Mathematical Tools
Problem Set 9

1 Convolution and Fourier Series

1.1 Linear Time-Invariant Systems

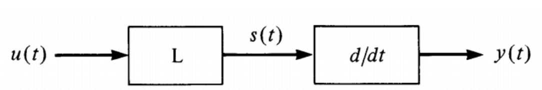
1. Define the following terms: convolution operator, impulse response, linear time-invariant systems, and convolution sum.

2. Consider an integrator that has the input-output relation [1]:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

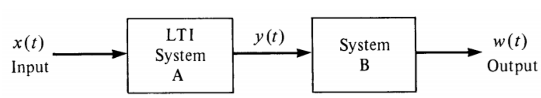
   Determine the input-output relation for the inverse system.

3. Consider the linear, time-invariant system in the following figure, which is composed of a cascade of two LTI systems. $u(t)$ is a unit step signal and $s(t)$ is the step response of system L [1].



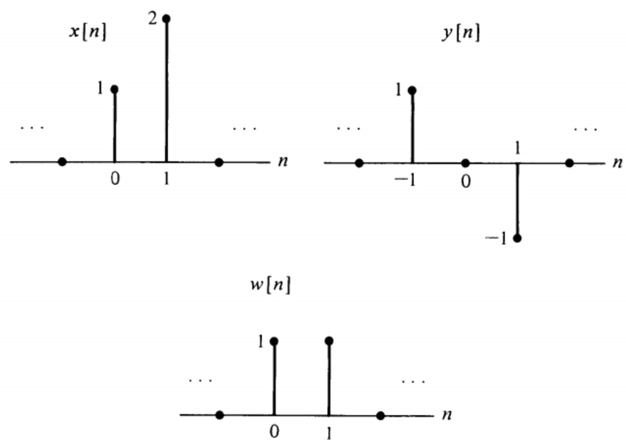
   Using the fact that the overall response of LTI systems in cascade is independent of the order in which they are cascaded, show that the impulse response of system L is the derivative of its step response, i.e., $h(t) = \frac{ds(t)}{dt}$.

4. Consider the cascade of two systems shown in the following figure. System B is the inverse of system A [1].



   (a) Suppose the input is $\delta(t)$. What is the output of $w(t)$?
   (b) Suppose the input is some more general signal $x(t)$. What is the output $w(t)$ in terms of $x(t)$?

5. Consider the three discrete-time signals shown in the following figure [1]



   (a) Verify the distributive law of convolution:

$$(x + w) * y = (x * y) + (w * y)$$

   (b) You may have noticed a similarity between the convolution operation and multiplication, but they are not equivalent. Verify that

$$(x * y) \cdot w \neq x * (y \cdot w)$$

6. Let $y(t) = x(t) * h(t)$. Show the following [1]:

   (a) $\frac{dy(t)}{dt} = x(t) * \frac{dh(t)}{dt} = \frac{dx(t)}{dt} * h(t)$.

   (b) $y(t) = (\int_{-\infty}^{t} x(\tau)d\tau) * h'(t)$.

   (c) $y(t) = \int_{-\infty}^{t} [x'(t) * h(t)]d\tau$.

   (d) $y(t) = x'(t) * \int_{-\infty}^{t} h(\tau)d\tau$.

7. Find the necessary and sufficient condition on the impulse response $h[n]$ such that for any input $x[n]$,

$$\max\{|x[n]|\} \geq \max\{|y[n]|\}$$

   where $y[n] = x[n] * h[n]$.

1.2 Fourier Series and Fourier Transform

1. Define Fourier series and Fourier transform [1].

2. Find the Fourier series coefficients for each of the following signals [1]:

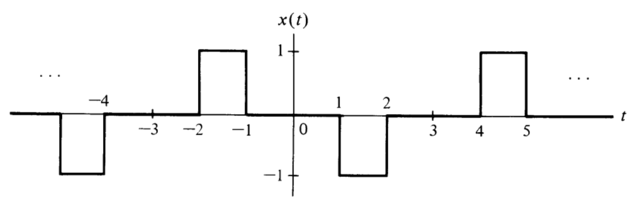
(a) $x(t) = \sin\left(10\pi t + \frac{\pi}{6}\right)$.

