Mathematical Tools
Solutions to Problem Set 9

1 Convolution and Fourier Series [1]

1.1 Linear Time-Invariant Systems

1. **Convolution** is a mathematical operation on two functions that yields a third function, showing how the shape of one function is modulated by the other. The convolution operation $(u * h)[t]$ is defined as follows:

$$x[t] = \sum_{\tau=-\infty}^{\infty} u[\tau]h[t-\tau] \equiv (u * h)[t]$$

   The output in response to a unit impulse is termed the **impulse response**, $h[t]$.

   Linear time-invariant systems (LTI systems) are a class of systems used in signals and systems that are both linear and time-invariant. Linear systems are systems whose outputs for a linear combination of inputs are the same as a linear combination of individual responses to those inputs. Time-invariant systems are systems where the output does not depend on when an input was applied.

2. The inverse system for a continuous-time accumulation (or integration) is a differentiator. This can be verified because

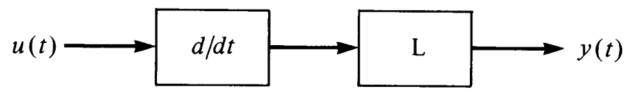
$$\frac{d}{dt}\Big[\int_{-\infty}^{t} x(\tau)d\tau\Big] = x(t)$$

   Therefore, the input-output relation for the inverse system is

$$x(t) = \frac{dy(t)}{dt}$$

3. By using the commutative property of convolution we can exchange the two systems to yield the system in the following figure.

[1] The problems and solutions are taken in literal words from [1]



Now we note that the input to system L is

$$\frac{du(t)}{dt} = \delta(t)$$

so $y(t)$ is the impulse response of system L. From the original diagram,

$$\frac{ds(t)}{dt} = y(t)$$

Therefore,

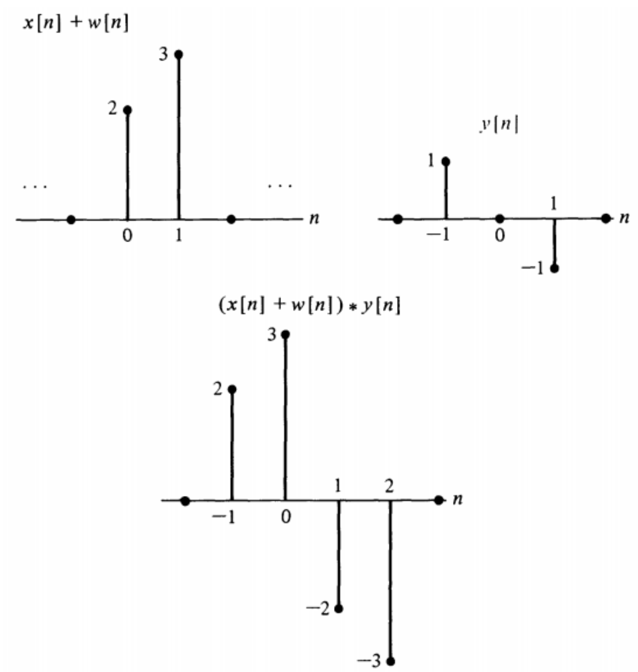
$$h(t) = \frac{ds(t)}{dt}$$

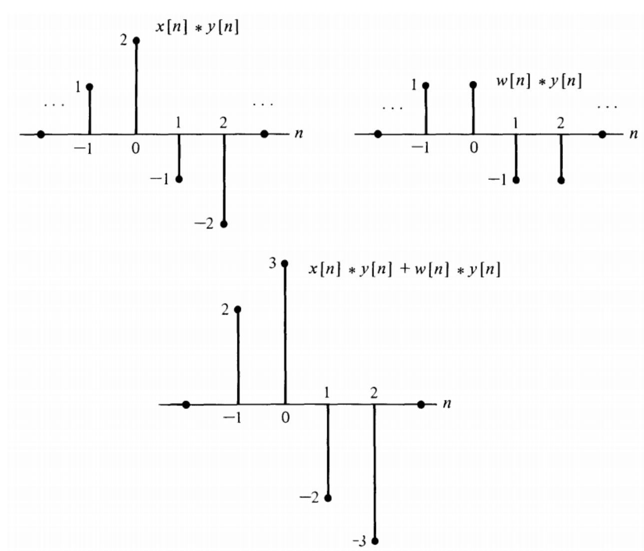
4. (a) By definition, an inverse system cascaded with the original system is the identity system, which has an impulse response $h(t) = \delta(t)$. Therefore, if the cascaded system has an input of $b(t)$, the output $w(t) = h(t) = \delta(t)$.

