Mathematical Tools
Solutions to Problem Set 3

1 Linear Algebra

1.1 Vectors, Norm and Unit Vectors, and Matrices

1. **Scalar:** a single-element value that represents some magnitude.
   Vector: is a collection of multiple scalars.
   Dot product: $u.v = \sum_{i=1}^{n} x_i v_i$.
   Norm: $\|u\| = \sqrt{\sum_{i=1}^{n} u_i^2}$.
   Matrix: m by n grid of scalars.
   Transpose: of W is the matrix that is given by flipping the elements of W across the diagonal.

2. The components of every $cv + dw$ add to zero because the components of v and of w add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$. There is no solution to $cv + dw = (3, 3, 6)$ because $3 + 3 + 6$ is not zero.

3. A four-dimensional cube has $2^4 = 16$ corners (each corner is a binary vector of length 4) and $2 \times 4 = 8$ three-dimensional faces (each 3D face has a normal vector that is parallel to one of the axes) and 24 two-dimensional faces (each 2D face projects to two of the axes, there are 12 ways of choosing two non-equal axes and for each selection we have two projection directions) and 32 edges (each node connects to 4 other nodes but each edge is counted exactly twice this way).

4. (a) Sum = zero vector.
   (b) Sum = $-2:00$ vector = $8:00$ vector.
   (c) $2:00$ is $30°$ from horizontal = $(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

5. For a line, choose $u = v = w =$ any nonzero vector. For a plane, choose u and v in different directions. A combination like $w = u + v$ is in the same plane.

6. (a) $v.(-v) = -1$.

(b) $(v+w).(v-w) = v.v + w.v - v.w - w.w = 1 + () - () - 1 = 0$ so $\theta = 90^\circ$ (notice $v.w = w.v$).

(c) $(v-2w).(v+2w) = v.v - 4w.w = 1 - 4 = -3$.

7. (a) False: v and w are any vectors in the plane perpendicular to u

   (b) True: $u.(v+2w) = u.v + 2u.w = 0$.

   (c) True, $\|u-v\|^2 = (u-v).(u-v)$ splits into $u.u + v.v = 2$ when $u.v = v.u = 0$.

8. $2v.w \leq 2\|v\|\|w\|$ leads to $\|v+w\|^2 = v.v + 2v.w + w.w \leq \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$. This is $(\|v\|+\|w\|)^2$. Taking square roots gives $\|v+w\| \leq \|v\| + \|w\|$.

9. For a specific example, pick $v = (1,2,-3)$ and then $w = (-3,1,2)$. In this example $\cos\theta = \frac{v.w}{\|v\|\|w\|} = \frac{-7}{\sqrt{14}\sqrt{14}} - 1/2$ and $\theta = 120^\circ$. This always happens when $x+y+z = 0$:

$$v.w = xy + yz + zx = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$$

$$\text{This is the same as } v.w = 0 - \frac{1}{2}\|v\|\|w\| \text{ then } \cos\theta = \frac{1}{2}$$

10. $3s_1 + 4s_2 + 5s_3 = (3,7,12)$. The same vector b comes from S times $x = (3,4,5)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \text{row } 1.x \\ \text{row } 2.x \\ \text{row } 3.x \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}$$

11. First of all, A times BC always equals AB times C. Parentheses are not needed in $A(BC) = (AB)C = ABC$. But we must keep the matrices in this order:

$$\textbf{Usually } AB \neq BA \quad AB = \begin{bmatrix} p & q \\ p & q+4 \end{bmatrix} \quad \begin{bmatrix} p+q & r \\ q & r \end{bmatrix}$$

$$\textbf{By chance } BC = CB \quad \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$

B and C happen to commute. Part of the explanation is that the diagonal of B is I, which commutes with all 2 by 2 matrices. When p, q, r, Z are 4 by 4 blocks and 1 changes to I, all these products remain correct. So the answers are the same.

12. •

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

- $$\text{If} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{then} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^3 = 0$$

1.2 Inverse Matrix, Determinant, Vector Spaces

1. **Identity matrix**: A n by n matrix with ones on the main diagonal and zeros elsewhere.
   Inverse matrix: For an n by n matrix W its inverse W^{-1} is another n by n matrix such that $WW^{-1} = W^{-1}W = I$.
   Determinant: The determinant of a matrix $\det(A)$ tells us the volume occupied by the columns of said matrix A.
   Vector spaces: A vector space, or linear space, V is a set of objects (called vectors or points) that is closed under linear combinations.

2. If $z = 2$ then $x + y = 0$ and $x - y = 2$ give the point $(x, y, z) = (1, -1, 2)$. If $z = 0$ then $x + y = 6$ and $x - y = 4$ produce $(5, 1, 0)$. Halfway between those is $(3, 0, 1)$.

3. 8 Four planes in 4-dimensional space normally meet at a point. The solution to $Ax = (3, 3, 3, 2)$ is $x = (0, 0, 1, 2)$ if A has columns $(1, 0, 0, 0)$, $(1, 1, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$. The equations are $x + y + z + t = 3, y + z + t = 3, z + t = 3, t = 2$. Solve them in reverse order!

4. (a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
   (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$
   (c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

5. $u_2 = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ and $u_3 \begin{bmatrix} .65 \\ .35 \end{bmatrix}$. The components add to 1. They are always positive.

6. A is singular when its third column w is a combination $cu + dv$ of the first columns. A typical column picture has b outside the plane of u, v, w. A typical row picture has the intersection line of two planes parallel to the third plane. Then no solution.

7. (a) Multiply $AB = AC$ by A^{-1} to find $B = C$ (since A is invertible).
   (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$.

8. A can be invertible with diagonal zeros (example to find). B is singular because each row adds to zero. The all-ones vector x has $Bx = 0$.

