Mathematical Tools
Solutions to Problem Set 8

1 Nonlinear Dynamics

1.1 Linearization

1. The points of $\frac{dx}{dt}$ that intersect x have a rate of change of 0. These points are called **fixed points** because at these points, no movement occurs. Any initial point x that is not at these fixed points will result in movement toward or away from the fixed points, usually represented by the arrows in the **phase line** diagram. The equivalent to phase line in higher dimension is called **phase plane** in which the arrows live in a higher dimensional space, and determine the trajectories along the variables of the dynamical system.

2. (a) We have an equilibrium point at the origin. The linearized system is now:

$$X' = \begin{bmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{bmatrix} X$$

which has eigenvalues $\frac{1}{2} \pm i$. All solutions of this system spiral away from the origin and toward ∞ in the counterclockwise direction, as is easily checked.

(b) Solving the nonlinear system looks formidable. However, if we change to polar coordinates, the equations become much simpler. We compute:

$$\begin{cases} r'\cos\theta - r(\sin\theta)\theta' = x' = \frac{1}{2}(r - r^3)\cos\theta - r\sin\theta \\ r'\sin\theta + r(\cos\theta)\theta' = y' = \frac{1}{2}(r - r^3)\sin\theta + r\cos\theta \end{cases}$$

from which we conclude, after equating the coefficients of $\cos\theta$ and $\sin\theta$,

$$\begin{cases} r' = \frac{r(1-r^2)}{2} \\ \theta' = 1 \end{cases}$$

We can now solve this system explicitly, since the equations are decoupled. Rather than do this, we will proceed in a more geometric fashion. From the equation $\theta = 1$, we conclude that all nonzero solutions spiral around the origin in the counterclockwise direction. From the first equation, we see that solutions do not spiral toward ∞. Indeed, we have $r = 0$ when $r = 1$, so all solutions that start on the unit circle stay there forever and move periodically around the circle. Since $r > 0$ when $0 < r < 1$, we conclude that nonzero solutions inside the circle spiral away from the origin and toward the unit circle. Since $r < 0$ when $r > 1$, solutions outside the circle spiral toward it.

3. The linearized system is

$$\begin{cases} x' = -y \\ y' = x \end{cases}$$

so we see that the origin is a center and all solutions travel in the counterclockwise direction around circles centered at the origin with unit angular speed.

In polar coordinates, this system reduces to

$$\begin{cases} r' = \epsilon r^3 \\ \theta' = -1 \end{cases}$$

Thus when $\epsilon > 0$, all solutions spiral away from the origin, whereas when $\epsilon < 0$, all solutions spiral toward the origin. The addition of the nonlinear terms, no matter how small near the origin, changes the linearized phase portrait dramatically; we cannot use linearization to determine the behavior of this system near the equilibrium point.

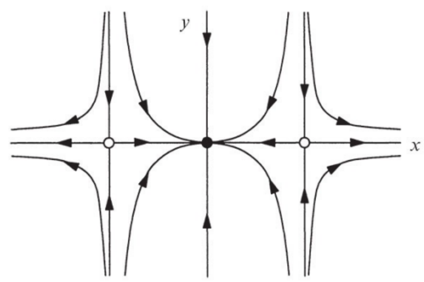
4. Fixed points occur where and simultaneously. Hence we need $x = 0$ or $x = \pm 1$, and $y = 0$. Thus, there are three fixed points: $(0, 0)$, $(1, 0)$, and $(1, 0)$. The Jacobian matrix at a general point (x, y) is:

$$A = \begin{bmatrix} \frac{\partial \dot{x}}{x} & \frac{\partial \dot{x}}{y} \\ \frac{\partial \dot{y}}{x} & \frac{\partial \dot{y}}{y} \end{bmatrix} = \begin{bmatrix} -1 + 3x^2 & 0 \\ 0 & -2 \end{bmatrix}$$

Next we evaluate A at the fixed points. At $(0, 0)$, we find $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ so $(0, 0)$ is a stable node. At $(\pm 1, 0)$, $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$, so both $(1, 0)$ and $(-1, 0)$ are saddle points.

Now because stable nodes and saddle points are not borderline cases, we can be certain that the fixed points for the full nonlinear system have been predicted correctly.

his conclusion can be checked explicitly for the nonlinear system, since the x and y equations are uncoupled; the system is essentially two independent first-order systems at right angles to each other. In the y-direction, all trajectories decay exponentially to $y = 0$. In the x-direction, the trajectories are attracted to $x = 0$ and repelled from $x = \pm 1$. The vertical lines $x = 0$ and $x = \pm 1$ are invariant, because on them; hence any trajectory that starts on these lines stays on them forever. Similarly, $y = 0$ is an invariant horizontal line. As a final observation, we note that the phase portrait must be symmetric in both the x and y axes, since the equations are invariant under the transformations $x \to -x$ and $y \to -y$. Putting all this information together, we arrive at the phase portrait shown below:



2 Stability

1. When the movement of a function is always toward the fixed point, that fixed point is **stable**. When the movement of a function is always away from the fixed point, that fixed point is **unstable**. When the movement of a function is towards the fixed point on one side and away from the fixed point on the other side, that fixed point is **half-stable**.

