Mathematical Tools
Problem Set 2

1 ODEs

1. **Basic definitions:**
   (a) Define the following terms: ODE, dependent and independent variables, and directional field.
   (b) Name 6 classes of ODEs and review the ideas that are used for solving them analytically.
2. **Directional field:**
   (a) Sketch the direction field for the following differential equation $y' = x^2 + y + 2 - 1$.
   (b) Use part (a) to sketch the solution curve that passes through the origin.
3. Use Euler's method with step size 0.1 to construct a table of approximate values for the solution of the following ODE:
   $y' = x + y \quad \text{with} \quad y(0) = 1$
4. Determine which class of ODEs each equation belongs to and then solve it.
   (a) $\frac{dy}{dx} = \frac{x^2}{y^2} \quad \text{with} \quad y(0) = 2$
   (b) $2xy\frac{dy}{dx} = 4x^2 + 3y^2$
   (c) $\frac{dy}{dx} = \frac{6x^2}{2y+\cos y}$
   (d) $\frac{dy}{dx} = (x + y + 3)^3$
   (e) $y' = x^2 y$
   (f) The membrane current of a neuron $I(t)$ is modeled by the following differential equation $L\frac{dI}{dt} + RI = E(t)$ where R is the membrane resistance, L is the membrane conductance, and $E(t)$ is the external input injected to the neuron by an electrode at time t. Find an expression for the membrane current where the membrane resistance is 12 V, the membrane inductance is 4 H, and we start injecting

current into the neuron with a constant voltage of 60 V at time $t = 0$. What is the limiting value of the current?

(g) $x\frac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$

2 Complex Numbers

1. Based on Euler's formula we have:

$$e^{i\theta} = cos(\theta) + isin(\theta)$$

Using this formula show that:

(a) $cos(s+t) = cos(s)cos(t) - sin(s)sin(t)$

(b) $sin(s+t) = sin(s)cos(t) + cos(s)sin(t)$

(c) Compute the value, absolute value and the conjugate of $z = (1+i)^6$.

(d) Find i^i.

2. Compute real and imaginary part of $z = \frac{i-4}{2i-3}$.

3. Compute the square roots of $z = -1 - i$.

4. Find $z \in \mathbf{C}$ such that $z^2 z = \bar{z}$.

5. Find all $z \in \mathbf{C}$ such that $z^2 \in \mathbf{R}$.

3 Taylor Series

1. Show the correctness of the following power series representations:

(a) $e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

(b) $\cos x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

(c) $\sin x = \sum\limits_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

(d) $\cosh x = \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

(e) $\sinh x = \sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x5}{5!} + \dots$

(f) $\ln(1+x) = \sum\limits_{n=0}^{\infty} \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$

(g) $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$

2. Given $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ we can show that the power representation series of functions $f(x)+g(x)$ and $f(x)g(x)$ is given by the following equations:

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$f(x)g(x) = \sum_{n=0}^{\infty} c_n x^n = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + \dots$$

Where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0$. Use these formulas and show the following hold:

(a) $\sin x \cos x = \sin 2x$

(b) $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

3. If the power series representation $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$ converges on an interval I, then f is differentiable on I and

$$f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

Using this show that $(\frac{1}{1-x})' = \frac{1}{(1-x)^2}$

4. In Einstein's theory of special relativity the mass of an object moving with velocity v is:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m_0 is the mass of the object when at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest:

$$K = mc^2 - m_0 c^2$$

(a) Show that when v is very small compared with c, this expression for K agrees with classical Newtonian physics: $K = \frac{1}{2} m_0 v^2$.

(b) Use Taylor's Inequality to estimate the difference in these expressions for K when $|v| < 100 \frac{\mathrm{m}}{\mathrm{s}}$.

5. Consider the following function:

$$\begin{cases} f : \mathbf{R} \to \mathbf{R} \\ f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x < 0 \end{cases} \end{cases}$$

(a) Use the chain rule to show by mathematical induction that for any order k,

$$f^k(x) = \begin{cases} \frac{p_k(x)}{x^{3k}} e^{-\frac{1}{x^2}} & x > 0 \\ 0 \quad x < 0 \end{cases}$$

for some polynomial p_k of degree $2(k-1)$.

(b) Show that the Taylor series of f converges uniformly to the zero function $T_f(x) = 0$. Discuss why the function f is not equal to its Taylor series.