MATH 2135 Linear Algebra

Chapter 1 Vector Spaces

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Contents

1 1.A Complex Numbers

1.1 Complex numbers

1.1.1 Definition

A complex number is a pair (a, b) where $a, b \in \mathbb{R}$. We write $\mathbb{C}$ for the set of complex numbers. The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}.$$

With the following operations:

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b) \cdot (c, d) &= (ac - bd, ad + bc)\end{aligned}$$

We also define:

$$0 = (0, 0) \qquad 1 = (1, 0).$$

Claim. The set $\mathbb{C}$, together with $0, 1 \in \mathbb{C}$ and the operations $+$ and $\cdot$ defined above, is a field.

1.1.2 Notation

- We write $i = (0, 1) \in \mathbb{C}$.
- If a is a real number, we will also write $a = (a, 0) \in \mathbb{C}$.

Note, if a, b are real numbers, then

$$\begin{aligned}a + bi &= (a, 0) + (b, 0) \cdot (0, 1)\\ &= (a, 0) + (0, b)\\ &= (a, b).\end{aligned}$$

The notation $a + bi$ is what everybody uses for complex numbers. With this notation, the rules of addition and multiplication become easier to understand and remember.

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i\\ (a + bi)(c + di) &= ac + adi + bci + bdi^2\\ &= ac + adi + bci - bd\\ &= (ac - bd) + (ad + bc)i.\end{aligned}$$

Note: $i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0) = -1$.

1.1.3 Terminology

Given a complex number

$$z = a + bi = (a, b)$$

the real number a is called the *real part* of z, and the real number b is called the *imaginary part* of z.

The complex number $\bar{z} = a - bi$ is called the *complex conjugate* of z.

$$\begin{aligned} z\bar{z} &= (a + bi)(a - bi) \\ &= a^2 - abi + abi - b^2 i^2 \\ &= a^2 + b^2 \qquad \text{which is } \textit{real} \end{aligned}$$

1.1.4 Division Operation, Part I

It is easy to divide a complex number by a real number.

$$\frac{a + bi}{c} = \frac{a}{c} + \frac{b}{c} i.$$

1.1.5 Division Operation, Part II

How do we divide a complex number by a complex number? For instance,

$$\frac{a + bi}{z} \qquad \text{where } z = c + di.$$

We can simply take the conjugate of z and then we have

$$\frac{a + bi}{z} = \frac{(a + bi)\bar{z}}{z\bar{z}} = \frac{(a + bi)(c - di)}{c^2 + d^2} \qquad \text{where } c^2 + d^2 \text{ is a real.}$$

1.1.6 Multiplicative Inverse

The multiplicative of a complex number $z = a + bi$ is

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i.$$

1.1.7 Argument of a complex number

[1] The argument of z is the angle between the positive real axis and the line joining the point to the origin. For each point on the plane, arg is the function which returns the angle ϕ. The numeric value is given by the angle in radians, and is positive if measured counterclockwise.
Algebraically, as any real quantity ϕ, such that

$$z = r(cos\phi + isin\phi) = re^{i\phi}$$

for some positive real r. The quantity r is the modulus (or absolute value) of z, denoted $|z|$.

$$r = \sqrt{x^2 + y^2}$$

Some identities are

$$\arg(zw) = \arg(z) + \arg(w) \mod (-\pi, \pi]$$

$$\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w) \mod (-\pi, \pi]$$

If $z \neq 0$ and n is any integer, then

$$\arg(z^n) \equiv n \arg(z) \mod (-\pi, \pi]$$

Example.

$$\arg\left(\frac{-1-i}{i}\right) = \arg(-1-i) - \arg(i) = -\frac{3\pi}{4} - \frac{\pi}{2} = -\frac{5\pi}{4}.$$

Complex logarithm. From $z = |z|e^{i\arg(z)}$ or $z = |z|e^{i\theta}$, it easily follows that

$$\arg(z) = -i \ln \frac{z}{|z|}.$$

[1] Taken from the Wikipedia.

1.2 Fundamental Theorem of Algebra

Over the complex numbers, every non-constant polynomial has a root.

