MATH 2135 Linear Algebra

Chapter 2 Finite-Dimensional Vector Spaces

Alyssa Motas

March 8, 2021

Contents

1 2.A Span and Linear Independence

1.1 Linear Combinations and Span

1.1.1 List of vectors

A "list" is an ordered finite sequence of things. For example, $1, 2, 5, 2, 3$ is a list of numbers, and $1, 2, 2, 5, 3$ is a different list. There can be repetitions. We usually write lists without parentheses. With lists, we do not usually specify ahead of time how many entries the list has.

1.1.2 Linear combination

A *linear combination* of a list $v_1, \ldots, v_m$ of vectors in V is a vector of the form

$$a_1 v_1 + \cdots + a_m vm,$$

where $a_1, \ldots, a_m \in \mathbf{F}$.

Examples.

- $(17, -4, 2)$ is a linear combination of $(2, 1, -3), (1, -2, 4)$ because

$$(17, -4, 2) = 6(2, 1, -3) + 5(1, -2, 4).$$

- $(17, -4, 5)$ is not a linear combination of $(2, 1, -3), (1, -2, 4)$ because there do not exist numbers $a_1, a_2 \in \mathbf{F}$ such that

$$(17, -4, 5) = a_1(2, 1, -3) + a_2(1, -2, 4).$$

1.1.3 Span

The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the span of $v_1, \ldots, v_m$, denoted $span(v_1, \ldots, v_m)$. In other words,

$$span(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m \mid a_1, \ldots, a_m \in \mathbf{F}\}.$$

The span of the empty list $()$ is defined to be $\{0\}$.

Examples.

From the previous examples, we have

- $(17, -4, 2) \in span((2, 1, -3), (1, -2, 4))$;
- $(17, -4, 5) \notin span((2, 1, -3), (1, -2, 4))$.

1.1.4 Span is the smallest containing subspace

The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

Remark. What this means is the following:

(1) $span(v_1, \dots, v_m)$ is a subspace of V containing $v_1, \dots, v_m$.

(2) If W is any subspace of V containing $v_1, \dots, v_m$, then $span(v_1, \dots, v_m) \subseteq W$.

Proof. Let us prove the two remarks above.

(1) To show that $span(v_1, \dots, v_m)$ is a subspace, we must prove the three subspace properties.

   (a) $0 = 0v_1 + \cdots + 0v_m$, so we have $0 \in span(v_1, \dots, v_m)$.

   (b) To show that it is closed under addition, let $v, w \in span(v_1, \dots, v_m)$ be arbitrary. We must show that $v + w \in span(v_1, \dots, v_m)$. By assumption,

$$v = a_1 v_1 + \cdots + a_m v_m$$

   for some $a_1, \dots, a_m$, and

$$w = b_1 v_1 + \cdots + b_m v_m$$

   for some $b_1, \dots, b_m$. Therefore,

$$\begin{aligned} v + w &= (a_1 v_1 + \cdots + a_m v_m) + (b_1 v_1 + \dots b_m v_m) \\ &= (a_1 + b_1) v_1 + \cdots + (a_m + b_m) v_m. \end{aligned}$$

   So the span is closed under addition.

   (c) To show that it is closed under scalar multiplication, assume $v \in span(v_1, \dots, v_m)$ and $k \in F$. We need to show that $kv \in span(v_1, \dots, v_m)$. There exists $a_1, \dots, a_m$ such that $v = a_1 v_1 + \cdots + a_m v_m$. Then

$$\begin{aligned} kv &= k(a_1 v_1 + \cdots + a_m v_m) \\ &= (ka_1) v_1 + \cdots + (ka_m) v_m. \end{aligned}$$

   Therefore, the span is closed under scalar multiplication.

Next, we show that $span(v_1, \dots, v_m)$ contains $v_1, \dots, v_m$. But

$$\begin{aligned} v_1 &= 1v_1 + 0v_2 + \cdots + 0v_m \\ v_2 &= 0v_1 + 1v_2 + \cdots + 0v_m \\ &\vdots \\ v_m &= 0v_1 + 0v_2 + \cdots + 1v_m. \end{aligned}$$

So each of $v_1, \dots, v_m \in span(v_1, \dots, v_m)$.

