CSCI/MATH 2113 Discrete Structures

Chapter 11 An Introduction to Graph Theory

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Contents

1 11.1 Definitions and Examples

Let V be a finite nonempty set, and let $E \subseteq V \times V$. The pair (V, E) is then called a *directed graph* or *digraph*, where V is the set of vertices, or nodes, and E is its set of (directed) edges or arcs. We write $G = (V, E)$ to denote such a graph.

When there is no concern about the direction of any edge, we still write $G = (V, E)$. But now E is a set of unordered pairs of elements taken from V (cardinality ≤ 2), and G is called an *undirected* graph.

Whether $G = (V, E)$ is directed or undirected, we often call V the vertex set of G and E the edge set of G.

If (a, b) is an edge in a directed graph, then a and b are the *source* and *target* of the edge. We also say if $\{a, b\}$ or (a, b), then a and b are *adjacent* and that $\{a, b\}$ (or (a, b)) is incident to a and b.

A loop-free graph is a graph in which no vertex is adjacent to itself.

1.1 $x - y$ walk

Let x, y be (not necessarily distinct) vertices in an undirected graph $G = (V, E)$. An $x - y$ *walk* in G is a (loop-free) finite alternating sequence

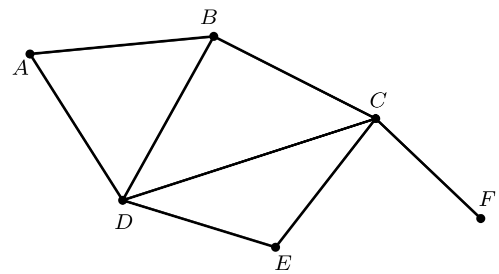
$$x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = y$$

of vertices and edges from G, starting at vertex x and ending at vertex y and involving the n edges $e_i = \{x_{i-1}, x_i\}$, where $1 \leq i \leq n$.

The *length* of this walk si n, the number of edges in the walk. (When $n = 0$, there are no edges; $x = y$, and the walk is called *trivial*).

Any $x - y$ walk where $x = y$ (and $n > 1$) is called a *closed walk*. Otherwise, the walk is called *open*.

1.2 Example of a walk



$\{a,b\},\{b,d\},\{d,c\},\{c,e\},\{e,d\},\{d,b\}$ is an $a-b$ walk. This walk has length 6.

1.3 Trail, circuit, path, and cycles

An $x-y$ walk is a *trail* if no edge is repeated. A closed trail is a *circuit*. A *path* is a trail in which no vertex occurs more than once. A closed path is a *cycle*. (=trail where the only repeated vertices are the first and last one).

Theorem: Let $G=(V,E)$ be an undirected graph. Let $a,b \in V$ with $a \neq b$. If there is a trail between a and b then there is a path between a and b.

Proof. Suppose there is a trail from a to b. Take one of smallest length.

- If the trail is a path, we are done.
- If not, then it is of the form

  $$\{a,x_1\},\{x_1,x_2\},\ldots,\{x_{k-1},x_k\},\{x_{k+1},x_{k+2}\},\ldots,\{x_{m-1},x_m\},\{x_m,x_{m+1}\},\ldots,\{x_m,b\}$$

  where $x_k = x_m$. But then

  $$\{a,x_1\},\{x_1,x_2,\ldots,\{x_{k-1}\},x_k\},\{x_m,x_{m+1}\},\ldots,\{x_n,b\}$$

  is a shorter trail from a to b which is a contradiction.

□

Repeated Vertex (Vertices)	Repeated Edge(s)	Open	Closed	Name
Yes	Yes	Yes		Walk (open)
Yes	Yes		Yes	Walk (closed)
Yes	No	Yes		Trail
Yes	No		Yes	Circuit
No	No	Yes		Path
No	No		Yes	Cycle

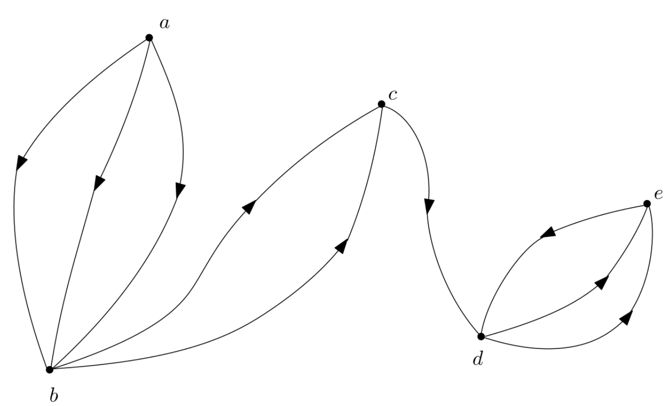
1.4 Connect and disconnect

A graph G is *connected* if there is a path between any two distinct vertices of G. A graph that is not connected is *disconnected*.

For any graph $G = (V, E)$, the number of (connected) components of G is denoted by $\kappa(G)$.

1.5 Multigraph

Let V be a set (non-empty). Then (V, E) is a *multigraph* if there are $a, b \in V$ such that E contains more than one edge between a and b. (Strictly speaking, E is a multiset (set with repetition)).

