Mathematical Tools
Solutions to Problem Set 7

1 Multivariate Calculus [1]

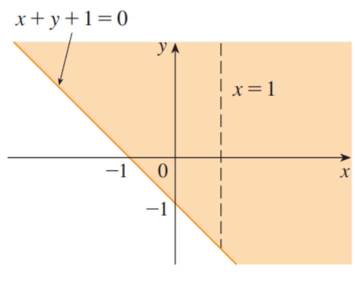
1. A **vector-valued function** is a function where $f : \mathbb{R} \to \mathbb{R}^m$. For example, $\vec{r} = \vec{f}(t)$. A **scalar field** takes a vector as input and gives a scalar as output. Scalar fields can be visualized in two ways. The first takes weights w_1, w_2 as making a horizontal plane, then plots the scalar values for loss L in the third dimension. Each cross-section of this bowl-like shape has the same scalar function output value. These lines are called **level curves**. A **contour plot** is very similar to a topographic map. We take our weights w_1, w_2 as our two axes and draw contour lines or level curves to indicate depth.

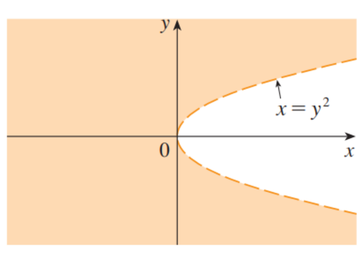
2. (a) $f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{2}}{2}$. The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is:

$$D = \{(x,y)|x + y + 1 \geq 0, x \neq 1\}$$

The inequality $x + y + 1 \geq 0$, or $y \geq -x - 1$, describes the points that lie on or above the line $y = -x - 1$, while $x \neq 1$ means that the points on the line $x - 1$ must be excluded from the domain.

   (b) $f(3,2) = 3\ln(2^2 - 3) = 3\ln 1 = 0$. Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is $x < y^2$, the domain of f is $D = \{(x,y)|x < y^2\}$. This is the set of points to the left of the parabola $x = y^2$.





[1] All problems and solutions are taken from [1]

3. The domain of g is

$$D = \{(x, y)|9 - x^2 - y^2 \geq 0\} = \{(x, y)|x^2 + y^2 \leq 9\}$$

Which is the disk with center $(0, 0)$ and radius 3. THe range of g is

$$\{z|z = \sqrt{9 - x^2 - y^2 \geq 9}\} = \{(x, y) \in D\}$$

Since z is a positive square root, $z \geq 0$. Also because $9 - x^2 - y^2 \leq 9$, we have $\sqrt{9 - x^2 - z^2} \leq 3$, so the range is $\{z|0 \leq z \leq 3\} = [0, 3]$.

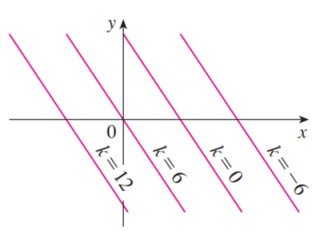
4. The point $(1, 3)$ lies partway between the level curves with z-values 70 and 80. We estimate that (and similarly for $(4, 5)$):

$$f(1, 3) \approx 73$$
$$f(4, 5) \approx 56$$

5. The level curves are

$$6 - 3x - 2y = k \quad \text{or} \quad 3x + 2y + (k - 6) = 0$$

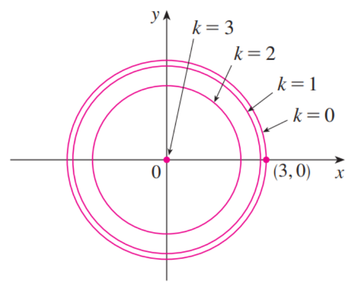
This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6,$ and 12 are $3x+2y-12 = 0$, $2x+2y-6 = 0$, $3x+2y = 0$, and $3x+2y+6 = 0$. They are sketched below. The level curves are equally spaces parallel lines because the graph of f is a plane.



