Indirect Proofs

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Implications

Defn. An implication is a statement of the form

$$\underbrace{\text{If } p \text{ is true,}}_{\text{antecedent}} \quad \underbrace{\text{then } q \text{ is true.}}_{\text{consequent}}$$

ex.

If n is an even integer, then n^2 is an even integer.

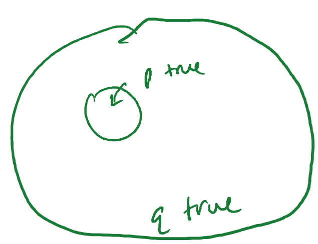
Restrictions. If antecedent, then consequent.

→ Directionality: only implies as read, not "if consequent, then antecedent"

→ Only says anything about consequent when antecedent is true

→ Does not imply causality.

Diagram



$p \to q$

Negation.

Defn. A proposition is a statement that is either true or false.

The negation of a proposition x is a proposition that is true when x is false and false when x is true.

The negation of a universal statement

for all p, then q is true

is the existential statement

there exists a p such that q is false

The negation of an existential statement

there exists a p such that q is true

is the universal statement

for all p, then q is false

The negation of the implication

if p, then q

is the implication

if p, then q is false

Proof by contrapositive

Defn. The contrapositive of an implication

if p is true, then q is true

is the implication

if q is false, then p is false.

These statements are <u>equivalent</u>.

Theorem. $\forall n.\ n \in \mathbb{Z}$ n^2 even $\rightarrow$ n even.

Proof. Assume n is odd. Then there exists some k such that $n = 2k+1$. Note $n^2 = (2k+1)^2$

$= 4k^2 + 4k + 1$. Note that there exists some m such that $m = 2k^2 + 2k$, so $n^2 = 2m+1$. Therefore n^2 is odd, so we have proved the contrapositive.

Biconditionals.

Defn. The phase "if and only if" ($\leftrightarrow$) is used as follows:

$$(p \to q) \wedge (q \to p) \to (p \leftrightarrow q)$$

and $(p \leftrightarrow q) \to (p \to q) \wedge (q \to p)$.

To prove, need to prove $p \to q$ and $q \to p$.

Proof by contradiction.

Defn. A proof by contradiction shows that some statement p is true by showing it cannot be false.

Structure First, assume p is false.
Next, show this leads to an impossible statement.
This means that p cannot be false, so p must be true.

Theorem. There is no set that is larger than every other set.

Proof. We prove by contradiction. Assume there exists such a largest set, S. Now consider the power set $\mathcal{P}(S)$. By Cantor's Theorem, we know that $|S| < |\mathcal{P}(S)|$, so $\mathcal{P}(S)$ is a larger set than S. This contradicts the assumption that there is no set larger than S, so there exists no largest set.