Mathematical Tools
Solutions to Problem Set 4

1 Linear Algebra

1.1 Linear Transformations

1. A function $T : V \to W$, where V, W are vector spaces is a linear transformation if and only if

   (a) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

   (b) $T(a\vec{x}) = aT(\vec{x})$

2. Which of the following transformations are linear? Why?

   (a) This is linear. The inputs v come from three-dimensional space, so $V = R^3$. The outputs are just numbers, so the output space is $W = R^1$. We are multiplying by the row matrix A and therefore $T(v) = Av$.

   (b) The length $T(v) = \|v\|$ is not linear. Requirement (a) for linearity would be $\|v + w\| = \|v\|\cdot$ Requirement (b) would be $\|cv\| = c\,\|v\|$. Both are false! Not (a): The sides of a triangle satisfy an inequality $\|v + w\| \leq \|v\| + \|w\|$. Not (b): The length $\|-v\|$ is not $-\|v\|$. For negative c, we fail.

   (c) Yes it is. We can rotate two vectors and add the results. The sum of rotations $T(v)+T(w)$ is the same as the rotation $T(v+w)$ of the sum. The whole plane is turning together, in this linear transformation.

3. (d) is not linear

4. For these transformations of $V = \mathbb{R}^2$ to $W = \mathbb{R}^2$, find $T(T(v))$. Is this transformation T^2 linear?

   (a) $T(T(v)) = v$ is linear.

   (b) $T(T(v)) = v + (2, 2)$ is not linear.

   (c) $T(T(v)) = -v$ is linear.

   (d) $T(T(v)) = T(v)$ is linear.

5. If $T(v1, v2, v3) = (v2, v3, v1)$ then $T(T(v)) = (v3, v1, v2)$; and $T^3(v) = v$; therefore $T^{100}(v) = T(v)$.

6. Linear transformations keep straight lines straight! And two parallel edges of a square (edges differing by a fixed v) go to two parallel edges (edges differing by $T(v)$). So the output is a parallelogram.

1.2 Eigenvalues and Eigenvectors

1. In linear algebra, an eigenvector of a linear transformation is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it. The corresponding eigenvalue is the factor by which the eigenvector is scaled.

2. (a) Each column of P adds to 1, so $\lambda = 1$ is an eigenvalue. P is singular, so $\lambda = 0$ is an eigenvalue. P is symmetric, so its eigenvectors $(1, 1)$ and $(1, -1)$ are perpendicular.

   (b) The reflection matrix has eigenvalues 1 and -1. The eigenvector $(1,)$ is unchanged by R. The second eigenvector is $(1, -1)$-its signs are reversed by R. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P, because reflection = 2(projection) - I (If $Px = \lambda x$ then $2Px = 2\lambda x$. The eigenvalues are doubled when the matrix is doubled. Now subtract $Ix = x$. The result is $(2P - I)x = (2\lambda - 1)x$. When a matrix is shifted by I, each λ is shifted by 1. No change in eigenvectors).

3. (a) Starting from $Ax = \lambda x$ first move λx to the left side. Write the equation $Ax = \lambda x$ as $(A - \lambda I)x = O$. The matrix $A - \lambda I$ times the eigenvector x is the zero vector. The eigenvectors make up the nullspace of $A - \lambda I$. When we know an eigenvalue A, we find an eigenvector by solving $(A - \lambda I)x = 0$. If $(A - \lambda I)x = 0$ has a nonzero solution, $A - \lambda I$ is not invertible. The determinant of $A - \lambda I$ must be zero. This is how to recognize an eigenvalue A: The number λ is an eigenvalue of A if and only if $A - \lambda I$ is singular. The equation for the eigenvalues is therefore $\det(A - \lambda I) = 0$.

   For the matrix in this problem, $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$. Take the determinant "$ad - bc$" of this 2 by 2 matrix.

   $$\det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda$$

   Set this determinant $\lambda^2 - 5\lambda$ to zero. One solution is $\lambda = 0$ (as expected, since A is singular). Factoring into λ times $\lambda - 5$, the other root is $\lambda = 5$.

