MATH 2135 Linear Algebra

3.A The Vector Space of Linear Maps

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1 Definition and Examples of Linear Maps

The words "map," "mapping," "function" all mean exactly the same thing. Let X, Y be sets. A *function* $f : X \to Y$ is an operation that assigns to each element $x \in X$ a unique element $y \in Y$, called $y = f(x)$. X is called the *domain* of f, and Y is called the *codomain* of f. We also say that f is a "map" from X to Y.

1.1 Definition of a linear map

A *linear map* or *linear transformation* from V to W is a function $T : V \to W$ with the following properties:

1. Additivity.
$$T(u + v) = Tu + Tv \text{ for all } u, v \in V$$
2. Homogeneity.
$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbf{F} \text{ and all } v \in V.$$

1.2 Notation $\mathcal{L}(V, W)$

The set of all linear maps from V to W is denoted by

$$\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is linear } \}.$$

In the case that $V = W$, a linear function $T : V \to V$ is called an *operator* on V. In other words, $\mathcal{L}(V, V)$ is the set of operators on V.

1.3 Examples of linear maps

1. **zero function** The zero function is denoted by $0 \in \mathcal{L}(V, W)$ and defined with
$$0v = 0 \quad \text{or} \quad f(v) = 0.$$
The 0 on the left side is a function from $V to W$, whereas the right one is the additive identity in W. The zero function is linear.
2. **identity function** The identity map, denoted by I, is the function that takes each element to itself. It is defined as $I \in \mathcal{L}(V, V)$ such that
$$Iv = v.$$
The identity map is linear.

3. **differentiation** Define $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ by

$$Dp = p'.$$

D is a linear function because: $(f+g)' = f' + g'$ and $(\lambda f)' = \lambda f'$ whenever f, g are differentiable and λ is a constant.

4. **integration (antiderivative)** Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ such that

$$T(p) = \int_0^x p(t)dt.$$

T is a linear function since the integral of the sum of two functions equals the sum of the integrals, and the integral of a constant times a function equals the constant times the integral of the function.

5. **integration (definite integrals)** Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ such that

$$T(p) = \int_0^1 p(x)dx.$$

T is also linear with similar reasons as the previous example.

6. **multiplication by** x^2 Define $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ such that

$$(Tp)(x) = x^2 p(x).$$

7. **backward shift** Recall that $\mathbf{F}^\infty$ is the vector space of all sequences of elements of $\mathbf{F}$. Define $B \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ such that

$$B(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots).$$

8. **forward shift** Define $F \in \mathcal{L}(\mathbf{F}^\infty, \mathbf{F}^\infty)$ such that

$$F(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots).$$

9. **recurrence relation** Consider the Fibonacci sequence such taht

$$a_{n+2} = a_n + a_{n+1}, \qquad n \geq 3.$$

For example, we have $(a_1, a_2, a_3, a_4, a_5, \dots) = (1, 1, 2, 3, 5, \dots)$. We can also write the reccurence in terms of the backward (and forward) shfit operators:

$$\begin{aligned} a &= (a_1, a_2, a_3, a_4, a_5, \dots) \\ Ba &= (a_2, a_3, a_4, a_5, a_6, \dots) \\ B^2 a = B(Ba) &= (a_3, a_4, a_5, a_6, a_7, \dots) \end{aligned}$$

Cosnider the equation $B^2a = Ba + a$. We can do algebra with such equation:

$$B^2a - Ba - a = 0 \qquad \Leftrightarrow \qquad (B^2 - B - I)a = 0.$$

10. **from $\mathbb{R}^3$ to $\mathbb{R}^2$** Define $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ by

$$T(x, y, z) = (2x - y + 3z, 7x + 5y - 6z).$$

11. **from $\mathbf{F}^n$ to $\mathbf{F}^m$** Let m and n be positive integers, and let $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, and define $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ by

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1, \cdots + A_{m,n}x_n).$$

1.4 Linear maps and basis of domain

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$.

