Mathematical Tools
Problem Set 6

1 Probability

1.1 Independence, Conditional Variables, Bayes

1. Describe subjective and frequentist perspectives in probability.

2. Define the following terms: random experiment, sample space, event, probability measure, conditional probability, independent events, and Bayes' rule.

3. Suppose that A, B and C are three events such that A and B are disjoint, A and C are independent, and B and C are independent. Suppose also that $4P(A) = 2P(B) = P(C) > 0$ and $P(A \cup B \cup C) = 5P(A)$. Determine the value of $P(A)$.

4. Suppose 3 coins are tossed. Each coin has an equal probability of head or tail, but are not independent.

   (a) What are the minimum and maximum values of the probability of three heads?

   (b) Now assume that all pairs of coins are mutually independent. What are the minimum and maximum values of the probability of three heads?

5. In the following figure , the probability of being disconnected from network for each node is independent from others and is equal to p. A can be connected to B only when there is a path in which all the nodes are on.

   (a) Suppose 5 nodes are disconnected , what is the probability of A being connected to B?

   (b) Suppose h, a, d are disconnected and we don't have any information about the state of other nodes, what is the probability of connection between A and B?

6. For any three events A, B and D, such that $P_r(D) > 0$, prove that:

$$P(A \cup B|D) = P(A|D) + P(B|D) - P(A \cap B|D)$$

7. In each day stock price goes up or down by one unit with probabilities p and $1-p$ respectively, independent of its behavior on the other days. Given that the stock prices has increased by one unit after 3 days, what is the probability that the stock went up at the end of first day?

8. In a test for detecting a particular type of genetic disorder, the probability that a person who has this type of disorder have a positive reaction to the test is 0.95 and the probability that the person have a negative reaction is 0.05. If the test is applied to a person who does not have this type of genetic disorder, the probability of positive reaction is 0.05 and the probability of negative reaction is 0.95. Suppose that in the general population, one person out of every 10,000 people has this type of genetic disorder. If a person selected at random has a positive reaction to the test, what is the probability that he has this type of genetic disorder?

1.2 Discrete Random Variables

1. Define the following terms: probability space, random variable, simple and discrete variables, probability mass function (PMF), cumulative distribution function (CDF).

2. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus:

   (a) Which of $\mathbb{E}[X]$ or $\mathbb{E}[Y]$ do you think is larger? Why?

   (b) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

3. An urn contains $m+n$ balls, numbered $1, 2, \dots, m+n$. A set of size n is drawn. If we let X denote the number of balls drawn having numbers that exceed each of the numbers of those remaining, compute the probability mass function of X.

4. A random variable X is said to be a Poisson random variable with parameter $\lambda > 0$ if:

$$P(X = i) = e^{\lambda} \frac{\lambda^i}{i!}, \quad i = 0, 1, 2, \dots$$

   Poisson random variables are usually used in neuroscience for modeling the spike counts of a neuron in repeated experiments. Let X be a Poisson random variable with parameter λ. Show that:

$$P(X \text{ is even}) = \frac{1}{2}(1 + e^{-2\lambda})$$

5. The non-negative continuous random variable X is said to have an exponential distribution with parameter λ if its density function is:

$$f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

Let Y_1, Y_2 be independent exponential random variables with parameters λ_1 and λ_2 respectively. Find the value of $P(Y_1 < Y_2)$.

1.3 Continuous Random Variables and Joint

1. What is a probability density function and how is it different from probability mass function?

2. Let X be a random variable with probability density function:

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of c?

   (b) What is the cumulative distribution function of X?

3. The probability density function of X, the lifetime of a certain type of bacteria (measured in hours), is given by,

$$f(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \leq 10 \end{cases}$$

   (a) Find the probability of this lifetime being more than 20 hours.

   (b) What is the cumulative distribution function of X?

4. The random variables X and Y have a joint density function given by:

$$f(x,y) = \begin{cases} \frac{2e^{-2x}}{x} & 0 \leq x, y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Compute $\text{Cov}(X, Y)$ defined as $\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$.

5. The joint density function of X and Y is given by:

$$f(x,y) = \frac{1}{y} e^{y + \frac{x}{y}}$$

Find $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and show that $\text{Cov}(X, Y) = 1$.

6. The random vector (X, Y) is said to be uniformly distributed over a region R in the plane if, for some constant c, its joint density is:

$$f(x) = \begin{cases} c & \text{if } (x,y) \in R \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $\frac{1}{c}$ = area of region R.

(b)) Suppose that (X, Y) is is uniformly distributed over the square centered at $(0, 0)$, whose sides are of length 2. Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.

(c) What is the probability that that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P(X^2 + Y^2 \leq 1)$.

1.4 Expectation

1. What is the expectation of a random variable? Describe both in intuitive terms and using formal language. What properties do you know about expectation?

2. A player throws a fair die and simultaneously flips a fair coin. If the coin lands heads, then she wins twice, and if tails, the one-half of the value that appears on the die. Explain her expected winnings.

3. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find:

   (a) The expected number of urns that are empty.

   (b) The probability that none of the urns is empty.

4. Compute $\mathbb{E}[X]$ if X has the density function given by

   (a)

$$f(x) = \begin{cases} \frac{1}{4} x e^{-\frac{x}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   (b)

$$f(x) = \begin{cases} c(1 - x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

5. The density function of X is given by:

$$f(x) = \begin{cases} a + bx^2 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   If $\mathbb{E}[X] = \frac{3}{5}$, find a and b.