CSCI/MATH 2113 Discrete Structures

Chapter 7 Relations The Second Time Around

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Contents

1 7.1 Properties of Relations

1.1 Recall: Binary Relation

For sets A, B, any subset of $A \times B$ is called a (*binary*) *relation* from A to B. Any subset of $A \times A$ is called a (*binary*) *relation* on A.

1.2 Properties of Relations

1.2.1 Reflexive property

A relation R on a set A is called *reflexive* if for all $x \in A$, $(x, x) \in R$. This means that each element x of A is related to itself.

Example. For $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3)\}$, we know it is not reflexive since $(3, 3) \notin R$.

Remark. A relation R on A is reflexive if and only if $\{(a, a) \mid a \in A\} \subseteq R$.

Counting. Let A be a set with n elements. How many relations on A are reflexive? There are 2^{n^2} relations on A (2 choices for each of the n^2 pairs in $A \times A$). In a reflexive relation, n of these pairs are decided. Hence, we have $n^2 - n$ chocies to make and so there are

$$2^{n^2-2}$$

reflexive relations.

1.2.2 Symmetric property

A relation R on a set A is called *symmetric* if $(x, y) \in R \Rightarrow (y, x) \in R$, for all, $x, y \in A$.

Counting. If $|A| = n \leq 0$, how many relations on A are symmetric? Write $A \times A = A_1 \cup A_2$ where

$$A_1 = \{(a_i, a_i) \mid 1 \leq i \leq n\}$$
$$A_2 = \{(a_i, a_j) \mid 1 \leq i, j \leq n, i \neq j\}.$$

We have

$$|A_1| = n \qquad |A_2| = n^2 - n.$$

For each element of A_1 and for half of the elements of A_2, we choose whether it belongs to R. Thus, intotal, there are

$$2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$$

such relations. In counthing those relations on A that are both reflexive and symmetric, we have only one choice for each ordered pair in A_1. So we have

$$2^{\frac{n^2-n}{2}}$$

relations on A that are both reflexive and symmetric.

1.2.3 Transitive property

For a set A, a relation R on A is called *transitive* if, for all $x, y, z \in A, (x, y), (y, z) \in R \Rightarrow (x, z) \in R$. So if x "is related to" y, and y "is related to" z, we want x "related to" z, with y playing the role of "intermediary."

Counting. There is no known general formula for the total number of transitive relations on a finite set.

1.2.4 Antisymmetric property

Given a relation R on a set A, R is called *antisymmetric* if for all $a, b \in A$, $(aRb$ and $bRa) \Rightarrow a = b$. Here, the only way we can have both a "related to" b and b "related to" a is if a and b are one and the same element from A.

Examples. Let $A = \mathbb{Z}$ and $R = \leq$ is an antisymmetric relation.

Antisymmetric is different than "not symmetric." Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (2, 3)\}$. Then, R is *not* symmetric and it is *not* antisymmetric.

Counting. How relations that are antisymmetric? Suppose that $|A| = n > 0$. By the rule of product, the number of antisymmetric relations are

$$(2^n)(3^{\frac{n^2-n}{2}}).$$

1.3 Partial order

A relation R on a set A is called a *partial order*, or a *partial ordering relation*, if R is reflexive, antisymmetric, and transitive.

1.3.1 Example

- $\leq$ on $\mathbb{N}$. Partial order and total implies total order.
- $\subseteq$ on $\mathcal{P}(S)$ is not total order.

1.4 Total order

A relation R on a set A is a *total order* if R is a partial order and for every $x, y \in A$, we have $(x, y) \in R$ or $(y, x) \in R$.

1.5 Equivalence relation

An *equivalence relation* on A is a relation that is reflexive, symmetric, and transitive.

