Mathematical Tools
Solutions to Problem Set 1

1 Sets and Functions

1. **Basic definitions:**

   (a) A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

   (b) In part (a), D is the domain and E is the co-domain.

   (c) If f is a function with domain D, then its graph is the set of ordered pairs

   $$\{(x, f(x)|x \in D\}$$

2. **Four ways of representing a function:**

   (a)
   - Verbally: Domain and co-domain are either implicit or specified in the verbal statement, for example: area of a circle with a **real** radius r.
   - Numerically: Domain is the set of values in the left column. Co-domain is the set of values in the right column.
   - Visually: Domain is the projection of the curve to the x-axis. Co-domain is the projection of the curve to the y-axis.
   - Algebraically: Usually specified in the following form: $f : D \to E$ where D is domain and E is co-domain

   (b) There is only one property that needs to be checked for a function. Each function maps $x \in D$ to one unique $f(x) \in E$. Translating this property into different representations we have:

   - Curve: Each line parallel to y-axis must intersect the curve at most once.
   - Numerical table: every time that x appears in the left column the same unique $f(x)$ has to appear in its corresponding cell in the right column.

- Algebraic formula: the mapping should map x to a unique value $f(x)$. An example of a rule that is not a function is

$$f(x) = \begin{cases} x-2 & x>1 \\ x+2 & x<2 \end{cases}$$

- Verbal rule: same property must be checked in the verbal rule.

(c) The example in the problem statement is a valid example.

3. **One-to-one functions:**

(a)
- Verbal: All pairs unequal pairs of (x_1, x_2) must yield unequal function values $(f(x_1), f(x_2))$
- Numerical: No repetition of the values in the right column.
- Visual: Every parallel line to the x-axis intersects with the function curve at most once.
- Algebraic: $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$.

(b)
- If $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$ (two different numbers can't have the same cube). Therefore, by definition 1, $f(x) = x^3$ is one-to-one.
- This function is not one-to-one because, for instance, $g(1) = g(-1) = 1$ and so 1 and -1 have the same output.

4. **Inverse of a function:**

(a)
- Graph: Reflecting the curve of the function about the line $y = x$.
- Table: Switching the right and left columns will provide the inverse of the function.

(b)

$$\begin{aligned} y &= x^3 + 2 \\ \Rightarrow x^3 &= y - 2 \\ \Rightarrow y &= \sqrt[3]{x-2} \end{aligned}$$

Finally, we interchange x and y:

$$\begin{aligned} x &= \sqrt[3]{y-2} \\ \Rightarrow f^{-1}(x) &= \sqrt[3]{x-2} \end{aligned}$$

2 Limits

1. (a) $\lim_{x \to 2} (8 - 3x + 12x^2) = 8 - 3(2) + 12(4) = 50$

   (b) $\lim_{t \to -3} (\frac{6+4t}{t^2+1}) = \frac{-6}{10} = \frac{3}{5}$

(c) $\lim_{x\to -5}(\frac{x^2-25}{x^2+2x-15}) = \lim_{x\to -5}\frac{(x-5)(x+5)}{(x-3)(x+5)} = \lim_{x\to -5}\frac{x-5}{x-3} = \frac{5}{4}$

(d) $\lim_{z\to 4}(\frac{\sqrt{z}-2}{z-4}) = \lim_{z\to 4}\frac{\sqrt{z}-2}{z-4}\frac{\sqrt{z}+2}{\sqrt{z}+2} = \lim_{z\to 4}\frac{z-4}{(z-4)(\sqrt{z}+2)} = \frac{1}{4}$

2. (a) 3
   (b) 1.05
   (c) Does not exist because from left the limit is -2 and from right the limit 0. If the right and left limits are not equal then the limit does not exist.