   (b) Because the system is an identity system, an input of $x(t)$ produces an output $w(t) = x(t)$.

5. (a) The following signals are obtained by addition and graphical convolution

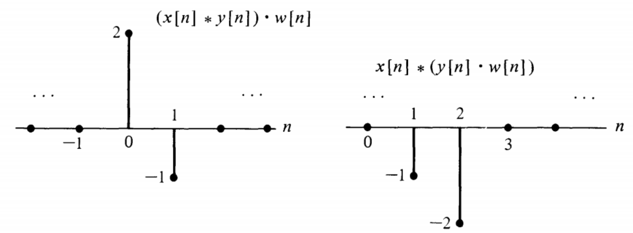
$$(x[n] + w[n]) * y[n]$$
$$x[n] * y[n] + w[n] * y[n]$$





Therefore, the distributive property $(x + w) * y = x * y + w * y$ is verified

(b) The following figure shows the required convolutions and multiplications.



Note, therefore, that $(x[n] * y[n]) - w[n] \neq x[n] * (y[n] - w[n])$.

6. Consider

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

(a)

$$\begin{aligned} y'(t) &= \int_{-\infty}^{\infty} x'(t-\tau)h(\tau)d\tau = x'(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)h'(t-\tau)d\tau = x(t) * h'(t) \end{aligned}$$

where the primes denote $\frac{d}{dt}$.

(b)

$$\begin{aligned} y(t) &= x(t) * h(t), \\ y(t) &= x(t) * u_{-1}(t) * u_1(t) * h(t), \\ y(t) &= \int_{-\infty}^{t} x(\tau)d\tau * h'(t) \end{aligned}$$

(c)

$$\begin{aligned} y(t) &= x(t) * h(t), \\ y(t) &= x(t) * u_1(t) * h(t) * u_{-1}(t), \\ y(t) &= \int_{-\infty}^{t} x'(\tau) * h(\tau)d\tau \end{aligned}$$

(d)

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= x(t) * u_1(t) * h(t) * u_{-1}(t), \\ y(t) &= x'(t) * \int_{-\infty}^{t} h(\tau)d\tau \end{aligned}$$

7. We are given that $y[n] = x[n] * h[n]$

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$|y[n]| = |\sum_{k=-\infty}^{\infty} x[n-k]h[k]|$$

$$\max\{|y[n]|\} = \max\{|\sum_{k=-\infty}^{\infty} x[n-k]h[k]|\}$$

$$\leq \max \sum_{k=-\infty}^{\infty} |x[n-k]h[k]|$$

$$= \max\{|x[n]|\} \sum_{k=-\infty}^{\infty} |h[k]|$$

We see from the inequality

$$\max\{|y[n]|\} \leq \max\{|x[n]|\} \sum_{k=-\infty}^{\infty} |h[k]|$$

that $\sum_{k=-\infty}^{\infty} |h[k]| \leq 1 \Rightarrow \max\{|y[n]|\} \leq \max\{|x[n]|\}$. This means that $\sum_{k=-\infty}^{\infty} |h[k]| \leq 1$ is a sufficient condition. It is necessary becasue some $x[n]$ always exists that yields $y[n] = \sum_{k=-\infty}^{\infty} |h[k]|$. ($x[n]$ consists of a sequence of $+1$'s and -1's). Therefore, since $\max\{x[n]\} = 1$, it is necessary that $\sum_{k=-\infty}^{\infty} |h[k]| \leq 1$ to ensure that $y[n] \leq \max\{|x[n]|\} = 1$.

1.2 Fourier Series and Fourier Transform

1. **Fourier Series** is writing functions as scales of exponential functions.

