MATH 2135 Linear Algebra

Chapter 6 Inner Product Spaces

Alyssa Motas

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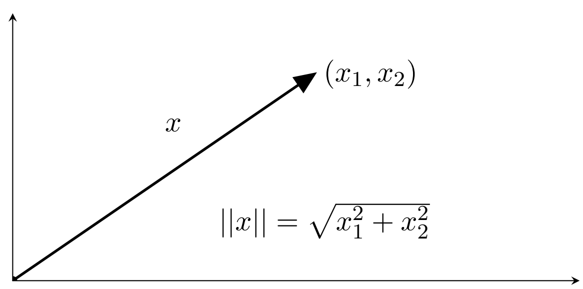
1 6.A Inner Products and Norms

To motivate the concept of inner product, think of vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ as arrows with initial point at the origin. The length of a vector x in $\mathbb{R}^2$ or $\mathbb{R}^3$ is called the *norm* of x, denoted $||x||$. Thus for $x = (x_1, x_2) \in \mathbb{R}^2$, we have $||x|| = \sqrt{x_1^2 + x_2^2}$. The generalization to $\mathbb{R}^n$ is: we defined the norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ by

$$||x|| = \sqrt{x_1^2 + \cdots + x_n^2}.$$



The norm is not linear on $\mathbb{R}^n$.

1.1 Definition of dot product

For $x, y \in \mathbb{R}^n$, the ***dot product*** of x and y, denoted $x \cdot y$, is defined by

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Note that the dot product of two vectors in $\mathbb{R}^n$ is a number, not a vector.

An inner product is a generalization of the dot product. Recall that if $\lambda = a + bi$, where $a, b \in \mathbb{R}$, then

- the absolute value of λ, denoted $|\lambda|$, is defined by $|\lambda| = \sqrt{a^2 + b^2}$;
- the complex conjugate of λ, denoted $\overline{\lambda}$, is defined by $\overline{\lambda} = a - bi$;
- $|\lambda|^2 = \lambda\overline{\lambda}$.

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define the norm of z by

$$||z|| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

The absolute values are needed because we want $||z||$ to be nonegative number. Note that

$$||z||^2 = z_1\overline{z_1} + \cdots + z_n\overline{z_n}.$$

We want to think of $||z||^2$ as the inner product of z with itself. The equation above suggests that the inner product of $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ with z should equal

$$w_1\overline{z_1} + \cdots + w_n\overline{z_n}.$$

If the roles of w and z were interchanged, the expression above would be its complex conjugate. We should expect that the inner product of w with z equals the complex conjugate of the inner product of z with w.

1.2 Definition of inner product

An ***inner product*** on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties:

positivity

$$\langle v, v\rangle \geq 0 \text{ for all } v \in V;$$

definiteness

$$\langle v, v\rangle = 0 \text{ if and only if } v = 0;$$

additivity in first slot

$$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle \text{ for all } u, v, w \in V;$$

homogeneity in first slot

$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle \text{ for all } \lambda \in \mathbf{F} \text{ and all } u, v \in V;$$

conjugate symmetry

$$\langle u, v\rangle = \overline{\langle v, v\rangle} \text{ for all } u, v \in V.$$

Every real number equals its complex conjugate. If we are dealing with a real vector space, then the last condition can be $\langle u, v\rangle = \langle v, u\rangle$ for all $v, w \in V$.

1.2.1 Examples

(a) The ***Euclidean inner product*** on $\mathbf{F}^n$ is defined by

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}.$$

(b) If $c_1, \ldots, c_n$ are positive numbers, then an inner product can be defined on $\mathbf{F}^n$ by

$$\langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle = c_1 w_1 \overline{z_1} + \cdots + c_n w_n \overline{z_n}.$$

(c) An inner product can be defined on the vector space of continuous real-valued functions on the interval $[-1, 1]$ by

$$\langle f, g \rangle = \int_{-1}^{1} f(x) \overline{g(x)} dx.$$

This is an inner product since for example: additivity in the left slot is defined as

$$\begin{aligned} \langle f + h, g \rangle &= \int_{-1}^{1} (f(x) + h(x)) \overline{g(x)} dx \\ &= \int_{-1}^{1} f(x) \overline{g(x)} + \int_{-1}^{1} h(x) \overline{g(x)} dx \\ &= \langle f, g \rangle + \langle h, g \rangle. \end{aligned}$$