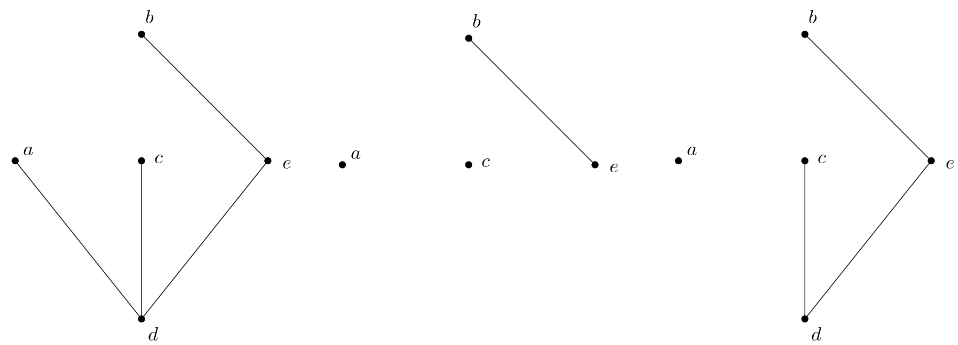


Here, the edge (a, c) has *multiplicity* 2.

2 Subgraphs, Complements and Graph Isomorphism

2.1 Subgraph

If $G = (V, E)$ is a graph (directed or undirected), then $G_1 = (V_1, E_1)$ is called a *subgraph* of G if $\varnothing \neq V_1 \subseteq V$ and $E_1 \subseteq E$, where each edge in E_1 is incident with vertices in V_1.

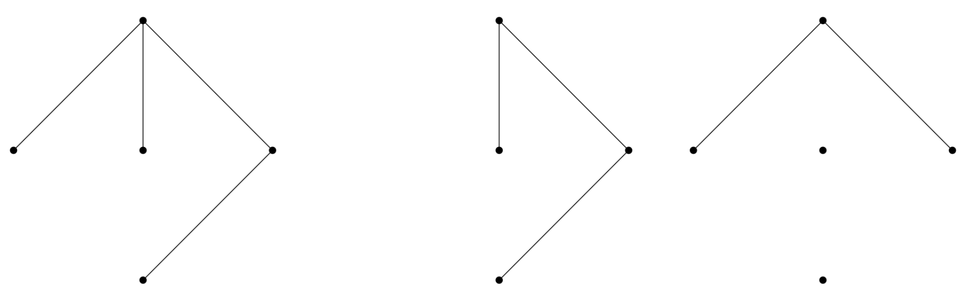


G $\qquad$ G_1 is a subgraph of G $\qquad$ G_2 is a subgraph of G

Note that G_1 is a subgraph of G_2.

2.2 Spanning subgraph

If $G = (V, E)$ and $G_1 = (V_1, E_1)$ and G_1 is a subgraph of G, then G_1 is a *spanning* subgraph if $V_1 = V$.



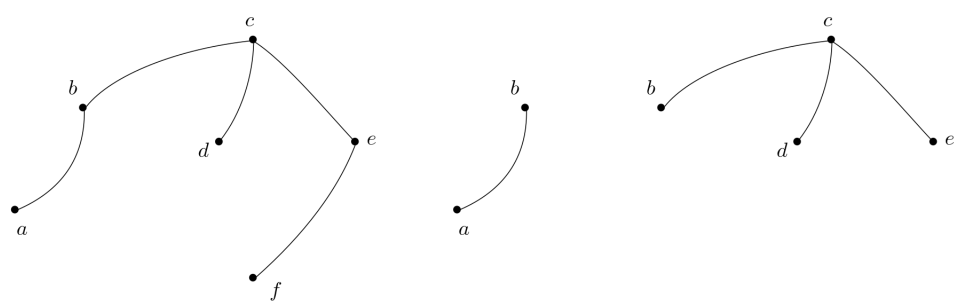
G $\qquad$ G_1 $\qquad$ G_2

G_1, G_2 are subgraphs of G and G_2 is a spanning subgraph of G.

2.3 Induced subgraph

Let $G = (V, E)$ and $V' \subseteq V$ with $V' \neq \varnothing$. The subgraph *induced* by V' is the subgraph whose vertex set is V' and whose edge set contains all of the edges of E between elements of V'. We denote this subgraph by $\langle v' \rangle$.

2.3.1 Example



G $\langle\{a, b\}\rangle$ $\langle\{b, c, d, e\}\rangle$

2.4 Deleted vertex or edge

Let $G = (V, E)$ be a graph and let $w \in V$. Then

$$G - w = (V', E')$$

where

$$V' = V \setminus \{w\}$$

and E' contains all the edges in E except those incident to w.

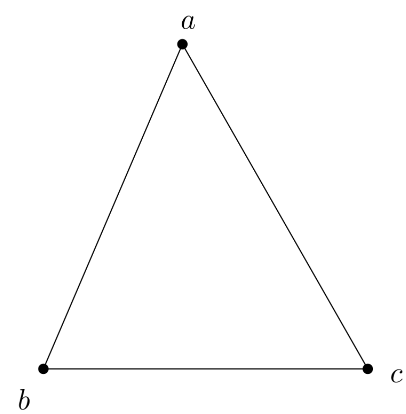
2.5 Complete graph

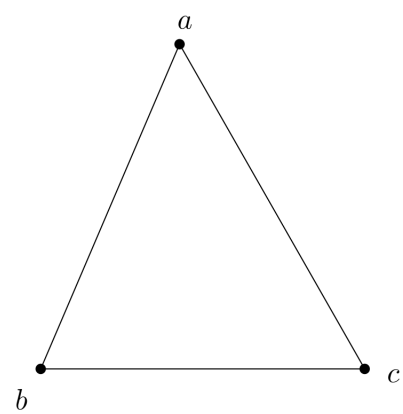
Let V be a set of $n \geq 1$ vertices. The *complete graph* on n vertices is denoted K_n and has V as vertices and all edges $\{a, b\}$ for $a, b \in V$.

