Mathematical Tools
Solutions to Problem Set 2

1 ODEs

1. **Basic definitions:**

   (a)
   - In mathematics, an ordinary differential equation (ODE) is a differential equation containing one or more functions of one independent variable and the derivatives of those functions
   - In the ODE of the form $\frac{dx}{dt} = f(x, t)$ since t evolves independently we call it the independent variable and similarly since x depends on t we call it dependent variable.
   - The solutions of a first-order differential equation of a scalar function $y(x)$ can be drawn in a 2-dimensional space with the x in horizontal and y in vertical direction. Sometimes it is too cumbersome solving the differential equation analytically. Then one can still draw the tangents of the function curves e.g. on a regular grid.

   (b) Separable, Linear, Bernoulli, Ricatti, Non-linear Homogeneous, Linear Homogeneous.

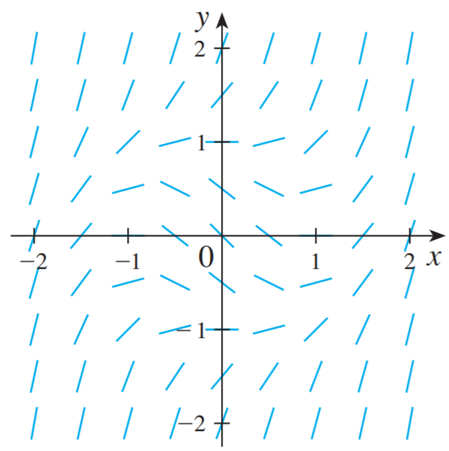
2. **Directional field:**

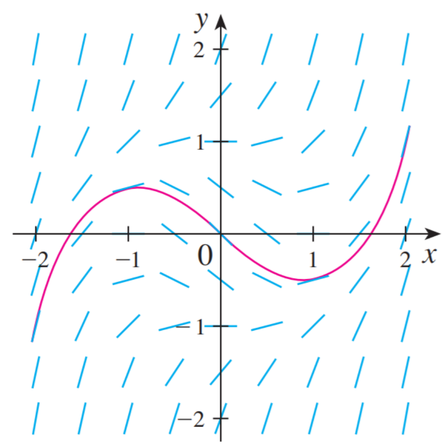
   (a) We start by computing the slope at several points in the following chart Now we draw short line segments with these slopes at these

x	-2	-1	0	1	2	-2	-1	0	1	2	...
y	0	0	0	0	0	1	1	1	1	1	...
$y' = x^2 + y^2 - 1$	3	0	-1	0	3	4	1	0	1	4	...

   points. The result is the direction field shown in Figure 2a.

   (b) We start at the origin and move to the right in the direction of the line segment (which has slope 21). We continue to draw the solution curve so that it moves parallel to the nearby line segments. The resulting solution curve is shown in Figure 2b. Returning to the origin, we draw the solution curve to the left as well.





3. We are given that $h = 0.1, x_0 = 0, y_0 = 1$ and $F(x, y) = x + y$. So we have

$$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1(0 + 1) = 1.1$$
$$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1(0.1 + 1.1) = 1.22$$
$$y_3 = y_2 + hF(x_2, y_2) = 1.22 + 0.1(0.2 + 1.22) = 1.362$$

Proceeding with similar calculations, we get the values in the table:

n	x_n	y_n	n	x_n	y_n
1	0.1	1.100000	6	0.6	1.943122
2	0.2	1.220000	7	0.7	2.197434
3	0.3	1.362000	8	0.8	2.487178
4	0.4	1.528200	9	0.9	2.815895
5	0.5	1.721020	10	1.0	3.187485

4. (a) Separable

$$\begin{aligned} y^2 dy &= x^2 dx \\ \int y^2 dy &= \int x^2 dx \\ \frac{1}{3}y^3 &= \frac{1}{3}x^3 + C \\ y &= \sqrt[3]{x^3 + C} \end{aligned}$$

(b) Homogeneous

$$\frac{dy}{dx} = \frac{4x^2 + 3y^2}{2xy} = 2\left(\frac{x}{y}\right) + \frac{3}{2}\left(\frac{y}{x}\right)$$

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}, \quad v = \frac{y}{x} \quad \text{and} \quad \frac{1}{v} = \frac{x}{y}$$

