CSCI/MATH 2113 Discrete Structures

5.6 Function Composition and Inverse Functions

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Contents

1 Bijective functions

If $f : A \to B$, then f is said to be *bijective*, or to be a *one-to-one corespondence*, if f is both one-to-one and onto.

2 Identity function

The function $1_A : A \to A$, defined by $1_A(a) = a$ for all $a \in A$, is called the *identity function*.

3 Equality of functions

If $f, g : A \to B$, we say that f and g are *equal* and write $f = g$, if $f(a) = g(a)$ for all $a \in A$.
A common pitfall in dealing with the equality of functions occurs when f and g are functions with a common domain A and $f(a) = g(a)$ for all $a \in A$. It may *not* be the case that $f = g$. The pitfall results from not paying attention to the codomains of the functions.

3.1 Example

Let $f : \mathbb{Z} \to \mathbb{Z}, g : \mathbb{Z} \to \mathbb{Q}$ where $f(x) = x = g(x)$, for all $x \in \mathbb{Z}$. Then, f, g share the common domain $\mathbb{Z}$, have the same range $\mathbb{Z}$, and act the same on every element of $\mathbb{Z}$. Yet $f \neq g$ because f is injective and g is injective but surjective; so the codomains do not make a difference.

4 Composite functions

If $f : A \to B$ and $g : B \to C$, we define the *composite function*, which is denoted $g \circ f : A \to C$, by $(g \circ f)(a) = g(f(a))$, for each $a \in A$. f and g are composable. However, if $C \neq A$ then $f \circ g$ is not defined.

The definition and examples for composite functions required that the codomain of f = domain of g. If range of $f \subseteq g$, this will actually be enough to yield the composite function $g \circ f : A \to C$. Also, for any $f : A \to B$, we observe that $f \circ 1_A = f = 1_B \circ f$.

4.1 Theorem

Let $f : A \to B$ and $g : B \to C$.

(a) If f and g are one-to-one, then $g \circ f$ is one-to-one.

(b) If f and g are onto, then $g \circ f$ is onto.

Proof. Let us prove the following theorem above.

(a) Let $a_1, a_2 \in A$ with $(g \circ f)(a_1) = (g \circ f)(a_2)$. Then

$$(g \circ f)(a_1) = (g \circ f)(a_2) \Rightarrow g(f(a_1)) = g(f(a_2)) \Rightarrow f(a_1) = f(a_2)$$

since g is one-to-one. Also, $a_1 = a_2$ because f is one-to-one. Consequently, $g \circ f$ is one-to-one.

(b) Let $z \in C$. Since g is onto, there exists $y \in B$ with $g(y) = z$. With f onto and $y \in B$, there exists $x \in A$ with $f(x) = y$. Hence, $z = g(y) = g(f(x)) = (g \circ f)(x)$, so the range of $g \circ f = C =$ the codomain of $g \circ f$, and $g \circ f$ is onto.

□

Function composition is not commutative, but it is associative.

4.2 Collection of functions

If A is a set then

$$A^A = \{f \mid f : A \to A\}$$

is the collection of functions $A \to A$. So the function composition is a binary operation on A^A.

4.3 Theorem

If $f : A \to B, g : B \to C$, and $h : C \to D$, then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof. We have

$$(h \circ g) \circ f(x) = h(g(f(x)))$$

and

$$h \circ (g \circ f)(x) = h(g(f(x))).$$

Therefore, we have $(h \circ g) \circ f = h \circ (g \circ f)$. □

4.4 Powers of functions

If $f : A \to A$, we define $f^1 = 1$, and for $n \in \mathbb{Z}^+$, $f^{n+1} = f \circ f(^n)$.

This definition is another example wherein the result is defined *recursively*. With $f^{n+1} = f \circ (f^n)$, we see the dependence of f^{n+1} on a previous power, namely, f^n.

5 Invertible functions

5.1 Converse of a relation

For sets A, B, if R is a relation from A to B, then the *converse* of R, denoted R^c, is the relation from B to A defined by

$$R^c = \{(b, a) \mid (a, b) \in R\}.$$

We simply interchange the components of each ordered pair in R.

5.2 Invertible function

If $f : A \to B$, then f is said to be *invertible* if there is a function $g : B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

5.3 Uniqueness

If a function $f : A \to B$ is invertible and a function $g : B \to A$ satisfies $g \circ f = 1_A$ and $f \circ g = 1_B$, then this function g is unique.

Proof. If g is not unique, then there is another function $h : B \to A$ with $h \circ f = 1_A$ and $f \circ h = 1_B$. Consequently,

$$h = h \circ 1_B = h \circ (f \circ g) = (h \circ f) \circ g = 1_A \circ g = g.$$

□

As a result of this theorem, we shall call the function g the inverse of f and shall adopt the notation $g = f^{-1}$. Note that $f^{-1} = f^c$ and $(f^{-1})^{-1} = f$.

5.4 Theorem

$f : A \to B$ is invertible if and only if f is bijective.

Proof. Assuming that f is invertible, we have a unique function $g : B \to A$ with $g \circ f = 1_A, f \circ g = 1_B$. If $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$, then $g(f(a_1)) = g(f(a_2))$. It follows that $a_1 = a_2$, so f is one-to-one. For the onto property, let $b \in B$, then $g(b) \in A$. We have $b = 1_B(b) = (f \circ g)(b) = f(g(b))$, so f is onto.

For the other direction, suppose $f : A \to B$ is bijective. Since f is onto, for each $b \in B$, there is an $a \in A$ with $f(a) = b$. Consequently, we define the function $g : B \to A$ by $g(b) = a$, where $f(a) = b$. Our definition of g such that $g \circ f = 1_A$ and $f \circ g = 1_B$, so we find that f is invertible, with $g = f^{-1}$. $\square$

5.5 Theorem

If $f : A \to B, g : B \to C$ are invertible functions, then $g \circ f : A \to C$ is invertible and

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

6 Inverse image

If $f : A \to B$ and $B_1 \subseteq B$, then $f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$. The set $f^{-1}(B_1)$ is called the *preimage or inverse image* of B_1 under f.

Note. $f^{-1}(B_1)$ is defined even if f is not invertible.

6.1 Examples

- For $f : \mathbb{Z} \to \mathbb{Z}$, we have

$$f^{-1}[\{2\}] = \{2\}.$$

- For $f : \mathbb{Z} \to \mathbb{Z}$, we have

$$f^{-1}[\{0\}] = \{x \in \mathbb{Z} \mid f(x) \in \{0\}\} = \{x \in \mathbb{Z} \mid f(x) = 0\}$$

  and

$$f^{-1}[\{1, 2\}] = \varnothing.$$

- For $f : \mathbb{Z} \to \mathbb{Z}_2$, we have

$$f^{-1}[\{0\}] = 2\mathbb{Z} \qquad \text{even integers}$$

and

$$f^{-1}[\{1\}] = 2\mathbb{Z} + 1 \qquad \text{odd integers}$$

7 Theorem

If $f : A \to B$ and $B_1, B_2 \subseteq B$, then

(a) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$;

(b) $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$;

(c) $f^{-1}(\overline{B_1}) = \overline{f^{-1}(B_1)}$.

8 Finite sets

Let $f : A \to B$ for finite sets A and B, where $|A| = |B|$. Then the following statements are equivalence: (a) f is one-to-one; (b) f is onto; and (c) f is invertible.