- There are two solutions (namely $x = -2, 2$) over $\mathbb{R}$:

$$x^2 - 4 = 0 \Leftrightarrow (x-2)(x+2) = 0$$

- Has no roots over $\mathbb{R}$:

$$x^2 + 4 = 0.$$

- Has two solutions over $\mathbb{C}$:

$$z^2 + 4 = 0 \Leftrightarrow z^2 = -4 \Leftrightarrow z = \pm\sqrt{-4} = \pm 2i$$

$$z^2 + 4 = (z - 2i)(z + 2i)$$

- Only one root over $\mathbb{R}$:

$$x^3 = 1$$

- Three distinct solutions over $\mathbb{C}$:

$$z^3 = 1$$

1.3 Lists

1.3.1 Definition

Suppose n is a nonnegative integer. A *list* of *length* n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \ldots, x_n).$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

1.3.2 Examples

- The set $\mathbb{R}^2$ is the set of all ordered pairs of real numbers:

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}.$$

- The set $\mathbb{R}^3$ is the set of all ordered triples of real numbers:

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}.$$

- A list of length 0 looks like this: ().
- Lists differ from sets in two ways: in lists, order matters and repetitions have meaning; in sets, order and repetitions are irrelevant.

1.4 $\mathbf{F}^n$

1.4.1 Definition

$\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$:

$$\mathbf{F}^n = \{(x_1, \ldots, x_n \mid x_j \in \mathbf{F} \text{ for } j = 1, \ldots, n)\}.$$

For $(x_1, \ldots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the jth *coordinate* of $(x_1, \ldots, x_n)$.

1.4.2 Addition

Addition is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

It is also commutative.

1.4.3 0

Let 0 denote the list of length n whose coordinates are all 0:

$$0 = (0, \ldots, 0).$$

1.4.4 Additive Inverse

For $x \in \mathbf{F}^n$, the *additive inverse* of x, denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that

$$x + (-x) = 0.$$

In other words, if $x = (x_1, \ldots, x_n)$ then $-x = (-x_1, \ldots, -x_n)$.

1.4.5 Scalar Multiplication

The *product* of a number λ and a vector in $\mathbf{F}^n$ is computed by multiplying each coordinate of the vector by λ:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n);$$

here $\lambda \in \mathbf{F}$ and $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

1.5 Digression on Fields

A *field* is a set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication. Thus, $\mathbb{R}$ and $\mathbb{C}$ are fields, as is the set of rational numbers along with the usual operations. Another example of a field is the set $\{0, 1\}$ with the usual operations of addition and multiplication except that $1 + 1$ is defined to equal 0.

2 1.B Definition of Vector Space

2.1 Definition

Let $\mathbf{F}$ be a field. A *vector space* over $\mathbf{F}$ is a set V, together with a distinguished element $0 \in V$ and with operations

$$\begin{aligned} \text{addition} \quad & + : V \times V \to V \\ \text{scalar multiplication} \quad & \cdot : \mathbf{F} \times V \to V. \end{aligned}$$

Satisfies the following 8 axioms:

(A1) Commutativity of addition.

$$\forall v, w \in V, v + w = w + v$$

(A2) Associativity of addition.

$$\forall v, w, u \in V, (v + w) + u = v + (w + u)$$

(A3) Additive identity.

$$\forall v \in V, 0 + v = v$$

(A4) Additive inverse.

$$\forall v \in V, \exists w \in V, v + w = 0$$

(M1) Multiplicative identity.

$$\forall v \in V, 1v = v$$

(M2) Left distributivity.

$$\forall a \in F, \forall v, w \in V, a(v + w) = av + aw$$

(M3) Right distributivity.

$$\forall a, b \in F, \forall v \in V, (a + b)v = av + bv$$

(M4) Associativity of multiplication.

$$\forall a, b \in F, \forall v \in V, (ab)v = a(bv)$$

2.1.1 Terminology

- The elements of $\mathbf{F}$ are called *scalars.*
- The elements of V are called *vectors or points.*
- A vector space over $\mathbb{R}$ is called a *real vector space.*
- A vector space over $\mathbb{C}$ is called a *complex vector space.*

2.2 Examples of Vector Spaces

(1) $\mathbf{F}^n$ is the set of column vectors (sometimes row vectors) with elements from $\mathbf{F}$. For instance, $\mathbb{R}^n$ and $\mathbb{C}^n$ are such vector spaces.

$$\begin{aligned}\mathbf{F}^n &= \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \mid a_1, \dots, a_n \in \mathbf{F} \right\} \\ &= \{(a_1, a_2, \dots, a_n) \mid a_1, \dots, a_n \in \mathbf{F}\}\end{aligned}$$

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \qquad k \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix}$$

The properties of $\mathbf{F}^n$ makes it a vector space.