(2) We must show that $span(v_1, \dots, v_m)$ is the smallest. So let W be any subspace of V containing $v_1, \dots, v_m$. We must show $span(v_1, \dots, v_m) \subseteq W$.

Take any arbitrary $v \in span(v_1, \dots, v_m)$. We must show that $v \in W$. There exists $a_1, \dots, a_m$ such that

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Then, $v_1, \dots, v_m \in W$. But W is a subspace, so it is closed under scalar multiplication.

$$\Rightarrow a_1 v_1, \dots a_m v_m \in W.$$

Also, W is closed under addition, then

$$a_1 v_1 + \cdots + a_m v_m \in W$$

$$\Rightarrow v \in W.$$

Because $v \in span(v_1, \dots, v_m)$ was arbitrary, it follows

$$span(v_1, \dots, v_m) \subseteq W$$

as had to be shown.

$\square$

1.1.5 Definition of spanning

If $span(v_1, \ldots, v_m)$ equals V, we say that $v_1, \ldots, v_m$ spans V.

1.1.6 Definition of finite-dimensional vector space

A vector space is called *finite-dimensional* if some list of vectors in it spans the space. In other words, there exist $k \leq 0$, $v_1, \ldots, v_k$ such that

$$span(v_1, \ldots, v_k) = V.$$

Otherwise, V is called *infinite-dimensional.*

Examples.

- Recall that $P_m(F)$ is the vector space of polynomials (with coefficients in F) of degree at most m.

  $$P_m(F) = \{a_0 + a_1x + \cdots + a_mx^m \mid a_0, \cdots + a_m \in F\}.$$

  Then $P_m(F)$ is finite-dimensional. It is spanned by the following list of $m + 1$ polynomials: $1, x, x^2, \ldots, x^m$.

- Let $P(F)$ be the set of all polynomials with coefficients in F, regardless of degree. Then $P(F)$ is an infinite-dimensional vector space.

  Proof. Assume, for the sake of obtaining a contradiction, that $P(F)$ is finite dimensional. Then $P(F)$ is spanned by a finite list of polynomials, say $P_1, \ldots, P_k \in P(F)$. Let m be the largest degree of any of $P_1, \ldots, P_k$.

  Then clearly, any linear combination

  $$a_1P_1 + \cdots + a_kP_k$$

  also has degree at most m.

  So therefore, $x^{m+1} \in P(F)$ is not a linear combination of $P_1, \ldots, P_k$, contradicting the fact that $P_1, \ldots, P_k$ span $P(F)$. $\square$

1.2 Linear Independence

1.2.1 Definition

- A list $v_1, \ldots, v_m$ of vectors in V is called *linearly independent* if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1v_1 + \cdots + a_mv_m$ equal 0 is $a_1 = \cdots = a_m = 0$.

- The empty list () is also declared to be linearly independent.

1. To prove that some $v_1, \ldots, v_m$ are linearly independent, you have to do the following:

    - Take arbitrary $a_1, \ldots, a_m$
    - Assume $a_1v_1 + \cdots + a_mv_m = 0$
    - Must show $a_1, \ldots, a_m = 0$

2. To prove that some $v_1, \ldots, v_m$ are linearly dependent, you have to do the following:

    - Find specific $a_1, \ldots, am$, not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$.

1.2.2 Proposition

Suppose $v_1, \ldots, v_m$ are linearly independent, and let $w \in span(v_1, \ldots, v_m)$. Then, there exists a unique list of scalars $a_1, \ldots, a_m$ such that

$$w = a_1v_1 + \cdots + a_mv_m.$$

Proof. Existence. By definition of span, it exists.
Uniqueness. Suppose that w in two ways as a linear combination of $v_1, \ldots, v_m$:

