MATH 2135 Linear Algebra

2.A Span and Linear Independence

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Contents

1 Linear Combinations and Span

1.1 List of vectors

A "list" is an ordered finite sequence of things. For example, $1, 2, 5, 2, 3$ is a list of numbers, and $1, 2, 2, 5, 3$ is a different list. There can be repetitions. We usually write lists without parentheses. With lists, we do not usually specify ahead of time how many entries the list has.

1.2 Linear combination

A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m vm,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

1.2.1 Example

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ because

$$(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4).$$

- $(17, -4, 5)$ is not a linear combination of $(2, 1, -3), (1, -2, 4)$ because there do not exist numbers $a_1, a_2 \in \mathbf{F}$ such that

$$(17, -4, 5) = a_1(2, 1, -3) + a_2(1, -2, 4).$$

1.3 Span

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the span of $v_1, \ldots, v_m$, denoted $\mathit{span}(v_1, \ldots, v_m)$. In other words,

$$\mathit{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \ldots, a_m \in \mathbf{F}\}.$$

The span of the empty list $()$ is defined to be $\{0\}$.

1.3.1 Example

From the previous examples, we have

- $(17, -4, 2) \in \mathit{span}((2, 1, -3), (1, -2, 4))$;
- $(17, -4, 5) \notin \mathit{span}((2, 1, -3), (1, -2, 4))$.

1.4 Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Remark. What this means is the following:

(1) $span(v_1, \ldots, v_m)$ is a subspace of V containing $v_1, \ldots, v_m$.

(2) If W is any subspace of V containing $v_1, \ldots, v_m$, then $span(v_1, \ldots, v_m) \subseteq W$.

Proof. Let us prove the two remarks above.

(1) To show that $span(v_1, \ldots, v_m)$ is a subspace, we must prove the three subspace properties.

   (a) $0 = 0v_1 + \cdots + 0v_m$, so we have $0 \in span(v_1, \ldots, v_m)$.

   (b) To show that it is closed under addition, let $v, w \in span(v_1, \ldots, v_m)$ be arbitrary. We must show that $v + w \in span(v_1, \ldots, v_m)$. By assumption,
   $$v = a_1v_1 + \cdots + a_mv_m$$
   for some $a_1, \ldots, a_m$, and
   $$w = b_1v_1 + \cdots + b_mv_m$$
   for some $b_1, \ldots, b_m$. Therefore,
   $$\begin{aligned} v + w &= (a_1v_1 + \cdots + a_mv_m) + (b_1v_1 + \ldots b_mv_m) \\ &= (a_1 + b_1)v_1 + \cdots + (a_m + b_m)v_m. \end{aligned}$$
   So the span is closed under addition.

   (c) To show that it is closed under scalar multiplication, assume $v \in span(v_1, \ldots, v_m)$ and $k \in F$. We need to show that $kv \in span(v_1, \ldots, v_m)$. There exists $a_1, \ldots, a_m$ such that $v = a_1v_1 + \cdots + a_mv_m$. Then
   $$\begin{aligned} kv &= k(a_1v_1 + \cdots + a_mv_m) \\ &= (ka_1)v_1 + \cdots + (ka_m)v_m. \end{aligned}$$
   Therefore, the span is closed under scalar multiplication.

Next, we show that $span(v_1, \ldots, v_m)$ contains $v_1, \ldots, v_m$. But

$$\begin{aligned} v_1 &= 1v_1 + 0v_2 + \cdots + 0v_m \\ v_2 &= 0v_1 + 1v_2 + \cdots + 0v_m \\ &\vdots \\ v_m &= 0v_1 + 0v_2 + \cdots + 1v_m. \end{aligned}$$

So each of $v_1, \ldots, v_m \in span(v_1, \ldots, v_m)$.

(2) We must show that $span(v_1, \ldots, v_m)$ is the smallest. So let W be any subspace of V containing $v_1, \ldots, v_m$. We must show $span(v_1, \ldots, v_m) \subseteq W$.

