ let

$$\begin{aligned} d_1(a, b) &= d_X(x_1, x_2) + d_Y(y_1, y_2), \\ d_2(a, b) &= \left(d_X(x_1, x_2)^2 + d_Y(y_1, y_2)^2\right)^{1/2}, \\ d_\infty(a, b) &= \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}. \end{aligned}$$

These are all metrics on $X \times Y$.

Proposition 5.4. For any $p, q \in X \times Y$,

$$d_\infty(p,q) \leqslant d_2(p,q) \leqslant d_1(p,q) \leqslant 2d_\infty(p,q).$$

Example. Let $\mathcal{B}\langle[a,b],\mathbb{R}\rangle$ be the set of all bounded real-valued functions $f : [a,b] \to \mathbb{R}$. Given two points f, g in $\mathcal{B}\langle[a,b],\mathbb{R}\rangle$ let

$$d_\infty(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|.$$

This is a metric and it is called the **sup metric** or the **uniform metric**. The resulting metric space is denoted $(\mathcal{B}\langle[a,b],\mathbb{R}\rangle, d_\infty)$. This notation, however, is not universally agreed.

Any continuous function $f : [a,b] \to \mathbb{R}$ is bounded. Therefore, the set of all such continuous functions forms a subspace of $\mathcal{B}\langle[a,b],\mathbb{R}\rangle$. We will write this resulting subspace as $\mathcal{C}[a,b]$.

Example. Let X be the set of all continuous functions $f : [a,b] \to \mathbb{R}$ but now let

$$d_1(f,g) = \int_a^b |f(t) - g(t)| \, \mathrm{d}t$$

This is a metric and it is called the **L^1 metric**.

Example. Let X be the set of all continuous functions $f : [a,b] \to \mathbb{R}$ and let

$$d_2(f,g) = \left\{ \int_a^b |f(t) - g(t)|^2 \, \mathrm{d}t \right\}^{1/2}$$

This is a metric and it is called the **L^2 metric**.

5.2 Distances between sets, Bounded sets.

Definition 5.3 (Point-set distance). Let d be a metric on a set X. The *distance* between a point $x \in X$ and a non-empty subset A of X is defined by

$$d(x, A) = \inf\{d(x, a) \mid a \in A\}$$

i.e. the *infimum* of the distances from x to points of A.

Definition 5.4 (Set-set distance). Let d be a metric on a set X. The *distance* between two non-empty subsets A and B of X is defined by

$$d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$$

i.e. the *infimum* of the distances from points in x in A to points in B.

Definition 5.5 (Diameter). If S is a non-empty bounded subset of a metric space with metric d, then the **diameter of S** is defined to be the set $\sup\{d(x, y) : x, y \in S\}$. The diameter of the empty set is 0.

Remark. For the empty set, we adopt the following convention:

(i) $d(x, \varnothing) = \infty$;

(ii) $d(A, \varnothing) = d(\varnothing, A) = \infty$;

(iii) $\operatorname{diam} \varnothing = 0$.

Example (Discrete Metric on Sets). Let d be the discrete metric on a non-empty set X. Then for $x \in X$ and $A, B \subset X$,

$$d(x, A) = \begin{cases} 1, & \text{if } p \neq A \\ 0, & \text{if } p \in A \end{cases} \qquad d(A, B) = \begin{cases} 1, & \text{if } A \cap B = \varnothing \\ 0, & \text{if } A \cap B \neq \varnothing \end{cases}$$

Example. Consider the following intervals on $\mathbb{R}$: $A = [0, 1)$ and $B = (1, 2]$. Let d be the usual metric and ρ be the discrete metric. Then, $d(A, B) = 0$ whereas $\rho(A, B) = 1$ since A and B are disjoint.

The following proposition comes straight from definitions.

Proposition 5.5. Let A and B be non-empty subset of X; let $x \in X$. Then:

(a) $d(x, A)$, $d(A, B)$ and $\operatorname{diam} A$ are non-negative real numbers.

(b) If $x \in A$, then $d(x, A) = 0$.

(c) If $A \cap B$ is non-empty, then $d(A, B) = 0$.

(d) If A is finite, then A is bounded.

Proposition 5.6. Let d be a metric on a set X. Then for any subset $A, B \subset X$:

(1) $d(A \cup B) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B) + d(A, B)$.

(2) $\operatorname{diam}(\bar{A}) = \operatorname{diam}(A)$.

Here, we state a not so surprising result.

Theorem 5.1. Let d be a metric on a set X, and let $A \subset X$. Then the function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.

Now we define the notion of boundedness. Of course, we could have defined this earlier using the notion of diameters. We put that as a proposition and instead choose the following definition:

Definition 5.6 (Bounded). A subset S of a metric space (X, d) is **bounded** if there exist $c \in X$ and $K \in \mathbb{R}$ such that $d(x, c) \leqslant K$ for all $x \in S$.

It is easy to see the following proposition.

Proposition 5.7. A subset S of a metric space (X, d) is bounded *if and only if* $\operatorname{diam} S$ is finite.

Definition 5.7 (Bounded functions). If $f : S \to X$ is a function from a set S to a metric space X, then we say **f is bounded** if the subset $f(S)$ of X is bounded.

Here we come to an obvious result.

Proposition 5.8. The union of any finite number of bounded subsets of a metric space is bounded.

Now, we state more results regarding bounded sets on metric spaces.

Proposition 5.9. Let (X, d) be a metric space; and $A \subset X$. Then A is bounded *if and only if* there is some constant $K > 0$ such that $d(a, b) \leqslant K$ for all $a, b \in A$.

Proposition 5.10. Suppose that $A \subset B$ where B is a bounded subset of a metric space X. Then A is bounded and $\operatorname{diam} A \leqslant \operatorname{diam} B$.

Proposition 5.11. Suppose that A is bounded a subset of a metric space X. Then $\bar{A}$ is bounded and $\operatorname{diam} \bar{A} = \operatorname{diam} A$.

Proposition 5.12. Let A and B be bounded subsets of a metric space X such that $A \cap B \neq \varnothing$. Then

$$\operatorname{diam}(A \cup B) \leqslant \operatorname{diam} A + \operatorname{diam} B$$

Finally we state a bounded property of subsets in $\mathbb{R}$.

Proposition 5.13. Suppose that A is a non-empty bounded subset of $\mathbb{R}$. Then $\sup A$ and $\inf A$ are in $\bar{A}$.

5.3 Open balls in metric spaces

Definition 5.8 (Open Ball). Let d be a metric on a set X. For any point $x_0 \in X$ and any positive real number $\varepsilon > 0$, consider the set

$$B_d(x_0, \varepsilon) = \{y : d(x_0, y) < \varepsilon\}$$

of all points y whose distance from x_0 is strictly less than ε. This set is called the **d-open ball** with center x_0 and radius ε. When there is no confusion, we will just call it an open ball and denote it without the subscript d.

Example. In $\mathbb{R}$ (with its usual metric), $B(x_0, \varepsilon)$ is the open interval $(x_0 - \varepsilon, x_0 + \varepsilon)$.

Example. Consider the Euclidean 2-space, $(\mathbb{R}^2, d_2)$. Then $B(x_0, \varepsilon)$ is the open disc of radius ε centred at x_0.

Example. Consider the Euclidean 3-space, $(\mathbb{R}^2, d_2)$. Then $B(x_0, \varepsilon)$ is the open ball of radius ε centred at x_0.

Example. Let $X = \mathbb{R}^2$, but now give it the metric $d_1(x, y) = \sum |x_i - y_i|$. Then $B(x_0, \varepsilon)$ is the inside of the square centred at x_0 with diagonals[7] of length 2ε parallel to the axes.

[7] *not sides!*

Example. Let d denote the discrete metric on some set X; let $x_0 \in X$. Then

$$B(x_0, \varepsilon) = \begin{cases} X, & \text{if } \varepsilon > 1 \\ \{x_0\}, & \text{if } \varepsilon \leqslant 1 \end{cases}$$

Example. Let X be the set $\mathcal{B}([0, 1], \mathbb{R})$ of all bounded real-valued functions on $[0, 1]$, and let d be the uniform metric d_∞. Then for $f_0 \in X$ and $\varepsilon > 0$ a real number, $B(f_0, \varepsilon)$ is the set of all functions $f \in X$ whose graph lie inside a ribbon of vertical width $2r$ centred at the graph of f_0 i.e. the area bounded by $f_0 - \varepsilon$ and $f_0 + \varepsilon$.

As we have seen in some of the examples above, *balls* are not always round. Moreover, we have seen that balls do depend on the metric in general. We will now show that it may also depend on the underlying set.

Example. Let $A = [0, 1] \subset \mathbb{R}$ with the Euclidean metric on $\mathbb{R}$ and the induced metric d_A on A. Then we have $B_d(1, 1) = (0, 2)$ while on the other hand $B_{d_A}(1, 1) = (0, 1]$.

We can rephrase the language of boundedness by using open balls.

Proposition 5.14 (Boundedness II). A subset S of a metric space X is bounded iff $S \subset B(x_0, \varepsilon)$ for some $x_0 \in X$ and $r > 0$.

Corollary 5.1. If S is a bounded set in $\mathbb{R}^n$ (Euclidean metric), then S is contained in the product $[a, b]^n$ for some $a, b \in \mathbb{R}$.

$[a, b]^n = [a, b] \times [a, b] \times \cdots \times [a, b]$ where the product is done n times.

We have the following interesting fact.

Topological Fact. Any metric space equipped with the discrete metric is bounded.

Proof. Let (X,d) be a metric space endowed with the discrete metric. Then for any $x \in X$, we have

$$B_d(x,\varepsilon) = \begin{cases} X & \text{if } \varepsilon > 1 \\ \{x\} & \text{if } \varepsilon \leqslant 1, \end{cases}$$

and $X \subset X$. ■

An important property of open balls in metric spaces is highlighted in the following lemma. It says that for every point in an open ball, you can find a (sufficiently small) open ball around that point as well.

Lemma 5.2. For every point $p \in B(x,\varepsilon)$, there is $\delta > 0$ such that $B(p,\delta) \subset B(x,\varepsilon)$.

this lemma is basically saying that "I can find another ball inside a ball".

Proof. Let $p \in B(x,\varepsilon)$. Then by definition, $d(x,p) < \varepsilon$. Set

$$\delta = \varepsilon - d(x,p) > 0$$

We claim that $B(p,\delta) \subset B(x,\varepsilon)$. Suppose $y \in B(p,\delta)$. Then by definition $d(p,y) < \delta$. Now, observe that by the triangle inequality we have

$$d(x,y) < d(x,p) + d(p,y) < d(x,p) + \delta = d(x,p) + [\varepsilon - d(x,p)] = \varepsilon$$

So, $y \in B(x,\varepsilon)$ as required. ■

In general, the intersection of two open balls need not be an open ball. However, every point in the intersection of two open balls does belong to an open ball contained in the intersection.

Lemma 5.3. Let B_1 and B_2 be open balls and let $p \in B_1 \cap B_2$. Then, there is an open ball B_p centred at p such that $B_p \subset B_1 \cap B_2$.

Proof. Let B_1 and B_2 be open balls and let $p \in B_1 \cap B_2$. Then $p \in B_1$ and $p \in B_2$. By the preceding lemma, there are positive real numbers δ_1 and δ_2 such that $B(p,\delta_1) \subset B_1$ and $B(p,\delta_2) \subset B_2$. Taking $\delta = \min\{\delta_1,\delta_2\}$, we have $B(p,\delta) \subset B_1 \cap B_2$, as desired. ■

Lemma 5.4. Let X be a metric space and $x,y \in X$; and let $\varepsilon > 0$. If $y \in B(x,\varepsilon/2)$, then $B(y,\varepsilon/2) \subset B(x,\varepsilon)$.

Another useful notion of balls are the closed balls.

Definition 5.9 (Closed Balls). Let d be a metric on a set X. For any point $x_0 \in X$ and any positive real number $\varepsilon > 0$, the set

$$B_d[x_0,\varepsilon] = \{y : d(x_0,y) \leqslant \varepsilon\}$$

of all points y whose distance from x_0 is less than (or equal to) ε, is called the **closed ball** with center x_0 and radius ε.

A rookie mistake is to think that the closure of an open ball $B(x,\varepsilon)$ is the closed ball $B[x,\varepsilon]$. This is not always true!

Since for every $x \in X$ (which is equipped with a metric d), x is an element of the ball $B(x,\varepsilon)$ for every $\varepsilon > 0$; and together with Lemma (5.3) above, we have indeed shown that the collection of all open balls is a base for a topology on X.

Theorem 5.5. Let d be a metric on a non-empty set X. Then the collection of open balls is a base for a topology on X.

5.4 Metric Topology (part I)

We will now start a series of three parts of the metric topology. This first part is short and simple, we will discuss how to get a topology from metrics.

Definition 5.10 (Metric Topology). Let d be a metric on a non-empty set X. The topology, $\mathcal{T}$ generated by the collection of open balls in X is called the **metric topology induced by d** or simply, the **metric topology**.

Here we give an alternative definition of a metric space.

Definition 5.11 (Metric Space). Let d be a metric on a non-empty set X. The set X together with the metric topology $\mathcal{T}$ induced by d is called a **metric space** and is denoted (X, d).

Basically, a metric space is a topological space in which the topology is induced by a metric.

Since a metric space is a topological space, all concepts defined for topological spaces are also defined for metric spaces. In fact, we will see that the correspondence is quite natural (after all, topological spaces are generalization of *metric-like* property of $\mathbb{R}$).

Example. The usual metric on $\mathbb{R}$ induces the usual topology on $\mathbb{R}$. The usual metric on the plane $\mathbb{R}^2$ induces the usual topology on $\mathbb{R}^2$.

Example. Let d be the discrete metric on some set X. Now observe that for any $x \in X$, we have $B(x, 1/2) = \{x\}$. So, open balls in the discrete metric are singletons (and X itself). Hence, every singleton set is open; and thus every set is open. In other words, the discrete metric on X induces the discrete topology on X.

Theorem 5.6. If the topological space X is homeomorphic to a metric space Y, then X is metrizable.

5.5 Metric Topology (part II)

Frankly speaking, part II of this three-series is superfluous. However, we feel that there is some merit to making things explicit and focusing on examples that are purely metric spaces related. Here, we will try to redefine open, closed, closure etc. purely using the language of open balls (which forms a basis for the metric topology).

If the definition transcends verbatim from topological spaces, we will omit it if needed. A strong remark to the amateur readers that everything written here is consistent with every definition made when discussing topological spaces. These are just special cases, because a metric space is, after all, a topological space.

Definition 5.12. Let (X, d) be a metric space and $U \subset X$. We say that U is **d-open** or **open** in X if for every $x \in U$, there is $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subset U$.

A set being open is like saying that "there is an elbow-room around each point" of the set. Finally, we have something intuitive! As this is a generalization of Lemma (5.2), the next proposition comes as no surprise at all.

Proposition 5.15. Any open ball in a metric space X is open in X.

Proof. This is Lemma (5.2). We have proven it there. ■

Before we look at examples below, we give an equivalent definition of being open in a metric space.

Theorem 5.7. A subset of a metric space is open *if and only if* it is a union of open balls.

Example. Any open interval (a, b) in $\mathbb{R}$ is open in $\mathbb{R}$. On the other hand, intervals in $\mathbb{R}$ such as $[a, b]$, $[a, b)$, $(a, b]$ are not open in $\mathbb{R}$:

For $a \in [a, b)$, no matter how small a positive ε we take, $B(a, \varepsilon)$ contains points such as $a - \varepsilon/2$ to the left of a, which are not in $[a, b)$.

Note that not every open set is an open ball!

Example. In $\mathbb{R}^2$ (with the Euclidean metric d_2) let U be the interior of a rectangle say

$$U = \{(x_1, x_2) \in \mathbb{R}^2 : a < x_1 < b, c < x_2 < d\}.$$

If $x = (x_1, x_2) \in U$ and we set $\varepsilon = \min\{x_1 - a, b - x_1, x_2 - c, d - x_2\}$, it can be easily seen that $B(x, \varepsilon) \subset U$. Therefore U is open, but it is not an open ball of $(\mathbb{R}^2, d_2)$.

Example. For any metric space X, the whole set X and the empty set $\varnothing$ are both open in X.

this follows trivially from the definition of open.

We can give a better perspective of why the whole set is open in itself.

Example. Let (X, d) be a metric space and consider $a \in X$. Then for each $x \in X$, we have

$$x \in \{\flat : d(\flat, a) < d(a, x) + 1\} = B_d(a, d(a, x) + 1),$$

convince yourself by taking $X = \mathbb{R}$, $a = 0$ and, say, $x = 1$.

so that

$$X = \bigcup \{B_d(a, \varepsilon) \mid \varepsilon > 0\}$$

Since X is a union of open balls, it must be open.

Example. In a metric space X endowed with the **discrete metric**, any subset $A \subset X$ is open in X. If $x \in A$, we can choose $\varepsilon_x = 1$ (or even $1/3$), and then $B(x, \varepsilon_x) = \{x\} \subset A$.

Example. The open upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is open as it can be expressed as the union of open balls

$$\mathbb{H} = \bigcup_{z \in \mathbb{H}} B_d(z, \operatorname{Im} z)$$

where d is the Euclidean metric inherited from $\mathbb{C}$.

We have to be careful about which metric we are using when we want to say that a (sub)set is open in that space. This is why it pays to favour brevity over precision as the information really lies in both the set and the metric.

Example. A singleton subset such as $\{0\}$ is open in $\mathbb{R}$ with the discrete metric, but not in $\mathbb{R}$ with the usual (Euclidean) metric d_2.

The interval $[a, b]$ is open in $[a, b]$ with the inherited Euclidean metric, but not in the ambient space $\mathbb{R}$.

The interval (a, b) is open in $\mathbb{R}$, but not in $\mathbb{R}^2$ when we identify (a, b) with the line $(a, b) \times \{0\}$: if we take any $x_0 \in (a, b) \times \{0\}$, there is no $\varepsilon > 0$ such that the disc

$B(x_0, \varepsilon)$ in $\mathbb{R}^2$ is contained in $(a, b) \times \{0\}$ — any such disc contains points which are off the x-axis.

Now, we revisit the notion of "closed" from topological spaces. Turns out, there's nothing new here but we will state it for completion. What will be interesting are the examples below.

Definition 5.13. Let X be a metric space. A subset A of X is **closed** in X if A^c is open in X.

Proposition 5.16. Any closed ball in a metric space X is closed in X.

Now, we discuss some examples of closed sets in a metric space.

Example. $[a, b]$, $(-\infty, 0]$, $\{0\}$ and $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ are all closed sets in $\mathbb{R}$ with the usual Euclidean metric.

Example. The *closed unit disc* $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$ is closed in $\mathbb{R}^2$.

Example. The *closed rectangle* $\{(x_1, x_2) \in \mathbb{R}^2 : a \leqslant x_1 \leqslant b, c \leqslant x_2 \leqslant d\}$ is closed in $\mathbb{R}^2$.

Example. Let X be any (non-empty) set equipped with the discrete metric. Then any subset of X is closed in X.

Example. In the space $\mathcal{C}([0, 1])$ of continuous real-valued functions on $[0, 1]$ with the uniform metric d_∞, the subset

$$\{f \in \mathcal{C}([0, 1]) : f(1) = 0\}$$

is closed because its complement is open:

If $f(1) \neq 0$, then $g(1) \neq 0$ for all $g \in \mathcal{C}([0, 1])$ close enough to f in the uniform metric. For example, $g \in \mathcal{C}([0, 1])$ such that

$$d_\infty(g, f) < \frac{|f(1)|}{2}$$

Let A be a subset of a metric space X. As usual we denote the closure of A in X to be $\bar{A}$, the interior of A in X to be $\operatorname{int}(A)$, the set of limit points of A in X to be A' and the boundary of A in X to be ∂A.

Definition 5.14. Let X be a metric space and $A \subset X$. The **closure** of A is the intersection of all closed sets containing A; the **interior** of A is the union of all open subsets of A;and the **boundary** of A is the set $\bar{A} \backslash \operatorname{int}(A)$.

Definition 5.15. Let X be a metric space and $A \subset X$; let $x \in X$. Then x is a **limit point** of A if every open neighbourhood of x contains a point of A different from x.

As the reader can see, none of the definitions really changed from the ones defined for topological spaces. However, what is interesting is the following. We can consider an open ball criterion for all of these concepts. First, we give the limit points an open ball criterion. This is straight from its definition.

Proposition 5.17 (Limit points using balls). Let (X,d) be a metric space and $A \subset X$; let $x \in X$. Then

$$x \in A' \iff (B_d(x,\varepsilon)\backslash\{x\}) \cap A \neq \varnothing \text{ for every } \varepsilon > 0.$$

Next, we give the closure an open ball criterion using the open set criterion for closure — Proposition (1.5).

Proposition 5.18 (Closure using balls). Let (X,d) be a metric space and $A \subset X$; let $x \in X$. Then

$$x \in \bar{A} \iff B_d(x,\varepsilon) \cap A \neq \varnothing \text{ for every } \varepsilon > 0.$$

Likewise, we can give its dual, the interior, an open ball criterion. Unlike closure however, this hails directly from definition.

Proposition 5.19 (Interior using balls). Let (X,d) be a metric space and $A \subset X$; let $x \in X$. Then

$$x \in \operatorname{int}(A) \iff \text{there exists an } \varepsilon > 0 \text{ such that } B_d(x,\varepsilon) \subset A.$$

Example. Let $\mathbb{R}$ be given the usual metric and consider $\mathbb{Q} \subset \mathbb{R}$. Then $\operatorname{int}(\mathbb{Q}) = \varnothing$.