6. The level curves are

$$\sqrt{9 - x^2 - y^2} = k \quad \text{or} \quad x^2 + y^2 = 9 - k^2$$

This is a family of cocentric circles with center $(0, 0)$ and radius $\sqrt{9 - k^2}$. The cases $k = 0, 1, 2, 3$ are shown below.



2 Derivatives and Gradients

1. **Partial derivatives** are defined as follows:

$$\frac{\delta f}{\delta x_k} \equiv \lim_{\Delta x_k \to 0} \frac{f(x_1, \cdots, x_k + \Delta x_k, \cdots, x_n) - f(x_1, \cdots, x_k, \cdots, x_n)}{\Delta x_k}$$

   The **gradient** is essentially a vector containing the partial derivatives of some function. It is a vector field defined as the following:

$$\nabla f = \begin{bmatrix} \frac{\delta f}{\delta x_1} \\ \vdots \\ \frac{\delta f}{\delta x_n} \end{bmatrix}$$

   If we take an infinitesimally small step h in the u direction, we're taking the **directional derivative**.

2. We can take the derivative of that partial derivative function. This is called the **second partial derivative**. If we organize these second order partial derivatives into a matrix, that matrix is called a **Hessian**, where each i, j entry is equivalent to the j, i entry. This matrix of partial derivatives is called a **Jacobian**. To make things clear by comparison, the Jacobian of a vector field is analogous to the gradient of a scalar field is analogous to the derivative of a real valued function. **Critical points**, also termed stationary points, are the minima, maxima, and saddle points of functions. We can find these points within some spread of data.

3. Let $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$. First let's approach $(0, 0)$ along the x-axis. Then $y = 0$ gives $f(x, 0) = \frac{x^2}{x^2} = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0, 0) \quad \text{along the } x\text{-axis}$$

   We now approach along the y-axis by putting $x = 0$. Then $f(0, y) = \frac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0, 0) \quad \text{along the } y\text{-axis}$$

Since f has two deferment limits along two different lines, the given limit does not exist.

4. If $y = 0$ then $f(x, 0) = \frac{0}{x^2} = 0$. Therefore

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \quad \text{along the } x\text{-axis}$$

If $x = 0$ then $f(0, y) = \frac{0}{y^2} = 0$, so

$$f(x, y) \to 0 \quad \text{as} \quad (x, y) \to (0, 0) \quad \text{along the } y\text{-axis}$$

Although we have obtained identical limits along the axes, that does not show that the given limit is 0. Let's now approach $(0, 0)$ along another line, say $y = x$. For all $x \neq 0$

$$f(x, y) = \frac{x^2}{x^2 + x^2} = \frac{1}{2}$$

Therefore

$$f(x, y) \to \frac{1}{2} \quad \text{as} \quad (x, y) \to (0, 0) \quad \text{along } y = x$$

Since we have obtained different limits along different paths, the given limit does not exist.

5. Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so

$$f_x(2, 1) = 3 \times 2^2 + 2 \times 2 \times 1^3 = 16$$

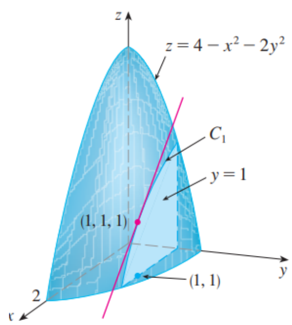
Holding x constant and differentiating with respect to y, we get

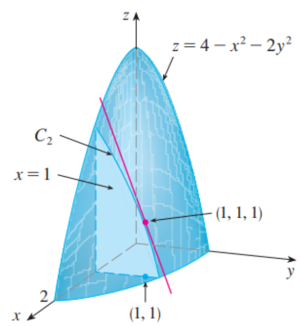
$$f_y(x, y) = 3x^2y^2 - 4y$$
$$f_y(2, 1) = 3 \times 2^2 \times 1^2 - 4 \times 1 = 8$$

6. We have

$$f_x(x, y) = -2x \quad f_y(x, y) = -4y$$
$$f_x(1, 1) = -2 \quad f_y(1, 1) = -4$$

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2, y = 1$. The slope of the tangent line to this parabola at the point $(1, 1, 1)$ is $f_x(1, 1) = -2$. Similarly the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2, x = 1$, and the slope of the tangent line at $(1, 1, 1)$ is $f_x(1, 1) = -4$.





7. Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$
$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{(1+y)^2}\right) \cdot \frac{1}{1+y}$$

8. We first compute the needed second-order partial derivatives

$$u_x = e^x \sin y \quad u_y = e^x \cos y$$
$$u_{xx} = e^x \sin y \quad u_{yy} - e^x \sin y$$
$$u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$$

Therefore u is a harmonic function.

9. Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \quad f_y(x, y) = 2y$$
$$f_x(1, 1) = 4 \quad f_y(1, 1) = 2$$

We use the following formula to find the tangent plane at $(1, 1, 3)$: *Tangent to the surface $z = f(x, y)$ at point $(x_0, y_0, z_0$ is given by*

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Therefore

$$z - 3 = 4(x - 1) + 2(y - 1)$$
$$z = 4x + 2y - 3$$

10. The partial derivatives are

$$f_x(x, y) = e^{xy} + xte^{xy} \quad f_y(x, y) = x^2 e^{xy}$$
$$f_x(1, 0) = 1 \quad f_y(1, 3) = 1$$

Both f_x and f_y are continuous functions, so f is differentiable, The linearization is

$$
\begin{aligned}
L(x, y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\
&= 1 + 1(x-1) + 1 \cdot y = x + y
\end{aligned}
$$

The corresponding approximation is

$$xe^{xy} \approx x + y \Rightarrow f(1.1, -0.1) \approx 1.1 - 0.1 = 1$$

Compare this with the actual value of $f(1.1, -0.1) = 1.1e^{-0.11} \approx 0.98542$.

11. (a)

$$dx = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = (2x+3y)dx + (3x-2y)dy$$

(b) Putting $x = 2$, $dx = \Delta x = 0.05$, $y = 3$, and $dy = \Delta y = -0.04$ we get

$$dz = [2(2) + 3(3)] \times 0.05 + [3(2) - 2(3)] \times (-0.04) = 0.65$$

The increment of z is

$$
\begin{aligned}
\Delta z &= f(2.05, 2.96) - f(2,3) \\
&[(2.05)^2 + 3(2.05)(2.96) - (2.96)^2] - [2^2 + 3(2)(3) - 3^2] \\
&= 0.6449
\end{aligned}
$$

Notice that $\Delta z \approx dz$ but dz is easier to compute.

12. The volume V of a cone with base radius r and height h is $V = \pi r^2 \frac{h}{3}$. So the differential of V is

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = \frac{2\pi rh}{3}dr + \frac{\pi r^2}{3}dh$$

Since each error is at most 0.1 cm, we have $|\Delta r| \leq 0.1$, $|\Delta h| \leq 0.1$. To estimate largest error in the volume we take the largest error in the measurement of r and of h. Therefore we take $dr = 0.1$ and $dh = 0.1$ along with $r = 10$, $h = 25$. This gives

$$dV = \frac{500\pi}{3}(0.1) + \frac{100\pi}{3}(0.1) = 20\pi$$

Thus the maximum error in the calculated volume is about $20\pi\text{cm}^3 \approx 63\text{cm}^3$.

13. The Chain Rule gives

$$
\begin{aligned}
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial t}\frac{dy}{dt} \\
&-(2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)
\end{aligned}
$$

It's not necessary to substitute the expressions for x and y in terms of t. We simply observer that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$. Therefore

$$\frac{dz}{dt}|_{t=0} = (0+3)(2\cos 0) + (0+0)(-\sin 0) = 6$$

14. Again by Chain Rule we have

$$\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\
&= t^2 e^{st^2} \sin(s^2) + 2ste^{st^2} \cos(s^t) \\
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\
&= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^t)
\end{aligned}$$

15.

$$\begin{aligned}
\frac{\partial w}{\partial u} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial u} \\
\frac{\partial w}{\partial v} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}
\end{aligned}$$