3. To prove this, we need to show that for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for all x with $|x - 5| < \delta$ we get $|f(x) - 25| < \epsilon$. Since we are considering the numbers in the neighbourhood of 5 we can restrict our attention to only positive numbers. Let us find the δ for a given ϵ:

$$|x-5| < \delta \iff -\delta < x-5 < \delta \iff 5-\delta < x < 5+\delta \iff$$
$$(5-\delta)^2 < x^2 < (5+\delta)^2 \iff 25+\delta^2-10\delta < f(x) < 25+\delta^2+10\delta$$
$$\iff \delta^2-10\delta < f(x)-25 < \delta^2+10\delta$$

If we select the δ in a way that $-\epsilon < \delta^2 - 10\delta < \delta^2 + 10\delta < \epsilon$ then we have:

$$-\epsilon < \delta^2-10\delta < f(x)-25 < \delta^2+10\delta < \epsilon \iff |f(x)-25| < \epsilon$$

And then the proof is complete. If we set $\delta = \frac{\epsilon}{N}$ then we need to choose N such that: $-\epsilon < (\frac{\epsilon}{N})^2 - 10\frac{\epsilon}{N} < (\frac{\epsilon}{N})^2 + 10\frac{\epsilon}{N} < \epsilon \iff -N^2 < \epsilon^2 - 10N < \epsilon^2 + 10N < N^2$. Notice that the two functions $N^2 - 10N - \epsilon$ and $N^2 - 10N + \epsilon^2$ are quadratic functions of N and we can choose N arbitrarily large so that both become positive. Therefore, both LHS and RHS of the inequality holds and the proof is complete.

3 Derivatives

1. Find the derivative of the following functions:
   (a) $f'(x) = 18x^2 - 9$
   (b) $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{8}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-\frac{3}{4}}$
   (c) $h'(x) = 8x - 8x^{-2}$
   (d) $g'(y) = 3y^2 - 4y - 8$

2. (a) We need to find all points x satisfying $f'(x) = 0$.

$$f'(x) = 6x^2 + 6x - 12 = 0$$
$$\Rightarrow (x+2)(x-1) = 0$$
$$\Rightarrow x = 1 \quad \text{or} \quad x = -2$$

(b) We need to find all points x satisfying $g'(x) = 0$.

$$\begin{aligned} g'(x) &= 2xe^{-2x} + x^2e^{-2x}(-2) = 0 \\ &\Rightarrow 2xe^{-2x}(1-x) = 0 \\ &\Rightarrow x(1-x) = 0 \\ &\Rightarrow x = 0 \quad \text{or} \quad x = 1 \end{aligned}$$

3. To find the absolute minimum and absolute maximum of a function defined on a closed interval we need to evaluate the function at the critical points and at the endpoint of the intervals, and then choose the biggest value for the maximum and the smallest for the minimum. First we find the critical points, i.e., points x satisfying $f'(x) = 0$. We have that:

$$\begin{aligned} f'(x) &= \frac{1}{x} - \frac{2}{x^2} = 0 \\ &\Rightarrow x - 2 = 0 \Rightarrow x = 2 \end{aligned}$$

Since $x = 2$ belongs to the interval $(1, e)$ it is a critical point. Now we evaluate $f(x) = \ln x + \frac{2}{x}$ at the critical point 2 and at the endpoints of the domain 1 and e. We have:

$$\begin{aligned} f(2) &= \ln 2 + \frac{2}{2} = \ln 2 + 1 \approx 1.69 \\ f(1) &= \ln 1 + \frac{2}{1} = 2 \\ f(e) &= \ln e + \frac{2}{e} = 1 + \frac{2}{e} \approx 1.73 \end{aligned}$$

Therefore f has the absolute maximum at $x = 1$ with value 2, and the absolute minimum at $x = e$ with value $1 + \frac{2}{e}$.

4 Euler's Number

Starting with the analytic function $g(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$ we will show that all the definitions are equivalent to $g(1)$. (a) by definition is equal to $g(1)$. In order to show that (c) is equaivalent to $g(1)$ we need to show that function g is equal to its derivative. And then by the uniqueness property of the solution to the ODEs we conclude that $f = g$ and therefore $f(1) = g(1)$. To show that $g'(x) = g(x)$ let us take the derivative of g:

$$\begin{aligned} g'(x) &= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots)' \\ &= 0 + 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \dots \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = g(x) \end{aligned}$$

We will finally show that (a) and (b) are equal. To show this we will use the binomial expansion:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

$$= x^n + nx^{n-1}y + \frac{n(n-1)}{2}x^{n-2}y^2 + \cdots + \frac{n(n-1)}{2}x^2y^{n-2} + nxy^{n-1} + y^n$$

$$\Rightarrow (1+\frac{1}{n})^n = 1 + n(\frac{1}{n}) + \frac{n(n-1)}{2}(\frac{1}{n})^2 + \cdots + (\frac{1}{n})^n$$

$$= \frac{1}{0!} + \frac{1}{1!} + (1-\frac{1}{n})\frac{1}{2!} + \cdots + (1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{n-1}{n})\frac{1}{n!}$$

