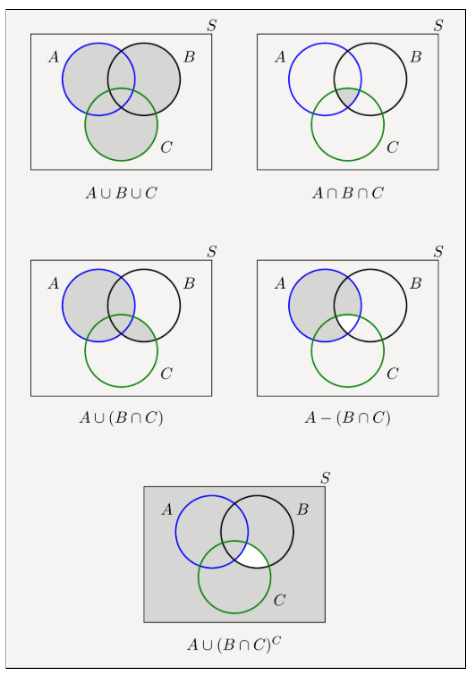
Mathematical Tools
Solutions to Problem Set 5

1 Set Algebra

1. (a) $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
   (b) $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
   (c) If S is the reference set then $A^c = \{x \in S | x \notin A\}$.
   (d) Sets A and B are disjoint if $A \cap B = \emptyset$.
   (e) $A \times B = \{(a, b) | a \in A, b \in B\}$.
2. Shaded areas are shown in the following diagram:



2 Combinatorics

2.1 Rule of Product

1. The Rule of Product states that given n ways for doing the first part of a task, and m ways for doing the second part of a task, there are mn ways for doing part 1 **and** part 2 of the task. More formally, if we have a mapping between the elements of set X_1 and the ways of doing the first part of the task and another mapping between the elements of X_2 and the ways of doing the second part of the task, then each way of doing both parts will be mapped to a pair $(x_1, x_2) \in X_1 \times X_2$. Since $X_1 \times X_2$ has $|X_1| \times |X_2|$ elements then there are mn ways for doing both parts of the task.

2. 6×8.

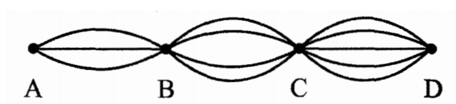
3. $2 \times 5 \times 3$.

4. (a) 6^4.
   (b) $6 \times 5 \times 4 \times 3$.

5. 7×6^9.

6. (a) 64×14.
   (b) $64 \times 49 \times 36$.

7. (a) $3 \times 4 \times 5$.



   (b) $(3 \times 4 \times 5)^2$.
   (c) $3 \times 4 \times 5 \times 4 \times 3 \times 2$.

8. There are 5 ways for selecting two numbers in this set such that their difference is 5. There are 2^4 ways that we can select numbers in between the two largest and smallest numbers that we have already selected. Therefore in total there are 5×2^4 ways.

9. $2 \times 3 \times 3 \times 2$.

10. The first person has 9 choices for selecting his/her teammate. The next person now has 7 ways to select his/her teammates (2 people are already in a team), etc. Therefore the solution is $9 \times 7 \times 5 \times 3 \times 1$.

11. $6 \times 5 \times 2$ (the factor 2 is because of horizontal and vertical).

2.2 Rule of Sum

1. The Rule of Sum states that given n ways for doing a task using method 1 and m ways for doing the same task using method 2 if the methods are non-overlapping there are $m+n$ ways of doing that task using method 1 **or** method 2. If they are overlapping we should subtract the number of elements in the overlap. In other words using set theoretical notation, if each way of doing the task using method 1 (or 2) is an element of a set X_1 (or X_2) the number of ways for doing the task using either method is given by the following formula: $|x_1| + |x_2| - |x_1 \cap x_2|$.

2. There are three types of rectangles with area 16 in a chessboard which are $2 \times 8, 4 \times 4, 8 \times 2$. We have 7×1 of type one, 5×5 of type two, and 1×7 of type three which is 39 in sum.

3. The a in the word can appear in the first, second, or third position. There are 25 words in each category, therefore we have written the letter a $25 + 25 + 25 = 75$ times.

