CSCI/MATH 2113 Discrete Structures

5.1 Cartesian Products and Relations

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1 Cartesian Product

1.1 Definition

For sets A, B the *Cartesian product*, or *cross product*, of A and B is denoted by $A \times B$ and equals $\{(a, b) \mid a \in A, b \in B\}$.

1.2 Example

Suppose that $A = \{2, 3, 4\}$ and $B = \{4, 5\}$ then we have

$$A \times B = \{(2,4), (2,5), (3,4), (3,5), (4,4), (4,5)\}.$$

Note that

$$A \times B \neq B \times A.$$

Another example of a Cartesian product is the real plane $\mathbb{R} \times \mathbb{R}$.

1.3 Notation

$$A^n = \underbrace{A \times A \times A \times \cdots \times A}_{n \text{ times}}.$$

2 Relations

2.1 Definition

For sets A, B, any subset of $A \times B$ is called a (*binary*) *relation* from A to B. Any subset of $A \times A$ is called a (*binary*) *relation* on A.

2.2 Notation

If R is a relation on A and $(a, a') \in R$, then we write aRa'.

2.3 Examples

Suppose that $A = \{2, 3, 4\}$ and $B = \{4, 5\}$. Then,

- $\{(2, 5), (2, 4)\}$
- $A \times B$
- $\varnothing$

are relations from A to B.

2.4 Counting

For finite sets A, B with $|A| = m$ and $|B| = n$, there are 2^{mn} relations from A to B, including the empty relation as well as the relation $A \times B$ itself.

There are also $2^{nm} (= 2^{mn})$ relations from B to A, one of which is also $\varnothing$ and another of which is $B \times A$. The reason we get the same number of relations from B to A as we have from A to B is that any relation R_1 from B to A can be obtained from a unique relation R_2 from A to B by simply reversing the components of each ordered pair in R_2 (and vice versa).

2.5 Standard Relations

Standard relations can be expressed in this way:

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b = a + n \text{ for } n \in \mathbb{N}\}.$$

This is the relation "is less than or equal to." Indeed

$$(a, b) \in R \text{ if and only if } a \leq b.$$

Suppose that $A = \{1\}$ and let $R \subseteq \mathcal{P}(A)^2$ defined by

$$R = \{(\varnothing, \varnothing), (\varnothing, \{1\}), (\{1\}, \{1\})\}.$$

This is the relation "is a subset of." Indeed,

$$(S, S') \in R \text{ if and only if } S \subseteq S'.$$

2.6 Theorem

For sets A, B, and C:

- $A \times \varnothing = \varnothing = \varnothing \times A$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $(B \cap C) \times A = (B \times A) \cap (C \times A)$
- $(B \cup C) \times A = (B \times A) \cup (C \times A)$

Proof. Let $x \in A \times (B \cap C)$. Then $x = (a, d)$ when $a \in A, d \in B \cap C$. So, $x = (a, d)$ with $a \in A$ and $d \in B$ which implies that $x \in A \times B$. But $x = (a, d)$ with $a \in A$ and $d \in C$, which implies $x \in A \times C$. Hence, we have $x \in (A \times B) \cap (A \times C)$ which implies $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$. The converse inclusion is shown similarly. Hence we have the desired equality. The other statements are proved similarly as well. $\square$

2.7 Recursive Relation

An example of a recursively defined relation is on $\mathbb{N} \times \mathbb{N}$:

1. $(0, 0) \in R$
2. If $(s, t) \in R$ then $(s + 1, t + 7) \in R$.

In fact, we have

$$R = \{(m, n) \mid n = 7m\}.$$