2.6 Complement

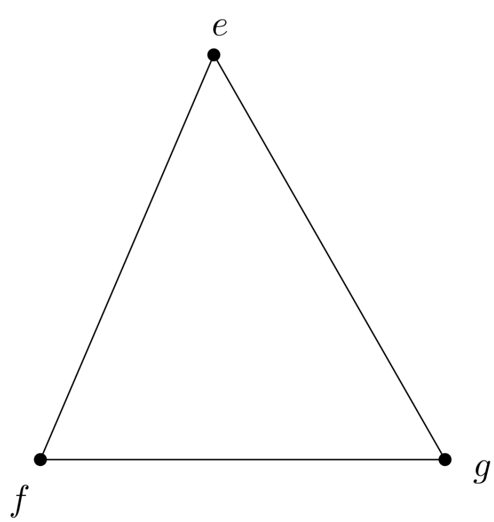
Let G be a graph with $G = (V, E)$. The *complement* of G, written $\overline{G}$, has the same vertices as G but all of the edges not in G.

2.7 Comparing graphs





are two ways of representing the same graph (different embeddings in the plane). However



is a different graph. But they are *isomorphic*.

2.8 Homomorphism and isomorphism

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. A function

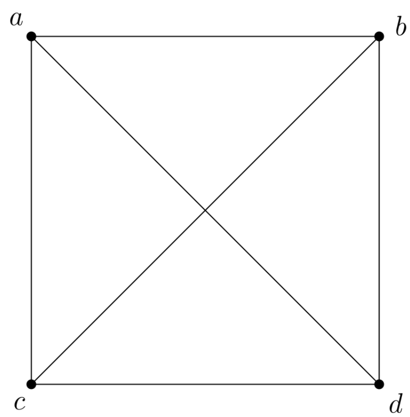
$$f : V_1 \to V_2$$

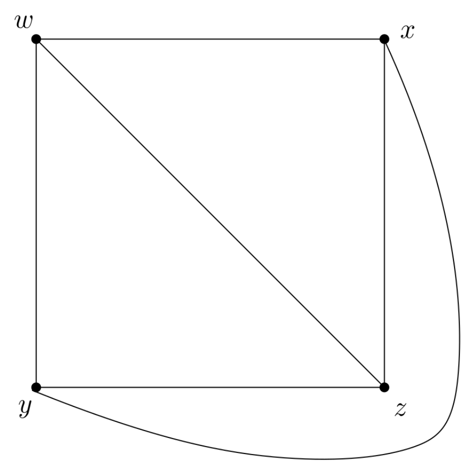
which satisfies:

$$a \text{ and } b \text{ adjacent in } G_1 \text{ implies } f(a) \text{ and } f(b) \text{ adjacent in } G_2$$

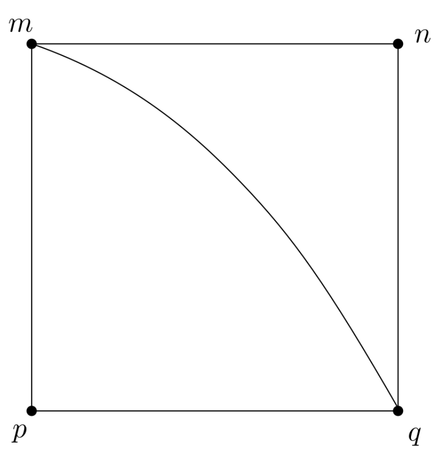
is called a graph *homomorphism*. If f is also a bijection, then f is a graph *isomorphism*. When there is an isomorphism between G_1 and G_2, we say that G_1 and G_2 are *isomorphic*.

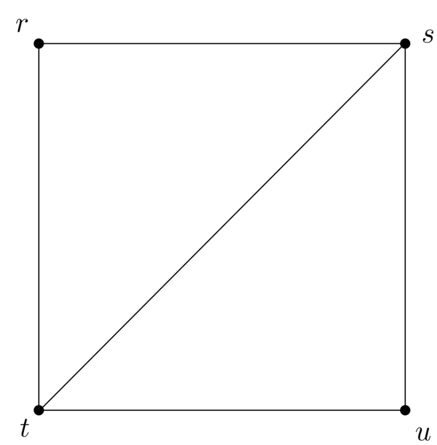
2.8.1 Examples of isomorphism





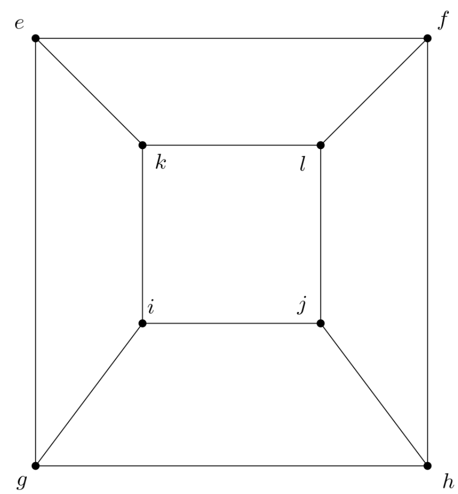
are isomorphic since $a \to w, b \to x, c \to y, d \to z$ is an isomorphism. Here, any bijection $\{a, b, c, d\} \to \{w, x, y, z\}$ would do.





