CSCI/MATH 2113 Discrete Structures

5.2 Functions: Plain and One-to-One

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1 Defintion of a function

For nonempty sets A, B, a *function*, or *mapping*, f from A to B, denoted $f : A \to B$, is a relation from A to B in which every lement of A appears exactly once as the first component of an ordered pair in the relation.

We have $f \subseteq A \times B$ and

(1) Existence:

$$\forall a \in A, \exists b \in B, (a, b) \in f.$$

(2) Uniqueness: If $(a, b) \in f$ and $(a, b') \in f$ then

$$b = b'.$$

1.1 Image and Preimage

If $(a, b) \in f$, we write $f(a) = b$. We then say that b is the *image* of a under f, and that a is the *preimage* of b under f.

Example. The absolute value is the function $|x| : \mathbb{R} \to \mathbb{R}$. Here, 2 and -2 are two preimages of 2 since

$$|2| = 2 = |-2|.$$

So a given element can have more than one preimage.

1.2 Domain and codomain

For the function $f : A \to B$, A is called the *domain* of f and B the *codomain* of f. The subset of B consisting of those elements that appear as second components in the ordered pairs of f is called the *range* of f and is also denoted by $f(A)$ because it is the set of images (of the elements of A) under f.

Note: Range does not imply that it is equal to codomain.

1.3 Examples

1. The *greatest integer function*, or *floor function*. This function $f : \mathbb{R} \to \mathbb{R}$, is given by

$$f(x) = \lfloor x \rfloor = \text{ the greatest integer less than or equal to } x$$

Example. $\lfloor 7.7 + 8.4 \rfloor = \lfloor 16.1 \rfloor = 16$.

2. The *ceiling function.* This function $g : \mathbb{R} \to \mathbb{Z}$ is defined by

$$g(x) = \lceil x \rceil = \text{ the least integer greater than or equal to } x.$$

Example. $\lceil 3.3 + 4.2 \rceil = \lceil 7.5 \rceil = 8$.

3. The function trunc (for truncation). It deletes the fractional part of a real number. Note that trunc(3.78) $= \lfloor 3.78 \rfloor = 3$ while trunc(-3.78) $= \lceil -3.78 \rceil = -3$.

4. The *access* function. In storing a matrix in a one-dimensional array, many computer languages use the *row major* implementation. If $A = (a_{ij})_{m\times n}$ is an $m \times n$ matrix, to determine the location of an entry a_{ij} from A, where $1 \leq i \leq m, 1 \leq j \leq n$, we can use the formula for the access function:
$$f(a_{ij}) = (i-1)n + j.$$

1.4 Counting functions

Let A, B be nonempty sets with $|A| = m, |B| = n$. How many functions are there in $f : A \to B$?

Suppose that $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$, then a typical function can be described by $\{(a_1, x_1), (a_2, x_2), \ldots, (a_m, x_m)\}$. We can select any n elements of B for x_1 then do the same for x_2. We continue this selection until one of the n elements of B is finally selected for x_m. Using the product rule, there are

$$n^m = |B|^{|A|}$$

functions from A to B.

Example. Suppose that $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$. There are $4^3 = 64$ functions from A to B.

In general, we do not expect $|A|^{|B|} = |B|^{|A|}$.

2 One-to-one Correspondence

2.1 Definition

A function $f : A \to B$ is called *one-to-one*, or *injective*, if each element of B appears at most once as the image of an element of A.

If $f : A \to B$ is one-to-one, with A, B finite, we must have $|A| \leq |B|$. For arbitrary sets A, B, $f : A \to B$ is one-to-one if and only if for all $a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2$. Consequently, we also have $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$.

2.2 Examples

1. Suppose that $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 3x + 7$ for all $x \in \mathbb{R}$. Then for all $x_1, x_2 \in \mathbb{R}$, we find that

   $$f(x_1) = f(x_2) \Rightarrow 3x_1 + 7 = 3x_2 + 7 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2,$$

   so the given function f is one-to-one.

2. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is the function defined by $g(x) = x^4 - x$ for each real number x. Then

   $$g(0) = (0)^4 - 0 = 0 \text{ and } g(1) = (1)^4 - (1) = 1 - 1 = 0.$$

   Consequently, g is not one-to-one since $g(0) = g(1)$ but $0 \neq 1$.

2.3 Counting injective functions

Let A, B be nonempty sets with $|A| = m, |B| = n$. How many injective functions are there in $f : A \to B$?

Suppose that $A = \{a_1, \ldots, a_m\}$, $B = \{b_1, \ldots, b_n\}$, and that $m \leq n$. The one-to-one function has the form $\{(a_1, x_1), \ldots, (a_m, x_m)\}$, where there are n choices for x_1, $n-1$ choices for x_2, $n-2$ choices for x_3, and so on, finishing with $n - (m-1) = n - m + 1$ choices for x_m. By the rule of product, we have

$$n(n-1)(n-2)\ldots(n-m+1) = \frac{n!}{(n-m)!} = P(n,m) = P(|B|, |A|).$$

2.4 Direct Image

If $f : A \to B$ and $A_1 \subseteq A$, then

$$f(A_1) = f[A_1] = \{b \in B \mid b = f(a), \text{ for some } a \in A_1\},$$

and $f(A_1)$ is called the *direct image* of A_1 under f.

Example. Let $g : \mathbb{R} \to \mathbb{R}$ be given by x^2. Then $g(\mathbb{R}) =$ the range of g $= [0, +\infty)$. The *image* of $\mathbb{Z}$ under g is

$$g(\mathbb{Z}) = \{0, 1, 4, 9, 16, \dots\}$$

and for $A_1 = [-2, 1]$ we get

$$g(A_1) = [0, 4].$$

2.5 Set Operations

Let $f : A \to B$, with $A_1, a_2 \subseteq A$. Then

(a) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$;

(b) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$;

(c) $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ when f is one-to-one.

Proof. Part (b): For each $b \in B, b \in f(A_1 \cap A_2) \Rightarrow b = f(a)$, for some $a \in A_1 \cap A_2 \implies [b = f(a)$ for some $a \in A_1]$ and $[b = f(a)$ for some $a \in A_2] \Rightarrow b \in f(A_1)$ and $b \in f(A_2) \Rightarrow b \in f(A_1) \cap f(A_2)$, so $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$. $\square$

2.6 Restriction and Extension

2.6.1 Restriction

If $f : A \to B$ and $A_1 \subseteq A$, then $f|_{A_1} : A_1 \to B$ is called the *restriction of f to A_1* if $f|_{A_1}(a) = f(a)$ for all $a \in A_1$.

2.6.2 Extension

Let $A_1 \subseteq A$ and $f : A_1 \to B$. If $g : A \to B$ and $g(a) = f(a)$ for all $a \in A_1$, then we call g an *extension of f to A*.