(2) Let $\mathbf{F}^\infty = \{(x_1, x_2, x_3, x_4, \dots) \mid x_1, x_2, \dots \in \mathbf{F}\}$ be the set of inifnite sequences of scalars. We define the following:

- $0 = (0, 0, 0, \dots)$ is the constant zero sequences.
- If $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ then we define

$$x + y = (x_1 + y_1, x_2 + y_2, \dots).$$

- If $k \in \mathbf{F}$ and $x = (x_1, x_2, x_3, \dots)$, then we define

$$kx = (kx_1, kx_2, \dots).$$

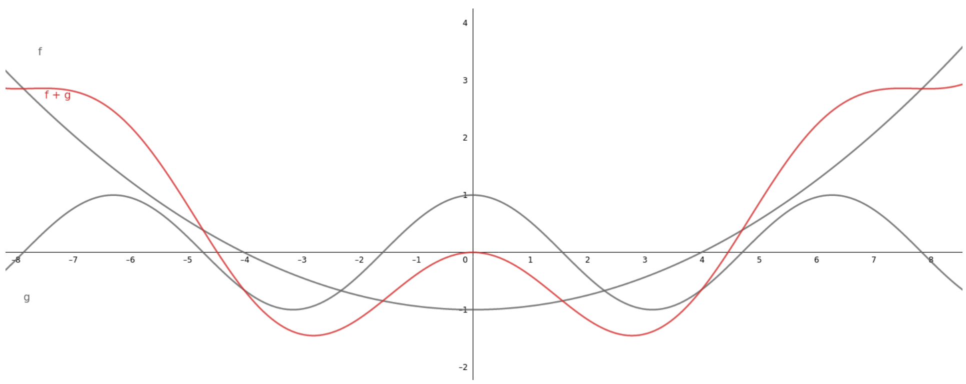
Then $\mathbf{F}^\infty$ is a vector space.

(3) Let $\mathbf{F}$ be a field and let S be a set. Define $\mathbf{F}^S = \{f : S \to \mathbf{F} \mid f \text{ is a function from } S \text{ to } \mathbf{F}\}$.

Define $0 \in \mathbf{F}^S$ by $0(x) = 0$. The f is the zero function, x is any element in S, which gives the output of 0.

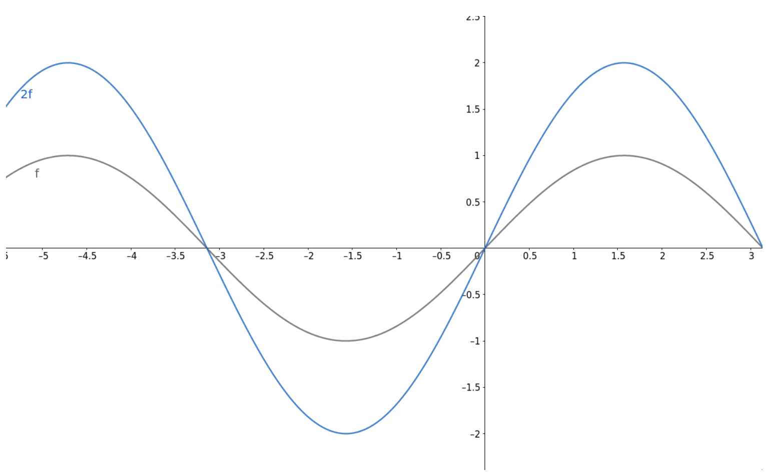
If $f, g \in \mathbf{F}^S$, define $f + g \in \mathbf{F}^S$ as

$$(f+g)(x) = f(x) + g(x).$$



If $k \in \mathbf{F}$ and $f \in \mathbf{F}^S$, define $kf \in \mathbf{F}^S$ as

$$(kf)(x) = k(f(x)).$$



Then $\mathbf{F}^S$ is a vector space.

Note: The functions $F, G : X \to Y$ are equal if

$$\forall x \in X, F(x) = G(x).$$

Proof. Take arbitrary $f, g \in \mathbf{F}^S$. We have to show that $f + g = g + f$ or $\forall x \in S, (f+g)(x) = (g+f)(x)$. Suppose we take an arbitrary $x \in S$, then we have

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) \\
&= g(x) + f(x) \quad \text{by properties of fields} \\
&= (g+f)(x).
\end{aligned}$$

This finishes the proof of (A1). The other field axioms are similar. $\square$

(4) For a field $\mathbf{F}$, we define $\mathcal{P}(\mathbf{F})$ as the set of all formal polynomials with variable x and coefficients in $\mathbf{F}$.

$$\mathcal{P}(\mathbf{F}) = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid n \leq 0, a_0, \ldots, a_n \in \mathbf{F}\}.$$

An example would be $\mathcal{P}(x) = 3 + 5x - 7x^2$.

