MATH 2135 Linear Algebra

Chapter 3 Linear Maps

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Contents

1 The Vector Spaces of Linear Maps (3.A)

1.1 Definition and Examples of Linear Maps

The words "map," "mapping," "function" all mean exactly the same thing. Let X, Y be sets. A *function* $f : X \to Y$ is an operation that assigns to each element $x \in X$ a unique element $y \in Y$, called $y = f(x)$. X is called the *domain* of f, and Y is called the *codomain* of f. We also say that f is a "map" from X to Y.

1.1.1 Definition of a linear map

A *linear map* or *linear transformation* from V to W is a function $T : V \to W$ with the following properties:

1. Additivity.
$$T(u+v) = Tu + Tv \text{ for all } u, v \in V$$

2. Homogeneity.
$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbf{F} \text{ and all } v \in V.$$

1.1.2 Notation $\mathcal{L}(V, W)$

The set of all linear maps from V to W is denoted by
$$\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is linear }\}.$$

In the case that $V = W$, a linear function $T : V \to V$ is called an *operator* on V. In other words, $\mathcal{L}(V, V)$ is the set of operators on V.

1.1.3 Examples of linear maps

1. **zero function** The zero function is denoted by $0 \in \mathcal{L}(V, W)$ and defined with
$$0v = 0 \quad \text{or} \quad f(v) = 0.$$
The 0 on the left side is a function from $V to W$, whereas the right one is the additive identity in W. The zero function is linear.

2. **identity function** The identity map, denoted by I, is the function that takes each element to itself. It is defined as $I \in \mathcal{L}(V, V)$ such that
$$Iv = v.$$
The identity map is linear.

3. **differentiation** Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by

$$Dp = p'.$$

D is a linear function because: $(f+g)' = f' + g'$ and $(\lambda f)' = \lambda f'$ whenever f, g are differentiable and λ is a constant.

4. **integration (antiderivative)** Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ such that

$$T(p) = \int_0^x p(t)dt.$$

T is a linear function since the integral of the sum of two functions equals the sum of the integrals, and the integral of a constant times a function equals the constant times the integral of the function.

5. **integration (definite integrals)** Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ such that

$$T(p) = \int_0^1 p(x)dx.$$

T is also linear with similar reasons as the previous example.

6. **multiplication by** x^2 Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ such that

$$(Tp)(x) = x^2 p(x).$$

7. **backward shift** Recall that $\mathbf{F}^\infty$ is the vector space of all sequences of elements of $\mathbf{F}$. Define $B \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ such that

$$B(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

8. **forward shift** Define $F \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ such that

$$F(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

9. **recurrence relation** Consider the Fibonacci sequence such taht

$$a_{n+2} = a_n + a_{n+1}, \qquad n \geq 3.$$

For example, we have $(a_1, a_2, a_3, a_4, a_5, \dots) = (1, 1, 2, 3, 5, \dots)$. We can also write the reccurence in terms of the backward (and forward) shfit operators:

$$\begin{aligned} a &= (a_1, a_2, a_3, a_4, a_5, \dots) \\ Ba &= (a_2, a_3, a_4, a_5, a_6, \dots) \\ B^2 a = B(Ba) &= (a_3, a_4, a_5, a_6, a_7, \dots) \end{aligned}$$

Cosnider the equation $B^2a = Ba + a$. We can do algebra with such equation:

$$B^2a - Ba - a = 0 \qquad \Leftrightarrow \qquad (B^2 - B - I)a = 0.$$

10. **from $\mathbb{R}^3$ to $\mathbb{R}^2$** Define $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

11. **from $\mathbf{F}^n$ to $\mathbf{F}^m$** Let m and n be positive integers, and let $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, and define $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ by

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1, \cdots + A_{m,n}x_n).$$

1.1.4 Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

Proof. Let us prove the existence of a linear map T with the desired property. Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n,$$

where $c_1, \ldots, c_n$ are arbitrary elements of $\mathbf{F}$. Since $v_1, \ldots, v_n$ is a basis of V, the equation does define a function T since each element of V can be uniquely written in the form of $c_1v_1 + \cdots + c_nv_n$. Suppose that $c_j = 1$ and the other c's being equal to 0, then we have $Tv_j = w_j$.