Proof. Let us prove the existence of a linear map T with the desired property. Define $T : V \to W$ by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n,$$

where $c_1, \ldots, c_n$ are arbitrary elements of $\mathbf{F}$. Since $v_1, \ldots, v_n$ is a basis of V, the equation does define a function T since each element of V can be uniquely written in the form of $c_1v_1 + \cdots + c_nv_n$. Suppose that $c_j = 1$ and the other c's being equal to 0, then we have $Tv_j = w_j$.

If $u, v \in V$ with $u = a_1v_1 + \cdots + a_nv_n$ and $v = c_1v_1 + \cdots + c_nv_n$, then

$$\begin{aligned} T(u + v) &= T((a_1 + c_1)v_1 + \cdots + (a_n + c_n)v_n) \\ &= (a_1 + c_1)w_1 + \cdots + (a_n + c_n)w_n \\ &= (a_1w_1 + \cdots + a_nw_n) + (c_1w_1 + \cdots + c_nw_n) \\ &= Tu + Tv. \end{aligned}$$

If $\lambda \in \mathbf{F}$ and $v = c_1v_1 + \cdots + c_nv_n$, then

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1v_1 + \cdots + \lambda c_nv_n) \\ &= \lambda c_1w_1 + \cdots + \lambda c_nw_n \\ &= \lambda(c_1w_1 + \cdots + c_nw_n) \\ &= \lambda Tv. \end{aligned}$$

Thus, T is a linear map from V to W. Now we need to prove that it is unique. Suppose that $T \in \mathcal{L}(V, W)$ and that $Tv_j = w_j$ for $j = 1, \ldots, n$. Let $c_1, \ldots, c_n \in \mathbf{F}$. The homogeneity property of T implies that $T(c_jv_j) = c_jw_j$. The additivity of T now implies that

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n.$$

Then T is unqiuely determined on $\text{span}(v_1, \ldots, v_n)$. Because $v_1, \ldots, v_n$ is a basis of V, this implies that T is uniquely determined on V. $\square$

2 Algebraic Operations on $\mathcal{L}(V, W)$

2.1 Addition and scalar multiplication on $\mathcal{L}(V, W)$

Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The *sum* $S + T$ and the *product* λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \qquad \text{and} \qquad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

2.2 $\mathcal{L}(V, W)$ is a vector space

With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space (it holds all the 8 laws of vector space). Note that the additive identity is the zero linear map or function.

2.3 Product of linear maps

If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the *product* $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$. In other words, ST is the composition $S \circ T$ of two functions. Note that ST is defined only when T maps into the domain of S.

2.4 Algebraic properties of products of linear maps

1. Associativity:
$$(T_1T_2)T_3 = T_1(T_2T_3)$$
where T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1.

2. Identity:
$$TI = IT = T$$
where $I \in \mathcal{L}(W, W)$ and $T \in \mathcal{L}(V, W)$.

3. Distributive properties:
$$(S_1 + S_2)T = S_1T + S_2T \qquad \text{and} \qquad S(T_1 + T_2) = ST_1 + ST_2$$
where $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$.

4. Powers: If $T \in \mathcal{L}(V, V)$, we define $T^n \in \mathcal{L}(V, V)$ by
$$\begin{aligned} T^2 &= TT \qquad \text{(i.e. } T^2v = T(Tv)) \\ T^3 &= TTT \\ &\vdots \\ T^n &= TT^{n-1}. \end{aligned}$$

2.5 Example

Suppose that $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the differentiation map and $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ is the multiplication by x^2. Show that $TD \neq DT$.

We have
$$((TD)p)(x) = x^2p'(x)$$
but
$$((DT)p)(x) = x^2p'(x) + 2xp(x).$$
In other words, differentiating and then multiplying by x^2 is not the same as multiplying by x^2 and then differentiating. $\square$

2.6 Linear maps take 0 to 0

Suppose T is a linear map from V to W. Then $T(0) = 0$.

Proof. We can prove this by additivity or homogeneity.

- By additivity: $T(0) = T(0 + 0) = T(0) + T(0)$. Then add the additive inverse of $T(0)$ to each side of the equation to conclude $T(0) = 0$.
- By homogeneity: Suppose that $\lambda = 0$ and $u = 0$. Then $T(0 \cdot 0) = 0 \cdot T(0) \Rightarrow T(0) = 0$.

$\square$