Mathematical Tools
Problem Set 3

1 Linear Algebra

1.1 Vectors, Norm and Unit Vectors, and Matrices

1. Review the following definitions; scalar, vector, dot product, norm, matrix, and transpose.

2. Every combination of $v = (1, -2, 1)$ and $w = (0, 1, -1)$ has components that add to ———. Find c and d so that $cv + dw = (3, 3, -6)$ [1].

3. How many corners does a cube have in 4 dimensions? How many 3D faces? How many edges? A typical corner is $(0, 0, 1, 0)$. A typical edge goes to $(0, 1, 0, 0)$ [1].

4. (a) What is the sum V of the twelve vectors that go from the center of a clock to the hours 1:00, 2:00, . . . , 12:00?
   (b) If the 2:00 vector is removed, why do the 11 remaining vectors add to 8:00?
   (c) What are the components of that 2:00 vector $v = (\cos\theta, \sin\theta)$?

5. Draw vectors u, v, w so that their combinations $cu + dv + ew$ fill only a line. Find vectors u, v, w so that their combinations $cu + dv + ew$ fill only a plane.

6. For any unit vectors v and w, find the dot products (actual numbers) of:
   (a) v and $-v$.
   (b) $v + w$ and $v - w$.
   (c) $v - 2w$ and $v + 2w$.

7. True or false (give a reason if true or a counterexample if false):
   (a) If u is perpendicular (in three dimensions) to v and w, those vectors v and w are parallel.
   (b) If u is perpendicular to v and w, then u is perpendicular to $v + 2w$,
   (c) If u and v are perpendicular unit vectors then $\|u - v\| = \sqrt{2}$.

8. The triangle inequality says: (length of $v+w$) $\leq$ (length of v) + (length of w). Use the Schwarz inequality ($v.w \leq \|v\|\,\|w\|$) to show that $\|$**side 3**$\|$ of a triangle can not exceed $\|$**side 2**$\|$ + $\|$**side 1**$\|$:

$$\textbf{Triangle Inequality:} \quad \|v+w\| \leq \|v\| + \|w\|$$

9. Pick any numbers that add to $x+y+z=0$. Find the angle between your vector $v=(x,y,z)$ and the vector $w=(z,x,y)$. Challenge question: Explain why $\frac{v.w}{\|v\|\|w\|}$ is always $\frac{-1}{2}$.

10. Find the linear combination $2s_1 + 3s_2 + 4s_3 = b$. Then write b as a matrix-vector multiplication Sx. Compute the dot products (row of S).x:

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, s_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{go into the columns of } S.$$

11. For these matrices, when does $AB = BA$? When does $BC = CB$? When does A times BC equal AB times C? Give the conditions on their entries p, q, r, z:

$$A = \begin{bmatrix} p & 0 \\ q & r \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$

- Find a nonzero matrix A for which $A^2 = 0$.
- Find a matrix that has $A^2 \neq 0$ but $A^3 = 0$.

1.2 Inverse Matrix, Determinant, Vector Spaces

1. Review the following definitions: inverse matrix, identity matrix, determinant, and vector spaces.

2. Find a point with $z=2$ on the intersection line of the planes $x+y+3z=6$ and $x-y+z=4$. Find the point with $z=0$. Find a third point halfway between.

3. Normally 4 "planes" in 4-dimensional space meet at a ———. Normally 4 column vectors in 4-dimensional space can combine to produce b. What combination of $(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1)$ produces $b = (3,3,3,2)$? What 4 equations for x, y, z, t are you solving?

4. (a) What 2 by 2 matrix R rotates every vector by 90°? R times $\begin{bmatrix} x \\ y \end{bmatrix}$, is $\begin{bmatrix} y \\ -x \end{bmatrix}$.

   (b) What 2 by 2 matrix R_2 rotates every vector by 180°?

(c) What 2 by 2 matrix E subtracts the first component from the second component? What 3 by 3 matrix does the same?

$$E\begin{bmatrix}3\\5\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix} \qquad E\begin{bmatrix}3\\5\\7\end{bmatrix}=\begin{bmatrix}3\\2\\7\end{bmatrix}$$

5. Start with the vector $u_0 = (1,0)$. Multiply again and again by the same "Markov matrix" $A = \begin{bmatrix}.8 & .3\\ .2 & .7\end{bmatrix}$. The next three vectors are u_1, u_2, u_3: $u_1 = Au_0, u_2 = Au_1, u_3 = Au_2$. What property do you notice for all four vectors $u_0, \ldots, u_3$?

6. Suppose u and v are the first two columns of a 3 by 3 matrix A. Which third columns w would make this matrix singular? Describe a typical column picture of $Ax = b$ in that singular case, and a typical row picture (for a random b).