9. (a) True (If A has a row of zeros, then every AB has too, and $AB = I$ is impossible).
   (b) False (the matrix of all ones is singular even with diagonal 1's.
   (c) True (the inverse of A^{-1} is A and the inverse of A^2 is $(A^{-1})^2$).

10. Inverting the identity $A(I + BA) = (I + AB)A$ gives $(I + BA)^{-1}A^{-1} = A^{-1}(I+AB)^{-1}$. So $I+BA$ and $I+AB$ are both invertible or both singular when A is invertible. (This remains true also when A is singular.

11. (a) V_1 starts with three vectors. A subspace S comes from all combinations of the first two vectors $(1, 1, 0, 0)$ and $(1, 1, 1, 0)$. A subspace SS of S comes from all multiples $(c, C, 0, 0)$ of the first vector. So many possibilities.
   (b) A subspace S of V_2 is the line through $(1, -1, 1)$. This line is perpendicular to u. The vector $x = (0, 0, 0)$ is in S and all its multiples cx give the smallest subspace $SS = Z$.
   (c) The diagonal matrices are a subspace S of the symmetric matrices. The multiples cI are a subspace SS of the diagonal matrices.
   (d) V_4 contains all cubic polynomials $y = a + bx + cx^2 + dx^3$, with $\frac{d^4y}{dx^4}$. The quadratic polynomials give a subspace S. The linear polynomials are one choice of SS. The constants could be SSS.

    In all four parts we could take $S = V$ itself, and $SS =$ the zero subspace Z. Each V can be described as all combinations of .... and as all solutions of .... :
    $V_1 =$ all combinations of the 3 vectors $\quad V_1 =$ all solutions of $V_1 - V_2 = 0$
    $V_2 =$ all combinations of $(1, 0, -1)$ and $(1, -1, 1)$ are solutions of $u.v = 0$.
    $V_3 =$ all combinations of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. $\quad V_3 =$ all solutions $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of b = c
    $V_4 =$ all combinations of 1, x, x^2, x^3 $\quad V_4 =$ all solutions to $\frac{d^4y}{dx^4}$.

1.3 Span, Independence, Basis

1. **Independent vectors:** Vectors $\vec{v}_1, \cdots, \vec{v}_n$ are linearly independent if no vector is a linear combination of the others.
   Spanning a space: A span is the set of all linear combinations of a set of vectors
   $$\underbrace{\text{span}(\vec{v}_1, \cdots, \vec{v}_n)}_{\text{a vector space}} = \sum_{i=1}^{n} a_i\vec{v}_i : a_i \in \mathbb{R}$$
   Basis for a space: The basis of a vector space is a linearly independent spanning set, or the smallest set of vectors you can use to get around the space.

Dimension of a space: The dimension of a vector space V is the cardinality (i.e. the number of vectors) of a basis of V.

2. Singular matrices have dependent rows/columns. Invertible matrices have independent rows/columns.

3. In matrix language: Put the basis vectors $V_1, \ldots, V_n$ in the columns of an invertible matrix V. Then $Av_1, \ldots, Av_n$ are the columns of AV. Since A is invertible, so is AV and its columns give a basis. In vector language: Suppose $c_1Av_1 + \cdots + c_nAv_n = 0$. This is $Av = 0$ with $V = c_1v_1 + \cdots + c_nv_n$. Multiply by $A - I$ to reach $V = 0$. By linear independence of the v's, all $c_i = 0$. This shows that the Av's are independent. To show that the Av's span $\mathbb{R}^n$, solve $c_1Av_1 + \cdots + c_nAv_n = b$ which is the same as $c_1v_1 + \cdots + c_nv_n = A^{-1}b$. Since the v's are a basis, this must be solvable.

4. (a) v_1 and v_2 are independent the only combination to give 0 is $0v_1 + 0v_2$.
   (b) Yes, they are a basis for the space they span.
   (c) That space V contains all vectors $(x, y, 0)$. It is the $x - y$ plane in $\mathbb{R}^3$.
   (d) The dimension of V is 2 since the basis contains two vectors.
   (e) This V is the column space of any 3 by n matrix A of rank 2, if every column is a combination of v_1 and v_2. In particular A could just have columns v_1 and v_2.
   (f) This V is the nullspace of any m by 3 matrix B of rank 1, if every row is a multiple of $(0, 0, 1)$. In particular take $B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. Then $Bv_1 = 0$ and $Bv_2 = 0$.
   (g) Any third vector $v_3 = (a, b, c)$ will complete a basis for $\mathbb{R}^3$ provided $c \neq 0$.

5. (a) Line in $\mathbb{R}^3$.
   (b) Plane in $\mathbb{R}^3$.
   (c) All of $\mathbb{R}^3$.
   (d) All of $\mathbb{R}^3$.

6. n independent columns $\Rightarrow$ rank n. Columns span $\mathbb{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbb{R}^m \Rightarrow$ rank $= m = n$. The rank counts the number of independent columns.

7. $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

8. We can also write down an equation for the steady-state or fixed-point output activity pattern b^{FP} for a given input activity pattern h: by definition, a steady state or fixed point is a point where $\frac{db}{dt} = 0$. Thus, the fixed point is determined by:

$$(1 - B)b^{\text{FP}} = h$$

If the matrix $(1-B)$ has an inverse, $(1-B)^{-1}$, then we can multiply both sides by this inverse to obtain

$$b^{\mathrm{FP}} = (1-B)^{-1}h$$

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References

[1] Kenneth Miller. Linear Algebra for Theoretical Neuroscience. https://ctn.zuckermaninstitute.columbia.edu/sites/default/files/content/Miller/math-notes-1.pdf, 2008. [Online].