If $u, v \in V$ with $u = a_1v_1 + \cdots + a_nv_n$ and $v = c_1v_1 + \cdots + c_nv_n$, then

$$\begin{aligned} T(u + v) &= T((a_1 + c_1)v_1 + \cdots + (a_n + c_n)v_n) \\ &= (a_1 + c_1)w_1 + \cdots + (a_n + c_n)w_n \\ &= (a_1w_1 + \cdots + a_nw_n) + (c_1w_1 + \cdots + c_nw_n) \\ &= Tu + Tv. \end{aligned}$$

If $\lambda \in \mathbf{F}$ and $v = c_1 v_1 + \cdots + c_n v_n$, then

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1 v_1 + \cdots + \lambda c_n v_n) \\ &= \lambda c_1 w_1 + \cdots + \lambda c_n w_n \\ &= \lambda (c_1 w_1 + \cdots + c_n w_n) \\ &= \lambda T v. \end{aligned}$$

Thus, T is a linear map from V to W. Now we need to prove that it is unique. Suppose that $T \in \mathcal{L}(V, W)$ and that $Tv_j = w_j$ for $j = 1, \ldots, n$. Let $c_1, \ldots, c_n \in \mathbf{F}$. The homogeneity property of T implies that $T(c_j v_j) = c_j w_j$. The additivity of T now implies that

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n.$$

Then T is unqiuely determined on $\operatorname{span}(v_1, \ldots, v_n)$. Because $v_1, \ldots, v_n$ is a basis of V, this implies that T is uniquely determined on V. $\square$

1.2 Algebraic Operations on $\mathcal{L}(V, W)$

1.2.1 Addition and scalar multiplication on $\mathcal{L}(V, W)$

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \qquad \text{and} \qquad (\lambda T)(v) = \lambda (Tv)$$

for all $v \in V$.

1.2.2 $\mathcal{L}(V, W)$ is a vector space

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space (it holds all the 8 laws of vector space). Note that the additive identity is the zero linear map or function.

1.2.3 Product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$. In other words, ST is the composition $S \circ T$ of two functions. Note that ST is defined only when T maps into the domain of S.

1.2.4 Algebraic properties of products of linear maps

1. Associativity:
$$(T_1T_2)T_3 = T_1(T_2T_3)$$
where T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1.

2. Identity:
$$TI = IT = T$$
where $I \in \mathcal{L}(W, W)$ and $T \in \mathcal{L}(V, W)$.

3. Distributive properties:
$$(S_1 + S_2)T = S_1T + S_2T \qquad \text{and} \qquad S(T_1 + T_2) = ST_1 + ST_2$$
where $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

4. Powers: If $T \in \mathcal{L}(V, V)$, we define $T^n \in \mathcal{L}(V, V)$ by
$$\begin{aligned} T^2 &= TT \qquad (\text{i.e. } T^2v = T(Tv)) \\ T^3 &= TTT \\ &\vdots \\ T^n &= TT^{n-1}. \end{aligned}$$

1.2.5 Example

Suppose that $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map and $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the multiplication by x^2. Show that $TD \neq DT$.

We have
$$((TD)p)(x) = x^2p'(x)$$
but
$$((DT)p)(x) = x^2p'(x) + 2xp(x).$$
In other words, differentiating and then multiplying by x^2 is not the same as multiplying by x^2 and then differentiating. $\square$

1.2.6 Linear maps take 0 to 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof. We can prove this by additivity or homogeneity.

- By additivity: $T(0) = T(0 + 0) = T(0) + T(0)$. Then add the additive inverse of $T(0)$ to each side of the equation to conclude $T(0) = 0$.

- By homogeneity: Suppose that $\lambda = 0$ and $u = 0$. Then $T(0 \cdot 0) = 0 \cdot T(0) \Rightarrow T(0) = 0$.

□

2 Null Spaces and Ranges (3.B)

2.1 Null Space and Injectivity

2.1.1 Definition of null space

For $T \in \mathcal{L}(V, W)$, the *null space* or *kernel* of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0:

$$\text{null } T = \{v \in V \mid Tv = 0\}.$$

2.1.2 Examples of null spaces

- If $T : V \to W$ is the zero map where $Tv = 0$ for every $v \in V$, then null $T = V$.
- If $T : V \to V$ is the identity function, then null $T = \{0\}$.
- Suppose $\phi \in \mathcal{L}(\mathbb{R}^3, \mathbf{F})$ is defined by $\phi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$. Then null $\phi = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + 2z_2 + 3z_3 = 0\}$. A basis of null ϕ is $(-2, 1, 0), (-3, 0, 1)$.
- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. The only functions whose derivative equals the zero function are the constant functions. Then, the null space of D equals the set of constant functions.

  $$\text{null } D = \{a_0 \mid a_0 \in \mathbb{R}\}.$$

- Suppose $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the multiplication by x^2 map defined by $(Tp)(x) = x^2 p(x)$. The only polynomial p such that $x^2 p(x) = 0$ for all $x \in \mathbb{R}$ is the 0 polynomial. Then null $T = \{0\}$.
- Suppose $B \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ is the backward shift defined by

  $$B(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

  Then $B(x_1, x_2, x_3, \dots) = 0$ if and only if $x_2 = x_3 = \cdots = 0$. So we have null $B = \{(a, 0, 0, \dots) \mid a \in \mathbf{F}\}$.

