CSCI/MATH 2113 Discrete Structures

7.1 Relations Revisited: Properties of Relations

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Contents

1 Recall: Binary Relation

For sets A, B, any subset of $A \times B$ is called a (*binary*) *relation* from A to B. Any subset of $A \times A$ is called a (*binary*) *relation* on A.

2 Properties of Relations

2.1 Reflexive property

A relation R on a set A is called *reflexive* if for all $x \in A$, $(x, x) \in R$. This means that each element x of A is related to itself.

Example. For $A = \{1, 2, 3\}$ and $R = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3)\}$, we know it is not reflexive since $(3, 3) \notin R$.

Remark. A relation R on A is reflexive if and only if $\{(a, a) \mid a \in A\} \subseteq R$.

Counting. Let A be a set with n elements. How many relations on A are reflexive? There are 2^{n^2} relations on A (2 choices for each of the n^2 pairs in $A \times A$). In a reflexive relation, n of these pairs are decided. Hence, we have $n^2 - n$ chocies to make and so there are

$$2^{n^2-2}$$

reflexive relations.

2.2 Symmetric property

A relation R on a set A is called *symmetric* if $(x, y) \in R \Rightarrow (y, x) \in R$, for all, $x, y \in A$.

Counting. If $|A| = n \leq 0$, how many relations on A are symmetric? Write $A \times A = A_1 \cup A_2$ where

$$\begin{aligned} A_1 &= \{(a_i, a_i) \mid 1 \leq i \leq n\} \\ A_2 &= \{(a_i, a_j) \mid 1 \leq i, j \leq n, i \neq j\}. \end{aligned}$$

We have

$$|A_1| = n \qquad |A_2| = n^2 - n.$$

For each element of A_1 and for half of the elements of A_2, we choose whether it belongs to R. Thus, intotal, there are

$$2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n^2+n}{2}}$$

such relations. In counthing those relations on A that are both reflexive and symmetric, we have only one choice for each ordered pair in A_1. So we have

$$2^{\frac{n^2-n}{2}}$$

relations on A that are both reflexive and symmetric.

2.3 Transitive property

For a set A, a relation R on A is called *transitive* if, for all $x, y, z \in A, (x, y), (y, z) \in R \Rightarrow (x, z) \in R$. So if x "is related to" y, and y "is related to" z, we want x "related to" z, with y playing the role of "intermediary."

Counting. There is no known general formula for the total number of transitive relations on a finite set.

2.4 Antisymmetric property

Given a relation R on a set A, R is called *antisymmetric* if for all $a, b \in A$, $(aRb$ and $bRa) \Rightarrow a = b$. Here, the only way we can have both a "related to" b and b "related to" a is if a and b are one and the same element from A.

Examples. Let $A = \mathbb{Z}$ and $R = \leq$ is an antisymmetric relation.

Antisymmetric is different than "not symmetric." Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 1), (2, 3)\}$. Then, R is *not* symmetric and it is *not* antisymmetric.

Counting. How relations that are antisymmetric? Suppose that $|A| = n > 0$. By the rule of product, the number of antisymmetric relations are

$$(2^n)(3^{\frac{n^2-n}{2}}).$$

3 Partial order

A relation R on a set A is called a *partial order*, or a *partial ordering relation*, if R is reflexive, antisymmetric, and transitive.

3.1 Example

- $\leq$ on $\mathbb{N}$. Partial order and total implies total order.
- $\subseteq$ on $\mathcal{P}(S)$ is not total order.

4 Total order

A relation R on a set A is a *total order* if R is a partial order and for every $x, y \in A$, we have $(x, y) \in R$ or $(y, x) \in R$.

5 Equivalence relation

An *equivalence relation* on A is a relation that is reflexive, symmetric, and transitive.