Take any arbitrary $v \in span(v_1, \ldots, v_m)$. We must show that $v \in W$. There exists $a_1, \ldots, a_m$ such that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Then, $v_1, \ldots, v_m \in W$. But W is a subspace, so it is closed under scalar multiplication.

$$\Rightarrow a_1 v_1, \ldots a_m v_m \in W.$$

Also, W is closed under addition, then

$$a_1 v_1 + \cdots + a_m v_m \in W$$
$$\Rightarrow v \in W.$$

Because $v \in span(v_1, \ldots, v_m)$ was arbitrary, it follows

$$span(v_1, \ldots, v_m) \subseteq W$$

as had to be shown.

□

1.5 Definition of spanning

If $span(v_1, \ldots, v_m)$ equals V, we say that $v_1, \ldots, v_m$ spans V.

1.6 Definition of finite-dimensional vector space

A vector space is called *finite-dimensional* if some list of vectors in it spans the space. In other words, there exist $k \leq 0$, $v_1, \ldots, v_k$ such that

$$span(v_1, \ldots, v_k) = V.$$

Otherwise, V is called *infinite-dimensional*.

1.6.1 Example

- Recall that $P_m(F)$ is the vector space of polynomials (with coefficients in F) of degree at most m.

$$P_m(F) = \{a_0 + a_1x + \cdots + a_mx^m \mid a_0, \cdots + a_m \in F\}.$$

  Then $P_m(F)$ is finite-dimensional. It is spanned by the following list of $m+1$ polynomials: $1, x, x^2, \ldots, x^m$.

- Let $P(F)$ be the set of all polynomials with coefficients in F, regardless of degree. Then $P(F)$ is an infinite-dimensional vector space.

  Proof. Assume, for the sake of obtaining a contradiction, that $P(F)$ is finite dimensional. Then $P(F)$ is spanned by a finite list of polynomials, say $P_1, \ldots, P_k \in P(F)$. Let m be the largest degree of any of $P_1, \ldots, P_k$.

  Then clearly, any linear combination

$$a_1P_1 + \cdots + a_kP_k$$

  also has degree at most m.

  So therefore, $x^{m+1} \in P(F)$ is not a linear combination of $P_1, \ldots, P_k$, contradicting the fact that $P_1, \ldots, P_k$ span $P(F)$. $\square$

2 Linear Independence

2.1 Definition

- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1v_1 + \cdots + a_mv_m$ equal 0 is $a_1 = \cdots = a_m = 0$.
- The empty list () is also declared to be linearly independent.

1. To prove that some $v_1, \ldots, v_m$ are linearly independent, you have to do the following:
    - Take arbitrary $a_1, \ldots, a_m$
    - Assume $a_1v_1 + \cdots + a_mv_m = 0$
    - Must show $a_1, \ldots, a_m = 0$
2. To prove that some $v_1, \ldots, v_m$ are linearly dependent, you have to do the following:
    - Find specific $a_1, \ldots, am$, not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$.

2.2 Proposition

Suppose $v_1, \ldots, v_m$ are linearly independent, and let $w \in span(v_1, \ldots, v_m)$. Then, there exists a unique list of scalars $a_1, \ldots, a_m$ such that

$$w = a_1 v_1 + \cdots + a_m v_m.$$

Proof. Existence. By definition of span, it exists.
Uniqueness. Suppose that w in two ways as a linear combination of $v_1, \ldots, v_m$:

$$w = a_1 v_1 + \ldots a_m v_m$$

$$w = b_1 v_1 + \cdots + b_m v_m.$$

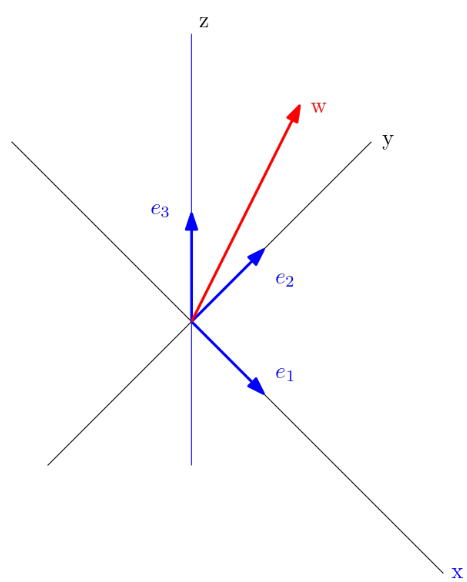
Then $(w - w) = (a_1 v_1 + \cdots + a_m v_m) - (b_1 v_1 + \cdots + b_m v_m)$. By vector space axioms,