(d) An inner product can be defined on $\mathcal{P}(\mathbb{R})$ by

$$\langle p, q \rangle = \int_{0}^{\infty} p(x) q(x) e^{-x} dx.$$

(e) The dot product on $\mathbb{R}^n$

$$\langle v, w \rangle = v \cdot w = x_1 y_1 + \cdots + x_n y_n$$

and

$$\langle v, v \rangle = v \cdot v = x_1^2 + \cdots + x_n^2 \geq 0.$$

1.3 Definition of inner product space

An ***inner product space*** is a vector space V along with an inner product on V. For the rest of this chapter, V denotes an inner product space over $\mathbf{F}$.

1.4 Basic properties of an inner product

(a) For each fixed $u \in V$, the function that takes v to $\langle v, u \rangle$ is a linear map from V to $\mathbf{F}$.

Proof.

- $f(v + v') = \langle v + v', u \rangle = \langle v, u \rangle + \langle v', u \rangle = f(v) + f(v')$
- $f(\lambda v) = \cdots = \lambda f(v)$.

$\square$

(b) $\langle 0, u \rangle = 0$ for every $u \in V$.

(c) $\langle u, 0 \rangle = 0$ for every $u \in V$.

(d) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.

Proof. This is additivity in the second slot.

$$\begin{aligned}
\langle u, v + w \rangle &= \overline{\langle v + w, u \rangle} \\
&= \overline{\langle v, u \rangle + \langle w, u \rangle} \\
&= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle} \\
&= \langle u, v \rangle + \langle u, w \rangle.
\end{aligned}$$

$\square$

(e) $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and $u, v \in V$.

Proof. This is homogeneity in the second slot.

$$\begin{aligned}
\langle u, \lambda v \rangle &= \overline{\langle \lambda v, u \rangle} \\
&= \overline{\lambda \langle v, u \rangle} \\
&= \overline{\lambda} \overline{\langle v, u \rangle} \\
&= \overline{\lambda} \langle u, v \rangle.
\end{aligned}$$

$\square$

1.5 Definition of norm, $||v||$

For $v \in V$, the ***norm*** of v, denoted $||v||$, is defined by

$$||v|| = \sqrt{\langle v, v \rangle} \geq 0.$$

Note that $||v||^2 = \langle v, v \rangle$.

1.6 Basic properties of the norm

Suppose $v \in V$.

(a) $||v|| = 0$ if and only if $v = 0$.

(b) $||\lambda v|| = |\lambda|||v||$ for all $\lambda \in \mathbf{F}$.

Proof. (a) The desired result holds because $\langle v, v \rangle = 0$ if and only if $v = 0$.

(b) Suppose $\lambda \in \mathbf{F}$. then

$$\begin{aligned} ||\lambda v||^2 &= \langle \lambda v, \lambda v \rangle \\ &= \lambda \langle v, \lambda v \rangle \\ &= \lambda \overline{\lambda} \langle v, v \rangle \\ &= |\lambda|^2 ||v||2. \end{aligned}$$

Taking square roots now gives the desired equality.

□

1.7 Definition of orthogonal

Two vectors $u, v \in V$ are called ***orthogonal*** if $\langle u, v \rangle = 0$. We write $u \perp v$ to mean "u is orthogonal to v."

