MATH 2135 Linear Algebra

Sets and Logic Notations

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Contents

1 Sets

1.1 Finite sets

A set is an unordered collection of things. A finite set would look something like $\{1, 2, 3\}$. “Unordered” means that the order does not matter, i.e. $\{1, 2, 3\}$ and $\{2, 3, 1\}$ are the same set.

1.2 Infinite sets

An example of an infinite set is the set of natural numbers.

$$\mathbb{N} = \{0, 1, 2, 3, \dots\}$$

and its respective *set comprehension notation* would look like

$$\mathbb{N} = \{x \mid \text{ x is a natural number }\}.$$

Another example of an inifnite set would be

$$\{x \in \mathbb{N} \mid \text{ x is prime }\} = \{2, 3, 5, 7, 11, 13, 17, \dots\}.$$

1.3 Membership

The notation $\in$ implies membership such as “x is an element of A” and conversely, the notation $\notin$ implies “x is not an element of A.”

1.4 Equality

Two sets A and B are *equal* if they have the same elements.

$$A = B \Leftrightarrow (\forall x, x \in A \Leftrightarrow x \in B).$$

We say that A is a *subset* of B, in symbols $A \subseteq B$, if all elements of A are elements of B.

$$A \subseteq B \Leftrightarrow (\forall x, x \in A \Rightarrow x \in B).$$

1.5 Empty set

The empty set ($\varnothing$) is the set with no elements.

1.6 Cartesian product

If A and B are sets, we define the *cartesian product* of A and B, in symbols $A \times B$, as

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

Note that (x, y) is a *pair* or *2-tuple* which is an *ordered pair*, i.e. $(1, 2) \neq (2, 1)$.

2 Logic Notations

2.1 Propositional logic (Boolean logic)

A *proposition* is a statement that can be true or false.

Let P and Q be propositions.

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	$\neg P$
T	F
F	T

2.2 Predicate logic

A *predicate* is a proposition that depends on some "thing" x.

$$\begin{aligned}
P(x) &= \text{“}x \text{ is a prime number”} \\
P(5) &= \text{“5 is a prime number”} \ \textit{TRUE} \\
P(6) &= \text{“6 is a prime number”} \ \textit{FALSE} \\
&\vdots \\
Q(x,y) &\to \text{“x is greater or equal to y”} \\
Q(3,7) &\to \textit{FALSE} \\
Q(7,7) &\to \textit{TRUE} \\
Q(19,7) &\to \textit{TRUE} \\
&\vdots
\end{aligned}$$

2.3 Quantifiers

2.3.1 Universal Quantifier

If $P(x)$ is a predicate, then "for all x, $P(x)$" is a proposition that is either true or false.

Example. Let $A = \{3, 5, 7, 8, 11\}$, $P(x) =$"x is prime," $Q(x) =$"x is even."

- For all $x \in A, P(x) \rightarrow$ *FALSE.* This is because $P(3), P(5), P(7), P(11)$ are all true but $P(8)$ is false.

- For all $x \in A$, $(x \leq 7 \Rightarrow P(x)) \rightarrow$ *TRUE.*

x	$x \leq 7$	$P(x)$	$x \leq 7 \Rightarrow P(x)$
3	T	T	T
5	T	T	T
7	T	T	T
8	F	F	T
11	F	T	T

The notation for "for all" is $\forall$.

2.3.2 Existential Quantifer

The notation for "there exists" is $\exists$.

Example. Let $A = \{3, 5, 7, 8, 11\}$.

- There exists an $x \in A$ such that $P(x)$. *TRUE*

- There exists $x \in A$ such that $x \leq 7$ and $P(x)$. *TRUE*

2.3.3 Nested Quantifiers

Suppose that $\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}$.

- $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, x < y \rightarrow$ *TRUE*

- $\exists y \in \mathbb{N}, \forall x \in \mathbb{N}, x < y \rightarrow$ *FALSE*

- $\forall x \in \mathbb{N}, (x \geq 3$ and $(\forall y \in \mathbb{N}, \forall z \in \mathbb{N}, (x = yz \Rightarrow y = 1$ or $z = 1)) \Rightarrow$ x is odd$)$

2.4 "Vacuously true"

Question: Is the empty set a subset of every set? Yes. For $A \subseteq B$ it means $\forall x \in A, x \in B$. If A is empty, this is *vacuously* true.

What does "vacuously true" mean? Suppose we have the following sets and statements:

- $A = \{3, 5, 7, 8, 11\}$, $\forall x \in A, P(x) \rightarrow$ *FALSE*
- $A = \{3, 5, 7\}$, $\forall x \in A, P(x) \rightarrow$ *TRUE*
- $A = \{3\}$, $\forall x \in A, P(x) \rightarrow$ *TRUE*
- $A = \varnothing$, $\forall x \in A, P(x) \rightarrow$ *"vacuously true"*

If A is the empty set, the statement $\forall x \in A, P(x)$ is *always* true, no matter what $P(x)$ is.

Another example would be: "All unicorns are green." This is true because there are 0 unicorns to check.