MATH 2135 Linear Algebra

3.B Null Spaces and Ranges

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Contents

1 Null Space and Injectivity

1.1 Definition of null space

For $T \in \mathcal{L}(V, W)$, the *null space* or *kernel* of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null } T = \{v \in V \mid Tv = 0\}.$$

1.2 Examples of null spaces

- If $T : V \to W$ is the zero map where $Tv = 0$ for every $v \in V$, then null $T = V$.
- If $T : V \to V$ is the identity function, then null $T = \{0\}$.
- Suppose $\phi \in \mathcal{L}(\mathbb{R}^3, \mathbf{F})$ is defined by $\phi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$. Then null $\phi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + 2z_2 + 3z_3 = 0\}$. A basis of null ϕ is $(-2, 1, 0), (-3, 0, 1)$.
- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. The only functions whose derivative equals the zero function are the constant functions. Then, the null space of D equals the set of constant functions.

  $$\text{null } D = \{a_0 \mid a_0 \in \mathbb{R}\}.$$
- Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the multiplication by x^2 map defined by $(Tp)(x) = x^2 p(x)$. The only polynomial p such that $x^2 p(x) = 0$ for all $x \in \mathbb{R}$ is the 0 polynomial. Then null $T = \{0\}$.
- Suppose $B \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ is the backward shift defined by

  $$B(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

  Then $B(x_1, x_2, x_3, \dots) = 0$ if and only if $x_2 = x_3 = \cdots = 0$. So we have null $B = \{(a, 0, 0, \dots) \mid a \in \mathbf{F}\}$.

1.3 The null space is a subspace

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

Proof. Since T is a linear map, we know that $T(0) = 0$ then $0 \in \text{null } T$. Suppose $u, v \in \text{null } T$, then

$$T(u + v) = Tu + Tv = 0 + 0 = 0.$$

Since $u + v \in \text{null } T$, then it is closed under addition. Suppose that $u \in \text{null } T$ and $\lambda \in \mathbf{F}$. Then

$$T(\lambda u) = \lambda Tu = \lambda 0 = 0.$$

Hence $\lambda u \in \text{null } T$ and it is closed under scalar multiplication. Thus, null T is a subspace of V. $\square$

1.4 Injective

A function $T : V \to W$ is called *injective* or *one-to-one* if $Tu = Tv$ implies $u = v$. This can be rephrased to say that T is injective if $u \neq v$ implies that $Tu \neq Tv$.

1.5 Injectivity is equivalent to null space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

Proof. Suppose T is injective. We want to prove that null $T = \{0\}$. We already know that $\{0\} \subset \text{null } T$ since $0 \in \text{null } T$. To prove that null $T \subset \{0\}$, suppose $v \in \text{null } T$. Then

$$T(v) = 0 = T(0).$$

Since T is injective, the equation above implies $v = 0$, then we can conclude that null $T = \{0\}$, as desired.
Suppose that null $T = \{0\}$ and we need to show that T is injective. Suppose $u, v \in V$ and $Tu = Tv$. Then

$$0 = Tu - Tv = T(u - v).$$

Then $u - v \in \text{null } T$, which equals $\{0\}$. Hence $u - v = 0 \Rightarrow u = v$ which implies T is injective, as desired. $\square$

2 Range and Surjectivity

2.1 Definition of range

For T a function from V to W, the *range* of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\text{range } T = \{Tv \mid v \in V\}.$$

2.2 Examples of range

- If T is the zero map from V to W, in other words if $Tv = 0$ for every $v \in V$, then range $T = \{0\}$.
- Suppose $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is defined by $T(x, y) = (2x, 5y, x + y)$, then range $T = \{(2x, 5y, x + y) \mid x, y \in \mathbb{R}\}$. A basis of range T is $(2, 0, 1), (0, 5, 1)$.
- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map by $Dp = p'$. Because for every polynomial $q \in \mathcal{P}(\mathbb{R})$ there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that $p' = q$, the range of D is $\mathcal{P}(\mathbb{R})$.
- Suppose $B \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ is the backshift operator defined by $B(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$. Then
$$\begin{aligned}\text{range } B &= \{B(x_1, x_2, \dots) \mid (x_1, x_2, dots) \in \mathbf{F}^\infty\} \\ &= \{(x_2, x_3, x_4, \dots) \mid x_1, x_2, \dots \in \mathbf{F}\} \\ &= \mathbf{F}^\infty.\end{aligned}$$
- Let $F \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ be the forward shift operator defined by $F(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$. Then
$$\begin{aligned}\text{range } F &= \{F(x_1, x_2, \dots) \mid (x_1, x_2, \dots) \in \mathbf{F}^\infty\} \\ &= \{(0, x_1, x_2, \dots) \mid x_1, x_2, \dots \in \mathbf{F}\}.\end{aligned}$$
This is a proper subspace of $\mathbf{F}^\infty$.

