MATH 2135 Linear Algebra
1.C Subspaces

Alyssa Motas

February 27, 2021

Contents

1 Definition

Let V be a vector space over a field F. A subset U of B is called a *subspace* of V if U is also a vector space in its own right, using the same zero, addition, and scalar multiplication as V.

1.1 Characterization of Subspaces

A subset $U \subseteq V$ is a subspace if and only if U satisfies the following three conditions:

(1) Additive identity.
$$0 \in U$$

(2) Closed under addition.
$$\forall v, w, v, w \in U \Rightarrow v + w \in U$$

(3) Closed under scalar multiplication.
$$\forall a, v, a \in F, v\ inU \Rightarrow av \in U$$

Proof. "$\Rightarrow$" Given $U \subseteq V$, assume U is a subspace of V. We want to show that U satisfies all three conditions above.

(1) By definition of subspaces, the zero vector of V is the zero vector of U. So $0 \in U$.

(2) Since U is a vector space, the sume of two vectors in U is a vector in U. Also, U uses the same addition operation as V. So whenever $v, w \in U$, then $v + w \in U$.

(3) Similar to (2).

$\square$

Proof. "$\Leftarrow$" Another proof is this: To show that U is a vector space, we first need an element $0 \in U$ and operations
$$+ : U \times U \to U \qquad \text{and} \qquad \cdot : F \times U \to U.$$

Second, we must show axioms (A1) - (M4).

(1) By assumption, $0 \in U$, where 0 is the additive identity of V. So we can use 0 as the additive identity of U.

(2) By assumption, U is closed under addition, so the addition function $+ : V \times V \to V$ restricts to a function $+ : U \times U \to U$. We can use the same function as the addition function on U.

(3) We do the same with scalar multiplication.

Second: We must show (A1) - (M4) hold. We only do (A1) since the rest are similar. To prove (A1), take arbitrary $u, v \in U$. We need to show that

$$u + v = v + u$$

in U. But since V is a vector space, we know that

$$u + v = v + u$$

in V. This automatically holds.

The other parts of the definition of a vecotr space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V. Thus, U is a vector space and hence is a subspace of V. □

2 Examples of Subspaces

(a) Let $V = \mathbb{R}^4$ and let

$$W = \{(x, y, z, w) \mid x = 3y + 2z\}.$$

Then W is a subspace of V.

Proof. We need to show that W is a subspace of V.

(1) $(0, 0, 0, 0) \in W$.

(2) Assume $v = (x, y, z, w) \in W$ and $v' = (x', y', z', w') \in W$. We need to show that

$$x + x' = 3(y + y') + 2(z + z').$$

We know that $v \in W$ implies

$$x = 3y + 2z.$$

And we know that $v' \in W$ implies

$$x' = 3y' + 2z'.$$

Add these two equations together and we get

$$x + x' = 3(y + y') + 2(z + z').$$

(3) Similar proof for scalar multiplication.

$\square$

(b) Recall that $V = \mathbb{R}^{[0,1]}$ is the vector space of functions from the unit interval

$$[0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

to $\mathbb{R}$. Let $W = \{f : [0,1] \to \mathbb{R} \mid \ f \text{ is continuous } \}$. Then W is a subspace of V.

Proof. We need to show that W is a subspace of V.

(1) The zero function $f(x) = 0$ is continuous. From Calculus, any constant function is continuous.

(2) The sum of two continuous functions is continuous. This is from Calculus.

(3) If f is continuous, then so is kf, for any $k \in \mathbb{R}$. This is also from Calculus.

$\square$

(c) Again, let $V = \mathbb{R}^{[0,1]}$, and $U = \{f : [0,1] \to \mathbb{R} \mid \ f \text{ is differentiable } \}$. Then U is a subspace of V.

Proof. From Calculus, we know these hold true:

(1) The zero function $f(x) = 0$ is differentiable with the derivative of

$$f'(x) = 0.$$

(2) If f, g are differentiable then so is $f + g$ and

$$(f + g)' = f' + g'.$$

(3) If f is differentiable and k is a scalar, then kf is differentiable.

$$(kf)' = kf'.$$

For example, the derivative of $(0+2\sin(x)-3\cos(x))' = 0+2\cos(x)-3(-\sin(x))$.