4. In 9^4 numbers only the left-most digit is 3. In $4 \times 8 \times 9^3$ numbers one of the 4 digits on the right is 3. Therefore the solution is $9^4 + 4 \times 8 \times 9^3$.

5. For each $-100 \leq x \leq +100$ the number of ys that satisfy the equation is $202 - 2|x|$. Therefore the solution is:

$$1 + 3 + \cdots + 201 + 199 + 197 + \cdots + 3 + 1$$

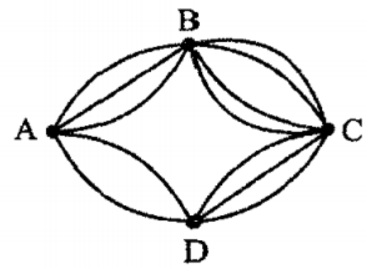
6. We divide the cells on a chessboard into 4 layers. Layer one contains the border cells from which we have $8^2 - 6^2 = 28$. Layer two contains the border cells of the 6×6 grid after removing the layer one cells. This layer has $6^2 - 4^2$ cells. Similarly we can define layer 3 and 4. Now depending on which layer the white queen is in there are $21, 23, 25, 27$ choices for the black queen. Therefore the solution is:

$$28 \times 21 + 20 \times 23 + 12 \times 25 + 4 \times 27$$

7. Number i is the largest element of 2^{i-1} subsets. Therefore the solution is:

$$\sum_{i=0}^{10} i2^i$$

8. (a) $3 \times 4 + 2 \times 3$.



(b) $(3 \times 4 + 2 \times 3)^2$.

(c) $1 \times 2 \times 3 \times 2 + 3 \times 4 \times 3 \times 2 + 2 \times 3 \times 4 \times 3 + 2 \times 3 \times 4 \times 3$.

2.3 Permutations

1. The solution is the number of permutations of *log*, a, r, i, t, h, and m which is 7!.

2. We consider 7 positions in a row. There are 4 ways for choosing two positions such that there are two empty positions between them and there are 2 ways that we can put t and k in them. Finally, there are 5! ways for inserting the remaining letters in the empty positions. Therefore the solution is $4 \times 2 \times 5!$.

3. Each permutation is in one of the following forms:

$$cvccvccv \quad cvccvcvc \quad cvcvccvc \quad vccvccvc$$

   Where c is consonant and v is vowel. Therefore the solution is $4 \times 3! \times 5!$.

4. 11!.

   (a) $6! \times 6!$.

   (b) $5! \times 6!$.

   (c) $2! \times 6! \times 5!$.

5. There are $10 \times 6!$ permutations in which a is between r and t. There are $8 \times 6!$ permutations in which a is not between r and t. Therefore the solution is $18 \times 6!$.

6. There are $P(7,4)$ 4-permutations and $P(6,4)$ of them don't contain f. Therefore the solution is $P(7,4) - P(6,4)$.

7. P(15,9)

8. There are 7! ways that the students can sit around the table. Now there are 8 slots between them that the teachers can use. This can be done in P(8,3) ways therefore the soltuon is $7!P(8,3)$.

9. Mirror symmetry does not change necklaces, therefore there are $\frac{4!}{2}$ possible necklaces.

10. $4! \times 2^5$.

2.4 Combinations and Binomial Coefficients

1. In a counting problem, if we count each desired state exactly k times, then to get the correct count we need to divide the total count by k. The Rule of division states that if $f : A \to B$ is a k-to-1 function, then: $|A| = k|B|$.

2. We divide such subsets into 3 categories: subsets with 3, 4, and 5 odd elements respectively. The number of subsets in these categories are $\binom{5}{2}\binom{5}{3}, \binom{5}{1}\binom{5}{4}, \binom{5}{0}\binom{5}{5}$. Therefore the solution is $\binom{5}{2}\binom{5}{3} + \binom{5}{1}\binom{5}{4} + \binom{5}{0}\binom{5}{5} = 126$.

3. First we place all the letters except the two ls. There are 5! ways for doing that. The we choose 2 spots from 6 spots in between the letters which can be done in $\binom{6}{2}$. Therefore the solution is $5!\binom{6}{2}$.