Proof. Let $x \in \mathbb{R}$. By the preceding proposition, we require an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset \mathbb{Q}$ i.e. **wholly contained in** $\mathbb{Q}$ if we want $x \in \operatorname{int}(\mathbb{Q})$. However, we can't do this as $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$ i.e. we can always find an irrational number in $(x-\varepsilon, x+\varepsilon)$ for any $\varepsilon > 0$. Accordingly, it is impossible for $(x-\varepsilon, x+\varepsilon) \subset \mathbb{Q}$. So, $x \notin \operatorname{int}(\mathbb{Q})$ and since x was arbitrary, $\operatorname{int}(\mathbb{Q}) = \varnothing$.

We can run the same argument (with a little tweak) to prove that $\operatorname{int}(\mathbb{R}\backslash\mathbb{Q}) = \varnothing$ — of course, $\mathbb{R}$ is again given the usual metric in this case as well.

the *little tweak* is to replace the fact that $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$ with $\mathbb{Q}$ is dense in $\mathbb{R}$.

Finally, we give the boundary an open ball criterion. This is a special case of the open set criterion for boundary — Theorem (1.18).

Proposition 5.20 (Boundary using balls). Let (X,d) be a metric space and $A \subset X$; let $x \in X$. Then

$$x \in \partial A \iff B_d(x,\varepsilon) \cap A \neq \varnothing \text{ and } B_d(x,\varepsilon) \cap A^c \neq \varnothing \text{ for every } \varepsilon > 0.$$

5.6 Metric Topology (part III)

As the topology of a metric space X is induced by the metric on X, it is natural to expect that topological properties of X are related to distance properties of X.

Theorem 5.8. Let (X,d) be metric space and $A \subset X$. Then

$$\bar{A} = \{x \in X \mid d(x,A) = 0\}$$

that is, the set of points whose distance from A is zero.

Proof. Suppose $x \in \{x \in X \mid d(x, A) = 0\}$. Then $d(x, A) = 0$. Observe that every open ball $B(x, \varepsilon)$ centred at x contains at least one point of A. Consequently, every open set U containing x, also contains at least one point of A. Therefore, $x \in A$ or $x \in A'$, and thus $x \in \bar{A}$.

Conversely, we prove the contrapositive. Suppose $x \notin \{x \in X \mid d(x, A) = 0\}$. Then, $d(x, A) = \varepsilon > 0$. Now observe[8] that the open ball $B(x, \varepsilon/2)$ centred at x contains no point of A. By Proposition (1.5), $x \notin \bar{A}$, as desired. ■

[8] *because $d(x, A) = \inf d(x, a) = \varepsilon > \varepsilon/2$, so you can't possibly find points of A in the open ball with radius $\varepsilon/2$.*

Corollary 5.2. In a metric space X, all finite sets are closed.

Proof. Let $p \in X$ and consider the singleton subset $\{p\}$ of X. Observe that $d(x, \{p\}) = \inf\{d(x, p)\} = d(x, p)$. $d(x, \{p\}) = 0$ *if and only if* $d(x, p) = 0$ which by positive-definiteness of metric, is true *if and only if* $x = p$. By the preceding Theorem (5.8), the closure of the singleton of $\{p\}$ is thus itself, so $\{p\}$ is closed. Since $p \in X$ was arbitrary we conclude that singletons in X are closed. But since finite sets are finite union of singletons, and union of closed sets is closed, we get that finite sets are closed. ■

Corollary 5.3. Let (X, d) be a metric space and $F \subset X$. Then

$$F \text{ is closed} \iff \{x \in X : d(x, F) = 0\} \subset F$$

Proof. This is obvious. A subset F of a topological space X is closed if and only if $F = \bar{F}$. By the preceding Theorem (5.8), we have $\bar{F} = \{x \in X : d(x, F) = 0\}$. So, F is closed if and only if $F = \{x \in X : d(x, F) = 0\}$, in particular, if and only if $F \supset \{x \in X : d(x, F) = 0\}$ as required. ■

Corollary 5.4. Let (X, d) be a metric space and $x \in X$; let $F \subset X$. If F is closed and $x \notin F$, then $d(x, F) \neq 0$.

Proof. If $d(x, F) = 0$ and F is closed, by the preceding Corollary (5.3) we have that $x \in F$. But by hypothesis, $x \notin F$ so $d(x, F) \neq 0$ as desired. ■

This is quite a revealing moment for us. These corollaries tells us that metric spaces possess certain topological properties which do not hold for the general topological space. Below is even a more revealing truth.

Theorem 5.9 (Separation Axiom). Let X be a metric space; let A, B be closed disjoint subsets of X. Then there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Proof. Not too hard but omitted. ■

Remember how topology broke our minds and intuitions? We thought metric spaces should be better right? Not exactly. From the Separation Axiom, one would definitely inspect that the distance between two closed disjoint sets must be greater than 0. However, this is not true! WTF?

Example. Let $\mathbb{R}^2$ be equipped with usual metric. Consider the following subsets of $\mathbb{R}^2$:

$$A = \{(x, y) : xy \geqslant -1, x < 0\}, \qquad B = \{(x, y) : xy \geqslant 1, x > 0\}$$

Now observe that A and B are both closed (in the metric topology). However $d(A, B) = 0$.

5.7 Convergence in metric spaces

Again, this section is really superfluous. And again, we believe there is merit to making things explicit. However, it is good for the reader to keep in mind that the definition of convergence here is completely equivalent to the definition of convergent for sequences in topological spaces.

Definition 5.16 (Convergence). Let (X, d) be a metric space. A sequence $\langle x_n \rangle$ of points in X is said to **converge** to $x \in X$ if for every $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > N_\varepsilon$. We write $x_n \to x$ as $n \to \infty$ or simply just, $x_n \to x$.

We have our first result regarding convergence of sequences in metric spaces.

Proposition 5.21. Let (X, d) be a metric space and $x \in X$; let $\langle x_n \rangle$ be a sequence in X. Then

$$x_n \to x \quad \textit{if and only if} \quad d(x_n, x) \to 0$$

as n gets sufficiently large.

N.B. $\langle x_n \rangle$ is a sequence of points in X; whereas $\langle d(x_n, x) \rangle$ is a sequence of real numbers!

The following lemma gives a criteria for when a sequence converges.

Lemma 5.10. Let (X, d) be a metric space and $x \in X$; let $\langle x_n \rangle$ be a sequence of points in X. If $\langle R_n \rangle$ is a sequence of non-negative real numbers such that $R_n \to 0$ and $d(x_n, x) \leqslant R_n$ for all (sufficiently large) n, then $x_n \to x$.

Proposition 5.22 (Uniqueness of Limits). Let (X, d) be a metric space and $x, y \in X$. If $\langle x_n \rangle$ is a sequence in X such that $x_n \to x$ and $x_n \to y$ as $n \to \infty$, then $x = y$.

This allows us to write $x = \lim_{n\to\infty} x_n$ as we now know whenever a sequence a converge to a limit, it converges to the limit.

Theorem 5.11. Let (X, d) be a metric space; and $x, y \in X$; let $\langle a_n \rangle$ and $\langle b_n \rangle$ be sequences in X. If $a_n \to x$ and $b_n \to y$, then

$$d(a_n, b_n) \to d(x, y)$$

Finally, we list two important theorems which relates the notion of open and closed sets with convergence.

Theorem 5.12 (Open Sequence Criterion). Let X be a metric space and $A \subset X$. Then A is open *if and only if* all sequences $\langle x_n \rangle$ in X which converges to a point $a \in A$ are eventually[9] in A.

[9] *i.e. all but finitely many of* $\langle x_n \rangle$ *are in* A.

Theorem 5.13 (Closed Sequence Criterion). Let X be a metric space and $A \subset X$. Then A is closed *if and only if* for all sequences $\langle a_n \rangle$ in A with $a_n \to a$, we have $a \in A$.

5.8 Continuous functions on metric spaces

Here, we would like to show that the "ε-δ" definition of continuity carries over to general metric spaces, and so does the "convergent sequence definition" of continuity.

Definition 5.17 (Continuity). Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **continuous** at $c \in X$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$d_X(x, c) < \delta \implies d_Y(f(x), f(c)) < \varepsilon$$

Here we remind you that continuity is really a topological property.

Example. Let d_2 be the usual metric and d be the discrete metric. Let $f : (\mathbb{R}, d_2) \to (\mathbb{R}, d)$ be the function defined by $f(x) = \sin x$. Then f is not continuous! Suppose for contradiction that it is continuous, say, at $0 \in \mathbb{R}$. Then taking $\varepsilon = 1/2$, we can find a $\delta > 0$ such that

$$d_2(x, 0) = |x| < \delta \implies d(\sin x, 0) < 1/2.$$

But $d(\sin x, 0) < 1/2$ implies that $d(\sin x, 0) = 0$ and hence by property of metrics, $\sin x = 0$ for all $x \in \mathbb{R}$ [$\lightning$]; which is absurd!

Why is this the case? It is really because of the topology. The discrete metric induces the discrete topology. And it is super hard for non-constant functions $f : (\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{discrete})$ to guarantee preimages of every singleton (which is open in the discrete topology) in $\mathbb{R}$ to be open in $\mathbb{R}$.

For example, for $f(x) = \sin x$ above. Then $\{0\}$ is open in the discrete $\mathbb{R}$. But $f^{-1}(\{0\}) = \{n\pi : n \in \mathbb{N}\}$ is not open in the Euclidean $\mathbb{R}$.

Lemma 5.14 (Sequence Lemma). Let X be a topological space; let $A \subset X$. If there is a sequence $\langle a_n \rangle$ in A converging to $x \in X$, then $x \in \bar{A}$; the converse holds if X is metrizable.

note that the forward direction is a topological result, requiring no metric whatsoever.

Proof. ($\implies$). Let $\langle a_n \rangle$ be a sequence in A such that $a_n \to x$, where $x \in X$. Then for every open neighbourhood U of x, there is $N \in \mathbb{N}$ such that $a_n \in U$ whenever $n > N$. This implies that every open neighbourhood of x intersects A non-trivially. By Proposition (1.5), we have $x \in \bar{A}$.

Alternate proof. Suppose $x \notin \bar{A}$. Then $x \in V = X \backslash \bar{A}$. This is open in X because $\bar{A}$ is closed in X. If $a_n \to x$, then there is $n_0 \in \mathbb{N}$ such that $a_n \in V$ for all $n > n_0$ i.e. $a_n \notin \bar{A}$. But $A \subset \bar{A}$ by definition. So $a_n \notin A$ for $n > n_0$ [$\lightning$].

($\impliedby$). Let (X, d) be a metric space and $x \in \bar{A}$. Then Proposition (1.5) tells us that for every $\varepsilon > 0$ we have $B_d(x, \varepsilon) \cap A \neq \varnothing$. The idea is now to consider open balls of radius $\varepsilon_n = 1/n$ for $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, take the open neighbourhood $B_d(x, 1/n)$ of radius $1/n$ of x, and choose $\langle a_n \rangle$ to be a point of its intersection with A. We claim that $a_n \to x$:

again, we know $B_d(x, 1/n) \cap A \neq \varnothing$ by Proposition (1.5) so the points a_n do exist.

Any open neighbourhood U of x contains an open ball $B_d(x, \varepsilon)$ centred at x with radius ε. If we choose N so that $1/N < \varepsilon$, then U contains a_i for all $i \geqslant N$; as desired. ∎

In topological spaces, we know continuity implies sequential continuity, this is Theorem

(4.5). But the converse (which was true in real analysis) is not true in general. However, we can make the converse a truth by restricting X to be a metric space.

Theorem 5.15. Let (X, d) be a metric space and Y be a topological space; and consider the map $f : X \to Y$. Then f is continuous *if and only if* it is sequentially continuous.

please revise the definition of sequential continuous if you have forgotten, this should be on page 49.

Proof. The forward direction is Theorem (4.5). We are left to show the converse. Suppose f is sequentially continuous. Let A be a subset of a metric space X. We want to show that $f(\bar{A}) \subset \overline{f(A)}$. If $x \in \bar{A}$, the preceding Sequence Lemma tells us that there is a sequence $\langle a_n \rangle$ in A converging to x. By hypothesis, $f(a_n)$ converges to $f(x)$. Since $f(a_n) \in f(A)$ (i.e. it is a sequence of points in $f(A)$), the preceding Sequence Lemma implies that $f(x) \in \overline{f(A)}$. Hence, $f(\bar{A}) \subset \overline{f(A)}$, as desired. ■

Slogan. In **metric spaces**, sequential continuity is equivalent to continuity.

We can also rephrase continuity in terms of open balls.

Proposition 5.23 (Continuity II). Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a map. Then f is continuous at c iff given $\varepsilon > 0$, there is $\delta > 0$ such that

$$f(B_{d_X}(c, \delta)) \subset B_{d_Y}(f(c), \varepsilon)$$

Now, we will consider additional methods for constructing continuous functions (aside from the ones we already have). We will first need this lemma:

Lemma 5.16. The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and the quotient operation is a continuous function from $\mathbb{R} \times \mathbb{R} \backslash \{0\}$ into $\mathbb{R}$.

Theorem 5.17. If X is a topological space, and if $f, g : X \to \mathbb{R}$ are continuous functions, then $|f|$, $f + g, f - g$ and $f \cdot g$ are continuous. If $g(x) \neq 0$ for all x, then f/g is continuous.

This final result is really a theorem of topological spaces. We put it here because of... pedantry. If we agree that addition, subtraction, multiplication and quotient are all continuous functions (of metric spaces), then we could have proved this earlier.

5.8.1 Uniform convergence

Next, we come to the notion of uniform convergence.

Definition 5.18 (Uniform Convergence). Let X be a set and (Y, d_∞) be a metric space equipped with the **uniform metric**; let $f_n : X \to Y$ be a sequence of functions. We say that the sequence $\langle f_n \rangle$ **converges uniformly** to the function $f : X \to Y$ if given $\varepsilon > 0$, there is N_ε such that

$$d_\infty(f_n(x), f(x)) < \varepsilon$$

for all $n > N_\varepsilon$ and all $x \in X$.

said differently, we say that f_n converges uniformly to f, if it converges with the uniform metric d_∞.

Uniformity of convergence depends not only on the topology of Y but also on its metric. We have the following theorem about uniformly convergent sequences (of functions):

Theorem 5.18 (Uniform Limit Theorem). Let X be a topological space and (Y, d) be a metric space; Let $f_n : X \to Y$ be a sequence of continuous functions. If $\langle f_n \rangle$ converges uniformly to f, then f is continuous.

Theorem 5.19. Let X be a topological space and let (Y, d) be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions. Let $\langle x_n \rangle$ be a sequence of points in X converging to x. Then if $\langle f_n \rangle$ converges uniformly to f, then $\langle f_n(x_n) \rangle$ converges to $f(x)$.

Definition 5.19 ((LN)Pointwise Convergence). Let $B(X)$ be a metric space, and $\langle f_n \rangle$ be a sequence of real (or complex)-valued functions in $B(X)$; let $f \in B(X)$. Then f_n **converges pointwise** to f if for every $x \in X$, there is $N_{\varepsilon,x} \in \mathbb{N}$ such that $|f(x) - f_n(x)| \leqslant \varepsilon$ for all $n > N_{\varepsilon,x}$.

Definition 5.20 ((LN)Uniform Convergence). Let $\langle f_n \rangle$ be a sequence of functions in $B(X)$; and $f \in B(X)$. We say f_n **converges uniformly** to f if $f_n \to f$ with respect to the uniform metric d_∞ i.e. the metric

$$d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|$$

the sequence can also be in some subspace of $B(S)$.

Lemma 5.20 ((LN)Uniform Convergence II). Let $\langle f_n \rangle$ be a sequence of functions in $B(X)$; and $f \in B(X)$. Then $f_n \to f$ in $B(X)$ *if and only if* for every $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N_\varepsilon$ and $x \in X$.

the difference is that N_ε is independent of x whereas the ones in the definition of pointwise convergence may depend on x.

5.8.2 Continuous functions on product metric spaces

Finally we look at results concerning product spaces.

Proposition 5.24. Suppose that $f : X \to X'$, $g : Y \to Y'$ are maps of metric spaces which are continuous at $a \in X, b \in Y$ respectively. Then the map $f \times g : X \times Y \to X' \times Y'$ given by

$$(f \times g)(x, y) = (f(x), g(y))$$

for all $(x, y) \in X \times Y$, is continuous at (a, b).

Proposition 5.25. The projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ of a metric product onto its factors, defined by

$$p_X(x, y) = x \text{ and } p_Y(x, y) = y$$

are continuous.

Definition 5.21 (Diagonal Map). The **diagonal map** $\Delta : X \to X \times X$ of any set X is the map defined by $\Delta(x) = (x, x)$.

Proposition 5.26. The diagonal map $\Delta : X \to X \times X$ of any metric space X is continuous.

5.9 Equivalent metrics

We introduced metrics to study continuity in more detail. So it is reasonable to call two metrics on a set equivalent if they make the same maps in to and out of that set continuous. That is the motivation for the following definition:

Definition 5.22 (Equivalent). Two metrics d and ρ on a set X are said to be **topologically equivalent** or just **equivalent** if they induce the same topology on X.

That is, the *collection of all d-open balls* and the *collection of all ρ-open balls* in X are bases for the same topology on X.

two metrics are equivalent if they make the same sets open.

Example. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be members of $\mathbb{R}^2$. Consider the following metrics on $\mathbb{R}^2$:

$$\begin{aligned} d_1(x, y) &= \text{usual Euclidean metric}, \\ d_2(x, y) &= |x_1 - y_1| + |x_2 - y_2| \\ d_3(x, y) &= \max\{|x_1 - y_1|, |x_2 - y_2|\} \end{aligned}$$

Then, all d_1, d_2 and d_3 induce the usual topology on the plane $\mathbb{R}^2$ because the collection of open balls of each metric is a base for the usual topology on $\mathbb{R}^2$. So, d_1, d_2 and d_3 are equivalent metrics.

Example. Consider the metric d on a non-empty set X defined by

$$d(x, y) = \begin{cases} 2, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Observe that $B(p, 1) = \{p\}$ for any $p \in X$; so singleton sets are open and d induces the discrete topology on X.

Example. On the set $\mathcal{C}[a, b]$ of all continuous real-valued functions on $[a, b]$, the uniform metric d_∞ and the L^1 metric d_1 are not topologically equivalent.

The next proposition is clear from definition but still requires a proof.

Proposition 5.27. The relation "d is equivalent to ρ" is an equivalence relation in any collection of metrics on a set X.

Theorem 5.21 (Equivalent II). Let d_1 and d_2 be two metrics on a set X; let $U \subset X$. Then d_1 and d_2 are equivalent *if and only if*

$$U \text{ is } d_1\text{-open} \iff U \text{ is } d_2\text{-open}.$$

Example. Let σ be the discrete metric and d_2 be the usual Euclidean metric, and give both metrics to $\mathbb{R}$. Then of course, σ and d_2 are not equivalent as, for example, $\{0\}$ is σ-open but not d_2-open.

But now consider $\mathbb{Z}$ as a subset of $\mathbb{R}$. Interesting things arise:

Example. Let σ be the discrete metric and d_2 be the usual Euclidean metric, and give both metrics to $\mathbb{Z}$. The topology induced by the discrete metric σ is the discrete topology. But now, observe that d_2 induces the discrete topology on $\mathbb{Z}$ as well! To see this, let $x \in \mathbb{Z}$ be any point of $\mathbb{Z}$. Then consider the open ball centred at x of radius, say, 1/2. We have that

$$B_{d_2}(x, 1/2) = \{x\}$$

Therefore, every subset of $\mathbb{Z}$ is d_2-open in $\mathbb{Z}$. So in fact σ and d_2 are topologically equivalent.

Lemma 5.22. Let d_1 and d_2 be metrics on a set X. If for any point $x \in X$ and any radius $\varepsilon > 0$, there exists $\delta > 0$ such that $B_{d_2}(x, \delta) \subset B_{d_1}(x, \varepsilon)$, then the topology $\mathcal{T}_{d_1}$ induced by d_1 is coarser than the topology $\mathcal{T}_{d_2}$ induced by d_2 i.e. $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$.

informal slogan: $B_{d_2} \subset B_{d_1}$, then $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$.

Proof. Let $U \in \mathcal{T}_{d_1}$ i.e. U is d_1-open; *we want to show* $U \in \mathcal{T}_{d_2}$. Let $x \in U$ be arbitrary. Since $U \in \mathcal{T}_{d_1}$, there is an open ball $B_{d_1}(x, \varepsilon)$ centred at x with radius $\varepsilon > 0$ such that $B_{d_1}(x, \varepsilon) \subset U$. By hypothesis, there exists $\delta > 0$ such that $x \in B_{d_2}(x, \delta) \subset B_{d_1}(x, \varepsilon) \subset U$. Observe that we can write U as the following

$$U = \bigcup_{x \in U} B_{d_2}(x, \delta)$$

So U is the union of d_2-open balls, and hence is d_2-open i.e. $U \in \mathcal{T}_{d_2}$, as required. ■

Theorem 5.23 (Equivalent III). Let d_1 and d_2 be metrics on a set X. If for any point $x \in X$ and any radius $\varepsilon > 0$, there exist $\delta, \delta' > 0$ such that

$$B_{d_1}(x, \delta) \subset B_{d_2}(x, \varepsilon) \quad \text{and} \quad B_{d_2}(x, \delta') \subset B_{d_1}(x, \varepsilon)$$

Then d_1 and d_2 are equivalent metrics i.e. they induce the same topology on X.

the converse to this theorem is true as well.