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{2}{v} + \frac{3}{2}v \\ x\frac{dv}{dx} &= \frac{2}{v} + \frac{v}{2} = \frac{v^2 + 4}{2v} \\ \int \frac{2v}{v^2 + 4} dv &= \int \frac{1}{x} dx \\ \log(v^2 + 4) &= \log|x| + \log C \\ v^2 + 4 &= C|x| \\ \frac{y^2}{x^2} + 4 &= C|x| \\ y^2 + 4x^2 &= kx^3 \\ y &= \pm\sqrt{kx^3 - 4x^2} \end{aligned}$$

(c) Separable

$$\begin{aligned} (2y + \cos y)dy &= 6x^2 dx \\ \int (2y + \cos y)dy &= \int 6x^2 dx \\ y^2 + \sin y &= 2x^3 + C \end{aligned}$$

Equation 3 gives the general solution implicitly. In this case it's impossible to solve the equation to express y explicitly as a function of x.

(d) Let's try the substitution: $v = x + y + 3$ that is $y = v - x - 3$. Then:

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$
$$\frac{dv}{dx} = 1 + v^3$$

This is a separable equation, and we have no difficulty in obtaining its solution:

$$x = \int \frac{dv}{1 + v^2} = \tan^{-1}(v) + C$$

So $v = \tan(x - C)$ because $v = x + y + 3$, the general solution of the original equation $\frac{dy}{dx} = (x + y + 3)^2$ is $x + y + 3 = \tan(x - C)$ that is: $y(x) = \tan(x - C) - x - 3$.

(e) Separable, if $y \neq 0$ we can rewrite it in differential notation and integrate:

$$\frac{dy}{y} = x^2 dx \quad y \neq 0$$
$$\int \frac{dy}{y} = \int x^2 dx$$
$$\log |y| = \frac{x^3}{3} + C$$
$$y = \pm e^C e^{\frac{x^3}{3}}$$

We can easily verify that the function $y = 0$ is also a solution of the given differential equation. So we can write the general solution in the form $y = Ax^{\frac{x^3}{3}}$.

(f) With $L = 4, R = 12$, and $E(t) = 60$, the equation becomes:

$$4\frac{dI}{dt} + 12I = 60 \quad \text{or} \quad 15 - 3I$$

and the initial-value problem is:

$$\frac{dI}{dt} = 15 - 3I \quad I(0) = 0$$

We recognize this equation as being separable, and we solve it as follows:

$$\int \frac{dI}{15-3I} = \int dt$$
$$-\frac{1}{3}\log|15-3I| = t + C$$
$$|15-3I| = e^{-3(t+C)}$$
$$15-3I = \pm Ae^{-3t}$$
$$I = 5 - \frac{1}{3}Ae^{-3t}$$

Since $I(0) = 0$, we have $5 - \frac{1}{3}A = 0$, so $A = 15$ and the solution is:

$$I(t) = 5 - 5e^{-3t}$$

The limiting current, in amperes is:

$$\lim_{t\to\infty} I(t) = 5 - 5\lim_{t\to\infty} e^{-3t} = 5 - 0 = 5$$

(g) Bernoulli with $n = \frac{4}{3}$ and $1 - n = -\frac{1}{3}$

$$v = y^{-\frac{1}{3}} \quad \Rightarrow \quad y = v^{-3} \quad \text{and} \quad \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = -3v^{-4}\frac{dv}{dx}$$
$$-3xv^{-4}\frac{dv}{dx} + 6v^{-3} = 3xv^{-4}$$
$$\frac{dv}{dx} - \frac{2}{x}v = -1$$

Now we are back to the linear case. By using the integrating factor $\rho = e^{\int -\frac{2}{x}dx} = x^{-2}$ we have:

$$D_x(x^{-2}v) = -\frac{1}{x^2} \quad x^{-2}v = \frac{1}{x} + C \quad v = x + Cx^2$$
$$y(x) = \frac{1}{(x+Cx^2)^3}$$

2 Complex Numbers

1.

$$e^{i(s+t)} = \cos(s+t) + i\sin(s+t)$$
$$e^{i(s+t)} = e^{is}e^{it} = (\cos(s) + i\sin(s))(\cos(s) + i\sin(s))$$
$$= \cos(s)\cos(t) - \sin(s)\sin(t) + i(\cos(s)\sin(t) + \sin(s)\cos(t))$$

Therefore since the real parts and imaginary parts of the two derivations must be equal respectively we have:

$$\cos(s+t) = \cos(s)\cos(t) - \sin(s)\sin(t)$$
$$\sin(s+t) = \sin(s)\cos(t) + \cos(s)\sin(t)$$

(a) Showed above.