$$w = a_1v_1 + \ldots a_mv_m$$

$$w = b_1v_1 + \cdots + b_mv_m.$$

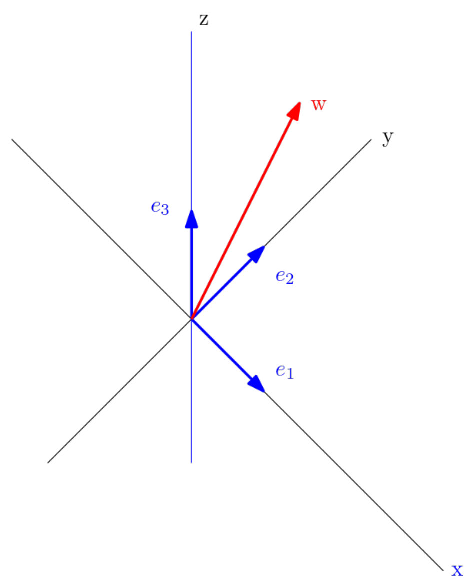
Then $(w - w) = (a_1v_1 + \cdots + a_mv_m) - (b_1v_1 + \cdots + b_mv_m)$. By vector space axioms,

$$0 = (a_1 - b_1)v_1 + \cdots + (a_m - b_m)v_m.$$

Because $v_1, \ldots, v_m$ are linearly independent, it follows that $a_1 - b_1 = 0, \ldots, a_m - b_m = 0$. Therefore, $a_1 = b_1, \ldots, a_m = b_m$. This proves uniqueness. $\square$

1.2.3 Coordinates

If $v_1, \ldots, v_m$ are linearly independent and $w = a_1 v_1 + \cdots + a_m v_m$, we call $a_1, \ldots, a_m$ the *coordinates* of w (with respect to the list $v_1, \ldots, v_m$).



$$w = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = ae_1 + be_2 + ce_3$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Examples.

In $P_3(\mathbb{R})$, let $P_1 = x^3 + 1, P_2 = x^3 - 2x, P_3 = 4x + 2$. Is P_1, P_2, P_3 linearly independent?
To answer the question, we attempt to rpove that they are linearly independent.

Proof. Take arbitrary $a_1, a_2, a_3 \in \mathbb{R}$. Assume $a_1 p_1 + a_2 p_2 + a_3 p_3 = 0$. In other words,

$$a_1(x^3 + 1) + a_2(x^3 - 2x) + a_3(4x + 2) = 0 \qquad (1)$$

From (1), we can make a system of equations and turn it into a matrix.

$$\begin{aligned} a_1 + a_2 &= 0 \\ -2a_2 + 4a_3 &= 0 \\ a_1 + 2a_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix} \simeq \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $a_3 = t, a_2 = 2t, a_1 = -2t$. Since we failed to show that $a_1, a_2, a_3 = 0$, then P_1, P_2, P_3 are linearly dependent.
For $t = 1$, we get $a_1 = -2, a_2 = 2, a_3 = 1$, and we find that $-2P_1+2P_2+1P_3 = 0$. □

1.2.4 Linear Dependence Lemma

Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such taht the following hold:

(a) $v_j \in span(v_1, \ldots, v_{j-1})$;

(b) if the jth term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $span(v_1, \ldots, v_m)$.

Proof. By assumption, $v_1, \ldots, v_m$ are linearly dependent. Therefore, there exist scalars $a_1, \ldots, a_m \in F$ such that $a_1v_1 + \cdots + a_mv_m = 0$ but not all of $a_1, \ldots, a_m$ are 0.
Let j be the largest index such that $a_j \neq 0$. Then $a_{j+1}, \ldots, a_m = 0$.
From our assumption, we have $a_1v_1 + \cdots + a_jv_j = 0$. We can write

$$a_jv_j = -a_1v_1 - a_2v_2 - \cdots - a_{j-1}v_{j-1}.$$

But since $a_j \neq 0$, it has a multiplicative inverse.

$$v_j = -\frac{a_1}{a_j}v_1 - \frac{a_2}{a_j}v_2 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}.$$

This proves (a).
For (b), note that

$$span(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \subseteq span(v_1, \ldots, v_m).$$