2 7.2 Zero-One Matrices and Directed Graphs

2.1 Relation Composition

If A, B and C are sets with $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$, then the *composite relation* $R_1; R_2$ is a relation from A to C defined by

$$R_1; R_2 = \{(x, z) \mid x \in A, z \in C, \exists y \in B \text{ with } (x, y) \in R_1, (y, z) \in R_2\}.$$

2.1.1 Example

Suppose that $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, and $C = \{5, 6, 7\}$ where

$$\begin{aligned}
R_1 &= \{(1, x), (2, x), (3, y), (3, z)\} \subseteq A \times B \\
R_2 &= \{(w, 5), (x, 6)\} \subseteq B \times C \\
R_3 &= \{(w, 5), (w, 6)\} \subseteq B \times C
\end{aligned}$$

Then we have

$$R_1; R_2 = \{(1, 6), (2, 6)\}$$

and

$$R_1; R_3 = \varnothing.$$

2.1.2 Theorem

Let A, B, C and D be sets with $R_1 \subseteq A \times B$, $R_2 \subseteq B \times C$, and $R_3 \subseteq C \times D$. Then $R_1; (R_2; R_3) = (R_1; R_2); R_3$.

Proof. If $(a, d) \in R_1; (R_2; R_3)$, then there is an element $b \in B$ with $(a, b) \in R_1$ and $(b, d) \in (R_2; R_3)$. Also, $(b, d) \in (R_2; R_3) \Rightarrow (b, c) \in R_2$ and $(c, d) \in R_3$ for some $c \in C$. Then $(a, b) \in R_1$ and $(b, c) \in R_2 \Rightarrow (a, c) \in R_1; R_2$. Finally, $(a, c) \in R_1; R_2$ and $(c, d) \in R_3 \Rightarrow (a, d) \in (R_1; R_2); R_3$, and $R_1; (R_2; R_3) \subseteq (R_1; R_2); R_3$. The opposite inclusion follows by similar reasoning. $\square$

2.1.3 Powers of a relation

Given a set A and a relation R on A, we define the *powers* of R recursively by

(a) $R^1 = R$;

(b) for $n \in \mathbb{Z}^+, R^{n+1} = R; R^n$.

Example. Suppose that $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (1, 3), (2, 4), (3, 2)\}$. Then we have

$$\begin{aligned} &R^2 = \{(1, 4), (1, 2), (3, 4)\} \\ &R^3 = \{(1, 4)\} \\ &R^n = \varnothing \text{ for } n \leq 4. \end{aligned}$$

2.2 Zero-one matrices

An $m \times n$ *zero-one matrix* $E = (e_{ij})_{m \times n}$ is a rectangular array of numbers arranged in m rows and n columns, where each e_{ij}, for $1 \leq i \leq m$ and $1 \leq j \leq n$, denotes the entry in the ith row and jth column of E, and each such entry is 0 or 1.

Example. The matrix

$$E = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is a 3×4 (0,1)-matrix where, for example, $e_{11} = 1, e_{23} = 0$, and $e_{31} = 1$.

0 is the matrix such that $0_{i,j} = 0$. And 1 is the matrix such that $1_{i,j} = 1$. I_n is the identity matrix where

$$(I_n)_{i,j} = S_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

If M is a matrix then M^{tr} is

$$(M^{tr})_{i,j} = M_{j,i}$$

We think of 0 and 1 as truth values and of $\cdot$ (multiplication) as conjunction and of $+$ (addition) as disjunction. This means that

$$\begin{array}{cc} 0 \cdot 0 = 0 & 0 + 0 = 0 \\ 0 \cdot 1 = 0 & 0 + 1 = 1 \\ 1 \cdot 0 = 0 & 1 + 0 = 1 \\ \underbrace{1 \cdot 1 = 1}_{\text{logical ``and''}} & \underbrace{1 + 1 = 1}_{\text{logical ``or''}} \end{array}$$

We use these operations when multiplying matrices.

Example.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

As well as multiplication, we define another operation on matrices.
Let M and N be $(0,1)$-matrices of the same size. Then $M \cap N$ is defined as

$$(M \cap N)_{i,j} = M_{i,j} \cdot N_{i,j}.$$

Example.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \cap \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let M and N be $(0,1)$-matrices of the same size. Then

$$M \leq N$$

if $M_{i,j} \leq N_{i,j}$ for all i, j. In this case, we say that M *precedes* N.

