

MATH 2135 Linear Algebra

1.C Subspaces

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Contents

1	Definition	3
1.1	Characterization of Subspaces	3
2	Examples of Subspaces	4
3	Intersection of Subspaces	9
3.1	Theorem	9
3.2	Notations	10
4	Sums of Subspaces	10
5	Direct Sums	10

1 Definition

Let V be a vector space over a field F . A subset U of V is called a *subspace* of V if U is also a vector space in its own right, using the same zero, addition, and scalar multiplication as V .

1.1 Characterization of Subspaces

A subset $U \subseteq V$ is a subspace if and only if U satisfies the following three conditions:

- (1) Additive identity.

$$0 \in U$$

- (2) Closed under addition.

$$\forall v, w, v, w \in U \Rightarrow v + w \in U$$

- (3) Closed under scalar multiplication.

$$\forall a, v, a \in F, v \in U \Rightarrow av \in U$$

Proof. “ \Rightarrow ” Given $U \subseteq V$, assume U is a subspace of V . We want to show that U satisfies all three conditions above.

- (1) By definition of subspaces, the zero vector of V is the zero vector of U . So $0 \in U$.
- (2) Since U is a vector space, the sum of two vectors in U is a vector in U . Also, U uses the same addition operation as V . So whenever $v, w \in U$, then $v + w \in U$.
- (3) Similar to (2).

□

Proof. “ \Leftarrow ” Another proof is this: To show that U is a vector space, we first need an element $0 \in U$ and operations

$$+ : U \times U \rightarrow U \quad \text{and} \quad \cdot : F \times U \rightarrow U.$$

Second, we must show axioms (A1) - (M4).

- (1) By assumption, $0 \in U$, where 0 is the additive identity of V . So we can use 0 as the additive identity of U .
- (2) By assumption, U is closed under addition, so the addition function $+: V \times V \rightarrow V$ restricts to a function $+: U \times U \rightarrow U$. We can use the same function as the addition function on U .
- (3) We do the same with scalar multiplication.

Second: We must show (A1) - (M4) hold. We only do (A1) since the rest are similar. To prove (A1), take arbitrary $u, v \in U$. We need to show that

$$u + v = v + u$$

in U . But since V is a vector space, we know that

$$u + v = v + u$$

in V . This automatically holds.

The other parts of the definition of a vector space, such as associativity and commutativity, are automatically satisfied for U because they hold on the larger space V . Thus, U is a vector space and hence is a subspace of V . \square

2 Examples of Subspaces

- (a) Let $V = \mathbb{R}^4$ and let

$$W = \{(x, y, z, w) \mid x = 3y + 2z\}.$$

Then W is a subspace of V .

Proof. We need to show that W is a subspace of V .

- (1) $(0, 0, 0, 0) \in W$.
- (2) Assume $v = (x, y, z, w) \in W$ and $v' = (x', y', z', w') \in W$. We need to show that

$$x + x' = 3(y + y') + 2(z + z').$$

We know that $v \in W$ implies

$$x = 3y + 2z.$$

And we know that $v' \in W$ implies

$$x' = 3y' + 2z'.$$

Add these two equations together and we get

$$x + x' = 3(y + y') + 2(z + z').$$

(3) Similar proof for scalar multiplication.

□

(b) Recall that $V = \mathbb{R}^{[0,1]}$ is the vector space of functions from the unit interval

$$[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

to \mathbb{R} . Let $W = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is continuous}\}$. Then W is a subspace of V .

Proof. We need to show that W is a subspace of V .

- (1) The zero function $f(x) = 0$ is continuous. From Calculus, any constant function is continuous.
- (2) The sum of two continuous functions is continuous. This is from Calculus.
- (3) If f is continuous, then so is kf , for any $k \in \mathbb{R}$. This is also from Calculus.

□

(c) Again, let $V = \mathbb{R}^{[0,1]}$, and $U = \{f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is differentiable}\}$. Then U is a subspace of V .

Proof. From Calculus, we know these hold true:

- (1) The zero function $f(x) = 0$ is differentiable with the derivative of

$$f'(x) = 0.$$

- (2) If f, g are differentiable then so is $f + g$ and

$$(f + g)' = f' + g'.$$

(3) If f is differentiable and k is a scalar, then kf is differentiable.

$$(kf)' = kf'.$$

For example, the derivative of $(0 + 2\sin(x) - 3\cos(x))' = 0 + 2\cos(x) - 3(-\sin(x))$.

