MATH 2135 Linear Algebra

1.B Definition of Vector Space

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1 Definition

Let $\mathbf{F}$ be a field. A *vector space* over $\mathbf{F}$ is a set V, together with a distinguished element $0 \in V$ and with operations

$$\begin{aligned} \text{addition} \quad & + : V \times V \to V \\ \text{scalar multiplication} \quad & \cdot : \mathbf{F} \times V \to V. \end{aligned}$$

Satisfies the following 8 axioms:

(A1) Commutativity of addition.

$$\forall v, w \in V, v + w = w + v$$

(A2) Associativity of addition.

$$\forall v, w, u \in V, (v + w) + u = v + (w + u)$$

(A3) Additive identity.

$$\forall v \in V, 0 + v = v$$

(A4) Additive inverse.

$$\forall v \in V, \exists w \in V, v + w = 0$$

(M1) Multiplicative identity.

$$\forall v \in V, 1v = v$$

(M2) Left distributivity.

$$\forall a \in F, \forall v, w \in V, a(v + w) = av + aw$$

(M3) Right distributivity.

$$\forall a, b \in F, \forall v \in V, (a + b)v = av + bv$$

(M4) Associativity of multiplication.

$$\forall a, b \in F, \forall v \in V, (ab)v = a(bv)$$

1.1 Terminology

- The elements of $\mathbf{F}$ are called *scalars*.
- The elements of V are called *vectors or points*.
- A vector space over $\mathbb{R}$ is called a *real vector space*.
- A vector space over $\mathbb{C}$ is called a *complex vector space*.

2 Examples of Vector Spaces

(1) $\mathbf{F}^n$ is the set of column vectors (sometimes row vectors) with elements from $\mathbf{F}$. For instance, $\mathbb{R}^n$ and $\mathbb{C}^n$ are such vector spaces.

$$\begin{aligned}\mathbf{F}^n &= \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \mid a_1, \dots, a_n \in \mathbf{F} \right\} \\ &= \{(a_1, a_2, \dots, a_n) \mid a_1, \dots, a_n \in \mathbf{F}\}\end{aligned}$$

$$0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \qquad k \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ka_1 \\ \vdots \\ ka_n \end{bmatrix}$$

The properties of $\mathbf{F}^n$ makes it a vector space.

(2) Let $\mathbf{F}^\infty = \{(x_1, x_2, x_3, x_4, \dots) \mid x_1, x_2, \dots \in \mathbf{F}\}$ be the set of inifnite sequences of scalars. We define the following:

- $0 = (0, 0, 0, \dots)$ is the constant zero sequences.
- If $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ then we define

$$x + y = (x_1 + y_1, x_2 + y_2, \dots).$$

- If $k \in \mathbf{F}$ and $x = (x_1, x_2, x_3, \dots)$, then we define

$$kx = (kx_1, kx_2, \dots).$$

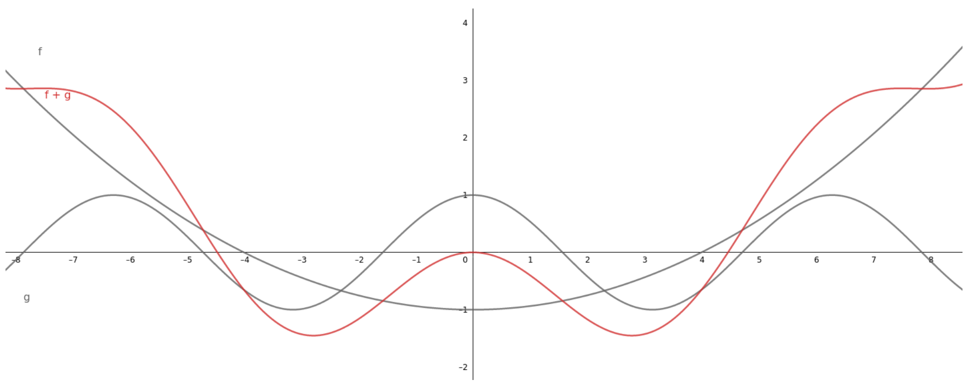
Then $\mathbf{F}^\infty$ is a vector space.

(3) Let $\mathbf{F}$ be a field and let S be a set. Define $\mathbf{F}^S = \{f : S \to \mathbf{F} \mid f \text{ is a function from } S \text{ to } \mathbf{F}\}$.

Define $0 \in \mathbf{F}^S$ by $0(x) = 0$. The f is the zero function, x is any element in S, which gives the output of 0.