To show the opposite inclusion, consider some arbitrary $w \in span(v_1, \dots, v_m)$. We must show $w \in span(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$. By assumption, $w = b_1v_1 + \cdots + b_mv_m$ for such scalars $b_1, \dots, b_m$. By combining this equation to (a), we get:

$$
\begin{aligned}
w &= b_1v_1 + \cdots + b_{j-1}v_{j-1} + b_jv_j + b_{j+1}v_{j+1} + \cdots + b_mv_m \\
&= b_1v_1 + \cdots + b_{j-1}v_{j-1} + bj\left(-\frac{a_1}{a_j}v_1 - \frac{a_2}{v_j}v_2 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}\right) \\
&\quad + b_{j+1}v_{j+1} + \cdots + b_mv_m.
\end{aligned}
$$

So $w \in span(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$ as required. □

1.2.5 Replacement lemma

Let V be a finite-dimensional vector space. Consider two lists of vectors:

$u_1, \dots, u_m$	linearly independent in V
$w_1, \dots, w_n$	spans V

Then we have

(a) $m \leq n$.

(b) It is possible to extend the list $u_1, \dots, u_m$ with $n-m$ additional vectors from the list $w_1, \dots, w_n$ such that the resulting list spans V.

Proof. We shall prove by induction.

(1) If $w_1, \dots, w_n$ spans V, then $u_1, w_1, \dots, w_n$ also spans V. Also, $u_1, w_1, \dots, w_n$ is linearly dependent because $u_1 \in V = span(w_1, \dots, w_n)$, therefore u_1 is a lienar combination of $w_1, \dots, w_n$.

By the linear dependence lemma, one of $u_1, w_1, \dots, w_n$ can be written as a linear combination of preceding vectors.

This cannot be u_1 (by linear independence), so some w_j can be written as a linear combination of preceding vectors.

By linear dependence lemma part (b),

$$span(u_1, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_n) = V.$$

(2) The list of n vectors from (1) spans V. Add u_2 to the list:

$$u_1, u_2, w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n.$$

Then this list is linearly dependent. One of the w's is redundant and can be removed, without changing the span.

(3) Add u_3 to the list and it becomes linearly dependent. We remove one of the w's in the list.

$\vdots$

(m) Add u_m to the list and it becomes linearly dependent. We remove one of the w's. Now we have a spanning list of n vectors of the form

$$u_1, \ldots, u_m,$$

followed by $n - m$ of the original w's.

Then $m \leq n$, because the n-vector list starts with $u_1, \ldots, u_m$. So we have proved (a) and (b).

□

1.2.6 Theorem

Every subspace of a finite-dimensional vector space is finite-dimensional.

Proof. Let V be finite-dimensinoal and let W be a subspace of V. we must show that W is finite dimensional.

(1) If $W = \{0\}$, then W is finite-dimensional and we are done. Otherwise, there exists at least one non-zero vector in W. Let $v_1 \in W$ be such a non-zero vector. Notice that the list of vectors "v_1" is linearly independent.

(2) If $W = span(v_1)$, then W is finite-dimensional and we are done. Otherwise, there exists at least one vector $v_2 \in W$ such that $v_2 \notin span(v_1)$. Notice that the list of vectors v_1, v_2 is linearly independent.

(3) If $W = span(v_1, v_2)$, then W is finite-dimensional and we are done. Otherwise, there exists at least one vector $v_3 \in W$ such that $v_3 \notin span(v_1, v_2)$. Notice that the list of vectors v_1, v_2, v_3 is linearly independent.

And so on.

(k) If $W = span(v_1, \ldots, v_{k-1})$, then W is finite-dimensional and we are done. Otherwise, there exists at least one vector $v_k \in W$ such that $v_k \notin span(v_1, \ldots, v_{k-1})$. Notice that the list of vectors $v_1, \ldots, v_k$ is linearly independent.

If W were infinite-dimensional, we could repeat this step any number of times, because the "If" part would never be done, so in each step, we'd be doing the "otherwise."
This contradicts the assumption that V is finite-dimensional.
Namely, since V is finite-dimensional, $V = span(w_1, \ldots, w_n)$ is spanned by some finite list of vectors.
If we repeat the above proceduce $n+1$ times, we get $n+1$ linearly independent vectors $v_1, \ldots, v_{n+1}$.
This contradicts the replacement lemma. Therefore, W is finite-dimensional.