We also know from Calculus that every differentiable function is continuous.

$$U \subseteq W \subseteq V$$

where U is the set of all differentiable functions, W is the set of all continuous functions, and V is the set of all functions. □

(d) Let $V = \mathbb{R}^{[0,1]}$ and define

$$X = \left\{ f : [0,1] \to \mathbb{R} \mid \ f \text{ is differentiable and } f'\left(\frac{1}{2}\right) = 0 \right\}.$$

We claim that X is a subspace of V.

Note. We already know that U, the set of differentiable functions, is a subspace of V.

Proof. Effectively, it is sufficient to show that X is a subspace of U.

(1) The zero function $f(x) = 0$ is differentiable since

$$f'(x) = 0 \text{ and } f'\left(\frac{1}{2}\right) = 0.$$

So $f \in X$.

(2) Given $f, g \in X$, we must check that $f+g \in X$. Clearly $f+g$ is differentiable. We must check that $(f+g)'\left(\frac{1}{2}\right) = 0$. From Calculus, we have

$$\begin{aligned}(f+g)'\left(\frac{1}{2}\right) &= f'\left(\frac{1}{2}\right) + g'\left(\frac{1}{2}\right) \\ &= 0+0 \\ &= 0.\end{aligned}$$

So we have $f+g \in X$.

(3) Similar proof as follows for scalar multiplication:

$$(kg)'\left(\frac{1}{2}\right) = k \cdot f'\left(\frac{1}{2}\right) = 0.$$

$\square$

(e) Let $V = \mathbb{R}^\infty$, the set of infinite sequences of real numbers.

$$\mathbb{R}^\infty = \{(a_0, a_1, a_2, a_3, \dots) \mid a_0, a_1, \dots \in \mathbb{R}\}.$$

Recall that V is a vector space. Let $W \subseteq V$ be the set of *convergent* sequences. From Calculus, we know that some sequences converge and some do not.

Some examples are:

- $a_i = i \Rightarrow (0, 1, 2, 3, 4, 5, 6, 7, \dots)$

$$\lim_{i \to \infty} a_i = \text{ does not exist, so it does not converge}$$

- $b_i = \frac{1}{i} \Rightarrow (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots)$

$$\lim_{i \to \infty} b_i = \text{ converges to } 0$$

- $c_i = (-1)^i \Rightarrow (1, -1, 1, -1, 1, -1, \dots)$ does not converge.
- $d_i = 2 + \left(-\frac{1}{2}\right)^i \Rightarrow (3, 1.5, 2.25, 1.875, \dots)$

$$\lim_{i \to \infty} d_i = \text{ converges to } 2$$

Then W is a subspace of V.

Proof. We must show that this is true.

(1) The zero sequence $a_i = 0 \Rightarrow (0, 0, 0, \dots)$ converges to 0.

(2) From Calculus, the sum of two convergent sequences converges. In fact,

$$\lim_{i \to \infty} (a_i + b_i) = \lim_{i \to \infty} a_i + \lim_{i \to \infty} b_i$$

(3) Similar proof to scalar muiltiplication.

$$\lim_{i \to \infty} (ka_i) = k \lim_{i \to \infty} a_i$$

$\square$

Let U be the set of sequences that converge to 0. Then U is a subspace of V (and of W).

(f) A reccurence relation. Consider the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Let F_n be the nth element of this sequence. Then, we have the following recurrence:

$$\begin{aligned} \text{base case } F_0 &= 1 \\ \text{base case } F_1 &= 1 \\ \text{reccurence } F_{n+2} &= F_n + F_{n+1} \qquad \text{for all } n \geq 0. \end{aligned}$$

If we forget the base cases, we can consider the set of *all* sequences satisfying the reccurence

$$U = \{(a_0, a_1, a_2, \ldots) \mid \text{ for all } n \geq 0, a_{n+2} = a_n + a_{n+1}\}$$

A sequence of numbers is called a *generalized Fibonacci sequence* if it satisfies this reccurence, i.e. if it is a member of the set U.

Examples.

$$\begin{aligned} &1, 2, 3, 5, 8, 13, \ldots \\ &7, -3, 4, 1, 5, 6, 11, 17, 28, 45, \ldots \\ &0, 0, 0, 0, 0, 0, 0, 0, \ldots \end{aligned}$$

Claim. U is a subspace of $\mathbb{R}^\infty$.