(b) Again $Ax = \lambda x$ yields that $(A - \lambda I)x = 0$. Therefore to find the eigenvector associated with λ we need to solve the abvoe equation. For this problem let's solve separately for $\lambda_1 = 0$ and $\lambda_2 = 5$:

$$A - 0I = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$A - 5I = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

4. (a)

$$\begin{aligned} \det(A - \lambda I) = p(\lambda) &= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) \\ &= (-1)(\lambda - \lambda_1)(-1)(\lambda - \lambda_2) \dots (-1)(\lambda - \lambda_n) \\ &= (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda) \end{aligned}$$

The first equality follows from the factorization of a polynomial given its roots; the leading (highest degree) coefficient $(-1)^n$ can be obtained by expanding the determinant along the diagonal. Now, by setting λ to zero (simply because it is a variable) we get on the left side $\det(A)$, and on the right side $\lambda_1 \lambda_1 \dots \lambda_1$, that is, we indeed obtain the desired result: $\det(A) = \lambda_1 \lambda_1 \dots \lambda_1$.

(b) We have

$$p(\lambda) = \det(A - \lambda I) = (-1)^n (\lambda^n - (\mathrm{tr} A)\lambda^{n-1} + \dots + (-1)^n \det A)$$

On the other hand, $p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$, where the. So, comparing coefficients, we have $\mathrm{tr} A = \lambda_1 + \dots + \lambda_n$.

5. For simplicity, consider the 3×3 case. If A is upper triangular, then

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

Therefore λ is an eigenvalue of A $\iff$ The equation $(A - \lambda I)x = 0$ has a nontrivial solution. $\iff$ $(A - \lambda I)x = 0$ has a free variable $\iff$ Because of the zero entries in $A - \lambda I$, at least one of the entries on the diagonal of $A - \lambda I$ is zero. $\iff$ λ equals one of the entries a_{11}, a_{22}, a_{33} in A. For the lower triangular case the exact same reasoining works.

6. (a)) Multiply by A: $A(Ax) = A(\lambda x) = \lambda Ax$ gives $A^2 x = \lambda^2 x$.
   (b) Multiply by A^{-1}: $x = A^{-1}Ax = A{-}1\lambda x = \lambda A^{-1}x$ gives $A^{-1}x = \frac{1}{\lambda}x$.
   (c) Add $Ix = x$: $(A + I)x = (\lambda + 1)x$.

7. (a) This is simply because the $\det(A - \lambda I) = \det(A^T - \lambda I)$.

(b) Pick $M = A - \lambda I$:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ have different eigenvalues.}$$

8. **Eigenvalue Decomposition:**

(a) We know that for the eigenvalue λ_i and eigenvector q_i we have $Aq_i = \lambda_i q_i$ therefore AQ is a matrix that has Aq_i in its i-th column. Therefore we can replace the i-th column by $\lambda_i q_i$ and hence AQ is equal to a matrix whose i-th column is $\lambda_i q_i$. Now if we multiply a diagonal matrix whose i-th main diagonal entry is λ_i by the Q matrix from the left side we will get a matrix whose i-th column is $\lambda_i q_i$. Therefore $AQ = Q\Lambda$, multiplying both sides by Q^{-1} from the right side (given that Q is invertible) we get $A = Q\Lambda Q^{-1}$.

(b) Let's substitute $x = Qy$ into the dynamics equation:

$$\frac{dQy}{dt} = AQy \Rightarrow Q\frac{dy}{dt} = AQy = Q\Lambda y \Rightarrow \frac{dy}{dt} = \Lambda y$$

Since Λ is a diagonal matrix then y (which is a linear transformation of x) has linear dynamics represented by a diagonal matrix meaning that y is a network of decoupled variables.

(c) Let u, v be eigenvectors corresponding to distinct eigenvalues α, β, respectively. Namely we have $Au = \alpha u$ and $Av = \beta v$. To prove that u and v are orthogonal, we show that the inner product $u.v = 0$. Keeping this in mind, we compute:

$$\begin{gathered} \alpha(u.v) = (\alpha u).v \\ = Au.v = (Au)^T v = u^T A^T v \\ = u^T Av \text{ (since A is symmetric)} \\ = u^T \beta v = \beta(u^T v) = \beta(u.v) \\ \alpha(u.v) = \beta(u.v) \Rightarrow (\alpha - \beta)(u.v) = 0 \end{gathered}$$

Since α and β are distinct, $\alpha - \beta \neq 0$. Hence we must have $u.v = 0$. If the eigenvalues are not distinct, eigenvectors corresponding to different eigenvalues are orthogonal. But eigenvectors corresponding to the same eigenvalue are not necessarily orthogonal unless we construct them or modify them so that they are. For any eigenvalue λ whose eigenspace $\{x|Ax = \lambda x\}$ has dimension greater than 1 (which occurs if we have "repeated eigenvalues"), we use the Gram-Schmidt procedure to find an orthonormal basis for that eigenspace.