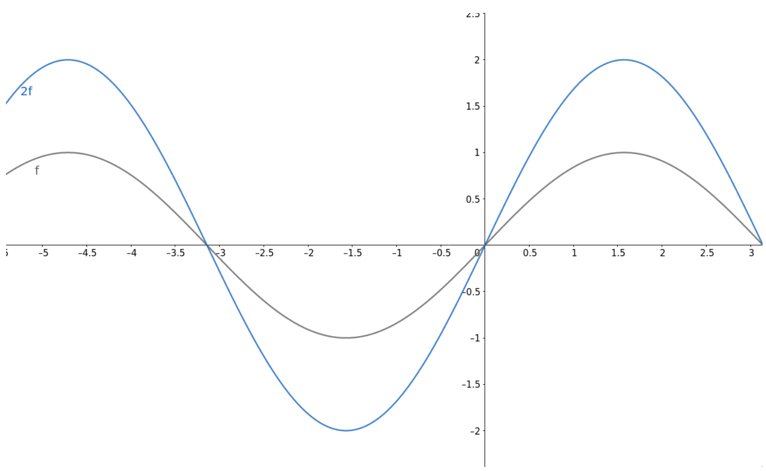
If $f, g \in \mathbf{F}^S$, define $f + g \in \mathbf{F}^S$ as

$$(f+g)(x) = f(x) + g(x).$$



If $k \in \mathbf{F}$ and $f \in \mathbf{F}^S$, define $kf \in \mathbf{F}^S$ as

$$(kf)(x) = k(f(x)).$$



Then $\mathbf{F}^S$ is a vector space.

Note: The functions $F, G : X \to Y$ are equal if

$$\forall x \in X, F(x) = G(x).$$

Proof. Take arbitrary $f, g \in \mathbf{F}^S$. We have to show that $f + g = g + f$ or $\forall x \in S, (f+g)(x) = (g+f)(x)$. Suppose we take an arbitrary $x \in S$, then we have

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ &= g(x) + f(x) \quad \text{by properties of fields} \\ &= (g+f)(x).\end{aligned}$$

This finishes the proof of (A1). The other field axioms are similar. □

(4) For a field $\mathbf{F}$, we define $\mathcal{P}(\mathbf{F})$ as the set of all formal polynomials with variable x and coefficients in $\mathbf{F}$.

$$\mathcal{P}(\mathbf{F}) = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid n \leq 0, a_0, \ldots, a_n \in \mathbf{F}\}.$$

An example would be $\mathcal{P}(x) = 3 + 5x - 7x^2$.

There are two ways to think about a polynomial: "formal" or "function." For example, define the two following polynomials over $\mathbb{Z}_2$

$$p(x) = x + 1 \qquad q(x) = x^2 + 1.$$

As a function, it would be equal since:

x	$p(x)$	$q(x)$
0	1	1
1	0	0

As a formal polynomial, it would be different because it has the following form:

$$\begin{aligned}p(x) &= 0x^2 + 1x + 1 \\ q(x) &= 1x^2 + 0x + 1.\end{aligned}$$

To prove that $\mathcal{P}(\mathbf{F})$ is a vector space, let us define the following:

- Zero polynomial.
$$\mathcal{P}(x) = 0$$
- Addition.
$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \\ q(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n \end{aligned}$$

$$(p+q)(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$
- Scalar multiplication.
$$kp(x) = (ka_0) + (ka_1)x + (ka_2)x^2 + \cdots + (ka_n)x^n$$

With these operations, $\mathcal{P}(\mathbf{F})$ is a vector space.

3 Properties of Vector Spaces

Let V be a vector space over a field $\mathbf{F}$.

- The additive identity is unique. In other words, if $u \in V$ is an additive identity (satisfying $\forall v \in V, u + v = v$) then $u = 0$.

- Additive inverses are unique. Therefore, we can write $-v$ for the additive inverse of v. We also use related notations such as $v - w$ to mean $v + (-w)$.

- Cancellation of addition.

$$v + w = u + w \Rightarrow v = u$$

- For all $v \in V$, we have
$$0v = 0.$$
The 0 is a scalar being multiplied by v (vector), and the 0 on the right is the zero vector.

  Proof. We have
$$\begin{aligned} 0v + 0v &= (0+0)v && \text{by (M3)} \\ &= 0v && \text{by properties of scalars} \\ &= 0 + 0v && \text{by (A3)} \end{aligned}$$
So $0v = 0$ follows by cancellation. □

- For all scalars $a \in \mathbf{F}$, we have
$$a0 = 0.$$

  Proof. We have
$$\begin{aligned} a0 + a0 &= a(0+0) && \text{by (M2)} \\ &= a0 && \text{by (A3)} \\ &= 0 + a0 && \text{by (A3)} \end{aligned}$$
So $a0 = 0$ follows by cancellation. □

- For all $v \in V$, we have
$$(-1)v = -v.$$

Proof. We have

$$\begin{aligned}(-v)+v &= v+(-v) && \text{by (A1)}\\ &= 0. && \text{by (A4)}\end{aligned}$$

We also have

$$\begin{aligned}(-1)v+v &= (-1)v+1v && \text{by (M1)}\\ &= (-1+1)v && \text{by (M3)}\\ &= 0v && \text{by properties of scalars}\\ &= 0. && \text{by a previously proved property}\end{aligned}$$

In particular,

$$(-v)+v=(-1)v+v.$$

Then the claim $-v=(-1)v$ follows by cancellation. □