Mathematical Tools
Problem Set 7

1 Multivariate Calculus

1.1 Functions of Several Variables [1]

1. Define the following terms: vector-valued function, scalar field, level curves, contour plots, vector fields.

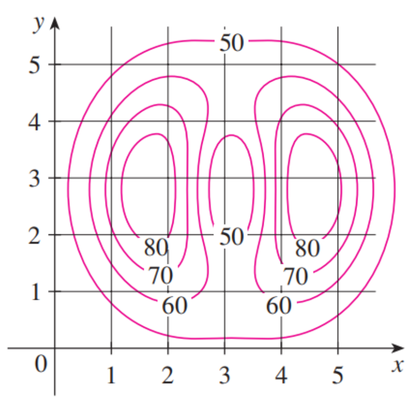
2. For each of the following functions, evaluate $f(3, 2)$ and find and sketch the domain.

   (a) $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$

   (b) $f(x, y) = x \ln(y^2 - x)$

3. Find the domain and co-domain of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$

4. A contour map for a function f is shown below. Use it to estimate the values of $f(1, 3)$, $f(4, 5)$.



5. Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = 26, 0, 6, 12$.

6. Sketch some level curves of the function $h(x, y) = \sqrt{4x^2 + y^2 + 1}$.

1.2 Derivatives and Gradients

1. Describe the differences between the following: partial derivatives, gradients, and directional derivatives.

2. Review the following definitions: second partial derivative, Hessian, Jacobian, and critical points.

3. Show that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exits.

4. If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

5. If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2,1)$ and $f_y(2,1)$.

6. If $f(x,y) = 4 - x^2 - 2y^2$, find $f_x(1,1)$ and $f_y(1,1)$ and interpret these numbers as slopes.

7. If $f(x,y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

8. A function $u(x,y)$ is called harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Show that the function $u(x,y) = e^x \sin y$ is a harmonic function.

9. Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1,1,3)$.

10. Show that $f(x,y) = xe^{xy}$ is differentiable at $(1,0)$ and find its linearization there. Then use it to approximate $f(1.1,-0.1)$.

11. (a) If $z = f(x,y) = x^2 + 3xy - y^2$, find the differential dz.
    (b) If x changes from 2 to 2.05 and y changes from 3 to 2.96 compare the values of Δz and dz.

12. The base radius and height of a right circular cone are measured as 10 cm and 25 cm, respectively, with a possible error in measurement of as much as 0.1 cm in each. Use differentials to estimate the maximum error in the calculated volume of the cone.

13. If $z = x^2 + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find $\frac{dz}{dt}$ when $t = 0$.

14. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

15. Considering a neural network with inputs u, v, output w, and hidden layer nodes x, y, z, t write out the Chain Rule for the case where $w = f(x,y,z,t)$ and $x = x(u,v)$, $y = y(u,v)$, $z = z(u,v)$, and $t = t(u,v)$ and compute $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

16. If $g(s,t) = f(s^2 - t^2, t^2 - s^2)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

17. Find the directional derivative $D_u f(x,y)$ if

$$f(x,y) = x^3 - 3xy + 4y^2$$

18. Find the gradient of the following functions:
    (a) $f(x,y) = \sin x + e^{xy}$.
    (b) $f(x,y) = x^2y^3 - 4y$.
    (c) $f(x,y,z) = x \sin yz$.

19. (a) If $f(x,y) = xe^y$ fnd the rate of change of f at the point $P(2,0)$, in the direction from P to $Q(\frac{1}{2}, 2)$.
    (b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

1.3 Optimization

1. We have two neuron types A and B. Neuron A computes the squared value of its input and neuron B subtracts its first input from the second. Considering that we have two neurons of type A (call them A_1 and A_2) and a neuron of type B the goal is to find inputs to neurons A_1 and A_2 that maximizes or minimizes the output of neuron B. If the inputs to A_1 and A_2 are represented by y and x the output of neuron B is just $f(x,y) = y^2 - x^2$. Find the extreme values of $f(x,y) = y^2 - x^2$.

2. Find the local maximum and minimum values and saddle points of $f(x,y) = x^4 + y^4 - 4xy + 1$.

3. Find and classify the critical points of the function $f(x,y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$.

4. Find the shortest distance from the point $(1, 0, -2)$ to the plane $x+2y+z = 4$.

5. A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box. Provide two solutions, one with and one without using lagrange multipliers.

6. Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y)| \quad 0 \le x \le 3, 0 \le y \le 2\}$.

7. Find the extreme values of the function $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

References

[1] James Stewart. *Single variable calculus: Early transcendentals*. Cengage Learning, 2011.