There are two ways to think about a polynomial: "formal" or "function." For example, define the two following polynomials over $\mathbb{Z}_2$

$$p(x) = x + 1 \qquad q(x) = x^2 + 1.$$

As a function, it would be equal since:

x	$p(x)$	$q(x)$
0	1	1
1	0	0

As a formal polynomial, it would be different because it has the following form:

$$\begin{aligned}
p(x) &= 0x^2 + 1x + 1 \\
q(x) &= 1x^2 + 0x + 1.
\end{aligned}$$

To prove that $\mathcal{P}(\mathbf{F})$ is a vector space, let us define the following:

- Zero polynomial.
$$\mathcal{P}(x) = 0$$
- Addition.
$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$
- Scalar multiplication.
$$kp(x) = (ka_0) + (ka_1)x + (ka_2)x^2 + \cdots + (ka_n)x^n$$

With these operations, $\mathcal{P}(\mathbf{F})$ is a vector space.

2.3 Properties of Vector Spaces

Let V be a vector space over a field $\mathbf{F}$.

- The additive identity is unique. In other words, if $u \in V$ is an additive identity (satisfying $\forall v \in V, u + v = v$) then $u = 0$.
- Additive inverses are unique. Therefore, we can write $-v$ for the additive inverse of v. We also use related notations such as $v - w$ to mean $v + (-w)$.
- Cancellation of addition.
$$v + w = u + w \Rightarrow v = u$$
- For all $v \in V$, we have
$$0v = 0.$$
The 0 is a scalar being multiplied by v (vector), and the 0 on the right is the zero vector.

Proof. We have
$$\begin{aligned} 0v + 0v &= (0+0)v && \text{by (M3)} \\ &= 0v && \text{by properties of scalars} \\ &= 0 + 0v && \text{by (A3)} \end{aligned}$$
So $0v = 0$ follows by cancellation. $\square$

- For all scalars $a \in \mathbf{F}$, we have

$$a0 = 0.$$

Proof. We have

$$\begin{aligned} a0 + a0 &= a(0+0) && \text{by (M2)} \\ &= a0 && \text{by (A3)} \\ &= 0 + a0 && \text{by (A3)} \end{aligned}$$

So $a0 = 0$ follows by cancellation. $\square$

- For all $v \in V$, we have

$$(-1)v = -v.$$

Proof. We have

$$\begin{aligned} (-v) + v &= v + (-v) && \text{by (A1)} \\ &= 0. && \text{by (A4)} \end{aligned}$$

We also have

$$\begin{aligned} (-1)v + v &= (-1)v + 1v && \text{by (M1)} \\ &= (-1+1)v && \text{by (M3)} \\ &= 0v && \text{by properties of scalars} \\ &= 0. && \text{by a previously proved property} \end{aligned}$$

In particular,

$$(-v) + v = (-1)v + v.$$

Then the claim $-v = (-1)v$ follows by cancellation. $\square$

3 1.C Subspaces

3.1 Definition

Let V be a vector space over a field F. A subset U of B is called a *subspace* of V if U is also a vector space in its own right, using the same zero, addition, and scalar multiplication as V.

3.1.1 Characterization of Subspaces

A subset $U \subseteq V$ is a subspace if and only if U satisfies the following three conditions:

(1) Additive identity.
$$0 \in U$$

(2) Closed under addition.
$$\forall v, w, v, w \in U \Rightarrow v + w \in U$$

(3) Closed under scalar multiplication.
$$\forall a, v, a \in F, v \; inU \Rightarrow av \in U$$

Proof. "$\Rightarrow$" Given $U \subseteq V$, assume U is a subspace of V. We want to show that U satisfies all three conditions above.

(1) By definition of subspaces, the zero vector of V is the zero vector of U. So $0 \in U$.

(2) Since U is a vector space, the sume of two vectors in U is a vector in U. Also, U uses the same addition operation as V. So whenever $v, w \in U$, then $v + w \in U$.

(3) Similar to (2).

□

Proof. "$\Leftarrow$" Another proof is this: To show that U is a vector space, we first need an element $0 \in U$ and operations
$$+ : U \times U \to U \qquad \text{and} \qquad \cdot : F \times U \to U.$$

Second, we must show axioms (A1) - (M4).

(1) By assumption, $0 \in U$, where 0 is the additive identity of V. So we can use 0 as the additive identity of U.

(2) By assumption, U is closed under addition, so the addition function $+ : V \times V \to V$ restricts to a function $+ : U \times U \to U$. We can use the same function as the addition function on U.

(3) We do the same with scalar multiplication.

Second: We must show (A1) - (M4) hold. We only do (A1) since the rest are similar. To prove (A1), take arbitrary $u, v \in U$. We need to show that

$$u + v = v + u$$

in U. But since V is a vector space, we know that

$$u + v = v + u$$

in V. This automatically holds.