(b) Showed above.

(c)

$$(1+i)^6 = \sqrt{2}\big(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\big)^6 = 8\big(\cos(\frac{3\pi}{2}) + i\sin(\frac{3\pi}{2})\big) = 8i$$

Therefore $|z| = 8$ and $\bar{z} = 8i$.

(d) Find i^i.

2.

$$z = \frac{i-4}{2i-3} = \frac{i-4}{2i-3}\frac{2i+3}{2i+3} = \frac{-2+3i-8i-12}{-4-9} = \frac{14}{13} + i\frac{5}{13}$$

3. Every $z \in C$ has n distinct roots of order n, which correspond (in the complex plane) to the vertices of a regular n-agon inscribed in the circle of radius $\sqrt[n]{|z|}$ centered at the origin. When $z = \rho(\cos\theta + i\sin\theta) = \rho e^{i\theta}$ then the roots of order n of z are:

$$\sqrt[n]{\rho}\big(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n}\big)$$

Therefore the squared roots of $z = -1-i = \sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$ are

$$z_1 = \sqrt[4]{2}(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}) \quad \text{and} \quad z_2 = \sqrt[4]{2}(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8})$$

4. The equation becomes $z(z\bar{z}-1) = 0$. Hence a first solution is $z = 0$, while the others satisfy $z\bar{z} = |z|^2 = 1$. Then also all the points of the circle of radius 1 centered at the origin satisfies the equation.

5. If $z = a+ib$, $a, b \in \mathrm{R}$ then $z^2 \in \mathrm{R}$ if and only if $a^2 - b^2 + 2iab \in \mathrm{R}$, that is if and only if $ab = 0$. Hence $z^2 \in \mathrm{R}$ if and only if $z \in \mathrm{R}$ ($b = 0$) or if z is a pure imaginary number ($a = 0$).

3 Taylor Series

1. Show the correctness of the following power series representations:

(a) $(e^x)^{(n)}\big|_0 = e^x\big|_0 = 1 \Rightarrow c_n = \frac{1}{n!}$

(b)

$$(\cos x)^{(2n)} = (-1)^n \cos x \quad \text{and} \quad (\cos x)^{(2n+1)} = (-1)^n \sin x$$

$$(\cos x)^{(2n)}\big|_0 = (-1)^n \quad \text{and} \quad (\cos x)^{(2n+1)}\big|_0 = 0$$

$$c_{2n} = \frac{(-1)^n}{2n!} \quad \text{and} \quad c_{2n+1} = 0$$

(c)

$$(\sin x)^{(2n)} = (-1)^n \sin x \quad \text{and} \quad (\sin x)^{(2n+1)} = (-1)^n \cos x$$
$$(\cos x)^{(2n)}\big|_0 = 0 \quad \text{and} \quad (\sin x)^{(2n+1)}\big|_0 = (-1)^n$$
$$c_{2n} = 0 \quad \text{and} \quad c_{2n+1} = \frac{(-1)^n}{(2n+1)!}$$

(d)

$$(\cosh x)^{(2n)} = \cosh x \quad \text{and} \quad (\cosh x)^{(2n+1)} = \sinh x$$
$$(\cosh x)^{(2n)}\big|_0 = 0 \quad \text{and} \quad (\cosh x)^{(2n+1)}\big|_0 = 0$$
$$c_{2n} = \frac{1}{(2n)!} \quad \text{and} \quad c_{2n+1} = 0$$

(e)

$$(\sinh x)^{(2n)} = \sinh x \quad \text{and} \quad (\sinh x)^{(2n+1)} = \cosh x$$
$$(\sinh x)^{(2n)}\big|_0 = 0 \quad \text{and} \quad (\sinh x)^{(2n+1)}\big|_0 = 1$$
$$c_{2n} = 0 \quad \text{and} \quad c_{2n+1} = \frac{1}{(2n+1)!}$$

(f)

$$\frac{1}{1+x} = 1 - x + x^2 - \dots$$
$$\log(1+x) = \int (1 - x + x^2 - \dots)dx = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{4} + \dots$$

(g) Showed in the class.

