MATH 2135 Linear Algebra

Chapter 5 Eigenvalues, Eigenvectors, and Invariant Subspaces

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Contents

1 5.A Invariant Subspaces

1.1 Definition of Invariant Subspace

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called ***invariant*** under T if $u \in U$ implies $Tu \in U$.

1.1.1 Example

Suppose $T \in \mathcal{L}(V)$. Show that each of the following subspaces of V is invariant under T:

1. $\{0\}$: If $u \in \{0\}$, then $u = 0$ and hence $Tu = 0 \in \{0\}$. Thus $\{0\}$ is invariant under T.
2. V: If $u \in V$, then $Tu \in V$. Thus V is invariant under T.
3. null T: If $u \in \text{null } T$, then $Tu = 0$, hence $Tu \in \text{null } T$. Thus null T is invariant under T.
4. range T: If $u \in \text{range } T$, then $Tu \in \text{range } T$. Thus range T is invariant under T.
5. Suppose that $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is defined by $Tp = p'$. Then $\mathcal{P}_4(\mathbb{R})$, which is a subspace of $\mathcal{P}(\mathbb{R})$, is invariant under T because if $p \in \mathcal{P}(\mathbb{R})$ has degree at most 4, then p' also has degree at most 4.

1.2 Eigenvalues and Eigenvectors

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbf{F}$ is called an ***eigenvalue*** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$. The vector $v \in V$ is called an ***eigenvector*** of T corresponding to λ.

1.2.1 Equivalent conditions to be an eigenvalue

Recall that I is the identity operator. Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Then the following are equivalent:

1. λ is an eigenvalue of T;
2. $T - \lambda I$ is not injective;
3. $T - \lambda I$ is not surjective;
4. $T - \lambda I$ is not invertible.

1.2.2 Example

Suppose $T \in \mathcal{L}(\mathbf{F}^2)$ is defined by

$$T(w, z) = (-z, w).$$

Find the eigenvalues and eigenvectors of T if $\mathbf{F} = \mathbb{C}$.

Solution: To find eigenvalues of T, we must find the scalars λ such that

$$T(w, z) = \lambda(w, z)$$

has some solution other than $w = z = 0$. The equation above is equivalent to

$$-z = \lambda w, \quad w = \lambda z.$$

Substituting the value for w, we get

$$-z = \lambda^2 z.$$

Now z cannot equal to 0, so we have

$$-1 = \lambda^2$$

and the solutions are $\lambda = i$ and $\lambda = -i$. The eigenvectors corresponding to the eigenvalue i are the vectors of the form $(w, -wi)$, with $w \in \mathbb{C}$ and $w \neq 0$. For the eigenvalue $-i$ are the vectors of the form w, wi, with $w \in \mathbb{C}$ and $w \neq 0$. $\square$

1.2.3 Linearly independent eigenvectors

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and $v_1, \dots, v_m$ are corresponding eigenvectors. Then $v_1, \dots, v_m$ is linearly independent.

Proof. Suppose $v_1, \dots, v_m$ are linearly dependent. We will derive a contradiction. Let k be the smallest index such that

$$v_k \in \operatorname{span}(v_1, \dots, v_{k-1}).$$

Then there exist $a_1, \dots, a_k$ such that

$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}.$$

Appply T to both sides of this equation, getting

$$\lambda_k v_k = a_1\lambda_1 v_1 + \cdots + a_{k-1}\lambda_{k-1}v_{k-1}.$$

Multiply both sides of v_k by λ_k and then subtract the equation above, getting

$$0 = a_1(\lambda_k - \lambda_1)v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1}.$$

Since k was the smallest index satisfying $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$, then $v_1, \ldots, v_{k-1}$ is linearly independent. Thus all the a's are 0. Since $\lambda_1, \ldots, \lambda_m$ were assumed to be distinct, then $\lambda_1 - \lambda_k \neq 0, \ldots, \lambda_{k-1} - \lambda_k \neq 0$. We have

$$a_1, \ldots, a_{k-1} = 0$$

and we get $v_k = 0$ which contradicts our assumption. □

1.2.4 Number of eigenvalues

Suppose V is finite-dimensional. Then each opeartor on V has at most dim V distinct eigenvalues.

Proof. Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Let $v_1, \ldots, v_m$ be corresponding eigenvectors. Then the previous theorem implies that the list $v_1, \ldots, v_m$ is linearly independent. Thus $m \leq \dim V$ as desired. □

2 5.B Eigenvectors and Upper-Triangular Matrices

2.1 Polynomials Applied to Operators

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- T^m is defined by
$$T^m = \underbrace{T \ldots T}_{m \text{ times}}$$
- T^0 is defined to be the identity operator I on V.
- If T is invertible with inverse T^{-1}, then T^{-m} is defined by
$$T^{-m} = (T^{-1})^m.$$
- If T is an operator, then
$$T^m T^n = T^{m+n} \quad \text{and} \quad (T^m)^n = T^{mn}.$$

2.1.1 Definition of $p(T)$

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for $z \in \mathbf{F}$. Then $p(T)$ is the operator defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

2.1.2 Example

Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation operator defined by $Dq = q'$ and p is the polynomial defined by $p(x) = 7 - 3x + 5x^2$. Then $p(D) = 7I - 3D + 5D^2$; thus

$$(p(D))q = 7q - 3q' + 5q''$$

for every $q \in \mathcal{P}(\mathbb{R})$.

