CSCI/MATH 2113 Discrete Structures

Chapter 12 Trees

Alyssa Motas

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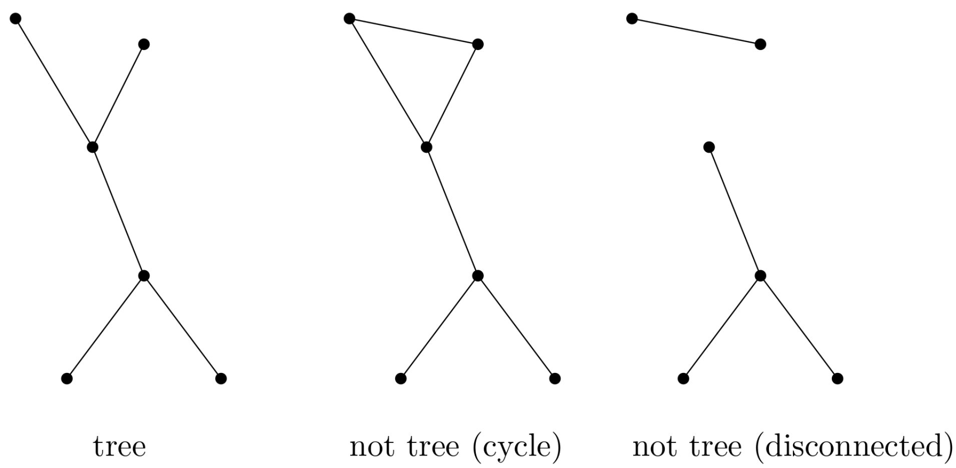
Contents

1 12.1 Definitions, Properties, and Examples

1.1 Tree

Let $G = (V, E)$ be a loop-free undirected graph. The graph G is called a *tree* if G is connected and contains no cycles.
Remark: Often simply a connected cycle-free graph.

1.1.1 Example



A collection of disconnected trees is a *forest*.

1.2 Spanning tree

A *spanning tree* for a graph G is a spanning subgraph of G that is a tree.

1.3 Unique path between two distinct vertices in a tree

If a, b are distinct vertices in a tree $T = (V, E)$, then there is a unique path that connects these vertices.

Proof. There is a path since T is connected. There is at most one path since otherwise T would contain a cycle. $\square$

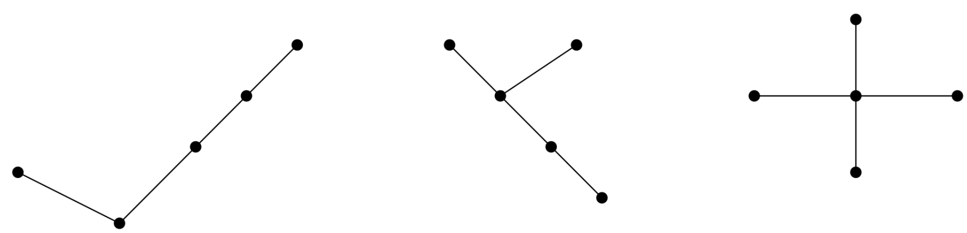
1.4 Condition of a connected graph

If $G = (V, E)$ is an undirected graph, then G is connected if and only if G has a spanning tree.

Proof. $\Leftarrow$ If G has a spanning tree, then there is a path between any two vertices of G along a subgraph of G (and thus in G).
$\Rightarrow$ Assume that G is connected. Consider the subgraph containing no loops. If G (without loops) is a tree then we are done. Ohterwise, G has a cycle and we can consider an edge e_1 in this cycle. Then consider the subgraph $G_1 = G - e_1$. If G_1 is a tree then we are done. Otherwise, we repeat this process. $\square$

1.5 Non-isomorphic trees

Three trees with 5 vertices:

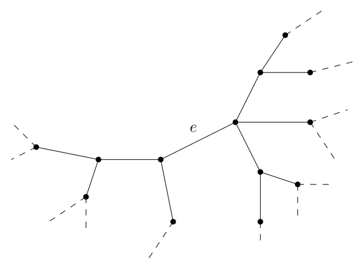


None of these graphs are isomorphic (e.g., the third graph has a vertex of degree 4 whereas none of the others do.)