   For a signal $x(t)$ its **Fourier Transform** is defined as:

$$\hat{x}[n] = \frac{1}{T} \int_0^T x(t) e^{-in\omega t} dt$$

2. (a)

$$x(t) = \sin\left(10\pi t + \frac{\pi}{6}\right)$$
$$= \frac{e^{\frac{j\pi}{6}}}{2j} e^{j2\pi t5} - \frac{e^{-\frac{j\pi}{6}}}{2j} e^{-j2\pi t5}$$

   We choose w_0, the fundamental frequency to be 2π.

$$x(t) = \sum_k a_k e^{jkw_0 t}$$

where

$$a_5 = \frac{e^{\frac{j\pi}{6}}}{2j}, \quad a_{-5} = \frac{-e^{-\frac{j\pi}{6}}}{2j}$$

Otherwise $a_k = 0$.

(b)

$$\begin{aligned} x(t) &= 1 + \cos(2\pi t) \\ &= 1 + \frac{e^{j2\pi t}}{2} + \frac{-j2\pi t}{2} \end{aligned}$$

For $w_0 = 2\pi$, $a_{-1} = a_1 = \frac{1}{2}$, and $a_0 = 1$. All other $a_k's$ are 0.

(c)

$$\begin{aligned} x(t) &= [1 + \cos(2\pi t)]\left[\sin\left(10\pi t + \frac{\pi}{6}\right)\right] \\ &= \sin\left(10\pi t + \frac{\pi}{6}\right) + \cos(2\pi t)\sin\left(10\pi t + \frac{\pi}{6}\right) \end{aligned}$$

$$= \left(\frac{e^{\frac{j\pi}{6}}}{2j}e^{j2\pi t5} - \frac{e^{-\frac{j\pi}{6}}}{2j}e^{-j2\pi t5}\right) + (\frac{1}{2}e^{j2\pi t} + \frac{1}{2}e^{-j2\pi t})\left(\frac{e^{\frac{j\pi}{6}}}{2j}e^{j2\pi t5} - \frac{e^{-\frac{j\pi}{6}}}{2j}e^{-j2\pi t5}\right)$$

$$= \frac{e^{\frac{j\pi}{6}}}{2j}e^{j2\pi t5} - \frac{e^{-\frac{j\pi}{6}}}{2j}e^{-j2\pi t5} + \frac{e^{\frac{j\pi}{6}}}{4j}e^{j2\pi t6} - \frac{e^{-\frac{j\pi}{6}}}{4j}e^{-j2\pi t4} + \frac{e^{\frac{j\pi}{6}}}{4j} - \frac{e^{-\frac{j\pi}{6}}}{4j}$$

Therefore

$$x(t) = \sum_k a_k e^{jkw_0 t}$$

where $w_0 = 2\pi$.

$$\begin{aligned} a_4 &= \frac{e^{\frac{j\pi}{6}}}{4j} \quad a_{-4} = \frac{-e^{-\frac{j\pi}{6}}}{4j} \\ a_5 & \frac{e^{\frac{j\pi}{6}}}{2j} \quad a_{-5} = \frac{-e^{-\frac{j\pi}{6}}}{2j} \\ a_6 &= \frac{e^{\frac{j\pi}{6}}}{4j} \quad a_{-6} = \frac{-e^{-\frac{j\pi}{6}}}{4j} \end{aligned}$$

All other a_k's are 0.

3. (a) Note that the period is $T_0 = 6$. Fourier coefficients are given by

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jkw_0 t}dt$$

We take $w_0 = \frac{2\pi}{T_0} = \frac{\pi}{3}$. Choosing the period of integration as -3 to 3, we have

$$a_k = \frac{1}{6}\int_{-2}^{-1} e^{-jk(\pi/3)t}dt - \frac{1}{6}\int_{1}^{2} e^{-jk(\pi/s)t}dt$$

$$= \frac{1}{6}\frac{1}{-jk(\pi/3)}e^{-jk(\pi/3)t}\Big|_{-2}^{-1} - \frac{1}{6}\frac{1}{-jk(\pi/3)}e^{-jk(\pi/3)t}\Big|_{1}^{2}$$

$$\frac{\cos(2\pi/3)k}{j\pi k} - \frac{\cos(\pi/3)k}{j\pi k}$$

Therefore

$$x(t) = \sum_k a_k e^{jkw_0 t}, \quad w_0 = \frac{\pi}{3}$$

and

$$a_k = \frac{\cos(2\pi/3)k - \cos(\pi/3)k}{j\pi k}$$

Note that $a_0 = 0$, as can be determined either by applying L'Hopital's rule or by noting that