Proof. Let $\mathcal{T}_{d_1}$ be the topology induced by d_1 and $\mathcal{T}_{d_2}$ be the topology induced by d_2. By the preceding Lemma (5.22), we have $\mathcal{T}_{d_1} \subset \mathcal{T}_{d_2}$ and $\mathcal{T}_{d_2} \subset \mathcal{T}_{d_1}$. ■

This theorem allows us to have an alternative definition for metric equivalence using notion of convergence of sequences. Of course proving equivalence of III and IV is super trivial. For this reason, we will prove equivalence with definition Equivalent II instead. Readers who want the equivalence proof between III and IV can look at the proof below in the ($\Longleftarrow$) part where we show IV implies III which implies a partial of II.

Theorem 5.24 (Equivalent IV). Let d_1 and d_2 be two metrics on a set X; let $x \in X$ Then d_1 and d_2 are equivalent *if and only if*

$$x_n \to x \text{ in } (X, d_1) \iff x_n \to x \text{ in } (X, d_2)$$

for every sequences $\langle x_n \rangle$ in X.

if two metrics are equivalent, they generate the same convergent sequences.

Proof. Let d_1, d_2 be two metrics on a set X; let $x \in X$.

($\Longrightarrow$). Assume Equivalent II, that is, for any subset A of X, we have A is d_1-open iff d_2-open. It is sufficient to show one implication. Let $\langle x_n \rangle$ be any sequence in X which converges to x w.r.t d_1. We want to show $\langle x_n \rangle$ converges to x w.r.t d_2. Suppose U is an arbitrary d_1-open set containing x. By convergence of $\langle x_n \rangle$ to x, there is $N \in \mathbb{N}$ such that $x_n \in U$ for $n > N$. But that is exactly what it means for $\langle x_n \rangle$ to converge to x w.r.t d_2 because U is d_2-open by hypothesis.

($\Longleftarrow$). Now assume Equivalent IV. It is sufficient to show that every d_1-open set is d_2-open. Since U is d_1-open, for every $x \in U$ there is $\varepsilon > 0$ such that $B_{d_1}(x, \varepsilon) \subset U$. Equivalent IV tells us that any sequence $\langle x_n \rangle$ converging to this x in U w.r.t d_1, also converges to x w.r.t d_2. In other words, there is $\delta > 0$ such that $B_{d_2}(x, \delta) \subset B_{d_1}(x, \varepsilon) \subset U$. So U is d_2-open, as desired. ■

the convergent criterion from Equivalent IV tells us for any radius $\varepsilon > 0$, we can choose positive numbers δ, δ' such that $B_{d_1}(x, \delta) \subset B_{d_2}(x, \varepsilon)$ and $B_{d_2}(x, \delta') \subset B_{d_1}(x, \varepsilon)$.

Example. Consider the uniform metric, d_∞ and the L^1 metric, ρ on $C[a, b]$:

$$d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|, \qquad \rho(f, g) = \int_a^b |f(x) - g(x)| \, \mathrm{d}x$$

Then the topology induced by the L^1 metric is coarser than the topology induced by the uniform metric.

Finally we give a sufficient but **not necessary** condition of metric equivalence. However, we will first need to make a definition.

Definition 5.23. Two metrics d_1, d_2 on a set X will be called **Lipschitz equivalent** if there are positive constants α, β such that for any $x, y \in X$,

$$\alpha d_2(x, y) \leqslant d_1(x, y) \leqslant \beta d_2(x, y)$$

Theorem 5.25. Lipschitz equivalent metrics are topologically equivalent.

Example. Let $X = \mathbb{R}^n$ and let d_1, d_2, d_∞ be the *taxicab metric*, *Euclidean metric* and the *square metric* respectively. Then as proven before in Proposition (5.3), for all $x, y \in \mathbb{R}^n$ we have

$$d_\infty(x, y) \leqslant d_2(x, y) \leqslant d_1(x, y) \leqslant n d_\infty(x, y)$$

So d_1, d_2 and d_∞ are Lipschitz equivalent and hence topologically equivalent.

Although the open balls of these metrics differ in shape, the open sets are the same. In a particular situation, it may be more convenient to use one of these metrics rather than the others.

5.10 Isometric metric spaces

Definition 5.24 (Isometry). Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is called an **isometry** or **distance preserving** if

$$d_Y(f(a), f(b)) = d_X(a, b)$$

for any $a, b \in X$.

Definition 5.25 (Isometric Isomorphism). Let X and Y be metric spaces. A **bijective isometry** from X to Y is called an **isometric isomorphism**. If there is an isometric isomorphism from X to Y, we say that X and Y are **isometrically isomorphic** or simply **isometric**.

Example. Let $f : \mathbb{R}^2 \to \mathbb{C}$ be given by $f(x_1, x_2) = x_1 + ix_2$. Then f is an isometry from $\mathbb{R}^2$ with the Euclidean metric to $\mathbb{C}$ with the metric $d(z, w) = |z - w|$ for $z, w \in \mathbb{C}$. Therefore $\mathbb{R}^2$ is isometrically isomorphic to $\mathbb{C}$.

Proposition 5.28. An isometry is continuous and injective.

Proposition 5.29 (Isometry is an Equivalence Relation). Let X and Y be metric spaces. Then,

(1) X is isometric to X.

(2) If X is isometric to Y, then Y is isometric to X.

(3) If X is isometric to Y, and Y is isometric to Z, then X is isometric to Z.

Theorem 5.26 (Isometric $\implies$ Homeomorphic)**.** Let X and Y be metric spaces. If X and Y are isometrically isomorphic, then X and Y are homeomorphic as topological spaces (with the metric topology).

The converse to this theorem is not always true.

Example. Let d be the **discrete metric** on a set X and let ρ be the metric on a set Y defined by

$$\rho(a, b) = \begin{cases} 2, & \text{if } a \neq b \\ 0, & \text{if } a = b \end{cases}$$

Suppose $|X| = |Y| > 1$. Then (X, d) and (Y, ρ) are not isometrically isomorphic since distances between points in each space are different. But both d and ρ induce the discrete topology, and two discrete spaces with the same cardinality are homeomorphic; so $X \equiv Y$.

5.11 Metrization problem

We know that given any metric space, there exists a topology (the metric topology) which makes it into a topological space. A natural question to ask then is: can we go in reverse? That is, given any topological space $(X, \mathcal{T})$, does there always exist a metric d on X which induces the topology $\mathcal{T}$.

Definition 5.26 (Metrizable). Let $(X, \mathcal{T})$ be a topological space. X is said to be **metrizable** if there exists a metric d on the set X that induces the topology $\mathcal{T}$ on X.

Using the notion of metrizability, we can redefine the notion of metric space in the following way.

Definition 5.27 (Metric Spaces II). Let $(X, \mathcal{T})$ be a metrizable space. X is said to be a **metric space** if it is equipped with a specific metric d that gives the topology $\mathcal{T}$ on X.

Example. Consider $\mathbb{R}$ endowed the usual topology $\mathcal{U}$. Observe that $(\mathbb{R}, \mathcal{U})$ is metrizable since the usual metric on $\mathbb{R}$ induces the usual topology on $\mathbb{R}$. Similarly $\mathbb{R}^2$ endowed with the usual topology is metrizable.

Topological Fact (Indiscrete). Any indiscrete space X with $|X| > 1$ is not metrizable.

Proof. Suppose X is an indiscrete space with $|X| > 1$. Now, the only closed sets in X are, $\varnothing$ and X itself. If $|X| > 1$, it follows that not all finite subsets of X are not closed. By Corollary (5.2), X and $\varnothing$ cannot be the only closed sets in a topology on X induced by a metric. So, X is not metrizable. ■

Topological Fact (Discrete). Every discrete space is metrizable.

Proof. The discrete metric on X induces the discrete topology $\mathcal{P}(X)$. ■

The *Metrization Problem* in topology is basically the problem of finding sufficient and necessary topological conditions for a topological space to be metrizable. We call such theorems that solve this problem *Metrization Theorems*.

6 Completeness

6.1 Cauchy sequences

Definition 6.1 (Cauchy sequence). A sequence $\langle x_n \rangle$ in a metric space (X, d) is called a **Cauchy sequence** if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geqslant N$.

Proposition 6.1. Any convergent sequence in a metric space is a Cauchy sequence.

Slogan. If $\langle s_n \rangle$ is Cauchy, any of its subsequence is Cauchy.

6.2 Complete metric spaces

Definition 6.2 (Completeness). A metric space X is **complete** if every Cauchy sequence in X converges (to a point of X).

Strong remark: *completeness* is a metric space property.

Example. $\mathbb{R}$ is complete (in the usual metric).

Example. $\mathbb{Q}$ is not complete (in the usual metric). The sequence of rational approximations to $\sqrt{2}$ converges to $\sqrt{2}$ which is not in $\mathbb{Q}$.

Example. $(0, 1) \subset \mathbb{R}$ is not complete (in the usual metric). The sequence $(1/n)$ is a Cauchy sequence in $(0, 1)$, but it does not converge to any point in $(0, 1)$.

Definition 6.3 (Subset Completeness). A subset of a metric space is said to be complete if it is complete as a metric subspace of X.

Lemma 6.1. A metric space X is complete if every Cauchy sequence in X has a convergent subsequence.

Proof. Let $\langle x_n \rangle$ be a Cauchy sequence in (X, d). We show that if $\langle x_n \rangle$ has a subsequence $\langle x_{n_k} \rangle$ that converges to a point x, then the sequence $\langle x_n \rangle$ itself converges to x.

Given $\varepsilon > 0$, choose $N \in \mathbb{N}$ large enough so that

such an N exists because $\langle x_n \rangle$ is Cauchy.

$$d(x_n, x_m) < \frac{\varepsilon}{2}$$

for all $n, m \geqslant N$. Then choose an integer k large enough so that $n_k \geqslant N$ and

such a k exists because $n_1 < n_2 < \ldots$ is an increasing sequence of integers and by assumption this subsequence converges to x.

$$d(x_{n_k}, x) < \frac{\varepsilon}{2}$$

Putting these together, we have that for $n \geqslant N$,

$$d(x_n, x) \leqslant d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon,$$

as desired. ■

$\mathbb{R}$ being complete while $(0, 1)$ is not complete shows that completeness is not a topological property as $\mathbb{R} \equiv (0, 1)$. However, it is invariant under *uniform equivalence*.

Proposition 6.2. Suppose that X, Y are metric space and there exists a bijective map $f : X \to Y$ such that both f and f^{-1} are **uniformly continuous**. Then X is complete *if and only if* Y is complete.

Observe that the only difference between a homeomorphism and the function f as defined in the preceding proposition is their *type* of continuity. The former is usual continuity whereas the latter is of uniform continuity.

Here is another good reminder that completeness is really a metric space property and not a topological one.

Topological Fact. Any metric space endowed with the discrete metric is complete. However, any metric space whose metric induces the discrete topology need not be complete!

i.e. any (discrete) metric space need not be complete; where remember a *discrete space* means a topological space endowed with the discrete topology.

Observe that they are topologically equivalent, but as metric spaces they are not equivalent at all as one is complete but the other is not.

Proof. Consider the subset $S = \{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$ which inherits the usual metric $d(x, y) = |x = y|$ on $\mathbb{R}$. Then observe that for any $n \in \mathbb{N}$ we have

$$d\left(\frac{1}{n+1}, \frac{1}{n}\right) = \frac{1}{n(n+1)} \quad \text{and} \quad d\left(\frac{1}{n-1}, \frac{1}{n}\right) = \frac{1}{n(n-1)}$$

Clearly, $\frac{1}{n(n-1)} > \frac{1}{n(n+1)}$. So, the open ball $B(\frac{1}{n}, \varepsilon_n)$ centred at $1/n$ with radius $\varepsilon_n = \frac{1}{n(n+1)}$ is just $\{1/n\}$. Therefore, singletons are open in S i.e. d induces the discrete topology on S. And indeed, S is not complete as the Cauchy sequence $(1/n)$ converges to 0 which is not in S. ■

this is super clear if you draw it on paper. It is just the usual metric, so it's super easy to draw and convince yourself of its truth.

Corollary 6.1. If two metrics d_1, d_2 on a set X are Lipschitz equivalent, then

$$(X, d_1) \text{ is complete} \iff (X, d_2) \text{ is complete.}$$

Proof. Since d_1 and d_2 are Lipschitz equivalent, Proposition (9.9) implies that the identity map from (X, d_1) to (X, d_2) and its inverse are both uniformly continuous. Therefore, by the preceding proposition we get the result as desired. ■

Proposition 6.3. Let X be a metric space; and $Y \subset X$. If Y is complete, then Y is closed in X.

Proof. Suppose $x \in \bar{Y}$. Since X is a metric space and hence metrizable, the converse to the Sequence Lemma (5.14) tells us that there is a sequence (y_n) in Y converging to $x \in X$. Since (y_n) is convergent, it is Cauchy by Proposition (6.1). So (y_n) is a Cauchy sequence in Y. By hypothesis, Y being complete implies that (y_n) must converge to a point in Y. By uniqueness of limits this would imply that $x \in Y$. We have therefore proven $\bar{Y} \subset Y$ i.e. Y is closed in X, as desired. ■

Proposition 6.4. Let X be a **complete** metric space; and $Y \subset X$. If Y is closed in X, then Y is complete.

Proof. Suppose that (y_n) is a Cauchy sequence in Y. Since X is complete, there is an $x \in X$ such that (y_n) converges to x. Since Y is closed, the Sequence Lemma (5.14) tells us that $x \in \bar{Y} = Y$. So, Y is complete. ■

In the case when the superspace X is complete, we therefore have:

Theorem 6.2. Let X be a complete metric space and $Y \subset X$. Then

$$Y \text{ is complete} \iff Y \text{ is closed in } X.$$

Example. $[a, b]$, $(-\infty, b]$, $[a, \infty)$ (in the usual subspace metric) are complete.

Next, we prove some more basic results on completeness.

Lemma 6.3. Let X be a metric space. If A, B are complete subspaces of X, then $A \cup B$ is complete.

Proof. Let $\langle s_n \rangle_{n \in \mathbb{N}}$ be a Cauchy sequence in $A \cup B$. Then $\langle s_n \rangle$ has a subsequence $\langle s_{n_k} \rangle_{k \in \mathbb{N}}$ which is fully contained in either A or B. The subsequence $\langle s_{n_k} \rangle$ is Cauchy since $\langle s_n \rangle$ is Cauchy. Assume wlog that the subsequence is fully contained in A. By completeness of A, this subsequence converges to a point $a \in A \subset A \cup B$. By the preceding Lemma (6.1), $\langle s_n \rangle$ also converges to $a \in A \cup B$. So $A \cup B$ is complete. ■

Proposition 6.5. Let X be a metric space. The union of a finite number of complete subspaces of X is complete.

Proof. Either use the preceding lemma and proof by induction or run the same argument again. ■

The **union** of an **infinite** number of complete subspaces of a metric space need not be complete!

Example. Consider $\mathbb{R}$ equipped with the usual metric. Observe that for each $n \in \mathbb{N}$, the singleton subset $\{1/n\}$ of $\mathbb{R}$ is complete. But the union

$$\bigcup_{n \in \mathbb{N}} \frac{1}{n} = \{1/n : n \in \mathbb{N}\}$$

is not complete as the Cauchy sequence $1/n$ tends to 0 which is not in that union.

"union of an infinite number of complete subspaces is not complete" counterexample.

Proposition 6.6. Let X be a metric space. The intersection of an arbitrary number of complete subspaces of X is complete.

Proof. Let $\{A_i\}_{i \in I}$ be an arbitrary collection of complete subspaces of X. Let $\langle s_n \rangle$ be a Cauchy sequence in $\bigcap A_i$. Then $\langle s_n \rangle$ is a Cauchy sequence in every A_i for all $i \in I$. By completeness of each A_i, $\langle s_n \rangle$ converges to a point of A_i for every $i \in I$. By uniqueness of limits in X, the unique limit is in A_i for every $i \in I$. Therefore, $\langle s_n \rangle$ converges in $\bigcap A_i$ (i.e. to a point of this intersection), as required. ■

Proposition 6.7. Let X and Y be two metric spaces. Then the cartesian product $X \times Y$ is complete *if and only if* X and Y are both complete.

Corollary 6.2. A finite cartesian product of complete metric spaces is complete.

Corollary 6.3. $\mathbb{R}^n$ is complete for each $n \in \mathbb{N}$.

6.3 Complete function spaces

Now we consider examples of complete metric spaces among function spaces.

Theorem 6.4. The function space $(\mathcal{B}(I, \mathbb{R}), d_\infty)$ of bounded real-valued functions on $I \subset \mathbb{R}$ equipped with the uniform metric is complete.

Corollary 6.4. The function space $(\mathcal{C}([a,b], \mathbb{R}), d_\infty)$ of continuous real-valued functions on $[a,b]$ equipped with the uniform metric d_∞ is complete.

More generally we may consider the space $\mathcal{B}(S, X)$ of all bounded functions from a set S to a metric space X, with the uniform metric d_∞.

Proposition 6.8. The space $(\mathcal{B}(S, X), d_\infty)$ is complete *if and only if* X is complete.

S here is any set and X here is any metric space.

Even more generally, we have the following claim.

Theorem 6.5. Let X be any topological space and (Y, d) be any complete metric space. Then the space of all continuous bounded functions from X to Y with the uniform metric d_∞ is complete.

Below is an important counter example of reminding us that completeness is really a property of the metrics.

Example. The space $(C([0,1], \mathbb{R}), d_1)$ of continuous real-valued functions on $[0,1]$ with the L^1 metric, d_1 is **not** complete.

6.4 Completion of a metric space

If a metric space X is not complete, then it has Cauchy sequences which do not converge. An obvious idea should spark in your head now: if we add the limits of all the Cauchy sequences which did not converge to the space X, then all these Cauchy sequences can now converge and therefore X plus these limits is complete. This process is called completing X.

Theorem 6.6 (Completion Theorem). Let (X, d) be a metric space. Then there exists a complete metric space $(\hat{X}, \hat{d})$ such that

(1) $X \subset \hat{X}$ and $\hat{d}(x, y) = d(x, y)$ whenever $x, y \in X$.

(2) For every $\hat{x} \in \hat{X}$, there is a sequence $\langle x_n \rangle$ of points in X such that $\langle x_n \rangle$ converges to $\hat{x}$ in the space $(\hat{X}, \hat{d})$.

The metric space $(\hat{X}, \hat{d})$ is said to be the **completion** of (X, d). If (X, d) is already complete, then necessarily $X = \hat{X}$ and $d = \hat{d}$.

Note that we can talk about **the** completion instead of **a** completion. A fact that you can prove is that if Y and Z both satisfy (1) and (2) of the Completion Theorem, then Y and Z are isometrically isomorphic.

Theorem 6.7. Every metric space X has a completion and all completions of X are isometrically isomorphic.

Proposition 6.9. Let A be a subspace of a complete metric space X; and let $\bar{A}$ be the closure of A in X. Then $\bar{A}$ is the completion of A.

Example. $\mathbb{R}$ is the completion of $\mathbb{Q}$ (in the usual metric).

6.5 Banach's fixed point theorem

We now come to one of metric space theory's most attractive result. The so-called Banach's fixed point theorem is useful both theoretically and applied-ly.

Definition 6.4 (Fixed Point). Given any self-map $f : S \to S$ of a set S, a **fixed point** of f is a point $s \in S$ such that $f(s) = s$.

Definition 6.5 (Contraction). Let (X, d) be a metric space. A self-map $T : X \to X$ is a **contraction** if for some constant $0 \leqslant c < 1$ we have

$$d(Tx, Ty) \leqslant cd(x, y) \text{ for all } x, y \in X.$$

where here we denote Tx for $T(x)$.

Remark. A contraction depends on the choice of metric d on X!

It is possible that d and d' are uniformly equivalent metrics on X (in which case (X, d) is complete iff (X, d') is complete) but then $f : X \to X$ is a d-contraction but **not** a d'-contraction.

Our first result of contractions is an obvious one.

Lemma 6.8. Any contraction of a metric space X is uniformly continuous, and hence continuous.

Proof. Given any $\varepsilon > 0$, take $\delta = \varepsilon/K$. Then for any $x, y \in X$ such that $d(x, y) < \delta$, we have $d(f(x), f(y)) \leqslant Kd(x, y) < \varepsilon$ as required. ■

Theorem 6.9 (Banach's Fixed Point Theorem). If $T : X \to X$ is a contraction of a **complete** metric space X, then T has a unique fixed point $x \in X$. Moreover, for any $x_0 \in X$, the sequence $x_n = T^n x_0$ converges to x.

this theorem is also known as the Contraction Mapping Theorem.

Proof. There are two parts to this theorem: existence and uniqueness.

Existence. Let x_1 be chosen arbitrarily in X and inductively let $x_n = Tx_{n-1}$ for $n > 1$. We shall prove that $\langle x_n \rangle$ is a Cauchy sequence.

