MATH 2135 Linear Algebra

Chapter 7 Operators Inner Product Spaces

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Contents

1 7.A Self-Adjoint and Normal Operators

Definition An $n \times n$-matrix A is called *symmetric* if $A = A^T$.

1.1 Definition of adjoint, T^*

Suppose $T \in \mathcal{L}(V, W)$. The ***adjoint*** of T is the function $T^* : W \to V$ such that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and every $w \in W$. Or in other words, let A be a complex $n \times n$-matrix. The adjoint of A is the complex conjugate of the tranpose of A, written

$$A^* = \overline{A^T}.$$

1.1.1 Example

$$A = \begin{bmatrix} i & 5 & 1+i \\ 0 & 2i & -i \\ -7 & 3-i & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} i & 0 & -7 \\ 5 & 2i & 3-i \\ 1+i & -i & 0 \end{bmatrix}$$

$$A^* = \overline{A^T} = \begin{bmatrix} -i & 0 & -7 \\ 5 & -2i & 3+i \\ 1-i & i & 0 \end{bmatrix}$$

1.2 Definition of self-adjoint

An operator $T \in \mathcal{L}(V)$ is called ***self-adjoint*** if $T = T^*$. In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle$$

for all $v, w \in V$. Alternative, an $n \times n$-matrix A is called self-adjoint or hermitian if

$$A = A^*.$$

1.2.1 Example

$$A = \begin{bmatrix} 1 & i & 1+i \\ -i & 2 & 5 \\ 1-i & 5 & 3 \end{bmatrix}$$

A is self-adjoint if for all j, k, $a_{jk} = \overline{a_{jk}}$.
Remark: If A is self-adjoint, the diagonal entries of A are *real* numbers.

1.3 Orthogonal Property

A *real* $n \times n$-matrix A is called ***orthogonal*** if A is invertible and $A^{-1} = A^T$.

Remark: The following are equivalent:

1. A is orthogonal
2. $AA^T = I$
3. $A^T A = I$
4. The columns of A form an orthonormal basis of $\mathbb{R}^n$.
5. The rows of A form an orthonormal basis of $\mathbb{R}^n$.

Proof. For (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), it is obvious. To prove (3) $\Leftrightarrow$ (4), we write

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & \dots & a_{n1} \\ a_{12} & \dots & \vdots \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix}.$$

Condition (3) states that $A^T A = I$.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \dots & a_{n1} \\ a_{12} & \dots & \vdots \\ \vdots & \ddots & \vdots \\ a_{1n} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \cdot \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} = 1, \text{i.e.} \left\| \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \right\|$$

$$\begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} \cdot \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} = 0.$$

So if

$$v_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix}, v_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix}, \dots, v_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{nn} \end{bmatrix},$$

the condition $A^T A = I$ means exactly that

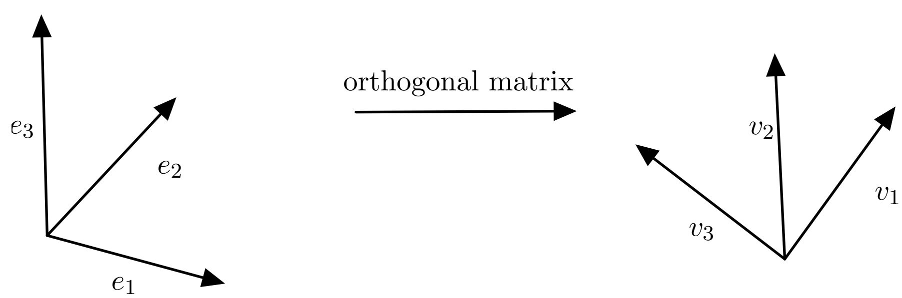
$$\langle v_i, v_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

This ist he case if and only if $v_1, \ldots, v_n$ are orthonormal. $\square$

1.3.1 Example

The following matrices are orthogonal:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



2 7.B The Spectral Theorem

2.1 The real Spectral Theorem

Let A be a real symmetric $n \times n$-matrix. Then

1. There exists an *orthonormal* basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
2. A is diagonalizable as $D = S^{-1}AS$ where S is an orthogonal matrix.

Remark: The columns of S are the orthonormal eigenvectors of A. Since S is orthogonal, we have

$$S^{-1} = S^T$$

so we have $D = S^T AS$.

2.2 The complex Spectral Theorem

"Unitary" is the complex version of "orthogonal" (for matrices). A *complex* $n \times n$-matrix A is called *unitary* if A is invertible and $A^{-1} = A^*$.

Remark: A is unitary if and only if the columns of A form an orthonormal basis of $\mathbb{C}^n$.

Examples:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$
$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad A^* = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$

2.2.1 The complex Spectral Theorem, Version 1

Let A be a *complex self-adjoint* $n \times n$-matrix. Then

1. There exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.
2. A is diagonalizable as $D = S^{-1}AS$ where S is a unitary matrix.
3. The eigenvalues of A are real.

2.2.2 The complex Spectral Theorem, Version 2

Let A be unitary $n \times n$-matrix. Then

1. There exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.
2. Therefore, A is diagonalizable as $D = S^{-1}AS$ where S is unitary.
3. All the eigenvalues (the diagonal entries of D) satisfy $|\lambda| = 1$.

2.2.3 The complex Spectral Theorem, Version 3

A matrix A is called *normal* if

$$A^*A = AA^*.$$

Remark: Self-adjoint matrices are normal because $A = A^* \Rightarrow A^*A = A^2 = AA^*$. Unitary matrices are normal because $A^{-1} = A^* = A^*A = A^{-1}A = I = AA^{-1} = AA^*$.

Let A be a *normal* matrix. Then

1. There exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A.

2. Therefore, A is diagonalizable as $D = S^{-1}AS$ where S is unitary.