1.3 Linear Dynamics

1. The solution to this linear system is described by

$$\begin{bmatrix} e^{at}x_1(0) \\ e^{-t}x_2(0) \end{bmatrix}$$

If $a \leq 0$ then the system is stable and has an attractor point at $(0,0)$ because when $t \to \infty$, $\begin{bmatrix} e^{at}x_1(0) \\ e^{-t}x_2(0) \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

If $a > 0$ then x_1 repels the points but x_2 still attracts the points around it. More formally, when $t \to \infty$, $\begin{bmatrix} e^{at}x_1(0) \\ e^{-t}x_2(0) \end{bmatrix} \to \begin{bmatrix} \pm\infty \\ 0 \end{bmatrix}$. Therefore $(0,0)$ is a saddle point in this case.

2. The complete solution $u(t)$ is built from pure solutions $e^{At}x$. If the eigenvalue A is real, we know exactly when $e^{\lambda t}$ will approach zero: The number λ must be negative. If the eigenvalue is a complex number $\lambda = r + is$, the real part r must be negative. When $e^{\lambda t}$ splits into e^{rt} and e^{ist}, the factor e^{ist} has absolute value fixed at 1:

$$e^{ist} = \cos st + i \sin st \quad \text{has} \quad |e^{ist}|^2 = \cos^2 st + \sin^2 st = 1$$

Now the question is: Which matrices have negative eigenvalues? More accurately, when are the real parts of the λ's all negative? If the λ's are real and negative, their sum is negative. This is the trace T. Their product is positive. This is the determinant D.

3. (a) Since t is a scalar for every matrix Bt^k, the scalar t^k is multiplied by all the elements of the matrix B. Therefore its time derivative is simply given by $\frac{dBt^k}{dt} = kBt^{k-1}$. Using this fact we can compute the derivative of the infinite sum in the definition of e^{At}:

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots \\ \Rightarrow \frac{de^{At}}{dt} &= A + A^2t + \frac{1}{2!}A^3t^2 + \dots \\ &= A(I + At + \frac{1}{2!}(At)^+ \dots = Ae^{At} \end{aligned}$$

$\frac{de^{At}}{dt} = A + A^2t + \frac{1}{2}A^3t^2 + \dots = Ae^{At}$.

(b) If λ is an eigenvalue of A then $Ax = \lambda x$ therefore we have $A^k x = \lambda A^{k-1}x = \lambda^2 A_{k-2}x = \dots = \lambda^k x$. It means that λ^k is an eigenvalue of A^k. We can write:

$$\begin{aligned}
&(I + At + \frac{1}{2!}(At)^2 + \dots)x \\
&= x + Axt + \frac{1}{2!}A^2xt^2 + \dots \\
&= x + \lambda xt + \frac{1}{2}\lambda^2xt^2 + \dots \\
&= (1 + \lambda t + \frac{1}{2}\lambda^2t^2 + \dots)x \\
&= e^{\lambda t}x
\end{aligned}$$

(c) First we show if matrices A and B commute (meaning that $AB = BA$) then $e^{A+B} = e^Ae^B$.

$$\begin{aligned}
e^Ae^B &= (I + A + \frac{1}{2!}A^2 + \dots)(I + B + \frac{1}{2!}B^2 + \dots) \\
&= I + (A+B) + \frac{1}{2!}(A^2 + AB + BA + B^2) + \dots \\
&= I + (A+B) + \frac{1}{2!}(A+B)^2 + \frac{1}{3!}(A+B)^3 + \dots \text{ (since } A \text{ and } B \text{ commute)} \\
&= e^{A+B}
\end{aligned}$$

Now since $(At)(-At) = (-At)(At) = -A^2t^2$ we can write: $e^{At}e^{-At} = e^{At-At} = e^0 = I$ and hence e^{-At} is the inverse of e^{At}.

(d) (First you need to show that $Ae^BA^{-1} = e^{ABA^{-1}}$). Therefore substituting $y = Qu$ as showed before we get:

$$\begin{aligned}
&\frac{dy}{dt} = \Lambda y \Rightarrow y = e^{\Lambda}y(0) \Rightarrow u = Q^{-1}e^{\Lambda t}Qu(0) \\
&= Q^{-1}e^{QAtQ^{-1}}Qu(0) = Q^{-1}Qe^{At}Q^{-1}Qu(0) = e^{At}u(0)
\end{aligned}$$