Firstly, since T is a contraction, then there is $c \in [0, 1)$ such that $d(Tx, Ty) \leqslant cd(x, y)$ for all $x, y \in X$. Therefore, by an easy induction we thus have $d(x_{k+1}, x_k) \leqslant c^{k-1}d(x_2, x_1)$ for all $k > 1$. Now for $m > n$, by spamming the triangle inequality we have

$$d(x_m, x_n) \leqslant d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+1}, x_n).$$

Hence we have

$$\begin{aligned} d(x_m, x_n) &\leqslant (c^{m-2} + c^{m-3} + \ldots + c^{n-1})d(x_2, x_1) \\ &= c^{n-1}(c^{m-n-1} + c^{m-n-2} + \ldots + 1)d(x_2, x_1) \\ &= \frac{c^{n-1}(1 - c^{m-n})}{1 - c} d(x_2, x_1) \\ &< \frac{c^{n-1}}{1 - c} d(x_2, x_1). \end{aligned}$$

Now $c^{n-1} \to 0$ as $n \to \infty$, since $0 \leqslant c < 1$. Hence for any $\varepsilon > 0$ we have $d(x_m, x_n) < \varepsilon$ whenever $m \geqslant n$ and n is sufficiently large. This proves that $\langle x_n \rangle$ is a Cauchy sequence. Since X is complete, $\langle x_n \rangle$ converges to some point $x \in X$. By continuity of T at x, $Tx_n \to Tx$ as $n \to \infty$. But $Tx_n = x_{n+1} \to x$ as $n \to \infty$. So $Tx = x$.

Uniqueness. If $Tx = x$ and also $Ty = y$, then

$$d(x, y) = d(Tx, Ty) \leqslant cd(x, y).$$

Since $c < 1$ this is a contradiction unless $d(x, y) = 0$ i.e. $x = y$. So the fixed point is unique. ∎

We have the following proposition which have practical interests.

Proposition 6.10. Let X be a complete metric space and $f : X \to X$ be a contraction. Suppose $p \in X$ is the fixed point of f. Then

$$d(p, x_n) \leqslant \frac{K^{n-1}}{1 - K} d(x_2, x_1)$$

It is basically an error estimate of how far x_n is from p. The right-hand side can be calculated in specific cases without knowing p in advance.

6.6 Applications of Banach's fixed point theorem

Theorem 6.10. Suppose that $K : [a, b] \times [a, b] \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous. Then the (Volterra) equation

$$\phi(x) = f(x) + \int_a^x K(x, y)\phi(y) \, \mathrm{d}y$$

has a unique continuuos solution ϕ on $[a, b]$.

Theorem 6.11 (Picard's Theorem). Let $\mathcal{D} = [x_0 - a, x_0 + a] \times [y_0 - b, y_0 + b]$. Suppose that $f : \mathcal{D} \to \mathbb{R}$ is continuous and satisfies the Lipschitz condition

$$f(x, y_1) - f(x, y_2)| \leqslant K|y_1 - y_2|$$

for all $(x, y_1), (x, y_2) \in \mathcal{D}$ and some $K > 0$. Let M be an upper bound for $|f(x, y)|$

on $\mathcal{D}$, and let $\delta = \min\{a, b/M\}$. Then on $I = [x_0 - \delta, x_0 + \delta]$, there exists a unique solution y of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ such that $y(x_0) = y_0$.

7 Hausdorffness

Hausdorfness is a funny property, at least for non-topologists. We went too general when generalizing the idea of open sets of the line and the plane; and then realize, aha, why not we define a condition that "*makes sense*" to the brain. After all, if two points are distinct, we should be able to see and tell that they are **indeed** distinct.

Our pursuit for generality leads us to getting funny results such as any sequence in an indiscrete space converges to anything in the space. We would want to step back, and say, "well, we want to at least have sequences that have unique limits". With these words, arise from the ashes, the Hausdorff space.

7.1 Separation conditions

Definition 7.1 (Hausdorff condition). A topological space X satisfies the **Hausdorff condition** if for any two distinct points $x, y \in X$, there exist disjoint open sets U, V of X such that $x \in U$, $y \in V$.

A topological space satisfying the Hausdorff condition is said to be **Hausdorff** and may be called a **Hausdorff space**.

Proposition 7.1. In a Hausdorff space, any given convergent sequence has a unique limit.

Proof. Suppose that $\langle x_n \rangle$ is a sequence of points of X that converses to x. If $y \neq x$, let U and V be disjoint open neighbourhoods of x and y, respectively — which exist by the Hausdorff condition. Since U contains $\langle x_n \rangle$ for all but finitely many values of n, the set V cannot. Therefore, x_n cannot converge to y. ■

Lemma 7.1. Suppose that x, y are distinct points in a metric space (X, d) and let $\varepsilon = d(x, y)/2$. Then

$$B(x, \varepsilon) \cap B(y, \varepsilon) = \varnothing$$

Proof. Suppose for contradiction that the intersection is non-empty, then let $z \in B(x, \varepsilon) \cap B(y, \varepsilon)$. Then $d(x, z) < \varepsilon$ and $d(y, z) < \varepsilon$. By triangle inequality, we have

$$d(x, y) \leqslant d(x, z) + d(z, y) < 2\varepsilon$$

which is a contradiction to the fact that $d(x, y) = 2\varepsilon$. ■

Theorem 7.2. Any metrizable space X is Hausdorff.

since any metrizable space is a metric space, it follows that every metric space is Hausdorff.

Proof. Let $(X, \mathcal{T})$ be a metrizable space; so suppose d is a metric such that it induces the topology $\mathcal{T}$. If $x, y \in X$ with $y \neq x$, then $d(x, y) > 0$. Take $\varepsilon = d(x, y)/2$. Then by the preceding Lemma (7.1), the open balls $B(x, \varepsilon)$ and $B(y, \varepsilon)$ are disjoint open sets containing x, y respectively. ■

This gives us a criteria to check whether a space is not metrizable by considering its converse.

Lemma 7.3. In a Hausdorff space, every singleton subset is closed.

Proof. Let X be a Hausdorff space; and let $x \in X$. Then for every $y \in X$ distinct from x, the Hausdorff condition guarantees the existence of disjoint open sets U_x and V_y such that $x \in U_x$ and $y \in V_y$. Now consider

$$V = \bigcup_{y \in X \setminus \{x\}} V_y.$$

Then clearly V is open in X because it is a union[10] of open sets in X. Therefore, $X \setminus V = \{x\}$ is closed in X. Thus singletons are closed in X. ■

[10] *it doesn't really matter if X is infinite, as the union of an arbitrary collection of open sets is open.*

Topological Fact (Indiscrete). Any indiscrete space with more than one points is **not** Hausdorff.

Proof. Let X be an indiscrete space such that $|X| > 1$; and suppose for contradiction that X is Hausdorff. Then for any two distinct points $x, y \in X$, there exist disjoint open sets U, V in X such that $x \in U$ and $y \in V$. Since X is an indiscrete space, the candidates for U, V are either X or $\varnothing$. But since U, V are non-empty, U, V have to be X itself [↯]. This contradicts the fact that U and V are disjoint. ■

Topological Fact (Discrete). A finite Hausdorff space must be a discrete space.

Proof. Let $(X, \mathcal{T})$ be a finite Hausdorff space; and let $x \in X$. By Lemma (7.3), singletons are closed in X. Since a union of a finite collection of closed sets is closed, the union of finitely many singletons are closed in X. Therefore, any finite subset of X is closed. But since X is itself finite by hypothesis, the set $X \setminus \{x\}$ must be finite and therefore closed. Hence, $\{x\}$ is open in X, i.e. singletons are open in X. Therefore, $\mathcal{T}$ must be the discrete topology. ■

Topological Fact (Cofinite). Any infinite space X with the co-finite topology is **not** Hausdorff.

Proof. Let X be an infinite set endowed with the co-finite topology; and suppose for contradiction that X is Hausdorff. Then for any two distinct points $x, y \in X$, there exist disjoint open sets U, V of X such that $x \in U$ and $y \in V$. Since U, V are open, it follows that U^c and V^c are finite. Now because $U \cap V = \varnothing$, we have

$$U^c \cup V^c = (U \cap V)^c = \varnothing^c = X$$

The LHS is a union of finite sets so it must be finite; but the RHS is infinite by hypothesis [↯]. ■

Proposition 7.2. Any subspace of a Hausdorff space is Hausdorff.

Proof. (1). Let X be a Hausdorff space and consider the subspace A of X. Let $a, b \in A$ be any two distinct points, then by the Hausdorff condition, there are disjoint open sets U, V in X such that $a \in U$ and $b \in V$. Since U, V are open in X, by definition of the subspace topology, $U \cap A$ and $V \cap A$ are both open in A. Moreover, we have that $a \in U \cap A$ and $b \in V \cap A$. If $U \cap A$ and $V \cap A$ are disjoint, we are done.

$$(U \cap A) \cap (V \cap A) = (U \cap V) \cap A = \varnothing \cap A = \varnothing,$$

So indeed they are disjoint. Therefore A is Hausdorff. ∎

Proposition 7.3. Let X and Y be topological spaces. If $f : X \to Y$ is an injective continuous map and Y is Hausdorff. Then X is Hausdorff.

Proof. Let a and b be two distinct points in X. Since f is a map from X to Y, we have $f(a), f(b) \in Y$. Since f is injective, $f(a) \neq f(b)$. Since Y is Hausdorff, there exist disjoint open sets U, V in Y such that $f(a) \in U$ and $f(b) \in V$. By definition of preimages, $a \in f^{-1}(U)$ and $b \in f^{-1}(V)$. Since f is continuous both $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. If we can show they are disjoint we are done:

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(\varnothing) = \varnothing,$$

so X is Hausdorff, as desired. ∎

Corollary 7.1. Let X, Y be topological spaces such that $X \equiv Y$. Then X is Hausdorff *if and only if* Y is Hausdorff.

i.e. Hausdorffness is a topological property

Proof. Suppose there exists a homeomorphism $f : X \to Y$; and assume that Y is Hausdorff. By the preceding Proposition (7.3), X is Hausdorff.

Conversely, suppose X is Hausdorff. $f^{-1} : Y \to X$ is also bijective (hence, injective) and continuous, so Y is Hausdorff. ∎

Proposition 7.4. The cartesian product $X \times Y$ is Hausdorff $\iff$ X and Y are both Hausdorff.

Here are two necessary conditions for Hausdorfness.

Proposition 7.5. Let X be a Hausdorff space and $x \in X$. Then

(1) The intersection of all open sets containing x equals the singleton $\{x\}$.

(2) The intersection of all closed sets containing x equals the singleton $\{x\}$.

Proof. (1). Let X be a Hausdorff space and $x \in X$. The fact that $\{x\}$ is contained in the intersection of all open sets containing x is trivial; we are left to show the reverse inclusion. Let $y \neq x$. Since X is Hausdorff, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$. Since y is not in one open set containing x, then it is certainly not in every open set containing x. Therefore, the reverse inclusion follows, as desired.

(2). Let X be a Hausdorff space. We can run the same argument. But we choose to use a proven lemma. By Lemma (7.3), any singleton subset $\{x\}$ is closed in X;

therefore it equals its closure. But by definition, the closure of $\{x\}$ is the intersection of all closed sets containing $\{x\}$ i.e. containing x; we are done. ■

Turns out, we get a sufficient and necessary condition for Hausdorffness if we strengthen the hypothesis.

Proposition 7.6. X is a Hausdorff space *if and only if* any singleton subset $\{x\}$ equals the intersection of **all** closed sets that contains an open neighbourhood of x.

Proof. Suppose that X is Hausdorff and let $x \in X$. Then for any $y \in X$ distinct from x, there exist disjoint open sets U_x, V_y in X such that $x \in U_x$ and $y \in V_y$. Let $C_y = X \backslash V_y$; then C_y is closed in X. Moreover, $x \in U_x \subset C_y$. So, C_y is a closed set that contains an open neighbourhood of x, which does not contain y. This leads us to the observation that

$$\bigcap_{y \in X \backslash \{x\}} C_y = \{x\}.$$

So the intersection of all closed sets that contains an open neighbourhood of x must be $\{x\}$.

For the reverse direction, the proof is the same but backwards. ■

8 Connectedness

8.1 Some definitions

Definition 8.1 (Separation). Let X be a topological space. A **separation** of X is a pair U, V of disjoint non-empty **open** subsets of X whose union is X i.e. the pair U, V satisfies:

(1) U and V are non-empty.

(2) $U \cup V = X$.

(3) $U \cap V = \varnothing$.

We may denote the pair U, V as $\langle U, V \rangle$ if they form a separation of X.

N.B. Since U, V are non-empty, we can never have U or V to equal X because of (3).

Alternatively, a *separation* is called a *disconnection*. We shall not use the term here.

Definition 8.2 (Connectedness). A topological space X is said to be **connected** if there does not exist a separation of X.

Since any subset of a topological space is naturally endowed with a topology — the subspace topology, we can define connectedness for subsets.

Definition 8.3 (Subset Connectedness). A subset of a topological space X is said to be connected if it is connected in the subspace topology.

being connected in the subspace topology means the separation pair (U, V) consists of subsets of $Y \subset X$ that are relatively open in Y (need not be open in X itself).

This definition of subset connectedness shows that connectedness is an **absolute property** of sets. That is, it doesn't matter to say where is S is connected in. If we say S as subset is connected, we don't really care what's happening in the ambient space.

Connectedness is clearly a topological property (as it is defined entirely using collection of open sets). So, if X is connected, any space homeomorphic to X is also connected. To remind ourselves that indeed connectedness is a topological property, we look at the following example.

Example. Let $X = \{a, b\}$. X equipped with the Sierpinski topology $\mathcal{T} = \{X, \varnothing, \{a\}\}$ is connected. However, X equipped with the discrete topology is not connected as $\langle \{a\}, \{b\} \rangle$ forms a separation of X.

$\langle \{a\}, \{b\} \rangle$ does not form a separation in the Sierpinski topology because $\{b\}$ is not open.

Slogan. The Sierpinski Space is also called the connected two-point set.

Definition 8.4 (Clopen). Let X be a topological space. A subset $A \subset X$ is said to be **clopen** in X if it is both open and closed in X.

Remark. Since X and $\varnothing$ are clopen in any topological space X, we call them **trivial** clopen (sub)sets of X.

We can reformulate the definition of connectedness using the notion of *clopen* subsets.

Theorem 8.1 (Connectedness II). A topological space X is connected *if and only if* the only subsets of X that are clopen in X are the empty set $\varnothing$ and X itself.

in other words, X has only trivial clopen sets.

Proof. ($\implies$). We prove the contrapositive. Suppose A is a non-empty proper subset of X that is clopen in X. Consider $U = A$ and $V = A^c$. Then the pair $\langle U, V\rangle$ is a separation of X as they are non-empty, $U \cup V = X$ and $U \cap V = A \cap A^c = \varnothing$. So, X is not connected.

($\impliedby$). We also prove the contrapositive. Suppose X is not connected. Then there exist a separation $\langle U, V\rangle$ of X. By definition U, V are non-empty open subsets of X; and are different from X. Moreover, since $U^c = V$ and $V^c = U$, it follows that U, V are closed. So, U and V are clopen and are different from $\varnothing$ and X, as required. ■

Remark. In this characterization, we say a subset S of X is connected iff it is a connected topological space when it has its subspace topology. That is, S is connected iff the only subsets of S that are clopen in S are $\varnothing$ and S itself.

N.B. clopen in S and not in X!

Example. Consider $\mathbb{R}$ with the usual topology; and let $S = [0,1) \cup (1,2]$ be a subspace of $\mathbb{R}$. Then S is not connected:

$U = [0,1)$ is open in S. Moreover, $S \backslash U = (1,2]$ is also open in S, and so U is closed in S; and hence, is clopen in S. But U is neither S nor $\varnothing$.

Proposition 8.1. Let X be a topological space. Then the following are equivalent:

(i) There is a non-trivial clopen set.

(ii) There is a separation $\langle U, V\rangle$ of X.

(iii) There are $C, D \subset X$ disjoint non-empty **closed** subsets of X whose union is X.

Proof. (i) $\Leftrightarrow$ (ii) is basically Theorem (8.1). We will show (ii) $\Leftrightarrow$ (iii). Suppose there is a separation $\langle U, V\rangle$ of X. Then the pair U, V are disjoint non-empty open subsets of X. Accordingly, $U^c = V$ and $V^c = U$, so U and V are closed in X, taking $C = U$ and $D = V$ we are done.

The converse is very similar. Suppose C, D are non-empty closed subsets of X whose union is X. Then, C and D are also open; so in fact $\langle U, V\rangle$ is a separation of X. ■

The statement (iii) in the above proposition is basically the definition of separation with "open" replaced by "closed". But since by definition the separation pair U, V are disjoint but their union is the whole space X itself, it follows that their complements are also closed.

Turns out there is a third equivalent definition of connectedness.

Theorem 8.2 (Connectedness III). Let X be a topological space. Then X is connected *if and only if* there does not exist a continuous surjective map from X onto the discrete two-point space $\{0,1\}$.

equivalently, X is connected $\Leftrightarrow$ any continuous surjective map from X to a two-point discrete space $\{0,1\}$ is **constant**.

Proof. ($\implies$). We prove the contrapositive. Suppose there exists a continuous surjective map from X onto the discrete two-point space $\{0,1\}$. Now consider $A = f^{-1}(\{0\})$ and $B = f^{-1}(\{1\})$. First observe that A and B are non-empty as f is surjective. Secondly, observe that $\{0\}$ and $\{1\}$ are disjoint. We use this to our advantage to have

$$A \cap B = f^{-1}(\{0\}) \cap f^{-1}(\{1\}) = f^{-1}(\{0\} \cap \{1\}) = f^{-1}(\varnothing) = \varnothing;$$

and therefore A and B are disjoint. Finally, observe that

$$A \cup B = f^{-1}(\{0\}) \cup f^{-1}(\{1\}) = f^{-1}(\{0\} \cup \{1\}) = f^{-1}(\{0,1\}) = X$$

Thus we have shown that $\langle A, B\rangle$ forms a separation of X, and so X is not connected.

$(\Longleftarrow)$. We prove the contrapositive. Suppose X is not connected. Then there exists a separation $\langle A, B\rangle$ of X. Now define $f : X \to \{0, 1\}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \in B \end{cases}$$

Since A, B are non-empty, f is surjective onto $\{0, 1\}$. Now observe that since $\{0, 1\}$ is given the discrete topology, the open sets in $\{0, 1\}$ are $\varnothing, \{0\}, \{1\}$ and $\{0, 1\}$. The inverse images of these sets under f are $\varnothing$, A, B and X which are all open in X so f is continuous; as required. ■

For a subspace Y of a topological space X, there is an alternative and more useful way of formulating the definition of connectedness:

Lemma 8.3 (Subspace Lemma). Let X be a topological space; let Y be a subspace of X. Then

$\langle A, B\rangle$ is a separation of Y $\iff$ A and B are disjoint non-empty sets in X whose union is Y, neither of which contains a limit point of the other.

The space Y is connected if there does not exist a separation of Y.

Proof. $(\Longrightarrow)$. Suppose A and B forms a separation of Y. Then A is clopen (w.r.t subspace topology) in Y. The closure of A in Y is the set $\bar{A} \cap Y$ (where $\bar{A}$ is the closure of A in X). Since A is closed in Y, it equals its closure in Y, so $A = \bar{A} \cap Y$. This is equivalent to having $\bar{A} \cap B = \varnothing$. Since $\bar{A} = A \cup A'$, we have $A' \cap B = \varnothing$ which implies that B contains no limits points of A. By a similar argument, we conclude that A contains no limits points of B.

the equivalence is because of: $A = \bar{A} \cap Y = \bar{A} \cap (A \cup B) = (\bar{A} \cap A) \cup (\bar{A} \cap B) = A \cup (\bar{A} \cap B)$ which by identity law of sets is true iff $\bar{A} \cap B = \varnothing$.

$(\Longleftarrow)$. Suppose that A and B are disjoint non-empty sets whose union is Y, neither of which contains a limit point of the other. Then $\bar{A} \cap B = \varnothing$ and $A \cap \bar{B} = \varnothing$. Thus, we conclude that $\bar{A} \cap Y = A$ and $\bar{B} \cap Y = B$. This implies that both A and B are closed in Y. But then by hypothesis $A^c = B$ and $B^c = A$, so A and B are also open in Y. So, A and B forms a separation of Y. ■

Remark. For two subsets of $Y \subset X$, there is no distinction between *disjoint in* Y and *disjoint in* X. But observe that if we have two subsets of X and their intersections with Y are *disjoint in* Y, then it doesn't imply that they are *disjoint in* X.

Topological Fact. In any topological space, $\varnothing$ and singletons are connected.

reminder: these are subsets, so being connected means being connected in the subspace topology.

Proof. Let X be a topological space. The empty set $\varnothing$ as a subspace satisfies Theorem (8.1) trivially. Now let $x \in X$ and consider the singleton subset $\{x\}$ of X. The only (relatively open) sets in $\{x\}$ are the empty set $\varnothing$ and $\{x\}$ itself. So, the only (*relatively*) closed sets in $\{x\}$ are again $\varnothing$ and $\{x\}$. Therefore, the only clopen sets in $\{x\}$ are itself and $\varnothing$ i.e. $\{x\}$ is connected. ■

Topological Fact (Indiscrete). Every indiscrete space (and its subsets) are connected.

Proof. If X is an indiscrete space, then the only non-empty open set in X is X. So there is no separation of X i.e. X is connected. ■

Topological Fact (Discrete). Every discrete space (and its subsets) other than $\varnothing$ and singletons are **not** connected.

Topological Fact (Cofinite). Every infinite set with the cofinite topology is connected.

Example. Let Y denote the subspace $[-1,0) \cup (0,1]$ of the real line $\mathbb{R}$. Observe that each of the sets $[-1,0)$ and $(0,1]$ is non-empty and relatively open in Y (although not in $\mathbb{R}$). Therefore, the pair $[-1,0)$ and $(0,1]$ forms a separation of Y i.e. Y is not connected.

Alternatively, since $[-1,0)$ and $(0,1]$ are disjoint non-empty sets in $\mathbb{R}$ (whose union is Y), and neither of which contains a limit point of the other (they do have a limit point 0 in common, but that does not matter), we can use the Subspace Lemma to deduce that they form a a separation of Y.