$\square$

2 2.B Bases

2.1 Bases

2.1.1 Definition

A *basis* of V is a list of vectors in V that is linearly independent and spans V.

2.1.2 Examples

(a)

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ is a basis of } \mathbb{R}^3.$$

(b)

$$\begin{bmatrix}1\\3\\5\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\5\end{bmatrix} \text{ is a basis of } \mathbb{R}^3$$

(c)

$$1, x, x^2, x^3 \text{ is a basis of } \mathcal{P}_3(\mathbf{F})$$

2.1.3 Criterion for basis

A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n,$$

where $a_1, \ldots, a_n \in \mathbf{F}$.

Proof. Suppose $v_1, \ldots, v_n$ is a basis of V. Let $v \in V$. Since $v_1, \ldots, v_n$ spans V, there exists $a_1, \ldots, a_n \in \mathbf{F}$ such that

$$v = a_1 v_1 + \ldots a_n v_n.$$

To show that it is unique, suppose that $c_1, \ldots, c_n$ are scalars where $v = c_1 v_1 + \cdots + c_n v_n$. Subtracting this equation from the previous, we get

$$0 = (a_1 - c_1)v_1 + \cdots + (a_n - c_n)v_n.$$

This completes the proof for uniqueness.

In the other direction, suppose every $v \in V$ can be written uniquely in the form

$$v = a_1 v_1 + \cdots + a_n v_n.$$

This implies that $v_1, \ldots, v_n$ spans V. To show that $v_1, \ldots, v_n$ are linearly independent, suppose that $a_1, \ldots, a_n \in \mathbf{F}$. Then

$$0 = a_1 v_1 + \cdots + a_n v_n.$$

Thus $v_1, \ldots, v_n$ is linearly independent and hence is a basis of V. □

2.2 Coordinates

If $B = v_1, \ldots, v_n$ is a basis of V, and $v = a_1 v_1 + \cdots + a_n v_n$, then we say $a_1, \ldots, a_n$ are the *coordinates* of v with respect to the basis B.

2.2.1 Examples of coordinates

Suppose that $B = 1, x, x^2, x^3$ is the basis of $\mathcal{P}_3(\mathbb{R})$. Find the coordinates of $p = (1 + 2x)(3x + x^2)$ with respect to the basis B.

Solution:

$$\begin{aligned} p &= (1 + 2x)(3x + x^2) \\ &= 3x + x^2 + 6x^2 + 2x^3 \\ &= 3x + 7x^2 + 2x^3 \\ &= 0 \cdot 1 + 3 \cdot x + 7 \cdot x^2 + 2 \cdot x^3 \end{aligned}$$

The coordinates are: 0, 3, 7, and 2.

Another basis for $\mathcal{P}_3(\mathbb{R})$ is $B' = 1, (x-1), (x-1)^2, (x-1)^3$. Find the coordinates of $p = 3x + 6x^2 + 2x^3$ in the basis B'.

Solution: Suppose that $y = x - 1$ and $x = y + 1$. Then

$$\begin{aligned} p &= 3(y + 1) + 7(y + 1)^2 + 2(y + 1)^3 \\ &= 3y + 3 + 7y^2 + 14y + 7 + 2y^3 + 6y^2 + 6y + 2 \\ &= 12 + 23y + 13y^2 + 2y^3 \\ &= 12 + 23(x - 1) + 13(x - 1)^2 + 2(x - 1)^3 \end{aligned}$$

The coordinates of p with respect to B' are: 12, 23, 13, and 2.

2.3 Theorems about Bases

2.3.1 Spanning list contains a basis

Every spanning list in a vector space can be reduced to a basis of the vector space (by removing 0 or more vectors from the list).

Proof. Suppose $v_1, \ldots, v_n$ spans V. We want to remove some of the vectors from $v_1, \ldots, v_n$ so that the remaining vectors form a basis of V. We do this through induction.
Start with B equal to the list $v_1, \ldots, v_n$.