(1) The sequence $0, 0, 0, \ldots$ is a generealized Fibonacci sequence.

(2) U is closed under addition.

$$\begin{array}{l} 1, 2, 3, 5, 8, 13 \ldots \\ 7, -3, 4, 1, 5, 6, \ldots \\ \hline 8, -1, 7, 6, 13, 19, \ldots \end{array}$$

Proof. Suppose $a = (a_0, a_1, a_2, \dots) \in U$ and $b = (b_0, b_1, b_2, \dots) \in U$. Then $a + b = c = (c_0, c_1, c_2, \dots)$ where $c_i = a_i + b_i$.
We must show $c \in U$, i.e. we must show that c is a generalized Fibonacci sequence.
So take an arbitrary $n \geq 0$. We must show $C_{n+2} = C_n + C_{n+1}$. Indeed, we have:

$$\begin{aligned} C_{n+2} &= a_{n+2} + b_{n+2} \\ &= (a_n + a_{n+1}) + (b_n + b_{n+1}) \\ &= (a_n + b_n) + (a_{n+1} + b_{n+1}) \\ &= c_n + c_{n+1} \end{aligned}$$

□

(3) Closed under scalar multiplication: similar.

3 Intersection of Subspaces

3.1 Theorem

Let V be a vector space over a field F. Assume U and W are subspaces of V. Then $U \cap W$ is a subspace of V.

Proof. To show that $U \cap W$ is a subspace, we need to show the three properties.

(1) We must show that $0 \in U \cap W$. But by assumption, U is a subspace, so $0 \in U$. Also, W is a subspace, so $0 \in W$. By definiton of intersection, we have $0 \in U \cap W$.

(2) We must show that $U \cap W$ is closed under addition. Consider arbitrary $v, w \in U \cap W$ and we need to show that $v + w \in U \cap W$.

Indeed, we have:

- Since $v \in U \cap W$, we know $v \in U$.
- Since $w \in U \cap W$, we know $w \in U$.
- Since U is a subspace, it is closed under addition, so $v + w \in U$.

Similarly:

- Since $v \in U \cap W$, we know $v \in W$.

- Since $w \in U \cap W$, we know $w \in W$.
- Since W is a subspace, it is closed under addition, so $v + w \in W$.

From $v + w \in U$ and $v + w \in W$, by definition of intersection, we know

$$v + w \in U \cap W.$$

(3) We must show that $U \cap W$ is closed under scalar multiplication. So consider arbitrary $k \in F$ and $v \in U \cap W$. We must show that $kv \in U \cap W$.

Since $v \in U \cap W$, we have $v \in U$. Since U is a subspace of V, we know that U is closed under scalar multiplication, so $kv \in U$.

Similarly, since $v \in U \cap W$, we know $v \in W$. Since W is a subspace of V, we know that W is closed under scalar multiplication, so $kv \in W$.

From $kv \in U$ and $kv \in W$, it follows that $kv \in U \cap W$ (by definition of intersection), as desired.

$\square$

3.2 Notations

- $(x_1, \ldots, x_n)$ is called an n-tuple.
- $(x_i)_{i \in \{1,\ldots,n\}}$ is another notation for the same thing. This is called "family" notation. But, this notation also works for infinite index sets.

$$(x_i)_{i \in \mathbb{N}} = (x_0, x_1, x_2, \ldots)$$

$$\left(\frac{1}{i+1}\right)_{i \in \mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right)$$

- More generally, we can use the "family notation" for other families of things, not necessarily numbers.

U_1, U_2, U_3	3 subspaces of V
$(U_i)_{i \in \{1,2,3\}}$	Notation for the same thing.
$(U_i)_{i \in I}$	Some family of subspaces.

- Other notations that go along with these:
  - Summation.
  $$\sum_{i \in \mathbb{N}} x_i$$
  - Product.
  $$\prod_{i \in \mathbb{N}} x_i$$
  - Limit.
  $$\lim_{i \to \infty} x_i$$
  - Union.
  $$\bigcup_{i \in \{1,2,3\}} U_i$$
  - Intersection.
  $$\bigcap_{i \in I} U_i$$

3.3 Theorem

Let V be a vector space over a field F. Let I be a set. Let $(U_i)_{i \in I}$ be a *family* of subspaces of V. Then,

$$\bigcap_{i \in I} U_i$$

is a subspace of V.