7. (a) If A is invertible and $AB = AC$, prove quickly that $B = C$.

   (b) If $A = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$ find two different matrices such that $AB = AC$.

8. Could a 4 by 4 matrix A be invertible if every row contains the numbers $0, 1, 2, 3$ in some order? What if every row of B contains $0, 1, 2, -3$ in some order?

9. True or false (with a counterexample if false and a reason if true):

   (a) A 4 by 4 matrix with a row of zeros is not invertible.

   (b) Every matrix with 1 's down the main diagonal is invertible.

   (c) If A is invertible then A^{-1} and A^2 are invertible.

10. How does the identity $A(I + BA) = (I + AB)A$ connect the inverses of $I + BA$ and $I + AB$? Those are both invertible or both singular: not obvious.

11. Describe a subspace S of each vector space V, and then a subspace SS of S.

    (a) V_1 = all combinations of $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$ and $(1, 1, 1, 1)$.

    (b) V_2 = all vectors perpendicular to $u = (1, 2, 1)$, so $u.v = 0$.

    (c) V_3 = all symmetric 2 by 2 matrices (as a subspace of all the 2 by 2 real matrices).

    (d) V_4 = all solutions to the equation $\frac{d^4y}{dx^4} = 0$ (as subspace of the vector space of all real functions $f(x)$).

1.3 Span, Independence, Basis

1. Review the following definitions: independent vectors, spanning a space, basis for a space, and the dimension of a space.

2. Describe singular and invertible matrices in terms of the independence of their columns.

3. Suppose $v_1, \ldots, v_n$ is a basis for R^n and the n by n matrix A is invertible. Show that $Av_1, \ldots, Av_n$ is also a basis for R^n.

4. Start with the vectors $v_1 = (1, 2, 0)$ and $v_2 = (2, 3, 0)$.

   (a) Are they linearly independent?

   (b) Are they a basis for any space?

   (c) What space V do they span?

   (d) What is the dimension of V?

   (e) Which matrices A have V as their column space?

   (f) Which matrices have V as their nullspace?

   (g) Describe all vectors v_3 that complete a basis V_1, v_2, v_3 for R^3.

5. Describe the subspace of R^3 (is it a line or plane or R^3?) spanned by:

   (a) The two vectors $(1, 1, -1)$ and $(-1, -1, 1)$.

   (b) The three vectors $(0, 1, 1)$ and $(1, 1, 0)$ and $(0, 0, 0)$.

   (c) All vectors in R^3 with whole number components.

   (d) All vectors with positive components.

6. The columns of A are n vectors from R^m. If they are linearly independent, what is the rank of A? If they span R^m , what is the rank? If they are a basis for Rm , what then? Looking ahead: The rank r counts the number of columns.

7. The cosine space F_3 contains all combinations $y(x) = A\cos x + B\cos 2x + C\cos 3x$. Find a basis for the subspace with $y(0) = 0$.

8. **Activity in a network of neurons:** Consider two layers of N neurons each, an input layer and an output layer. Label the activities of the input layer neurons by $a_i, i = 0, \ldots, N1$, and similarly label the activities of the output layer neurons by b_i. Let W_{ij} by the strength of the synaptic connection from input neuron j to output neuron i. Also let there be synaptic connections between the output neurons: let B_{ij} be the strength of the connection from output neuron j to output neuron i (we can define $B_{ii} = 0$ for all i, if we want to exclude self-synapses). Let τ be a time constant of integration in the postsynaptic neuron. Then a very simple,

linear model of activity in the output layer, given the activity in the input layer, would be:

$$\tau \frac{db_i}{dt} = -b_i + \sum_j W_{ij} a_j + \sum_j B_{ij} b_j \tag{1}$$

The $-b_i$ term on the right just says that, in the absence of input from other cells, the neuron's activity b_i decays to zero (with time constant τ). Again, this is only a toy model, *e.g.* rates can go positive or negative and are unbounded in magnitude. The equation 1 can be written as a vector equation:

$$\tau \frac{db}{dt} = -b + Wa + Wb = -(1 - B)b + Wa$$

Write down an equation for the steady-state or fixed-point output activity pattern and solve it. [2]

References

[1] Gilbert Strang. *Introduction to Linear Algebra*. Wellesley-Cambridge Press, Wellesley, MA, fourth edition, 2009.

[2] Kenneth Miller. Linear Algebra for Theoretical Neuroscience. https://ctn.zuckermaninstitute.columbia.edu/sites/default/files/content/Miller/math-notes-1.pdf, 2008. [Online].