The other parts of the definition of a vecotr space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V. Thus, U is a vector space and hence is a subspace of V. □

3.2 Examples of Subspaces

(a) Let $V = \mathbb{R}^4$ and let

$$W = \{(x, y, z, w) \mid x = 3y + 2z\}.$$

Then W is a subspace of V.

Proof. We need to show that W is a subspace of V.

(1) $(0, 0, 0, 0) \in W$.

(2) Assume $v = (x, y, z, w) \in W$ and $v' = (x', y', z', w') \in W$. We need to show that

$$x + x' = 3(y + y') + 2(z + z').$$

We know that $v \in W$ implies

$$x = 3y + 2z.$$

And we know that $v' \in W$ implies

$$x' = 3y' + 2z'.$$

Add these two equations together and we get

$$x + x' = 3(y + y') + 2(z + z').$$

(3) Similar proof for scalar multiplication.

$\square$

(b) Recall that $V = \mathbb{R}^{[0,1]}$ is the vector space of functions from the unit interval

$$[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

to $\mathbb{R}$. Let $W = \{f : [0, 1] \to \mathbb{R} \mid \ f \text{ is continuous } \}$. Then W is a subspace of V.

Proof. We need to show that W is a subspace of V.

(1) The zero function $f(x) = 0$ is continuous. From Calculus, any constant function is continuous.

(2) The sum of two continuous functions is continuous. This is from Calculus.

(3) If f is continuous, then so is kf, for any $k \in \mathbb{R}$. This is also from Calculus.

$\square$

(c) Again, let $V = \mathbb{R}^{[0,1]}$, and $U = \{f : [0, 1] \to \mathbb{R} \mid \ f \text{ is differentiable } \}$. Then U is a subspace of V.

Proof. From Calculus, we know these hold true:

(1) The zero function $f(x) = 0$ is differentiable with the derivative of

$$f'(x) = 0.$$

(2) If f, g are differentiable then so is $f + g$ and

$$(f + g)' = f' + g'.$$

(3) If f is differentiable and k is a scalar, then kf is differentiable.

$$(kf)' = kf'.$$

For example, the derivative of $(0+2\sin(x)-3\cos(x))' = 0+2\cos(x) - 3(-\sin(x))$.

We also know from Calculus that every differentiable function is continuous.

$$U \subseteq W \subseteq V$$

where U is the set of all differentiable functions, W is the set of all continuous functions, and V is the set of all functions. $\square$

(d) Let $V = \mathbb{R}^{[0,1]}$ and define

$$X = \left\{ f : [0,1] \to \mathbb{R} \mid f \text{ is differentiable and } f'\left(\frac{1}{2}\right) = 0 \right\}.$$

We claim that X is a subspace of V.

Note. We already know that U, the set of differentiable functions, is a subspace of V.

Proof. Effectively, it is sufficient to show that X is a subspace of U.

(1) The zero function $f(x) = 0$ is differentiable since

$$f'(x) = 0 \text{ and } f'\left(\frac{1}{2}\right) = 0.$$

So $f \in X$.

(2) Given $f, g \in X$, we must check that $f+g \in X$. Clearly $f+g$ is differentiable. We must check that $(f+g)'\left(\frac{1}{2}\right) = 0$. From Calculus, we have

$$\begin{aligned}(f+g)'\left(\frac{1}{2}\right) &= f'\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right) \\ &= 0+0 \\ &= 0.\end{aligned}$$

So we have $f+g \in X$.

(3) Similar proof as follows for scalar multiplication:

$$(kg)'\left(\frac{1}{2}\right) = k \cdot f'\left(\frac{1}{2}\right) = 0.$$

$\square$

(e) Let $V = \mathbb{R}^\infty$, the set of infinite sequences of real numbers.

$$\mathbb{R}^\infty = \{(a_0, a_1, a_2, a_3, \dots) \mid a_0, a_1, \dots \in \mathbb{R}\}.$$

Recall that V is a vector space. Let $W \subseteq V$ be the set of *convergent* sequences. From Calculus, we know that some sequences converge and some do not.

Some examples are:

- $a_i = i \Rightarrow (0, 1, 2, 3, 4, 5, 6, 7, \dots)$

$$\lim_{i\to\infty} a_i = \text{ does not exist, so it does not converge}$$

- $b_i = \frac{1}{i} \Rightarrow (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots)$

$$\lim_{i\to\infty} b_i = \text{ converges to } 0$$

- $c_i = (-1)^i \Rightarrow (1, -1, 1, -1, 1, -1, \dots)$ does not converge.
- $d_i = 2 + \left(-\frac{1}{2}\right)^i \Rightarrow (3, 1.5, 2.25, 1.875, \dots)$

$$\lim_{i\to\infty} d_i = \text{ converges to } 2$$

Then W is a subspace of V.

