CSCI/MATH 2113 Discrete Structures

7.2 Computer Recognition: Zero-One Matrices and Directed Graphs

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Contents

1 Relation Composition

If A, B and C are sets with $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, then the *composite relation* $R_1; R_2$ is a relation from A to C defined by

$$R_1; R_2 = \{(x, z) \mid x \in A, z \in C, \exists y \in B \text{ with } (x, y) \in R_1, (y, z) \in R_2\}.$$

1.1 Example

Suppose that $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, and $C = \{5, 6, 7\}$ where

$$\begin{aligned} R_1 &= \{(1, x), (2, x), (3, y), (3, z)\} \subseteq A \times B \\ R_2 &= \{(w, 5), (x, 6)\} \subseteq B \times C \\ R_3 &= \{(w, 5), (w, 6)\} \subseteq B \times C \end{aligned}$$

Then we have

$$R_1; R_2 = \{(1, 6), (2, 6)\}$$

and

$$R_1; R_3 = \varnothing.$$

1.2 Theorem

Let A, B, C and D be sets with $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, and $R_3 \subseteq C \times D$. Then $R_1; (R_2; R_3) = (R_1; R_2); R_3$.

Proof. If $(a, d) \in R_1; (R_2; R_3)$, then there is an element $b \in B$ with $(a, b) \in R_1$ and $(b, d) \in (R_2; R_3)$. Also, $(b, d) \in (R_2; R_3) \Rightarrow (b, c) \in R_2$ and $(c, d) \in R_3$ for some $c \in C$. Then $(a, b) \in R_1$ and $(b, c) \in R_2 \Rightarrow (a, c) \in R_1; R_2$. Finally, $(a, c) \in R_1; R_2$ and $(c, d) \in R_3 \Rightarrow (a, d) \in (R_1; R_2); R_3$, and $R_1; (R_2; R_3) \subseteq (R_1; R_2); R_3$. The opposite inclusion follows by similar reasoning. $\square$

1.3 Powers of a relation

Given a set A and a relation R on A, we define the *powers* of R recursively by

(a) $R^1 = R$;

(b) for $n \in \mathbb{Z}^+, R^{n+1} = R; R^n$.

Example. Suppose that $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$. Then we have

$$\begin{aligned} R^2 &= \{(1, 4), (1, 2), (3, 4)\} \\ R^3 &= \{(1, 4)\} \\ R^n &= \varnothing \text{ for } n \leq 4. \end{aligned}$$

2 Zero-one matrices

An $m \times n$ *zero-one matrix* $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers arranged in m rows and n columns, where each e_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq n$, denotes the entry in the ith row and jth column of E, and each such entry is 0 or 1.

Example. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is a 3×4 (0,1)-matrix where, for example, $e_{11} = 1, e_{23} = 0$, and $e_{31} = 1$.

0 is the matrix such that $0_{i,j} = 0$. And 1 is the matrix such that $1_{i,j} = 1$. I_n is the identity matrix where

$$(I_n)_{i,j} = S_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If M is a matrix then M^{tr} is

$$(M^{tr})_{i,j} = M_{j,i}$$

We think of 0 and 1 as truth values and of $\cdot$ (multiplication) as conjunction and of $+$ (addition) as disjunction. This means that

$$\begin{array}{cc} 0 \cdot 0 = 0 & 0 + 0 = 0 \\ 0 \cdot 1 = 0 & 0 + 1 = 1 \\ 1 \cdot 0 = 0 & 1 + 0 = 1 \\ \underbrace{1 \cdot 1 = 1}_{\text{logical “and”}} & \underbrace{1 + 1 = 1}_{\text{logical “or”}} \end{array}$$

We use these operations when multiplying matrices.

Example.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

As well as multiplication, we define another operation on matrices.
Let M and N be $(0,1)$-matrices of the same size. Then $M \cap N$ is defined as

$$(M \cap N)_{i,j} = M_{i,j} \cdot N_{i,j}.$$

Example.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let M and N be $(0,1)$-matrices of the same size. Then

$$M \leq N$$

if $M_{i,j} \leq N_{i,j}$ for all i, j. In this case, we say that M *precedes* N.

