Mathematical Tools
Problem Set 4

1 Linear Algebra

1.1 Linear Transformations

1. Define a linear transformation. What properties need to hold for a transformation to be linear?

2. Which of the following transformations are linear? Why?

   (a) $T(\vec{v}) = \vec{a} \cdot \vec{v} = \sum_{i=1}^{n} a_i v_i$.
   (b) $T(v) = \|v\|$.
   (c) T is the transformation that rotates every vector by $30°$. The "domain" is the xy plane (all input vectors v). The "range" is also the xy plane (all rotated vectors $T(v)$).

3. Which of these transformations are not linear? The input is $v = (v_1, v_2)$:

   (a) $T(v) = (v_2, v_1)$.
   (b) $T(v) = (v_1, v_1)$.
   (c) $T(v) = (0, v_1)$.
   (d) $T(v) = (0, 1)$.
   (e) $T(v) = v_1 - v_2$.
   (f) $T(v) = v_1 v_2$.

4. For these transformations of $V = \mathbb{R}^2$ to $W = \mathbb{R}^2$, find $T(T(v))$. Is this transformation T^2 linear?

   (a) $T(v) = -v$.
   (b) $T(v) = v + (1, 1)$.
   (c) $T(v) = 90°\text{rotation} = (-v_2, v1)$.
   (d) $T(v) = \text{projection} = (\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2})$.

5. The "cyclic" transformation T is defined by $T(v_1, v_2, v_3) = (v_2, v_3, v_1)$. What is $T(T(v))$? What is $T^3(v)$? What is $T^{100}(v)$? Apply T a hundred times to v. In each case find the matrix corresponding to the transformation.

6. Why does every linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ take squares to parallelograms? Rectangles also go to parallelograms.

1.2 Eigenvalues and Eigenvectors

1. Define eigenvalues and eigenvectors.

2. Find the eigenvalues and eigenvectors of the following matrices:

   (a) The projection matrix $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$.

   (b) The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3. (a) The **"characteristic polynomial"** of a matrix A is defined as $P(\lambda) = \det(A - \lambda I)$. Show that the eigenvalues of matrix A are the roots of its characteristic polynomial. Use this to find the eigenvalues of $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

   (b) Show that for each eigenvalue λ its corresponding eigenvector can be found by solving $A - \lambda I = 0$. Use this to find the eigenvectors of A matrix in part (a).

4. Show that:

   (a) The product of the n eigenvalues of an n by n matrix equals the determinant.

   (b) The sum of the n eigenvalues of an n by n matrix equals the sum of the n diagonal entries.

5. Why do the eigenvalues of a triangular matrix lie on its diagonal?

6. What do you do to the equation $Ax = \lambda x$, in order to prove the following?

   (a) λ^2 is an eigenvalue of A^2.

   (b) λ^{-1} is an eigenvalue of A^{-1}.

   (c) $\lambda + 1$ is an eigenvalue of $A + I$.

7. (a) Show that the eigenvalues of A equal the eigenvalues of A^T.

   (b) Show by an example that the eigenvectors of A and A^T are *not* the same.

8. **Eigenvalue Decomposition:** Let A be the square $n \times n$ connectivity matrix between n neurons with n linearly independent eigenvectors q_i (where $i = 1, \ldots, n$).

   (a) Show that A can be factorized as
   $$A = Q\Lambda Q^{-1}$$
   where Q is the square $n \times n$ matrix whose i-th column is the eigenvector q_i of A, and Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues, $\Lambda_{ii} = \lambda_i$. Note that only diagonalizable matrices can be factorized in this way. For example, the singular matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ cannot be diagonalized.

   (b) Using this decomposition show that for a fully connected network of neurons that follows linear dynamics described by $\frac{dx}{dt} = Ax$ the following linear transformation decouples all the variables in the target space $y = Q^{-1}x$.

   (c) For a symmetric matrix A show that the eigenvectors can be chosen such that they are orthogonal to each other and therefore we have the following eigenvalue decomposition: $A = Q\Lambda Q^T$ where Q is an orthogonal matrix.

1.3 Linear Dynamics

1. Given two neurons and their connectivity matrix $A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$ the activity of the neurons are described by the following 2D differential equation $\frac{dx}{dt} = Ax$ where a is a parameter that determines the strength of the self-connection of neuron 1. Solve this linear system and graph the phase portrait as a varies from $-\infty$ to $+\infty$, showing the qualitatively different cases.

2. For an autonomous network of neurons the firing rates of neurons are described by the solution of $\frac{du}{dt} = Au$, there is a fundamental question. Does the solution approach $u = 0$ as $t \to \infty$? Is the network stable, by dissipating energy? Show that A is stable and $u(t) \to 0$ when all eigenvalues have negative real parts. More specifically, for a 2 by 2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that the system is stable if A passes the two following tests:

   - The trace of the matrix must be negative.
   - The determinant of the matrix must be positive.

3. The exponential of a matrix A is defined as:
   $$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$$

Show that:

(a) $\frac{de^{At}}{dt} = A + A^2t + \frac{1}{2}A^3t^2 + \cdots = Ae^{At}$.

(b) Eigenvalues of e^{At} are $e^{\lambda t}$ where λ is an eigenvalue of A. For this, you need to show that $(I + At + \frac{1}{2}(At)^2 + \ldots)x = (1 + \lambda t + \frac{1}{2}(\lambda t)^2 + \ldots)$.

(c) e^{At} always has the inverse e^{-At}.

(d) The solution to $\frac{d\vec{u}}{dt} = A\vec{u}$ is given by

$$e^{At}\vec{u}(0)$$