We also know from Calculus that every differentiable function is continuous.

$$U \subseteq W \subseteq V$$

where U is the set of all differentiable functions, W is the set of all continuous functions, and V is the set of all functions. \square

(d) Let $V = \mathbb{R}^{[0,1]}$ and define

$$X = \left\{ f : [0, 1] \rightarrow \mathbb{R} \mid f \text{ is differentiable and } f' \left(\frac{1}{2} \right) = 0 \right\}.$$

We claim that X is a subspace of V .

Note. We already know that U , the set of differentiable functions, is a subspace of V .

Proof. Effectively, it is sufficient to show that X is a subspace of U .

(1) The zero function $f(x) = 0$ is differentiable since

$$f'(x) = 0 \text{ and } f' \left(\frac{1}{2} \right) = 0.$$

So $f \in X$.

(2) Given $f, g \in X$, we must check that $f + g \in X$. Clearly $f + g$ is differentiable. We must check that $(f + g)' \left(\frac{1}{2} \right) = 0$. From Calculus, we have

$$\begin{aligned} (f + g)' \left(\frac{1}{2} \right) &= f' \left(\frac{1}{2} \right) + g' \left(\frac{1}{2} \right) \\ &= 0 + 0 \\ &= 0. \end{aligned}$$

So we have $f + g \in X$.

(3) Similar proof as follows for scalar multiplication:

$$(kg)' \left(\frac{1}{2} \right) = k \cdot f' \left(\frac{1}{2} \right) = 0.$$

□

(e) Let $V = \mathbb{R}^\infty$, the set of infinite sequences of real numbers.

$$\mathbb{R}^\infty = \{(a_0, a_1, a_2, a_3, \dots) \mid a_0, a_1, \dots \in \mathbb{R}\}.$$

Recall that V is a vector space. Let $W \subseteq V$ be the set of *convergent* sequences. From Calculus, we know that some sequences converge and some do not.

Some examples are:

- $a_i = i \Rightarrow (0, 1, 2, 3, 4, 5, 6, 7, \dots)$

$$\lim_{i \rightarrow \infty} a_i = \text{does not exist, so it does not converge}$$

- $b_i = \frac{1}{i} \Rightarrow (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots)$

$$\lim_{i \rightarrow \infty} b_i = \text{converges to } 0$$

- $c_i = (-1)^i \Rightarrow (1, -1, 1, -1, 1, -1, \dots)$ does not converge.
- $d_i = 2 + \left(-\frac{1}{2}\right)^i \Rightarrow (3, 1.5, 2.25, 1.875, \dots)$

$$\lim_{i \rightarrow \infty} d_i = \text{converges to } 2$$

Then W is a subspace of V .

Proof. We must show that this is true.

- (1) The zero sequence $a_i = 0 \Rightarrow (0, 0, 0, \dots)$ converges to 0.
- (2) From Calculus, the sum of two convergent sequences converges.
In fact,

$$\lim_{i \rightarrow \infty} (a_i + b_i) = \lim_{i \rightarrow \infty} a_i + \lim_{i \rightarrow \infty} b_i$$

- (3) Similar proof to scalar multiplication.

$$\lim_{i \rightarrow \infty} (ka_i) = k \lim_{i \rightarrow \infty} a_i$$

□

Let U be the set of sequences that converge to 0. Then U is a subspace of V (and of W).

- (f) A recurrence relation. Consider the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots$$

Let F_n be the n th element of this sequence. Then, we have the following recurrence:

$$\begin{aligned} \text{base case } F_0 &= 1 \\ \text{base case } F_1 &= 1 \\ \text{recurrence } F_{n+2} &= F_n + F_{n+1} \quad \text{for all } n \geq 0. \end{aligned}$$

If we forget the base cases, we can consider the set of *all* sequences satisfying the recurrence

$$U = \{(a_0, a_1, a_2, \dots) \mid \text{for all } n \geq 0, a_{n+2} = a_n + a_{n+1}\}$$

A sequence of numbers is called a *generalized Fibonacci sequence* if it satisfies this recurrence, i.e. if it is a member of the set U .