Examples.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\leq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\geq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

2.1 Relation matrices

Suppose we have $A = \{1,2,3,4\}$, $B = \{w,x,y,z\}$, $C = \{5,6,7\}$, $R_1 = \{(1,x),(2,x),(3,y),(3,z)\}$, and $R_2 = \{(w,5),(x,6)\}$. The matrix for R_1, R_2 is

$$M(R_1) = \begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M(R_2) = \begin{array}{c c} & \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

We have

$$M(R_1; R_2) = M(R_1) \cdot M(R_2).$$

So to figure out what the matrix for $R_1; R_2$ is, it suffices to multiply the matrices for R_1 and R_2.

$$M(R_1; R_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.2 Theorem

Let A be a set with $|A| = n \leq 1$, R be a relation on A, and M be the matrix for R. Then

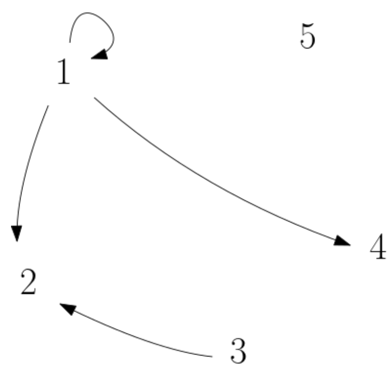
1. R is reflexive if and only if $I_n \leq M$.
2. R is symmetric if and only if $M = M^{tr}$.
3. R is transitive if and only if $M^2 \leq M$.
4. R is antisymmetric if and only if $M \cap M^{tr} \leq I_n$.

3 Directed graphs

Let V be a finite nonempty set. A *directed graph* (or *digraph*) G on V is made up of the elements of V, called *vertices* or *nodes* of G, anda subset E, of $V \times V$, that contains the (*directed*) *edges*, or *arcs*, of G. The set V is called the *vertex set* of G, and the set E is called the *edge set*. We then write $G = (V, E)$ to denote the graph.

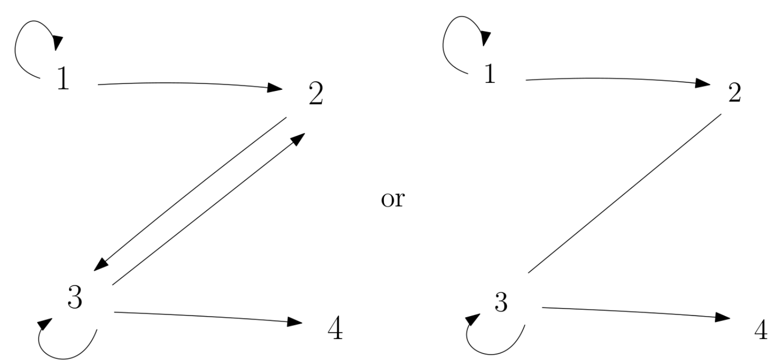
If $a, b \in V$ and $(a, b) \in E$, then there is an edge from a to b. Vertex a is called the *origin* or *source* of the edge, with b the *terminus*, or *terminating vertex*, and we say that b is *adjacent from* a and that a is *adjacent* to b. In addition, if $a \neq b$, then $(a, b) \neq (b, a)$. An edge of the form (a, a) is called a *loop* at a.

Example. Suppose that $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 1), (1, 2), (1, 4), (3, 2)\}$. Then



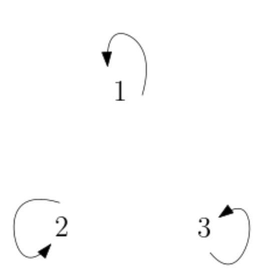
We can interpret a relation on a set A as a directed graph.

Example. Suppose that $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2)\}$. Then we have

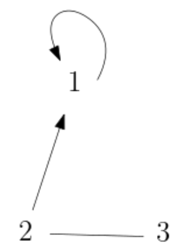


The relation matrix M is also called the adjacency matrix of the associated graph.

Remark. A relation is reflexive if and only if its directed graph has a loop at each vertex.

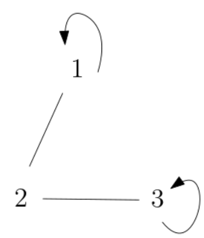


reflexive

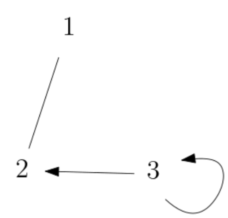


not reflexive

Remark. A relation is symmetric if and only if its directed graph contains only loops and undirected edges.



symmetric



not symmetric