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)dt$$

(b) The period is $T_0 = 2$ with $w_0 = \frac{2\pi}{2} = \pi$. The Fourier coefficients are

$$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jkw_0 t}dt$$

Choosing the period of integration as $-\frac{1}{2}$ to $\frac{3}{2}$ we have

$$a_k = \frac{1}{2}\int_{-1/2}^{3/2} x(t)e^{-jkw_0 t}dt$$

$$= \frac{1}{2}\int_{-1/2}^{3/2} [\delta(t) - 2\delta(t-1)]e^{-jkw_0 t}dt$$

$$= \frac{1}{2} - e^{-jkw_0} = \frac{1}{2} - (e^{-j\pi})^k$$

Therefore,

$$a_0 = -\frac{1}{2}, \quad a_k = \frac{1}{2} - (-1)^k$$

4. (a) (i) and (ii) we have

$$x(t - \frac{T}{2}) = -x(t)$$

which means odd harmonics. Since $x(t)$ is real and even, the waveform has real coefficients.

(b) (i) and (ii)

$$-x(t) = x(t - \frac{T}{2})$$

which means odd harmonics. Since $x(t)$ is real and odd, the waveform has imaginary coefficients.

(c) (i)

$$-x(t) = x(t - \frac{T}{2})$$

which means odd harmonics. Also, $x(t)$ is neither even nor odd.

5. The Fourier transform of $x(t)$ is

$$X(w) = \int -\infty^{\infty} x(t)e^{-jwt}dt = \int_{-\infty}^{\infty} e^{-t/2}u(t)e^{-jwt}dt$$

Since $u(t) = 0$ for $t < 0$, we can write the above equation as

$$X(w) = \int_{0}^{\infty} e^{(-1/2+jw)t}dt = \frac{2}{1 + j2w}$$

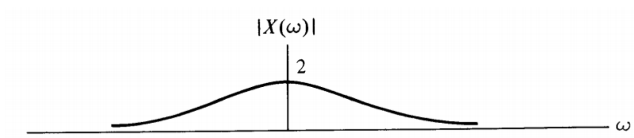
It is convenient to write $X(w)$ in terms of its real and imaginary parts:

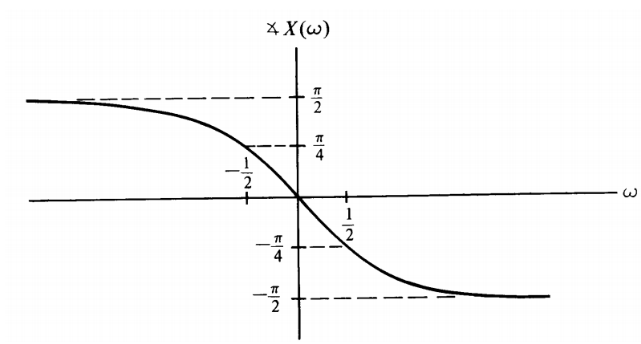
$$X(w) = \frac{2}{1+j2w}\frac{1-j2w}{1-j2w} = \frac{2-j4w}{1+4w^2} = \frac{2}{1+4w^2} - j\frac{4w}{1+4w^2}$$

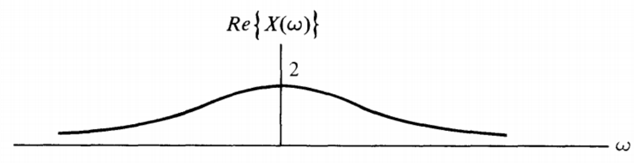
Magnitude of $X(w) = \frac{2}{\sqrt{1+4w^2}}$

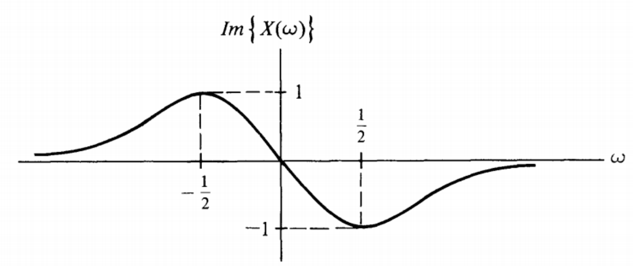
$$X(w) = \tan^{-1}(-2w) = -\tan^{-1}(2w)$$

$$\mathrm{Re}\{X(w)\} = \frac{2}{1+4w^2} \quad \mathrm{Im}\{X(w)\} = \frac{-4w}{1+4w^2}$$









6. The output of a discrete-time linear, time-invariant system is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

where $h[n]$ is the impulse response and $x[n]$ is the input. By substitution, we have the following.