$$0 = (a_1 - b_1) v_1 + \cdots + (a_m - b_m) v_m.$$

Because $v_1, \ldots, v_m$ are linearly independent, it follows that $a_1 - b_1 = 0, \ldots, a_m - b_m = 0$. Therefore, $a_1 = b_1, \ldots, a_m = b_m$. This proves uniqueness. □

2.3 Coordinates

If $v_1, \ldots, v_m$ are linearly independent and $w = a_1 v_1 + \cdots + a_m v_m$, we call $a_1, \ldots, a_m$ the *coordinates* of w (with respect to the list $v_1, \ldots, v_m$).



$$w = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = ae_1 + be_2 + ce_3$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

2.3.1 Example

In $P_3(\mathbb{R})$, let $P_1 = x^3 + 1, P_2 = x^3 - 2x, P_3 = 4x + 2$. Is P_1, P_2, P_3 linearly independent?
To answer the question, we attempt to rpove that they are linearly independent.

Proof. Take arbitrary $a_1, a_2, a_3 \in \mathbb{R}$. Assume $a_1p_1 + a_2p_2 + a_3p_3 = 0$. In other words,

$$a_1(x^3 + 1) + a_2(x^3 - 2x) + a_3(4x + 2) = 0 \qquad (1)$$

From (1), we can make a system of equations and turn it into a matrix.

$$\begin{aligned} a_1 + a_2 &= 0 \\ -2a_2 + 4a_3 &= 0 \\ a_1 + 2a_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $a_3 = t, a_2 = 2t, a_1 = -2t$. Since we failed to show that $a_1, a_2, a_3 = 0$, then P_1, P_2, P_3 are linearly dependent.
For $t = 1$, we get $a_1 = -2, a_2 = 2, a_3 = 1$, and we find that $-2P_1 + 2P_2 + 1P_3 = 0$. $\square$

2.4 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such taht the following hold:

(a) $v_j \in span(v_1, \ldots, v_{j-1})$;

(b) if the jth term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $span(v_1, \ldots, v_m)$.

Proof. By assumption, $v_1, \ldots, v_m$ are linearly dependent. Therefore, there exist scalars $a_1, \ldots, a_m \in F$ such that $a_1 v_1 + \cdots + a_m v_m = 0$ but not all of $a_1, \ldots, a_m$ are 0.
Let j be the largest index such that $a_j \neq 0$. Then $a_{j+1}, \ldots, a_m = 0$.
From our assumption, we have $a_1 v_1 + \cdots + a_j v_j = 0$. We can write

$$a_j v_j = -a_1 v_1 - a_2 v_2 - \cdots - a_{j-1} v_{j-1}.$$

But since $a_j \neq 0$, it has a multiplicative inverse.

$$v_j = -\frac{a_1}{a_j} v_1 - \frac{a_2}{a_j} v_2 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1}.$$

This proves (a).
For (b), note that

$$span(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \subseteq span(v_1, \ldots, v_m).$$

To show the opposite inclusion, consider some arbitrary $w \in span(v_1, \ldots, v_m)$. We must show $w \in span(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$. By assumption, $w = b_1 v_1 + \cdots + b_m v_m$ for such scalars $b_1, \ldots, b_m$. By combining this equation to (a), we get:

$$\begin{aligned} w &= b_1 v_1 + \cdots + b_{j-1} v_{j-1} + b_j v_j + b_{j+1} v_{j+1} + \cdots + b_m v_m \\ &= b_1 v_1 + \cdots + b_{j-1} v_{j-1} + bj \left(-\frac{a_1}{a_j} v_1 - \frac{a_2}{v_j} v_2 - \cdots - \frac{a_{j-1}}{a_j} v_{j-1} \right) \\ &\quad + b_{j+1} v_{j+1} + \cdots + b_m v_m. \end{aligned}$$