Examples.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \leq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\leq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \not\geq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

2.2.1 Relation matrices

Suppose we have $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$, $C = \{5, 6, 7\}$, $R_1 = \{(1, x), (2, x), (3, y), (3, z)\}$, and $R_2 = \{(w, 5), (x, 6)\}$. The matrix for R_1, R_2 is

$$M(R_1) = \begin{array}{c c} & \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \qquad M(R_2) = \begin{array}{c c} & \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{array}{c} w \\ x \\ y \\ z \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

We have
$$M(R_1; R_2) = M(R_1) \cdot M(R_2).$$

So to figure out what the matrix for $R_1; R_2$ is, it suffices to multiply the matrices for R_1 and R_2.

$$M(R_1; R_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2.3 Theorem

Let A be a set with $|A| = n \leq 1$, R be a relation on A, and M be the matrix for R. Then

1. R is reflexive if and only if $I_n \leq M$.
2. R is symmetric if and only if $M = M^{tr}$.

3. R is transitive if and only if $M^2 \leq M$.

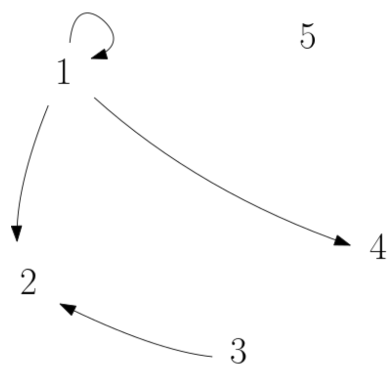
4. R is antisymmetric if and only if $M \cap M^{tr} \leq I_n$.

2.4 Directed graphs

Let V be a finite nonempty set. A *directed graph* (or *digraph*) G on V is made up of the elements of V, called *vertices* or *nodes* of G, anda subset E, of $V \times V$, that contains the (*directed*) *edges*, or *arcs*, of G. The set V is called the *vertex set* of G, and the set E is called the *edge set*. We then write $G = (V, E)$ to denote the graph.

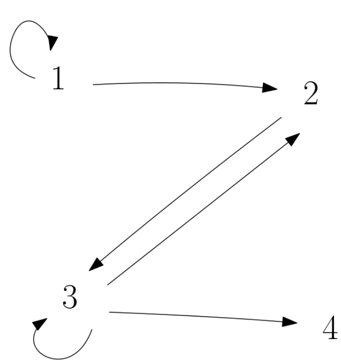
If $a, b \in V$ and $(a, b) \in E$, then there is an edge from a to b. Vertex a is called the *origin* or *source* of the edge, with b the *terminus*, or *terminating vertex*, and we say that b is *adjacent from* a and that a is *adjacent* to b. In addition, if $a \neq b$, then $(a, b) \neq (b, a)$. An edge of the form (a, a) is called a *loop* at a.

Example. Suppose that $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 1), (1, 2), (1, 4), (3, 2)\}$. Then

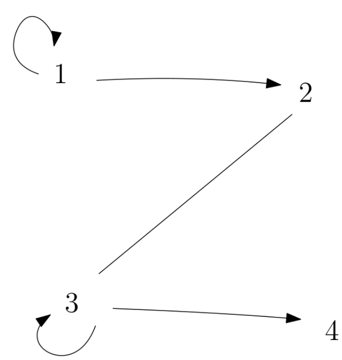


We can interpret a relation on a set A as a directed graph.

Example. Suppose that $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2)\}$. Then we have

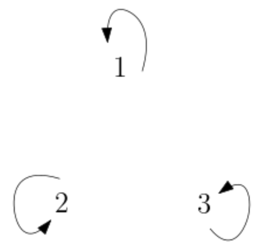


or

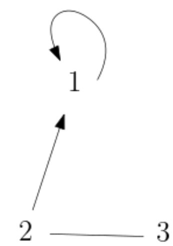


The relation matrix M is also called the adjacency matrix of the associated graph.

Remark. A relation is reflexive if and only if its directed graph has a loop at each vertex.