Proof. We must show that this is true.

(1) The zero sequence $a_i = 0 \Rightarrow (0, 0, 0, \dots)$ converges to 0.

(2) From Calculus, the sum of two convergent sequences converges. In fact,

$$\lim_{i\to\infty}(a_i + b_i) = \lim_{i\to\infty} a_i + \lim_{i\to\infty} b_i$$

(3) Similar proof to scalar muiltiplication.

$$\lim_{i\to\infty}(ka_i) = k \lim_{i\to\infty} a_i$$

$\square$

Let U be the set of sequences that converge to 0. Then U is a subspace of V (and of W).

(f) A reccurence relation. Consider the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

Let F_n be the nth element of this sequence. Then, we have the following recurrence:

$$\begin{aligned} \text{base case } F_0 &= 1 \\ \text{base case } F_1 &= 1 \\ \text{reccurence } F_{n+2} &= F_n + F_{n+1} \qquad \text{for all } n \geq 0. \end{aligned}$$

If we forget the base cases, we can consider the set of *all* sequences satisfying the reccurence

$$U = \{(a_0, a_1, a_2, \dots) \mid \text{ for all } n \geq 0, a_{n+2} = a_n + a_{n+1}\}$$

A sequence of numbers is called a *generalized Fibonacci sequence* if it satisfies this reccurence, i.e. if it is a member of the set U.

Examples.

$$\begin{aligned} &1, 2, 3, 5, 8, 13, \dots \\ &7, -3, 4, 1, 5, 6, 11, 17, 28, 45, \dots \\ &0, 0, 0, 0, 0, 0, 0, 0, \dots \end{aligned}$$

Claim. U is a subspace of $\mathbb{R}^\infty$.

(1) The sequence $0, 0, 0, \dots$ is a generealized Fibonacci sequence.

(2) U is closed under addition.

$$\begin{array}{l} 1, 2, 3, 5, 8, 13 \dots \\ 7, -3, 4, 1, 5, 6, \dots \\ \hline 8, -1, 7, 6, 13, 19, \dots \end{array}$$

Proof. Suppose $a = (a_0, a_1, a_2, \dots) \in U$ and $b = (b_0, b_1, b_2, \dots) \in U$. Then $a + b = c = (c_0, c_1, c_2, \dots)$ where $c_i = a_i + b_i$.

We must show $c \in U$, i.e. we must show that c is a generalized Fibonacci sequence.

So take an arbitrary $n \geq 0$. We must show $C_{n+2} = C_n + C_{n+1}$. Indeed, we have:

$$\begin{aligned} C_{n+2} &= a_{n+2} + b_{n+2} \\ &= (a_n + a_{n+1}) + (b_n + b_{n+1}) \\ &= (a_n + b_n) + (a_{n+1} + b_{n+1}) \\ &= c_n + c_{n+1} \end{aligned}$$

□

(3) Closed under scalar multiplication: similar.

3.3 Intersection of Subspaces

3.3.1 Theorem

Let V be a vector space over a field F. Assume U and W are subspaces of V. Then $U \cap W$ is a subspace of V.

Proof. To show that $U \cap W$ is a subspace, we need to show the three properties.

(1) We must show that $0 \in U \cap W$. But by assumption, U is a subspace, so $0 \in U$. Also, W is a subspace, so $0 \in W$. By definiton of intersection, we have $0 \in U \cap W$.

(2) We must show that $U \cap W$ is closed under addition. Consider arbitrary $v, w \in U \cap W$ and we need to show that $v + w \in U \cap W$.

   Indeed, we have:

   - Since $v \in U \cap W$, we know $v \in U$.
   - Since $w \in U \cap W$, we know $w \in U$.
   - Since U is a subspace, it is closed under addition, so $v + w \in U$.

   Similarly:

   - Since $v \in U \cap W$, we know $v \in W$.

- Since $w \in U \cap W$, we know $w \in W$.
- Since W is a subspace, it is closed under addition, so $v + w \in W$.

From $v + w \in U$ and $v + w \in W$, by definition of intersection, we know

$$v + w \in U \cap W.$$

(3) We must show that $U \cap W$ is closed under scalar multiplication. So consider arbitrary $k \in F$ and $v \in U \cap W$. We must show that $kv \in U \cap W$.

Since $v \in U \cap W$, we have $v \in U$. Since U is a subspace of V, we know that U is closed under scalar multiplication, so $kv \in U$.

