Mathematical Tools
Problem Set 1

1 Sets and Functions

1. **Basic definitions:**
   (a) What is a function?
   (b) What are its domain and co-domain?
   (c) What is the graph of a function?

2. **Four ways of representing a function:**

   There are four ways to represent a function described as following:

   - Verbally: the area of a circle with radius r.
   - Numerically:

r	$f(r)$
1	π
2	4π
3	9π
4	16π

   - Visually: [Fig. 1]

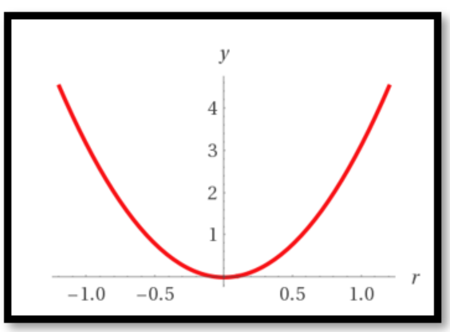

Figure 1: visual representation of a function.

- Algebraically:

$$f(r) = \pi r^2$$

(a) What is the domain and co-domain of a function is each representation?

(b) How can you tell whether a given curve, numerical table, algebraic formula, or a verbal rule is a function? What property needs to be checked in each case?

(c) Starting from a verbal rule, represent a function in other 3 forms.

3. **One-to-one functions:**

A function f is one-to-one if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever} \quad x_1 \neq x_2$$

(a) What does this property mean in terms of each of the four function representations described above?

(b) Which of the following functions are one-to-one?

$$f(x) = x^3$$
$$g(x) = x^2$$

4. **Inverse of a function:** Let f be a one-to-one function with domain A and co-domain B. Then its inverse function f^{-1} has domain B and co-domain A and is defined by:

$$f^{-1}(y) = x \iff f(x) = y$$

(a) What is the inverse of a function in its graph and table representations?

(b) Find the inverse of the following function and sketch the graph of f and f^{-1}:

$$f(x) = x^3 + 2$$

2 Limits

1. Using the limit laws [Fig. 2] evaluate the following limits if they exist:

(a) $\lim\limits_{x \to 2} (8 - 3x + 12x^2)$

(b) $\lim\limits_{t \to -3} (\frac{6+4t}{t^2+1})$

(c) $\lim\limits_{x \to -5} (\frac{x^2-25}{x^2+2x-15})$

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \qquad \text{and} \qquad \lim_{x \to a} g(x)$$

exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$

2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$

3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$

4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$

Figure 2: limit laws.

   (d) $\lim_{z \to 4} (\frac{\sqrt{z}-2}{z-4})$

2. One-sided limits are defined as in [Fig. 3].The graph f is given [Fig. 4]. Find each limit or explain why it does not exist.

   (a) $\lim_{x \to 2^+} f(x)$

   (b) $\lim_{x \to -2^+} f(x)$

   (c) $\lim_{x \to -3} f(x)$

3. For function $f(x) = x^2$, prove that $\lim_{x \to 5} f(x) = 25$ using epsilon-delta definition.

3 Derivatives

1. Find the derivative of the following functions:

   (a) $f(x) = 6x^3 - 9x + 4$

   (b) $y = \sqrt{x} + 8\sqrt[3]{x} - 2\sqrt[4]{x}$

   (c) $h(x) = \frac{4x^3 - 7x + 8}{x}$

   (d) $g(y) = (y - 2)(2y + y^2)$

2. We say that $x = c$ is a critical point of the function $f(x)$ if $f(c)$ exists and if either of the following are true:

2 Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit of $f(x)$ as x approaches a** [or the **limit of $f(x)$ as x approaches a from the left**] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x *less than* a.

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a. Similarly, if we require that x be greater than a, we get "the **right-hand limit of $f(x)$ as x approaches a** is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Figure 3: definition of one-sided limits.

$$f'(c) = 0 \quad \text{OR} \quad f'(c) \text{ does not exist.}$$

There is a theorem that states that for a function f defined on a closed interval, its local maxima or local minima occur either in the endpoints of the interval or the critical points. That is why in the optimization literature for maximizing or minimizing a function we set the derivative (or gradient) equal to zero and solve for the parameters.

Determine the critical points of:

(a) $f(x) = 2x^3 + 3x^2 - 12x + 5$

(b) $g(x) = x^2 e^{-2x}$

3. Find the absolute minimum and absolute maximum of the function:

$$f(x) = \ln x + \frac{2}{x}$$

4 Euler's Number

1. We define a function called factorial which for each positive integer n returns the product of all the integers from 1 to n. We represent factorial function by putting the sign "!" after integer n. Therefore, we have:

$$0! := 1 \quad \text{(notation := means "defined as")}$$
$$1! = 1$$
$$2! = 2(1) = 2$$
$$3! = 3(2)(1) = 6$$

Show that all the following definitions for Euler's number are equivalent:

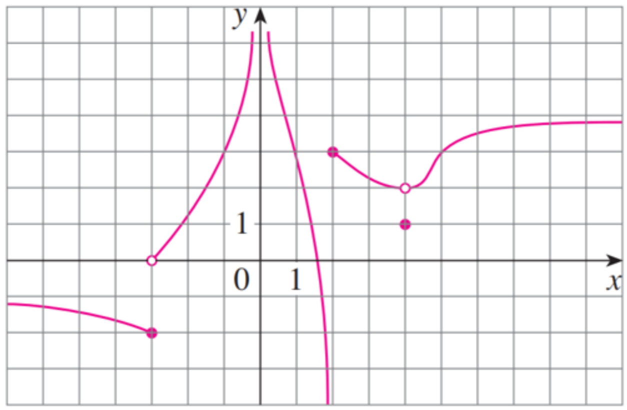

Figure 4: graph of function f.

(a) $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$

(b) A function $f(x)$ that equals its derivative, evaluated at $x = 1$.

(c) $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$

5 Integrals

1. Evaluate the following integrals by interpreting each in terms of areas:

   (a) $\int_0^1 \sqrt{1 - x^2} dx$

   (b) $\int_0^3 (x - 1) dx$

2. Using the table [Fig. 5] evaluate the following indefinite integrals:

   (a) $\int (4x^6 - 2x^3 + 7x - 4) dx$

   (b) $\int (12) dx$

   (c) $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3}$

   (d) $\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}}$

3. One of the simplest models of the changes in the membrane voltage of a neuron is a capacitor. Capacitors store energy for later use. The voltage and current of a capacitor are related. The relationship between a capacitor's voltage and current define its capacitance and its power. To see how the current and voltage of a capacitor are related, you need to take the derivative of the capacitance equation $q(t) = Cv(t)$, which is:

$$\frac{dq(t)}{dt} = C \frac{dv(t)}{dt}$$

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Figure 5: integral laws.

Because $\frac{dq(t)}{dt}$ is the current through the capacitor, you get the following $i - v$ relationship:

$$i(t) = C\frac{dv(t)}{dt}$$

Given that the current through a neuron follows the rule $i(t) = e^t + t^2$ what is the membrane voltage at time t.

References

[1] Online math tutorials and notes.

[2] Calculus Early Transcendentals [Ch 1-3].

[3] Capacitor Example