MATH 2135 Linear Algebra
3.C Matrices

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Contents

1 Representing a Linear Map by a Matrix

1.1 Definition of a matrix

Let m and n denote positive integers. An m-by-n *matrix* is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

1.2 Definition of the matrix of a linear map

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The *matrix* of T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \dots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_n))$ is used.

$$\mathcal{M}(T) = A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

1.3 Example

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by $T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$. Find the matrix of T with respect to the standard bases of $\mathbf{F}^2$ and $\mathbf{F}^3$.

Since $T(1, 0) = (1, 2, 7)$ and $T(0, 1) = (3, 5, 9)$, then

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{bmatrix}.$$

$\square$

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find the matrix of D with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$.

Since $(x^n)' = nx^{n-1}$, then we have

$$\mathcal{M}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\square$

2 Addition and Scalar Multiplication of Matrices

2.1 Definition of matrix addition

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} + \begin{bmatrix} C_{1,1} & \dots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \dots & C_{m,n} \end{bmatrix} = \begin{bmatrix} A_{1,1} + C_{1,1} & \dots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \dots & A_{m,n} + C_{m,n} \end{bmatrix}$$

In other words, $(A + C)_{j,k} = A_{j,k} + C_{j,k}$.

2.2 The matrix of the sum of linear maps

Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

2.3 Definition of scalar multiplication of a matrix

The product of a scalar and a matrix is the matrix obtained by mutliplying each entry in the matrix by the scalar:

$$\lambda \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} = \begin{bmatrix} \lambda A_{1,1} & \dots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \dots & \lambda A_{m,n} \end{bmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$

2.4 The matrix of a scalar times a linear map

Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

2.5 Notation of $\mathbf{F}^{m,n}$

For m and n positive integers, the set of all m-by$n$ matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$.

2.6 $\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space with dimension mn.

3 Matrix Multiplication

3.1 Definition of matrix multiplication

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then AC is defined to be the m-by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r}C_{r,k}.$$

In other words, the entry in row j, column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing. Matrix multiplication is not commutative, but it is associative and distributive.

3.2 The matrix of the product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

4 Isomorphism

4.1 Definition of isomorphism

Let V, W be vector spaces over $\mathbf{F}$ and let $T \in \mathcal{L}(V, W)$. We say that T is an *isomorphism* if T is bijective (i.e. injective and surjective.

4.2 Inverse function

If $T \in \mathcal{L}(V, W)$ is an isomorphism, then the inverse function $T^{-1} \in \mathcal{L}(W, V)$ exists and is linear.

Example. Let $V = \mathbb{R}^4$ and let $W = \mathcal{P}_3(\mathbb{R})$. A basis of V is $v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,0), v_4 = (0,0,0,1)$, and a basis of $\mathcal{P}_3(\mathbb{R})$ is $w_1 = 1, w_2 = x, w_3 = x^2, w_4 = x^3$. Define an isomorphism $T \in \mathcal{L}(\mathbb{R}^4 \to \mathcal{P}_3(\mathbb{R}))$, namely the unique linear map such that

$$T(v_1) = w_1, \ldots, T(v_4) = w_4.$$

The inverse $T^{-1} \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4)$ is the unique linear map such that

$$T^{-1}w_1 = v_1, \ldots, T^{-1}w_4 = v_4.$$

More concretely, we can describe T and T^{-1} like this:

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a + bx + cx^2 + dx^3$$

and

$$T^{-1}(a + bx + cx^2 + dx^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

4.3 Definition of isomorphic

Vector spaces V, W are *isomorphic* if there exists an isomorphism between them.

4.4 Finite-dimensional vector spaces are isomorphic

Finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.