   Around **stable limit cycles**, all trajectories approach the limit cycle asymptotically. By default, this means that an unstable fixed point $x*_u$ exists inside every stable limit cycle. Around **unstable limit cycles**, all trajectories move away from the limit cycle, meaning that by default, there exists some stable fixed point $x*_s$ inside every unstable limit cycle. Finally, on one side of **half-stable limit cycles**, trajectories approach the limit cycle, while on the other side, trajectories move away from the limit cycle. Thus, either a stable or unstable fixed point must exist within every half-stable limit cycle.

   Limit cycles are "like a fixed point, but they oscillate." More formally, limit cycles are an isolated closed trajectory, wherein adjacent trajectories are not closed.

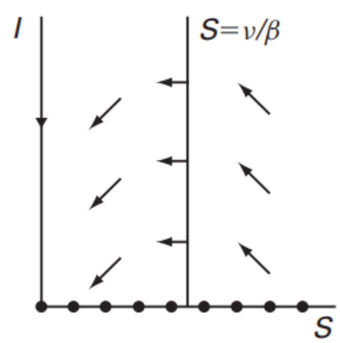
   The x_i-**nullcline** is the set of points where the derivative $\dot{x_i} = 0$ for the two-dimensional function, $\dot{\vec{x}} = f(\vec{x})$.

2. The system is a gradient system with potential function $V(x,y) = -x\sin y$, since $\dot{x} = \frac{\partial V}{\partial x}$ and $\dot{y} = \frac{\partial V}{\partial y}$ here we apply the following theorem: *Closed orbits are impossible in gradient systems.*

3. Suppose that there were a periodic solution $x(t)$ of period T. Consider the energy function $E(x,\dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$. After one cycle, x and $\dot{x}$ return to their starting values, and therefore $\Delta E = 0$ around any closed orbit. On the other hand, $\Delta E = \int_0^T \dot{E}dt$. If we can show this integral is nonzero, we've reached a contradiction. Note that $\dot{E} = \dot{x}(x + \ddot{x}) = \dot{x}(-\dot{x}^3) = -\dot{x}^4 \leq 0$. Therefore $\Delta E = \int_0^T (\dot{x})^4 dt \leq 0$ with equality only if $\dot{x} \equiv 0$. But $\dot{x} \equiv 0$ would mean the trajectory is a fixed point, contrary to the original assumption that it's a closed orbit. Thus ΔE is strictly negative, which contradicts $\Delta E = 0$. Hence there are no periodic solutions.

4. Consider $V(x,y) = x^2 + ay^2$, where a is a parameter to be chosen later. Then $\dot{V} = 2x\dot{x} + 2ay\dot{y} = 2x(-x+4y) + 2ay(-x-y^3) = -2x^2 + (8-2a)xy - 2ay^4$. If we choose $a = 4$, then xy term disappears and $\dot{V} = -2x^2 - 8y^4$. By inspection $V > 0$ and $\dot{V} < 0$ for all $(x,y) \neq (0,0)$. Hence $V = x^2 + 4y^2$ is a Liapunov function and so there are no closed orbits. In fact, all trajectories approach the origin as $t \to \infty$.

5. The S-nullclines are given by the S and I axes. On the I-axis, we have $I = -\nu I$, so solutions simply tend to the origin along this line. The I-nullclines are $I = 0$ and the vertical line $S = \frac{\nu}{\beta}$. Hence we have the nullcline diagram depicted below.



We can explicitly compute a function that is constant along solution curves. Note that the slope of the vector field is a function of S alone:

$$\frac{I'}{S'} = \frac{\beta SI - \nu I}{\beta SI} = -1 + \frac{\nu}{\beta S}$$

Hence we have

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt} = -1 + \frac{\nu}{\beta S}$$

which we can immediately integrate to find

$$I = I(S) = -S + \frac{\nu}{\beta} \log S + \text{constant}$$

Hence the function $I + S - \frac{\nu}{\beta} \log S$ is constant along solution curves. It then follows that there is a unique solution curve connecting each equilibrium point in the interval $\frac{\nu}{\beta} < S < \infty$ to one in the interval $0 < S < \frac{\nu}{\beta}$.

3 Bifurcations and Chaos

1. A **saddle-node bifurcation** is one in which the stable and unstable fixed point trajectories move toward the intersection in a parabolic manner they shape an unstable fixed point, this change in behavior in the dynamical system is called saddle-node bifurcation.