2.1.3 The null space is a subspace

Suppose $T \in \mathcal{L}(V, W)$. Then null T is a subspace of V.

Proof. Since T is a linear map, we know that $T(0) = 0$ then $0 \in$ null T. Suppose $u, v \in$ null T, then

$$T(u + v) = Tu + Tv = 0 + 0 = 0.$$

Since $u + v \in$ null T, then it is closed under addition. Suppose that $u \in$ null T and $\lambda \in \mathbf{F}$. Then

$$T(\lambda u) = \lambda Tu = \lambda 0 = 0.$$

Hence $\lambda u \in$ null T and it is closed under scalar multiplication. Thus, null T is a subspace of V. $\square$

2.1.4 Injective

A function $T : V \to W$ is called *injective* or *one-to-one* if $Tu = Tv$ implies $u = v$. This can be rephrased to say that T is injective if $u \neq v$ implies that $Tu \neq Tv$.

2.1.5 Injectivity is equivalent to null space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$.

Proof. Suppose T is injective. We want to prove that null $T = \{0\}$. We already know that $\{0\} \subset$ null T since $0 \in$ null T. To prove that null $T \subset \{0\}$, suppose $v \in$ null T. Then

$$T(v) = 0 = T(0).$$

Since T is injective, the equation above implies $v = 0$, then we can conclude that null $T = \{0\}$, as desired.
Suppose that null $T = \{0\}$ and we need to show that T is injective. Suppose $u, v \in V$ and $Tu = Tv$. Then

$$0 = Tu - Tv = T(u - v).$$

Then $u - v \in$ null T, which equals $\{0\}$. Hence $u - v = 0 \Rightarrow u = v$ which implies T is injective, as desired. $\square$

2.2 Range and Surjectivity

2.2.1 Definition of range

For T a function from V to W, the *range* of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\text{range } T = \{Tv \mid v \in V\}.$$

2.2.2 Examples of range

- If T is the zero map from V to W, in other words if $Tv = 0$ for every $v \in V$, then range $T = \{0\}$.

- Suppose $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ is defined by $T(x, y) = (2x, 5y, x + y)$, then range $T = \{(2x, 5y, x + y) \mid x, y \in \mathbb{R}\}$. A basis of range T is $(2, 0, 1), (0, 5, 1)$.

- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map by $Dp = p'$. Because for every polynomial $q \in \mathcal{P}(\mathbb{R})$ there exists a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that $p' = q$, the range of D is $\mathcal{P}(\mathbb{R})$.

- Suppose $B \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ is the backshift operator defined by $B(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$. Then

$$\begin{aligned}\text{range } B &= \{B(x_1, x_2, \dots) \mid (x_1, x_2, dots) \in \mathbf{F}^\infty\} \\ &= \{(x_2, x_3, x_4, \dots) \mid x_1, x_2, \dots \in \mathbf{F}\} \\ &= \mathbf{F}^\infty.\end{aligned}$$

- Let $F \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ be the forward shift operator defined by $F(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$. Then

$$\begin{aligned}\text{range } F &= \{F(x_1, x_2, \dots) \mid (x_1, x_2, \dots) \in \mathbf{F}^\infty\} \\ &= \{(0, x_1, x_2, \dots) \mid x_1, x_2, \dots \in \mathbf{F}\}.\end{aligned}$$

  This is a proper subspace of $\mathbf{F}^\infty$.

2.2.3 The range is a subspace

If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W.

Proof. Suppose $T \in \mathcal{L}(V, W)$, then $T(0) = 0$ which implies that $0 \in$ range T. If $w_1, w_2 \in$ range T, then there exist $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. So

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2.$$

Hence $w_1 + w_2 \in$ range T, so it is closed under addition. If $w \in$ range T and $\lambda \in \mathbf{F}$, then there exists $v \in V$ such that $Tv = w$. Thus

$$T(\lambda v) = \lambda T v = \lambda w.$$

Hence $\lambda w \in$ range T, and it is closed under scalar multiplication. Hence, range T is a subspace of W. □

2.2.4 Surjective

A function $T : V \to W$ is called *surjective* or *onto* if its range equals W.