16. Let $x = s^2 - t^2$ and $y = t^2 - s^2$. Then $g(s,t) = f(x,y)$ and the Chain Rule gives:

$$\begin{aligned}
\frac{\partial g}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = \frac{\partial f}{\partial x}(2s) + \frac{\partial f}{\partial y}(-2s) \\
\frac{\partial g}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial x}(-2t) + \frac{\partial f}{\partial y}(2t)
\end{aligned}$$

Therefore

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = (2st\frac{\partial f}{\partial x} - 2st\frac{\partial f}{\partial y}) + (-2st\frac{\partial f}{\partial x} + 2st\frac{\partial f}{\partial y}) = 0$$

17.

$$\begin{aligned}
D_{\boldsymbol{u}} f(x,y) &= f_x(x,y)\cos\frac{\pi}{6} + f_y(x,y)\sin\frac{\pi}{6} \\
&(3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2} \\
&\frac{1}{2}[3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y]
\end{aligned}$$

Therefore

$$D_{\boldsymbol{u}} f(1,2) = \frac{1}{2}[3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})(2)] = \frac{13 - 3\sqrt{3}}{2}$$

18. (a) $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{ex}, xe^{xy} \rangle \Rightarrow \nabla f(0,1) = \langle 2, 0 \rangle$
    (b) $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle \cos x + ye^{ex}, xe^{xy} \rangle$.
    (c) $\nabla f(x.y) = \langle 2xy^3, 3x^2y^2 - 4 \rangle$.
    (d) $\nabla f(x,y,z) = \langle \sin yz, xz \cos yz, xy \cos yz \rangle$.

19. (a) We first compute the gradient vector

$$\nabla f9x, y) = \langle e^y, xe^y \rangle \Rightarrow \nabla f(2,0) = \langle 1, 2 \rangle$$

The unit vector in the direction of $\vec{PQ} = \langle -\frac{3}{2}, 2 \rangle$ is $\boldsymbol{u} = \langle -\frac{3}{5}, \frac{4}{5} \rangle$ so the rate of change of f in the direction from P to Q is

$$D_{\boldsymbol{u}} f(2,0) = \nabla f(2,0).\boldsymbol{u} = \langle 1, 3 \rangle \cdot \langle -\frac{3}{5}, \frac{4}{5} \rangle = 1$$

    (b) f increases fastest in the direction of the gradient vector $\nabla f(2,0) = \langle 1, 2 \rangle$. The maximum rate of change is

$$|\nabla f(2,0)| = |\langle 1, 2 \rangle| = \sqrt{5}$$

3 Optimization

1. Since $f_x = -2x$ and $f_y = 2y$, the only critical point is $(0,0)$. Notice that for points on the x-axis we have $y = 0$, so $f(x,y) = -x^2 < 0$ (if $x \neq 0$). However, for points on the y-axis we have $x = 0$, so $f(x,y) = y^2 > 0$ (if $y \neq 0$). Thus every disk with center $(0,0)$ contains points where f takes positive values as well as points where f takes negative values. Therefore $f(0,0) = 0$ can't be an extreme value for f , so f has no extreme value.

2. We first locate the critical points

$$f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x$$

Setting these partial derivatives euqal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation to the second one. This gives

$$0 = x^9 - x = x(x^8 - 1) = x(x-1)(x+1)(x^2+1)(x^4+1)$$

so there are these real roots: $x = 0, 1, -1$. The three critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$. Next we calculate the second partial derivatives and $D(x,y)$

$$f_{xx} = 12x^2 \quad f_{xy} = -4 \quad f_{yy} = 12y^2 \tag{1}$$
$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 144x^2y^2 - 16 \tag{2}$$

Since $D(0,0) = -16 < 0$ the origin is a saddle point. Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$ then $f(1,1) = -1$ is a local minimum. Similarly since $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$ then $f(-1,-1) = -1$ is also a local minimum.