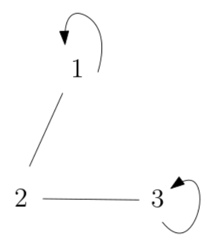


reflexive

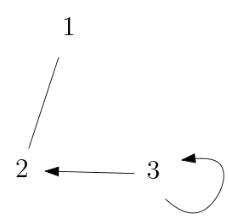


not reflexive

Remark. A relation is symmetric if and only if its directed graph contains only loops and undirected edges.



symmetric



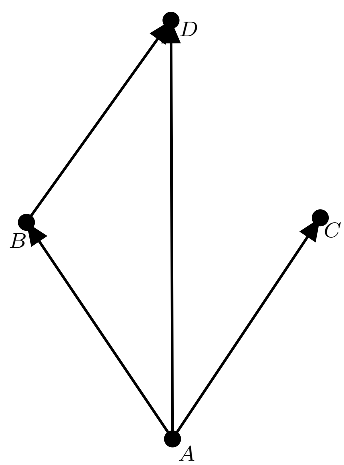
not symmetric

3 7.3 Partial Orders: Hasse Diagrams

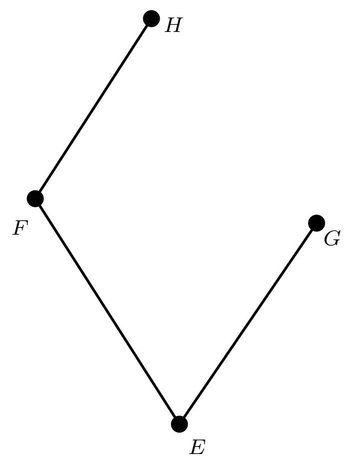
A partial order on a set A is a relation on A that is reflexive, antisymmetric, and transitive.

Example: Let $A = \{1, 2, 3, 4\}$ and let R be defined by xRy if and only if $x \mid y$. Then we have $R = \{(1,1), (2,2), (3,3), (4,4), (1,2), (1,3), (1,4), (2,4)\}$.

Directed Graph



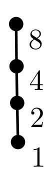
Hasse Diagram



If R is a partial order on A then we draw a line *up* from x to y if xRy and there is no z such that xRz and zRy.

Examples:

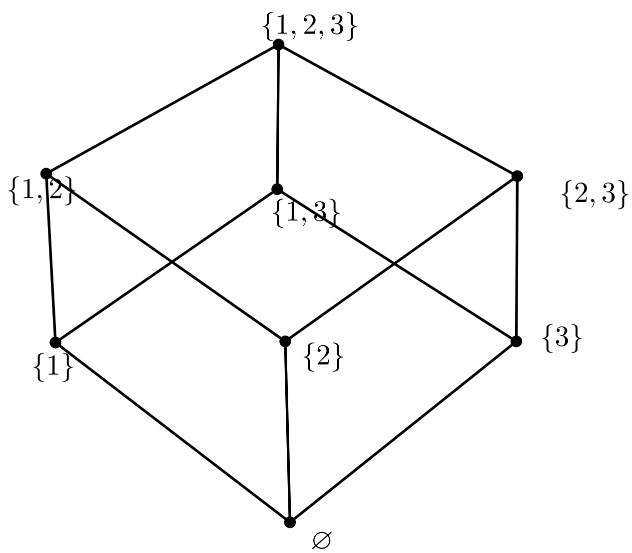
- If $A = \{1, 2, 4, 8\}$ and R is "divides."



- If $A = \{2, 3, 5, 7\}$ and R is "divides."

$$2 \quad 3 \quad 5 \quad 7$$

- If $B = \{1, 2, 3\}$, $A = \mathcal{P}(\{1, 2, 3\})$ and R is $\subseteq$.



3.1 Totally ordered poset

If (A, R) is a poset, we say that A is *totally ordered* (or, *lineraly ordered*) if for all $x, y \in A$ either xRy or yRx. In this case R is called *total order* (or, a *linear order*).

Examples

- $(\mathbb{Z}, \leq)$ is totally ordered.
- $(P(\{a, b\}), \subseteq)$ is not totally ordered. Indeed, $\{a\} \not\subseteq \{b\}$ and $\{b\} \not\subset \{a\}$.