Proof. Let $W = \bigcap_{i \in I} U_i$. To show that W is a subspace of V, we must show the three subspace properties.

(1) We must show that $0 \in W$. Take an arbitrary $i \in I$. By assumption, U_i is a subspace of V. Therefore, $0 \in U_i$.

   Since i was arbitrary, we have $0 \in U_i$ for all i, and therefore, by definition of intersection, $0 \in \bigcap_{i \in I} U_i = W$.

(2) Must show W is closed under addition. Take an arbitrary $v, u \in W$. We must show that $v + u \in W$ or $v + u \in \bigcap_{i \in I} U_i$.

   Take an arbitrary $i \in I$. We must show that $v + u \in U_i$. By assumption, $v \in W = \bigcap_{i \in I} U_i$, therefore $v \in U_i$.

Simiarly, by assumption, $u \in W = \bigcap_{i \in I} U_i$, therefore $u \in U_i$.

Also by assumption, U_i is a subspace of V, therefore it is closed under addition. So $v + u \in U_i$.

Since i was arbitrary, we therefore know for all $i \in I$ that $v + u \in U_i$. It follows that $v + u \in \bigcap_{i \in I} U_i$ as desired.

(3) Closed under scalar multiplication: similar proof.

□

3.4 Meta-theorem

Let V be a vector space over a field F. Let P be any property of subspaces of V such that P is closed under arbitrary intersections.
This means that whenever we have a family of subspaces, $(U_i)_{i \in I}$ of V,

- If, for all $i \in I$, U_i has the property P, then

$$\bigcap_{i \in I} U_i$$

has the property P.

Then there exists a *smallest* subspace of V with the property P.

Proof. Let P be such a property, i.e. a property of subspaces of V that is closed under intersections.
We want to showt hat there exists a *smallest* subspace W of V with property P. Specifically, this means:

(1) W is a subspace of V and has the property P.

(2) Whenever W' is a subspace of V that has the property P, then $W \subseteq W'$.

To show it, let $(U_i)_{i \in I}$ be the family of *all* subspaces of V satisfying property P. Define

$$W = \bigcap_{i \in I} U_i.$$

We have to show (1) and (2).

(1) W is a subspace of V by the previous theorem. Also, W has the property P because all U_i have the property P and P is closed under intersections.

(2) We must show that W is smallest. So consider any subspace W' with property P. We must show that $W \subseteq W'$.

But the family $(U_i)_{i\in I}$ contains *all* subspaces with property P. So $W' = U_i$ for all some $i \in I$. Then

$$W = \bigcap_{i\in I} U_i \subseteq U_i = W'.$$

□

3.4.1 Example

Consider $v_1, v_2, v_3 \in V$.
There exists a *smallest subspace* W of V such that $v_1, v_2, v_3 \in W$. We normally call W the *span* of v_1, v_2, v_3.

Proof. By the meta-theorem, the property "contains v_1, v_2, and v_3" is closed under intersections. □

4 Sums of Subspaces

4.1 Definition of Sum of Subsets

Suppose $U_1, \ldots, U_m$ are subsets of V. The **sum** of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$. More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_1 \in U_1, \ldots, u_m \in U_m\}.$$

4.1.1 Example

For $U = \{(x, 0, 0) \in \mathbf{F}^3 \mid x \in \mathbf{F}\}$ and $W = \{(0, y, 0) \in \mathbf{F}^3 \mid y \in \mathbf{F}\}$, we have

$$U + W = \{(x, y, 0) \mid x, y \in \mathbf{F}\}.$$

For $U = \{(x, x, y, y) \in \mathbf{F}^4 \mid x, y \in \mathbf{F}\}$ and $W = \{(x, x, x, y) \in \mathbf{F}^4 \mid x, y \in \mathbf{F}\}$, then

$$U + W = \{(x, x, y, z) \in \mathbf{F}^4 \mid x, y, z \in \mathbf{F}\}.$$

4.2 Sum of subspaces is the smallest containing subspace

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$.