3. The first order partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \quad f_y = 12x^2 - 8y - 8y^3$$

So to find the critical points we need to solve the equations

$$2x(10y - 5 - 2x^2) = 0$$
$$5x^2 - 4y - 4y^3 = 0$$

From the first equation we see that either

$$x = 0 \quad \text{or} \quad 10y - 5 - 2x^2 = 0$$

In the first case $x = 0$ second equation becomes $-4y(1 + y^2) = 0$ so $y = 0$ and we have the critical point $(0,0)$. In the second case $10y - 5 - 2x^2 = 0$ we get $x^2 = 5y - 2.5$ and putting in into the second equation we have to solve the cubic equation $4y^3 - 21y + 12.5 = 0$. Using a graphing calculator to compute the graph of the above function we get

$$y \approx -2.5452 \quad y \approx y = 0.6468 \quad y \approx 1.8985$$

The corresponding x values are $x = \pm\sqrt{5y - 2.5}$. If $y \approx -2.5452$ then x has no corresponding real values. If $y \approx 0.6468$ then $x = \pm 0.8567$. If $y \approx 1.8984$ then $x \approx \pm 2.6442$. The we have a total of five critical points and $(0,0)$ is local maximum, $(\pm 2.64, 1.90)$ is also local maximum, and $(\pm 0.86, 0.65)$ is a saddle point.

4. The distance from any point (x, y, z) to the point $(1, 0, -2)$ is $d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}$. But if (x, y, z) lies on the plane $x + 2y + z = 4$ then $z = 4 - x - 2y$ and so we have $d = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}$. We can minimize d by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14 = 0$$
$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24 = 0$$

we find that the only critical point is $(\frac{11}{6}, \frac{5}{3})$. Since $f_{xx} = 4$, $f_{xy} = 4$, and $f_{yy} = 10$, we have $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = 24 > 0$ and $f_{xx} > 0$ so f has a local minimum at this point where the distance is $d = \frac{5}{6}\sqrt{6}$.

5. Let the length, width, and height of the box be x, y, and z, the volume then is $V = xyz$. We can express V as a function of just two variables x and y by using the fact that the area of the four sides and the bottom of the box is $2xz + 2yz + xy = 12$. Again by computing partial derivatives and we need to solve the following $12 - 2xy - x^2 = 0$ and $12 - 2xy - y^2 = 0$ which gives $x, y = 2, z = 1$ and hence $V = 2 \cdot 2 \cdot 1 = 4$. So the maximum volume of the box is 4m^3.

6. The only critical point is $(1, 1)$ because it has to satisfy $f_x = 2x - 2y = 0$ and $f_y = -2x + 2 = 0$ and we have $f(1, 1) = 1$. On the boundary we have $f(x, 0) = x^2$ when $0 \leq x \leq 3$; $f(3, y) = 9 - 4y$ when $0 \leq y \leq 2$; $f(x, 2) = x^2 - 4x + 4$ when $0 \leq x \leq 3$; and $f(0, y) = 2y$ when $0 \leq y \leq 2$ with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$. Thus on the boundary, the minimum and maximum are 0, 9. Finally we compare these values with the critical point in which $f(1, 1) = 1$ and therefore the absolute maximum of the function f on D is $f(3, 0)$ and the absolute minimum is $f(0, 0) = f(2, 2) = 0$.

7. We use Lagrange multipliers and solve $\nabla f = \lambda \nabla g$ and $g(x, y) = 1$ which can be written as

$$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g(x, y) = 1$$

   or

$$2x = 2x\lambda \quad 4y = 2y\lambda \quad x^2 + y^2 = 1$$

   From the first equation we have $x = 0$ or $\lambda = 1$. If $x = 0$ then the third equation gives $y = \pm 1$. If $\lambda = 1$ then $y = 0$ from the second equation so then the third equation gives $x = \pm 1$. Therefore f has possible extreme values at $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$. Evaluating f at these points we get

$$f(0, 1) = 2 \quad f(0, -1) = 2 \quad f(1, 0) = 1 \quad f(-1, 0) = 1$$

   Therefore the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1, 0) = 1$.

References

[1] James Stewart. *Single variable calculus: Early transcendentals.* Cengage Learning, 2011.