Example. Let X be the subspace $[-1,1]$ of $\mathbb{R}$. The sets $[-1,0]$ and $(0,1]$ are disjoint and non-empty, but they do **not** form a separation of X, because the first set $[-1,0]$ is not relatively open in X.

Alternatively, since $[-1,0]$ contains the limit point 0 of the second set $(0,1]$, the Subspace Lemma tells us that the pair $([-1,0],(0,1])$ is not a separation of X.

Indeed, there exist **no** separation of the space $[-1,1]$ i.e. $[-1,1]$ is connected — this is a fact that we shall prove.

here we see how handy the Subspace Lemma truly is. imagine if we are dealing with a more complicated non-intuitive topological space, it's quite hard to tell when a subset is open or not, but it's not that hard to calculate limit points.

Example. The rationals $\mathbb{Q}$ are not connected.

A useful way to get a separation of $Y \subset X$ from a separation of X is the following:

Theorem 8.4. Let X be a topological space; and let Y be a subspace of X. Suppose $\langle A, B\rangle$ is a separation of X. Then

$$\langle A \cap Y, B \cap Y\rangle \text{ is a separation of } Y \iff A \cap Y \text{ and } B \cap Y \text{ are both non-empty.}$$

we implictly assumed X is not connected (as there is a separation of X).

Proof. ($\Longrightarrow$). This direction is trivial as suppose $\langle A \cap Y, B \cap Y\rangle$ is a separation of Y, then by definition $A \cap Y$ and $B \cap Y$ are non-empty.

($\Longleftarrow$). Suppose $A \cap Y$ and $B \cap Y$ are non-empty. Since $\langle A, B\rangle$ is a separation of X, A and B are open in X. Therefore, the two sets $A \cap Y$ and $B \cap Y$ are relatively open in Y. Moreover, observe that the two sets $A \cap Y$ and $B \cap Y$ are disjoint and their union is Y. Therefore, $\langle A \cap Y, B \cap Y\rangle$ forms a separation of Y. ■ ■

An obvious consequence of this theorem is the following:

Corollary 8.1. Let X be a topological space; and let Y be a subspace of X. Suppose $\langle A, B\rangle$ is a separation of X. Then

$$A \cap Y \text{ and } B \cap Y \text{ are both non-empty} \implies Y \text{ is not connected.}$$

Proof. This is trivial really. Since $\langle A, B\rangle$ is a separation of X, we can use the preceding Theorem (8.4). $A \cap Y$ and $B \cap Y$ being both non-empty implies that there is a separation of Y. Therefore Y is not connected. ■

Remark. The converse to this corollary is false i.e. if Y is not connected, it does not imply that $A \cap Y$ and $B \cap Y$ are both non-empty. Consider $\mathbb{R}$ endowed with the usual topology and consider the subspace $X = (-1, 0) \cup (0, 1)$. Then $A = (-1, 0)$ and $B = (0, 1)$ forms a separation of X. Now consider the subset $Y = (0, 1/2) \cup (1/2, 1)$ which is a subspace of X. By the Subspace Lemma, Y is **not connected** as the sets $(0, 1/2)$ and $(1/2, 1)$ forms a separation of Y. However, $A \cap Y = \varnothing$ but $B \cap Y \neq \varnothing$ i.e. **not both of them are non-empty**.

We have seen some example of spaces that are not connected. A natural question to ask is, *how can one construct spaces that are connected*? Or rather, *how can one construct new connected spaces from olds ones*?

Lemma 8.5. Let X be a topological space; and let Y be a connected subspace of X. If $\langle A, B\rangle$ is a separation of X, then either $Y \subset A$ or $Y \subset B$.

N.B. $\langle A, B\rangle$ is a separation of X, not of Y!

Proof. Since $\langle A, B\rangle$ is a separation of X, the sets A and B are open in X. Therefore, the sets $A \cap Y$ and $B \cap Y$ are relatively open in Y. Observe that the two sets $A \cap Y$ and $B \cap Y$ are disjoint and their union is Y. If both $A \cap Y$ and $B \cap Y$ are non-empty, then they would form a separation of Y. But since by hypothesis Y is connected, one of them must be empty. If $A \cap Y$ is empty, then $Y \subset B$. On the other hand, if $B \cap Y$ is empty, then $Y \subset A$, as desired. ■

Theorem 8.6 (Connectedness IV). A topological space X is connected *if and only if* every non-empty proper subset of X has non-empty boundary.

8.2 Some truths regarding connectedness

Theorem 8.7. Suppose A, B are connected subsets of a space X such that $A \cap \bar{B} = \varnothing$. Then $A \cup B$ is connected.

Theorem 8.8. Let X be a topological space. The union of an arbitrary collection of connected subspaces of X that have a point in common is connected.

Proof. Let $\{C_i\}_{i \in I}$ be a collection of connected subspaces of a topological space X; and let $p \in \bigcap C_i$. We want to show that $Y = \bigcup C_i$ is connected. Suppose $\langle A, B\rangle$ is a separation of Y, so we have $Y = A \cup B$. Since $p \in \bigcap C_i$, we have $p \in Y$. So, $p \in A$ or $p \in B$. Without loss of generality, suppose $p \in A$. Since C_i is connected, by Lemma

(8.5), $C_i \subset A$ or $C_i \subset B$. Since $p \in C_i$, we can't have $C_i \subset B$ as $p \in A$. So $C_i \subset A$ for every i, i.e. $\bigcup C_i \subset A$, but this implies that B is empty [ϟ]. ■

Remark. Using the notation as in the proof, we can reformulate the theorem as the following: Let X be a topological space. If $\bigcap C_i \neq \varnothing$, then $\bigcup C_i$ is connected. Note that $\bigcup C_i$ is connected in the subspace topology (that it inherits from X for being a union of subsets of X).

Theorem 8.9. Let A be a connected subspace of X. If $A \subset B \subset \bar{A}$, then B is also connected and hence, $\bar{A}$ is connected.

since $B \subset \bar{A} \subset X$, we implicitly assume that B is a subspace of X here.

Slogan. The closure of a connected set is connected.

Proof. Let A be connected and let $A \subset B \subset \bar{A}$. Suppose that $\langle C, D\rangle$ is a separation of B, so we have $B = C \cup D$. By Lemma (8.5), $A \subset C$ or $A \subset D$. Suppose without loss of generality that $A \subset C$. Then $\bar{A} \subset \bar{C}$. Now, by the Subspace Lemma (8.3), D does not contain the limit points of C i.e. $C' \cap D = \varnothing$. It follows that $\bar{C} \cap D = \varnothing$. Since $\bar{C}$ and D are disjoint, B cannot intersect D [ϟ]. This contradicts the fact that D is a non-empty subset of B. ■

Theorem 8.10. Suppose $f : X \to Y$ is a continuous map of topological spaces and that X is connected. Then $f(X)$ is connected.

the image of a connected space under a continuous map is connected.

Proof. Let $f : X \to Y$ be a continuous map; and that X is connected. By virtue of Theorem (4.7), it suffices to prove the theorem when f is surjective.

So, assume f is surjective (i.e. $f(X) = Y$). Suppose A is clopen in Y, then $f^{-1}(A)$ is clopen in X. Since X is connected, either $f^{-1}(A) = \varnothing$ or $f^{-1}(A) = X$ — this is Theorem (8.1). Now observe that $f(f^{-1}(A)) = A$ because f is surjective. Therefore either $A = f(\varnothing) = \varnothing$ or $A = f(X) = Y$. That is, A has to be trivial. By Theorem (8.1), $Y = f(X)$ is connected. ■

Why can we assume f surjective? Let $f : X \to Y$ be continuous. Consider $g : X \to f(X)$ defined by $g(x) = f(x)$ for all $x \in X$. By Theorem (4.7), g is continuous as f is continuous. Now clearly, g is surjective i.e. $g(X) = f(X)$. Because of this equality, to show $f(X)$ is connected, it suffices to show that $g(X)$ is connected. So in the statement of Theorem (8.10) we may replace "$f(X)$ is connected" by "$g(X)$ is connected"; and thus suffice to prove the theorem when f is surjective.

Remark. TLDR. The margin note tells us that if we just consider $g : X \to f(X)$ (i.e. a function of which f is surjective) and assuming the assumptions of Theorem (8.10), then $g(X)$ is connected. But then $g(X) = f(X)$ and so $f(X)$ is connected; i.e. we get the same thing.

Corollary 8.2. Connectedness is a topological property.

Proof. Suppose $f : X \to Y$ is a homeomorphism. If X is connected, then $f(X)$ is connected. But since f is bijective, $f(X) = Y$; we are done. ■

Lemma 8.11. Let X and Y be connected spaces. Then the cartesian product $X \times Y$ is connected.

Proof. Omitted. ■

We can generalize this to finite cartesian products.

Theorem 8.12. A finite cartesian product of connected spaces is connected.

Proof. Omitted. ■

8.3 Connectedness and $\mathbb{R}$

Definition 8.5 (Interval). A non-empty subset $I \subset \mathbb{R}$ is an **interval** if

$$x, y \in I \text{ and } z \in \mathbb{R} \text{ such that } x < z < y \implies z \in I.$$

Remark. With this definition, all of the following sets are *intervals*:

(1) $[a, b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$,

(2) $\{a\}$ where $a \in \mathbb{R}$,

(3) $(a, b) = \{x \in \mathbb{R} : a < x < b\}$,

(4) $(a, b] = \{x \in \mathbb{R} : a < x \leqslant b\}$,

(5) $[a, b) = \{x \in \mathbb{R} : a \leqslant x < b\}$,

(6) $(-\infty, b] = \{x \in \mathbb{R} : x \leqslant b\}$,

(7) $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$,

(8) $[a, \infty) = \{x \in \mathbb{R} : x \geqslant a\}$,

(9) $(a, \infty) = \{x \in \mathbb{R} : x > a\}$,

(10) $(-\infty, \infty) = \mathbb{R}$.

some people defined these to be intervals and prove as a theorem the equivalence of this definition with our definition.

Theorem 8.13. Any connected subspace S of $\mathbb{R}$ is an interval.

$\mathbb{R}$ is endowed with the usual topology here.

Proof. Suppose S is not an interval. Then there exist $x, y \in S$ and $z \in \mathbb{R}$ such that $x < z < y$ but $z \notin S$. Consider $A = (-\infty, z) \cap S$ and $B = (z, \infty) \cap S$. Then $\langle A, B \rangle$ is a separation of S and hence, S is not connected. ■

By definition of the subspace topology, both A, B are open in S; each is non-empty as $x \in A$ and $y \in B$; and they are disjoint and have union S because $z \notin S$.

Example. $\mathbb{Q}$ is not connected as it is not an interval.

Corollary 8.3. Suppose that $f : X \to \mathbb{R}$ is continuous and X is connected. Then $f(X)$ is an interval.

Proof. Suppose that $f : X \to \mathbb{R}$ is continuous and X is connected. By Theorem (8.10), $f(X)$ is connected. But $f(X) \subset \mathbb{R}$, so by virtue of Theorem (8.13), $f(X)$ is an interval. ■

Corollary 8.4 (Intermediate Value Theorem). Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then it has the intermediate value property i.e. for any d between[11] $f(a)$ and $f(b)$, there exists $c \in (a, b)$ such that $f(c) = d$.

[11] *we mean strictly between.*

Proof. Since $I := [a,b]$ is an interval, it is connected. Therefore, by the preceding Corollary (8.3), $f(I)$ is an interval. Without loss of generality, assume $f(a) < f(b)$. Since $f(I)$ is an interval, if d is any real number such that $f(a) < d < f(b)$, then $d \in f(I)$. This is equivalent to saying that $f(c) = d$ for some $c \in I$. Since $d \neq f(a)$ and $d \neq f(b)$, we have that $c \in (a,b)$, as required. ■

Theorem 8.14. Any interval I in $\mathbb{R}$ is connected.

again, $\mathbb{R}$ is endowed with the usual topology here.

Proof. Omitted. ■

Combining this theorem and the earlier theorem we have:

Theorem 8.15. Any subset I of $\mathbb{R}$ is connected iff it is an interval.

$\mathbb{R}$ endowed with the usual topology here!

Example. Let S^1 be the unit circle, viewed as a subspace of $\mathbb{R}^2$ with the Euclidean topology; and consider $\mathbb{R}$ with the Euclidean topology. We claim that there does not exist an injective continuous function $f : S^1 \to \mathbb{R}$.

Proof. Suppose that $f : S^1 \to \mathbb{R}$ is an injective continuous function. Then $f(S^1)$ is connected because S^1 is path-connected[12] and hence, connected. Therefore $f(S^1)$ is an interval as it is a subset of $\mathbb{R}$. Since f is injective, $f(S^1)$ cannot be a degenerate interval (i.e. a singleton). So there is $t_0 \in f(S^1)$ such that

[12] *we will prove this in the next section*

$$(t_0 - \varepsilon, t_0 + \varepsilon) \subset f(S^1).$$

Define $\mathbf{x}_0 = (x_0, y_0) \in S^1$ by $f(\mathbf{x}_0) = t_0$; this is well-defined since f is injective. Now consider the restriction of f to $S^1 \backslash \{\mathbf{x}_0\}$:

$$g := f|_{S^1 \backslash \{\mathbf{x}_0\}} : S^1 \backslash \{\mathbf{x}_0\} \to f(S^1) \backslash \{t_0\}.$$

Then g is still a continuous injective map. Now $S^1 \backslash \{\mathbf{x}_0\}$ is connected. So by injectivity and continuity

we use injectivity to deduce $f(A \backslash B) = f(A) \backslash f(B)$.

$$g(S^1 \backslash \{\mathbf{x}_0\}) = g(S^1) \backslash g(\{\mathbf{x}_0\}) = f(S^1) \backslash \{t_0\}$$

is connected [↯]. This is absurd, so no such f can exist. ■

We will give an even more beautiful proof of this when we meet the notion of compactness; in particular, when we meet with the Heine-Borel theorem.

8.4 Path-connectedness

There is a more intuitive type of connectedness.

Definition 8.6 (Path). Let X be a topological space; and let $x, y \in X$. A **path** in X from x to y is a continuous map $f : [0,1] \to X$ such that $f(0) = x$, $f(1) = y$. We say that such a path **joins** x and y.

N.B. Since the image of a path $f : [0,1] \to X$ lies in X, the path must take values in X.

Let's use the usual English meaning for a *path*. Intuitively, if we have a *path* from x to y and a *path* from y to z, then going along these *paths* should give a *path* from x to z. It's no coincidence that our defined path above has this same property (or else, why should we name it a path in the first place).

Lemma 8.16. Suppose that $f, g : [0,1] \to X$ are paths in a space X from x to y and from y to z, respectively. Let

$$h(x) = \begin{cases} f(2t) & t \in [0,1/2], \\ g(2t-1) & t \in [1/2,1]. \end{cases}$$

Then h is a path in X from x to z.

Proof. Prove it yourself. ■

Lemma 8.17. Suppose that $f : [0,1] \to X$ is a path in space X from x to y. Let $g : [0,1] \to X$ be the map

$$g(t) = f(1-t)$$

Then g is a path from y to x. We say g is the **reverse** of f.

Proof. Clearly, g is continuous because f is continuous, and therefore the map $t \mapsto 1-t$ of $[0,1]$. ■

Definition 8.7 (Path-connectedness). A topological space X is **path-connected** if any two points of X can be joined by a path in X.

Definition 8.8 (Subset Path-connectedness). A subset of a topological space X is said to be path-connected if it is path-connected in the subspace topology.

Example. For any $n \geqslant 1$, $\mathbb{R}^n$ is path-connected.

Any convex subset of C of $\mathbb{R}^n$ is path-connected as any two points in C may be joined by a straight line segment in C. ■

Example. $\mathbb{R}^n \setminus \{\mathbf{0}\}$ is path-connected for $n > 1$.

$\mathbf{0}$ is the origin in $\mathbb{R}^n$.

Given $\mathbf{x}$ and $\mathbf{y}$ different from $\mathbf{0}$, we can join $\mathbf{x}$ and $\mathbf{y}$ by the straight-line path between them if that path does not go through the origin. Otherwise, we can choose a point $\mathbf{z}$ not on the line joining $\mathbf{x}$ and $\mathbf{y}$, and take the broken-line path from $\mathbf{x}$ to $\mathbf{z}$, and then from $\mathbf{z}$ to $\mathbf{y}$. ■

Example. Define the unit ball B^n in $\mathbb{R}^n$ by the equation

$$B^n = \{\mathbf{x} : \|\mathbf{x}\| \leqslant 1\}.$$

Then B^n is path-connected.

Given any two points $\mathbf{x}$ and $\mathbf{y}$ of B^n, the straight-line path $f : [0,1] \to \mathbb{R}^n$ defined by

$$f(t) = (1-t)\mathbf{x} + t\mathbf{y}$$

lies in B^n. For if $\mathbf{x}$ and $\mathbf{y}$ are in B^n and $t \in [0,1]$, then

$$\|f(t)\| \leqslant (1-t)\|\mathbf{x}\| + t\|\mathbf{y}\| \leqslant 1$$

By a similar argument, every open ball $B_d(\mathbf{x}, \varepsilon)$ (and the closed ball) in $\mathbb{R}^n$ is path-connected.

The unit circle S^1 is obviously path-connected as the arcs and the whole circle itself (viewed as the image of a function in $\mathbb{R}^2$) are paths which connect any two points in S^1. But how do we prove this rigorously?

Example. The unit circle S^1 is path-connected.

Consider $f : [0, 2\pi] \to S^1$ defined by $f(x) = (\cos x, \sin x)$; this f is surjective by construction. Now pick $\mathbf{x} \neq \mathbf{y}$ on S^1. Since f is surjective, there is $a, b \in [0, 2\pi]$ such that $f(a) = \mathbf{x}$ and $f(b) = \mathbf{y}$. We may assume without loss of generality that $a < b$. Then consider a map $g : [0,1] \to [a,b]$ defined by $g(t) = a + t(b-a)$. Now the function $f \circ g : [0,1] \to S^1$ is continuous (because it is the composition of continuous functions) and $(f \circ g)(0) = f(a) = \mathbf{x}$ and $(f \circ g)(1) = f(b) = \mathbf{y}$. Thus $f \circ g$ is a path in S^1. ■

one could have of course take a smaller interval than $[0, 2\pi]$, then we would get an arc (in the image of f).

We have the more general fact and construction:

Example. Define the unit sphere S^{n-1} in $\mathbb{R}^n$ by the equation

$$S^{n-1} = \{\mathbf{x} : \|\mathbf{x}\| = 1\}.$$

$\mathbb{R}^n$ is given the Euclidean topology, so $\|\cdot\|$ here is the Euclidean norm.

If $n > 1$, then S^{n-1} is path-connected.

The map $g : \mathbb{R}^n \backslash \{\mathbf{0}\} \to S^{n-1}$ defined by $g(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$ is continuous and surjective. It is then easy to prove that the image of a path-connected space (here $\mathbb{R}^n \backslash \{\mathbf{0}\}$) under a continuous map is path-connected. ■

Proposition 8.2. Any path-connected space X is connected.

Proof. Suppose X is path-connected and $g : X \to \{0,1\}$ is continuous where $\{0,1\}$ is equipped with the discrete topology. Suppose for contradiction that g is not constant, so there exist $x, y \in X$ such that $g(x) = 0$ and $g(y) = 1$. Let $f : [0,1] \to X$ be a path in X from x to y. Then the composition $g \circ f : [0,1] \to \{0,1\}$ is continuous. Moreover, observe that

$$g(f(0)) = g(x) = 0 \quad \text{and} \quad g(f(1)) = g(y) = 1$$

So, $g \circ f$ is surjective [↯]. This contradicts the fact that $[0,1]$ is connected. ■

remember that g not constant is equivalent to g is not surjective; as the codomain is a two-point space.

This proposition gives a method of checking whether a space is connected — by showing that it is path-connected which is sometimes easier. But an important reminder is that the converse is false in general.

Example (Topologist's sine curve)**.** Let S denote the following subset of $\mathbb{R}^2$:

$$S = \{(x, \sin \tfrac{1}{x}) : x \in (0,1]\} \cup \{(0,0)\}$$

S is called the Topologist's sine curve and it is connected, but not path-connected.

However for $\mathbb{R}^n$ we do have a positive answer.

Proposition 8.3. A connected open subset U of $\mathbb{R}^n$ is path-connected.

we consider $\mathbb{R}^n$ in the usual topology.

We can push this boundary even further by making the observation that we have a positive answer for any normed vector space.

Proposition 8.4. A connected open subset U of a normed vector space V is path-connected.

Theorem 8.18. Suppose that $f : X \to Y$ is a continuous surjective map from a path-connected space X onto a space Y. Then Y is path-connected.

Theorem 8.19. Let $C[0,1]$ be the function space of all continuous real-valued functions on $[0,1]$ with the uniform metric, d_∞. Then $C[0,1]$ is path-connected and hence connected.

8.5 Connectedness and homeomorphisms

Theorem 8.20. $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic.

considered with the usual topology.

