# Mathematical Induction

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## Induction.

**Defn.** Let $P$ be some predicate.

The principle of mathematical induction states that if

$P(0)$ is true

and

$\forall k \in \mathbb{N}. \left(P(k) \to P(k+1)\right)$

then

$\forall n \in \mathbb{N}. P(n)$

## Proof by Induction.

**Defn.** A proof by induction is a way to use induction to show that some result is true for all natural numbers $n$.

**Procedure**

1. Prove that $P(0)$ is true
   (i.e., base case)
2. Prove that if $P(k)$ is true, then $P(k+1)$ is true
   (inductive hypothesis / inductive step)

3. Conclude that $P(n)$ is true for all $n \in \mathbb{N}$ by induction.

Example

**Theorem.** The sum of the first $n$ powers of two is $2^n - 1$.

**Proof.** Let $P(n)$ be the claim. We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Base case: We show that $P(0)$ is true; i.e, the sum of the first 0 powers of two is $2^0 - 1$. Since both entities are zero, we see that $P(0)$ is true.

Inductive step: Assume that for some arbitrary $k \in \mathbb{N}$, $P(k)$ holds, meaning that

$$\sum_{i=0}^{k-1} 2^i = 2^k - 1.$$

We will show that $P(k+1)$ holds; i.e. that the sum of the first $k+1$ powers of two is $2^{k+1} - 1$. To see this, note that

$$\sum_{i=0}^{k} 2^i = 2^k + \sum_{i=0}^{k-1} 2^i = 2^k + 2^k - 1$$

$$= 2(2^k) - 1 = 2^{k+1} - 1$$

as required.