Mathematical Proofs

Friday, September 18, 2020 5:42 PM

What is a proof?

Defn. A proof is an argument that demonstrates why a certain statement is true, subject to certain rules.

Elements of coming up with a proof.

Definitions : Terms, formal meaning

Intuitions : Meaning of theorem, why it "should" be true

Conventions : Standard format + techniques for proofs

Writing a first proof.

Theorem. If n is an even integer, then n^2 is even.

Defns: An integer n is even if there exists an integer k such that $n = 2k$.

Intuition:

$$2^2 = 4 = 2 \cdot \underline{2}$$

$$10^2 = 100 = 2 \cdot \underline{50}$$

$$(-2)^2 = 4 = 2 \cdot \underline{2}$$

$$n^2 = 2 \cdot p$$

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$$

Proof: Let n be an even integer.

Then, there exists an integer k such that $n = 2k$.

Then, $n^2 = (2k)^2 = 4k^2$.

Note that $n^2 = 4k^2 = 2(2k^2)$, so there exists an integer $m = 2k^2$ such that $n = 2m$.

Therefore, n is even.

Another proof

Theorem. For odd integers m, n, $m+n$ is even.

Proof. Let integers m, n be odd. Then there exist even integers p, q such that $m = 2p+1$ and $n = 2q+1$. Then, $m+n = 2p+2q+2$. Note that $m+n = 2(p+q+1)$, so there exists an integer $r = p+q+1$ such that $m+n = 2r$, so $m+n$ is even.

Universal and Existential Statements.

Theorem. For any odd integer n, there exist integers r, s such that
$r^2 - s^2 = n$.

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Defn. A universal statement is of the form

For all x, \$property holds for x.

An existential statement is of the form

There exists an x where \$property holds for x.

Proving existential statements.
→ Finding the x

Intuition.

$$1 = 1^2 - 0^2$$
$$3 = 2^2 - 1^2$$
$$5 = 3^2 - 2^2$$
$$7 = 4^2 - 3^2$$
$$9 = 5^2 - 4^2$$
$$\vdots$$
$$2k+1 = (k+1)^2 - k^2$$

Proof. Let n be an arbitrary odd integer.
Then there exists an integer k such that
$n = 2k+1$.
Now, let $r = (k+1)^2$, and $s = k^2$.

Then $r^2 - s^2 = (k+1)^2 - k^2$
$= k^2 + 2k + 1 - k^2$
$= 2k+1$
$= n.$

Therefore we have found $r = (k+1)^2$ and $s = k^2$ such that $n = r^2 - s^2$.

Proofs on sets.

Theorem. If A, B, C are sets, then for any $x \in (A \cap B) \cup C$, we have $x \in (A \cup C) \cap (B \cup C)$.

Note. Proofs on sets <u>almost always</u> focus on individual elements of those sets.

Defns. The set $S \cup T$ is defined such that for all x, if $x \in S$ <u>or</u> $x \in T$, then $x \in (S \cup T)$.

The set $S \cap T$ is defined such that for all x, if $x \in S$ <u>and</u> $x \in T$, then $x \in (S \cap T)$.

Proof. Consider an arbitrary element x such that $x \in (A \cap B) \cup C$.
Note that this means $x \in (A \cap B)$ or $x \in C$.
If $x \in (A \cap B)$, then $x \in A$ and $x \in B$,
so $x \in (A \cup C)$ and $x \in (B \cup C)$,
so $x \in (A \cup C) \cap (B \cup C)$.
If $x \in C$, then $x \in (A \cup C)$ and $x \in (B \cup C)$,
so $x \in (A \cup C) \cap (B \cup C)$.

Therefore, in both cases, $x \in (A \cup C) \cap (B \cup C)$.