Step 1 If $v_1 = 0$, delete v_1 from B. If $v_1 \neq 0$, leave B unchanged.

Step j If v_j is in $span(v_1, \ldots, v_{j-1})$, delete v_j from B. If v_j is not in $span(v_1, \ldots, v_{j-1})$, leave B unchanged.

Stop the process after step n, getting a list B. This list spans V because our original list spanned V and we have discarded vectors that were already in the span of the previous vectors. This process ensures that no vector in B is in the span of the previous ones. Thus, B is linearly independent, by the Linear Dependence Lemma. Hence B is a basis of V. □

2.3.2 Basis of finite-dimensional vector space

Every finite-dimensional vector space has a basis.

Proof. By definition, a finite-dimensional vector space has a spanning list. The previous result tells us that each spanning list can be reduced to a basis. □

2.3.3 Linearly independent list extends to a basis

Every linearly indepdendent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Proof. Suppose $u_1, \ldots, u_m$ is linearly independent in a finite-dimensional vector space V. Let $w_1, \ldots, w_n$ be a basis of V. Thus the list

$$u_1, \ldots, u_m, w_1, \ldots, w_n$$

spans V. Applying the produce of the proof of 3.1 to reduce this list to a basis of V produces a basis consisting of the vectors $u_1, \ldots, u_m$ (none of the u's get deleted because $u_1, \ldots, u_m$ is linearly independent) and some of the w's. □

2.3.4 Every subspace of V is part of a direct sum equal to V

Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.

Proof. Since V is finite-dimensional, so is U. Thus, there is a basis $u_1, \ldots, u_m$ of U and is linearly independent in V. Hence, this list can be extended to a basis $u_1, \ldots, u_m, w_1, \ldots, w_n$ of V. Let $W = span(w_1, \ldots, w_n)$.

To prove that $V = U \oplus W$, we need to show that

$$V = U + W \text{ and } U \cap W = \{0\}.$$

Proving the first equation, suppose $v \in V$ Then, since the list $u_1, \ldots, u_m, w_1, \ldots, w_n$ spans V, there exist $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that

$$v = \underbrace{a_1u_1 + \cdots + a_mu_m}_{u} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{w} \Rightarrow v = u + w, u \in U, w \in W.$$

Thus we have $v \in U + W$.
Proving the second equation, suppose $v \in U \cap W$. There exists scalars $a_1, \ldots, a_m, b_1, \ldots, b_n \in \mathbf{F}$ such that

$$v = a_1u_1 + \cdots + a_mu_m = b_1w_1 + \cdots + b_nw_n.$$

Thus

$$a_1u_1 = \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n = 0.$$

Since $u_1, \ldots, u_m, w_1, \ldots, w_n$ is linearly independent, this implies $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Thus $v = 0$ and this completes the proof. □

3 2.C Dimension

3.1 Basis length does not depend on basis

Any two bases of a finite-dimensional vector space have the same length.

Proof. Suppose V is finite-dimensional. Let B_1 and B_2 be two bases of V. Then B_1 is linearly independent in V and B_2 spans V, so the length of B_1 is at most length of B_2. Interchanging the roles, we also see that the length of B_2 is at most the length of B_1. Thus the length of B_1 equals the length of B_2, as desired. □

3.2 Definition of a dimension

- The *dimension* of a finite-dimensional vector space is the length of any basis of the vector space.
- The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

3.3 Examples of a dimension

1. $\dim \mathbf{F}^n = n$ because the standard basis of $\mathbf{F}^n$ has length n.
2. $\dim \mathcal{P}_m(\mathbf{F}) = m + 1$ because the basis $1, z, \ldots, z^m$ of $\mathcal{P}_m(\mathbf{F})$ has length $m + 1$.

3.4 Dimension of a subspace

If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$.

