Lecture 5: Error Estimation and Adaptive Mesh Refinment

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Aims for this module

- Measure the error of a given approximation
- Measure the convergence rate of a method
- Brief introduction to error estimation

How to measure the Error?

- Method of Manufactured Solutions
  - Take the “u” you want as a solution, plug in the equations, get the boundary conditions and the right hand side that force the given “u”
  - Integrate (with a fine quadrature formula) the difference between the exact solution and the computed one (VectorTools::integrate_difference)
  - Possibly integrate the difference between the gradients of the exact and computed solutions
  - Use one of the utility classes (like ParsedConvergenceTable) that computes convergence tables for you

Estimate the rate of convergence

- Once you have computed the error, how do we estimate if we get the correct *convergence ratio?*

$$\|u - u_h\|_1 \leq Ch^k |u|_{k+1}$$

$$\|u - u_h\|_0 \leq Ch^{k+1} |u|_{k+1}$$

Estimate the global rate of convergence

- Compute two successive solutions, on half the size of the mesh (i.e., after one global refinement):

$$\|u - u_h\| \sim \tilde{C}(h)^p$$

$$\|u - u_{2h}\| \sim \tilde{C}(2h)^p$$

$$\frac{\|u - u_{2h}\|}{\|u - u_h\|} \sim 2^p$$

$$p \sim \log_2\left(\frac{\|u - u_{2h}\|}{\|u - u_h\|}\right)$$

a-priori error estimation

$$-\Delta u = f$$
$$u|_{\partial\Omega} = 0$$

Find $u \in \mathcal{V}$ such that:

$$(\nabla u, \nabla v) = (f, v) \quad \text{(*)}$$
$$\forall v \in \mathcal{V}$$

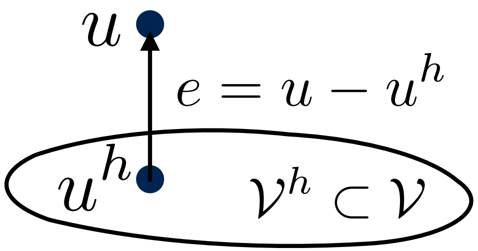
Find $u^h \in \mathcal{V}^h \subset \mathcal{V}$ such that:

$$(\nabla u^h, \nabla v^h) = (f, v^h) \quad \text{(**)}$$
$$\forall v^h \in \mathcal{V}^h \subset \mathcal{V}$$

By taking $v \in \mathcal{V}^h$ in (*) and subtracting (**) we get "Galerkin orthogonality":

$$\boxed{\left(\nabla \left[u - u^h\right], \nabla v^h\right) = 0 \qquad \forall v^h \in \mathcal{V}^h \subset \mathcal{V}}$$

"orthogonality" because the bilinear form defines a scalar/inner product



a-priori error estimation

…it then follows that

$$||\nabla\left[u-u^h\right]||^2=\left(\nabla\left[u-u^h\right],\nabla\left[u-I^hu\right]\right)$$

the last term on the right hand size is zero (thanks to Galerkin orthogonality)

Recall the **Cauchy-Schwarz** inequality

$$(f,g)\leq||f||\,||g||\quad\forall f,g\in L_2$$

(similar to trigonometry and the scalar product between two vectors)

and therefore

$$||\nabla\left[u-u^h\right]||^2\leq||\nabla\left[u-u^h\right]||\,||\nabla\left[u-I^hu\right]||$$

$$\boxed{||\nabla\left[u-u^h\right]||\leq||\nabla\left[u-I^hu\right]||}$$

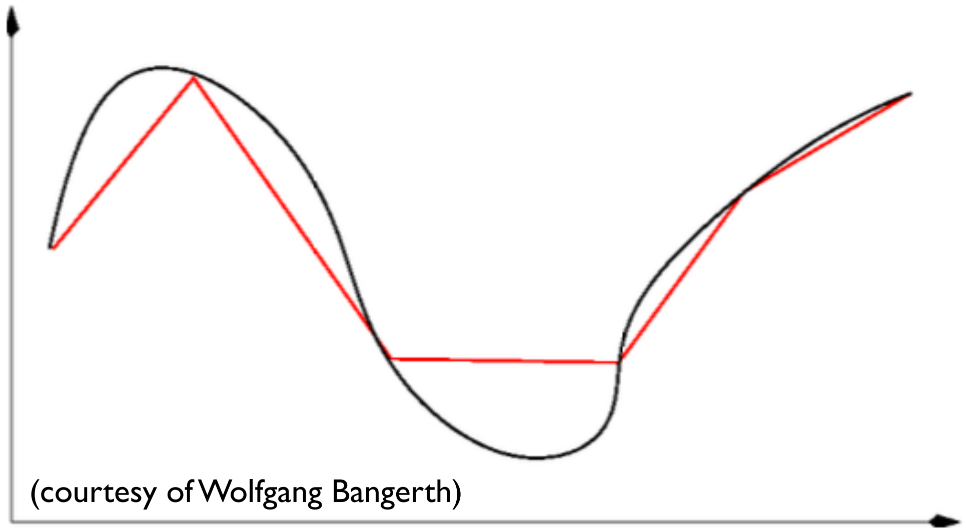
Interpretation: the **FE error** is **not larger** than the **interpolation error**. That is, it's closer (in energy norm) to the actual solution as compared to its interpolate onto FE space. Often called the "best approximation property".

