MATH 2135 Linear Algebra
2.C Dimension

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March 8, 2021

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1 Dimension

1.1 Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length.

Proof. Suppose V is finite-dimensional. Let B_1 and B_2 be two bases of V. Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is at most length of B_2. Interchanging the roles, we also see that the length of B_2 is at most the length of B_1. Thus the length of B_1 equals the length of B_2, as desired. $\square$

1.2 Definition of a dimension

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

1.3 Examples of a dimension

1. $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length n.
2. $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ because the basis $1, z, \ldots, z^m$ of $\mathcal{P}_m(\mathbf{F})$ has length $m + 1$.

1.4 Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Proof. Suppose V is finite-dimensional and U is a subspace of V. Think of a basis of U as a linearly independent list in V, and think of a basis of V as a spanning list in V. These linearly independent vectors $u_1, \ldots, u_m$ can be extended to a basis of V. That extended basis has at least m vectors, so $\dim V \geq \dim U$. $\square$

1.5 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof. Suppose $\dim V = n$ and $v_1, \ldots, v_n$ is linearly independent in V. The list $v_1, \ldots, v_n$ can be extended to a basis of V. However, every basis of V has length n, so in this case the extension is the trivial one, meaning that no elements are adjoined to $v_1, \ldots, v_n$. In other words, $v_1, \ldots, v_n$ is a basis of V, as desired. $\square$

1.6 Examples

1. Show that the list $(5,7),(4,3)$ is a basis of $\mathbf{F}^2$.

   Proof. The two vectors are linearly independent (because neither vector is a scalar multiple of the other). Note that $\mathbf{F}^2$ has dimension 2. Thus, Theorem 1.5 implies that the linearly independent list of length 2 is a basis of $\mathbf{F}^2$. $\square$

2. Show that $p(x) = x^2+1, q(x) = x^2+x, r(x) = x^2$ are a basis of $\mathcal{P}_2(\mathbf{F})$.

   Proof. Assume $a(x^2+1) + b(x^2+x) + c(x^2) = 0$, where $a, b, c \in \mathbf{F}$. Then we have $(a+b+c)x^2 + bx + a = 0 \Rightarrow a+b+c = 0$. We know that $a = b = 0$ so it follows that $c = 0$. Hence, p, q, r are linearly independent. Since we know that $\dim \mathcal{P}_2(\mathbf{F}) = 3$ then by Theorem 1.5, p, q, r are bases of $\mathcal{P}_2(\mathbf{F})$. $\square$

1.7 Spanning list of the right length is a basis

Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof. Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. The list $v_1, \ldots, v_n$ can be reduced to a basis of V (by removing 0 or more vectors from the list). However, every basis of V has length n, so the reduction is the trivial one, meaning that no elements are deleted from $v_1, \ldots, v_n$. In other words, $v_1, \ldots, v_n$ is a basis of V, as desired. $\square$

1.8 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof. Let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$; thus $\dim(U_1 \cap U_2) = m$. These basis are linearly independent in U_1 and can be extended to a basis $u_1, \ldots, u_m, v_1, \ldots, v_j$. Thus, $\dim U_1 = m + j$ Also, $u_1, \ldots, u_m, w_1, \ldots, w_k$ of U_2 and so $\dim U_2 = m + k$.
We need to show that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is a basis of $U_1 + U_2$. so

$$\begin{aligned}
\dim(U_1 + U_2) &= m + j + k \\
&= (m + j) + (m + k) - m \\
&= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).
\end{aligned}$$

Clearly $span(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ contains $U_1 + U_2$ which equals $U_1 + U_2$. To show that this list is a basis of $U_1 + U_2$, we need to show that it is linearly independent. Suppose that

$$a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k = 0$$

where $a, b, c \in \mathbf{F}$. Then

$$c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j.$$

This implies that $c_1w_1 + \cdots + c_kw_k \in U_1$ and consequently, $c_1w_1 + \cdots + c_kw_k \in U_1 \cap U_2$. Since $u_1, \ldots, u_m$ is a basis of $U_1 \cap U_2$, we can write

$$c_1w_1 + \cdots + c_kw_k = d_1u_1 + \cdots + d_mu_m$$

for some scalars $d \in \mathbf{F}$. But $u_1, \ldots, u_m, w_1, \ldots, w_k$ are linearly independent, so all c's and d's equal 0. Thus, our original equation becomes

$$a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j = 0.$$

Since $u_1, \ldots, u_m, v_1, \ldots, v_j$ are linearly independent, then all a's and b's equal 0. $\square$