   A **transcritical bifurcation** is one in which a fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point as the parameter is varied.

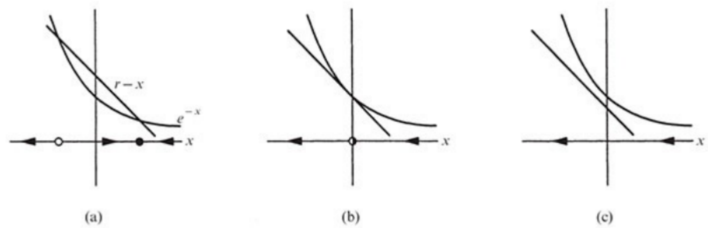
   A **pitchfork bifurcation** is a particular type of local bifurcation where the system transitions from one fixed point to three fixed points.

2. There is no strict definition for **chaos**. It is deterministic, but not predictable: we can write down a system of coupled nonlinear equations that describe chaos, but we do not know what will happen to them in the long-run. To be more precise, Strogatz defines chaos as:

   - deterministic (*i.e.* no random variables)
   - aperiodic longterm behavior (having long trajectories but never approaching a fixed point or limit cycle)
   - sensitivity to initial conditions (having exponential divergence of nearby trajectories)

3. The fixed points satisfy $f(x) = r - x - e^{-x} = 0$. But now we run into a difficulty, we can't find the fixed points explicitly as a function of r. Instead we adopt a geometric approach. One method would be to graph the function $f(x) = r - x - e^{-x}$ for different values of r, look for its roots x^*, and then sketch the vector field on the x-axis. This method is fine, but there's an easier way. The point is that the two functions $r - x$ and e^{-x} have much more familiar graphs than their difference $r - x - e^{-x}$. So we plot $r - x$ and e^{-x} on the same picture. Where the line $r - x$ intersects the curve e^{-x}, we have $r - x = e^{-x}$ and so $f(x) = 0$. Thus, intersections of the line and the curve correspond to fixed points for the system. This picture also allows us to read off the direction of flow on the x-axis: the flow is to the right where the line lies above the curve, since $r - x > e^{x}$ and

therefore $\dot{x} > 0$. Hence, the fixed point on the right is stable, and the one on the left is unstable. Now imagine we start decreasing the parameter r. The line $r - x$ slides down and the fixed points approach each other. At some critical value $r = r_c$, the line becomes tangent to the curve and the fixed points coalesce in a saddle-node bifurcation. For r below this critical value, the line lies below the curve and there are no fixed points.



4. Note that $x = 0$ is a fixed point for all (a, b). This makes it plausible that the fixed point will bifurcate transcritically, if it bifurcates at all. For small x, we find

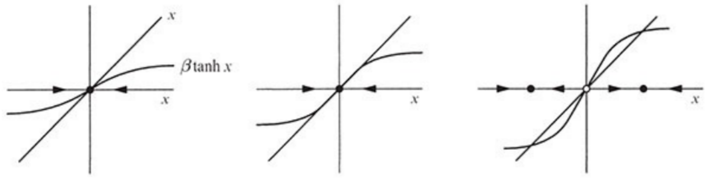
$$1 - e^{-bx} = 1 - [1 - bx + \frac{1}{2}b^2x^2 + \mathcal{O}(x^3)] = bx - \frac{1}{2}b^2x^2 + \mathcal{O}(x^3)$$

and so

$$\dot{x} = x - a(bx - \frac{1}{2}b^2x^2) + \mathcal{O}(x^3) = (1 - ab)x + (\frac{1}{2}ab^2)x^2 + \mathcal{O}(x^3)$$

Hence a transcritical bifurcation occurs when $ab = -1$; this is the equation for the bifurcation curve. The nonzero fixed point is given by the solution of $1 - ab + (\frac{1}{2}ab^2)x \approx 0$, i.e. $x \approx \frac{2(ab-1)}{ab^2}$. This formula is approximately correct only if x^* is small, since our series expansions are based on the assumption of small x. Thus the formula holds only when ab is close to 1, which means that the parameters must be close to the bifurcation curve.

5. The graphs of $y = x$ and $y = \beta \tanh x$ are shown in the following figure; their intersections correspond to fixed points. The key thing to realize is that as β increases, the tanh curve becomes steeper at the origin (its slope there is β). Hence for $\beta < 1$ the origin is the only fixed point. A pitchfork bifurcation occurs at $\beta = 1$, $x^* = 0$, when the tanh curve develops a slope of 1 at the origin. Finally, when $\beta > 1$, two new stable fixed points appear, and the origin becomes unstable.



$\beta < 1$ $\beta = 1$ $\beta > 1$