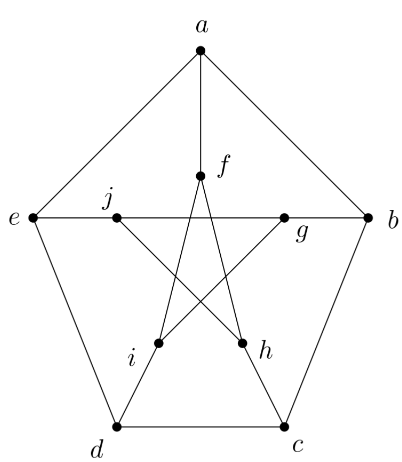
are isomorphic since $m \to s, q \to t, n \to r, p \to u$ is an isomorphism.

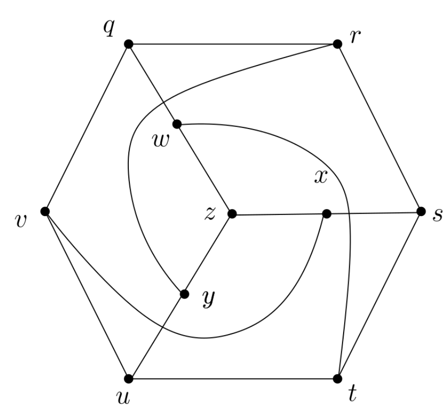
An example of a graph homomorphism that is not an isomorphism:



The function $a \to k, b \to l, c \to i, d \to j$ is such an example.

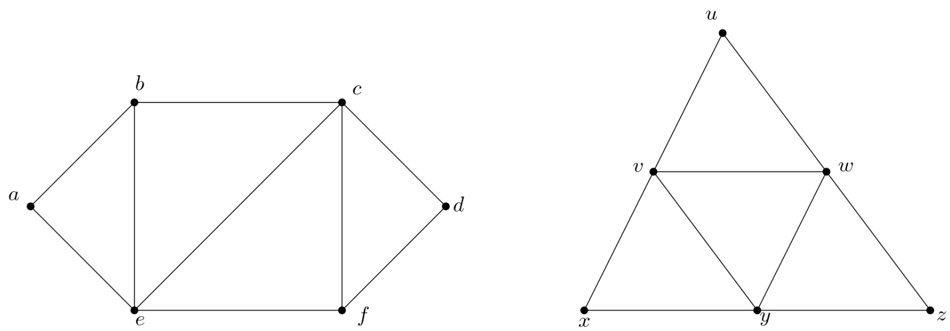
Are the graphs below isomorphic?





Yes, for example: $a \to q, c \to u, e \to r, g \to x, i \to z, b \to v, f \to w, u \to t, j \to s$.

What about these graphs?



Nope. In the left graph, a and d are adjacent to 2 vertices and all other vertices are adjacent to 4 vertices. This is not the case in the right graph.

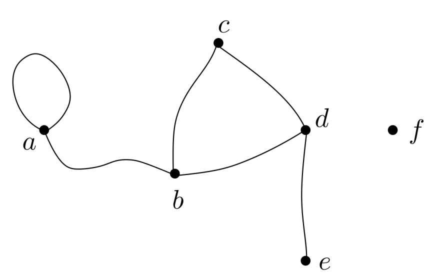
Remark: There currently is no method to decide efficiently if two graphs are isomorphic.

3 11.3 Vertex Degree: Euler Trails and Circuits

3.1 Degree

In an undirected graph (multigraph), the *degree* of a vertex v, written $\deg(v)$, is the number of edges incident to v. (A loop is considered as 2 edges incident to v).

Example:



We have $\deg(a) = \deg(b) = \deg(d) = 3, \deg(c) = 2, \deg(e) = 1, \deg(f) = 0$. Since e has degree 1, it is called a *pendant* vertex.

3.2 Sum of degrees

If $G = (V, E)$ is an undirected graph (or multigraph), then

$$\sum_{v \in V} \deg(v) = 2 \cdot |E|.$$

Proof. By induction.

- $|E| = 0$ then the equality holds.
- $|E| = 1$ then we have

  $$* \cdots * \,(\deg 0) \quad * - * \,(\deg 1)$$

  so $\sum \deg(v) = 2 = 2|E|$ and the equality holds.
- $|E| = n + 1$ then $E = \{\{a, b\}\} \cup E'$. Consider $G' = (V, E')$. By the induction hypothesis:

  $$\sum_{v \in V} \deg(v) \in G' = 2|E'|.$$

  Then

  $$\begin{aligned} \sum_{v \in V} \deg(v) &= \sum_{v \in V} \deg(v) + 2 \\ &= 2|E'| + 2 \\ &= 2|E|. \end{aligned}$$

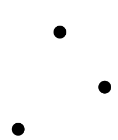
□

Corollary: The number of vertices of odd degree is even.