1.8 Orthogonality and 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

Proof. If $v \in V$ and $\langle v, v \rangle = 0$, then $v = 0$ (by definition of inner product). □

(c) $u \perp v \Leftrightarrow v \perp u$

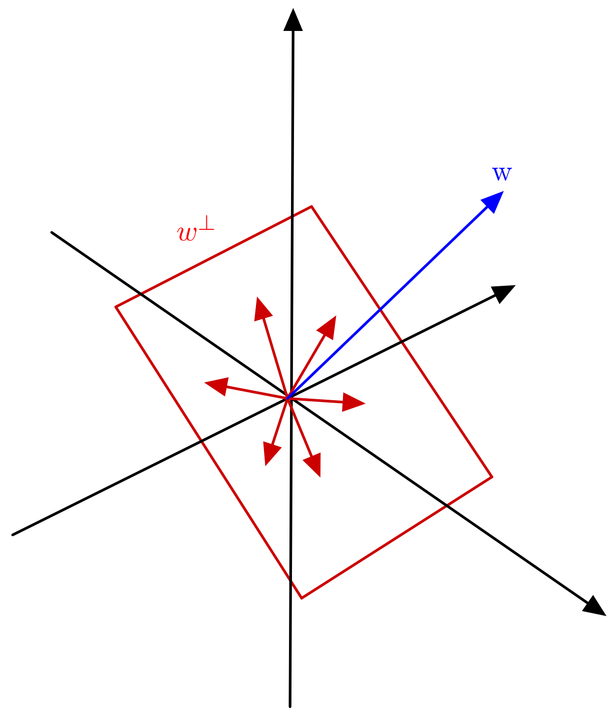
(d) $u \perp w$ and $v \perp w \Rightarrow (u + v) \perp w$.

(e) $u \perp w$ and $\lambda \in \mathbf{F} \Rightarrow (\lambda u) \perp w$.

The last two properties imply that the set

$$w^{\perp} = \{v \mid v \perp w\}$$

is a subspace of V, called the *orthogonal complement of* V.



1.9 Pythagorean Theorem

Suppose u and v are orthogonal vectors in V. Then

$$||u + v||^2 = ||u||^2 + ||v||^2.$$

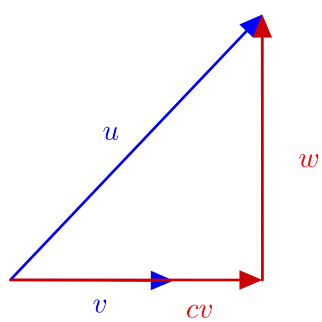
Proof. We have

$$\begin{aligned}||u+v||^2 &= \langle u+v, u+v\rangle \\ &= \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \underbrace{\langle u,v\rangle + \langle v,u\rangle}_{0} + \langle v,v\rangle \\ &= \langle u,u\rangle + \langle v,v\rangle \\ &= ||u||^2 + ||v||^2,\end{aligned}$$

as desired. ◻

1.10 Orthogonal Decomposition (Projection)

Given $u, v \in V$, assuming $v \neq 0$. Then we can write u as a sum of two vectors, the first of which is parallel to v and the second is orthogonal to v.



Let $c = \dfrac{\langle u,v\rangle}{||v||^2} = \dfrac{\langle u,v\rangle}{\langle v,v\rangle}$ and let $w = u - cv$. Then $\langle w, v\rangle = 0$ and $u = cv + w$.

Proof. We know $u = cv + w$ holds by the definition of w. We also know that cv is parallel to v by the definition of "parallel." To prove that w is orthogonal to v, we can calculate:

$$\begin{aligned}\langle w,v\rangle &= \langle u - cv, v\rangle \\ &= \langle u,v\rangle - c\langle v,v\rangle \\ &= \langle u,v\rangle - \frac{\langle u,v\rangle}{\langle v,v\rangle}\langle v,v\rangle \\ &= \langle u,v\rangle - \langle u,v\rangle = 0.\end{aligned}$$

Therefore, $w \perp v$. $\square$

1.11 Cauchy-Schwarz Inequality

Suppose $u, v \in V$. Then
$$|\langle u, v\rangle| \leq ||u||||v||.$$
This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

Proof. Consider two cases:

Case 1. $v = 0$ and in this case, $\langle u, v\rangle = 0, ||u|| \cdot ||v|| = ||u|| \cdot 0 = 0$. So the inequality holds.