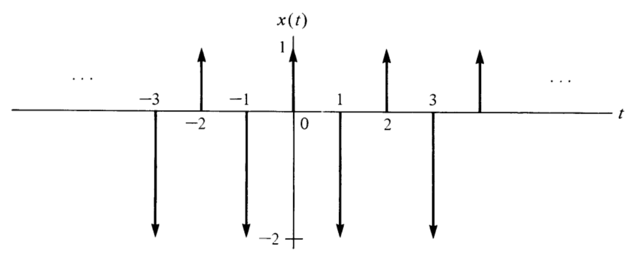
(b) $x(t) = 1 + \cos(2\pi t)$.

(c) $x(t) = [1 + \cos(2\pi t)]\left[\sin\left(10\pi t + \frac{\pi}{6}\right)\right]$.

Hint: You may want to first multiply the terms and then use appropriate trigonometric identities.

3. By evaluating the Fourier series analysis equation, determine the Fourier series for the following signals [1].



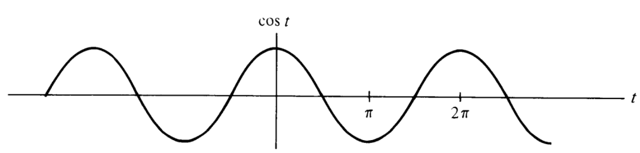


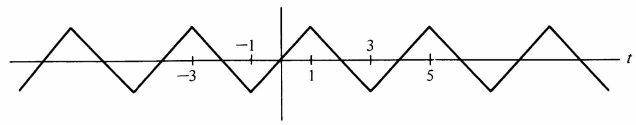
4. Without explicitly evaluating the Fourier series coefficients, determine which of the periodic waveforms in the following figures have Fourier series coefficients with the following properties [1]

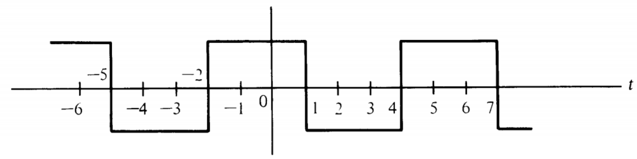
(i) Has only odd Harmonics.

(ii) Has only purely real coefficients.

(iii) Has only purely imaginary coefficients.



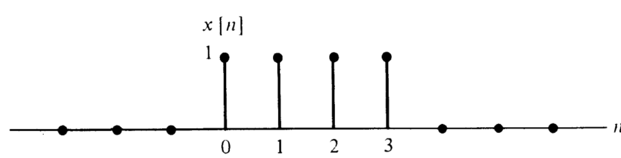




5. Determine the Fourier transform of $x(t) = e^{-\frac{t}{2}}u(t)$ and sketch [1]:
   (a) $|X(w)|$.
   (b) $\sphericalangle X(w)$.
   (c) $\text{Re}\{X(w)\}$.
   (d) $\text{Im}\{X(w)\}$
6. Consider a discrete-time system with impulse response [1]

$$h[n] = (\frac{1}{2})^n u[n]$$

   Determine the response to each of the following inputs:
   (a) $x[n] = (-1)^n = e^{j\pi n}$ for all n.
   (b) $x[n] = e^{j(\frac{\pi n}{4})}$ for all n.
   (c) $x[n] = \cos\left(\frac{\pi n}{4} + \frac{\pi}{8}\right)$ for all n.
7. Compute the discrete-time Fourier transform of the following signals [1].
   (a) $x[n] = (\frac{1}{4})^n u[n]$,
   (b) $x[n] = (a^n \sin \Omega_0 n)u(n)$ for $|a| < 1$.
   (c) $x[n]$ as show in the figure below.



References

[1] Alan V. Oppenheim. Signals and systems course. https://ocw.mit.edu/resources/res-6-007-signals-and-systems-spring-2011/, 2011.