Binary Relations continued

Thursday, October 1, 2020 11:35 AM

Equivalence classes.

Defn. Given an equivalence relation R over set A, for any $x \in A$, the equivalence class of x is the set

$$[x]_R = \{y \in A \mid xRy\}$$

i.e., the set of all elements related to x by R

Properties.

→ Given equiv. relation R, every element $a \in A$ belongs to exactly one equivalence class.

→ For any $x, y \in A$, we have xRy if and only if $[x]_R = [y]_R$.

Systems of representatives. If R is an equivalence relation over a set A, then a system of representatives for A is a set $X \subseteq A$ containing exactly one element from each equivalence class.

→ All systems of representatives for a given equivalence relation have the same cardinality

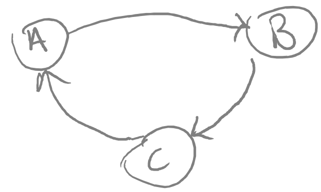
→ If X is a system of representatives for R, no two elements of X are related to each other by R

Equivalent definitions.

Cyclic relation. Given elements $a, b, c \in A$ and a cyclic relation R, then

$$\forall a,b,c \in A.\ (aRb \wedge bRc \rightarrow cRa)$$

ex.



Theorem. A binary relation R over a set A is an equivalence relation if and only if it is reflexive and cyclic.

Proof - **Lemma 1:** If R is an equivalence relation over A, then R is reflexive and cyclic.

Proof. R equivalence $\rightarrow$ R reflexive by defn.

For any $a, b, c \in A$, given aRb and bRc,

$aRb \wedge bRc \rightarrow aRc$ by transitivity.

$aRc \rightarrow cRa$ by symmetry.

Therefore $aRb \wedge bRc \rightarrow cRa$.

Lemma 2: If R is ... and reflexive and cyclic, then R is an equivalence relation

Proof. $\forall a, b, c \in A$, $aRb \wedge bRc \rightarrow cRa$ by definition.

From cyclic, reflexive property we have $aRb \wedge bRb \rightarrow bRa$,

$bRc \wedge cRc \rightarrow cRb$, $cRa \wedge aRa \rightarrow aRc$.

This implies $aRb \wedge bRc \rightarrow aRc$, same for other 3,

as well as $aRb \rightarrow bRa$, same for other 3.

Therefore R is reflexive, symmetric, transitive, so it is equivalence relation.

Prerequisite structures

"a must happen before b"

requires transitivity, irreflexivity, asymmetry

Called a <u>strict order</u>.