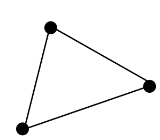
Proof. By considering the equality in the previous theorem modulo 2. □

3.3 k-regular

A graph is k-regular if all of its vertices have degree k.



is 0-regular



is 2-regular

Claim: There is no 4-regular graph with 15 edges. Indeed, if a graph is 4-regular then

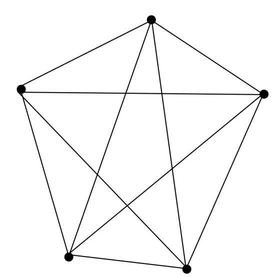
$$\sum_{v \in V} \deg(v) = 4|V|$$

and if there are 15 edges then

$$2|E| = 30$$

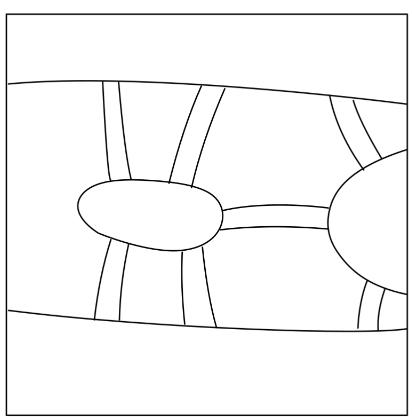
but $4|V| = 30$ has no integer solution.

Another example: 4-regular graph with 10 edges exist:

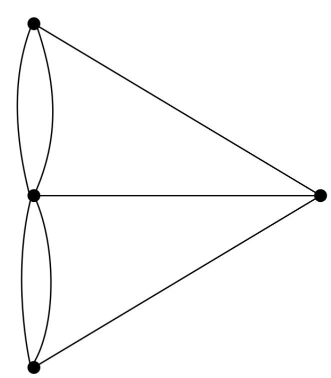


3.4 The seven bridges of Konigsberg

A map of Konigsberg:



$\Leftrightarrow$



Can you walk along all the bridges, using each bridge only once?

3.5 Euler circuit and trail

Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G has an *Euler circuit* if there is a circuit in G that traverses every edge. An *Euler trail* is an open trail that traverses every edge.

3.6 Euler circuit and trail condition

Let G be an undirected graph or multigraph with no isolated vertices. Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

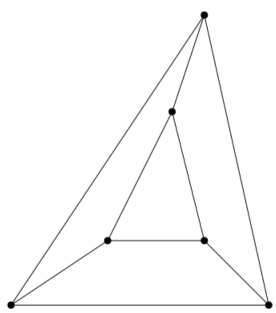
Proof. $\Rightarrow$ If G has an Euler circuit then it is connected. Moreover, every time the circuit reaches a node it must leave that node along another edge. Hence, the degree is even. $\square$

Corollary: If G is an undirected graph or multigraph, then G has an Euler trail if and only if G is connected and exactly two vertices have odd degree.
Remark: The Konigsberg graph has 4 nodes of odd degree.

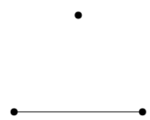
4 11.4 Planar Graphs

4.1 Definition of planar

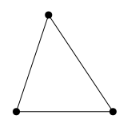
A graph G is *planar* if G can be drawn in the plane with its edges intersecting only at vertices.
(Recall: a drawing of G in the plane is an embedding.) A graph is planar if it has a planar embedding.



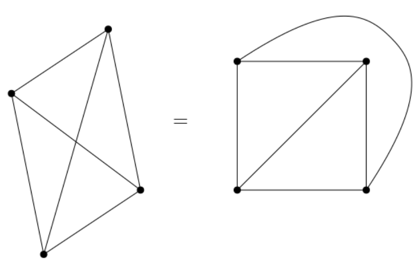
A 3-regular planar graph



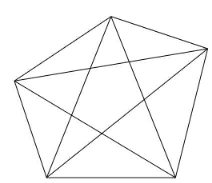
K_1 and K_2 are planar graphs



K_3 is a planar graph



K_4 is a planar graph

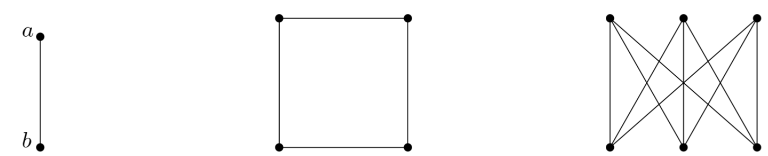


K_5 is not a planar graph (no planar embedding)

4.2 Bipartite

A graph $G = (V, E)$ is *bipartite* if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \varnothing$ and every edge in G is of the form $\{a, b\}$ where $a \in V_1$ and $b \in V_2$. If every vertex in V_1 is adjacent to every vertex in V_2 then the graph is a *complete bipartite* graph (denoted $K_{|V_1|,|V_2|}$).

4.2.1 Examples



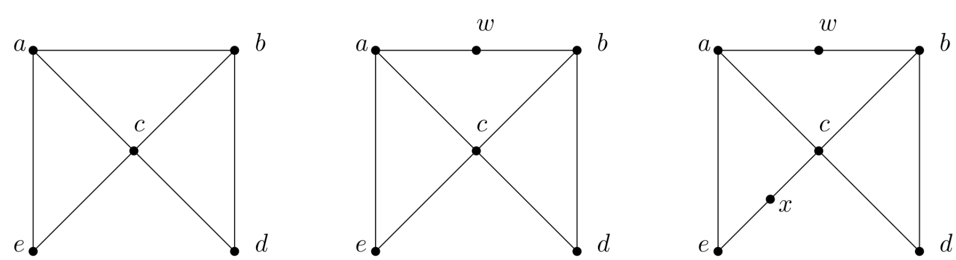
$K_{1,1}$ is bipartite and planar $K_{2,2}$ is bipartite and planar $K_{3,3}$ is bipartite but not planar

We know that K_5 and $K_{3,3}$ are not planar. As a result, if a graph G has K_5 or $K_{3,3}$ as a subgraph, then G is not planar.