Case 2. $v \neq 0$. Consider the orthogonal decomposition
$$u = cv + w$$
where $c = \dfrac{\langle u, v\rangle}{\langle v, v\rangle}$ and $w = u - cv$. We know that $w \perp v$. By Pythagoras' Theorem,
$$\begin{aligned} ||u||^2 &= ||cv||^2 + ||w||^2 \\ &\geq ||cv||^2 \\ &= |c|^2||v||^2 \\ &= \left|\frac{\langle u, v\rangle}{||v||^2}\right|^2 ||v||^2 \\ &= \frac{|\langle u, v\rangle|^2}{||v||^4} \cdot ||v||^2 \\ &= \frac{|\langle u, v\rangle|^2}{||v||^2}. \end{aligned}$$
We just proved that
$$||u||^2 \geq \frac{|\langle u, v\rangle|^2}{||v||^2}.$$
Multiply both sides of the equation by $||v||^2$ and we get
$$||u||^2||v||^2 \geq |\langle u, v\rangle|^2.$$
Take the square root of both sides of the equation and we get
$$||u|| \cdot ||v|| \geq |\langle u, v\rangle|$$
which is the Cauchy-Schwarz inequality. $\square$

1.11.1 Examples of the Cauchy-Schawrz Inequality

(a) If $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$ then

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

(b) If f, g are continuous real-valued functions on $[-1, 1]$, then

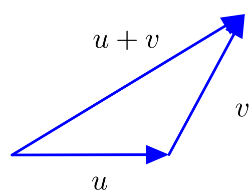
$$\left| \int_{-1}^{1} f(x)g(x)dx \right|^2 \leq \left(\int_{-1}^{1} (f(x))^2 dx \right) \left(\int_{-1}^{1} (g(x))^2 dx \right).$$

1.12 Triangle Inequality

The Triangle Inequality implies that the shortest path between two points is a line segment. Suppose $u, v \in V$. Then

$$||u + v|| \leq ||u|| + ||v||.$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.



Proof. We have

$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \langle v, u \rangle + \langle v, v \rangle \\ &= \langle u, u \rangle + \langle u, v \rangle + \overline{\langle u, v \rangle} + \langle v, v \rangle \\ &\leq \langle u, u \rangle + 2|\langle u, v \rangle| + \langle v, v \rangle \\ &\leq \langle u, u \rangle + 2||u||||v|| + \langle v, v \rangle \quad \text{(Cauchy-Schwarz)} \\ &= ||u||^2 + 2||u||||v|| + ||v||^2 \\ &= (||u|| + ||v||)^2 \end{aligned}$$

Taking the square roots:

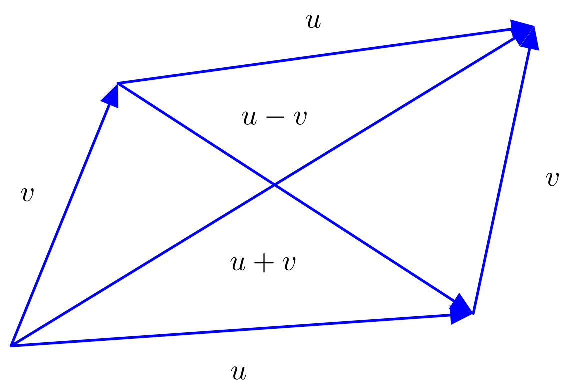
$$||u + v|| \leq ||u|| + ||v||,$$

thus we get the triangle inequality. □

1.13 Parallelogram Equality

In every parallelogram, the sum of the squares of the lengths of the diagonals equals the sum of the squares of the lengths of the four sides.
Suppose $u, v \in V$. Then

$$||u+v||^2 + ||u-v||^2 = 2(||u||^2 + ||v||^2).$$



Proof. We have

$$\begin{aligned} ||u+v||^2 + ||u-v||^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\ &= ||u||^2 + ||v||^2 + \langle u, v\rangle + \langle v, u\rangle \\ &\quad + ||u||^2 + ||v||^2 - \langle u, v\rangle - \langle v, u\rangle \\ &= 2(||u||^2 + ||v||^2), \end{aligned}$$

as desired. □

2 6.B Orthonormal Bases

3 6.C Orthogonal Complements and Minimization Problems