CSCI/MATH 2113 Discrete Structures

5.3 Onto Functions: Stirling Numbers of the Second Kind

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1 Surjective Functions

1.1 Definition

$f : A \to B$ is *onto* or *surjective* if $\forall b \in B, \exists a \in A$ such that $f(a) = b$.

Remark. A function is onto if its range is equal to the codomain.

If A, B are finite sets, then for an onto function $f : A \to B$ to possibly exist we must have $|A| \leq |B|$.

1.2 Examples

- For the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ we have: if $x \in \mathbb{R}$, then $f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$. This means that f is onto.
- For the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ we have: no such $y \in \mathbb{R}$ that satisfies $y^2 = -1$. For instance,

  $$g(x) = x^2 = -9 \Rightarrow r = 3i, -3i \in \mathbb{C}.$$

  Therefore, g is not onto.

1.3 Counting

Suppose that $A = \{x, y, z\}$ and $B = \{1, 2\}$. How many $f : A \to B$ are onto?

Proof. The function $f : A \to B$ is not onto if and only if $f(a) = 1$ for all $a \in A$ or $f(a) = 2$ for all $a \in A$. Hence, the number we seek is

$$|B|^{|A|} - 2 = 2^3 - 2 = 6.$$

□

In general, if A and B are sets with $|A| = m$ and $|B| = n$, then this quantity is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{n-k} (n-k)^m$$

onto functions from A to B.

2 Stirling Numbers

2.1 A combinatorial interpretation

In how many ways can you distribute 4 objects into 3 labelled cotainers with no container empty? We just need to count the number of surjections from A to B. By the previuos formula, this is

$$\sum_{k=0}^{3}(-1)^k\binom{3}{3-k}(3-k)^4 = 36.$$

What if we had the same situation but with containers that aren't labelled? The number of such distributions is

$$S(m,n) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{n-k}(n-k)^m$$

where the factor of $\frac{1}{n!}$ corrects for the fact that certain distributions are equivalent.
So $S(m,n)$ is the number of ways to distribute m objects into n identical containers with no container left empty.

2.2 Definition

$S(m,n)$ is a *Stirling number* of the 2nd kind.