Question: Given a partially ordered set, can we "extend" this relation so that the order becomes total? YES: topological sorting.

3.2 Maximal and minimal

If (A, R) is a psoet, then an element $x \in A$ is called a *maximal* element of A if for all $a \in A$, $a \neq x \Rightarrow (x, a) \notin R$. An element $y \in A$ is called *minimal* elment of A if whenever $b \in A$ and $b \neq y$, then $(b, y) \notin R$.

Example: Let $A = \mathcal{P}(\{1, 2, 3\})$ and $R = \subseteq$. Then $\{1, 2, 3\}$ is the maximal element and $\varnothing$ is the minimal element.

Let A be the collection of proper subsets of $\{1, 2, 3\}$ ordered by inclusion. Then. $\{1, 2\}$ is a maximal element and so is $\{2, 3\}$. Recall: for S and T sets, we have $S \subseteq T$ if $\forall x, x \in S \Rightarrow x \in T$.

3.3 Conditions for maximal and minimal element in a set

If (A, R) is a nonempty poset and A is finite, then A has a maximal element and a minimal element.

Proof. Let $a_1 \in A$. If there is no element $a \in A$ where $a \neq a_1$ and $a_1 R a$, then a_1 is maximal. Otherwise there is an element $a_2 \in A$ with $a_2 \neq a_1$ and $a_1 R a_2$. If no element $a \in A$, $a \neq a_2$, satisfies $a_2 R a$, then a_2 is maximal. Otherwise we can find $a_3 \in A$ so that $a_3 \neq a_2$, $a_3, \neq a_1$ while $a_1 R a_2$ and $a_2 R a_3$. Continuing in this manner, since A is finite, we get to an element $a_n inA$ with $(a_n, a) \notin R$ for all $a \in A$ where $a \neq a_n$, so a_n is maximal. The proof for a minimal element follows in a similar way. $\square$

3.4 Greatest and least element

If (A, R) is a poset, then an element $x \in A$ is called a *least* element if xRa for all $a \in A$. Element $y \in A$ is called a *greatest* element if aRy for all $a \in A$.

Examples

- $(\mathcal{P}(\{1, 2, 3\}), \subseteq)$ has a maximal and a greatest element of $\{1, 2, 3\}$.
- $(\mathcal{P}(\{1, 2, 3\}) \setminus \{\{1, 2, 3\}\}, \subseteq)$ has maximal elements (e.g., $\{1, 2\}$) but no greatest element.

3.5 Greatest and least elements are unique

If the poset (A, R) has a greatest (least) element, then that element is unique.

Proof. Suppose that $x, y \in A$ and that both are greatest elements. Since x is a greatest element, yRx. Likewise, xRy because y is a greatest element. As R is antisymmetric, it follows that $x = y$. The proof for the least element is similar. $\square$

3.6 Lower and upper bounds

Let (A, R) be a poset with $B \subseteq A$. An element $x \in A$ is called a *lower bound* of B if xRb for all $b \in B$. Likewise, an element $y \in A$ is called an *upper bound* of B if bRy for all $b \in B$.
An element $x' \in A$ is called a *greatest lower bound* (glb) of B if it is a lower bound of B and if for all other lower bounds x'' of B we have $x''Rx'$. Similarly $y' \in A$ is a *least upper bound* (lub) of B if it is an upper bound of B and if $y'Ry''$ for all other upper bounds y'' of B.

3.7 Unique lub/glb

If A, R is a poset and $B \subseteq A$, then B has at most one lub (glb).

3.8 Lattice

The poset (A, R) is called a *lattice* if for all $x, y \in A$ the elements $\text{lub}\{x, y\}$ and $\text{glb}\{x, y\}$ both exist in A.

Example: Let S be a set. Then $(\mathcal{P}(S), \subseteq)$ is a lattice.

4 7.4 Equivalence Relations and Partitions

Given a set A and index set I, let $\varnothing \neq A_i \subseteq A$ for each $i \in I$. Then $\{A_i\}_{i \in I}$ is a *partition* of A if

a) $A = \bigcup_{i \in I} A_i$ and

b) $A_i \cap A_j = \varnothing$ for all $i, j \in I$ where $i \neq j$.