4.3 Elementary subdivision

Let $G = (V, E)$ be loop-free and undirected with $E \neq \varnothing$. An *elementary subdivision* of G is obtained by removing an edge $\{a, b\}$ from E and adding the edges $\{a, c\}$ and $\{c, b\}$ to E and adding c in V (where $c \notin V$).

4.3.1 Example



G G_1 G_2

Here, G_1 is obtained through an elementary subdivision of G. Similarly, G_2 is obtained from G_1 (and from G as well).

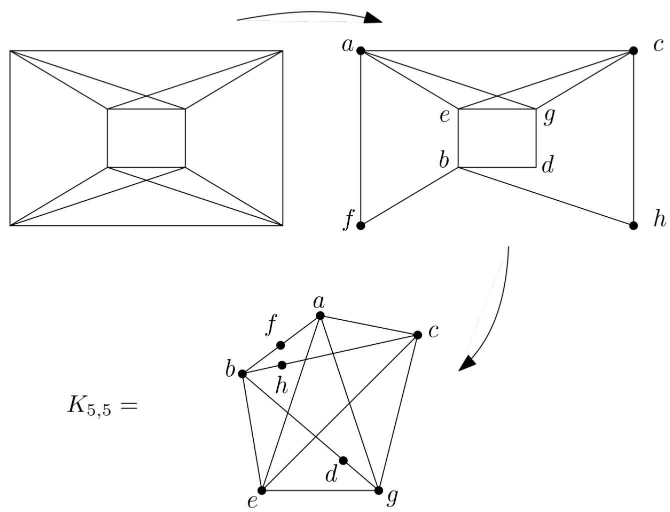
Remark: If G' is obtained from G by a single elementary subdivision then

$$|V'| = |V| + 1 \quad \text{and} \quad |E'| = |E| + 1.$$

Definition: If G and G' are two graphs such that G and G' can be obtained from the same graph through a sequence of elementary subdivisions.

4.4 Nonplanar homeomorphic property

A graph is nonplanar if and only if it contains a subgraph that is homeomorphic to K_5 or $K_{3,3}$.

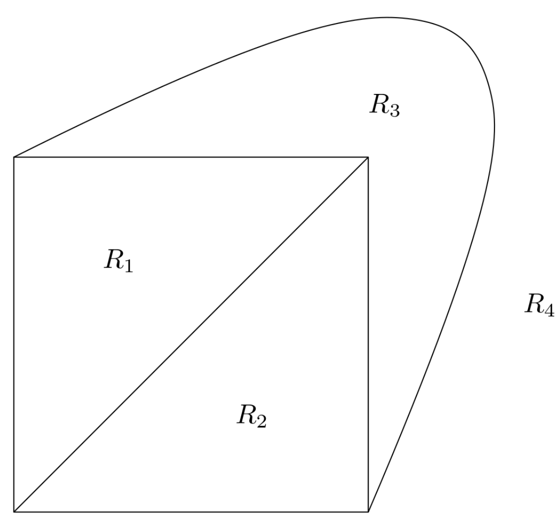


Hence, the graph above is not planar.

4.5 Regions

We can count the number of regions in the plane determined by a (planar embedding) of a planar graph.

$K_4 =$



Theorem: Let $G = (V, E)$ be a planar graph. Let r be the number of regions in the plane determined by G. Then

$$|V| - |E| + r = 2.$$

Corollary: For a connected loop-free planar graph with more than 2 edges, we have

$$3r \leq 2e \quad \text{and} \quad e \leq 3v - 6.$$

Example: K_5 has 10 edges and 5 vertices, so

$$3|V| - 6 = 15 - 6 = 9 \leq 10 = e.$$

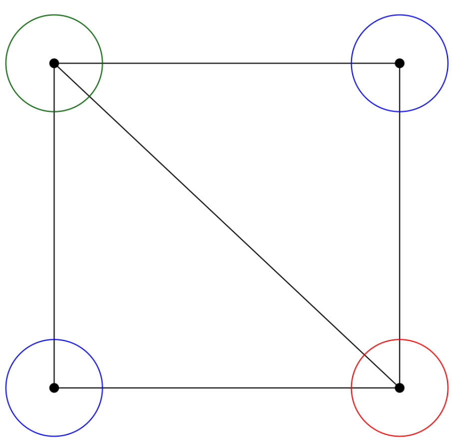
Hence, K_5 is not planar.

5 11.6 Graph Coloring and Chromatic Polynomials

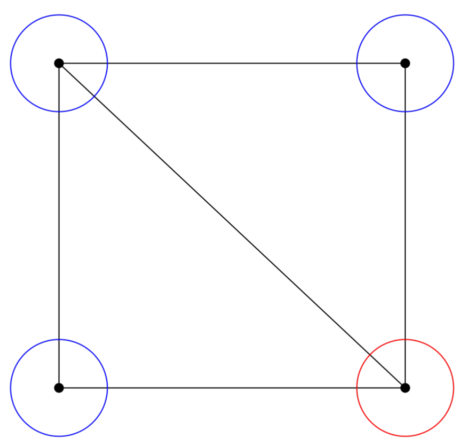
5.1 Proper colouring

If $G = (V, E)$ is an undirected graph, a *proper colouring* of G occurs when we color the vertices of G so that if $\{a, b\}$ is an edge in G, then a and b are coloured with different colors. (Hence adjacent vertices have different colors.)