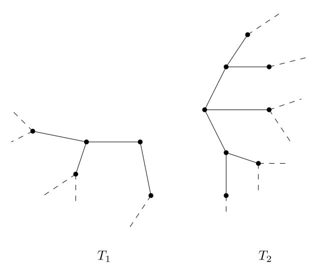
1.6 Number of edges in a tree

In every tree $T = (V, E)$, $|V| = |E| + 1$.

Proof. By induction on $|E|$: If $|E| = 0$, then T has one vertex and so $|V| = 1$, and the equation holds. If $|E| = 1$, then T has two vertices, so $|V| = 2$. If $|E| = 2$, then T has three vertices so $|V| = 3$. If $|E| = k + 1$ then T is of the form



Removing e from T gives us a forest



Then T_1 and T_2 are smaller trees, i.e. $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$. And

$$0 \leq |E_1| \leq k \quad \text{and} \quad 0 \leq |E_2| \leq k.$$

Hence

$$|V_1| = |E_1| + 1 \quad \text{and} \quad |V_2| = |E_2| + 1.$$

Thus

$$|V| = |V_1| + |V_2| = |E_1| + 1 + |E_2| + 1 = |E| + 1.$$

□

1.7 Pendant vertices

If $T = (V, E)$ is a tree with $|V| \geq 2$ then T has at least 2 pendant vertices.

Proof. We have $|E| = |V| - 1$ and

$$2|E| = \sum_{v \in V} \deg(v).$$

Hence $2(|V| - 1) = \sum_{v \in V} \deg(v)$. Let k be the number of pendant vertices. Then

$$\begin{aligned} 2(|V| - 1) &= \sum_{v \in V} \deg(v) \\ &\geq k + 2(|V| - k) \end{aligned}$$

since the graph is connected, there are no vertices of degree 0. Then $2(|V| - 1) \geq k + 2(|V| - k)$ implies $k \geq 2$. □

1.8 Properties of a tree

The following statements are equivalent for a loop-free undirected graph $G = (V, E)$.

1. G is a tree.
2. G is connected and removing an edge leaves a forest of two trees.
3. G contains no cycles and $|V| = |E| + 1$.
4. G is connected and $|V| = |E| + 1$.
5. G contains no cycles and if $a, b \in V$ with $\{a, b\} \notin E$ then adding $\{a, b\}$ to E yields exactly one cycle.

2 12.2 Rooted Trees

2.1 In- and out-degree

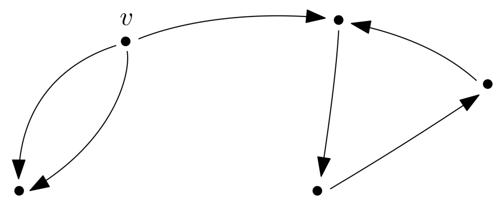
If $G = (V, E)$ is a directed graph and $v \in V$ then $in(v)$, the *in-degree* of v, is the number of edges ending at v. Similarly, $out(v)$, the *out-degree* of v, is the number of edges leaving v.

2.2 Underlying undirected graph

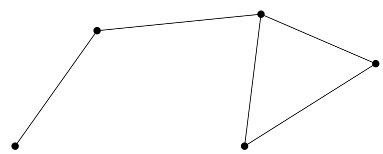
If $G = (V, E)$ is a directed graph then the *underlying undirected graph* is the undirected graph obtained by forgetting the orientation of the edges.