2.2.5 Example

The differentiation map $D \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_5(\mathbb{R}))$ defined by $Dp = p'$ is not surjective, because the polynomial x^5 is not in the range of D. However, the differentiation map $S \in \mathcal{L}(\mathcal{P}_5(\mathbb{R}), \mathcal{P}_4(\mathbb{R}))$ defined by $Sp = p'$ is surjective, because its range equals $\mathcal{P}_4(\mathbb{R})$, which is now the vector space into which S maps.

2.3 Fundamental Theorem of Linear Maps

Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Proof. Let $u_1, \ldots, u_m$ be a basis of null T, then dim null $T = m$. The linearly independent list $u_1, \ldots, u_m$ can be extended to a basis

$$u_1, \ldots, u_m, v_1, \ldots, v_n$$

of V, thus dim $V = m+n$. We need to show that range T is finite-dimensional and dim range $T = n$. We will do this by proving that $Tv_1, \ldots, Tv_n$ is a basis of range T.

Let $v \in V$. Since $u_1, \ldots, u_m, v_1, \ldots, v_n$ spans V, we can write

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

where $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$. Applying T to both sides, we get

$$Tv = b_1 Tv_1 + \cdots + b_n Tv_n,$$

where all the terms of Tu_j disappeared since $u_j \in \text{null } T$. The equation above implies that $Tv_1, \ldots, Tv_n$ spans range T, then range T is finite-dimensional.

To show that $Tv_1, \ldots, Tv_n$ is linearly independent, suppose $c_1, \ldots, c_n \in \mathbf{F}$ and $c_1 Tv_1 + \cdots + c_n Tv_n = 0$. Then

$$T(c_1 v_1 + \cdots + c_n v_n) = 0$$

and $c_1 v_1 + \cdots + c_n v_n \in \text{null } T$. Because $u_1, \ldots, u_m$ spans null T, we can write $c_1 v_1 + \cdots + c_n v_n = d_1 u_1 + \cdots + d_m u_m$ where $d_1, \ldots, d_m \in \mathbf{F}$. The equation implies all c's and d's are 0 (since $u_1, \ldots, u_m, v_1, \ldots, v_n$ is linearly independent). Thus $Tv_1, \ldots, Tv_n$ is linearly independent and hence is a basis of range T, as desired. □

2.3.1 A map to a smaller dimensional space is not injective

Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim \text{ null } T &= \dim V - \dim \text{ range } T \\ &\geq \dim V - \dim W \\ &> 0. \end{aligned}$$

The inequality states that dim null $T > 0$ which means null T contains vectors other than 0. Thus, T is not injective since null $T \neq \{0\}$. □

2.3.2 A map to a larger dimensional space is not surjective

Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim \text{ range } T &= \dim V - \dim \text{ null } T \\ &\leq \dim V \\ &< \dim W. \end{aligned}$$

The inequality states that $\dim \text{range } T < \dim W$. This means that range T $\neq W$, so T is not surjective. □

2.3.3 Homogenous system of linear equations

A homogenous system of linear equations with more variables than equations has nonzero solutions.

Proof. Consider $T : \mathbf{F}^n \to \mathbf{F}^m$ defined by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} x_k \right)$$

where $T(x_1, \ldots, x_n) = 0$ and 0 here is the additive identity in $\mathbf{F}^m$, which is the list of length m of all 0's. We have a homogenous system of m linear equations with n variables. If $\dim \mathbf{F}^n = n > \dim \mathbf{F}^m = m$, then T is not injective. We also have null $T \neq \{0\}$ which implies that there exists some $v \in$ null T such that $v \neq 0$. So, the system $Tv = 0$ has nonzero solutions. $\square$

2.3.4 Inhomogenous system of linear equations

An inhomogenous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

Proof. Define $T : \mathbf{F}^n \to \mathbf{F}^m$ by

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right)$$

where $T(x_1, \ldots, x_n) = (c_1, \ldots, c_m)$. We have a system of m equations with n variables $x_1, \ldots, x_n$. We see that T is not surjective if $n < m$ since range $T \neq \mathbf{F}^m$. $\square$

3 Matrices (3.C)

3.1 Representing a Linear Map by a Matrix

3.1.1 Definition of a matrix

Let m and n denote positive integers. An m-by-n *matrix* is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A.