5.1.1 Example:



proper



non-proper

Remark: We can think of a colouring as a function

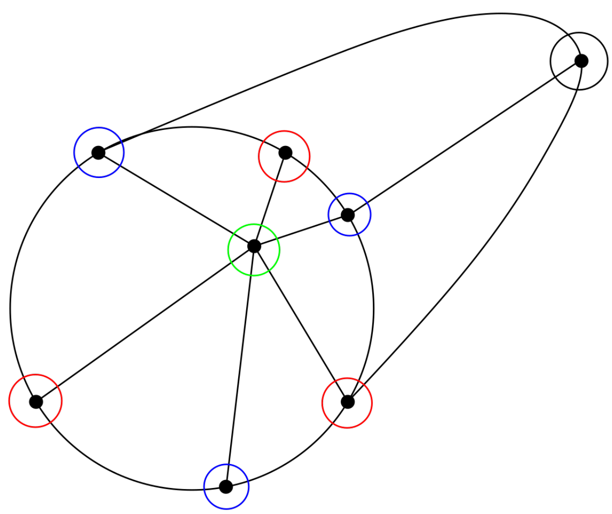
$$c : V \to \text{Colours}$$

where Colours is a set of colours.

5.2 Chromatic Number

The minimum number of colors needed to properly colour G is called the *chromatic number* of G and is written $\chi(G)$.

5.2.1 Example

$G =$ 

We have $\chi(G) = 3$.

Another example is: $\chi(K_n) = n$.

5.3 Chromatic Polynomial

Let G be a graph and let λ be the number of available colours. We write

$$P(G, \lambda)$$

for the number of ways that we could properly colour G using λ colours.
Note:

$$a \to b \qquad a \to b$$

are different colourings.

5.4 Examples

- Let $G = (\{a_1, \dots, a_n\}, \varnothing)$. So, we have for G:

  $$a_1 \quad a_2 \quad \dots \quad a_n$$

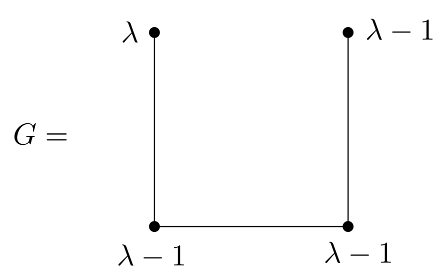
  What is $P(G, \lambda)$? We have λ options for a_1, λ options for a_2, ... Hence, $P(G, \lambda) = \lambda^n$.

- What is $P(K_n, \lambda)$? For the 1st vertex: λ and for the 2nd: $\lambda - 1$. So,

  $$P(K_n, \lambda) = \lambda \cdot (\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1) = \lambda^{(n)}.$$

  Note that if $\lambda < n$ then
  $$P(G, \lambda) = 0.$$
  Note also that $\lambda = n$ is the smallest integer for which
  $$P(G, \lambda) > 0$$
  which is $\chi(G)$.

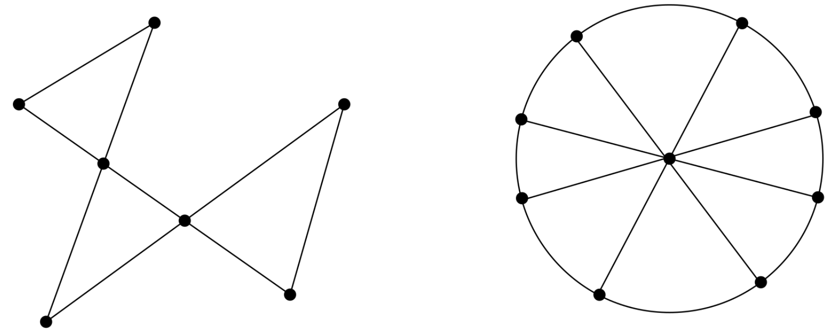
- For a graph G, we have

  which is $P(G, \lambda) = \lambda(\lambda - 1)^3$. More generally, for a path G on n vertices, we have
  $$P(G, \lambda) = \lambda(\lambda - 1)^{n-1}.$$

Remark:

G :



G_1 $\qquad$ G_2

We have $P(G, \lambda) = P(G_1, \lambda) \cdot P(G_2, \lambda)$.

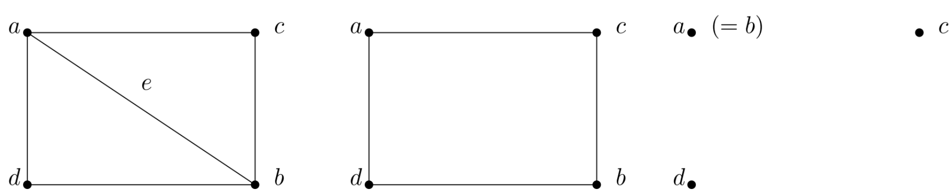
5.5 Coalescing vertices

Let $G = (V, E)$ be an undirected graph. For $e = \{a, b\} \in E$, let G_e denote the subgraph of G obtained by deleting e from G, without removing vertices a and b; that is,

$$G_e = G - e.$$

From G_e, a second subgraph of G is obtained by coalescing (or, identifying) the vertices a and b. This second subgraph is denoted by $G'e$.