Proof. Suppose V is finite-dimensional and U is a subspace of V. Think of a basis of U as a linearly independent list in V, and think of a basis of V as a spanning list in V. These linearly independent vectors $u_1, \ldots, u_m$ can be extended to a basis of V. That extended basis has at least m vectors, so $\dim V \geq \dim U$. □

3.5 Linearly independent list of the right length is a basis

Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Proof. Suppose $\dim V = n$ and $v_1, \ldots, v_n$ is linearly independent in V. The list $v_1, \ldots, v_n$ can be extended to a basis of V. However, every basis of V has length n, so in this case the extension is the trivial one, meaning that no elements are adjoined to $v_1, \ldots, v_n$. In other words, $v_1, \ldots, v_n$ is a basis of V, as desired. $\square$

3.6 Examples

1. Show that the list $(5,7),(4,3)$ is a basis of $\mathbf{F}^2$.

   Proof. The two vectors are linearly independent (because neither vector is a scalar multiple of the other). Note that $\mathbf{F}^2$ has dimension 2. Thus, Theorem 1.5 implies that the linearly independent list of length 2 is a basis of $\mathbf{F}^2$. $\square$

2. Show that $p(x) = x^2 + 1, q(x) = x^2 + x, r(x) = x^2$ are a basis of $\mathcal{P}_2(\mathbf{F})$.

   Proof. Assume $a(x^2+1) + b(x^2+x) + c(x^2) = 0$, where $a, b, c \in \mathbf{F}$. Then we have $(a+b+c)x^2 + bx + a = 0 \Rightarrow a + b + c = 0$. We know that $a = b = 0$ so it follows that $c = 0$. Hence, p, q, r are linearly independent. Since we know that $\dim \mathcal{P}_2(\mathbf{F}) = 3$ then by Theorem 1.5, p, q, r are bases of $\mathcal{P}_2(\mathbf{F})$. $\square$

3.7 Spanning list of the right length is a basis

Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V.

Proof. Suppose $\dim V = n$ and $v_1, \ldots, v_n$ spans V. The list $v_1, \ldots, v_n$ can be reduced to a basis of V (by removing 0 or more vectors from the list). However, every basis of V has length n, so the reduction is the trivial one, meaning that no elements are deleted from $v_1, \ldots, v_n$. In other words, $v_1, \ldots, v_n$ is a basis of V, as desired. $\square$

3.8 Dimension of a sum

If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).$$

Proof. Let $u_1, \ldots, u_m$ be a basis of $U_1 \cap U_2$; thus $\dim(U_1 \cap U_2) = m$. These basis are linearly independent in U_1 and can be extended to a basis $u_1, \ldots, u_m, v_1, \ldots, v_j$. Thus, $\dim U_1 = m + j$ Also, $u_1, \ldots, u_m, w_1, \ldots, w_k$ of U_2 and so $\dim U_2 = m + k$.
We need to show that $u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k$ is a basis of $U_1 + U_2$. so

$$\begin{aligned}
\dim(U_1 + U_2) &= m + j + k \\
&= (m + j) + (m + k) - m \\
&= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2).
\end{aligned}$$

Clearly $span(u_1, \ldots, u_m, v_1, \ldots, v_j, w_1, \ldots, w_k)$ contains $U_1 + U_2$ which equals $U_1 + U_2$. To show that this list is a basis of $U_1 + U_2$, we need to show that it is linearly independent. Suppose that

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j + c_1 w_1 + \cdots + c_k w_k = 0$$

where $a, b, c \in \mathbf{F}$. Then

$$c_1 w_1 + \cdots + c_k w_k = -a_1 u_1 - \cdots - a_m u_m - b_1 v_1 - \cdots - b_j v_j.$$

This implies that $c_1 w_1 + \cdots + c_k w_k \in U_1$ and consequently, $c_1 w_1 + \cdots + c_k w_k \in U_1 \cap U_2$. Since $u_1, \ldots, u_m$ is a basis of $U_1 \cap U_2$, we can write

$$c_1 w_1 + \cdots + c_k w_k = d_1 u_1 + \cdots + d_m u_m$$

for some scalars $d \in \mathbf{F}$. But $u_1, \ldots, u_m, w_1, \ldots, w_k$ are linearly independent, so all c's and d's equal 0. Thus, our original equation becomes

$$a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_j v_j = 0.$$

Since $u_1, \ldots, u_m, v_1, \ldots, v_j$ are linearly independent, then all a's and b's equal 0. $\square$