3.1.2 Definition of the matrix of a linear map

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The *matrix* of T with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \dots + A_{m,k}w_m.$$

If the bases are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_n))$ is used.

$$\mathcal{M}(T) = A = \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix}$$

3.1.3 Example

Suppose $T \in \mathcal{L}(\mathbf{F}^2, \mathbf{F}^3)$ is defined by $T(x, y) = (x + 3y, 2x + 5y, 7x + 9y)$. Find the matrix of T with respect to the standard bases of $\mathbf{F}^2$ and $\mathbf{F}^3$.

Since $T(1, 0) = (1, 2, 7)$ and $T(0, 1) = (3, 5, 9)$, then

$$\mathcal{M}(T) = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{bmatrix}.$$

$\square$

Suppose $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ is the differentiation map defined by $Dp = p'$. Find the matrix of D with respect to the standard bases of $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$.

Since $(x^n)' = nx^{n-1}$, then we have

$$\mathcal{M}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

□

3.2 Addition and Scalar Multiplication of Matrices

3.2.1 Definition of matrix addition

The *sum of two matrices of the same size* is the matrix obtained by adding corresponding entries in the matrices:

$$\begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} + \begin{bmatrix} C_{1,1} & \dots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \dots & C_{m,n} \end{bmatrix} = \begin{bmatrix} A_{1,1} + C_{1,1} & \dots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \dots & A_{m,n} + C_{m,n} \end{bmatrix}$$

In other words, $(A + C)_{j,k} = A_{j,k} + C_{j,k}$.

3.2.2 The matrix of the sum of linear maps

Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$.

3.2.3 Definition of scalar multiplication of a matrix

The product of a scalar and a matrix is the matrix obtained by mutliplying each entry in the matrix by the scalar:

$$\lambda \begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{bmatrix} = \begin{bmatrix} \lambda A_{1,1} & \dots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \dots & \lambda A_{m,n} \end{bmatrix}$$

In other words, $(\lambda A)_{j,k} = \lambda A_{j,k}$

3.2.4 The matrix of a scalar times a linear map

Suppose $\lambda \in \mathbf{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.

3.2.5 Notation of $\mathbf{F}^{m,n}$

For m and n positive integers, the set of all m-by$n$ matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$.

3.2.6 $\dim \mathbf{F}^{m,n} = mn$

Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space with dimension mn.

3.3 Matrix Multiplication

3.3.1 Definition of matrix multiplication

Suppose A is an m-by-n matrix and C is an n-by-p matrix. Then AC is defined to be the m-by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}.$$

In other words, the entry in row j, column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing. Matrix multiplication is not commutative, but it is associative and distributive.

3.3.2 The matrix of the product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.

3.4 Isomorphism

3.4.1 Definition of isomorphism

Let V, W be vector spaces over $\mathbf{F}$ and let $T \in \mathcal{L}(V, W)$. We say that T is an *isomorphism* if T is bijective (i.e. injective and surjective.

3.4.2 Inverse function

If $T \in \mathcal{L}(V, W)$ is an isomorphism, then the inverse function $T^{-1} \in \mathcal{L}(W, V)$ exists and is linear.

Example. Let $V = \mathbb{R}^4$ and let $W = \mathcal{P}_3(\mathbb{R})$. A basis of V is $v_1 = (1,0,0,0), v_2 = (0,1,0,0), v_3 = (0,0,1,0), v_4 = (0,0,0,1)$, and a basis of $\mathcal{P}_3(\mathbb{R})$ is $w_1 = 1, w_2 = x, w_3 = x^2, w_4 = x^3$. Define an isomorphism $T \in \mathcal{L}(\mathbb{R}^4 \to \mathcal{P}_3(\mathbb{R}))$, namely the unique linear map such that

$$T(v_1) = w_1, \ldots, T(v_4) = w_4.$$

The inverse $T^{-1} \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}) \to \mathbb{R}^4)$ is the unique linear map such that

$$T^{-1}w_1 = v_1, \ldots, T^{-1}w_4 = v_4.$$

More concretely, we can describe T and T^{-1} like this:

$$T \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = a + bx + cx^2 + dx^3$$

and

$$T^{-1}(a + bx + cx^2 + dx^3) = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

3.4.3 Definition of isomorphic

Vector spaces V, W are *isomorphic* if there exists an isomorphism between them.

3.4.4 Finite-dimensional vector spaces are isomorphic

Finite-dimensional vector spaces are isomorphic if and only if they have the same dimension.