2.2.1 Example

In



we have $in(v) = 0$ and $out(v) = 3$. And the underlying undirected graph is



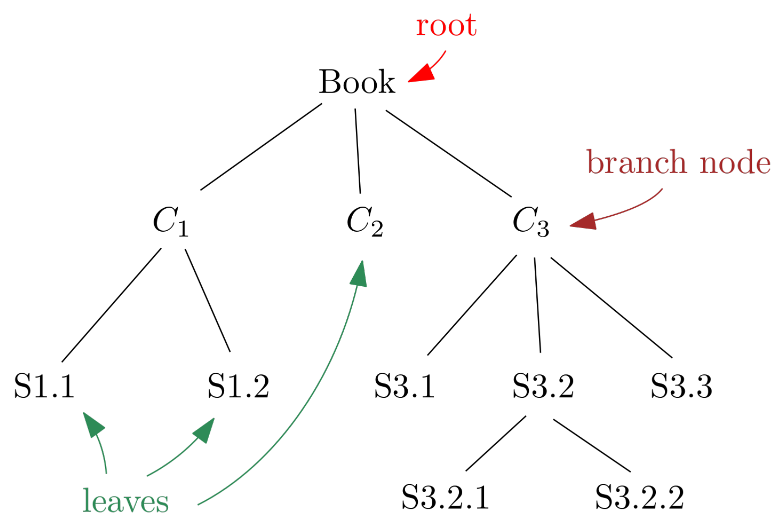
2.3 Directed tree and rooted tree

A directed graph G is a *directed tree* if the underlying graph is a tree. A directed tree G is *rooted* if there is a unique vertex r with $in(r) = 0$ and, for all other vertices v, $in(v) = 1$. In this case, r is the *root*.

Remark: The root is the only vertex without an incoming edge.

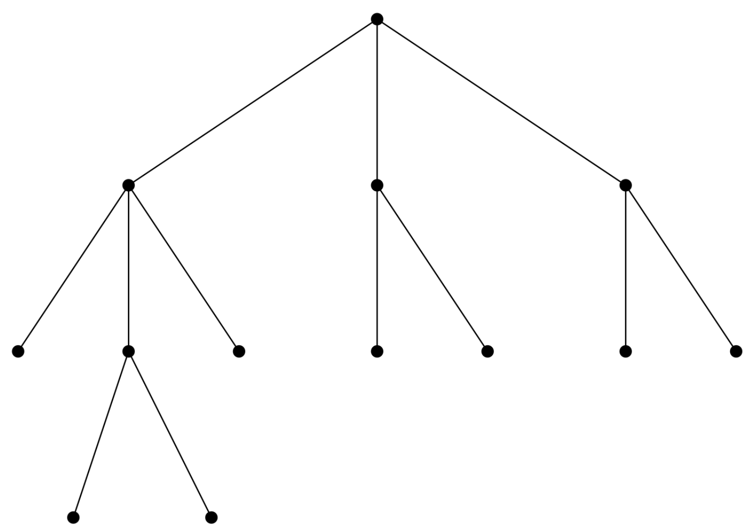
2.4 Terminology:

- the *root* has in-degree 0
- the *leaves* have out-degree 0
- the nodes that are neither root nor leaf are called *branch nodes*
- the length of the path from the root to a node is the *level* of that node
- If there is a directed path from a node n to a node m, we say that n is an *ancestor* of m and that m is a *descendant* of n
- If the length of the path is 1 then n is a *parent* of m and m is a *child* of n.



2.5 Lexicographic order

Consider the rooted tree



Here, we consider the nodes at a given level as ordered from left to right. And labelling the nodes as above yields a total order on the set of vertices called the *lexicographic order*.

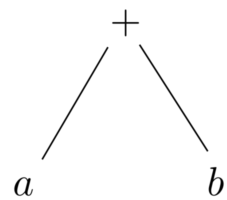
2.6 Binary rooted tree

A rooted tree is a *binary* rooted tree if the out-degree of a vertex is 0,1, or 2.

2.6.1 Example

Consider a binary operation such as + (addition). Then expressions involving + can be represented as binary trees.

$a + b :$



Similarly, an expression such as

$$((7 - a)/5) \times ((a + b)^3)$$

can be represented as