Similarly, since $v \in U \cap W$, we know $v \in W$. Since W is a subspace of V, we know that W is closed under scalar multiplication, so $kv \in W$.

From $kv \in U$ and $kv \in W$, it follows that $kv \in U \cap W$ (by definition of intersection), as desired.

□

3.3.2 Notations

- $(x_1, \ldots, x_n)$ is called an n-tuple.
- $(x_i)_{i \in \{1,\ldots,n\}}$ is another notation for the same thing. This is called "family" notation. But, this notation also works for infinite index sets.

$$(x_i)_{i \in \mathbb{N}} = (x_0, x_1, x_2, \ldots)$$

$$\left(\frac{1}{i+1}\right)_{i \in \mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$$

- More generally, we can use the "family notation" for other families of things, not necessarily numbers.

$$\begin{aligned} U_1, U_2, U_3 &\qquad \text{3 subspaces of } V \\ (U_i)_{i \in \{1,2,3\}} &\qquad \text{Notation for the same thing.} \\ (U_i)_{i \in I} &\qquad \text{Some family of subspaces.} \end{aligned}$$

- Other notations that go along with these:
  - Summation.
  $$\sum_{i \in \mathbb{N}} x_i$$
  - Product.
  $$\prod_{i \in \mathbb{N}} x_i$$
  - Limit.
  $$\lim_{i \to \infty} x_i$$
  - Union.
  $$\bigcup_{i \in \{1,2,3\}} U_i$$
  - Intersection.
  $$\bigcap_{i \in I} U_i$$

3.3.3 Theorem

Let V be a vector space over a field F. Let I be a set. Let $(U_i)_{i \in I}$ be a *family* of subspaces of V. Then,

$$\bigcap_{i \in I} U_i$$

is a subspace of V.

Proof. Let $W = \bigcap_{i \in I} U_i$. To show that W is a subspace of V, we must show the three subspace properties.

(1) We must show that $0 \in W$. Take an arbitrary $i \in I$. By assumption, U_i is a subspace of V. Therefore, $0 \in U_i$.

   Since i was arbitrary, we have $0 \in U_i$ for all i, and therefore, by definition of intersection, $0 \in \bigcap_{i \in I} U_i = W$.

(2) Must show W is closed under addition. Take an arbitrary $v, u \in W$. We must show that $v + u \in W$ or $v + u \in \bigcap_{i \in I} U_i$.

   Take an arbitrary $i \in I$. We must show that $v + u \in U_i$. By assumption, $v \in W = \bigcap_{i \in I} U_i$, therefore $v \in U_i$.

   Simiarly, by assumption, $u \in W = \bigcap_{i \in I} U_i$, therefore $u \in U_i$.

Also by assumption, U_i is a subspace of V, therefore it is closed under addition. So $v + u \in U_i$.

Since i was arbitrary, we therefore know for all $i \in I$ that $v + u \in U_i$. It follows that $v + u \in \bigcap_{i \in I} U_i$ as desired.

(3) Closed under scalar multiplication: similar proof.

$\square$

3.3.4 Meta-theorem

Let V be a vector space over a field F. Let P be any property of subspaces of V such that P is closed under arbitrary intersections.
This means that whenever we have a family of subspaces, $(U_i)_{i \in I}$ of V,

- If, for all $i \in I$, U_i has the property P, then

$$\bigcap_{i \in I} U_i$$

  has the property P.

Then there exists a *smallest* subspace of V with the property P.

Proof. Let P be such a property, i.e. a property of subspaces of V that is closed under intersections.
We want to showt hat there exists a *smallest* subspace W of V with property P. Specifically, this means:

(1) W is a subspace of V and has the property P.

(2) Whenever W' is a subspace of V that has the property P, then $W \subseteq W'$.

To show it, let $(U_i)_{i \in I}$ be the family of *all* subspaces of V satisfying property P. Define

$$W = \bigcap_{i \in I} U_i.$$

We have to show (1) and (2).

(1) W is a subspace of V by the previous theorem. Also, W has the property P because all U_i have the property P and P is closed under intersections.

(2) We must show that W is smallest. So consider any subspace W' with property P. We must show that $W \subseteq W'$.

But the family $(U_i)_{i \in I}$ contains *all* subspaces with property P. So $W' = U_i$ for all some $i \in I$. Then

$$W = \bigcap_{i \in I} U_i \subseteq U_i = W'.$$

□

3.3.5 Example

Consider $v_1, v_2, v_3 \in V$.
There exists a *smallest subspace* W of V such that $v_1, v_2, v_3 \in W$. We normally call W the *span* of v_1, v_2, v_3.