So $w \in span(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$ as required. $\square$

2.5 Replacement lemma

Let V be a finite-dimensional vector space. Consider two lists of vectors:

$$\begin{array}{ll} u_1, \ldots, u_m & \text{linearly independent in } V \\ w_1, \ldots, w_n & \text{spans } V \end{array}$$

Then we have

(a) $m \leq n$.

(b) It is possible to extend the list $u_1, \ldots, u_m$ with $n-m$ additional vectors from the list $w_1, \ldots, w_n$ such that the resulting list spans V.

Proof. We shall prove by induction.

(1) If $w_1, \ldots, w_n$ spans V, then $u_1, w_1, \ldots, w_n$ also spans V. Also, $u_1, w_1, \ldots, w_n$ is linearly dependent because $u_1 \in V = \mathit{span}(w_1, \ldots, w_n)$, therefore u_1 is a lienar combination of $w_1, \ldots, w_n$.

By the linear dependence lemma, one of $u_1, w_1, \ldots, w_n$ can be written as a linear combination of preceding vectors.

This cannot be u_1 (by linear independence), so some w_j can be written as a linear combination of preceding vectors.

By linear dependence lemma part (b),

$$\mathit{span}(u_1, w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n) = V.$$

(2) The list of n vectors from (1) spans V. Add u_2 to the list:

$$u_1, u_2, w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n.$$

Then this list is linearly dependent. One of the w's is redundant and can be removed, without changing the span.

(3) Add u_3 to the list and it becomes linearly dependent. We remove one of the w's in the list.

$\vdots$

(m) Add u_m to the list and it becomes linearly dependent. We remove one of the w's. Now we have a spanning list of n vectors of the form

$$u_1, \ldots, u_m,$$

followed by $n - m$ of the original w's.

Then $m \leq n$, because the n-vector list starts with $u_1, \ldots, u_m$. So we have proved (a) and (b).

□

2.6 Theorem

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof. Let V be finite-dimensinoal and let W be a subspace of V. we must show that W is finite dimensional.

(1) If $W = \{0\}$, then W is finite-dimensional and we are done. Otherwise, there exists at least one non-zero vector in W. Let $v_1 \in W$ be such a non-zero vector. Notice that the list of vectors "v_1" is linearly independent.

(2) If $W = span(v_1)$, then W is finite-dimensional and we are done. Otherwise, there exists at least one vector $v_2 \in W$ such that $v_2 \notin span(v_1)$. Notice that the list of vectors v_1, v_2 is linearly independent.

(3) If $W = span(v_1, v_2)$, then W is finite-dimensional and we are done. Otherwise, there exists at least one vector $v_3 \in W$ such that $v_3 \notin span(v_1, v_2)$. Notice that the list of vectors v_1, v_2, v_3 is linearly independent.

And so on.

(k) If $W = span(v_1, \ldots, v_{k-1})$, then W is finite-dimensional and we are done. Otherwise, there exists at least one vector $v_k \in W$ such that $v_k \notin span(v_1, \ldots, v_{k-1})$. Notice that the list of vectors $v_1, \ldots, v_k$ is linearly independent.

If W were infinite-dimensional, we could repeat this step any number of times, because the "If" part would never be done, so in each step, we'd be doing the "otherwise."
This contradicts the assumption that V is finite-dimensional.
Namely, since V is finite-dimensional, $V = span(w_1, \ldots, w_n)$ is spanned by some finite list of vectors.
If we repeat the above proceduce $n+1$ times, we get $n+1$ linearly independent vectors $v_1, \ldots, v_{n+1}$.
This contradicts the replacement lemma. Therefore, W is finite-dimensional.

□