2.2 Existence of Eigenvalues

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

Proof. Suppose V is a complex vector space with dimension $n > 0$ and $T \in \mathcal{L}(V)$. Choose a nonzero vector $v \in V$ then

$$v, Tv, T^2 v, \ldots, T^n v$$

is not linearly independent, because V has dimension n and we have $n+1$ vectors. Thus, there exist complex numbers $a_0, \ldots, a_n$ such that

$$0 = a_0 v + a_1 T v + \cdots + a_n T^n v.$$

Note that $a_1, \ldots, a_n$ cannot all be 0 because the equation becomes $a_0 v = 0$ which forces $a_0 = 0$. Make $a_0, \ldots, a_n$ be the coefficients of a polynomial, which by the Fundamental Theorem of Algebra has a factorization

$$a_0 + a_1 z + \cdots + a_n z^n = c(z - \lambda_1) \ldots (z - \lambda_m)$$

where c is a nonzero complex number, each λ_j is in $\mathbb{C}$, and the equation holds for all $z \in \mathbb{C}$. We then have

$$\begin{aligned} 0 &= a_0 v + a_1 T v + \cdots + a_n T^n v \\ &= (a_0 I + a_1 T + \cdots + a_n T^n) v \\ &= c(T - \lambda_1 I) \ldots (T - \lambda_m I) v. \end{aligned}$$

Thus $T - \lambda_j I$ is not injective for at least one j. In other words, T has an eigenvalue. $\square$

Note: The theorem above requires a *complex* vector space. Note that for a *real* vector space, this theorem would not hold.

3 5.C Eigenspaces and Diagonal Matrices

A ***diagonal matrix*** is a square matrix that is 0 everywhere except possibly along the diagonal.

Let V be a finite-dimensional vector space over a field $\mathbf{F}$. Let $B = v_1, \ldots, v_n$ be a basis of V and let $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. $\mathcal{M}(T, B, B)$ is diagonal,
2. Each $v_1, \ldots, v_n$ is an eigenvector of T.

Proof. (2) implies (1). Assume $v_1, \ldots, v_n$ are eigenvectors of T, with respective eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\mathcal{M}(T)$ is computed as follows:

$$\begin{aligned}
Tv_1 &= \lambda_1 v_1 = \lambda_1 v_1 + 0v_2 + \cdots + 0v_n \\
Tv_2 &= \lambda_2 v_2 = 0v_1 + \lambda_2 v_2 + \cdots + 0v_n \\
&\vdots \\
Tv_n &= \lambda_n v_n = 0v_1 + 0v_2 + \cdots + \lambda_n v_n.
\end{aligned}$$

Then

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

is diagonal.

(1) implies (2). Assume $\mathcal{M}(T)$ is diagonal. This means that

$$\begin{aligned}
Tv_1 &= a_n v_1 + 0v_2 + \cdots + 0v_n \\
&\vdots \\
Tv_n &=)v_1 + 0v_2 + \cdots + a_m v_n.
\end{aligned}$$

So we have

$$\begin{aligned} Tv_1 &= a_{11}v_1 \\ Tv_2 &= a_{22}v_2 \\ &\vdots \\ Tv_n &= a_{mm}v_n \end{aligned}$$

and all of $v_1, \ldots, v_n$ are eigenvectors. $\square$

3.1 Definition of Eigenspace, $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The ***eigenspace*** of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

$$E(\lambda, T) = \text{null}(T - \lambda I).$$

In other words, $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

3.2 Definition of Diagonalizable

An operator $T \in \mathcal{L}(V)$ is called ***diagonalizable*** if the operator has a diagonal matrix with respect to some basis of V.

3.2.1 Example

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$T(x, y) = (41x + 7y, -20x + 74y).$$

The matrix of T with respect to the standard basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} 41 & 7 \\ -20 & 74 \end{pmatrix}$$

which is not a diagonal matrix. However, T is diagonalizable, because the matrix of T with respect to the basis $(1, 4), (7, 5)$ is

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix}.$$

3.3 Conditions equivalent to diagonalizability

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

1. T is diagonalizable;
2. V has a basis consisting of eigenvectors of T;
3. $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Proof. (1) implies (2). By the previous proposition. (3) implies (2). Assume (3) and

$$\begin{aligned}
&\text{Let } V_{1,1}, \ldots, V_{1,k_1} \text{ be a basis of } E_{\lambda_1}(T).\\
&\text{Let } V_{2,1}, \ldots, V_{2,k_2} \text{ be a basis of } E_{\lambda_2}(T).\\
&\vdots\\
&\text{Let } V_{m,1}, \ldots, V_{m,k_m} \text{ be a basis of } E_{\lambda_m}(T).
\end{aligned}$$

Then all of the $V_{i,j}$ are linearly independent (by previous theorem). And by assumption (3), there are $n = \dim V$ of them. So they form a basis of V consisting of eigenvectors of T. So (2) holds. $\square$

3.4 Enough eigenvalues implies diagonalizability

If $T \in \mathcal{L}(V)$ has dim V distinct eigenvalues, then T is diagonalizable.

Proof. In this case, let $v_1, \ldots, v_n$ be the corresponding eigenvectors. They are linearly indepndent by theorem, so a basis. $\square$

3.4.1 Example

- For the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, there are 3 distinct eigenvalues and it is diagonalizable.
- For the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, there are 2 distinct eigenvalues, and 3 linearly independent eigenvectors. It is also diagonalizable.

- For the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, there is 1 distinct eigenvalue, 3 linearly independent eigenvectors, and it is diagonalizable.
- For the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$ there are 2 distinct eigenvalues and 2 linearly independent eigenvectors. However, it is not diagonalizable.