Proof. By the meta-theorem, the property "contains v_1, v_2, and v_3" is closed under intersections. □

3.4 Sums of Subspaces

3.4.1 Definition of Sum of Subsets

Suppose $U_1, \ldots, U_m$ are subsets of V. The **sum** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_1 \in U_1, \ldots, u_m \in U_m\}.$$

3.4.2 Example

For $U = \{(x, 0, 0) \in \mathbf{F}^3 \mid x \in \mathbf{F}\}$ and $W = \{(0, y, 0) \in \mathbf{F}^3 \mid y \in \mathbf{F}\}$, we have

$$U + W = \{(x, y, 0) \mid x, y \in \mathbf{F}\}.$$

For $U = \{(x, x, y, y) \in \mathbf{F}^4 \mid x, y \in \mathbf{F}\}$ and $W = \{(x, x, x, y) \in \mathbf{F}^4 \mid x, y \in \mathbf{F}\}$, then

$$U + W = \{(x, x, y, z) \in \mathbf{F}^4 \mid x, y, z \in \mathbf{F}\}.$$

3.4.3 Sum of subspaces is the smallest containing subspace

Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$.

Proof. We know that $0 \in U_1 + \cdots + U_m$ and $U_1 + \cdots + U_m$ is closed under addition and scalar multiplication. Thus, $U_1 + \cdots + U_m$ is a subspace of V. $U_1, \dots, U_m$ are contained in $U_1 + \cdots + U_m$, and every subspace of V containing $U_1, \dots, U_m$ contains $U_1 + \cdots + U_m$.
Thus, $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$. $\square$

Note. Sums of subspaces in theory of vector spaces are analogous to unions of subsets in set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union.

3.5 Direct Sums

Suppose $U_1, \dots, U_m$ are subspaces of V. Every element of $U_1 + \cdots + U_m$ can be written in the form

$$u_1 + \cdots + u_m$$

where each u_j is in U_j.

3.5.1 Definition of direct sum

- The sum $U_1 + \cdots + U_m$ is called a *direct sum* if each element of $U_1 + \cdots + U_m$ can eb written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

3.5.2 Examples

For $U = \{(x, y, 0) \in \mathbf{F}^3 \mid x, y \in \mathbf{F}\}$ and $W = \{(0, 0, z) \in \mathbf{F}^3 \mid z \in \mathbf{F}\}$, then

$$\mathbf{F}^3 = U \oplus W.$$

Suppose $U_j = \{(0, 0, 0, \dots, j) \in \mathbf{F}^n \mid x \in \mathbf{F}\}$. Then

$$\mathbf{F}^n = U_1 \oplus \cdots \oplus U_n.$$

Non-example: Let $U_1 = \{(x, y, 0) \in \mathbf{F}^3 \mid x, y \in \mathbf{F}\}$, $U_2 = \{(0, 0, z) \in \mathbf{F}^3 \mid z \in \mathbf{F}\}$, $U_3 = \{(0, y, y) \in \mathbf{F}^3 \mid y \in \mathbf{F}\}$. Then, $U_1 + U_2 + U_3$ is not a direct sum.

Proof. Clearly $\mathbf{F}^3 = U_1 + U_2 + U_3$ because every vector $(x, y, z) \in \mathbf{F}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0).$$

This is not equal to the direct sum because the vector $(0, 0, 0)$ can be written in two different ways.

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$$

and

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1).$$

□

3.5.3 Condition for a direct sum

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

Proof. Suppose $U_1 + \cdots + U_m$ is a direct sum. Then by definition, the only way to write 0 as a sum is by taking each u_j equal to 0. To show that $U_1 + \cdots + U_m$ is a direct sum, let $v \in U_1 + \cdots + U_m$. We can write

$$v = u_1 + \cdots + u_m$$

for some $u_1 \in U_1, \ldots, u_m \in U_m$. To show that this is unique, suppose we also have

$$v = v_1 + \cdots + v_m$$

where $v_1 \in U_1, \ldots, v_m \in U_m$. Subtracting them, we have

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

This implies that $u_j = v_j$. □

3.5.4 Direct sum of two subspaces

Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof. Suppose that $U + W$ is a direct sum. If $v \in U \cap W$, then $0 = v + (-v)$ where $v \in U$ and $-v \in W$. This implies $v = 0$ since it is unique.
To prove the other direction, suppose $U \cap W = \{0\}$. To prove that $U + W$ is a direct sum, suppose $u \in U, w \in W$, and

$$0 = u + w.$$

By the previous theorem, we know that $u = w = 0$. The equation above implies $u = -w \in W$, so $u \in U \cap W$ and $u = w = 0$ is true. $\square$

Note. Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint, because both contain 0. So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals $\{0\}$.