…but we can't use it yet as we don't have exact solution! We need to have more knowledge about the interpolant.

a-priori error estimation

Properties of the interpolant:

$$||\nabla \left[u - I^h u\right]||^2 = \int_\Omega |\nabla \left[u - I^h u\right]|^2 = \sum_K \int_K |\nabla \left[u - I^h u\right]|^2$$



Black: $u \in \mathcal{V}$

Red: $I^h u \in \mathcal{V}^h$

Intuitive observation: error is large where the second derivative of the exact solution is large.

The "Bramble-Hilbert Lemma" provides the following bound for **piecewise linear** elements:

$$||\nabla \left[u - I^h u\right]||_K^2 \equiv \int_K |\nabla \left[u - I^h u\right]|^2 \leq C^2 h_K^2 ||\nabla^2 u||_K^2$$

Consequently

$$||\nabla \left[u - I^h u\right]||_\Omega^2 \leq C^2 \sum_K h_K^2 ||\nabla^2 u||_K^2$$

(this is an a-priori error estimator)

derivation: FE error bound by **interpolation error** which is bound by terms with **mesh size** and **second derivatives**

from which it follows

$$\boxed{||e||_{H^1(\Omega)}^2 \leq C^2 \sum_K h_K^2 |u|_{H^2(K)}^2}$$

p.s. extension to degree p:

$$||\nabla \left[u - I^h u\right]||_\Omega^2 \leq C \sum_K h_K^{2p} ||\nabla^{p+1} u||_K^2 \equiv C \sum_K h_K^{2p} |u|_{H^{p+1}(K)}^2$$

■

a-priori error estimation

$$||u^h - u||^2_{H^1(\Omega)} := ||e||^2_{H^1(\Omega)} \leq C^2 \sum_K h_K^2 |u|^2_{H^2(K)} =: \sum_K e_K^2$$

- In practice not too useful as we rarely have any knowledge about the exact solution u.
- It proves that the FE solution will converge as we can always make the mesh size smaller and smaller

a-posteriori error estimation

$$||e||^2_{H^1(\Omega)} \leq C \sum_K e_K^2$$

$$e_K = h_K ||\nabla^2 u||_K$$

cell-wise error indicators

(wrong) idea:

$$e_K \approx h_K ||\nabla^2 u^h||_K$$

will not work as linear elements have zero second derivates within the element and first derivatives have jumps on the interfaces

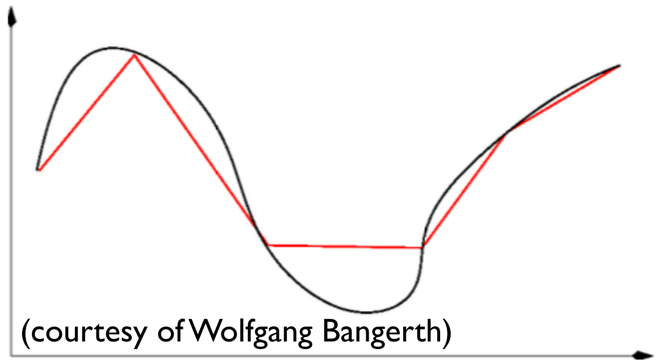
a better idea to approximate second derivatives at interface i:

$$\nabla^2 u \approx \frac{\nabla u^h(x^+) - \nabla u^h(x^-)}{h} =: \frac{[\![\nabla u^h]\!]_i}{h}$$

use jump in gradient as an indicator of the second derivative at vertices

can generalize to:

$$||\nabla^2 u||^2_K \approx \sum_{i \in \partial K} \frac{[\![\nabla u^h]\!]_i^2}{h}$$



(courtesy of Wolfgang Bangerth)

a-posteriori error estimation

As a result, the simplest and most widely used Kelly error **indicator** in 2D/3D follows:

$$e_K^2 = h_K^2 ||\nabla^2 u||_K^2 \approx h_K \int_{\partial K} |[\![\boldsymbol{\nabla} u \cdot \boldsymbol{n}]\!]|^2 \mathrm{ds} =: \eta_{\mathrm{K}}^2$$

For the Laplace equation, **Kelly, de Gago, Zienkiewicz, Babushka (1983)** proved that

$$||\nabla \left[u - u^h\right]||^2 \leq C \sum_K \eta_K^2$$ a-posteriori error estimator (*)

Note 1:

"**estimator**" is always a proven upper bound of error (*), whereas "**indicator**" is our best guess of error per cell which may not be an upper bound in the sense (*), but may still work well for considered equations and/or FE space.