MATH 2135 Linear Algebra

2.B Bases

Alyssa Motas

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1 Bases

1.1 Definition

A *basis* of V is a list of vectors in V that is linearly independent and spans V.

1.2 Examples

(a)

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ is a basis of } \mathbb{R}^3.$$

(b)

$$\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\5\end{bmatrix} \text{ is a basis of } \mathbb{R}^3$$

(c)

$$1, x, x^2, x^3 \text{ is a basis of } \mathcal{P}_3(\mathbf{F})$$

1.3 Criterion for basis

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

Proof. Suppose $v_1, \ldots, v_n$ is a basis of V. Let $v \in V$. Since $v_1, \ldots, v_n$ spans V, there exists $a_1, \ldots, a_n \in \mathbf{F}$ such that

$$v = a_1 v_1 + \ldots a_n v_n.$$

To show that it is unique, suppose that $c_1, \ldots, c_n$ are scalars where $v = c_1 v_1 + \cdots + c_n v_n$. Subtracting this equation from the previous, we get

$$0 = (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n.$$

This completes the proof for uniqueness.

In the other direction, suppose every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n.$$

This implies that $v_1, \ldots, v_n$ spans V. To show that $v_1, \ldots, v_n$ are linearly independent, suppose that $a_1, \ldots, a_n \in \mathbf{F}$. Then

$$0 = a_1 v_1 + \cdots + a_n v_n.$$

Thus $v_1, \ldots, v_n$ is linearly independent and hence is a basis of V. $\square$

2 Coordinates

If $B = v_1, \ldots, v_n$ is a basis of V, and $v = a_1 v_1 + \cdots + a_n v_n$, then we say $a_1, \ldots, a_n$ are the *coordinates* of v with respect to the basis B.

2.1 Examples of coordinates

Suppose that $B = 1, x, x^2, x^3$ is the basis of $\mathcal{P}_3(\mathbb{R})$. Find the coordinates of $p = (1 + 2x)(3x + x^2)$ with respect to the basis B.

Solution:

$$\begin{aligned} p &= (1 + 2x)(3x + x^2) \\ &= 3x + x^2 + 6x^2 + 2x^3 \\ &= 3x + 7x^2 + 2x^3 \\ &= 0 \cdot 1 + 3 \cdot x + 7 \cdot x^2 + 2 \cdot x^3 \end{aligned}$$

The coordinates are: 0, 3, 7, and 2.

Another basis for $\mathcal{P}_3(\mathbb{R})$ is $B' = 1, (x-1), (x-1)^2, (x-1)^3$. Find the coordinates of $p = 3x + 6x^2 + 2x^3$ in the basis B'.

Solution: Suppose that $y = x - 1$ and $x = y + 1$. Then

$$\begin{aligned} p &= 3(y+1) + 7(y+1)^2 + 2(y+1)^3 \\ &= 3y + 3 + 7y^2 + 14y + 7 + 2y^3 + 6y^2 + 6y + 2 \\ &= 12 + 23y + 13y^2 + 2y^3 \\ &= 12 + 23(x-1) + 13(x-1)^2 + 2(x-1)^3 \end{aligned}$$

The coordinates of p with respect to B' are: 12, 23, 13, and 2.

3 Theorems about Bases

3.1 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space (by removing 0 or more vectors from the list).

Proof. Suppose $v_1, \ldots, v_n$ spans V. We want to remove some of the vectors from $v_1, \ldots, v_n$ so that the remaining vectors form a basis of V. We do this through induction.
Start with B equal to the list $v_1, \ldots, v_n$.

Step 1 If $v_1 = 0$, delete v_1 from B. If $v_1 \neq 0$, leave B unchanged.

Step j If v_j is in $span(v_1, \ldots, v_{j-1})$, delete v_j from B. If v_j is not in $span(v_1, \ldots, v_{j-1})$, leave B unchanged.

Stop the process after step n, getting a list B. This list spans V because our original list spanned V and we have discarded vectors that were already in the span of the previous vectors. This process ensures that no vector in B is in the span of the previous ones. Thus, B is linearly independent, by the Linear Dependence Lemma. Hence B is a basis of V. $\square$

3.2 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Proof. By definition, a finite-dimensional vector space has a spanning list. The previous result tells us that each spanning list can be reduced to a basis. $\square$

3.3 Linearly independent list extends to a basis

Every linearly indepdendent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof. Suppose $u_1, \ldots, u_m$ is linearly independent in a finite-dimensional vector space V. Let $w_1, \ldots, w_n$ be a basis of V. Thus the list

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

spans V. Applying the produce of the proof of 3.1 to reduce this list to a basis of V produces a basis consisting of the vectors $u_1, \ldots, u_m$ (none of the u's get deleted because $u_1, \ldots, u_m$ is linearly independent) and some of the w's. $\square$

3.4 Every subspace of V is part of a direct sum equal to V

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof. Since V is finite-dimensional, so is U. Thus, there is a basis $u_1, \ldots, u_m$ of U and is linearly independent in V. Hence, this list can be extended to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$ of V. Let $W = \mathit{span}(w_1, \ldots, w_n)$.

To prove that $V = U \oplus W$, we need to show that

$$V = U + W \text{ and } U \cap W = \{0\}.$$

Proving the first equation, suppose $v \in V$ Then, since the list $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V, there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that

$$v = \underbrace{a_1u_1 + \cdots + a_mu_m}_{u} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{w} \Rightarrow v = u + w, u \in U, w \in W.$$

Thus we have $v \in U + W$.
Proving the second equation, suppose $v \in U \cap W$. There exists scalars $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that

$$v = a_1u_1 + \cdots + a_mu_m = b_1w_1 + \cdots + b_nw_n.$$

Thus

$$a_1u_1 = \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0.$$

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent, this implies $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Thus $v = 0$ and this completes the proof. $\square$