Each subset A_i is called a *cell* or *block* of the partition.

4.1 Examples

- Let $A = \{1, 2, \ldots, 10\} \subseteq \mathbb{Z}$. The following are partitions of A:

1. $\{A_1, A_2\}$ where
$$A_1 = \{1, \ldots, 5\} \quad \text{and} \quad A_2 = \{6, \ldots, 10\}.$$
2. $\{A_1, A_2, A_3\}$ where
$$A_1 = \{1, 2, 3\}, \quad A_2 = \{4, 5, 6\}, \quad \text{and} \quad A_3 = \{7, 8, 9, 10\}.$$
3. $\{A_i\}_{1 \le i \le 5}$ where
$$A_i = \{i, i+5\}.$$

- $\{A_i\}_{i \in \mathbb{Z}}$ where $A_i = [i, i+1) \subseteq R$ is a partition of $\mathbb{R}$.

4.2 Equivalence class

Let R be an equivalence relation on a set A. For each $x \in A$, the *equivalence class* of x, denoted $[x]$, is defined by $[x] = \{y \in A \mid yRx\}$.

4.3 Example

Consider the relation $\equiv_4$ on $\mathbb{Z}$ defined by
$$x \equiv_4 y \Leftrightarrow 4 \mid (x - y).$$

- $[0] = \{y \in \mathbb{Z} \mid y \equiv_4 = 0\} \Rightarrow \{y \in \mathbb{Z} \mid 4 \mid y\} \Rightarrow \{\ldots, -8, -4, 0, 4, 8, \ldots\}$.
- $[1] = \{\ldots, -3, 1, 5, 9, \ldots\}$.
- $[2] = \{\ldots, -2, 2, 6, \ldots\}$.
- $[3] = \{\ldots, -5, -1, 3, 7, \ldots\}$.

Remark: $\{[0], [1], [2], [3]\}$ is a partition of $\mathbb{Z}$.

Question: Is this always the case? That is, if R is an equivalence relation on A, does the collection of equivalence classes form a partition of A? Yes!

4.4 Equivalence relation

If R is an equivlanece relation on a set A, and $x, y \in A$ then

(a) $x \in [x]$;

(b) xRy if and only if $[x] = [y]$;

(c) $[x] = [y]$ or $[x] \cap [y] = \varnothing$.

Proof. (a) This result follows from the reflexive property of R.

(b) If xRy, let $w \in [x]$. Then wRx and because R is transitive, wRy. Hence $w \in [y]$ and $[x] \subseteq [y]$. With R symmetric, $xRy \Rightarrow yRx$. So if $t \in [y]$, then tRy and by the transitive property, tRx. Hence $t \in [x]$ and $[y] \subseteq [x]$. Consequently, $[x] = [y]$. Conversely, let $[x] = [y]$. Since $x \in [x]$ by part (a), then $x \in [y]$ or xRy.

(c) This property tells us that two equivalence classes can be related in only one of two possible ways. Either they are idenitcal or they are disjoint.

We assume that $[x] \neq [y]$ and show how it then follows that $[x] \cap [y] = \varnothing$. If $[x] \cap [y] \neq \varnothing$, then let $v \in A$ with $v \in [x]$ and $v \in [y]$. Then vRx, vRy, and, since R is symmetric, xRv. Now (xRv and vRy) $\Rightarrow xRy$, by the transitive property. Also $xRy \Rightarrow [x] = [y]$ by part (b). This contradicts the assumption that $[x] \neq [y]$, so we reject the supposition that $[x] \cap [y] \neq \varnothing$, and the result follows.

$\square$

Corollary: If R is an equivalence relation on A then

$$\{[x] \mid [x] \text{ is the equivalence class of } x\}$$

is a partition of A. This tells us that some partition arise through equivalence relations. In fact, all partitions arise in this way!

4.5 Equivalence classes and partition

If P is a partition of A and R_p is defined as

$$xR_py \Leftrightarrow x \text{ and } y \text{ belong to the same cell of } A$$

then P consists of the equivalence classes of R_p.
And for any set A, there is a one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A.