But we know that for small k and large n all of the terms $1-\frac{1}{n}, 1-\frac{2}{n}, \ldots, 1\frac{k}{n}$ are close to one and hence their product is close to one. Therefore $(1-\frac{1}{n})(1-\frac{2}{n})\ldots(1-\frac{k}{n})\frac{1}{k!} \to \frac{1}{k!}$ as $n \to \infty$. Therefore:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

And hence all the definitions are equivalent.

5 Integrals

1. (a) Since $f(x) = \sqrt{1-x^2} \geq 0$, we can interpret this integral as the area under the curve $y = \sqrt{1-x^2}$ from 0 to 1. But, since $y^2 = 1-x^2$, we get $x^2 + y^2 = 1$, which shows that the graph of f is the quarter-circle with radius 1 in Fig. 1 . Therefore: $\int\limits_0^1 \sqrt{1-x^2}dx = \frac{1}{4}\pi(1)^2$

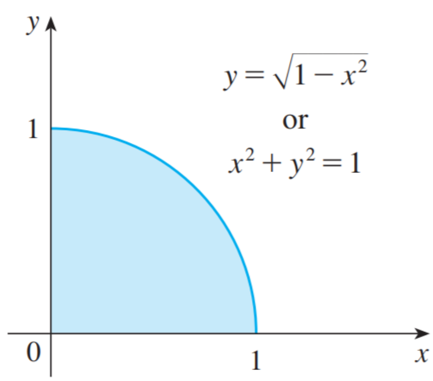

Figure 1:

(b) The graph of $y = x - 1$ is the line with slope 1 shown in Fig. 2. We compute the integral as the difference of the areas of the two triangles: $\int_0^3 (x-1)dx = A_1 - A_2 = \frac{1}{2}(2 \times 2) - \frac{1}{2}(1 \times 1) = 1.5$

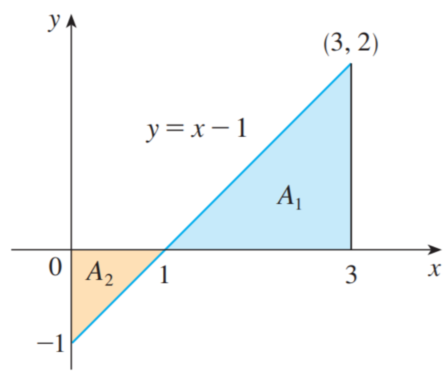

Figure 2:

2. (a) $\int (4x^6 - 2x^3 + 7x - 4)dx = \frac{4}{7}x^7 - \frac{2}{4}x^4 + \frac{7}{2}x^2 - 4x + c$

   (b) $\int (12)dx = 12x + c$

   (c) $\int \frac{4}{x^2} + 2 - \frac{1}{8x^3} = \int 4x^{-2} + 2 - \frac{1}{8}x^{-3}dx = 4\frac{1}{-1}x^{-1} + 2x - \frac{1}{8}\frac{1}{-2}x^{-2} + c$

   (d) $\int \frac{x^4 - \sqrt[3]{x}}{6\sqrt{x}} = \int \frac{x^4}{6x^{\frac{1}{2}}} - \frac{x^{\frac{1}{3}}}{6x^{\frac{1}{2}}}dx = \int \frac{1}{6}x^{\frac{7}{2}} - \frac{1}{6}x^{\frac{-1}{6}}dx = \frac{1}{27}x^{\frac{9}{2}} - \frac{1}{5}x^{\frac{5}{6}} + c$

3. We have $i(t) = C\frac{dv(t)}{dt}$ and $i(t) = e^t + t^2$. Therefore to compute the membrane voltage at time t we only need to take the integral from both sides. Therefore: $v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v(0) = e^t + \frac{1}{3}t^3 + v(0) - 1$

References

[1] Online math tutorials and notes.

[2] Calculus Early Transcendentals [Ch 1-3].

[3] Capacitor Example