Examples.

$$\begin{aligned} 1, 2, 3, 5, 8, 13, \dots \\ 7, -3, 4, 1, 5, 6, 11, 17, 28, 45, \dots \\ 0, 0, 0, 0, 0, 0, 0, \dots \end{aligned}$$

Claim. U is a subspace of \mathbb{R}^∞ .

- (1) The sequence $0, 0, 0, \dots$ is a generalized Fibonacci sequence.
- (2) U is closed under addition.

$$\begin{array}{r} 1, 2, 3, 5, 8, 13, \dots \\ 7, -3, 4, 1, 5, 6, \dots \\ \hline 8, -1, 7, 6, 13, 19, \dots \end{array}$$

Proof. Suppose $a = (a_0, a_1, a_2, \dots) \in U$ and $b = (b_0, b_1, b_2, \dots) \in U$. Then $a + b = c = (c_0, c_1, c_2, \dots)$ where $c_i = a_i + b_i$.

We must show $c \in U$, i.e. we must show that c is a generalized Fibonacci sequence.

So take an arbitrary $n \geq 0$. We must show $C_{n+2} = C_n + C_{n+1}$. Indeed, we have:

$$\begin{aligned} C_{n+2} &= a_{n+2} + b_{n+2} \\ &= (a_n + a_{n+1}) + (b_n + b_{n+1}) \\ &= (a_n + b_n) + (a_{n+1} + b_{n+1}) \\ &= c_n + c_{n+1} \end{aligned}$$

□

(3) Closed under scalar multiplication: similar.

3 Intersection of Subspaces

3.1 Theorem

Let V be a vector space over a field F . Assume U and W are subspaces of V . Then $U \cap W$ is a subspace of V .

Proof. To show that $U \cap W$ is a subspace, we need to show the three properties.

- (1) We must show that $0 \in U \cap W$. But by assumption, U is a subspace, so $0 \in U$. Also, W is a subspace, so $0 \in W$. By definition of intersection, we have $0 \in U \cap W$.
- (2) We must show that $U \cap W$ is closed under addition. Consider arbitrary $v, w \in U \cap W$ and we need to show that $v + w \in U \cap W$.

Indeed, we have:

- Since $v \in U \cap W$, we know $v \in U$.
- Since $w \in U \cap W$, we know $w \in U$.
- Since U is a subspace, it is closed under addition, so $v + w \in U$.

Similarly:

- Since $v \in U \cap W$, we know $v \in W$.

- Since $w \in U \cap W$, we know $w \in W$.
- Since W is a subspace, it is closed under addition, so $v + w \in W$.

From $v + w \in U$ and $v + w \in W$, by definition of intersection, we know

$$v + w \in U \cap W.$$

- (3) We must show that $U \cap W$ is closed under scalar multiplication. So consider arbitrary $k \in F$ and $v \in U \cap W$. We must show that $kv \in U \cap W$.

Since $v \in U \cap W$, we have $v \in U$. Since U is a subspace of V , we know that U is closed under scalar multiplication, so $kv \in U$.

Similarly, since $v \in U \cap W$, we know $v \in W$. Since W is a subspace of V , we know that W is closed under scalar multiplication, so $kv \in W$.

From $kv \in U$ and $kv \in W$, it follows that $kv \in U \cap W$ (by definition of intersection), as desired.

□

3.2 Notations

- (x_1, \dots, x_n) is called an n -tuple.
- $(x_i)_{i \in \{1, \dots, n\}}$ is another notation for the same thing. This is called “family” notation. But, this notation also works for infinite index sets.

$$(x_i)_{i \in \mathbb{N}} = (x_0, x_1, x_2, \dots)$$

$$\left(\frac{1}{i+1} \right)_{i \in \mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right)$$

- More generally, we can use the “family notation” for other families of things, not necessarily numbers.

U_1, U_2, U_3	3 subspaces of V
$(U_i)_{i \in \{1, 2, 3\}}$	Notation for the same thing.
$(U_i)_{i \in I}$	Some family of subspaces.

- Other notations that go along with these:

4 Sums of Subspaces

5 Direct Sums