2.

$$\begin{aligned}\sin x \cos x &= (x - \frac{1}{6}x^3 + \dots)(1 - \frac{1}{2}x^2 + \dots)\\ &= x + (-\frac{1}{6} - \frac{1}{2})x^3 + (\frac{1}{24} + \frac{1}{12} + \frac{1}{120})x^5\\ &= \frac{1}{2}(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \dots) = \frac{1}{2}\sin 2x\end{aligned}$$

(3)

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$
$$(\frac{1}{1-x})' = 1 + 2x + 3x^2 + \dots$$
$$\frac{1}{(1-x)^2} = (1 + x + x^2 + \dots)(1 + x + x^2 + \dots) = 1 + 2x + 3x^2 + \dots$$

4. (a) Using the expressions given for K and m, we get

$$K = mc^2 - m_0c^2 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} - m_0c^2 = m_0c^2\Big[(1-\frac{v^2}{c^2})^{-\frac{1}{2}} - 1\Big]$$

With $x = \frac{v^2}{c^2}$, the Taylor series for $(1+x)^{-\frac{1}{2}}$ is most easily computed as a binomial series with $k = -\frac{1}{2}$. (Notice that $|x| < 1$ because $v < c$). Therefore we have

$$\begin{aligned}(1+x)^{-\frac{1}{2}} &= 1 - \frac{1}{2}x + \frac{(-\frac{1}{2})(-\frac{3}{3})}{2!}x^2 + \dots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots\end{aligned}$$

and

$$K = m_0c^2\Big[(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \dots) - 1\Big]$$

If v is much smaller than c, then all terms after the first are very small when compared with the first term. If we omit them, we get: $K = m_0c^2(\frac{1}{2}\frac{v^2}{c^2}) = \frac{1}{2}m_0v^2$.

(b) If $x = -\frac{v^2}{c^2}$, $f(x) = m_0c^2[(1+x)^{-\frac{1}{2}} - 1]$, and M is a number such that $|f"(x)| \leq M$ then we can use Taylor's Inequality to write:

$$|R_1(x)| \leq \frac{M}{2!}x^2$$

We have $f"(x) = \frac{3}{4}m_0c^2(1+x)^{-\frac{5}{2}}$ and we are given $|v| < 100\frac{m}{s}$ so:

$$|f"(x)| = \frac{3m_0c^2}{4(1-\frac{v^2}{c^2})^{\frac{5}{2}}} \leq \frac{3m_0c^2}{4(1-\frac{100^2}{c^2})^{\frac{5}{2}}}$$

Thus, with $c = 3 \times 10^8\frac{m}{s}$:

$$|R_1(x)| \leq \frac{1}{2}\frac{3m_0c^2}{4(1-\frac{100^2}{c^2})^{\frac{5}{2}}}\frac{100^4}{c^4} < (4.17 \times 10^{-10})m_0$$

5. Consider the following function:

$$\begin{cases} f : \mathbf{R} \to \mathbf{R} \\ f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x < 0 \end{cases} \end{cases}$$

(a) It is easy to show that the following holds: if $f^{(n)}(x)$ is the n-th derivative of $f(x)$ and $f^{(n)}(x) = e^{-\frac{1}{x^2}}P_n(x)$ for $x > 0$ then P_n satisfies the following condition:

$$P_0(x) = 1$$

$$P_n(x) = (\frac{2}{x^3})P_{n-1}(x) + P'_{n-1}(x)$$

From this one can easily prove what the problem is asking for.

(b) The function $e^{-\frac{1}{x^2}}$ tends to zero faster than any polynomial as $x \to 0$, so f is infinitely many times differentiable and $f^{(k)}(0) = 0$ for every positive integer k. Now the estimates for the remainder for the Taylor polynomials show that the Taylor series of f converges uniformly to the zero function on the whole real axis. Nothing is wrong in here:

- The Taylor series of f converges uniformly to the zero function $T_f(x) = 0$.
- The zero function is analytic and every coefficient in its Taylor series is zero.
- The function f is infinitely many times differentiable, but not analytic.