(a)

$$\begin{aligned} y[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k e^{j\pi(n-k)} = e^{j\pi n} \sum_{k=0}^{\infty} \left(\frac{e^{-j\pi}}{2}\right)^k \\ &= \frac{e^{j\pi n}}{1 - \frac{1}{2}e^{-j\pi}} = \frac{2}{3}(-1)^n \end{aligned}$$

(b)

$$\begin{aligned} y[n] &= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k e^{j[\pi(n-k)/4]} = e^{j(\pi n/4)} \sum_{k=0}^{\infty} \left[\frac{e^{-j(\pi/4)}}{2}\right]^k \\ &\frac{e^{j(\pi n/4)}}{1 - \frac{2}{2}e^{-j(\pi/4)}} \end{aligned}$$

(c)

$$y[n] = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \left[\frac{1}{2}e^{j(\pi/8)}e^{j[\pi(n-k)/4]} + \frac{1}{2}e^{-j(\pi/8)}e^{-j[\pi(n-k)/4]}\right]$$

$$= \frac{1}{2}e^{j(\pi/8)}e^{j(\pi n/4)}\sum_{k=0}^{\infty}\left[\frac{e^{-j(\pi/4)}}{2}\right]^k + \frac{1}{2}e^{-j(\pi/8)}e^{-j(\pi n/4)}\sum_{k=0}^{\infty}\left[\frac{e^{j(\pi/4)}}{2}\right]^k$$

$$= \frac{\frac{1}{2}e^{j[(\pi/8)+(\pi n/4)]}}{1-\frac{1}{2}e^{-j(\pi/4)}} + \frac{\frac{1}{2}e^{-j[(\pi/8)+(\pi n/4)]}}{1-\frac{1}{2}e^{j(\pi/4)}}$$

$$= \frac{\cos[(\pi/4)n+\pi/8] - \frac{1}{2}\cos[(\pi/4)n+(3\pi/8)]}{\frac{5}{4}-\cos(\pi/4)}$$

7. (a)

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] - e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} \frac{1}{4}^n u[n]e^{-j\Omega n} = \sum_{n=0}^{\infty}(\frac{1}{4}e^{-j\Omega})^n = \frac{1}{1-\frac{1}{4}e^{-j\Omega}}$$

Here we have used the fact that

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad \text{for} \quad |a| < 1$$

(b)

$$x[n] = (a^m \sin\Omega_0 n)u[n]$$

We can use the modulation property to evaluate this signals. Since

$$\sin\Omega_0 n \iff \frac{2\pi}{2j}[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)]$$

periodically repeated, then

$$X(\Omega) = \frac{1}{2j}\left[\frac{1}{1-ae^{-j(\Omega-\Omega_0)}} - \frac{1}{1-ae^{-j(\Omega+\Omega_0)}}\right]$$

periodically repeated.

(c)

$$X(\Omega) = \sum_{n=0}^{3} e^{-j\Omega n} = \frac{1-e^{-j4\Omega}}{1-e^{-j\Omega}}$$

using the identity

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$$

(d)

$$x[n] = \frac{1}{4}^{n} u[n+2] = \frac{1}{4}^{n+2} \frac{1}{4}^{-2} u[n+2] = 16\frac{1}{4}^{n+2} u[n+2]$$

we know that

$$16\frac{1}{4}^{n}[n] \iff \frac{16}{1 - \frac{1}{4}e^{-j\Omega}}$$

so

$$16\frac{1}{4}^{n+2} u[n+2] \iff \frac{16e^{j2\Omega}}{1 - \frac{1}{4}e^{-j\Omega}}$$

References

[1] Alan V. Oppenheim. Signals and systems course. https://ocw.mit.edu/resources/res-6-007-signals-and-systems-spring-2011/, 2011.