CSCI/MATH 2113 Discrete Structures
5.4 Special Functions

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Contents

1 Binary operations

1.1 Definition

For any nonempty sets A, B, any function $f : A \times A \to B$ is called *binary operation* on A. If $B \subseteq A$, then the binrary operation is said to be *closed* (on A). (When $B \subseteq A$ we may also say that A is *closed under f*.)

Remark. Similarly, $f : A^n \to B$ is an *n-ary* operation on A. When $n = 1$, the operation is *unary* or *monary*.

1.2 Examples of Binary Operations

- For $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $f(a, b) = a - b$, it is a closed binary operation on $\mathbb{Z}$.
- For $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by $g(a, b) = a^b$, it is a non-closed binary operation.
- For $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ defined by $h(a, b) = a + b$, it is a binary operation.
- For $j : \mathcal{P}(A) \times \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $j(S, T) = S \cup T$, it is a closed binary operation.
- For $k : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by $k(S) = S^c$, it is a closed unary operation.

1.3 Commutativity and Associativity

Let $f : A \times A \to B$; that is, f is a binary operation on A.

(a) f is said to be *commutative* if $f(a, b) = f(b, a)$ for all $(a, b) \in A \times A$.

(b) When $B \subseteq A$ (that is, when f is closed), f is said tot be *associative* if for all $a, b, c \in A, f(f(a, b), c) = f(a, f(b, c))$.

1.4 Examples of Commutativity and Associativity

Binary operations that are both commutative and associative:

- $+$ on $\mathbb{Z}$: $n + m = m + n$ and $n + (m + r) = (n + m) + r$
- $\times$ on $\mathbb{Z}$
- $\cup$ on $\mathcal{P}(A)$

Binary operations that are associative but not commutative:

- $\times$ on $Mat_{n\times n}(\mathbb{R})$ which is the multiplication of $n \times n$ real matrices.

Binary operations that are both not commutative and associative:

- $-$ on $\mathbb{Z}$:

$$2 - 3 = -1 \neq 1 = 3 - 2$$

$$((3 - 3) - 2) = -2 \neq 2 = 3 - (3 - 2)$$

1.5 Symmetry

Suppose that $f : A \times A \to A$ is a binary operation where $A = \{a_1, \ldots, a_n\}$. We can represent f using a *table*.

f	a_1	a_2	$\ldots$	a_n
a_1	$f(a_1, a_1)$	$f(a_1, a_2)$		
a_2	$f(a_2, a_1)$			
$\vdots$				
a_n		$f(a_n, a_2)$		$f(a_n, a_n)$

If the operation is commutative, then the table is *symmetric*.
Now let $f : \{a, b, c\} \times \{a, b, c\} \to \{a, b, c\}$ be defined by the table:

f	a	b	c
a	b	a	a
b	a	c	a
c	a	a	c

Here we have

$$f(a, f(b, c)) = f(a, a) = b \neq a = f(a, c) = f(f(a, b), c)$$

so the operation is *not associative* but it is commutative since the table is symmetric.

2 Identity Element

2.1 Definition

Let $f : A \times A \to B$ be a binary operation on A. An element $x \in A$ is called an *identity* (or *identity element*) for f if $f(a, x) = f(x, a) = a$, for all $a \in A$.

2.2 Examples

- 0 for $+$ on $\mathbb{Z}$ since
$$a + 0 = 0 + a = a$$
for all $a \in \mathbb{Z}$.
- I_n (identity matrix) of x on $Mat_{n \times n}(\mathbb{R})$.
- $\varnothing$ for $\cup$ on $\mathcal{P}(A)$.
- A for $\cap$ on $\mathcal{P}(A)$.

2.3 Theorem

Let $f : A \times A \to B$ be a binary operation. If f has an identity, then that identity is unique.

Proof. If f has more than one identity, let $x_1, x_2 \in A$ with

$$\begin{aligned} f(a, x_1) &= a = f(x_1, a), &\quad& \text{for all } a \in A, \\ f(a, x_2) &= a = f(x_2, a), && \text{for all } a \in A. \end{aligned}$$

Consider x_1 as an element of A and x_2 as an identity. Then $f(x_1, x_2) = x_1$. Now reverse the roles of x_1 and x_2, that is, consider x_2 as an element of A and x_1 as an identity. We find that $f(x_1, x_2) = x_2$. Consequently, $x_1 = x_2$, and f has at most one identity. □

3 Projections

For sets A and B, if $D \subseteq A \times B$, then $\pi_A : D \to A$, defined by $\pi_A(a, b) = a$, is called the *projection* on the first coordinate. The function $\pi_B : D \to B$, defined by $\pi_B(a, b) = b$, is called the *projection* on the second coordinate.

4 Counting Binary Operations

- For the set $A = \{a, b, c, d\}$, how many closed binary operations are there on A?

  A binary operation is a function $A \times A \to A$. Hence this number is
  $$|A|^{|A| \times |A|} = 4^{16}.$$
  In other words, we need to fill the table below.

	a	b	c	d
a				
b				
c				
d				

  There are 4 choices for each cell.

- How many of these operations are commutative?

  Commutative operations correspond to symmetric tables.

	a	b	c	d
a				
b	x			
c	x	x		
d	x	x	x	

  Since only 10 cells need to be filled, there are
  $$4^{10}$$
  binary operations.

- How many of these operations have a as an identity?

	a	b	c	d
a	a	b	c	d
b	b			
c	c	x		
d	d	x	x	

  Since a is the identity, we have $f(d, a) = d$ for every $d \in A$. In total, there are 4^6 such operations since there are 6 cells to fill.