2.3 The range is a subspace

If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

Proof. Suppose $T \in \mathcal{L}(V, W)$, then $T(0) = 0$ which implies that $0 \in$ range T. If $w_1, w_2 \in$ range T, then there exist $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. So
$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2.$$
Hence $w_1 + w_2 \in$ range T, so it is closed under addition. If $w \in$ range T and $\lambda \in \mathbf{F}$, then there exists $v \in V$ such that $Tv = w$. Thus
$$T(\lambda v) = \lambda Tv = \lambda w.$$
Hence $\lambda w \in$ range T, and it is closed under scalar multiplication. Hence, range T is a subspace of W. $\square$

2.4 Surjective

A function $T : V \to W$ is called *surjective* or *onto* if its range equals W.

2.5 Example

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_5(\mathbb{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_4(\mathbb{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbb{R})$, which is now the vector space into which S maps.

3 Fundamental Theorem of Linear Maps

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof. Let $u_1, \ldots, u_m$ be a basis of null T, then dim null $T = m$. The linearly independent list $u_1, \ldots, u_m$ can be extended to a basis

$$u_1, \ldots, u_m, v_1, \ldots, v_n$$

of V, thus dim $V = m+n$. We need to show that range T is finite-dimensional and dim range $T = n$. We will do this by proving that $Tv_1, \ldots, Tv_n$ is a basis of range T.

Let $v \in V$. Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ spans V, we can write

$$v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$. Applying T to both sides, we get

$$Tv = b_1Tv_1 + \cdots + b_nTv_n,$$

where all the terms of Tu_j disappeared since $u_j \in$ null T. The equation above implies that $Tv_1, \ldots, Tv_n$ spans range T, then range T is finite-dimensional.

To show that $Tv_1, \ldots, Tv_n$ is linearly independent, suppose $c_1, \ldots, c_n \in \mathbf{F}$ and $c_1Tv_1 + \cdots + c_nTv_n = 0$. Then

$$T(c_1v_1 + \cdots + c_nv_n) = 0$$

and $c_1v_1 + \cdots + c_nv_n \in \text{null } T$. Because $u_1, \ldots, u_m$ spans null T, we can write $c_1v_1 + \cdots + c_nv_n = d_1u_1 + \cdots + d_mu_m$ where $d_1, \ldots, d_m \in \mathbf{F}$. The equation implies all c's and d's are 0 (since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is linearly independent). Thus $Tv_1, \ldots, Tv_n$ is linearly independent and hence is a basis of range T, as desired. $\square$

3.1 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned}\dim \text{ null } T &= \dim V - \dim \text{ range } T \\ &\geq \dim V - \dim W \\ &> 0.\end{aligned}$$

The inequality states that dim null $T > 0$ which means null T contains vectors other than 0. Thus, T is not injective since null $T \neq \{0\}$. $\square$

3.2 A map to a larger dimensional space is not surjective

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned}\dim \text{ range } T &= \dim V - \dim \text{ null } T \\ &\leq \dim V \\ &< \dim W.\end{aligned}$$

The inequality states that $\dim \text{range } T < \dim W$. This means that range $T \neq W$, so T is not surjective. $\square$

3.3 Homogenous system of linear equations

A homogenous system of linear equations with more variables than equations has nonzero solutions.

Proof. Consider $T : \mathbf{F}^n \to \mathbf{F}^m$ defined by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k}x_k, \ldots, \sum_{k=1}^{n} x_k \right)$$

where $T(x_1, \ldots, x_n) = 0$ and 0 here is the additive identity in $\mathbf{F}^m$, which is the list of length m of all 0's. We have a homogenous system of m linear equations with n variables. If $\dim \mathbf{F}^n = n > \dim \mathbf{F}^m = m$, then T is not injective. We also have null $T \neq \{0\}$ which implies that there exists some $v \in$ null T such that $v \neq 0$. So, the system $Tv = 0$ has nonzero solutions. $\square$

3.4 Inhomogenous system of linear equations

An inhomogenous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof. Define $T : \mathbf{F}^n \to \mathbf{F}^m$ by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right)$$

where $T(x_1, \ldots, x_n) = (c_1, \ldots, c_m)$. We have a system of m equations with n variables $x_1, \ldots, x_n$. We see that T is not surjective if $n < m$ since range $T \neq \mathbf{F}^m$. $\square$