5.5.1 Example



G $\qquad$ G_e $\qquad$ G'_e

5.6 Decomposition Theorem for Chromatic Polynomials

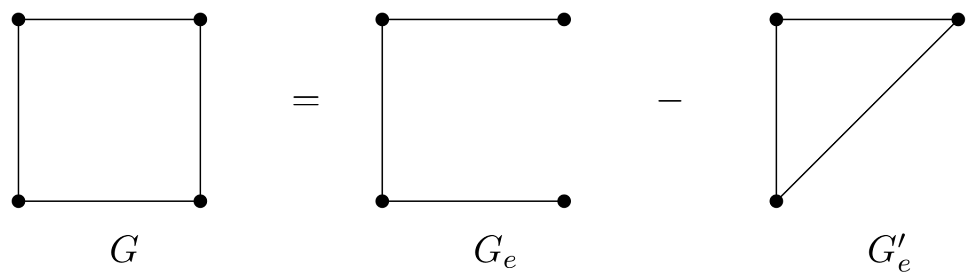
If $G = (V, E)$ is a connected graph and $e \in E$, then

$$P(G_e, \lambda) = P(G, \lambda) + P(G'_e, \lambda).$$

Proof. Let $e = \{a, b\}$. Consider a proper colouring of G_e. Either a and b have the same colour or not. In the first case we have a colouring of G'_e and in the second case a colouring of G. hence, a colouring of G_e is either a colouring of G or a colouring of G'_e. □

5.6.1 Examples

Write $[G]$ for the chromatic polynomial fo G.

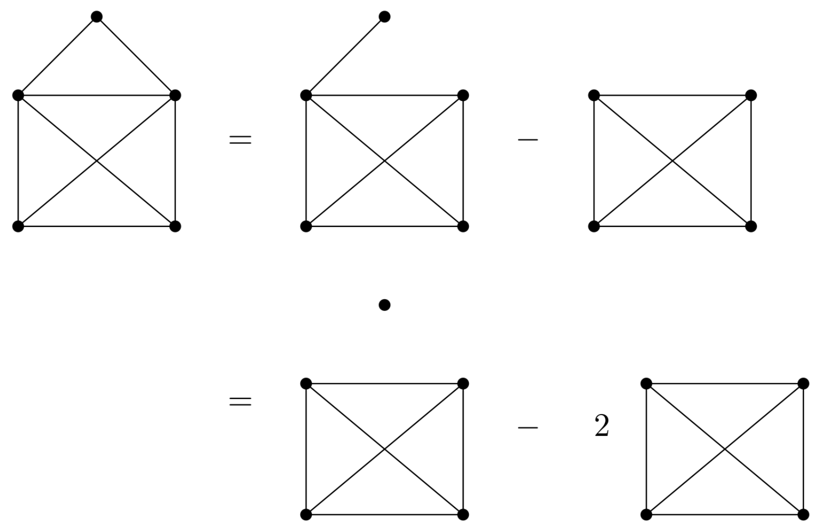


We have

$$\begin{aligned} G &= G_e - G'_e \\ &= P(G_e, \lambda) - P(G'_e, \lambda) \\ &= \lambda(\lambda-1)^3 - \lambda^{(3)} \\ &= \lambda^4 - 4\lambda^3 + 6\lambda^2 - 3\lambda. \end{aligned}$$

Then, we have $P(G, \lambda) > 0$ if and only if $\lambda \geq 2$ so $\chi(G) = 2$.

Another example:



Then we have

$$= \lambda(\lambda^{(4)}) - 2(\lambda^{(4)}) = \lambda(\lambda - 1)(\lambda - 2)^2(\lambda - 3)$$

5.7 Constant term

For every graph G, the constant term of $P(G, \lambda)$ is 0.

Proof. Evaluating $P(G, \lambda)$ at 0 gives the constant term of the polynomial. But there is no way to show a graph with 0 colours, so this term must be 0. $\square$

5.8 Sum of the coefficients

If $G = (V, E)$ and $|E| > 0$ then the sum of the coefficients in $P(G, \lambda)$ is 0.

Proof. Evaluating $P(G, \lambda)$ at 1 yields the sum of the coefficients. But G has at least one edge so 2 or more colours are needed to colour G. Hence, $P(G, \lambda)$ must be 0 when evaluated at 1. $\square$

5.9 Addition of an edge

Let $G = (V, E)$ with $a, b \in V$ and $\{a, b\} \notin E$. Adding the edge $\{a, b\}$ gives G_e^+ and identifying a and b in G gives G_e^{++}. Then

$$P(G, \lambda) = P(G_e^+, \lambda) + P(G_e^{++}, \lambda).$$

5.10 Union and intersection

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then

- the *union* of G_1 and G_2 is the graph $G_1 \cup G_2 = (V, E)$ where $V = V_1 \cup V_2$, $E = E_1 \cup E_2$.
- (when $V_1 \cap V_2 \neq \varnothing$) the *intersection* of V_1 and V_2 is the graph $G_1 \cap G_2 = (V, E)$ where $V = V_1 \cap V_2$, $E = E_1 \cap E_2$.

5.11 Ways to properly color K_n

Let G be an undirected graph with subgroups G_1, G_2. If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = K_n$ for some n then

$$P(G, \lambda) = \frac{P(G_1, \lambda) \cdot P(G_2, \lambda)}{\lambda^{(n)}}$$