Proof. We know that $0 \in U_1 + \cdots + U_m$ and $U_1 + \cdots + U_m$ is closed under addition and scalar multiplication. Thus, $U_1 + \cdots + U_m$ is a subspace of V. $U_1, \ldots, U_m$ are contained in $U_1 + \cdots + U_m$, and every subspace of V containing $U_1, \ldots, U_m$ contains $U_1 + \cdots + U_m$.
Thus, $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$. $\square$

Note. Sums of subspaces in theory of vector spaces are analogous to unions of subsets in set theory. Given two subspaces of a vector space, the smallest subspace containing them is their sum. Analogously, given two subsets of a set, the smallest subset containing them is their union.

5 Direct Sums

Suppose $U_1, \ldots, U_m$ are subspaces of V. Every element of $U_1 + \cdots + U_m$ can be written in the form

$$u_1 + \cdots + u_m$$

where each u_j is in U_j.

5.1 Definition of direct sum

- The sum $U_1 + \cdots + U_m$ is called a *direct sum* if each element of $U_1 + \cdots + U_m$ can eb written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j.
- If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

5.1.1 Examples

For $U = \{(x, y, 0) \in \mathbf{F}^3 \mid x, y \in \mathbf{F}\}$ and $W = \{(0, 0, z) \in \mathbf{F}^3 \mid z \in \mathbf{F}\}$, then

$$\mathbf{F}^3 = U \oplus W.$$

Suppose $U_j = \{(0, 0, 0, \ldots, j) \in \mathbf{F}^n \mid x \in \mathbf{F}\}$. Then

$$\mathbf{F}^n = U_1 \oplus \cdots \oplus U_n.$$

Non-example: Let $U_1 = \{(x, y, 0) \in \mathbf{F}^3 \mid x, y \in \mathbf{F}\}$, $U_2 = \{(0, 0, z) \in \mathbf{F}^3 \mid z \in \mathbf{F}\}$, $U_3 = \{(0, y, y) \in \mathbf{F}^3 \mid y \in \mathbf{F}\}$. Then, $U_1 + U_2 + U_3$ is not a direct sum.

Proof. Clearly $\mathbf{F}^3 = U_1 + U_2 + U_3$ because every vector $(x, y, z) \in \mathbf{F}^3$ can be written as

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0).$$

This is not equal to the direct sum because the vector $(0, 0, 0)$ can be written in two different ways.

$$(0, 0, 0) = (0, 0, 0) + (0, 0, 0) + (0, 0, 0)$$

and

$$(0, 0, 0) = (0, 1, 0) + (0, 0, 1) + (0, -1, -1).$$

□

5.2 Condition for a direct sum

Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j, is by taking each u_j equal to 0.

Proof. Suppose $U_1 + \cdots + U_m$ is a direct sum. Then by definition, the only way to write 0 as a sum is by taking each u_j equal to 0. To show that $U_1 + \cdots + U_m$ is a direct sum, let $v \in U_1 + \cdots + U_m$. We can write

$$v = u_1 + \cdots + u_m$$

for some $u_1 \in U_1, \ldots, u_m \in U_m$. To show that this is unique, suppose we also have

$$v = v_1 + \cdots + v_m$$

where $v_1 \in U_1, \ldots, v_m \in U_m$. Subtracting them, we have

$$0 = (u_1 - v_1) + \cdots + (u_m - v_m).$$

This implies that $u_j = v_j$. □

5.3 Direct sum of two subspaces

Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$.

Proof. Suppose that $U + W$ is a direct sum. If $v \in U \cap W$, then $0 = v + (-v)$ where $v \in U$ and $-v \in W$. This implies $v = 0$ since it is unique.
To prove the other direction, suppose $U \cap W = \{0\}$. To prove that $U + W$ is a direct sum, suppose $u \in U, w \in W$, and

$$0 = u + w.$$

By the previous theorem, we know that $u = w = 0$. The equation above implies $u = -w \in W$, so $u \in U \cap W$ and $u = w = 0$ is true. $\square$

Note. Sums of subspaces are analogous to unions of subsets. Similarly, direct sums of subspaces are analogous to disjoint unions of subsets. No two subspaces of a vector space can be disjoint, because both contain 0. So disjointness is replaced, at least in the case of two subspaces, with the requirement that the intersection equals $\{0\}$.