Proof. Suppose for contradiction that $\mathbb{R}^n$ and $\mathbb{R}$ are homeomorphic. Then by definition, there exists a homeomorphism $f : \mathbb{R}^n \to \mathbb{R}$. Denote the origin of $\mathbb{R}^n$ as $\mathbf{0}$; and let $\lambda = f(\mathbf{0})$. This is well-defined because f is injective. Now let $A = \mathbb{R}^n \backslash \{\mathbf{0}\}$ and observe that

$$g := f|_A : \mathbb{R}^n \backslash \{\mathbf{0}\} \to \mathbb{R} \backslash \{\lambda\}$$

is also a homeomorphism by Corollary (4.3). Since $\mathbb{R}^n \backslash \{\mathbf{0}\}$ is connected, it follows that

$$g(\mathbb{R}^n \backslash \{\mathbf{0}\}) = \mathbb{R} \backslash \{\lambda\}$$

the equality follows because g is a homeomorphism; hence bjiective.

is also connected [↯]. This is a contradiction as $\mathbb{R} \backslash \{\lambda\} = (-\infty, \lambda) \cup (\lambda, \infty)$: We have $I = (-\infty, \lambda)$ being open in $\mathbb{R}$ and $\mathbb{R} \backslash I = (\lambda, \infty)$ is open, so I is closed in $\mathbb{R}$. So, I is clopen in $\mathbb{R}$ but it is neither $\mathbb{R}$ nor $\varnothing$ — so it is a non-trivial clopen set. ■

Proposition 8.5. $[0,1]$ and the circle of unit radius, S^1 are not homeomorphic.

considered with the subspace topology inherited from $\mathbb{R}$ and $\mathbb{R}^2$ respectively.

Proof. Suppose $f : [0,1] \to S^1$ is a homeomorphism. Let $x = f(0)$ and $y = f(1)$. These are well-defined since f is bijective. Moreover, x and y are distinct points of S^1 by injectivity. Now let $A = (0,1) = [0,1] \backslash \{0,1\}$ and observe that

$$g := f|_A : (0,1) \to S^1 \backslash \{x, y\}$$

is also a homeomorphism by Corollary (4.3). But then $(0,1)$ is connected and $S^1 \backslash \{x, y\}$ is not connected which is a contradiction. ■

9 Compactness

The notion of compactness may be expressed easily using the language of covers and open covers.

Definition 9.1 (Cover). Suppose X is a set and $A \subset X$. A collection $\{U_i\}_{i \in I}$ of subsets of X is called a **cover** for A or **covers** A if

$$A \subset \bigcup_{i \in I} U_i.$$

Definition 9.2 (Subcover). A **subcover** of a cover $\{U_i\}_{i \in I}$ for A is a subcollection $\{U_j\}_{j \in J}$ for some subset $J \subset I$ such that $\{U_j\}_{j \in J}$ is still a cover for A. We call it a **finite subcover** if J is finite.

a subcover of a cover $\mathcal{U}$ of X is a subset of $\mathcal{U}$ that still covers X.

Definition 9.3 (Open Cover). If $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover for a subset A of a topological space X and if each U_i is open in X, then $\mathcal{U}$ is called an **open cover** for A.

Definition 9.4 (Compactness). A subset A of a topological space X is **compact** if every open cover for A has a finite subcover.

the keyword is **every** open cover. So to show non-compactness, one just need to find one open cover for A which does not have a finite subcover.

Since $X \subset X$, we can replace wherever you see A above with X. In particular, we have the definition of a topological space X itself being compact.

Example. The open interval $(0, 1)$ in $\mathbb{R}$ (with its usual topology) is not compact.

Let's prove this. If we begin with the open cover $\{(0, 1)\}$ then of course it has a finite subcover — it is itself finite. So we need to find for a weird one. Consider the collection

$$\mathcal{U} = \{(1/n, 1) : n \in \mathbb{N}, n > 1\}$$

This is indeed an open cover for $(0, 1)$: for any $x \in (0, 1)$, we have $x > 1/n$ for sufficiently large n so that $x \in (1/n, 1)$. But does it have a finite subcover? Any finite subcollection of $\mathcal{U}$, say

$$\{(1/n_1, 1), (1/n_2, 1), \ldots, (1/n_r, 1)\}$$

covers only $(1/N, 1)$ where $N = \max\{n_1, n_2, \ldots, n_r\}$. So $(0, 1)$ is **not** compact.

Example. The real line $\mathbb{R}$ is not compact.

Let's prove this. Consider the collection

$$\mathcal{U} = \{(n, n + 2)\}$$

This is indeed an open cover for $\mathbb{R}$. But it contains no finite subcollection that covers $\mathbb{R}$.

Topological Fact (Finite). Any finite subset A of a topological space X is compact.

in particular, the singletons and $\varnothing$ are compact.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of X. Suppose $\mathcal{U} = \{U_i\}_{i \in I}$ is any open cover of A. Then for each $r \in \{1, 2 \ldots, n\}$, we have $a_r \in U_{i_r}$ for some $i_r \in I$, and so $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}\}$ is a finite subcover of $\mathcal{U}$ for A. ■

Topological Fact (Indiscrete). Any indiscrete space is compact.

Proof. Suppose X is an indiscrete space, and let $\mathcal{U}$ be any open cover of X. Clearly, $\mathcal{U} = \{X\}$. Also, clearly, $\{X\}$ is a finite subscover of $\mathcal{U}$ for X. ■

Topological Fact (Discrete). Any discrete space is compact $\Leftrightarrow$ it is finite.

Proof. Let X be any discrete space. If X is finite, then we have shown earlier that it must be finite. So we are left to show the converse. Suppose X is compact but is infinite. Let $\mathcal{U} = \{\{x\} : x \in X\}$ be an open cover of X. Since X is compact, there is a finite subcover of $\mathcal{U}$ for X, say, $\mathcal{V} = \{\{x_1\}, \{x_2\}, \dots, \{x_n\}\}$ for some $n \in \mathbb{N}$. Since X is infinite by assumption, there is an element $y \in X$ such that $\{y\} \notin \mathcal{V}$ [$\lightning$]. This contradicts that $\mathcal{V}$ is a cover, so such a finite subcover cannot exist. ■

Topological Fact (Cofinite). Any cofinite space is compact.

Proof. Suppose X is a cofinite space, and let $\mathcal{U}$ be any open cover of X. Since X is non-empty[13] there is at least one element of $\mathcal{U}$, say, U_{i_0} that is non-empty. Since U_{i_0} is open, $X \backslash U_{i_0}$ must be finite, say

$$X \backslash U_{i_0} = \{x_1, x_2, \dots, x_n\}.$$

For each $r = 1, 2, \dots, n$ we have $x_r \in U_{i_r}$ for some $i_r \in I$. Therefore, $\{U_{i_0}, U_{i_2}, \dots, U_{i_n}\}$ is a finite subcover of $\mathcal{U}$ for X. ■

[13] *the cofinite topology is always given to non-empty set.*

to get a better idea of the proof, use $\mathbb{R}$ and take for example the open sets of the form $(-\infty, a) \cup (a, \infty)$. This is open in the cofinite topology.

Proposition 9.1. Let $\mathcal{C}, \mathcal{F}$ be topologies on a set X with $\mathcal{C} \subset \mathcal{F}$. If, $(X, \mathcal{F})$ is compact, then so is $(X, \mathcal{C})$.

9.1 Properties of compact spaces

Proposition 9.2. Any compact subset C of a metric space (X, d) is bounded.

Figure 1 would be useful to follow the proof of this proposition.

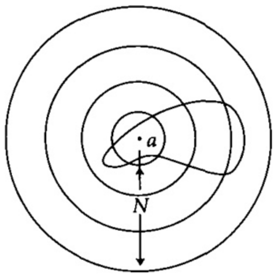

Figure 1. The blob is C while the contours are the open balls $B(a, n_i)$.

Proof. Let X be a metric space and C be a compact subset of X. Let $a \in X$. For any $c \in C$, we may choose an integer $n > d(a, c)$ so that $c \in B(a, n)$. This shows that $\{B(a, n) : n \in \mathbb{N}\}$ is an open cover of C. By compactness of C, there is some finite subcover of C, say $\{B(a, n_1), B(a, n_2), \dots, B(a, n_r)\}$ i.e.

$$C \subset \bigcup_{i=1}^{r} B(a, n_i).$$

But $\bigcup_{i=1}^{r} B(a, n_i) = B(a, N)$ where $N = \max\{n_1, n_2, \dots, n_r\}$. So, $C \subset B(a, N)$ i.e. C is bounded, as required. ■

Proposition 9.3. Any compact subset C of a Hausdorff space X is closed.

Proof. Let C be a compact subset of a Hausdorff space X. We claim that $X\backslash C$ is open in X. Fix a point $x \in X\backslash C$. We will show[14] that $X\backslash C$ is a neighbourhood of x i.e. that there is an open neighbourhood U_x of x such that $U_x \subset X\backslash C$.

For each $c \in C$, by the Hausdorff condition there exist disjoint sets U_c, V_c open in X and with $x \in U_c$, $c \in V_c$. Therefore the collection $\{V_c : c \in C\}$ is an open cover for C, and by compactness of C, there is a finite subcover for C, say $\{V_{c_1}, V_{c_2}, \ldots, V_{c_n}\}$, so we have

$$C \subset \bigcup_{i=1}^{n} V_{c_i}$$

Now let $U_x = \bigcap_{i=1}^{n} U_{c_i}$. This is open in X because it is the intersection of sets open in X. Clearly, $x \in U_x$ because $x \in U_{c_i}$ for each i. Then observe that for each $i = 1, 2, \ldots, n$, we have $U_x \subset U_{c_i}$, and so $U_x \cap V_{c_i} \subset U_{c_i} \cap V_{c_i} = \varnothing$. Therefore, we have

$$U_x \cap C \subset U_x \cap \bigcup_{i=1}^{n} V_{c_i} = \bigcup_{i=1}^{n} (U_x \cap V_{c_i}) = \varnothing.$$

Thus, $U_x \subset X\backslash C$, i.e. $X\backslash C$ is a neighbourhood of x. Since x was arbitrary, Theorem (1.23) tells us that $X\backslash C$ is open, as desired. ■

[14] *Idea: Since x is arbitrary, we can conclude that $X\backslash C$ is a neighbourhood of all of its points and hence is open by Theorem (1.23).*

Since every metrizable space (and hence, every metric space) is Hausdorff, the following theorem follows from the two propositions above.

Theorem 9.1. Any compact subset C of a metric space (X, d) is closed and bounded.

The converse to this theorem is false. As we will see, we can strengthen the notion of boundedness and get the converse as a truth.

There is one more notion related to compactness which is sometimes useful.

Definition 9.5. A subset A of a topological space X is said to be **relatively compact** in X if $\bar{A}$ is compact, where the closure is taken in X.

Example. $(0, 1)$ is relatively compact in $\mathbb{R}$ because $[0, 1]$ is compact.

Example. $(0, 1)$ is not relatively compact in $(0, 1)$ as $\overline{(0, 1)} = (0, 1)$ in $(0, 1)$ which is not compact (w.r.t subspace topology inherited from $\mathbb{R}$).

9.2 Continuous maps on compact spaces

Theorem 9.2. Suppose $f : X \to Y$ is a continuous map of topological spaces and that X is compact. Then $f(X)$ is compact.

Proof. Suppose that $\mathcal{U}$ is an open cover of $f(X)$. Since f is continuous, $f^{-1}(U)$ is open in X for every $U \in \mathcal{U}$. The collection $\{f^{-1}(U) : U \in \mathcal{U}\}$ covers X since $\mathcal{U}$ covers $f(X)$. Since X is compact by hypothesis, there is a finite subcover, say,

We have $f(X) \subset \bigcup U$, so by identity of sets, $X \subset \bigcup f^{-1}(U)$ i.e. $\{f^{-1}(U) : U \in \mathcal{U}\}$ covers X.

$\{f^{-1}(U_1), f^{-1}(U_2), \ldots, f^{-1}(U_r)\}$ for X i.e. we have

$$X \subset \bigcup_{i=1}^{r} f^{-1}(U_i).$$

By identity of sets, we thus have

$$f(X) \subset \bigcup_{i=1}^{r} f(f^{-1}(U_i)) \subset \bigcup_{i=1}^{r} U_i.$$

That is, the set $\{U_1, U_2, \ldots, U_r\}$ is a finite subcover of $f(X)$. So $f(X)$ is compact. ■

Corollary 9.1. Compactness is a topological property.

Proof. Suppose $f : X \to Y$ is a homeomorphism and that X is compact. Since f is bijective, $f(X) = Y$. By the preceding theorem, Y is thus compact. ■

Corollary 9.2. Any continuous map from a compact space to a metric space is bounded.

This is a generalization of the renowned local-to-global real analysis theorem: A continuous function $f : [a, b] \to \mathbb{R}$ is bounded on $[a, b]$.

9.3 Compactness of subspaces and products

Compactness is an absolute property of a set just like connectedness.

Theorem 9.3. Let K be a subset of a topological space $(X, \mathcal{T})$. Then K is a compact subset of X *if and only if* K is compact as a subset of itself.

i.e. with the subspace topology inherited from X.

Proof. Suppose K is a compact subset of X. Suppose $\mathcal{U} = \{U_i : i \in I\}$ is an open cover for K where U_i are open sets in X. Now $U_i \cap K$ is open in K for all $i \in I$; and observe that

$$K \subset \bigcup_{i \in I} (U_i \cap K),$$

so $\mathcal{O} = \{U_i \cap K : i \in I\}$ is an open cover for K (by sets open in K). Since K is a compact subset of X, $\mathcal{U}$ has a finite subcover $\mathcal{U}' = \{U_1, \ldots, U_n\}$. Observe then that the set $\mathcal{O}' = \{U_1 \cap K, \ldots, U_n \cap K\}$ is a finite subcollection of $\mathcal{O}$ such that

$$K \subset \bigcup_{i=1}^{n} (U_i \cap K).$$

That is, $\mathcal{O}'$ is a finite subcover for K. So K is a compact subset of itself.

The converse is very similar except we use the fact that if a set U is open in K, then there is a set G open in X such that $U = G \cap K$. ■

So it doesn't matter to say A is compact in the ambient space X or compact in A because it is the same thing.

Proposition 9.4. Any closed subset C of a compact space X is compact.

this is important!

Proof. Let C be a closed subset of the compact space X. Given an open cover $\mathcal{U}$ for C by sets open in X, we can form an open covering $\mathcal{O}$ of X by adjoining to $\mathcal{U}$ the single open set $X\backslash C$, that is,

$$\mathcal{O} = \mathcal{U} \cup \{X\backslash C\}.$$

Since X is compact, there is a finite subcollection of $\mathcal{O}$ which covers X. If this subcollection contains the set $X\backslash C$, throw it out; otherwise, leave the subcollection as it is. In both cases, the resulting collection is a finite subcollection of $\mathcal{U}$ that covers C. So, C is compact, as desired. ■

Proposition 9.5. If A and B are compact subsets of a space X, then so is $A \cup B$.

Proof. Let $\mathcal{U}, \mathcal{V}$ be an open cover for A, B respectively, by sets open in X. By identity of sets, $\mathcal{U} \cup \mathcal{V}$ is therefore a cover for $A \cup B$. Since A is compact, there is a finite subcollection of $\mathcal{U}$ which covers A, say, $\{U_1, U_2, \ldots, U_m\}$; likewise, there is a finite subcollection of $\mathcal{V}$ which covers B, say, $\{V_1, V_2, \ldots, V_n\}$. Therefore, by identity of sets we have that $\{U_1, \ldots, U_m, V_1, \ldots, V_n\}$ is a finite subcollection of $\mathcal{U} \cup \mathcal{V}$ which covers $A \cup B$, as desired. ■

Proposition 9.6. If A, B are compact subsets of a Hausdorff space, then $A \cap B$ is compact.

Proof. Since A, B are compact subsets of a Hausdorff space X, they are closed in X by Proposition (9.3). Therefore, the intersection $A \cap B$ is closed. But $A \cap B$ is a subset of A which is compact. By Proposition (9.4), it therefore must be compact. ■

We will not prove the following lemma and theorem. However, for the sake of completion, we state it here.

Lemma 9.4. A cartesian product $X \times Y$ of spaces X, Y is compact *if and only if* X and Y are both compact.

Theorem 9.5. The product of finitely many compact spaces is compact.

In fact, the product of infinitely many compact spaces is compact. It is called Tychonoff's theorem, an important, interesting (and difficult) result of topology.

9.4 Compactness results in $\mathbb{R}^n$

Every $\mathbb{R}^n$ here is endowed with the Euclidean topology. And of course, the metric which induces this topology is the Euclidean metric d_2 (or anything topologically equivalent to it). We need to remind ourselves of this fact as the results in this section are not wholly topological, but also depends on the metric; for example boundedness. We first characterize the compact subspaces of $\mathbb{R}^n$. This next theorem is really a corollary of the more general results.

Theorem 9.6. Any compact subset of $\mathbb{R}^n$ is closed and bounded.

Proof. By Proposition (9.3), any compact subset of $\mathbb{R}^n$ is closed because $\mathbb{R}^n$ is Hausdorff. By Proposition (9.2), any compact subset of $\mathbb{R}^n$ is bounded because $\mathbb{R}^n$ with the Euclidean metric d_2 is a metric space. ■

We then have a generalization of the extreme value theorem of real analysis.

Theorem 9.7 (Extreme Value Theorem). If $f : C \to \mathbb{R}$ is continuous and C is compact, then f attains its bounds on C.

i.e. there is at least one $c_0 \in C$ such that $f(c_0) = \inf f(C)$ and at least one $c_1 \in C$ such that $f(c_1) = \sup f(C)$.

Proof. Since f is continuous an C is compact, then $f(C)$ is compact and hence bounded and closed in $\mathbb{R}$ by Theorem (9.6). But for any non-empty bounded subset $A \subset \mathbb{R}$, we know that $\sup A, \inf A \in \bar{A}$. So $\sup f(C)$ and $\inf f(C)$ are in $\overline{f(C)} = f(C)$ i.e. f attains its bounds on C. ■

Taking $C = [a, b] \subset \mathbb{R}$, we arrive at the real analysis version of the extreme value theorem. So indeed, this theorem is a generalization of our well-known result.

Corollary 9.3. A continuous real-valued function on $[a, b]$ attains its bounds.

Slogan. Bounds of f on X is basically $\sup f(X)$ and $\inf f(X)$.

Proposition 9.7. If $X \subset \mathbb{R}$ is not compact, then there is a continuous function $f : X \to \mathbb{R}$ which is not bounded.

Proposition 9.8. If $X \subset \mathbb{R}$ is not compact, then there is a continuous function $f : X \to \mathbb{R}$ which is bounded but does not attain its bounds.

Finally, we meet with the Heine-Borel theorem which links our intuition to compactness.

Theorem 9.8. Any closed bounded interval $[a, b]$ in $\mathbb{R}$ is compact.

This is really a special case of a more general result.

Theorem 9.9 (Heine-Borel). Any closed and bounded subset C of $\mathbb{R}^n$ is compact.

We will revisit an example discussed before, but now view it from a different perspective. We will first use the majestic Heine-Borel theorem, and then show that it can be proven using an even simpler way.

Example. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ be the unit circle, viewed as a subspace of $\mathbb{R}^2$ with the Euclidean topology; and consider $\mathbb{R}$ with the Euclidean topology. The metric which induces the Euclidean topology on both $\mathbb{R}^2$ and $\mathbb{R}$ is of course the Euclidean metric d_2 — we will use this. Now, we claim that there does not exist an injective continuous function $f : S^1 \to \mathbb{R}$. The trouble really is to show that S^1 is compact. We will show this in two ways. First, we apply Heine-Borel directly to S^1, the other one we implictly apply Heine-Borel but not to S^1.

Showing S^1 is compact using Heine-Borel. Since $\mathbb{R}$ is Hausdorff, singleton subsets of $\mathbb{R}$ are closed in $\mathbb{R}$. In particular $\{1\}$ is closed in $\mathbb{R}$. Consider the map $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $(x, y) \mapsto x^2 + y^2$. This map is continuous because it is the sum[15] of two continuous functions. Therefore

$$g^{-1}(\{1\}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} = S^1$$

is closed in $\mathbb{R}^2$. Moreover, obviously we have, say, $S^1 \subset B_{d_2}(0, 2)$ so S^1 is bounded. By Heine-Borel, S^1 is compact. ■

[15] *it is the sum of the map $(x, y) \mapsto x^2$ and $(x, y) \mapsto y^2$.*

Showing S^1 is compact using continuous image. Consider the map $g : [0, 1] \to S^1$ defined by $g(x) = (\cos 2\pi x, \sin 2\pi x)$. This function is continuous and surjective. By Heine-Borel, $[0, 1]$ is compact. Since g is continuous, $g([0, 1])$ is compact. Since g is surjective, $g([0, 1]) = S^1$ and so S^1 is compact as required. ■

Finishing the proof. If there is an injective continuous function $f : S^1 \to \mathbb{R}$, then $f(S^1)$ must necessarily be compact as well. Since f is injective, it cannot be constant. It follows that $f(S_1) = [a, b]$ for some $a \neq b$ by Theorem (9.6). Pick a point $s \in S^1$ such that $f(s) \notin \{a, b\}$ i.e. not the endpoints. Now, it is not too hard to see that $S^1 \backslash \{s\}$ is path-connected and hence, connected. However $[a, b] \backslash \{f(s)\}$ is not connected as $\langle [a, f(s)), (f(s), b] \rangle$ forms a separation of $[a, b]$. ■

indeed $[a, f(s))$ and $(f(s), b]$ are open in the subspace topology given to $[a, b]$.

9.5 Compactness and uniform continuity

Definition 9.6. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is said to be **uniformly continuous** on X if given $\varepsilon > 0$, there is $\delta > 0$ such that for any $a, b \in X$,

$$d_X(a, b) < \delta \implies d_Y(f(a), f(b)) < \varepsilon$$

Remark. Uniform continuity is stronger than usual continuity. δ can only depend on ε but not on the point of the domain X — this is the significance of the word *uniformly.*

Usual continuity of f is a local property i.e. it says something about f in some neighbourhood of each point in X. However, uniform continuity is a **global property** since it says something about f over the whole space X. Since compactness allows us to pass from local to global, the next proposition shouldn't come as very surprising.

