Functions

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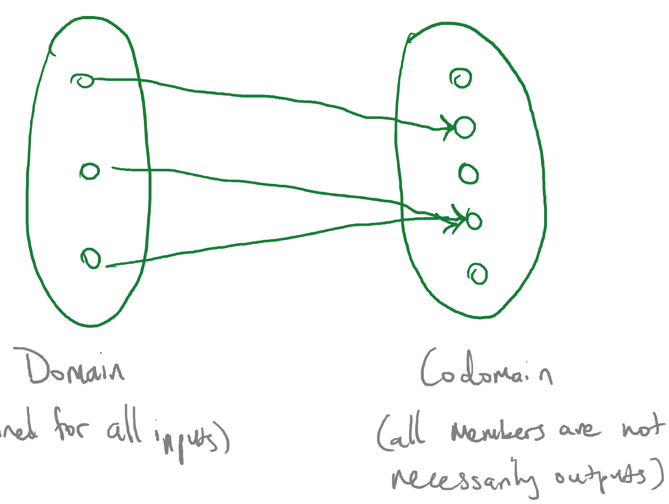
Functions

Rough Defn. A function is an object f that takes in exactly one input x and returns exactly one output $f(x)$.

Properties In mathematics, functions are <u>deterministic</u>.

A function f can only be applied to its domain set and the output will always be in its codomain set.

ex.



Notation: $f: A \to B$

where A is the domain and B is the codomain

Formal definition. We say $f: A \to B$ is a function if these two rules apply:

1. Domain/codomain rules

$$\forall a \in A. \left(\exists b \in B. \left(f(a) = b \right) \right)$$

("Every input in A maps to some input in B")

2. Determinism

$$\forall a_1 \in A. \left(\forall a_2 \in A. \left((a_1 = a_2) \to \left(f(a_1) = f(a_2) \right) \right) \right)$$

("Equal inputs produce equal outputs")

Declaring functions. Typically, declare by describing rule mapping domain to codomain. ex:

$$f(n) = n+1, \text{ where } f: \mathbb{Z} \to \mathbb{Z}$$

$$g(x) = \sin(x), \text{ where } f: \mathbb{R} \to \mathbb{R}$$

Combining Functions.

Function composition. Given two functions $f: A \to B$ and $g: B \to C$

We can write $h: A \to C$ as $h = g(f(x))$ or $(g \circ f)(x)$.

Special Types of Functions.

Injections. A function $f: A \to B$ is injective (or "one-to-one") if

$$\forall a_1 \in A. \left(\forall a_2 \in A. \left((a_1 \neq a_2) \to (f(a_1) \neq f(a_2)) \right) \right)$$

(if the inputs are different, the outputs are different)

Equivalently,

$$\forall a_1 \in A. \left(\forall a_2 \in A. \left((f(a_1) = f(a_2)) \to (a_1 = a_2) \right) \right)$$

(if the outputs are the same, the inputs are the same)

Surjections. A function $f: A \to B$ is surjective (or "onto") if

$$\forall b \in B. \left(\exists a \in A. (f(a) = b) \right)$$

("Every element in the codomain has a corresponding input)

Bijections. Functions that associate each element of the domain with unique elements of the codomain.

Formally, functions that are bijective are both injective and surjective.

Inverse functions A function $f: A \to B$ has an inverse $f^{-1}: B \to A$ if

$$\forall a \in A. \left(f^{-1}(f(a)) = a \right)$$

$$\forall b \in B. \left(f(f^{-1}(b)) = b \right).$$

Not all functions have inverses. If f has an inverse, then it is called <u>invertible</u>.