4. $\binom{9}{2}^2$ because we need to choose two pairs of parallel horizontal and vertical lines.

5. For each way of dividing 10 people into two groups of 5 we can name them A and B in two ways. Therefore the solution is $\frac{1}{2}\binom{10}{5}$.

6. We map each such subset to a binary code where i-th element is 1 if number i is in the subset and is 0 if it is not. Each such subset is mapped to a code where no two ones are adjacent. To count the number of such sequences we first place all the zeros in a row. There are $n - r + 1$ empty positions between them that we can place r ones. This can be done in $\binom{n-r+1}{r}$ ways.

7. (a) $\binom{10}{2}$?.
   (b) $\binom{10}{3}$.

8. $\binom{100}{25}\binom{75}{35}$.

9. $\binom{9}{2}5^7 6^1 0$.

10. $(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

11. $\binom{17}{5}3^5(-5)^{12}$.

12. Compute the following sums:
    (a) $(1 + 2)^n = 3^n$.
    (b) $(1 - 3)^n = (-2)^n$.

13. Using binomial coefficients prove that:

(a) Consider $(1+1)^n$.

(b) $0 = (1-1)^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}$ therefore $\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots = \binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots = 2^{n-1}$.

14. (a) Map each solution of this equation to a binary code in the following way. Place x_1 ones, then a zero, then x_2 ones, and a zero again, etc. Therefore the number of the solutions to this equation is the number of binary codes in which no two zeros are adjacent and the left-most and right-most numbers are also ones. To count this we first place all the n ones. There are $n-1$ slots between them and we have $k-1$ zeros to put in those slots. This can be done in $\binom{n-1}{k-1}$ ways.

    (b) Define $y_i = x_i + 1$ and therefore we have $y_i \in \mathbb{N}$ (natural numbers) and $y_1 + \ldots y_n = n + k$. Based on the previous part the number of solutions of this equation is $\binom{n+k-1}{k-1}$.

15. Each term is in the form of $x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}$ where $i_1+i_2+i_3+i_4 = 19$ and each $i_j \geq 0$. Based on previous problem the number of such selections is $\binom{22}{3}$.

2.5 Miscellaneous

1. Since each neuron is a binary element we can map its activity to 0 or 1. The number of sequences of length 302 from 0 and 1 is 2^{302} therefore the network can generate up to 2^{302} distinct patterns.

2. In a directed network with n neurons there are $n(n-1)$ possible synapses between the neurons (we have n selections for presynaptic neuron and if we do not allow self-connections there are $n-1$ choices for the postsynaptic neuron). Therefore since each connection can exist or not the total number of possible connectivity patterns is $2^{302(301)}$. If the network is undirected then the number of possible connections is $\frac{302(301)}{2}$ because we are counting each connection twice. Therefore the solution is $2^{\frac{302(301)}{2}}$ in the undirected case.

3. For the undirected case the solution is $2^{\binom{301}{2}}$ and the reason is that we can connect the neurons 1 to 301 arbitrarily and depending on the parity of the degree of each neuron we either connect it to neuron 302 or not uniquely (the degree of neuron 302 will automatically end up being even (why?).

4. Same as part one, but each connection has 3 states. Therefore the solutions are $2^{302(301)}$ and $2^{\frac{302(301)}{2}}$ in the directed and undirected cases respectively.

5. We have 3 choices for the activity of each neuron $(-1, 0, +1)$. Therefore the total number of possible activity patterns is equal to the number of sequences of length 302 with 3 elements which is 3^{302}.

6. The number of possible edges in this case is $2EI$ since each inhibitory $\rightarrow$ excitatory connection has I choices for the presynaptic and E choices for the postsynaptic neuron (EI in total) and each excitatory $\rightarrow$ inhibitory connection has E choices for the presynaptic and I choices for the postsynaptic neuron. Therefore the number of possible patterns are 2^{2EI} if we have two possible choices for each connection (existence or inexistence).

7. The number of possible connections in the excitatory network is $E(E-1)$ and the number of possible connections in the inhibitory network is $I(I-1)$ (assuming that both networks are directed). Therefore the total number of possible patterns is $2^{E(E-1)+I(I-1)}$.