Theorem 9.10 (Uniform Continuity Theorem). If $f : X \to Y$ is a continuous map of metric spaces and X is compact, then f is uniformly continuous on X.

Proposition 9.9. If the metrics d_1, d_2 on a set X are Lipschitz equivalent, then the identity map $\iota : (X, d_1) \to (X, d_2)$ is uniformly continuous, as its inverse.

9.6 An inverse function theorem

Proposition 9.10. Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof. It is left to show that $f^{-1} : Y \to X$ is continuous. Instead of considering open sets, we consider closed sets. Suppose V is closed in X. We want to show $(f^{-1})^{-1}(V)$ is closed in Y. By Proposition (B.4), we have $(f^{-1})^{-1}(V) = f(V)$.

Now since V is closed in X which is compact, Proposition (9.4) tells us that V is compact. Therefore by Theorem (9.2), $f(V)$ is compact. But since Y is Hausdorff, Proposition (9.3) tells us that $f(V) = (f^{-1})^{-1}(V)$ is closed; as required. ■

Corollary 9.4. Let $f : X \to Y$ be an injective continuous function. If X is compact and Y is Hausdorff, then f determines a homeomorphism of X onto $f(X)$.

Example. If $f : [a, b] \to \mathbb{R}$ is a continuous monotonic function, then it has a continuous inverse function $f^{-1} : f([a, b]) \to [a, b]$ which is also monotonic.

Proposition 9.11. Suppose $\mathcal{T}_1, \mathcal{T}_2$ are topologies on a set X such that $\mathcal{T}_1 \subset \mathcal{T}_2$; let $(X, \mathcal{T}_1), (X, \mathcal{T}_2)$ both be Hausdorff and compact. Then $\mathcal{T}_1 = \mathcal{T}_2$.

Example. There is no Hausdorff topology on $[0, 1]$ which is strictly coarser than the Euclidean topology.

9.7 Finite Intersection Property

There is another characterization for a space to be compact. Instead of using open sets (open covers), we we used closed sets.

Definition 9.7. A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** or **FIP** if for every finite subcollection

$$\{C_1, \ldots, C_n\}$$

of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is non-empty.

the collection $\mathcal{C}$ is an indexed collection

Theorem 9.11. Let X be a topological space. Then X is compact *if and only if* for every collection $\mathcal{C}$ of **closed** sets in X having the FIP, the intersection

$$\bigcap_{C \in \mathcal{C}} C$$

of all elements of $\mathcal{C}$ is non-empty.

the collection $\mathcal{C}$ is an indexed collection

Remark. A reformulation: X is compact *if and only if* any indexed collection

$$\{C_i : i \in I\}$$

of closed subsets of X such that

$$\bigcap_{j \in J} C_j$$

is non-empty for any finite subset $J \subset I$, then

$$\bigcap_{i \in I} V_i$$

is non-empty.

The formulation is not intuitive at all but its existence prove handy later on.

9.8 Compactness and metric spaces

Proposition 9.12. Let (X, d) be a compact metric space and suppose $T : X \to X$ satisfies the condition $d(Tx, Ty) < d(x, y)$ for all $x \neq y$ in X. Then the equation $Tx = x$ has a unique solution in X.

Proposition 9.13. Let (X, d) be a compact metric space and suppose $T : X \to X$ is continuous and satisfies the condition $d(Tx, Ty) \geqslant d(x, y)$ for all $x, y \in X$. Then $T(X) = X$.

Proposition 9.14. Any compact metric space X is complete.

The converse is not true in general. $\mathbb{R}$ is complete (with the usual metric) but is not compact.

10 Sequential Compactness

10.1 Sequential compactness for $\mathbb{R}$

Let us first recall the renowned Bolzano-Weierstrass theorem of real analysis.

Theorem 10.1. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

In this chapter, we generalize the Bolzano-Weierstrass theorem by introducing an alternative approach to compactness called sequential compactness.

Definition 10.1 (Sequential Compactness). A subset $S \subset \mathbb{R}$ is called **sequentially compact** if every sequence in S has at least one subsequence converging to a point in S.

Theorem 10.2. A subset $\mathbb{R}$ is sequentially compact *if and only if* it is closed and bounded in $\mathbb{R}$.
The forward direction of the proof is really Bolzano-Weierstrass. The converse is the only part that needs some hard work.

Theorem 10.3. A subset of $\mathbb{R}$ is compact *if and only if* it is sequentially compact.

This final theorem hints towards the fact that there is some relationship between compactness and sequentially compactness (as it should because of the name) when we generalize this notion to metric spaces.

10.2 Sequential compactness for metric spaces

Definition 10.2 (Sequential Compactness). A metric space X is **sequentially compact** if every sequence in X has at least one subsequence converging to a point of X.

Slogan. X is sequentially compact if it satisfies Bolzano-Weierstrass.

Definition 10.3 (Subset Sequential Compactness). A non-empty subset A of a metric space X is sequentially compact if it is sequentially compact with the subspace metric d_A.

Remark. Conventionally, the empty set $\varnothing$ is sequentiall compact.

Topological Fact. Any finite metric space is sequentially compact.

Proposition 10.1. Let $\langle x_n \rangle$ be a sequence in a metric space X and let $x \in X$. Suppose that for each $\varepsilon > 0$, the neighbourhood $B(x, \varepsilon)$ contains x_n for infinitely many values of n. Then $\langle x_n \rangle$ has a subsequence converging to x.

this proposition is really saying that "a compact metric space is sequentially compact."

Example. Consider the following sequence in $\mathbb{R}$:

$$\left\{1, 1, \frac{1}{2}, 3, \frac{1}{3}, \dots, n, \frac{1}{n}, \dots\right\}$$

Then 0 is such that for any $\varepsilon > 0$, the neighbourhood $B(0, \varepsilon)$ contains x_n for infinitely many n, since it contains every term of the form $1/n$ in the sequence for sufficiently large n. The subsequence of $\langle x_n \rangle$ formed by taking the second term is $(1/n)$ which converges to 0.

Corollary 10.1. Suppose that a sequence $\langle x_n \rangle$ in a metric space X has no convergent subsequences. Then for each $x \in X$, there exists $\varepsilon_x > 0$ such that $B(0, \varepsilon_x)$ contains x_n for only finitely many values of n.

Example. Consider the sequence (n) in $\mathbb{R}$ which has no convergent subsequence. Now for any $x \in \mathbb{R}$, we may take $\varepsilon_x = 1$ and the neighbourhood $B(x, \varepsilon_x)$ contains at most two terms in the sequence (n).

Our superstar result of this section is the following:

Theorem 10.4. A metric space is compact *if and only if* it is sequentially compact.

Proof. Omitted. Uses Lebesgue numbers and ε-nets which we will not discuss. ■

10.3 Verbatim extension from compactness

Proposition 10.2. A sequentially compact metric space is bounded.

Proposition 10.3. A closed subset of a sequentially compact metric space is sequentially compact.

Proposition 10.4. If $f : X \to Y$ is a continuous function of metric spaces and X is sequentially compact, then so if $f(X)$.

Proof. Since a metric space is compact if and only if it is sequentially compact, this result follows trivially from the fact that continuous image of compact spaces is compact. ■

Proposition 10.5. Let X and Y be metric spaces that are homeomorphic. Then

$$X \text{ is sequentially compact} \Leftrightarrow Y \text{ is sequentially compact}$$

Proposition 10.6. Any continuous function from a sequentially compact metric space to another metric space has bounded image.

Proposition 10.7. The product of two sequentially compact metric spaces is sequentially compact.

Proposition 10.8. A closed bounded subset of $\mathbb{R}^n$ is sequentially compact.

Proposition 10.9. A subspace C of a metric space X is relatively compact *if and only if* every sequence in C has a convergent subsequence.

10.4 Totally bounded subsets of a metric space

Let us first recall what it means for a metric space to be bounded.

Definition 10.4 (Bounded). A subset S of a metric space (X, d) is **bounded** if there exist $x_0 \in X$ and $r > 0$ such that $S \subset B(x_0, r)$.

We also recall the following theorem which was proven not too far ago.

Theorem 10.5. Any compact subset C of a metric space (X, d) is closed and bounded.

As we have said, the converse to this theorem is false. However, we will now define a stronger criteria so that we can get a converse.

Definition 10.5 (Totally Bounded). A metric space (X, d) is said to be **totally bounded** if for every $\varepsilon > 0$ there is a finite collection of open balls

$$\{B(x_1, \varepsilon), B(x_2, \varepsilon), \ldots, B(x_r, \varepsilon)\}$$

that covers X i.e.

$$X \subset \bigcup_{i=1}^{r} B(x_i, \varepsilon)$$

If X is totally bounded, we say X is a totally bounded space.

Definition 10.6 (Subset Totally Bounded). A subset A of a metric space (X, d) is said to be totally bounded if it is totally bounded in the subspace metric d_A.

Proposition 10.10. If a metric space X is totally bounded, then it is bounded.

Example. The closed interval $[a, b]$ in $\mathbb{R}$ with the usual metric is totally bounded.

The converse of the preceding proposition is true for $X = \mathbb{R}$.

Theorem 10.6. Any bounded subset of $\mathbb{R}$ with the usual metric is totally bounded.

What are examples of metric spaces which are bounded but **not** totally bounded?

Example (Bounded but not totally bounded). Consider $\mathbb{Z}$ with the discrete metric d; let $z \in \mathbb{Z}$ Then for any $r > 1$,

$$B_d(z, r) = \{y : d(z, y) < r\} = \{y : d(z, y) = 1 \text{ and } d(z, y) = 0\} = \mathbb{Z}$$

So in particular, $\mathbb{Z}$ is contained in, say, $B_d(0, 2020)$. So, $\mathbb{Z}$ is bounded.

On the other hand, take $\varepsilon = 1/2$. Then $B_d(z, 1/2) = \{z\}$. So for any finite collection of open balls

$$\{B(z_1, \varepsilon), B(z_2, \varepsilon), \ldots, B(z_n, \varepsilon)\},$$

we have that their union is $\{z_1, z_2, \ldots, z_n\}$ which is finite. So, we cannot have $\mathbb{Z}$ be in their union. So, $\mathbb{Z}$ is **not** totally bounded.

The superstar theorem of compactness of metric spaces is embedded in the following theorem.

Theorem 10.7 (TFAE Metric Spaces Compactness). Let X be a metric space. Then the following are equivalent:

(1) X is compact.

(2) X is sequentially compact.

(3) X is complete and totally bounded.

Proof. Omitted. ■

11 Quotient Spaces

11.1 Quotient of sets

We first start with some memory refresh.

Definition 11.1 (Equivalence relation). An **equivalence relation** on a set A is a relation $\sim$ such that it satisfies the properties: (i) $x \sim x$ for every $x \in A$; (ii) If $x \sim y$, then $y \sim x$; and (iii) If $x \sim y$ and $y \sim z$, then $x \sim z$.

Definition 11.2 (Equivalence class). Given an equivalence relation $\sim$ on a set A and an element x of A, we define a certain subset of A called the **equivalence class** of x under $\sim$ to be the set

$$[x] = \{y \in A : y \sim x\}$$

Remark. Note that if $x \sim y$, then $[x] = [y]$. This is not so hard to see, or to prove. Moreover, we have that two equivalence classes are either disjoint or equal. This allows us to partition a set using its equivalence classes.

By using the language of equivalance classes, we can now have the notion of *gluing* points in a set.

Definition 11.3 (Quotienting out). Let A be a set and $\sim$ be an equivalence relation on A. Then we define the **quotient of A modulo $\sim$** to be the set $A/\sim$ whose points are equivalence classes in A i.e.

$$A/\sim = \{[a] : a \in A\}$$

i.e. $A/\sim$ is the set of all equivalence classes in A.

Finally, we give a purely set-theoretic result about maps of quotients.

Proposition 11.1. Suppose that X, Y are sets and $\sim$ is an equivalence relation on X. Let $f : X \to Y$ be a map such that $f(x) = f(y)$ whenever $x \sim y$. Then there is a well-defined map $g : X/\sim \to Y$ where we define $g([x]) := f(x)$.

We say that f **respects the identifications on X** and we call g the **map induced by f**.

Proof. To see that g is well-defined, we need to check that $f(x') = f(x)$ whenever $x' \in [x]$. But if $x' \in [x]$, then $x' \sim x$ so $f(x') = f(x)$ by hypothesis. ■

11.2 Quotient topology

Definition 11.4 (Coinduced topology by f). Let $(X, \mathcal{T})$ be a topological space and let Y be any set. Then the collection of subsets

$$\tau_f = \{U \subset Y \mid f^{-1}(U) \text{ is open in } X\}$$

is a topology on Y and is called the **topology coinduced on Y by f**.

apparently Jerry used topology *induced* on Y by f. We will follow community wiki.

Let's prove that this topology is well-defined. This is an exercise of inverse images.

Proof. [T1]. We have $f^{-1}(Y) = X \in \mathcal{T}$ and so $Y \in \tau_f$. Also, $f^{-1}(\varnothing) = \varnothing$ which is in $\mathcal{T}$, so $\varnothing$ is in τ_f.

[T2]. Suppose $U, V \in \tau_f$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are open in X, and so is their intersection. And so because

$$f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V),$$

we therefore have $U \cap V \in \tau_f$.

[T3]. Suppose that U_i belongs to τ_f for each i in some indexing set I. Then

$$f^{-1}\left(\bigcup_{i\in I} U_i\right) = \bigcup_{i\in I} f^{-1}(U_i)$$

which belongs to $\mathcal{T}$ since each $f^{-1}(U_i)$ belongs to $\mathcal{T}$. Therefore, $\bigcup U_i$ belongs to τ_f. ■

Clearly, by giving Y this topology, the map f is continuous. We have also the following result regarding the continuity of f.

Proposition 11.2. Let $(X, \mathcal{T})$ be a topological space and Y be any set. The topology τ_f coinduced on Y by f is the finest topology such that $f : (X, \mathcal{T}) \to (Y, \tau_f)$ is continuous.

We now define quotient maps.

Definition 11.5 (Quotient map). A surjective map $f : X \to Y$ of topological spaces is called a **quotient map** if a subset U of Y is open in Y *if and only if* $f^{-1}(U)$ is open in X.

If $\sim$ is an equivalence relation on X, the natural map $\natural : X \to X/\sim$ defined by $x \mapsto [x]$ is a quotient map.

notice how this is stronger than continuity. A quotient map is a continuous surjective open map.

Definition 11.6 (Quotient topology). Let X be a topological space and $\sim$ be an equivalence relation on X. Then the **quotient topology** on $X/\sim$ is defined to be the topology coinduced on $X/\sim$ by the natural map $\natural : X \to X/\sim$.

The quotient set $X/\sim$ equipped with the quotient topology is called a **quotient space** of X modulo $\sim$.

Example. Let $X = [0, 1] \cup [2, 3]$. Define an equivalence relation on X by: $1 \sim 2$, but otherwise no two distinct points on X are equivalent i.e. $[1] = [2] = \{1, 2\}$ whereas $[x] = \{x\}$ for all $x \in X \backslash \{0, 1\}$. Then $X/\sim$ is homeomorphic to $[0, 1]$.

Recall that the unit circle is defined by $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Example. Let $X = [0, 1]$ and define an equivalence relation on X by: $0 \sim 1$, but otherwise no two distinct points on X are equivalent. Then $X/\sim$ is homeomorphic to the circle S^1. A homeomorphism that does the job is $f : X/\sim \to S^1$ defined by

$$x \mapsto (\cos 2\pi x, \sin 2\pi x)$$

This map is well-defined[16], bijective and its inverse is continuous.

The way we glue things in the two examples above can be generalized below.

[16] *the only problem here is that if it is multi-valuedness but this is taken care of as* $f(0) = f(1)$.

Example. If $A \subset X$, defined $\sim$ by $x \sim y$ iff $x = y$ or $x, y \in A$. This glues everything in A together and leaves everything else alone. We will write this as X/A to indicate that we *glued* or *quotiented out* points of A.

By taking $X = [0,1] \cup [2,3]$ and $A = \{1,2\}$, we recover our first example. By taking $X = [0,1]$ and $A = \{0,1\}$, we recover our second example.

Example. Let $X = \mathbb{R}$ and define an equivalence relation on X by: $x \sim y$ iff $x - y \in \mathbb{Z}$. Then $X/\sim := \mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1 given by the map $[x] \mapsto (\cos 2\pi x, \sin 2\pi x)$.

Finally, we note that even though X is Hausdorff, $X/\sim$ may not be. A classical example is considering $\mathbb{R}$.

Example. Let $X = \mathbb{R}$ and define an equivalence relation on X by: $x \sim y$ iff $x - y \in \mathbb{Q}$. Then $X/\sim := \mathbb{R}/\mathbb{Q}$ is not Hausdorff.

Super finally, we state a useful result of quotient spaces which is quite obvious.

Proposition 11.3 (Quotient spaces are connected/compact).

(1) If X is connected, then $X/\sim$ is connected.

(2) If X is compact, then $X/\sim$ is compact.

Proof. The natural map $\natural : X \to X/\sim$ is continuous and surjective. Therefore the image under $\natural$ is compact (and connected) by continuity. Since it is surjective, the image under $\natural$ is precisely $X/\sim$. ■

A Sets and Relations

A.1 Sets, subsets, supersets

Definition A.1. A set A is a **subset** of a set B, or equivalently, B is a **superset** of A, if and only if $x \in A$ implies $x \in B$ and we write this as $A \subset B$ or $B \supset A$.

We also say this as A is **contained** in B or B **contains** A.

Remark. Note that using this definition, $A \subset B$ does not *exclude* the possibility that $A = B$.

Remark. The negation of $A \subset B$ is written $A \not\subset B$ or $B \not\supset A$ and means that there is $x \in A$ such that $x \notin B$.

Definition A.2. Two sets A and B are equal *if and only if* $A \subset B$ and $B \subset A$.

Definition A.3. If $A \subset B$ but $A \neq B$, we say that A is a **proper subset** of B.

The following theorem then tells us that $\subset$ is an *almost equivalence relation*.

Theorem A.1. Let A, B and C be any sets. Then

(1). (*Reflexive*). $A \subset A$;

(2). If $A \subset B$ and $B \subset A$, then $A = B$;

(3). (*Transitive*). If $A \subset B$ and $B \subset C$, then $A \subset C$.

Definition A.4. (Universal Set). The set which is a superset of all sets under investigation is called the **universal set** and denoted ξ.

Definition A.5. (Empty Set). The set which contains no elements is called the **empty set** and denoted $\varnothing$.

The empty set is considered finite and a subset of every other set i.e. for any set A, $\varnothing \subset A$.

A.2 Set operations

Definition A.6. The **union** of two sets A and B, denoted by $A \cup B$, is the set of all elements which belong to either A or B i.e.

$$A \cup B := \{x : x \in A \ \text{ or } \ x \in B\}$$

Definition A.7. The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B i.e.

$$A \cap B := \{x : x \in A \ \text{ and } \ x \in B\}$$

Definition A.8. Two sets A and B are said to be **disjoint** or **non-intersecting** if $A \cap B = \varnothing$.

Definition A.9. The **relative complement** of a set B with respect to a set A denoted by $A\backslash B$, is the set

$$A\backslash B := \{x : x \in A,\ x \notin B\}$$

Remark. $A\backslash B$ and B are disjoint i.e. $(A\backslash B) \cap B = \varnothing$.

Definition A.10. The **absolute complement** or simply **complement** of a set A, denoted by A^c, is the set of elements which do not belong to A i.e.

$$A^c := \{x : x \in \xi,\ x \notin A\}$$

The absolute complement of A is the relative complement of A in ξ i.e. $A^c = \xi\backslash A$.

Everything you need to know about the algebra of sets.

Theorem A.2. (Algebra of Sets Cheat Sheet). Let A, B and C be sets.

(1). **Idempotent Laws.**

(a). $A \cup A = A$

(b). $A \cap A = A$

(2). **Associative Laws.**

(a). $(A \cup B) \cup C = A \cup (B \cup C)$,

(b). $(A \cap B) \cap C = A \cap (B \cap C)$.

(3). **Commutative Laws.**

(a). $A \cup B = B \cup A$,

(b). $A \cap B = B \cap A$.

(4). **Distributive Laws.**

(a). $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,

(b). $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(5). **Identity Laws.**

(a). $A \cup \varnothing = A$,

(b). $A \cap \xi = A$,

(c). $A \cup \xi = \xi$,

(d). $A \cap \varnothing = \varnothing$.

(6). **Complement Laws.**

(a). $A \cup A^c = \xi$,

(b). $A \cap A^c = \varnothing$,

(c). $(A^c)^c = A$,

(d). $\xi^c = \varnothing,\ \varnothing^c = \xi$.

(7). **De Morgan's Laws.**

(a). $(A \cup B)^c = A^c \cap B^c$,

(b). $(A \cap B)^c = A^c \cup B^c$.

Theorem A.3. (TFAE $A \subset B$). Let A and B be any two sets. Then, the following are equivalent.

(1). $A \subset B$,

(2). $A \cap B = A$,

(3). $A \cup B = B$,

(4). $B^c \subset A^c$,

(5). $A \cap B^c = \varnothing$,

(6). $A \backslash B = \varnothing$,

(7). $B \cup A^c = \xi$.

Proposition A.1 (Some complement rules). Let A, B be any two sets.

(1). $A \backslash B = A \cap B^c$,

(2). $(A \backslash B)^c = A^c \cup B$.

(3). $A^c \backslash B^c = B \backslash A$.

Proposition A.2. Let A, B be any two sets in the universe ξ. Then

$$A^c = B \iff B^c = A$$

Proposition A.3. Let A, B, C be any three sets. If $A = B \backslash C$ and $C \subset B$, then

$$B = A \cup C$$

draw a Venn diagram, this should be then super clear.

Theorem A.4. Let A, B, S be any three sets. Then

(1). $A \subset B \implies S \cap A \subset S \cap B$,

(2). $A \subset B \implies S \cup A \subset S \cup B$.

Theorem A.5. Let A, B, C be any three sets. Then

$$(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$$

Theorem A.6. Suppose that A, V are subsets of X. Then

$$A \backslash (A \cap B) = A \cap B^c$$

Theorem A.7. Suppose that A, B, X are sets with $A \subset B \subset X$. Suppose that U is a subset of X such that $B \backslash A = B \cap U$. Then

$$A = B \cap U^c$$

Theorem A.8. Suppose that $U \subset X$ and $V \subset Y$ where X, Y are sets. Then

$$U \times V = (X \times V) \cap (U \times Y)$$

A.3 Product sets

Definition A.11. Let A and B be two sets. The **product set** of A and B, denoted $A \times B$, is the set consisting of all *ordered pairs* (a, b) where $a \in A$ and $b \in B$ i.e.

$$A \times B := \{(a, b) : a \in A,\ b \in B\}$$

Remark. The notion *ordered pair*, (a, b) is defined rigorously by $(a, b) := \{\{a\}, \{a, b\}\}$. Using this definition, the *order* property may be proven:

$$(a, b) = (c, d) \implies a = c \text{ and } b = d$$

Remark. The product of a set with itself, say $A \times A$, will be denoted by A^2.

Remark. The concept of product set can be extended to any finite number of sets in an obvious way. The product set of the sets $A_1, \ldots, A_k$, denoted by

$$A_1 \times A_2 \times \cdots \times A_k \qquad \text{or} \qquad \prod_{i=1}^{k} A_i$$

consists of all k-tuples $(a_1, a_2, \ldots, a_k)$ where $a_i \in A_i$ for all i.

Theorem A.9. Let A, B and C be any three sets. Then

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

A.4 Relations

Definition A.12. (Relation). A **binary relation** or just **relation** R from a set A to a set B assigns to each pair (a, b) in $A \times B$ exactly one of the following statements:

(i). *a is related to b*, written aRb;

(ii). *a is not related to b* written $a \not R\, b$

will be continued.

A.5 Generalized operations

Theorem A.10. For any indexed class of sets $\{A_i\}_{i \in I}$ and any set B, we have

(1).
$$B \cup \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} B \cup A_i$$

(2).
$$B \cap \left(\bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} B \cap A_i$$

Theorem A.11. Let $\{A_i\}_{i\in I}$ be any indexed class of subsets of ξ. Then:

(1).
$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$$

(2).
$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$$

Theorem A.12. Let A be any set and, for each $p \in A$, let U_p be a subset of A such that $p \in U_p \subset A$. Then
$$A = \bigcup_{p\in A} U_p$$

Remark. In the case of any empty collection $\varnothing$ of subsets of a universal set ξ, it is convenient to define

$$\bigcup\{A : A \in \varnothing\} = \varnothing \quad \text{and} \quad \bigcap\{A : A \in \varnothing\} = \xi$$

and hence
$$\bigcup\{A_i : i \in \varnothing\} = \varnothing \quad \text{and} \quad \bigcap\{A_i : i \in \varnothing\} = \xi$$

which is equivalent to
$$\bigcup_{i\in\varnothing} A_i = \varnothing \quad \text{and} \quad \bigcap_{i\in\varnothing} A_i = \xi$$

B Functions

B.1 Functions, images, preimages

Definition B.1 (Function). Given two sets X, Y, a **function** $f : X \to Y$ sends each $x \in X$ to *exactly one* $f(x) \in Y$.

we may refer to a function as a **map** in this notes.

The set X is called the **domain** and the set Y is called the **co-domain**.

Example. The identity function and the inclusion map are important examples.

Remark. To each function $f : A \to B$, there corresponds a relation in $A \times B$ given by

$$\{(a, f(a)) : a \in A\}$$

We call this set the **graph** of f.

Definition B.2 (Image). Let $f : X \to Y$ be a function. Then **direct image** or **image** of any subset A of X under f is the set

$$f(A) = \{f(x) \in Y : x \in A\} \subset Y$$

Definition B.3 (Preimage). Let $f : X \to Y$ be a function. The **preimage** or **inverse image** of any subset B of Y under f is the set

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \subset X$$

Note that preimages has **nothing** to do with the inverse function of f. The preimage is defined whether f has an inverse or not! The notation is slightly misleading. But this is justified because of this reason: Let $f : X \to Y$ and consider $B \subset Y$. When f does have an inverse, then the preimage $f^{-1}(B)$ is exactly the image of B under the inverse map of f.

Definition B.4 (Fiber). Let $f : X \to Y$ be a function; let $y \in Y$. The inverse image of a singleton under f is called the **fiber** of y and it is the set

$$f^{-1}(\{y\}) = \{x \in X : f(x) = y\}$$

Alternatively, we may write it as $f^{-1}(y)$.

Slogan (A). Let $f : X \to Y$ and $B \subset Y$. Then $f(x) \in B \iff x \in f^{-1}(B)$.

this needs no proof, it is by definition.

Slogan (B). Let $f : X \to Y$ and $y \in Y$. Then $f(x) = y \iff x \in f^{-1}(y)$.

if f is constant, then $f^{-1}(y) = X$.

Slogan (C). Let $f : X \to Y$ and $A \subset X$. Then

$$y \in f(A) \iff \exists x \in A \text{ such that } f(x) = y.$$

again this needs no proof, it is by definition.

Slogan (D). Let $f : X \to Y$ and $A \subset X$. Then $x \in A \implies f(x) \in f(A)$.

this is clear from the preceding slogan C. the reverse direction is false in general.

Remark. We define preimages and images for subsets of the domain and codomain. What about the preimage of the whole codomain (and likewise, the image of the whole domain). Well, let $f : X \to Y$ be a function. Then we **always** have $f^{-1}(Y) = X$ directly by definition. Also, we have $f(X) = Y$.

Slogan (E). Let $f : X \to Y$. Then $f(X) = Y$ and $f^{-1}(Y) = X$.

This is always true no matter what sets X and Y are. or whether f is *continuous* (will be defined later) or not.

Proposition B.1. (Associativity). Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Definition B.5. (Injective). A function is **injective** if it hits everything at most once i.e. for all $x, y \in X$, $f(x) = f(y)$ implies that $x = y$.

Definition B.6. (Surjective). A function is **surjective** if it hits everything at least once. i.e. for all $y \in Y$, there is $x \in X$ such that $f(x) = y$.

the keyword is everything. In the definition of a function, we do not require every element of Y to be hit.

Theorem B.1. Let $f : X \to Y$ and $g : Y \to Z$. Then

(1). If f and g are surjective, then $g \circ f : X \to Z$ is surjective;

(2). If f and g are injective, then $g \circ f : X \to Z$ is injective.

Definition B.7. (Bijective). A function is **bijective** if it hits everything exactly one i.e. if it is both injective and surjective.

Theorem B.2. Let $f : X \to Y$ and $g : Y \to Z$ be bijective. Then, $(g \circ f)^{-1} : Z \to X$ exists and equals $f^{-1} \circ g^{-1} : Z \to X$.

Theorem B.3. Let $f : X \to Y$ be a function. Then, for any subsets A and B of X,

(1). $f(A \cup B) = f(A) \cup f(B)$,

(2). $f(A \cap B) \subset f(A) \cap f(B)$,

(3). $f(A \backslash B) \supset f(A) \backslash f(B)$,

(4). $A \subset B$ implies $f(A) \subset f(B)$.

More generally, for any indexed class $\{A_i\}$ of subsets of X,

(i). $f(\bigcup_i A_i) = \bigcup_i f(A_i)$,

(ii). $f(\bigcap_i A_i) \subset \bigcap_i f(A_i)$

if f injective $f(A \cap B) = f(A) \cap f(B)$ and hence $f(A \backslash B) = f(A) \backslash f(B)$.

The inverse set function is much more *well-behaved* in the sense that equality holds for most of the cases.

Theorem B.4. Let $f : X \to Y$. Then, for any subsets A and B of Y,

(1). $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, (2). $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$,

(3). $f^{-1}(A \backslash B) = f^{-1}(A) \backslash f^{-1}(B)$,

(4). $A \subset B$ implies $f^{-1}(A) \subset f^{-1}(B)$.

More generally, for any indexed class $\{A_i\}$ of subsets of X,

(i). $f^{-1}(\bigcup_i A_i) = \bigcup_i f^{-1}(A_i)$,

(ii). $f^{-1}(\bigcap_i A_i) = \bigcap_i f^{-1}(A_i)$

We will see that continuous functions will be defined using inverse images rather than images because of this well-behavedness.

Corollary B.1. Let $f : X \to Y$ and let $A \subset Y$. Then

$$f^{-1}(A^c) = (f^{-1}(A))^c$$

This follows from the fact that $f^{-1}(Y) = X$ so this corollary is really a special case of (3) above.

Theorem B.5. Let $f : X \to Y$ and let $A \subset X$ and $B \subset Y$. Then:

(1). $A \subset f^{-1}(f(A))$.

(2). $B \supset f(f^{-1}(B))$.

Proof. (1) Let $A \subset X$. Suppose $x \in A$. By definition, $f(x) \in f(A)$. Therefore by definition also we have $x \in f^{-1}(f(A))$.

(2) Let $B \subset Y$. Suppose $y \in f(f^{-1}(B))$. By definition, there is $x \in f^{-1}(B)$ such that $y = f(x)$. Clearly $y = f(x) \in B$ (because $x \in f^{-1}(B)$), as desired. ■

The inclusion in (1) and 2 cannot in general be replaced by equality i.e. the reverse inclusion fails. Below is an example when it fails.

Example. Let $X = \{1, 2\}$, $Y = \{a\}$, and consider the subset $A = \{1\}$ of X; let $f : X \to Y$ be a function defined by $f(1) = f(2) = a$. Then $f(A) = Y$, so

$$f^{-1}(f(A)) = f^{-1}(Y) = X \not\subset A$$

Here we give conditions for which equality can be achieved.

Theorem B.6. A function $f : X \to Y$ is injective *if and only if* $A = f^{-1}(f(A))$ for every subset $A \subset X$.

Proof. It remains to prove the reverse inclusion $f^{-1}(f(A)) \subset A$. Suppose $f : X \to Y$ is injective and let $x \in f^{-1}(f(A))$. Then by definition of preimages, $f(x) \in f(A)$. This is true if and only if there is some $y \in A$ such that $f(x) = f(y)$. But since f is injective, it follows that $x = y$. So, $x \in A$ as desired.

The converse is left as an exercise. ■

Theorem B.7. A function $f : X \to Y$ is surjective *if and only if* $B = f(f^{-1}(B))$ for every subset $B \subset Y$.

Proof. It remains to prove the reverse inclusion $B \subset f(f^{-1}(B))$. Suppose $f : X \to Y$ is surjective. Let $y \in B$. Since f is surjective, there is $x \in X$ such that $f(x) = y$ i.e. $x \in f^{-1}(y) \subset f^{-1}(B)$. Therefore $x \in f^{-1}(B)$. Since there is $x \in f^{-1}(B)$ such that $f(x) = y$, by definition, we therefore have $y \in f(f^{-1}(B))$ as required. ∎

We have the crazy theorem that:

Theorem B.8. Let $f : X \to Y$ be a function; and let $A \subset X$. Then

$$f(f^{-1}(f(A))) = f(A)$$

B.2 Inverse functions

Definition B.8. A function $f : X \to Y$ is said to be **invertible** if there exists a function $g : Y \to X$ such that the composition $g \circ f$ is the identity map of X and the composition $f \circ g$ is the identity map of Y. The function g is called the **inverse of f** and is denoted as f^{-1}.

Proposition B.2. A function $f : X \to Y$ is invertible *if and only if* it is bijective.

Proposition B.3. When f is invertible, the inverse of f is unique.

Proposition B.4. Suppose that $f : X \to Y$ is a bijective function and that $V \subset X$. Then the inverse image of V under the inverse map $f^{-1} : Y \to X$ equals the image set $f(V)$.

Slogan. $(f^{-1})^{-1}(V) = f(V)$

The conclusion is baffling notationally. The inner superscript -1 indicates the function f^{-1} is inverse to f. The outer superscript -1 indicates the inverse image of the set V under that inverse function.

C Topology of the line and plane

Many concepts in point-set topology are abstractions of properties of $\mathbb{R}$. So let's discuss about properties of $\mathbb{R}$.

C.1 Open sets in $\mathbb{R}$

Definition C.1. Let $A \subset \mathbb{R}$. A point $p \in A$ is an **interior point** of A if and only if p belongs to some open interval I_p which is contained in A i.e.

$$p \in I_p \subset A$$

Definition C.2. The set A is **open** if and only if each of its points is an interior point.

Example. Some examples of *open sets*.

1. An open interval $A = (a, b)$ is an open set as we can choose $I_p = A$ for each $p \in A$;
2. $\mathbb{R}$ itself is open since any open interval I_p must be a subset of $\mathbb{R}$ i.e. $p \in I_p \subset \mathbb{R}$.
3. The empty set $\varnothing$ is open since there is no point in $\varnothing$ which is not an interior point.
4. The infinite open intervals (a, ∞), $(-\infty, a)$ or $(-\infty, \infty) = \mathbb{R}$ are open sets.

Example. Some non-examples.

1. The closed interval $B = [a, b]$ is not an open set as a, b are not interior points of B.
2. The infinite closed intervals $[a, \infty)$ and $(-\infty, a]$ are not open sets for the same reason.

Two important theorems about open sets.

Theorem C.1. (Unions and Intersections).

(1). The union of any number of open sets in $\mathbb{R}$ is open.

(2). The intersection of any *finite* number of open sets in $\mathbb{R}$ is open.

Remark. We need the finiteness condition for intersections. Consider the collection of open intervals $\{A_n\}$ given by

$$A_n = \left(\frac{-1}{n}, \frac{1}{n}\right), \quad n \in \mathbb{N}$$

Observe that the intersection

$$\bigcap_{n=1}^{\infty} A_n = \{0\}$$

and $\{0\}$ is not open.

C.2 Limit points

Definition C.3. Let $A \subset \mathbb{R}$. A point $p \in \mathbb{R}$ is a **limit point of A** if every open set U containing p contains a point of A different from p i.e.

$$U \text{ open}, \ p \in U \implies A \cap (U \backslash \{p\}) \neq \varnothing$$

The set of limit points of A will be denoted A'.

learn to distinguish limit points of sets and limit points of sequences.

Example. Some examples.

1. Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. The point 0 is a limit point of A.

   In fact, it is ***the*** limit point of A. Why? The open sets in $\mathbb{R}$ are all the open intervals. So, I can for example take the open interval $(\frac{1}{2}, \frac{3}{2})$ about 1 and this contains only 1 and none other points of A. Similarly, I can take the open interval $(\frac{5}{12}, \frac{3}{4})$ about 1/2 and this contains only 1/2 and none other points of A. The only point p which I can always guarantee to find other points of A in an open interval (no matter how big or small) about p is $p = 0$.

2. Every real number $p \in \mathbb{R}$ is a limit point of $\mathbb{Q}$ since every open set contains rational numbers. In other words, $\mathbb{Q}' = \mathbb{R}$.

3. $\mathbb{Z}$ does not have any limit points. In other words, $\mathbb{Z}' = \varnothing$.

C.3 Bolzano-Weierstrass theorem

Theorem C.2. (Bolzano-Weierstrass). Let A be a bounded, infinite subset of $\mathbb{R}$. Then, A has at least one limit point.

C.4 Closed sets

Definition C.4. Let $A \subset \mathbb{R}$. We say A is a **closed set** if A^c is open.

Theorem C.3. Let $A \subset \mathbb{R}$. A is closed *if and only if* A contains all of its limit points.

Example. Examples of closed sets:

1. $[a, b]$ is a closed set since its complement $(\infty, a) \cup (b, \infty)$ is open.

2. $\mathbb{R}$ and $\varnothing$ are both closed since their complements are open.

Example. Non-examples of closed sets:

1. $A = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not closed as 0 is a limit point of A but $0 \notin A$.

Some sets may neither be open nor closed.

Example. Examples of sets that are both not open and not closed.

The half-open interval $A = (a, b]$ is not open since $b \in A$ is not an interior point of A. It is not closed either since $a \notin A$ but it is a limit point of A.

C.5 Heine-Borel theorem

Definition C.5. (Cover). A collection of sets $\mathcal{C} = \{C_i\}$ is said to be a **cover** of a set X if X is contained in the union of the members of $\mathcal{C}$ i.e. $X \subset \cup_i C_i$.

Definition C.6. (Subcover). Let $\mathcal{C}$ be a cover for X. A subset of $\mathcal{C}$ that still covers X is said to be a **subcover**.

Theorem C.4. (Heine-Borel). Every open cover of a closed and bounded interval admits a finite subcover.

C.6 Sequences

Definition C.7. A **sequence**, denoted $\langle s_n \rangle$, is a function whose domain is $\mathbb{N}$ i.e. a sequence assigns a point s_n to each positive integer $n \in \mathbb{N}$. The image s_n of $n \in \mathbb{N}$ is called the **n-th term** of the sequence.

Remark. Alternative notations to write a sequence $\langle s_n \rangle$ are (s_n), $(s_1, s_2, s_3, \ldots)$ and $(s_n : n \in \mathbb{N})$. In $\mathbb{R}$-analysis, it is written using the notation $\{s_n\}_{n=1}^{\infty}$ but we reserve this for the image under $\langle s_n \rangle$.

Definition C.8. (Bounded). A sequence $\langle s_n \rangle$ is said to be **bounded** if its image $\{s_n : n \in \mathbb{N}\}$ is a bounded set.

if we are talking about a sequence of real numbers, then we can write this as $\langle s_n \rangle$ is bounded if there exists $M > 0$ such that $|s_n| \leqslant M$ for all $n \in \mathbb{N}$.

C.7 Convergent sequences

Definition C.9. The sequence $\langle s_n \rangle$ of real numbers converges to $\ell \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $|s_n - \ell| < \varepsilon$.

We write this as $\lim_{n\to\infty} s_n = \ell$, $\lim s_n = \ell$ or just $s_n \to \ell$.

We can rewrite this definition.

Definition C.10. The sequence $\langle s_n \rangle$ converges to ℓ if every open set containing ℓ contains *almost all*[17] of the terms of the sequence.

[17] *almost all means all but a finite number.*

C.8 Subsequences

Definition C.11. Let $\langle s_n \rangle$ be a sequence. If (i_n) is a sequence of positive integers such that $i_1 < i_2 < \cdots$, then

$$\langle s_{n_k} \rangle = (s_{i_1}, s_{i_2}, s_{i_3}, \ldots)$$

is called a **subsequence** of $\langle s_n \rangle$.

Below is a reformulation of Bolzano-Weierstrass.

Theorem C.5. Every bounded sequence of real numbers contains a convergent subsequence.

C.9 Cauchy sequences

Definition C.12. A sequence $\langle s_n \rangle$ of real numbers is a **Cauchy sequence** if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $n, m > N$ implies $|s_n - s_m| < \varepsilon$.

C.10 Completeness

Definition C.13. Let $A \subset \mathbb{R}$. Then, A is said to be **complete** if every Cauchy sequence in A converges to a point in A.

Theorem C.6. (Cauchy). Every Cauchy sequence in $\mathbb{R}$ converges in $\mathbb{R}$.

C.11 Continuous functions

Definition C.14. A function $f : A \to \mathbb{R}$ is **continuous** at a point $x_0 \in A$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

The function f is a continuous function in A if f is continuous at every point in A.

Remark. Note that
$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$$
is equivalent to
$$x \in (x_0 - \delta, x_0 + \delta) \implies f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$$
which is equivalent to
$$f((x_0 - \delta, x_0 + \delta)) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$$

This remark leads us to the following reformulation of the definition.

Definition C.15. A function $f : A \to \mathbb{R}$ is continuous at a point $p \in A$ if for any open set $V_{f(p)}$ containing $f(p)$, there is an open set U_p containing p such that $f(U_p) \subset V_{f(p)}$.

Theorem C.7. A function is continuous *if and only if* the inverse image of every open set is open.

Theorem C.8. (IVT). Let $f : [a, b] \to \mathbb{R}$ be continuous. Then the function assumes every value between $f(a)$ and $f(b)$.

C.12 Topology of the plane

We denote the usual distance between two points $p, q \in \mathbb{R}^2$ as $d(p, q)$ i.e. if $p = (p_1, p_2)$ and $q = (q_1, q_2)$, then $d(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$.

Definition C.16. (Open Disc). An open disc centred at $a \in \mathbb{R}^2$ with radius $\delta > 0$ is the set
$$D(a, \delta) := \{x \in \mathbb{R}^2 : d(a, x) < \delta\}$$

Definition C.17. (Closed Disc). A closed disc centred at $a \in \mathbb{R}^2$ with radius $\delta > 0$ is the set
$$\overline{D(a,\delta)} := \{x \in \mathbb{R}^2 : d(a,x) \leqslant \delta\}$$

The open disc plays a role in the topology of the plane $\mathbb{R}^2$ that is analogous to the role of the open interval in the topology of the line $\mathbb{R}$. Due to this, every definition in the topology of $\mathbb{R}$ can be extended naturally to $\mathbb{R}^2$.